Supplement to

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~~Information Reviewed and Reaffirmed~~

~~March 1956~~ INFORMATION REVIEWED AND REAFFIRMED 1962

No. 1583-C

This Report is One of a Series
Issued in Cooperation with the
AIR FORCE-NAVY-CIVIL SUBCOMMITTEE
on
AIRCRAFT DESIGN CRITERIA
Under the Supervision of the
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MUNITIONS BOARD

FOREST PRODUCTS LABORATORY
MADISON 5, WISCONSIN

UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE
In Cooperation with the University of Wisconsin

Supplement to

EFFECTS OF SHEAR DEFORMATION IN THE CORE OF A FLAT RECTANGULAR SANDWICH PANEL

Deflection Under Uniform Load of Sandwich Panels Having Facings of Unequal Thickness[1]

By

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Introduction

The present report is a discussion of the problem of determining the deflection of a sandwich panel under normal uniform load. It is assumed that the core and facing materials are orthotropic and that the thicknesses of the two facings are different. Two types of edge conditions are considered, namely, all edges simply supported and all edges clamped.

The analysis used to determine the effect of the transverse shear deformations in the core is similar to that which was applied in the treatment of the buckling problem in Forest Products Laboratory Report No. 1583-B (1).[3] This approach consists in taking the components of displacement in the core as those in which normal plane sections, parallel to each of the two edges of the panel, remain plane but rotate as the panel undergoes deflection. In the case of simple support this type of analysis leads to results which

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[1]This progress report is one of a series prepared and distributed by the Forest Products Laboratory under U. S. Navy Bureau of Aeronautics Order No. NAer 01044 and U. S. Air Force No. USAF-(33-038)(51-4066-E). Results here reported are preliminary and may be revised as additional data become available.

[2]Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[3]Underlined numbers in parenthesis refer to Literature Cited at the end of this report.

are the same as those obtainable by use of the equations of Libove and Batdorf (5). In the case of clamped edges, explicit approximations are obtained.

The present problem was treated for the case of facings of equal thickness in Forest Products Laboratory Report No. 1583 (7). A subsequent publication in the same series, Report No. 1583-A (3), gave comparisons between predicted central deflections and those measured in tests. These comparisons, showing satisfactory agreement between prediction and test, indicated that the approximate method used in Report No. 1583 for analyzing the effect of shear deformation in the core upon the central deflection was adequate.

In the analysis of Report No. 1583 (7) it was assumed that the elastic properties of the orthotropic materials of the sandwich were not greatly different in the two directions parallel to the edges of the panel. This restriction, which does not seriously limit the applicability of the results in the range of present practical constructions, does nevertheless exclude the consideration of extreme cases. In the present analysis this limitation is removed.

The problem under consideration was solved for the case of a simply supported panel with isotropic facings and core, the facings being of equal thickness, by Hopkins and Pearson (2) and Reissner (10).

Notation

a, b	dimensions of the panel.
c	thickness of the core.
f_1, f_2	thicknesses of the facings.
x, y, z	coordinate and orthotropic axes.
E_f	Young's modulus of isotropic facings.
E_x, E_y	Young's modulus of orthotropic facings.
p	uniform normal load per unit area.
$\lambda = 1 - \sigma_{xy}\sigma_{yx}$	
$\lambda_f = 1 - \sigma^2$	
μ'	shear modulus of isotropic core.
μ_{xy}	shear modulus of facings.
μ'_{zx}, μ'_{yz}	shear moduli of orthotropic core.
σ	Poisson's ratio of isotropic facings.
σ_{xy}, σ_{yx}	Poisson's ratios of orthotropic facings.

Results and Discussion

Formulas for determining the deflection of a uniformly loaded sandwich panel are given in this section. These formulas are derived in Appendix A for the case of simple support and in Appendix B for the case of clamped edges.

1. All Edges Simply Supported

The deflection, under uniform load, of a sandwich panel with orthotropic facings and core depends upon five physical constants:

$$\alpha = \sqrt{\frac{E_x}{E_y}} \tag{1}$$

$$\beta = \frac{\lambda}{\sqrt{E_x E_y}} \left\{ \frac{E_x \sigma_{yx}}{\lambda} + 2\mu_{xy} \right\} \tag{2}$$

$$\gamma = \frac{\lambda \mu_{xy}}{\sqrt{E_x E_y}} \tag{3}$$

$$S_x = \frac{c f_1 f_2 \pi^2 \sqrt{E_x E_y}}{(f_1 + f_2) a^2 \lambda \mu'_{zx}} \tag{4}$$

$$S_y = \frac{c f_1 f_2 \pi^2 \sqrt{E_x E_y}}{(f_1 + f_2) a^2 \lambda \mu'_{yz}} \tag{5}$$

and the two quantities

$$I_f = \frac{f_1^3 + f_2^3}{12} \tag{6}$$

$$I = \frac{f_1 f_2}{f_1 + f_2} \left(c + \frac{f_1 + f_2}{2}\right)^2 \tag{7}$$

If the panel is simply supported, the deflection is given by the formula

$$w = \frac{16\, a^4\, p\, \lambda}{\pi^6\, I \sqrt{E_x\, E_y}} \sum_{\substack{m=1}}^{\infty} \sum_{\substack{n=1}}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{m\, n\, \left(V_{mn}^{(f)} + V_{mn}\right)} \tag{8}$$

m, n odd

with

$$V_{mn}^{(f)} = \frac{I_f}{I} \left\{ \alpha\, m^4 + \frac{2\, \beta\, m^2 n^2 a^2}{b^2} + \frac{n^4 a^4}{\alpha\, b^4} \right\} \tag{9}$$

and

$$V_{mn} = \frac{\alpha m^4 + \frac{2\, \beta\, m^2 n^2 a^2}{b^2} + \frac{n^4 a^4}{\alpha\, b^4} + \left\{ \frac{S_x n^2 a^2}{b^2} + S_y m^2 \right\} F_{mn}}{1 + S_x \left(\alpha m^2 + \frac{\gamma n^2 a^2}{b^2} \right) + S_y \left(\frac{n^2 a^2}{\alpha\, b^2} + \gamma m^2 \right) + S_x S_y F_{mn}}$$

with (10)

$$F_{mn} = (1 - \beta^2) \frac{m^2 n^2 a^2}{b^2} + \gamma \left(\alpha m^4 + \frac{2\, \beta\, m^2 n^2 a^2}{b^2} + \frac{n^4 a^4}{\alpha\, b^4} \right)$$

If the thickness of the core is large as compared with the thickness of each facing, the expression $\underline{V_{mn}^{(f)}}$ in formula (8) can normally be neglected.
If the facings are isotropic,

$$V_{mn}^{(f)} = \frac{I_f}{I} \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 , \tag{11}$$

$$V_{mn} = \frac{\left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 \left\{ 1 + \gamma \left(S_x \frac{n^2 a^2}{b^2} + S_y m^2 \right) \right\}}{1 + S_x m^2 + S_y \frac{n^2 a^2}{b^2} + \gamma \left\{ S_x \frac{n^2 a^2}{b^2} + S_y m^2 + S_x S_y \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 \right\}} \tag{12}$$

If both the facings and the core are isotropic, $\underline{V_{mn}^{(f)}}$ remains as defined by formula (11) and

$$V_{mn} = \frac{\left(m^2 + \frac{n^2a^2}{b^2}\right)^2}{1 + S\left(m^2 + \frac{n^2a^2}{b^2}\right)} \tag{13}$$

with

$$S = \frac{c\, f_1\, f_2\, \pi^2\, E_x}{(f_1 + f_2)\, a^2 \lambda_f \mu'} \tag{14}$$

The central deflection is given by the formula

$$w_{max} = \frac{16\, a^4\, p\, \lambda}{\pi^6 I \sqrt{E_x\, E_y}} \sum_{\substack{m=1 \\ m,\, n \text{ odd}}}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n-2}{2}}}{m\, n \left(V_{mn}^{(f)} + V_{mn}\right)} \tag{15}$$

where $V_{mn}^{(f)}$ and V_{mn} are taken from the preceding formulas. In the range

$$1 \leq \frac{b}{a}\left(\frac{E_x}{E_y}\right)^{\frac{1}{4}} \leq 1.4 \tag{16}$$

the term $m = n = 1$ is expected to give satisfactory results (7).

The central deflection of an infinitely long panel, obtained from formula 15 by neglecting $V_{mn}^{(f)}$ and summing with respect to m and n, is

$$w_{max} = \frac{5\, a^4\, p\lambda}{384\, I\, E_x}\left\{1 + \frac{48\, E_x\, c\, f_1\, f_2}{5\, a^2\, \lambda\, \mu'_{zx}\, (f_1 + f_2)}\right\} \tag{17}$$

In the event that both the facings and core materials are isotropic and the facings are considered as membranes, the central deflection of a panel of any aspect ratio may be obtained by using the formula

$$w_{max} = \frac{p\, a^4\, \lambda_f}{I\, E_f} \alpha_1 \left\{1 + S\, \alpha_2\right\} \tag{18}$$

in conjunction with the curves given in figure 1. For panels of this type, the bending and twisting moments, the transverse shear stress resultants, and the reactions at the edges and corners are independent of S, and the formulas for these quantities are identical with those for homogeneous isotropic plates. Such formulas are given in reference (11) together with a table of maximum values. With the use of the formulas for the moments and shear stress resultants, the fiber stress in the facings and the shear stress in the core can be estimated.

2. All Edges Clamped

For panels with all edges clamped, formulas have been derived only for the deflection at the center of the panel. The formulas which follow give approximate results and those which apply to a rectangular panel are limited in applicability to the range (16).

In the event that both the facings and core are orthotropic, the central deflection is given in terms of the quantities 1 to 7 above by the formulas

$$w_{max} = \frac{p\, a^4 \lambda}{3 \pi^4 I \sqrt{E_x E_y}\,\left(V^{(f)} + V\right)} \tag{19}$$

with

$$V^{(f)} = \frac{I_f}{I}\left\{\alpha + \frac{2 a^2 \beta}{3 b^2} + \frac{a^4}{\alpha b^4}\right\} \tag{20}$$

and

$$V = \frac{\alpha + \frac{2 a^2 \beta}{3 b^2} + \frac{a^4}{\alpha b^4} + 4\left\{\frac{S_x a^2}{b^2} + S_y\right\} F}{1 + 4 S_x\left(\alpha + \frac{a^2 \gamma}{3 b^2}\right) + 4 S_y\left(\frac{a^2}{\alpha b^2} + \frac{\gamma}{3}\right) + 16 S_x S_y F}$$

with

$$F = \frac{a^2}{b^2}\left(1 - \frac{\beta^2}{9}\right) + \frac{\gamma}{3}\left(\alpha + \frac{2 a^2 \beta}{3 b^2} + \frac{a^4}{\alpha b^4}\right) \tag{21}$$

The term $\underline{V}^{(f)}$ is negligible if the core is thick relative to the thickness of each facing.

If the facings are isotropic,

$$w_{max} = \frac{p\, a^4 \lambda_f}{3 \pi^4 I E_f \left(V^{(f)} + V\right)} \tag{22}$$

with

$$V^{(f)} = \frac{I_f}{I}\left\{1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4}\right\} \tag{23}$$

and

$$V = \frac{1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4} + 4\left\{\frac{8\,a^4}{9\,b^4} + \frac{\gamma\,a^2}{3\,b^2}\left(1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4}\right)\right\}\left\{S_x + \frac{S_y b^2}{a^2}\right\}}{1 + 4\,S_x\left(1 + \frac{\gamma\,a^2}{3\,b^2}\right) + 4\,S_y\left(\frac{a^2}{b^2} + \frac{\gamma}{3}\right) + 16\,S_xS_y\left\{\frac{8\,a^4}{9\,b^4} + \frac{\gamma}{3}\left(1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4}\right)\right\}} \tag{24}$$

with

$$\gamma = \frac{1 - \sigma}{2}$$

If both the facings and core are isotropic, w_{max} is given by formula (22), V^f by formula (23), and

$$V = \frac{1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4} + 4\,S\left\{\frac{8\,a^4}{9\,b^4} + \frac{\gamma\,a^2}{3\,b^2}\left(1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4}\right)\right\}\left\{1 + \frac{b^2}{a^2}\right\}}{1 + 4\,S\left(1 + \frac{\gamma}{3}\right)\left(1 + \frac{a^2}{b^2}\right) + 16\,S^2\left\{\frac{8\,a^4}{9\,b^4} + \frac{\gamma}{3}\left(1 + \frac{2\,a^2}{3\,b^2} + \frac{a^4}{b^4}\right)\right\}} \tag{25}$$

with S defined by (16).

The maximum deflection of an infinitely long panel is given by the formula

$$w_{max} = \frac{p\,a^4\,\lambda}{4\,\pi^4\,E_x\,I\left\{\frac{I_f}{I} + \frac{1}{1 + \frac{4\,c\,f_1f_2\,\pi^2\,E_x}{(f_1 + f_2)\,a^2\,\lambda\,\mu'_{zx}}}\right\}} \tag{26}$$

APPENDIX A

Panel Under Uniform Load

A1. Axes of Reference, Notation for Dimensions

The axes of reference x and y are taken in the undeformed surface of separation of the core and the facing of thickness denoted by f_1, and in coincidence with two of the edges of the panel. The z-axis is then perpendicular to the facings and is directed as shown in figure 2. It is assumed that the orthotropic core and facing materials are so oriented that these axes are perpendicular to their planes of elastic symmetry. The material in the two facings is considered the same and similarly oriented.

The dimensions of the panel are designated by a and b, with a measured along the x-axis as indicated in figure 3, while c, f_1 and f_2 denote the thicknesses of the core and of the two facings, respectively. It is convenient to designate a facing as 1 or 2 according as its thickness if f_1 or f_2.

A2. The Strain Energy in the Sandwich

The increase in the deflection of a uniformly loaded rectangular panel, associated with shear deformation in the core, is to be determined by the method used in Forest Products Laboratory Report No. 1583-B (1).[4] In the case of simple support, the deflection is taken as

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x, y) \qquad \text{(A1)}$$

where

$$w_{mn} = C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{(A2)}$$

This expression is taken as the deflection throughout the core and the two facings.

The analysis of the strain energy in the sandwich which was used in Forest Products Laboratory Report No. 1583-B (1) was based on the assumption that in the core, plane elements initially parallel to the x, z plane remained plane under deformation but rotated about their intersections with the

[4] The method of analysis is one which was used by Williams, Leggett, and Hopkins (12), and other British writers (4), (2). It was first used in the present type of problem by Hopkins and Pearson (2).

surface $z = q$ by an amount specified by a parameter k. Elements initially parallel to the y, z plane were treated similarly with two other parameters, r and h, determining their fixed lines and amounts of rotation respectively. In the present problem, where the series (A1) is used to describe the deflection, the displacement associated with each term is analyzed in this manner, using sets of parameters k_{mn}, q_{mn}, h_{mn}, and r_{mn} for each term, and the components of displacement in the core are assumed to be

$$\left.\begin{aligned} u_c &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (z - q_{mn})\, k_{mn} \frac{\partial w_{mn}}{\partial x} \\ v_c &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (z - r_{mn})\, h_{mn} \frac{\partial w_{mn}}{\partial y} \\ w_c &= w\,(x, y) \end{aligned}\right\} \qquad \text{(A3)}$$

In facing 1 the components of displacement are taken in the forms

$$\left.\begin{aligned} u_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (k_{mn}\, q_{mn} - z) \frac{\partial w_{mn}}{\partial x} \\ v_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (h_{mn}\, r_{mn} - z) \frac{\partial w_{mn}}{\partial y} \\ w_1 &= w\,(x, y) \end{aligned}\right\} \qquad \text{(A4)}$$

and the forms

$$u_2 = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ k_{mn}\,(c - q_{mn}) + z - c \right\} \frac{\partial w_{mn}}{\partial x} \qquad \text{(A5)}$$

$$\left.\begin{aligned} v_2 &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ h_{mn} (c - r_{mn}) + z - c \right\} \frac{\partial w_{mn}}{\partial y} \\ w_2 &= w(x, y) \end{aligned}\right\} \quad \text{(A5)}$$

are assumed for those in facing 2.

Love's (6) notation will be used for the components of strain, with the superscripts c, 1, and 2 used to denote components in the core, in facing 1, and in facing 2 respectively. The components of transverse shear strain in the core, as obtained for expressions (A3), are

$$\left.\begin{aligned} e_{zx}^{(c)} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1 - k_{mn}) \frac{\partial w_{mn}}{\partial x} \\ e_{yz}^{(c)} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1 - h_{mn}) \frac{\partial w_{mn}}{\partial y} \end{aligned}\right\} \quad \text{(A6)}$$

It is assumed that the bending strains associated with the displacement components (A3) contribute a negligible amount to the total strain energy and therefore need not be considered.

The state of strain in the facings is considered as the superposition of two states of strain. The first of these consists of the membrane strains or strains in their middle surfaces. According to formulas (A4) and (A5), the components of this type are

$$\begin{aligned} e_{xx}^{(1)} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(k_{mn} q_{mn} + \frac{f_1}{2}\right) \frac{\partial^2 w_{mn}}{\partial x^2} \\ e_{yy}^{(1)} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(h_{mn} r_{mn} + \frac{f_1}{2}\right) \frac{\partial^2 w_{mn}}{\partial y^2} \end{aligned} \quad \text{(A7)}$$

$$e_{xy}^{(1)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ k_{mn}\, q_{mn} + h_{mn}\, r_{mn} + f_1 \right\} \frac{\partial^2 w_{mn}}{\partial x \partial y} \qquad \text{(A7)}$$

and

$$\left.\begin{aligned} e_{xx}^{(2)} &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ k_{mn}\, (c - q_{mn}) + \frac{f_2}{2} \right\} \frac{\partial^2 w_{mn}}{\partial x^2} \\ e_{yy}^{(2)} &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ h_{mm}\, (c - r_{mn}) + \frac{f_2}{2} \right\} \frac{\partial^2 w_{mn}}{\partial y^2} \\ e_{xy}^{(2)} &= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ k_{mn}\, (c - q_{mn}) + h_{mn}\, (c - r_{mn}) + f_2 \right\} \frac{\partial^2 w_{mn}}{\partial x \partial y} \end{aligned}\right\} \qquad \text{(A8)}$$

The second state of strain in the facings is that associated with the bending of the facings about their own middle surfaces. This state, in either facing has the components

$$\left.\begin{aligned} e'_{xx} &= - z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^2 w_{mn}}{\partial x^2} \\ e'_{yy} &= - z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^2 w_{mn}}{\partial y^2} \\ e'_{xy} &= - 2z' \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial^2 w_{mn}}{\partial x \partial y} \end{aligned}\right\} \qquad \text{(A9)}$$

where $\underset{\sim}{z'}$ is measured from the middle surface of the facing under consideration.

The strain energy for any of the above states of strain is computed from the expression, (8) (9)

$$U = \frac{1}{2\lambda} \iiint_V \Big[E_x e_{xx}^2 + E_y e_{yy}^2 + 2 E_x \sigma_{yx} e_{xx} e_{yy} + \lambda \mu_{xy} e_{xy}^2 + \lambda \mu_{zx} e_{zx}^2 + \lambda \mu_{yz} e_{yz}^2 \Big] dz\, dy\, dx \tag{A10}$$

where E denotes a Young's modulus, μ a shear modulus, σ a Poisson's ratio, and $\lambda = 1 - \sigma_{yx} \sigma_{xy}$. The subscripts associate these constants with appropriate orthotropic axes. Primed letters will be used to denote the elastic constants of the core and unprimed letters those of the facings. The indicated integration is to be carried out over the entire volume of the core or facing under consideration.

With the substitution of formulas (A2) into (A6), and (A6) in turn into (A10), the strain energy in the core is expressed as

$$U_c = \frac{c}{2} \int_0^a \int_0^b \Bigg[\mu'_{zx} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1 - k_{mn}) \frac{m\pi}{a} C_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\}^2 + \mu'_{yz} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1 - h_{mn}) \frac{n\pi}{b} C_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right\}^2 \Bigg] dydx \tag{A11}$$

Now formally

$$\int_0^a \int_0^b \Bigg[\left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\} \left\{ \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} B_{rs} \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b} \right\} \Bigg] dydx \tag{A12}$$

$$= \frac{ab}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} B_{mn}$$

and this result remains unchanged for the integral of the product two cosine-cosine series or two cosine-sine series. When this formula is applied to (A11),

$$U_c = \frac{abc\,\pi^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[(1 - k_{mn})^2 \frac{m^2 \mu'_{zx}}{a^2} + (1 - h_{mn})^2 \frac{n^2}{b^2} \mu'_{yz} \right] C_{mn}^2 \qquad \text{(A13)}$$

The energy associated with the membrane strains in the facings is determined from formula (A10), using formulas (A7) and (A8) with w_{mn} given by formula (A2). After integrating with respect to z and applying formula (A12) for the integrations with respect to x and y, this component of the strain energy is given by

$$\begin{aligned} U_M = \frac{ab\pi^4}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg[& \frac{E_x}{\lambda} \left\{ f_1 \left(k_{mn}q_{mn} + \frac{f_1}{2}\right)^2 + f_2 \left(k_{mn} [c - q_{mn}] + \frac{f_2}{2}\right)^2 \right\} \frac{m^4}{a^4} \\ & + \frac{E_y}{\lambda} \left\{ f_1 \left(h_{mn}r_{mn} + \frac{f_1}{2}\right)^2 + f_2 \left(h_{mn} [c - r_{mn}] + \frac{f_2}{2}\right)^2 \right\} \frac{n^4}{a^4} \\ & + \frac{2 E_x \sigma_{yx}}{\lambda} \Bigg\{ f_1 \left(k_{mn}q_{mn} + \frac{f_1}{2}\right) \left(h_{mn}r_{mn} + \frac{f_1}{2}\right) \\ & \qquad + f_2 \left(k_{mn} [c - q_{mn}] + \frac{f_2}{2}\right)\left(h_{mn} [c - r_{mn}] + \frac{f_2}{2}\right) \Bigg\} \frac{m^2 n^2}{a^2 b^2} \\ & + \mu_{xy} \Big\{ f_1 (k_{mn}q_{mn} + h_{mn}r_{mn} + f_1)^2 \\ & \qquad + f_2 (k_{mn} [c - q_{mn}] + h_{mn} [c - r_{mn}] + f_2)^2 \Big\} \frac{m^2 n^2}{a^2 b^2} \Bigg] C_{mn}^2 \end{aligned} \qquad \text{(A14)}$$

In a similar manner the strain energy in bending of the two facings is found from formulas (A9) and (A10) to be

$$U_F = \frac{ab\pi^4}{8} \left(\frac{f_1^3 + f_2^3}{12}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{E_x m^4}{\lambda a^4} + \frac{E_y n^4}{\lambda b^4} + 2\left(\frac{E_x \sigma_{xy}}{\lambda} + 2\mu_{xy}\right) \frac{m^2 n^2}{a^2 b^2} \right\} C_{mn}^2 \qquad \text{(A15)}$$

The total strain energy in the sandwich

$$U = U_c + U_M + U_F \tag{A16}$$

is now written in the form

$$U = \frac{b\pi^2}{8a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}\, C_{mn}^2 \tag{A17}$$

with $\underline{T_{mn}}$ expressed as a quadratic in $\underline{(k_{mn}\, q_{mn})}$, $\underline{(h_{mn}\, r_{mn})}$, $\underline{k_{mn}}$ and $\underline{h_{mn}}$ as follows

$$\begin{aligned} T_{mn} = {} & B_{mn}^{(1)}\,(k_{mn}q_{mn})^2 + 2\,B_{mn}^{(2)}\,(k_{mn}q_{mn})(h_{mn}r_{mn}) \\ & + B_{mn}^{(3)}\,(h_{mn}r_{mn})^2 + 2\,B_{mn}^{(4)}\,(k_{mn}q_{mn})\,k_{mn} \\ & + 2\,B_{mn}^{(5)}\left\{(k_{mn}q_{mn})\,h_{mn} + (h_{mn}r_{mn})\,k_{mn}\right\} \\ & + 2\,B_{mn}^{(6)}\,(h_{mn}r_{mn})\,h_{mn} + B_{mn}^{(7)}\,k_{mn}^2 + 2\,B_{mn}^{(8)}\,k_{mn}h_{mn} \\ & + B_{mn}^{(9)}\,h_{mn}^2 + 2\,B_{mn}^{(10)}\,(k_{mn}q_{mn}) + 2\,B_{mn}^{(11)}\,(h_{mn}r_{mn}) \\ & + 2\,B_{mn}^{(12)}\,k_{mn} + 2\,B_{mn}^{(13)}\,h_{mn} + B_{mn}^{(14)} + B_{mn}^{(15)} \end{aligned} \tag{A18}$$

with $\underline{B_{mn}^{(i)}}$, $i = 1$ — 15 obtained from formulas (A13), (A14), and (A15) in the form

$$\left.\begin{aligned} & B_{mn}^{(1)} = (f_1 + f_2)\,A_{mn}^{(1)}, \quad B_{mn}^{(2)} = (f_1 + f_2)\,A_{mn}^{(2)}, \quad B_{mn}^{(3)} = (f_1 + f_2)\,A_{mn}^{(3)} \\ & B_{mn}^{(4)} = -\,cf_2\,A_{mn}^{(1)}, \quad B_{mn}^{(5)} = -\,cf_2\,A_{mn}^{(2)}, \quad B_{mn}^{(6)} = -\,cf_2\,A_{mn}^{(3)} \\ & B_{mn}^{(7)} = cA_{mn}^{(4)} + c^2f_2\,A_{mn}^{(1)}, \quad B_{mn}^{(8)} = c^2f_2\,A_{mn}^{(2)}, \quad B_{mn}^{(9)} = cA_{mn}^{(5)} + c^2f_2\,A_{mn}^{(3)} \end{aligned}\right\} \tag{A19}$$

$$
\left.
\begin{aligned}
B_{mn}^{(10)} &= \left(\frac{f_1{}^2 - f_2{}^2}{2}\right)\left(A_{mn}^{(1)} + A_{mn}^{(2)}\right), \quad B_{mn}^{(11)} = \left(\frac{f_1{}^2 - f_2{}^2}{2}\right)\left(A_{mn}^{(2)} + A_{mn}^{(3)}\right) \\
B_{mn}^{(12)} &= -\,cA_{mn}^{(4)} + \frac{cf_2{}^2}{2}\left(A_{mn}^{(1)} + A_{mn}^{(2)}\right) \\
B_{mn}^{(13)} &= -\,cA_{mn}^{(5)} + \frac{cf_2{}^2}{2}\left(A_{mn}^{(2)} + A_{mn}^{(3)}\right) \\
B_{mn}^{(14)} &= c\left(A_{mn}^{(4)} + A_{mn}^{(5)}\right) + \frac{f_1{}^3 + f_2{}^3}{4}\left\{A_{mn}^{(1)} + 2\,A_{mn}^{(2)} + A_{mn}^{(3)}\right\} \\
B_{mn}^{(15)} &= \left(\frac{f_1{}^3 + f_2{}^3}{12}\right)\left\{A_{mn}^{(1)} + 2\,A_{mn}^{(2)} + A_{mn}^{(3)}\right\}^{5}
\end{aligned}
\right\} \quad \text{(A19)}
$$

where

$$
\begin{aligned}
A_{mn}^{(1)} &= \frac{\pi^2}{a^2}\left(\frac{E_x}{\lambda}m^2 + \mu_{xy}\frac{n^2a^2}{b^2}\right)m^2 \\
A_{mn}^{(2)} &= \frac{\pi^2}{a^2}\left(\frac{E_x\sigma_{yx}}{\lambda} + \mu_{xy}\right)\frac{m^2n^2a^2}{b^2} \\
A_{mn}^{(3)} &= \frac{\pi^2}{a^2}\left\{\frac{E_y}{\lambda}\frac{n^4a^4}{b^4} + \mu_{xy}\frac{m^2n^2a^2}{b^2}\right\} \\
A_{mn}^{(4)} &= m^2\,\mu'_{zx} \\
A_{mn}^{(5)} &= \frac{n^2a^2}{b^2}\,\mu'_{yz}
\end{aligned}
\quad \text{(A20)}
$$

[5] The terms $B_{mn}^{(15)}$ are those obtained from (A15). These terms are written separately because they often have a negligible effect upon the deflection and may therefore be dropped.

The work done by the applied uniform load of p pounds per unit area is

$$U_L = p \int_0^a \int_0^b w \, dydx \qquad \text{(A21)}$$

Upon substituting the series (A1) for w and integrating,

$$U_L = \frac{4\,ab\,p}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{mn} \qquad \text{(A22)}$$

m, n odd

The total potential energy of the sandwich, W, is

$$W = U - U_L$$

With the substitution of formulas (A17) and (A22), this takes the form

$$W = \frac{b\pi^2}{8a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn} C_{mn}^2 - \frac{4\,ab\,p}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{mn} \qquad \text{(A23)}$$

m, n odd

The parameters k_{mn}, q_{mn}, h_{mn}, r_{mn}, and C_{mn} are now determined by the condition that they minimize the potential energy. An equivalent minimization is obtained by minimizing with respect to $(k_{mn}\,q_{mn})$, $(h_{mn}\,r_{mn})$, k_{mn}, h_{mn}, and and C_{mn}. With the use of expression (A18), the conditions for the vanishing of the partial derivatives of W with respect to these parameters are respectively,

$$\left.\begin{aligned}
B_{mn}^{(1)} (k_{mn}q_{mn}) + B_{mn}^{(2)} (h_{mn}r_{mn}) + B_{mn}^{(4)} k_{mn} + B_{mn}^{(5)} h_{mn} + B_{mn}^{(10)} &= 0 \\
B_{mn}^{(2)} (k_{mn}q_{mn}) + B_{mn}^{(3)} (h_{mn}r_{mn}) + B_{mn}^{(5)} k_{mn} + B_{mn}^{(6)} h_{mn} + B_{mn}^{(11)} &= 0 \\
B_{mn}^{(4)} (k_{mn}q_{mn}) + B_{mn}^{(5)} (h_{mn}r_{mn}) + B_{mn}^{(7)} k_{mn} + B_{mn}^{(8)} h_{mn} + B_{mn}^{(12)} &= 0 \\
B_{mn}^{(5)} (k_{mn}q_{mn}) + B_{mn}^{(6)} (h_{mn}r_{mn}) + B_{mn}^{(8)} k_{mn} + B_{mn}^{(9)} h_{mn} + B_{mn}^{(13)} &= 0
\end{aligned}\right\} \qquad \text{(A24)}$$

and

$$\left.\begin{aligned} T_{mn}\, C_{mn} &= \frac{16\, a^2\, p}{\pi^4\, mn} \text{ if m and n are odd} \\ &= 0 \text{ if m or n is even} \end{aligned}\right| \qquad \text{(A24)}$$

The first four of these equations express the condition that $\underline{T_{mn}}$ be a minimum with respect to the four variables in terms of which it is written. Designate this minimum by $\underline{T'_{mn}}$. Then, solving the first four equations for $(\underline{k_{mn}}\ \underline{q_{mn}})$, $(\underline{h_{mn}}\ \underline{r_{mn}})$, $\underline{k_{mn}}$, and $\underline{h_{mn}}$ and substituting into formula (A18), it is found that

$$T'_{mn} = \frac{\begin{vmatrix} B^{(1)}_{mn} & B^{(2)}_{mn} & B^{(4)}_{mn} & B^{(5)}_{mn} & B^{(10)}_{mn} \\ B^{(2)}_{mn} & B^{(3)}_{mn} & B^{(5)}_{mn} & B^{(6)}_{mn} & B^{(11)}_{mn} \\ B^{(4)}_{mn} & B^{(5)}_{mn} & B^{(7)}_{mn} & B^{(8)}_{mn} & B^{(12)}_{mn} \\ B^{(5)}_{mn} & B^{(6)}_{mn} & B^{(8)}_{mn} & B^{(9)}_{mn} & B^{(13)}_{mn} \\ B^{(10)}_{mn} & B^{(11)}_{mn} & B^{(12)}_{mn} & B^{(13)}_{mn} & B^{(14)}_{mn} \end{vmatrix}}{\begin{vmatrix} B^{(1)}_{mn} & B^{(2)}_{mn} & B^{(4)}_{mn} & B^{(5)}_{mn} \\ B^{(2)}_{mn} & B^{(3)}_{mn} & B^{(5)}_{mn} & B^{(6)}_{mn} \\ B^{(4)}_{mn} & B^{(5)}_{mn} & B^{(7)}_{mn} & B^{(8)}_{mn} \\ B^{(5)}_{mn} & B^{(6)}_{mn} & B^{(8)}_{mn} & B^{(9)}_{mn} \end{vmatrix}} + B^{(15)}_{mn} \qquad \text{(A25)}$$

When the expressions (A19) are substituted for the elements of the determinants in this formula it is found that

$$T'_{mn} = I_f \left\{ A^{(1)}_{mn} + 2\, A^{(2)}_{mn} + A^{(3)}_{mn} \right\}$$

$$+ \frac{I \left[A^{(1)}_{mn} + 2\, A^{(2)}_{mn} + A^{(3)}_{mn} + \left\{ A^{(1)}_{mn}\, A^{(3)}_{mn} - \left(A^{(2)}_{mn} \right)^2 \right\} \left\{ \frac{\phi}{A^{(4)}_{mn}} + \frac{\phi}{A^{(5)}_{mn}} \right\} \right]}{1 + \frac{\phi\, A^{(1)}_{mn}}{A^{(4)}_{mn}} + \frac{\phi\, A^{(3)}_{mn}}{A^{(5)}_{mn}} + \frac{\phi^2 \left\{ A^{(1)}_{mn}\, A^{(3)}_{mn} \quad \left(A^{(2)}_{mn} \right)^2 \right\}}{A^{(4)}_{mn}\, A^{(5)}_{mn}}} \qquad \text{(A26)}$$

with

$$I_f = \frac{f_1^3 + f_2^3}{12} \tag{A27}$$

$$I = \frac{f_1 f_2}{f_1 + f_2}\left(c + \frac{f_1 + f_2}{2}\right)^2 \tag{A28}$$

$$\phi = \frac{c\, f_1 f_2}{f_1 + f_2} \tag{A29}$$

The last of conditions (A24) may now be written

$$\left.\begin{aligned} C_{mn} &= \frac{16\, a^2 p}{\pi^4\, mn\, T'_{mn}} \text{ if m and n are odd} \\ &= 0 \text{ if m or n is even} \end{aligned}\right\} \tag{A30}$$

and the deflection of the panel is determined by the expression

$$w = \frac{16\, a^2 p}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn\, T'_{mn}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A31}$$

$$\text{m, n odd}$$

A formula for the deflection, w, which is somewhat simpler is obtained by introducing the parameters

$$\left.\begin{aligned} \alpha &= \sqrt{\frac{E_x}{E_y}} \\ \beta &= \frac{\lambda}{\sqrt{E_x E_y}}\left\{\frac{E_x \sigma_{yx}}{\lambda} + 2\mu_{xy}\right\} \\ \gamma &= \frac{\lambda\, \mu_{xy}}{\sqrt{E_x E_y}} \\ S_x &= \frac{\phi\, \pi^2 \sqrt{E_x E_y}}{a^2 \lambda\, \mu'_{zx}} \end{aligned}\right\} \tag{A32}$$

$$S_y = \frac{\phi \pi^2 \sqrt{E_x E_y}}{a^2 \lambda \mu'_{yz}} \tag{A32}$$

When the facing material is isotropic

$$\alpha = \beta = 1 \text{ and } \gamma = \frac{1 - \sigma}{2} \tag{A33}$$

If both the facing and core materials are isotropic, then in addition to the reductions, (A33), $\underline{S_x}$ and $\underline{S_y}$ both reduce to

$$S = \frac{\phi \pi^2 E_f}{a^2 \lambda_f \mu'} \tag{A34}$$

With the use of formulas (A32) and (A20), $\underline{T'_{mn}}$, formula (A26), may in the orthotropic case be written

$$T'_{mn} = \frac{\pi^2 I \sqrt{E_x E_y}}{a^2 \lambda} \left\{ V^{(f)}_{mn} + V_{mn} \right\} \tag{A35}$$

with

$$V^{(f)}_{mn} = \frac{I_f}{I} \left\{ \alpha m^4 + \frac{2 \beta m^2 n^2 a^2}{b^2} + \frac{n^4 a^4}{\alpha b^4} \right\} \tag{A36}$$

and

$$V_{mn} = \frac{\alpha m^4 + \dfrac{2 \beta m^2 n^2 a^2}{b^2} + \dfrac{n^4 a^4}{\alpha b^4} + \left\{ \dfrac{S_x n^2 a^2}{b^2} + S_y m^2 \right\} F_{mn}}{1 + S_x \left(\alpha m^2 + \dfrac{\gamma n^2 a^2}{b^2} \right) + S_y \left(\dfrac{n^2 a^2}{\alpha b^2} + \gamma m^2 \right) + S_x S_y F_{mn}}$$

with

$$F_{mn} = (1 - \beta^2) \frac{m^2 n^2 a^2}{b^2} + \gamma \left(\alpha m^4 + \frac{2 \beta m^2 n^2 a^2}{b^2} + \frac{n^4 a^4}{\alpha b^4} \right) \tag{A37}$$

With the substitution of (A35) into (A31),

$$w = \frac{16\, a^4\, p\, \lambda}{\pi^6\, I\, \sqrt{E_x E_y}} \sum_{\substack{m=1 \\ m,\, n \text{ odd}}}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn\left(V_{mn}^{(f)} + V_{mn}\right)} \tag{A38}$$

The expression $V_{mn}^{(f)}$ brings in the effect of the bending of the facings about their own middle surfaces. When the thickness of the core is large as compared with the thickness of each facing, this expression can be neglected with no practical effect upon the results.

With the omission of the expression $V_{mn}^{(f)}$, formula (A38) is identical with a result obtainable in a different manner by the use of the equations of Libove and Batdorf (5),[6] who neglect the effect of the bending of the facings. This fact indicates that the steps taken in the preceding formal analysis are justified.

The central deflection is determined by the formula

$$w_{max} = \frac{16\, a^4\, p\, \lambda}{\pi^6\, I\, \sqrt{E_x E_y}} \sum_{\substack{m=1 \\ m,\, n \text{ odd}}}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n-2}{2}}}{mn\,(V_{mn}^{(f)} + V_{mn})} \tag{A39}$$

For panels which are square or nearly so, the term $m = n = 1$ of this series often gives a good approximation to the complete sum. If the ratio $\frac{a}{b}$ is not near 1, however, a number of terms must be used to obtain satisfactory results. In the extreme case that the side b is infinitely long, the formula yields

$$w_{max} = \frac{4\, a^4\, p\lambda}{\pi^5\, I\, E_x} \sum_{m=1^m}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{\left\{\frac{I_f}{I} m^4 + \frac{m^4}{1 + S_x \alpha m^2}\right\}}$$

---

[6] To assure the identity of the two results, the physical constants of Libove and Batdorf are interpreted in terms of those of the present report as in Appendix C of reference (1).

If the expression $\frac{I_f}{I} \underline{m}^4$ in this formula is neglected, the summation with respect to $\underline{m}$ can be carried out to obtain

$$w_{max} = \frac{5\, a^4\, p\, \lambda}{384\, I\, E_x} \left\{ 1 + \frac{48\, E_x\, \phi}{5\, a^2\, \lambda\, \mu'_{zx}} \right\}, \text{ b infinite} \tag{A40}$$

If both the facings and the core are isotropic, it is found with the use of expressions (A33) and (A34) that formulas (A36) and (A37) reduce to

$$V_{mn}^{(f)} = \frac{I_f}{I} \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2 \tag{A41}$$

and

$$V_{mn} = \frac{\left(m^2 + \frac{n^2 a^2}{b^2} \right)^2}{1 + S \left(m^2 + \frac{n^2 a^2}{b^2} \right)} \tag{A42}$$

respectively. The formula

$$w = \frac{16\, a^4\, p\, \lambda_f}{\pi^6\, I\, E_f} \sum_{\substack{m=1 \\ m,\, n \text{ odd}}}^{\infty} \sum_{n=1}^{\infty} \frac{\left\{ 1 + S \left(m^2 + \frac{n^2 a^2}{b^2} \right) \right\}}{mn \left(m^2 + \frac{n^2 a^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{A43}$$

obtained by substituting (A42) into (A38) and neglecting $\underline{V_{mn}^{(f)}}$ in the latter is similar to that which has been derived by Reissner ((10 page 31) for the case of equal face thickness. Formula (A43) reduced to the case of equal face thickness is identical with that obtained by Reissner provided $\underline{S}$ is replaced by $\frac{c + f}{c}$ $\underline{S}$. This slight discrepancy, which affects only the additional deflection due to shear deformation in the core, arises from the fact that Reissner assumes that the stresses transmitted from the core to the facings act upon the middle surfaces of the respective facings.

A formula that is generally more suitable for use in computations can be derived from formula (A43) by making use of the following expansions, all of which are valid in the internal $0 < y < b$:

$$1 = \frac{4}{\pi} \sum_{n = 1, 3, 5 \cdots}^{\infty} \frac{\sin \frac{n\pi y}{b}}{n} \tag{A44}$$

$$\cosh \frac{m\pi}{a}\left(y - \frac{b}{2}\right) = \frac{4}{\pi} \frac{a^2}{b^2} \cosh \alpha_m \sum_{n = 1, 3, 5 \cdots}^{\infty} \frac{n \sin \frac{n\pi y}{b}}{\left(m^2 + \frac{n^2a^2}{b^2}\right)} \tag{A45}$$

and

$$\frac{m\pi}{a}\left(y - \frac{b}{2}\right) \sinh \frac{m\pi}{a}\left(y - \frac{b}{2}\right) = \frac{4\ a^2}{\pi\ b^2} \alpha_m \sinh \alpha_m \sum_{n = 1, 3, 5 \cdots}^{\infty} \frac{n \sin \frac{n\pi y}{b}}{\left(m^2 + \frac{n^2a^2}{b^2}\right)}$$

$$- \frac{8\ a^2}{\pi\ b^2} m^2 \cosh \alpha_m \sum_{n = 1, 3, 5}^{\infty} \frac{n \sin \frac{n\pi y}{b}}{\left(m^2 + \frac{n^2a^2}{b^2}\right)^2} \tag{A46}$$

where

$$\alpha_m = \frac{m\pi b}{2a} \tag{A47}$$

From expansions (A44) and (A45) it is found that

$$\frac{\pi}{4\ m^3}\left[1 - \frac{\cosh \frac{m\pi}{a}\left(y - \frac{b}{2}\right)}{\cosh \alpha_m}\right] = \sum_{n = 1, 3, 5}^{\infty} \frac{\sin \frac{n\pi y}{b}}{mn\left(m^2 + \frac{n^2a^2}{b^2}\right)} \tag{A48}$$

and from this expression together with (A45) and (A46)

$$\frac{\pi}{4m^5}\left[1-\frac{\cosh\frac{m\pi}{a}\left(y-\frac{b}{2}\right)}{\cosh\alpha_m}-\left\{\frac{\alpha_m\tanh\alpha_m\cosh\frac{m\pi}{a}\left(y-\frac{b}{2}\right)-\frac{m\pi}{a}\left(y-\frac{b}{2}\right)\sinh\frac{m\pi}{a}\left(y-\frac{b}{2}\right)}{2\cosh\alpha_m}\right\}\right]$$

$$= \sum_{n}^{\infty}\frac{\sin\frac{n\pi y}{b}}{mn\left(m^2+\frac{n^2a^2}{b^2}\right)^2} \qquad \text{(A49)}$$

With the use of the last two expansions the summation with respect to $\underline{n}$ in (A43) is accomplished and

$$w = \frac{4\,a^4\,p\,\lambda_f}{\pi^5\,I\,E_f}\sum_{m=1,3,5}^{\infty}\frac{1}{m^5}\left[1-\left\{\frac{2+\alpha_m\tanh a_m}{2\cosh\alpha_m}\right\}\cosh\frac{m\pi}{a}\left(y-\frac{b}{2}\right)\right.$$

$$\left.+\frac{\frac{m\pi}{a}\left(y-\frac{b}{2}\right)\sinh\frac{m\pi}{a}\left(y-\frac{b}{2}\right)}{2\cosh\alpha_m}\right]\sin\frac{m\pi x}{a}$$

$$+\frac{4\,a^4\,p\,\lambda_f\,S}{\pi^5\,I\,E_f}\sum_{m=1,3,5}^{\infty}\frac{1}{m^3}\left[1-\frac{\cosh\frac{m\pi}{a}\left(y-\frac{b}{2}\right)}{\cosh\alpha_m}\right]\sin\frac{m\pi x}{a} \qquad \text{(A50)}$$

Here the first expression is recognized as that obtained by the method of M. Levy for the deflection of homogeneous isotropic plates ((11), page 128) $\underline{I}$ being interpreted as the moment of inertia of a section. The second expression gives the additional deflection due to transverse shear deformation in the core.

By carrying out the summation of the first series in each expression of formula (A50), the central deflection, $x = \frac{a}{2}$, $y = \frac{b}{2}$, is obtained in the form

$$w_{max} = \frac{p\,a^4\,\lambda_f\,\alpha_1}{I\,E_f}\left\{1 + S\,\alpha_2\right\} \qquad \text{(A51)}$$

where

$$\alpha_1 = \frac{5}{384} - \frac{2}{\pi^5} \sum_{m=1,3,5}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^5} \left\{ \frac{2 + \alpha_m \tanh \alpha_m}{\cosh \alpha_m} \right\} \tag{A52}$$

and

$$\alpha_2 = \frac{1}{\alpha_1} \left\{ \frac{1}{8\pi^2} - \frac{4}{\pi^5} \sum_{m=1,3,5}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^3 \cosh \alpha_m} \right\} \tag{A53}$$

The parameters $\underline{\alpha_1}$ and $\underline{\alpha_2}$ are plotted in figure 1 as functions of $\frac{a}{b}$. The curve representing $\underline{\alpha_1}$ was constructed from values taken from reference (11), table 5, and converted for use in formula (A51). These values were used in computing $\underline{\alpha_2}$ by means of (A53).

In his analysis of the deflection of a uniformly loaded, simply supported, isotropic sandwich panel Reissner ((10), page 32) has demonstrated that the bending and twisting moments, M_x, M_y, and M_{xy}, as well as the transverse shear stress resultants, Q_x and Q_y, are independent of the transverse shear deformations in the core (they are independent of S) provided the boundary conditions are of the type

$$w = M_x = \frac{\partial w}{\partial y} - \frac{Q_y}{(c + f)\,\mu'} = 0 \text{ at } x = 0,\ a.$$

It then follows that the reactions at the edges and corners, V_x, V_y, and R are also independent of S and are therefore the same as those obtained in the theory for homogeneous isotropic plates. Timoshenko (11) has given formulas for M_x, M_y, Q_x, Q_y, V_x, V_y, and R for simply supported isotropic plates under uniform load and has tabulated the maximum values of the first six, together with R, as functions of $\frac{b}{a}$ in table 5 of the same reference.

APPENDIX B

Deflections Under Uniform Load, Edges Clamped

The deflection in the case of clamped edges will be determined approximately by assuming the expression

$$w = C \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} \tag{B1}$$

for the deflection surface. Results obtained on the basis of this assumption are considered applicable only in determining the central deflection of sandwich panels that are square or nearly so.

For the case under consideration, the formulas for the components of displacement and strain given in Appendix A apply, using the single term (B1) in place of the series (A1) and a single set of values k, q, h, and r in place of a series of sets k_{mn}, q_{mn}, h_{mn}, and r_{mn}. With the substitution of expressions for the components of transverse shear strain in the core obtained from (A6) in this manner into (A10), it is found that the strain energy in the core is given by

$$U_c = \frac{3\, bc\, \pi^2}{32a} C^2 \left[\mu'_{zx} (1 - k)^2 + \frac{\mu'_{yz}\, a^2}{b^2} (1 - h)^2 \right] \tag{B2}$$

Similarly, the strain energy associated with the membrane strains in the facing is obtained in the form

$$\begin{aligned} U_M = \frac{3\, C^2 \pi^4 b}{8\, a^3 \lambda} \Bigg[& E_x \left\{ f_1 \left(kq + \frac{f_1}{2} \right)^2 + f_2 \left[k (c - q) + \frac{f_2}{2} \right]^2 \right\} \\ & + E_y \frac{a^4}{b^4} \left\{ f_1 \left(hr + \frac{f_1}{2} \right)^2 + f_2 \left[h (c - r) + \frac{f_2}{2} \right]^2 \right\} \\ & + \frac{2E_x \sigma_{yx} a^2}{3\, b^2} \left\{ f_1 \left(kq + \frac{f_1}{2} \right) \left(hr + \frac{f_1}{2} \right) + f_2 \left[k (c - q) + \frac{f_2}{2} \right] \left[h (c-r) + \frac{f_2}{2} \right] \right\} \\ & + \frac{\lambda \mu_{xy} a^2}{3\, b^2} \left\{ f_1 (kq + hr + f_1)^2 + f_2 \left[k (c - q) + h (c - r) + f_2 \right]^2 \right\} \Bigg] \end{aligned} \tag{B3}$$

from the states of strain (A7) and (A8), using (B1) in place of (A1). In the same way the strain energy in bending the facings about their own middle planes,

$$U_f = \frac{3\, C^2 \pi^4 b}{8\, a^3 \lambda} \left(\frac{f_1^3 + f_2^3}{12}\right)\left[E_x + \frac{E_y a^4}{b^4} + \frac{2\, a^2}{b^2}\left(\frac{E_x \sigma_{yx}}{\lambda} + 2\, \mu_{xy}\right)\right] \tag{B4}$$

is derived from (A9).

The total strain energy, U, in the sandwich is taken as

$$U = U_c + U_M + U_f$$

With the substitution of expressions (B2), (B3), and (B4) into this formula, U can be expressed in the form

$$U = \frac{b\, \pi^2}{8a} T\, C^2 \tag{B5}$$

corresponding to formula (A17), with T obtained from (A18) and (A19) by suppressing the subscripts mn throughout. The quantities $A^{(i)}$ are in the present case defined as follows:

$$\left.\begin{aligned}
A^{(1)} &= \frac{3\, \pi^2}{a^2}\left\{\frac{E_x}{\lambda} + \frac{\mu_{xy}\, a^2}{3\, b^2}\right\} \\
A^{(2)} &= \frac{3\, \pi^2}{a^2}\left\{\frac{E_x \sigma_{yx}\, a^2}{3\, b^2 \lambda} + \frac{\mu_{xy}\, a^2}{3\, b^2}\right\} \\
A^{(3)} &= \frac{3\, \pi^2}{a^2}\left\{\frac{E_y a^4}{\lambda\, b^4} + \frac{\mu_{xy}\, a^2}{3\, b^2}\right\} \\
A^{(4)} &= \frac{3\, \mu'_{zx}}{4} \\
A^{(5)} &= \frac{3\, \mu'_{yz}\, a^2}{4\, b^2}
\end{aligned}\right\} \tag{B6}$$

The work done by the applied uniform load of p pounds per unit area is obtained by substituting formula (B1) into (A21). After integration

$$U_L = \frac{p\, C\, ab}{4} \tag{B7}$$

The total potential energy of the sandwich,

$$W = U - U_L$$

is now obtained in the form

$$W = \frac{b\,\pi^2}{8a}\,T\,C^2 - \frac{p\,C\,ab}{4} \tag{B8}$$

by the substitution of equations (B5) and (B7). The parameters $\underline{k}$, $\underline{q}$, $\underline{h}$, $\underline{r}$, and $\underline{C}$ are determined by the conditions

$$\frac{\partial W}{\partial kq} = 0, \quad \frac{\partial W}{\partial hr} = 0, \quad \frac{\partial W}{\partial k} = 0, \quad \frac{\partial W}{\partial h} = 0, \quad \frac{\partial W}{\partial C} = 0 \tag{B9}$$

Since $\underline{T}$ in the present case is given by formula (A18) with the subscripts $\underline{mn}$ suppressed, the first four of the above conditions yield the first four equations of (A24) with the subscripts $\underline{mn}$ suppressed. If $\underline{T'}$ is used to denote the expression for $\underline{T}$ after imposing these conditions, it follows that $\underline{T'}$ is obtained from formula (A26) with the suppression of the subscripts $\underline{mn}$, namely:

$$T' = I_f \left\{A^{(1)} + 2\,A^{(2)} + A^{(3)}\right\} + \frac{I\left[A^{(1)} + 2\,A^{(2)} + A^{(3)} + \left\{A^{(1)}\,A^{(3)} - \left(A^{(2)}\right)^2\right\}\left\{\frac{\phi}{A^{(4)}} \quad \frac{\phi}{A^{5}}\right\}\right]}{1 + \frac{\phi\,A^{(1)}}{A^{(4)}} + \frac{\phi\,A^{(3)}}{A^{(5)}} + \frac{\phi^2\left\{A^{(1)}\,A^{(3)} \quad \left(A^{(2)}\right)^2\right\}}{A^{(4)}\,A^{(5)}}} \tag{B10}$$

with $A^{(i)}$, $i = 1$ — 5 given by (B6). The condition $\frac{\partial W}{\partial C} = 0$, applied to (B8) yields

$$C = \frac{p\,a^2}{T\,\pi^2}$$

or, after imposing all of conditions (B9)

$$C = w_{max} = \frac{p\,a^2}{T'\,\pi^2} \tag{B11}$$

That this formula gives the central deflection of the panel can be seen by reference to formula (B1).

With the introduction of the parameters (A32) into formula (B10), the central deflection can be given in the form

$$w_{max} = \frac{p\,a^4\,\lambda}{3\,\pi^4\,I\,\sqrt{E_x E_y}\,\left(V^{(f)} + V\right)} \tag{B12}$$

with

$$V^{(f)} = \frac{I_f}{I}\left\{\alpha + \frac{2\,a^2\,\beta}{3\,b^2} + \frac{a^4}{\alpha\,b^4}\right\} \tag{B13}$$

and

$$\left.\begin{aligned} V &= \frac{\alpha + \dfrac{2\,a^2\,\beta}{3\,b^2} + \dfrac{a^4}{\alpha\,b^4} + 4\left\{\dfrac{S_x a^2}{b^2} + S_y\right\} F}{1 + 4\,S_x\left(\alpha + \dfrac{a^2\,\gamma}{3\,b^2}\right) + 4\,S_y\left(\dfrac{a^2}{\alpha\,b^2} + \dfrac{\gamma}{3}\right) + 16\,S_x\,S_y\,F} \\ &\text{with} \\ F &= \frac{a^2}{b^2}\left(1 - \frac{\beta^2}{9}\right) + \frac{\gamma}{3}\left(\alpha + \frac{2\,a^2\,\beta}{b^2} + \frac{a^4}{\alpha\,b^4}\right) \end{aligned}\right\} \tag{B14}$$

The term $\underline{V^{(f)}}$ is negligible if the thickness of the core is large as compared with the thickness of either facing.

In the event that the ratio $\frac{b}{a}$ is large, the function

$$w = C\,\sin^2\frac{\pi x}{a} \tag{B15}$$

leads to better results than those obtained on the basis of formula (B1). This representation of the displacement is one which has been used with good results in the treatment of an infinite plate in ordinary plate theory. With the use of this function in place of (B1) the expressions for the energies per unit length of sandwich are

$$U_c = \frac{c\,\pi^2\,\mu'_{zx}}{4a}\,(1 - k)^2\,C^2 \tag{B16}$$

$$U_M = \frac{\pi^4\,E_x}{a^3\lambda}\left[f_1\left(kq + \frac{f_1}{2}\right)^2 + f_2\left\{k\,(c - q) + \frac{f_2}{2}\right\}^2\right]C^2 \tag{B17}$$

and

$$U_f = \frac{\pi^4\,E_x}{a^3\lambda}\left[\frac{f_1^{\,3} + f_2^{\,3}}{12}\right]C^2 \tag{B18}$$

which replace (B2), (B3), and (B4) respectively, and

$$U_L = \frac{p\,Ca}{2} \tag{B19}$$

is obtained in place of (B7). When (B16), (B17), and (B18) are compared with (B2), (B3), and (B4), respectively, it is seen that the former can be derived by the latter by first multiplying each by $\frac{8}{3b}$ and then taking the limit as b becomes infinite. On this basis formula (B5) is replaced by

$$U = \frac{\pi^2}{3a} C^2 \, T_\infty \tag{B20}$$

where T_∞ denotes the limit of T as $b \rightarrow \infty$. According to the discussion following formula (B5) T is obtained from formulas (A18) and (A19) by suppressing the subscripts mn. After taking the limit as b becomes infinite, the quantities $A^{(i)}$, $i = 1 — 5$, in terms of which T is given, reduce to

$$A_\infty^{(1)} = \frac{3\,\pi^2}{a^2}\,\frac{E_x}{\lambda'}, \quad A_\infty^{(4)} = \frac{3\,\mu'_{zx}}{4}$$

$$A_\infty^{(2)} = A_\infty^{(3)} = A_\infty^{(5)} = 0 \tag{B21}$$

Now from (B19) and (B20)

$$W = \frac{\pi^2 C^2}{3a} T_\infty - \frac{p\,a\,C}{2}$$

and from the last of conditions (B9)

$$C = \frac{3\,p\,a^2}{4\,\pi^2\,T_\infty} \tag{B22}$$

The remaining conditions (B9) express the condition that T_∞ be a minimum with respect to (kq) and k. This minimum, which is denoted by T'_∞, is obtained from (B10) by taking the limit as $b \rightarrow \infty$. Thus

$$T'_\infty = I_f\, A_\infty^{(1)} + \frac{I\, A_\infty^{(1)}}{1 + \dfrac{\phi\, A_\infty^{(1)}}{A_\infty^{(4)}}}$$

The central deflection obtained by substituting this expression for $\underline{T_\infty}$ in (B22) and making use of (B21) is

$$w_{max} = \frac{p\, a^4\, \lambda}{4\, \pi^4\, E_x\, I \left\{ \frac{I_f}{I} + \frac{1}{1 + \frac{4\phi\, \pi^2\, E_x}{a^2\, \lambda\, \mu'_{zx}}} \right\}} \tag{B23}$$

Again, the ratio $\frac{I_f}{I}$ is usually so small that it can be neglected.

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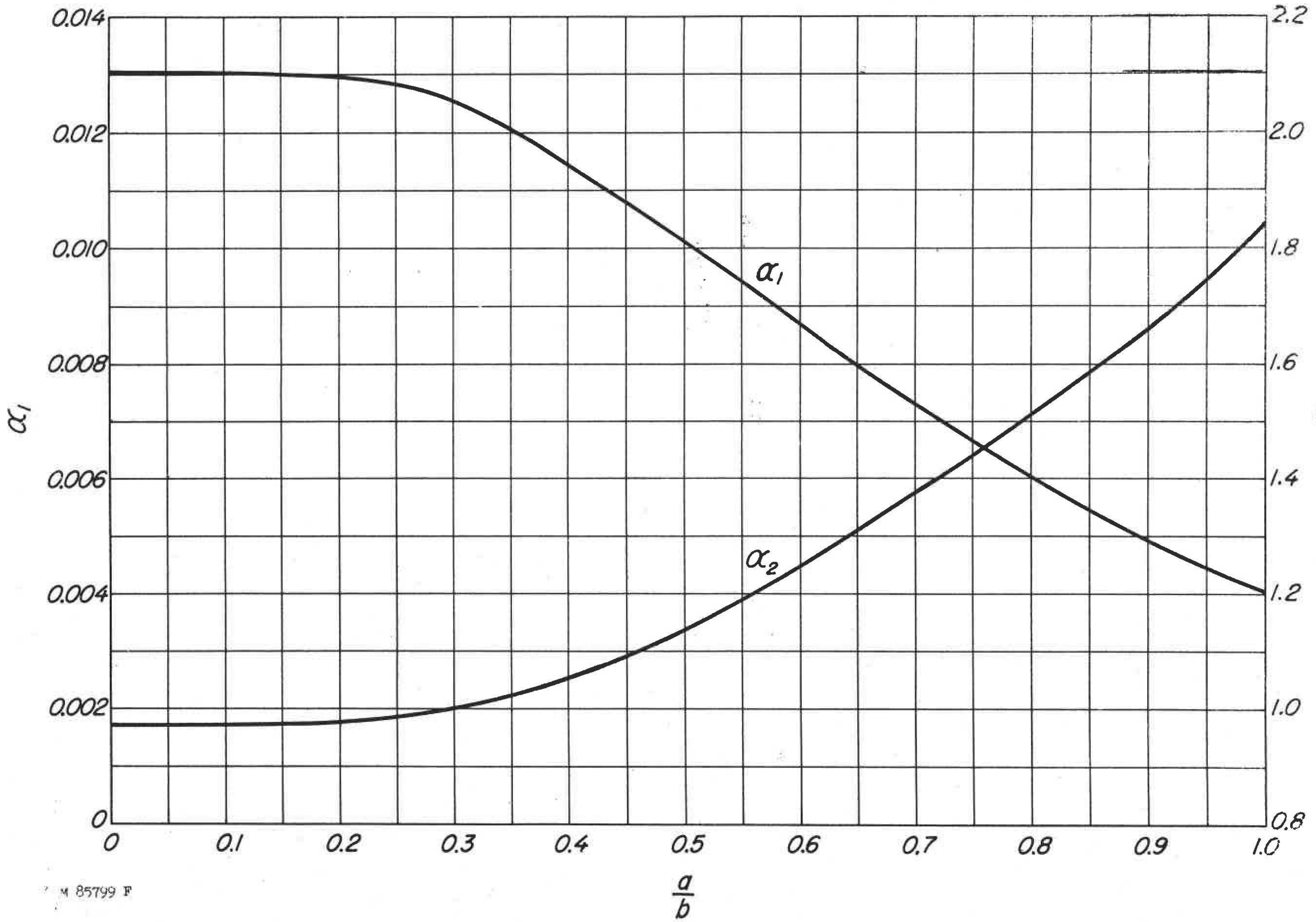
Figure 1.--Parameters for determining the central deflection of a uniformly loaded isotropic sandwich panel. Edges simply supported.

$$w_{max} = \frac{p\ a^4\ \lambda_f}{I\ E_f}\ \alpha_1 \{1 + S\ \alpha_2\}$$

(M 85799 F)

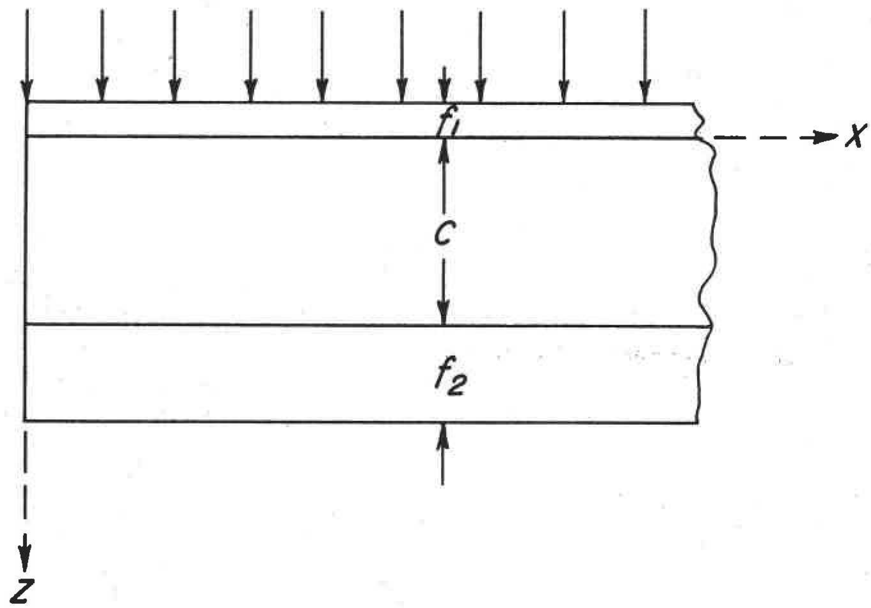


0.014
0.012
0.010
0.008
0.006
0.004
0.002
0
α1
α2
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
a/b

M 85799 F

Figure 2.--Cross section of loaded sandwich panel.

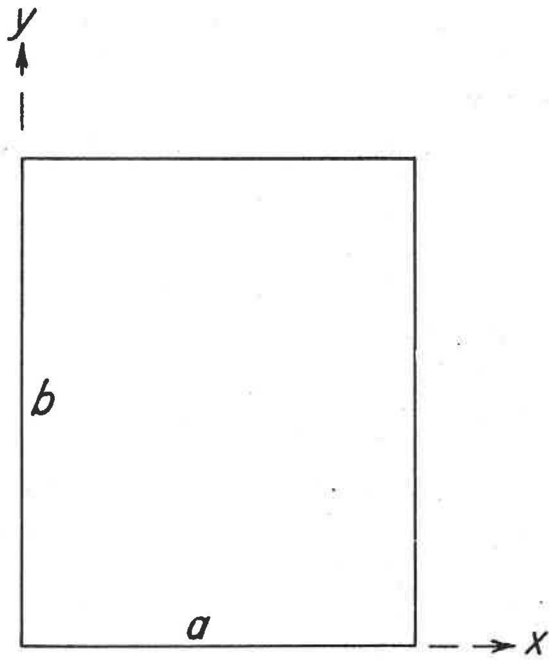
(M 85797 F)



Z M 85797 F

Figure 3.--Section of panel parallel to facings.

(M 85798 F)



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