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A standard tool in general relativity is the 3+1 or ADM point of view, namely slicing spacetime into spacelike hypersurfaces of constant time and then describing physics in terms of time-dependent quantities on a typical such hypersurface. Much less well-known is the 1+3 point of view, in which one foliates spacetime with timelike curves, then describes physics in terms of the surfaces "locally orthogonal" to the given foliation. This is precisely the description of physics as seen by a single observer. However, in many instances there do not exist such orthogonal hypersurfaces. One may instead attempt to describe physics on the manifold of orbits defined by the timelike curves, but one must then develop a parametric theory to handle the time dependent objects defined on the manifold of orbits.

I will present two equivalent descriptions of parametric manifolds. The first is based on a generalized Gauss-Codazzi formalism which involves projection to a lower-dimensional "surface". The second is an intrinsic description which involves redefining the action of vector fields on functions. In either description one is lead to generalized notions of connections, Lie bracket, and exterior differentiation.

Unique to a parametric theory of geometry is the *deficiency*. Although independent of the torsion, the deficiency behaves like torsion in the parametric direction. We will show how the deficiency emerges as a result of the above generalizations.

The 3+1 formalism arises naturally in considering initial-value formulations both for fields on a fixed background spacetime and for the spacetime itself. The applicability of parametric manifolds to such problems will be discussed.

Parametric Manifolds

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I would also like to extend a handshake to my parents, Al and Mickey, grandfather, Dr. Frank Boersma, and my wife's parents, Bob and Anita. Everyone seemed to understand my short and infrequent visits during the last few months while I finished my dissertation. Their patience and compassion is greatly appreciated.

Although my Basset Hound, Maggie, is unable to read, she deserves some thanks as well. She was a welcome sight on many occasions when I came home late and exhausted.

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1. Introduction

1.1 Statement of the Problem

The fundamental premise underlying general relativity is the unification of space and time into *spacetime*, a four-dimensional manifold with local Lorentzian geometry.[1]

While the unification of space and time is one of the great achievements of this century, occasionally it is quite useful to split spacetime back into space and time! Certainly such a splitting would depend upon the notion of "time". For example, given a family of observers, one may have surfaces of "constant time" which are used to synchronize the observers' clocks. Alternatively, each observer has an independent notion of time as measured along their world lines (proper time). In such a situation, what the observer sees at any given "time" is determined by the subspace perpendicular to its world line; the *local rest space* of the observer. Even in such simple cases as flat spacetime with constantly rotating observers, these two notions of time disagree. In an attempt to emphasize the fundamental nature of what observers see we turn to parametric manifolds.

A *parametric manifold* can be thought of as a manifold in which all of the geometric objects (*e.g.,* tensors) are allowed to depend on an extra parameter such as time. A *parametric structure* is then given by specifying a one-parameter family of one-form fields. This parametric structure affects the action of (parametric)

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[1] The theory of parametric manifolds is not dependent on the four-dimensional Lorentzian structure of spacetimes. Parametric manifolds of arbitrary dimension and signature may be defined.

vector fields on (parametric) functions, the notion of covariant differentiation, as well as other derivative operators. While this extra structure leads to generalized notions of connections and curvature, it also brings into existence a new operator known as the *deficiency*. As will be shown later in this dissertation, the deficiency measures the failure of the parametric theory to agree with a more traditional theory of tensor analysis on manifolds. While initially quite bothersome, the deficiency of a parametric structure may be easily defined and handled in an elegant way, much like torsion. In fact, the deficiency and torsion behave so similarly that the deficiency can easily be confused with torsion. They are, however, separate objects.

This dissertation will give a complete and comprehensive description of all of the objects and tools necessary for a rigorous study of parametric manifolds. This goal will be accomplished by examining parametric manifolds from two primary points of view. First, parametric manifolds will be discussed in terms of projected higher-dimensional quantities. This approach will be called the *extrinsic approach*. Second, parametric manifolds will be described entirely in terms of geometric quantities that are intrinsic to the manifold, including the given parametric structure. This approach will be called the *intrinsic approach*.

The extrinsic approach is closely connected with the historical development of parametric theories. While many of the tools and ideas associated with parametric manifolds have been around since the early days of general relativity, in 1993 Zoltán Perjés began a more abstract approach and succeeded in identifying the fundamental feature of a parametric structure (a one-parameter family of one-form fields). Historically, as we shall see, the parametric theories began to emerge as a consequence of decomposing (both four-dimensional and five-dimensional) spacetimes with respect to a preferred congruence of curves. This so called 1+3 (or 1+4) decomposition of spacetime has been overshadowed in the recent literature by the dual formalism of a 3+1 splitting of spacetime, also known as the ADM formalism. In either case, one is dealing with projected theories of geometry. In the ADM setting, one is primarily decomposing the higher-dimensional spacetime quantities in terms of lower-dimensional quantities which are thought of as geometric objects on embedded hypersurfaces. The study of embedded hypersurfaces is a classical topic in differential geometry known as the Gauss-Codazzi formalism, and there are standard ways of relating the geometry of a higher-dimensional manifold to the geometry of an embedded surface (of any dimension). Thus, the projected theory induced by a 3+1 decomposition of spacetime is the geometry induced by the well known Gauss-Codazzi formalism.

However, the 1+3 decomposition is quite different. As the 1+3 decomposition is based upon a given congruence of curves, there are no guarantees that these curves will be orthogonal to a family of hypersurfaces. That is, the three dimensional quantities defined by the 1+3 decomposition have no natural place to live! Without the existence of these orthogonal hypersurfaces, the standard Gauss-Codazzi formalism does not apply. This dissertation will generalize the standard Gauss-Codazzi formalism to the case where such surfaces do not exist. This generalization will lead to projected objects (as does the standard Gauss-Codazzi approach) which will be interpreted as living on the manifold of orbits, Σ, which will in turn be identified with a parametric manifold. Specifically, such a generalization will lead to a natural (affine) connection on Σ, a generalized notion of extrinsic curvature, as well as a unique curvature operator on Σ which is related to the Riemann curvature tensor of the original manifold by the standard Gauss equation. The induced notion of extrinsic curvature is no longer a symmetric tensor. However, its failure to be symmetric is directly related to the deficiency of the parametric structure defined on Σ. Given a projected theory of geometry based on the generalized Gauss-Codazzi theory, one may give a complete description of parametric manifolds in terms of the original higher dimensional manifold.

While the extrinsic approach to parametric manifolds allows one to develop a parametric theory based upon a given congruence, the intrinsic approach describes the notion of a parametric manifold without any reference to a higher-dimensional space. Thus, although one may use the insight gained from studying the extrinsic projected quantities, working intrinsically does not allow one the use of projection operators. In order to overcome the absence of projection operators, one is led, in particular, to consider a generalized notion of connection. This generalized connection has its roots in the parametric action of vector fields on functions which (in a coordinate basis) is **not** partial differentiation. The notions of Lie bracket and torsion must also be carefully defined in this setting.

One interesting development evolving from an intrinsic description of parametric manifolds involves the concept of a generalized exterior derivative operator d_*. It differs from a standard exterior derivative operator in the sense that $d_*^2 \neq 0$! However, the non-vanishing behavior of d_*^2 is completely characterized by the aforementioned deficiency of the parametric structure.

Having a complete mathematical description of parametric manifolds allows one an alternate viewpoint on initial-value problems in general relativity. By studying the decomposition of the spacetime Laplacian, I will demonstrate how parametric manifolds may be used to develop a procedure to quantize the scalar field in arbitrary background spacetimes. Also, I will address the initial-value problem of general relativity itself. As the standard initial-value formulation of general relativity relies on the Gauss-Codazzi equations, the generalized Gauss-Codazzi equations can be used with less restrictive sets of initial-date.

1.2 History of the Problem

In 1921 A. Einstein presented a paper by T. Kaluza [14] in which Kaluza introduced a five-dimensional theory of spacetime, based on Einstein's general theory of relativity, as an attempt to unify gravity and electromagnetism. In his paper, Kaluza assumed that the five-metric was independent of the fifth dimension. Central to Kaluza's analysis was his 1+4 decomposition of this stationary spacetime. In an attempt to add some physical interpretation to Kaluza's fifth dimension, Einstein and Bergmann generalized Kaluza's framework [8]. Their generalization contained two ideas central to the theory of parametric manifolds.

First, Einstein and Bergmann recognized only certain classes of five-dimensional coordinate transformations as having any relevance to the four-dimensional "physical world". As a result, Einstein and Bergmann were led to consider a four-dimensional metric (the natural metric associated with a 1+4 decomposition of spacetime) which was allowed to depend (with some restrictions) on the fifth dimension. In the context of this dissertation, Einstein and Bergmann were considering a one-parameter family of metrics, *i.e.*, parametric metrics!

Second, Einstein and Bergmann presented derivative operators which were covariant with respect to the restricted class of coordinate transformations, rather than the entire class of five-dimensional coordinate transformations. These analytic tools are precisely the analytical operators one needs to define a coherent parametric theory of spacetime. Einstein and Bergmann constructed an additional covariant derivative operator (partial derivative with respect to the "parameter"), a new notion of partial differentiation (parametric differentiation), a generalized covariant derivative operator, as well as a generalized notion of curvature (later called the Zel'manov curvature). In the language of parametric manifolds, these objects are the most natural geometric operators which are invariant with respect to a reparameterization of the manifold. Later, Bergmann presented one of the most complete descriptions of a parametric theory of spacetime to be found [3].

Around 1955 the Russian mathematician A. Zel'manov rediscovered many of the analytical tools developed by Einstein and Bergmann. In [31], Zel'manov discusses physical spacetime quantities which transform covariantly with respect to certain coordinate transformations. While he is led to consider the same family of coordinate transformations that Einstein and Bergmann were, Zel'manov was not working in an extra (fifth) dimension. Instead, Zel'manov was interested in discussing coordinate transformations which related coordinate systems which were at rest with respect to the same frame of reference. Zel'manov also made use of a derivative operator very similar to the one introduced by Einstein and Bergmann.

In 1958 the Italian mathematician C. Cattaneo, while studying equations of motions of free test particles in general relativity, made use of a transverse differential operator [5]. This differential operator is the same reparameterization invariant differential operator used by Einstein, Bergmann, and Zel'manov. Cattaneo explicitly recognized this operator as a projected partial differential operator, thus setting the stage for a projected theory of geometry leading to the notions surrounding parametric manifolds.

In 1993, Z. Perjés presented what he called a "parametric manifold picture of spacetime". While Perjés makes use of all the previous analytical quantities introduced by Einstein and Bergmann, he was the first to give an abstract definition of a parametric manifold. Central to such an abstract notion, Perjés successfully describes the additional structure necessary for an elegant description of parametric manifolds. In [26], Perjés provided a brief introduction to the subject of parametric manifolds before presenting a complete description of a parametric theory of spinors. Together with Gy. Fodor, Perjés applied this parametric theory to a canonical analysis of relativistic gravitation [9].

At this point, I believe a complete description of parametric manifolds is still missing. In this dissertation I will build upon Perjés' work in several ways. First, the generalized Gauss-Codazzi formalism will not only reproduce many of Perjés' analytical tools, but it will also help lead to the notion of a parametric Lie bracket. Such a definition is crucial both in providing a complete interpretation of the deficiency, as well as in understanding the similarities and differences between deficiency and torsion. Second, this dissertation will provide the necessary definitions for a complete intrinsic description of parametric manifolds. As we will see, an underlying theme to such a description involves re-defining the action of a vector field on a function. Changing the action of a vector field on a function yields generalized notions of "partial" derivative, covariant derivative, Lie bracket, exterior derivative, and curvature. Thus, one may think of the theory of parametric manifolds as a special case of a generalized differential geometry in which one simply changes the natural action of a vector field on a function. Again, as a result of this non-standard action, one is forced to introduce the notion of deficiency.

Although the literature abounds with descriptions of 1+3 and 1+4 splitting of spacetime, I have tried to concentrate the above chronology so as to include those authors who actually began to introduce some sort of reparameterization invariant objects and operators. In [13], Jantzen and Carini offer a more complete listing of references central to the 1+3 decomposition of spacetime.

1.3 Dissertation Summary

Chapter 2 sets the stage for the formal constructions by summarizing previous work. It begins with an introduction to the 3+1 and 1+3 (*slicing* and *threading* respectively) splittings of spacetime. In each case, the decomposition and construction of the spacetime metric is discussed. After the summary of the different spacetime splittings, we examine Perjés' recent work on parametric manifolds. Although Perjés was not considering an extra spacetime dimension, as was mentioned above, many of Perjés' definitions are identical to those first introduced by Einstein and Bergmann in 1938. However, Perjés interprets these objects as being *reparameterization invariant* rather than transforming under some higher-dimensional coordinate transformation. Furthermore, Perjés identifies the parametric structure as being carried by a one-parameter family of one-form fields. Such an observation lays the groundwork for a thorough description of parametric manifolds. In the light of spacetime splittings, one can better understand many of Perjés' definitions and observations. Central to the 1+3 decomposition of spacetime is the *threading metric* and the *threading shift one-form*. In this context, one sees that Perjés' parametric metric agrees with the threading metric, and Perjés' candidate for the structure one-form agrees with the threading shift one-form. Thus, we see how a threading decomposition of spacetime naturally leads one to consider a parametric manifold picture of spacetime. In addition, one may start to view Perjés' parametric operators (∂_* and ∇_*) as projected spacetime operators. The discussion of spacetime splittings and parametric manifolds concludes with a brief example illustrating the various viewpoints.

Chapter 3 offers a more thorough description and exhaustive discussion of parametric manifolds. Precise definitions of curvature, torsion, and deficiency are given. Two separate approaches are presented. First, the extrinsic approach focuses on the relationship between parametric manifolds and projected spacetime quantities. As was mentioned earlier, attempting to expand upon this relationship requires a generalization of the standard Gauss-Codazzi formalism to the case were there are no orthogonal hypersurfaces. The generalized Gauss-Codazzi approach then allows one to define a metric and an affine connection on the manifold of orbits which agree with Perjés' parametric metric and connection. In addition, one has the notion of deficiency, which measures the fact that the projected spacetime bracket operator is not closed (since we have generalized the Gauss-Codazzi approach to the case without orthogonal hypersurfaces). While this generalized approach reproduces Perjés' parametric metric and parametric connection, it leads to a curvature operator which differs from Perjés'. This new curvature operator is the unique curvature operator satisfying a generalized version of Gauss' equation. As Perjés' curvature operator was the same curvature operator found in the previously cited literature, the curvature operator induced from the generalized Gauss-Codazzi approach also differs from the curvature operators of Zel'manov, Einstein, and Bergmann. The relationship between the two curvature operators will be given explicitly and discussed.

Following the complete extrinsic description of parametric manifolds is the intrinsic description. Using all of the information gathered from examining spacetime splittings, projections, and the generalized Gauss-Codazzi formalism, we define parametric manifolds in terms of an abstract manifold together with some additional structure which transforms correctly under a reparameterization. Central to all of the parametric definitions is the action of parametric vector fields on parametric functions. Defining an action which differs from the standard action of vector fields on functions leads to an entirely different type of geometry; a parametric geometry! As was mentioned above, we will consider generalized notions of Lie bracket, covariant differentiation, torsion, curvature, and exterior differentiation. Such generalizations give rise to a new operator: the deficiency. In essence, the deficiency measures the deviation from a standard geometric theory.

Chapter 4 begins with a more formal treatment of foliations. Although the discussions of slicing and threading revolved around the notions of foliations, a detailed discussion of foliations was omitted so as not to overshadow the introductory nature of Chapter 2. Indeed, the majority of this dissertation concerns itself with understanding the relationships between different types of foliations rather then concentrating on the existence of such foliations. Of particular interest is the case where the leaves of the foliation corresponds to the fibres of a fibre bundle. While examining conditions which allow a foliated Riemannian manifold to be a fibre bundle over the manifold of orbits, Reinhart introduced the notion of a *bundle-like metric*. In the case of threading, it turns out that in order for the spacetime metric to be bundle-like, the threading metric must be independent of the time coordinate t. Thus, the notion of bundle-like is slightly less restrictive then requiring the coordinate vector field $\frac{\partial}{\partial t}$ to be Killing. Since we have already identified the threading metric with the parametric metric, we see how allowing for a parametric metric generalizes Reinhart's bundle-like condition.

In this chapter, I also discuss how a metric on the total space of a fibre bundle induces metrics on the typical fibre and the base space. Interpreting spacetime as a fibre bundle and choosing the fibre and base space correctly, one may easily reproduce the slicing and threading decompositions of spacetime. By letting the fibres represent the leaves of the slicing or threading foliation, one is naturally handed the slicing and threading decompositions respectively. Thus, the fibre bundle picture provides a single mathematical framework in which to discuss both decompositions simultaneously. Since the parametric manifold viewpoint naturally comes from the threading viewpoint, the fibre bundle which reproduces the threading decomposition may also be used to describe the parametric viewpoint. In such a setting, parametric functions are realized as functions on the total space, so that the action of parametric vector fields on parametric functions is achieved by considering their horizontal lifts. In this way, the parametric derivative operator ∂_* is naturally recovered. This action may also be recovered by interpreting it as an action of a jet field on a function. Under such an interpretation, the parametric exterior derivative operator d_* is seen to correspond to the notion of a *total derivative* in a jet bundle.

Chapter 5 concludes with some preliminary results concerning the quantization of the Klein-Gordon equation for arbitrary background spacetimes. By examining the decomposition of the spacetime Laplacian, one begins to see the usefulness of a parametric theory of spacetime. I will also show how the generalized Gauss-Codazzi formalism may be used to study the initial-value problem of general relativity itself.

With such strong positive results, I hypothesize the applicability of parametric manifolds to other initial-value problems.

2. Slicing, Threading, and Parametric Manifolds

2.1 Introduction

The phrase *parametric manifold* refers to a smooth manifold, Σ, possessing additional structure which remains invariant under a notion of reparameterization. This additional structure leads one to consider one-parameter families of tensor fields on Σ together with a parametric theory of tensor analysis. Although there are occasional references to some of these ideas in the literature, the most recent work on this subject is to be found in the current work of Zoltán Perjés, [26]. In [26], Perjés is concerned with the dynamics of spacetime as viewed in general relativity. Thus, this chapter concerns itself mainly with four-dimensional Lorentzian manifolds. However, there is certainly no need to restrict the definition of parametric manifolds to such situations.

Perjés' approach to parametric manifolds is closely related to a formalism which involves decomposing spacetimes which admit a preferred congruence of (non-lightlike) curves. Such an approach leads to a mathematical framework which concerns itself with a splitting of spacetime into two portions; one discusses the geometry of spacetime in a direction tangent to the (one-dimensional) congruence, while the other describes the (three-dimensional) geometry orthogonal to the curves. Although this (1+3)-decomposition of spacetime, called the *threading viewpoint* in [13], has its roots in classical texts ([16] and [20]), it is not as widely used as the so-called (3+1)-decomposition.

The (3+1)-decomposition, or ADM formalism, also refers to a mathematical framework for describing the four-geometry of spacetime in terms of a three-geometry and a one-geometry. However, the fundamental idea behind the (3+1)-decomposition is the existence of a foliation of the spacetime by spacelike hypersurfaces. Hence the (3+1)-decomposition refers to the splitting of spacetime in terms of the geometry of the hypersurfaces and the geometry of the spacetime orthogonal to the surfaces. In [13] such an approach is referred to as the *slicing viewpoint.*

As one might imagine, the slicing and threading viewpoints involve many of the same mathematical considerations. In [13], Robert Jantzen and Paolo Carini provide an elegant description and a much needed comparison of these two similar viewpoints. In Chapter 4, I show how to obtain both slicing and threading decompositions as special cases of a more general construction involving foliations.

The primary goal of this chapter is to introduce the notion of a parametric structure. While the slicing viewpoint is more widely used, it is the threading viewpoint which provides a natural parametric structure. Thus, in this chapter I will briefly introduce the two approaches to spacetime splittings, exhibit the central notions surrounding a parametric theory of spacetime, and describe how such a parametric theory inherits many of its ideas from the threading viewpoint. This unified summary of previous work by others will set the stage for a more detailed discussion in Chapter 3. The chapter concludes with an example involving rotating coordinates in three-dimensional Minkowski space. The example discusses the difference between the slicing and threading viewpoints and offers the reader a brief glimpse into some of the geometrical difficulties involved in developing a precise parametric theory of spacetime.

2.2 Slicing

2.2.1 Introduction

A $(3+1)$-decomposition of a four-dimensional spacetime has proved to be a successful framework for formulating the dynamics of geometry (*c.f.*, [19]). There exist two standard approaches to such a splitting, both of which yield the standard definitions of *lapse* and *shift*; one being a construction process and the other a decomposition process. For the construction, one begins with three-dimensional surfaces and attempts to "fill in" between these surfaces to construct a spacetime which admits the original three-dimensional surfaces as a foliation of spacelike hypersurfaces. The spacetime metric is thus constructed out of the three-metric of the hypersurfaces as well as additional bits of information. Alternatively, one could start with a spacetime which admits a one-parameter family of spacelike hypersurfaces and then decompose all of the original four dimensional geometrical information (*e.g.,* tensor fields) into two pieces; one tangent to the surfaces and one normal to the surfaces. As we shall see, both approaches yield an equivalent (3+1)-interpretation of spacetime.

2.2.2 Notation

Throughout the next two sections we will be working with complete spacetimes foliated by spacelike hypersurfaces. Furthermore, let us assume that the hypersurfaces are all diffeomorphic to each other. Thus, one may simplify the notation by working in extremely nice coordinate neighborhoods. Begin by introducing a global time function t which can be regarded as the parameter which labels the hypersurfaces. Furthermore, we will work in a neighborhood small enough so that the intersection of each hypersurface with the neighborhood admits coordinates $\{x^i\} = \{x^1,\ x^2,\ x^3\}$ on the hypersurface Σ_t. Thus, $p \in \Sigma_t$ can be given the coordinates (x^i, t). To simplify the notation, I will use Greek letters as indices which take on all four spacetime dimensions with $x^0 \equiv t$. Thus, $\{x^\alpha\} = \{t, x^i\}$ and we can write the spacetime metric g in terms of its components given by $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$. Throughout this dissertation, repeated indices are to be summed over.

2.2.3 The Construction

Suppose one had a spacetime foliated by a one-parameter family of spacelike hypersurfaces, Σ_t. One would like to realize the four-geometry of this spacetime as arising from the three-geometries of these surfaces. Thus, one can "construct" the spacetime metric out of the spatial metrics of the surfaces. Of course, additional information must also be provided. Following the description in [19], let us assume that the three-geometry of two infinitesimally close surfaces is known. Label these surfaces by Σ_t and $\Sigma_{t+\Delta t}$. Each of these surfaces has an associated spatial metric, k_t and $k_{t+\Delta t}$. At the risk of de-emphasizing the dependence on the coordinate t, I will use the same notation to refer to both spatial metrics and write $k_{ij}\, dx^i dx^j$ for the three-metrics on both surfaces.

We now describe the four-geometry that fills in between these slices. Given a point $p_0 = (x^i, t) \in \Sigma_t$ and a nearby point $q_0 = (x^i + \Delta x^i, t + \Delta t) \in \Sigma_{t+\Delta t}$, we are interested in calculating the coordinate distance between p_0 and q_0, $d(p_0, q_0)$. By taking advantage of the existence of a metric in the slice Σ_t, it seems most natural to use the Pythagorean Theorem of Lorentzian geometry and write

$$d(p_0, q_0)^2 = d(p_0, p_1)^2 - d(p_1, q_0)^2 \tag{2.1}$$

where $p_1 \in \Sigma_t$ is chosen so that $d(p_1, q_0)$ is the orthogonal distance between the two hypersurfaces. See figure 1.

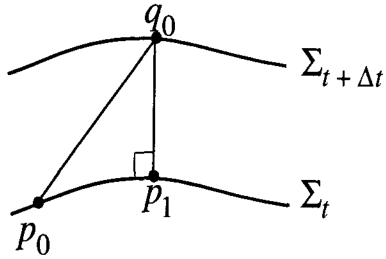

Figure 1: Calculating Distances

It is now apparent that in order to fully construct a spacetime metric much information must still be specified. As we have no *a priori* knowledge of what it means to move orthogonally to the surfaces, the location of p_1 is not fully determined. As Δt approaches zero, the point q_0 should approach p_1. However, there is no reason to assume that the coordinates of p_1 are $(x^i + \Delta x^i, t)$. Rather, the point p_1 could be "shifted" in any of the three spatial directions. Thus, we assign the coordinates $p_1 = (x^i + \Delta x^i + N^i \Delta t, t)$. The three functions N^i depend on the coordinates of Σ_t as well as the parameter t. As the functions N^i describe how the nearby surfaces are shifted with respect to one another, they are commonly referred to as the components of the *shift vector*. The given metric k_{ij} may now be used to measure $d(p_0, p_1)$.

The quantity $d(p_1, q_0)$ is still not determined. In order to fix this distance, one must know the relationship between the proper time ("distance") from Σ_t to $\Sigma_{t+\Delta t}$ and the arbitrary parameter t. Again, this distance may depend on the coordinates in Σ_t as well as t. Define the *lapse function* N by

$$d(p_1, q_0) = N(x^i, t)\,\Delta t.$$

One may now describe the four-geometry in terms of the lapse function and shift vector. Adding these newly defined quantities to the earlier figure yields the following picture:

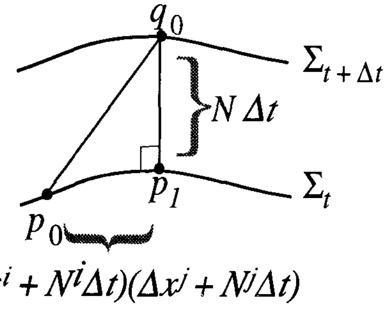

Figure 2: Slicing Lapse Function and Shift Vector

Using equation (2.1) and letting $\Delta t \to 0$ we see that the four-geometry of the spacetime can be represented by the line element

$$\begin{aligned} ds^2 &= k_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - N^2 dt \\ &= (N_i N^i - N^2)dt^2 + 2N_i dt\, dx^i + k_{ij} dx^i\, dx^j \end{aligned} \tag{2.2}$$

where I have used the three-metric k_{ij} to define $N_i = k_{ij}N^j$. As mentioned earlier, the functions N^i are thought of as the component functions of a vector field "tangent" to each hypersurface. The *shift vector field* is defined by

$$N^i \frac{\partial}{\partial x^i}.$$

Thus each hypersurface has a fully spatial metric, k_{ij}; a tangent vector field, $N^i \frac{\partial}{\partial x^i}$; and a function N. As we have seen, these three spatial quantities may be used to construct the four-dimensional spacetime metric g.

In matrix notation, one can write the components of the spacetime metric tensor in terms of N, N^i, and k_{ij} as follows:

$$(g_{\alpha\beta}) = \begin{pmatrix} -(N^2 - N_m N^m) & N_j \\ N_i & k_{ij} \end{pmatrix}$$

with inverse

$$(g^{\alpha\beta}) = \begin{pmatrix} -N^{-2} & N^{-2}N^j \\ N^{-2}N^i & k^{ij} - N^{-2}N^i N^j \end{pmatrix}$$

where k^{ij} is the inverse of k_{ij} defined by

$$k_{il}k^{lj} = \delta_i^{\ j}.$$

Our parameter t now takes the role of a spacetime coordinate whose coordinate vector field $\frac{\partial}{\partial t}$ may be interpreted as representing the "flow of time" in the newly constructed spacetime. Since the coordinates x^i are constant along integral curves of $\frac{\partial}{\partial t}$, one may think of the lapse and shift as the means of identifying points on different hypersurfaces. See figure 3.

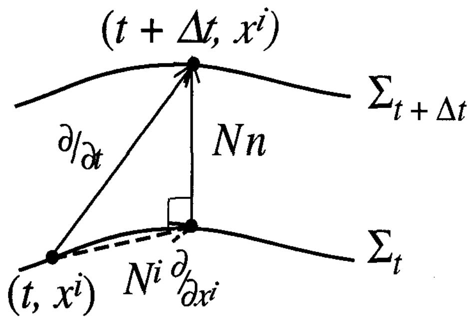

Figure 3: Decomposition of $\frac{\partial}{\partial t}$

The shift vector field $N^i \frac{\partial}{\partial x^i}$ was defined to account for the fact that there was no *a priori* knowledge of directions orthogonal to the surfaces Σ_t. However, in terms of the shift vector and lapse function we may now easily describe a future pointing unit spacetime vector field normal to each surface. Call this normal vector field n. Using $\langle\ ,\ \rangle$ to represent the four-metric we just constructed, we observe that $\langle N dt,\ N dt\rangle = -1$. Therefore, n is given by the metric-dual of the one-form $N\,dt$. Explicitly,

$$n = \frac{1}{N}\frac{\partial}{\partial t} - \frac{1}{N}N^i\frac{\partial}{\partial x^i}. \tag{2.3}$$

Written in this manner, it is now apparant how the functions N^i describe the "shifting" of neighboring surfaces. The case of vanishing shift corresponds to the scenario where coordinate time is flowing orthogonally to the hypersurfaces (*i.e.*, the directions of n and $\frac{\partial}{\partial t}$ agree).

2.2.4 The Decomposition

As stated earlier, the above construction is simply an orthogonal splitting of spacetime geared towards an initial value formulation of spacetime. To see the obvious, begin as above with a spacetime which admits a foliation of spacelike hypersurfaces. If we let n^α be the components of the future pointing unit vector field n normal to the hypersurfaces Σ_t, the naturally induced metric on each hypersurface may be obtained from the projection tensor

$$k_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta \tag{2.4}$$

where $n_\alpha = g_{\alpha\beta} n^\beta$ (*c.f.*, [30]) Now, for vector fields $X = X^\alpha \frac{\partial}{\partial x^\alpha}$ and $Y = Y^\beta \frac{\partial}{\partial x^\beta}$ tangent to Σ_t,

$$\begin{aligned} k_{\alpha\beta} X^\alpha Y^\beta &= g_{\alpha\beta} X^\alpha Y^\beta \\ &= g_{ij} X^i Y^j. \end{aligned}$$

Therefore, the functions $k_{ij} = g_{ij}$ may be thought of as the components of a three-dimensional metric on each hypersurface. One must not lose sight of the fact that the functions k_{ij} depend on the spacetime coordinate t (as do the $g_{\alpha\beta}$). We will refer to the functions k_{ij} as the components of the *slicing metric.*

The slicing metric on Σ_t is the naturally induced metric in the following sense: for each imbedding $\iota_t : \Sigma_t \hookrightarrow \mathcal{M}$, $k = \iota_t{}^*(g)$, where $\iota_t{}^*$ refers to the natural map on the co-tangent spaces $T^*\mathcal{M}$ and $T^*\Sigma_t$. Thus, k is both the projection of g to Σ via (2.4) and the pullback of g to Σ. Using ι_{t*} to denote the natural map between the tangent spaces, we may work out the relationship explicitly. For tangent vector fields X and Y on Σ_t we can write $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$, and

$$\begin{aligned} k_{ij} X^i Y^j &= \iota_t{}^*(g)_{ij} X^i Y^j \\ &= g_{\alpha\beta} \left(\iota_{t*}(X)\right)^\alpha \left(\iota_{t*}(Y)\right)^\beta \\ &= g_{ij} X^i Y^j. \end{aligned}$$

One may decompose the coordinate vector field $\frac{\partial}{\partial t}$ into vector fields normal and tangent to each surface Σ_t. Thus one has

$$\frac{\partial}{\partial t} = Nn + N^i \frac{\partial}{\partial x^i}. \tag{2.5}$$

Equation (2.5) defines the *slicing lapse function* N and the *slicing shift vector field* $N^i \frac{\partial}{\partial x^i}$ and can be seen to agree with the earlier definitions by comparing equation (2.5) with (2.3). In this scenario, the shift vector measures the tilting of $\frac{\partial}{\partial t}$ away from the direction normal to the hypersurfaces.

Since $N^i \frac{\partial}{\partial x^i}$ is tangent to each hypersurface, we will use the slicing metric to define $N_i = k_{ij} N^j$.

2.3 Threading

2.3.1 Introduction

In a $(3+1)$-decomposition (or *slicing*) of spacetime, one has a foliation of spacetime by spacelike hypersurfaces labeled by a global time function t. This time function together with the earlier definitions of the lapse function and shift vector gives one a way of identifying points on different hypersurfaces. In effect, one has, in addition to a foliation of spacetime by hypersurfaces, a congruence of curves given by the integral curves of the coordinate vector field $\frac{\partial}{\partial t}$. While the spacelike nature of the hypersurfaces are an integral part of the standard $(3+1)$-decomposition, there are no similar causality conditions on the congruence of curves. Although we usually think of the parameter t as a local time coordinate, no formal causality restriction is necessary. When one adopts the dual ansatz of a foliation of spacetime by timelike curves together with a foliation of hypersurfaces (with no causality conditions imposed upon them), one is led to consider a $(1+3)$-decomposition (or *threading*) of spacetime (see [13]).

In such a setting, the timelike congruence may be interpreted as the world-lines of a family of observers, while the hypersurfaces play the fundamental role of synchronizing the clocks of the different observers.

As with the last section, I will introduce the threading point of view from two different perspectives. First, I will address the issue of constructing a spacetime from a given family of curves. Second, I will illustrate the threading point of view by considering a certain decomposition of spacetime. One should notice the similarities between the slicing and threading points of view.

2.3.2 The Construction

In the previous section we saw how one would construct a spacetime metric from a one-parameter family of three-dimensional Riemannian manifolds. The resulting four-metric was easily described in terms of the given metrics on the surfaces, the slicing lapse function, and the slicing shift vector field. Suppose one is given a family of timelike one-manifolds (curves) in place of the three-manifolds. How would one go about constructing a spacetime which realized the original family of curves as a congruence of timelike curves? We will proceed as we did in the case of slicing.

In the earlier (3+1)-construction we had a parameter which labeled each hypersurface. Let us assume we have parameters $x^i, i = 1, 2, 3$ which label each curve L_{x^i}. On each curve suppose we have a coordinate t as well as a metric l, which can thus be expressed as

$$l = -M^2\, dt^2.$$

We will interpret these curves as being world-lines of observers, and hence requite that they be timelike. Consider the same measuring problem as before, that is, letting $p_0 = (x^i, t) \in L_{x^i}$ and $q_0 = (x^i + \Delta x^i, t + \Delta t) \in L_{x^i + \Delta x^i}$ we are interested in measuring the coordinate distance between p_0 and q_0. Since we are assuming we can measure distances in each curve L_{x^i}, again use the Pythagorean Theorem to write

$$d(p_0, q_0)^2 = -d(p_0, q_1)^2 + d(q_1, q_0)^2.$$

Here $d(q_1, q_0)$ is meant to refer to the orthogonal distance between two nearby curves. See figure 4

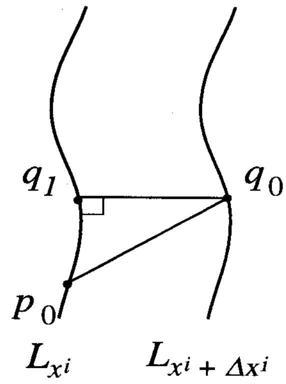

Figure 4: Calculating Distances

Since we do not have any notion of traveling "orthogonally" to the curves L_{x^i}, the t-coordinate of q_1 is not determined. The position of q_1 along L_{x^i} is affected by each of the Δx^i. Assign coordinates to q_1 by $q_1 = (x^i, t + \Delta t - M_i \Delta x^i)$. Again, the M_i record the amount of "shifting" of q_1 with respect to nearby curves. That is, the M_i may be thought of recording how $L_{x^i + \Delta x^i}$ has been shifted with respect to L_{x^i} in the construction process. Therefore, we have

$$d(p_0, q_1) = M(\Delta t - M_i \Delta x^i).$$

The three functions M_i and the function M depend on the parameters x^i as well as the coordinate t.

We now need to specify the relationship between the parameters x^i and the proper coordinate distance between neighboring curves. We thus introduce a "spatial metric" of the form $h_{ij} \Delta x^i \Delta x^j$ which gives the distance between L_{x^i} and $L_{x^i + \Delta x^i}$ for various choices of Δx^i. While we assume that $h_{ij} = h_{ji}$, the functions h_{ij} may otherwise be chosen arbitrarily. We continue our construction of the four-metric by assuming that this distance is precisely $d(q_1, q_0)$ (*i.e.*, measured orthogonally). We have:

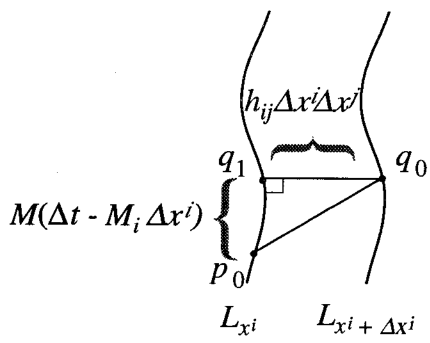

Figure 5: Threading Lapse Function and Shift One-Form

Thus, the Pythagorean Theorem implies that the spacetime metric may be written as

$$\begin{aligned} ds^2 &= -M^2(dt - M_i\,dx^i)^2 + h_{ij}\,dx^i\,dx^j \\ &= -M^2\,dt^2 + 2M^2M_i\,dx^i\,dt + (h_{ij} - M^2M_iM_j)dx^i\,dx^j. \end{aligned} \tag{2.6}$$

The component M of the original metric along each curve is referred to as the *threading lapse function.* As we see in the above representation of the metric (equation (2.6)), the functions M_i are most naturally associated with the one-form M_idx^i. The three functions M_i are referred to as the components of the *threading shift one-form* $M_i\,dx^i$. Finally, the functions h_{ij} may be thought of as the components of a metric, the *threading metric.*

In the case of slicing, one thought of the slicing shift vector as a three dimensional spatial vector field on the surfaces Σ_t and, hence, raised and lowered its indices with the slicing metric. In the present case of threading, one may again adopt the convention that the threading shift one-form be treated as a three dimensional tensor. Under such a convention, the threading metric h_{ij} may be used to raise and lower its indices.

The threading shift one-form was defined in order to introduce some notion of traveling "orthogonally" to the curves L_{x^i}. The unit one-form which annihilates the space of vectors orthogonal to the threading vector field may be written

$$m = -M(dt - M_idx^i).$$

One should compare this equation with the analogous equation for slicing (equation (2.3)).

2.3.3 The Decomposition

The threading lapse function and shift one-form field may also be described as arising from a simple orthogonal decomposition of spacetime. Analogous to the (3+1)-decomposition, the so-called (1+3)-decomposition attempts to decompose spacetime quantities into pieces orthogonal to the given congruence of curves, and pieces tangent to the congruence. As above, I will work in coordinates (t, x^i) where t acts as a parameter along the integral curves of $\frac{\partial}{\partial t}$ (the *threading curves*) and x^i are coordinates on each hypersurface $\Sigma_{t_0} = \{t \equiv t_0\}$. As in [13], I will refer to $\frac{\partial}{\partial t}$ as the *threading vector field.*

The normalization of the threading vector field is used to define the *threading lapse function* M:

$$\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = -M^2,$$

where $\langle\ ,\ \rangle$ refers to the spacetime metric g. If one views the threading curves as the world-lines of a family of observers, the threading lapse function measures the rate of change of the observed proper time with respect to the coordinate time function t.

In the slicing point of view one had to describe the discrepancy between $\frac{\partial}{\partial t}$ and the direction normal to the hypersurfaces. Analogously, in the present scenario one wishes to measure the amount of tilting of the local rest spaces of the observers with respect to each coordinate direction $\frac{\partial}{\partial x^i}$, keeping in mind that the local rest spaces of the observers need not constitute a hypersurface! Following the flavor of equation (2.5), we decompose the coordinate one-form dt into a piece which annihilates the local rest spaces and a piece which is in the co-tangent space of each surface. Letting m represent the metric dual of the unit vector field tangent to the threading curves, we have $m(\frac{\partial}{\partial t}) = -M$, so that we obtain

$$dt = -\frac{1}{M}m + M_i dx^i. \tag{2.7}$$

The functions M_i are the components of the *threading shift 1-form.*

Using the above definitions of M and M_i, the $(1+3)$-decomposition of the components of the four-metric takes the following form:

$$(g_{\alpha\beta}) = \begin{pmatrix} -M^2 & M^2 M_j \\ M^2 M_i & h_{ij} - M^2 M_i M_j \end{pmatrix} \tag{2.8}$$

with inverse

$$(g^{\alpha\beta}) = \begin{pmatrix} -(M^{-2} - M_m M^m) & M^j \\ M^i & h^{ij} \end{pmatrix}$$

where I have defined $M^i = h^{ij} M_j$.

The 3×3 matrix h_{ij} is defined by equation (2.8) and has inverse h^{ij}. The functions h_{ij} are the components of the *threading metric.*

Although one may take equation (2.8) as the definition of h_{ij}, historically it has a more familiar definition. For instance, in [16] and [20] one is given a physical interpretation of the threading metric. In general relativity, to calculate the spatial distance between an observer and an infinitesmly close event, one may direct a light signal from the observer to the event and back and calculate the "time" of propagation. One finds the spatial distance dl to be given by

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta$$

where

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta}}{g_{00}}.$$

Note that $\gamma_{00} = \gamma_{0\alpha} = \gamma_{\alpha 0} \equiv 0$ and that $\gamma_{ij} \equiv h_{ij}$ $(i, j = 1, 2, 3)$ as defined earlier (in our adapted coordinate system). [2]

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[2] In [8], Einstein and Bergmann used a similar argument during their attempts to generalize Kaluza's theory of electricity. Einstein and Bergmann, however, were

One can see that the threading metric simply measures what Cattaneo [5] refers to as the *space norm* of any 4-vector. That is, h_{ij} measures the norm of the component perpendicular to the threading curves. Specifically, for any 4-vector V^α write $V^\alpha = V_\parallel{}^\alpha + V_\perp{}^\alpha$ where $V_\parallel{}^\alpha$ is parallel to the threading curves and $V_\perp{}^\alpha$ is perpendicular. Letting m^α be the unit vector tangent to $\frac{\partial}{\partial t}$ one has

$$\begin{aligned} m_\alpha &= g_{\alpha\beta} m^\beta \\ &= \frac{1}{\sqrt{-g_{00}}} g_{0\alpha} . \end{aligned}$$

Thus,

$$\begin{aligned} g_{\alpha\beta} V_\perp{}^\alpha V_\perp{}^\beta &= (g_{\alpha\beta} + m_\alpha m_\beta) V^\alpha V^\beta \\ &= (g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta}}{g_{00}}) V^\alpha V^\beta \\ &= (g_{ij} + M^2 M_i M_j) V^i V^j \\ &= h_{ij} V^i V^j . \end{aligned}$$

At this point one notices a crucial difference between the slicing and threading pictures of spacetime. When slicing spacetime with spacelike hypersurfaces one defines the slicing metric which naturally lives on these hypersurfaces. While the threading metric arises in an analogous way, there exists no corresponding space (hypersurface) on which it naturally exists (since the local rest spaces of the observers may not be surface forming). One therefore constructs an abstract three-manifold with the threading metric as its Riemannian metric. By identifying each threading curve with the point (t_0, x^i) at which it pierces the slice $\{t \equiv t_0\}$ one constructs the manifold of orbits, Σ, with respect to the threading. Eventhough one may have that Σ is diffeomorphic to the surfaces Σ_t, Σ is not given the same geometry (metric) as any of the Σ_t. One gives Σ the threading metric in an attempt to recapture some of the spacetime geometry associated with the local rest spaces. Thus, one may think of Σ as a smooth model for the collection of local rest spaces. Σ comes equipped with the coordinates x^i and the threading metric as a function of not only

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working in a five-dimensional space, so that in their formalism the four-dimensional spacetime metric took the role of the threading metric in the above discussion.

the points of Σ, but also an additional parameter (the parameter along the original threading curves). Thus Σ has a one-parameter family of Riemannian metrics! This is the beginning of the *parametric manifold* picture of spacetime.

2.4 Parametric Manifolds

2.4.1 A Brief Introduction to Parametric Manifolds

Zoltán Perjés has written a series of papers using a parametric theory of spacetime. In [26], Perjés introduces the phrase "parametric manifold" to describe a certain type of reparameterization-invariant geometric structure. Perjés then describes a decomposition of spacetime based upon a preferred vector field whose integral curves provide a foliation of the spacetime. While the slicing approach has become the standard framework for studying the dynamics of spacetime, not all spacetimes admit such spacelike foliations. The fact that a threading decomposition of spacetime does not depend on the existence of a foliation by spacelike hypersurfaces, gives the theory of parametric manifolds an advantage over the standard ADM, or slicing, formalism. While spacetimes such as Gödel's universe (as described in [10]) are not causally stable, Perjés argues that at the quantum level such acausal contributions must be taken into account. Thus, the theory of parametric manifolds offers one the ability to improve upon the ADM formalism. Moreover, Perjés uses parametric spinor techniques to show that the ADM formalism emerges as a limiting case of the parametric theory.

This section will provide a brief introduction to parametric manifolds by summarizing some of the work done by Perjés in [26]. I will present some of the definitions central to Perjés' parametric manifold picture of spacetime. Although I will offer some of my own comments, most of the language and notation which follows is due to Perjés. In the following section I will describe the similarities between spacetime threadings and Perjés' parametric viewpoint of spacetime. The examination of the relationships between the threading and parametric viewpoints will help provide motivation for many of the definitions in [26].

According to [26], a parametric manifold is represented by a differentiable manifold Σ together with a smooth one-parameter family of one-form fields on Σ.

Call this family of one-forms $\omega(t)$. The family $\omega(t)$ is furthermore required to behave properly under a reparameterization. That is, given a smooth function $F : \Sigma \to \mathbb{R}$ and the reparameterization

$$t' = t + F(x) \qquad \text{for } x \in \Sigma, \tag{2.9}$$

we require

$$\omega'(t') = \omega(t) - dF. \tag{2.10}$$

The pair $(\Sigma, \omega(t))$ constitutes a parametric manifold.

Continuing with the definitions in [26], define a *parametric* (p,q)*-tensor field* to be a one-parameter family of (p,q)-tensor fields on Σ which are invariant under a reparameterization (equation (2.9)). Denote the set of parametric (p,q)-tensors by $\mathcal{T}^p_q$. Note that the original family $\omega(t)$ does not constitute a parametric one-form field.

Tensor analysis on parametric manifolds begins with the introduction of the parametric differential, $d_*\phi$, of a parametric function. For a parametric $(0,0)$-tensor (*i.e.*, a one parameter family of functions) $\phi(t)$, define

$$\begin{aligned} d_*\phi &= d\phi - \omega\partial_t\phi \\ &= d\phi - \omega\dot{\phi} \end{aligned} \tag{2.11}$$

where we have introduce a dot to denote differentiation with respect to the parameter t. Expanding $d_*\phi$ in a coordinate basis gives rise to the parametric derivative operator ∂_{*i}. Write

$$d_*\phi = \phi_{*i}\, dx^i$$

thus defining,

$$\begin{aligned} \phi_{*i} &= \frac{\partial}{\partial x^i}\phi - \omega_i\dot{\phi} \\ &= \phi_{,i} - \omega_i\dot{\phi} \\ &= \partial_{*i}\phi. \end{aligned} \tag{2.12}$$

I have used the last two lines of (2.12) to introduce some notation that will be used throughout this dissertation.

Claim 2.1 *$d_*\phi$ is a parametric $(0,1)$-tensor.*

Proof: It is clear that $d_*\phi : \mathcal{T}_0^1 \to \mathcal{T}_0^0$. In fact, for a parametric vector field $X \in \mathcal{T}_0^1$

$$\begin{aligned} d_*\phi\,(X) &= (d\phi - \omega\dot{\phi})(X) \\ &= d\phi(X) - \dot{\phi}\,\omega(X) \end{aligned}$$

which represents a one-parameter family of functions on $\mathcal{M}$. In terms of a coordinate basis, we can write

$$\begin{aligned} d_*\phi\,(X) &= \phi_{*i}\, dx^i(X) \\ &= (\phi_{,i} - \omega_i\dot{\phi})X^i. \end{aligned}$$

To conclude that $d_*\phi$ is a parametric tensor, we need to show that it is invariant under reparameterizations. Under a change of the form (2.9), we have: $\omega' = \omega - dF$ and $\partial_{t'} = \partial_t$. The exterior differential operator d on Σ does not change, but since ϕ is a parametric function, $d\phi$ is affected by a reparameterization. That is, the object $d\phi$ is understood to mean $d\phi_t$ where $\phi_t : \Sigma \to \mathbb{R}$ is ϕ restricted to a single value of t. Therefore, $\phi_a \equiv \phi\big|_{t=a}$. Let ϕ' represent ϕ after a reparameterization. That is, define

$$\begin{aligned} \phi'_b &= \phi\big|_{t'=b} \\ &= \phi\big|_{t=b-F(x)} \\ &= \phi(b - F(x), x). \end{aligned}$$

Therefore,

$$\begin{aligned} d\phi &\equiv d\phi_t \\ &= \frac{\partial\phi}{\partial x}(t,x)\,dx \end{aligned}$$

and

$$\begin{aligned} d\phi' &\equiv d\phi'_{t'} \\ &= \left(-\frac{\partial\phi}{\partial t}\frac{\partial F}{\partial x} + \frac{\partial\phi}{\partial x}\right)(t' - F(x), x)\,dx \\ &= d\phi - \dot{\phi}\,dF \end{aligned}$$

and, hence, $d_*\phi = d_*\phi'$. ♠

Covariant differentiation of parametric tensors is accomplished by means of a parametric connection (derivative operator), ∇_*. As introduced in [26], the action of ∇_* depends on the given one-form field ω and can be characterized by many of the familiar properties

1. $\nabla_* : \mathcal{T}^p_q \to \mathcal{T}^p_{q+1}$;

2. ∇_* is linear and commutes with contraction;

3. ∇_* satisfies the Leibnitz rule;

4. ∇_* is torsion-free;

and one slight variation

5. for $(0,0)$ tensors ϕ, $\nabla_* \phi = d_* \phi$.

Aside from the action of ∇_* on functions, ∇_* is just an ordinary derivative operator. However, because of the fifth condition, ∇_* has been called a *generalized connection* in [26].

As with ordinary derivative operators, if the manifold possesses a metric, h, there exists a unique operator satisfying properties 1–5 in addition to

$$\nabla_* h = 0.$$

For the rest of the discussion, we will assume Σ is a Riemannian manifold with parametric metric h. That is, h is a parametric $(0,2)$-tensor such that for each t, $h(t)$ is a Riemannian metric on $\mathcal{M}$. Working in a coordinate neighborhood with coordinates x^i, one may expand h in terms of its components

$$h = h_{ij}\, dx^i dx^j$$

where one must remember that the component functions h_{ij} are functions of the coordinates x^i as well as the parameter t. Since h is assumed to be a parametric tensor, h is invariant under reparameterization.

Let ∇_* be the unique parametric derivative operator associated with (Σ, ω, h). Expanding ∇_* in terms of the parametric derivative ∂_* Perjés gives the action of ∇_* on a parametric vector field:

$$\nabla_{*i} X^k = X^k{}_{*i} + \gamma^k{}_{ji} X^j$$

where

$$\gamma^i{}_{jk} = \frac{1}{2} h^{im} (h_{mj*k} + h_{mk*j} - h_{jk*m}). \tag{2.13}$$

The action of ∇_* on parametric tensors of rank (p,q) is analogous.

Notice the similarities (and differences!) between such a parametric action and, say, the classical Levi-Civita connection. On a basic level, the partial derivative operator ∂_i has been replaced by the parametric derivative operator ∂_{*i}. In the special case where the functions X^i and h_{ij} do not depend on t, the two actions agree. In the case where the threading curves are integral curves of a Killing vector field, the threading metric components h_{ij} will be independent of t. Thus, we have that $h_{ij*k} \equiv h_{ij,k}$ making the parametric Christoffel symbols agree with the standard Christoffel symbols.

There are several differences between a generalized connection and an ordinary connection. As mentioned earlier, the action of ∇_* on functions is perhaps the most obvious difference. However, this parametric action has several subtle but interesting repercussions.

First, one must decide the correct action of parametric vector fields on parametric functions. Although for any given value of the parameter t, a parametric vector field is simply a tangent vector field of Σ and a parametric function is just a function from Σ to $\mathbb{R}$, the ordinary action does not seem appropriate in the parametric setting. As the parametric derivative operator ∂_{*i} is the fundamental derivative operator (in a given basis), it seems most natural to introduce the action

$$\begin{aligned} X(\phi)_* &= X^i \partial_{*i} \phi \\ &= X^i \left(\phi_{,i} - \omega_i \dot{\phi} \right) \\ &= X^i \phi_{*i} \end{aligned}$$

where X is a parametric vector field and ϕ is a parametric function. Although it is not clear that Perjés adopts such a convention in [26], it greatly simplifies the notation as well as the analysis (as I will show later). Adopting this convention also allows one the standard approach of viewing vector fields as directional derivatives on scalar fields (although in this case the "direction" is not parallel to the manifold Σ). That is, we may write

$$X(\phi)_* = X^i \nabla_{*i} \phi.$$

From now on, the action of a parametric vector field on a function is assumed to be the above "parametric" action and I will dispense with the cumbersome notation of $X(\phi)_*$.

Second, one needs to understand what it means for a generalized connection to be torsion-free. Perhaps the greatest difference between generalized connections and classical connections lies buried in property 4. While it is not uncommon (at least in general relativity) to require a derivative operator to be torsion-free, the property of vanishing torsion is a bit different in the parametric case. As Perjés mentions, even though ∇_* is required to be torsion-free, it does admit a non-vanishing *deficiency*[3]. This is realized by the fact that

$$(\nabla_{*i} \nabla_{*k} - \nabla_{*k} \nabla_{*i})\phi = (\omega_{i*k} - \omega_{k*i})\dot{\phi}.$$

Thus, one must take note that condition 4 **does not** take the usual form (c.f. [30]) of

$$\nabla_{*i} \nabla_{*k} \phi = \nabla_{*k} \nabla_{*i} \phi. \tag{2.14}$$

In Chapter 3 I will give precise definitions of torsion and deficiency for parametric connections.

While the commutator of ∇_* on functions leads to the notion of deficiency, in [26] Perjés shows how the commutator of ∇_* on higher order tensors leads to a

[3] Perjés credits Lottermoser [18] with a similar term *Defekt*.

definition of curvature. The *Zel'manov curvature*[4] $Z^i{}_{jkl}$ is defined by

$$\left[\nabla_{*k}\nabla_{*j} - \nabla_{*j}\nabla_{*k} + \left(\omega_{j*k} - \omega_{k*j}\right)\frac{\partial}{\partial t}\right] X_i = Z^r{}_{ijk} X_r \tag{2.15}$$

and possess many of the familiar symmetries of the Riemann tensor.

We have:

$$Z^i{}_{jkl} = Z^i{}_{j[kl]} \qquad \text{and} \qquad Z^i{}_{[jkl]} = 0.$$

One can show that the components of Z^m_{ijk} may be expressed in terms of the symbols $\gamma^i{}_{jk}$ and ∂_* by

$$Z^m_{ijk} = \gamma^m_{ik*j} - \gamma^m_{ij*k} + \gamma^m_{nj}\gamma^n_{ki} - \gamma^m_{nk}\gamma^n_{ji}. \tag{2.16}$$

A more in-depth discussion of curvature, torsion, and deficiency appears in the next chapter.

2.4.2 Parametric Manifolds and Spacetime Threadings

The above section offered a brief introduction to the definitions and tools that Perjés associates with the theory of parametric manifolds. As some of the notation may have suggested, the above introduction may be further enhanced given our previous discussion of the slicing and threading pictures of spacetime. The language and viewpoint associated with the threading decomposition lends the necessary insight to fully appreciate the above definitions. We may use a threading decomposition of spacetime to provide excellent motivation for the definitions Perjés introduced in [26].

As Perjés mentions in [26], the parametric manifold structure of spacetime is induced by a non-null vector field. Hence, in the case of a spacetime which

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[4] Perjés traces this definition back to the Russian mathematician Zel'manov [31].

admits a threading decomposition, the threading vector field may be used to infuse the spacetime with a parametric structure. Thus, the ideas associated with the threading viewpoint should be closely related to the central ideas of parametric manifolds.

Consider a spacetime consisting of the manifold $\mathcal{M}$ together with the spacetime metric denoted by $g_{\alpha\beta}$. Further assume that the spacetime admits a threading decomposition. As in (2.8), one can write the spacetime metric in terms of the threading versions of the lapse, shift, and metric. As before, the spacetime coordinates are given by $x^\alpha = (t, x^i)$ where $\frac{\partial}{\partial t}$ is the threading vector field and the x^i are coordinates on each hypersurface $\Sigma_{t_0} = \{t \equiv t_0\}$. By choosing one of the slices, one is able to use the x^i as coordinates on the manifold of orbits by identifying each threading curve with the point (t_0, x^i) at which it pierces the slice Σ_{t_0}. Let Σ be the manifold of orbits with coordinates x^i. For each value of t, we also have a Riemannian metric on Σ; the threading metric h_{ij}. Σ will be the parametric manifold. As each point of Σ represents an equivalence class of points of the spacetime $\mathcal{M}$, one may wish to think of general parametric manifolds as collections of equivalence classes.

An attempt to develop a geometric theory on Σ will lead to many of the definitions we saw above. For instance, any tensors defined on Σ (parametric or otherwise) must transform correctly under a legitimate change of coordinates (keeping the parameter unchanged). Furthermore, our theory should not depend on the parameterization. That is, the original coordinates on Σ were defined with the choice of a specific value of t. Such a choice should not alter the theory. Consider the following definitions [5]:

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[5] These definitions are borrowed from [8]. The similarities between a parametric theory and Kaluza-Klein theory will be made explicit in Chapter 4.

Definition 2.2. *A regular coordinate transformation is a coordinate transformation of the form*

$$\begin{aligned} y^i &= y^i(x^j) \\ t' &= t. \end{aligned} \tag{2.17}$$

Definition 2.3. *A reparameterization is a coordinate transformation of the form*

$$\begin{aligned} y^i &= x^i \\ t' &= t + F(x^i). \end{aligned} \tag{2.18}$$

For any spacetime one-form $W = W_\alpha \, dx^\alpha$, if we consider the components of W after a coordinate transformation of the form (2.18), we have

$$\begin{aligned} W_0' &= W_0 \\ W_i' &= -W_0 \frac{\partial F}{\partial x^i} + W_i. \end{aligned}$$

The threading decomposition naturally led to the threading shift one-form $M_i dx^i$. Consider the spacetime one-form field $\bar{\omega} = dt - M_i dx^i$. Thus the threading shift one-form is just the pullback (up to a sign) of $\bar{\omega}$ to a slice Σ_{t_0}. Under a reparameterization we have

$$\omega_i' = \omega_i - \frac{\partial F}{\partial x^i}.$$

The spacetime one-form field $\bar{\omega}$ may be associated with a one parameter family of one-forms on Σ by identifying $\bar{\omega}$ with

$$\omega(t) = -M_i(t) dx^i.$$

Furthermore, a reparameterization of $\bar{\omega}$ would result in $\bar{\omega}'$ being identified with

$$\omega'(t') = \left(-M_i - \frac{\partial F}{\partial x^i} \right) dx^i.$$

Thus, a reparameterization of the spacetime one-form $\bar{\omega}$, equation (2.18), gives us a way of describing the result of $\omega(t)$ after a reparameterization of the parametric manifold (equation (2.9)). Therefore $\omega(t)$ satisfies equation (2.10) and can be used to give Σ a parametric structure.

In classical differential geometry, one can define a tensor field (locally) by defining it in a given coordinate basis, and then require that the components of the tensor transform correctly under an accepted change of coordinates. A similar approach will be used to get a working definition of parametric tensors. However, not only must we concern ourselves with coordinate transformations, but also with the notion of reparameterization.

Any coordinate transformation on Σ can be identified with a regular coordinate transformation of spacetime (equation (2.17)). Under such a transformation spacetime tensors transform according the their variance as usual. Since reparameterization of Σ corresponds to the freedom of choosing a slice Σ_t in order to give Σ coordinates, only those spacetime tensor fields which remain "tangent" to the local rest spaces at different values of t will be identified with parametric tensors. More precisely,

Definition 2.4 *A spacetime vector field V will be called a parametric vector field provided that it is orthogonal to $\frac{\partial}{\partial t}$.*

Definition 2.5 *A spacetime one-form field W will be called a parametric one-form field provided that $W(\frac{\partial}{\partial t}) \equiv 0$.*

Tensor products of parametric vector fields and parametric one-form fields will result in spacetime tensor fields which may be identified with parametric tensor fields (thus guaranteeing reparameterization invariance). That is, we will call a spacetime tensor field T a parametric tensor provided T contracted with $\frac{\partial}{\partial t}$ (on any index) is zero.

Definition 2.6 *A spacetime tensor field T will be called a parametric tensor field provided*

$$T(\frac{\partial}{\partial t}, X, Y, \ldots, Z, \sigma, \ldots, \tau) \equiv 0$$
$$T(X, Y, \ldots, \frac{\partial}{\partial t}, \ldots, Z, \sigma, \ldots, \tau) \equiv 0$$
$$T(X, Y, \ldots, \frac{\partial}{\partial t}, \sigma, \ldots, \tau) \equiv 0$$
$$T(X, Y, \ldots, Z, \bar{\omega}, \sigma, \ldots, \tau) \equiv 0$$
$$T(X, Y, \ldots, Z, \sigma, \ldots, \bar{\omega}, \ldots, \tau) \equiv 0$$
$$T(X, Y, \ldots, Z, \sigma, \tau, \ldots, \bar{\omega}) \equiv 0$$

where $\bar{\omega}$ is the metric dual of $\frac{\partial}{\partial t}$ and these above expressions are identically zero for all vector fields X,Y,Z and one-form fields σ, τ.

Proposition 2.7. *The threading metric is a parametric tensor.*

Proof: If the threading is induced by a foliation of timelike curves, then equation (2.8) relates the threading metric and the spacetime metric. The threading metric can be naturally associated with the spacetime tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + M^2 A_\alpha A_\beta$$

where $A_0 = -1$ and $A_i = M_i$, the components of the threading shift form as above. Note that $h_{00} = h_{0i} = 0$, and that restricting the Greek indices to Latin indices yields the threading metric as introduced before. In our coordinate system, h will be a parametric tensor if and only if its "spatial" components remained unchanged after a reparameterization. Under a reparameterization (2.18) one uses the fact that $h_{0\alpha} = 0$, thus yielding

$$\begin{aligned} h'_{ij} &= h_{mn} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} \\ &= h_{mn} \delta^m{}_i \delta^n{}_j \\ &= h_{ij}. \end{aligned}$$

Therefore the components of the threading metric remain unchanged after a reparameterization. Thus, the threading metric is a parametric tensor. ♠

For comparison,

Proposition 2.8. *The slicing metric is not a parametric tensor.*

Proof: First of all, since the slicing metric is just the spacetime metric with restricted index values, I really mean to say that the spacetime metric is not a parametric tensor. The problem lies in how the "spatial" part of the spacetime metric changes under a reparameterization. Consider a transformation of the form (2.18). The resulting change can be computed:

$$\begin{aligned} g'_{ij} &= g_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j} \\ &= g_{00}\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j} - g_{0j}\frac{\partial F}{\partial x^i} - g_{i0}\frac{\partial F}{\partial x^j} + g_{ij} \end{aligned}$$

Thus, in general $g'_{ij} \neq g_{ij}$ after a reparameterization. In this sense, the slicing metric is not a parametric tensor. ♠.

Theorem 2.9. *To any spacetime rank-one tensor, there corresponds a parametric rank-one tensor (and a scalar) via projection.*

Proof: Using the above information we simply project any spacetime vector (or covector) onto a slice Σ_t. To accomplish this we will use the threading metric to define a projection operator $P_\alpha{}^\beta \equiv h_\alpha{}^\beta$. That is

$$P_\alpha{}^\beta = \delta_\alpha{}^\beta + M^2 A_\alpha A^\beta$$

where $A^\alpha = \frac{1}{M^2}(\frac{\partial}{\partial t})^\alpha$, guaranteeing that $P_\alpha{}^\beta$ projects out the $\frac{\partial}{\partial t}$ component on spacetime vector fields.

Example 2.10.

A. For any spacetime vector field V^α

$$\begin{aligned} P_\alpha{}^\beta V^\alpha &= V^\beta + \frac{1}{M^2}\left(\frac{\partial}{\partial t}\right)^\beta \left(\frac{\partial}{\partial t}\right)_\alpha V^\alpha \\ &= V^\beta + \frac{1}{M^2}\left(\frac{\partial}{\partial t}\right)^\beta (-M^2 V^0 + M^2 M_i V^i), \end{aligned}$$

which implies that

$$P_\alpha{}^0 V^\alpha = M_i V^i \qquad \text{and} \qquad P_\alpha{}^i V^\alpha = V^i.$$

B. For one-form fields one finds

$$P_0{}^\beta W_\beta = 0 \qquad P_i{}^\beta W_\beta = W_i + M_i W_0.$$

Applying $P_\alpha{}^\beta$ to each slot of the spacetime metric tensor yields the tensor $h_{\mu\nu}$ which we have already shown possesses the characteristics of a parametric tensor:

$$\begin{aligned} P_\mu{}^\alpha P_\nu{}^\beta g_{\alpha\beta} &= (\delta_\mu{}^\alpha + M^2 A^\alpha A_\mu)(\delta_\nu{}^\beta + M^2 A^\beta A_\nu) g_{\alpha\beta} \\ &= g_{\mu\nu} + 2M^2 A_\mu A_\nu + M^2 A^\alpha A_\alpha A_\mu A_\nu \\ &= g_{\mu\nu} + M^2 A_\mu A_\nu \\ &= h_{\mu\nu}. \end{aligned}$$

As we mentioned earlier, the natural derivative operator on parametric manifolds is the parametric derivative (2.12). In this case this derivative operator takes the familiar form of a projected ordinary partial derivative. We have:

$$\begin{aligned} P_\alpha{}^\beta (\partial_i)^\alpha &= (\partial_i)^\beta + M^2 A^\beta M_i \\ &= (\partial_i)^\beta + M_i \left(\frac{\partial}{\partial t}\right)^\beta \\ &= (\partial_i)^\beta - \omega_i \left(\frac{\partial}{\partial t}\right)^\beta. \end{aligned} \tag{2.19}$$

Since $\frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t}$ spans the local rest space of the observers, we are taking derivatives in directions orthogonal to $\frac{\partial}{\partial t}$. One should also note that ∂_{*i} is more then just an ad-hoc projected derivative operator. This parametric derivative operator justifies its name by its invariance under reparameterization.

Theorem 2.11 *The action of ∂_{*i} on parametric functions is invariant under a reparameterization.*

Proof: This is a restatement of the earlier observation that $d_*\phi$ was a parametric (0,1)-tensor. Consider a parametric function f and a reparameterization of the form (2.18). First of all we have that the components of the tensor A^α transform according to

$$\begin{aligned} A_i' &= A_0\left(-\frac{\partial F}{\partial x^i}\right) + A_i \\ &= A_i + \frac{\partial F}{\partial x^i} \end{aligned}$$

which gives us

$$\begin{aligned} \frac{\partial f}{\partial y^i} &= \frac{\partial f}{\partial x^i} - \frac{\partial F}{\partial x^i}\frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial x^i} - (A_i' - A_i)\frac{\partial f}{\partial t}. \end{aligned}$$

Thus, since $\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}$,

$$\begin{aligned} \frac{\partial f}{\partial y^i} + A_i'\frac{\partial f}{\partial t'} &= \frac{\partial f}{\partial y^i} + A_i'\frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial x^i} + A_i\frac{\partial f}{\partial t}. \end{aligned}$$

♠

The above theorem illustrates the underlying principal which allows one to pass from a classical differential geometric setting to a parametric structure. Replacing $\frac{\partial}{\partial x^i}$ by ∂_{*i} leads to a theory which, while remaining true to regular coordinate transformations, also remains invariant under reparameterization.

Thus, as one attempts to define a natural covariant derivative operator, one finds that the first naive choice of

$$\nabla_i V^j = V^j_{\ ,i} + \frac{1}{2}h^{jn}(h_{nm,i} + h_{ni,m} - h_{mi,n})V^m$$

is not invariant under a reparameterization. However, replacing the ordinary partial derivative operator by the parametric derivative operator results in

$$
\begin{aligned}
\nabla_{*i} V^j &= V^j_{\ *i} + \frac{1}{2} h^{jn} (h_{nm*i} + h_{ni*m} - h_{mi*n}) V^m \\
&= V^j_{\ *i} + \gamma^j_{\ mi} V^m
\end{aligned}
\tag{2.20}
$$

which is invariant under reparameterization.

As was mentioned earlier, under any reparameterization, $\partial_{t'} = \partial_t$. Therefore differentiation with respect to the parameter is a covariant operation and may be denoted ∇_0.

2.4.3 Concluding Remarks

This section was meant to provide some motivation for the parametric definitions introduced earlier. When the threading decomposition of spacetime was first introduced, I emphasized the fact that such a decomposition was simply an orthogonal splitting of spacetime with respect to the threading curves. As we never assumed the existence of hypersurfaces orthogonal to the threading curves, all decompositions were only pointwise.

However, by working on the manifold of orbits, equipping it with the threading metric, and introducing the parametric derivative operators ∂_* and ∇_* we succeed in modeling much of the behavior of the spaces orthogonal to the threading curves. Moreover, the manifold of orbits provides us with a smooth structure allowing the existence and analysis of tensor *fields*. Thus, when analyzing a manifold (spacetime) from the threading point of view, it seems most natural (if not necessary!) to incorporate the parametric structure into one's approach.

The stage is now set for a more formal treatment of parametric manifolds. In the next chapter, I will give two different formal approaches to parametric manifolds (an extrinsic and an intrinsic approach). Through a more detailed discussion, one sees how the concept of deficiency emerges as an attempt to generalize many of the basic concepts of differential geometry on manifolds.

2.5 Rotating Coordinates

2.5.1 Setting Up

Rotating cylindrical coordinates provide a nice example of all of the different frameworks discussed. This three-dimensional example is especially illustrative in the sense that the threading vector field is not orthogonal to a family of hypersurfaces, so that the slicing and threading viewpoints are not equivalent. Although in this simple example the parametric metric is independent of the parameter and the Zel'manov curvature tensor reduces to the Riemann curvature tensor, a definite parametric structure is still present.

We begin by considering the stationary (non-static), axisymmetric spacetime given by flat Minkowski three-space in cylindrical coordinates (r, θ, z). The spacetime metric is of the form:

$$ds^2 = dr^2 + r^2\, d\theta^2 - dz^2.$$

Now, perform the following change of coordinates:

$$t = z \qquad \rho = r \qquad \psi = \theta + \Omega z$$

where Ω is some constant. One has the following relationships:

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial z} - \Omega\frac{\partial}{\partial \theta} & \qquad dt &= dz \\ \frac{\partial}{\partial \rho} &= \frac{\partial}{\partial r} & \qquad d\rho &= dr \\ \frac{\partial}{\partial \psi} &= \frac{\partial}{\partial \theta} & \qquad d\psi &= \Omega\, dz + d\theta \end{aligned}$$

which yield:

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2(d\psi - \Omega\, dt)^2 - dt^2 \\ &= d\rho^2 + \rho^2 d\psi^2 - 2\rho^2\Omega\, d\psi\, dt + (\rho^2\Omega^2 - 1)dt^2 \end{aligned}$$

The vector field $\frac{\partial}{\partial t}$ is timelike if and only if $\rho^2 < \frac{1}{\Omega^2}$.

These coordinates may be used to define a family of hypersurfaces $\Sigma_t \equiv \{t = \text{constant}\} = \{z = \text{constant}\}$. Each Σ_t is spanned by the coordinate vector fields $\partial_r = \partial_\rho$ and $\partial_\theta = \partial_\psi$.

As mentioned earlier:

Proposition 2.1 *The orthogonal subspace to $\frac{\partial}{\partial t}$ is* **not** *surface- forming.*

Proof: We will show that the orthogonal subspace, W, is not involutive. We have that $\{\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \psi} + \frac{\rho^2\Omega}{\rho^2\Omega^2-1}\frac{\partial}{\partial t}\}$ span W. But,

$$\left[\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \psi} + \frac{\rho^2\Omega}{\rho^2\Omega^2 - 1}\frac{\partial}{\partial t}\right] = \left(\frac{\rho^2\Omega}{\rho^2\Omega^2 - 1}\right)_{,\rho}\frac{\partial}{\partial t}$$
$$\notin W.$$

Therefore, W is not involutive. By the vector-field formulation of Frobenius' theorem (*c.f.* [30]) W is not surface forming. ♠

2.5.2 Slicing with Rotating Coordinates

In order to set up the slicing formalism, one needs a (future pointing) unit vector field normal to the hypersurfaces Σ_t. Since $dt(\partial_\rho) = 0 = dt(\partial_\psi)$, this is accomplished by calculating the metric dual of the one-form field dt:

$$(dt)^\# = -\frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial \psi}.$$

Since this vector field has norm -1, we conclude that $n = -\frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial \psi}$. The lapse function and the components of the slicing shift vector field may now be obtained by writing

$$\frac{\partial}{\partial t} = -n - \Omega\frac{\partial}{\partial \psi}$$

and comparing with equation (2.5). Thus

$$N^\psi = -\Omega \qquad \text{and} \qquad N = -1.$$
$$N^\rho = 0$$

This information may be interpreted in two different ways depending on one's viewpoint of slicing. In the construction scenario, the vector field $\frac{\partial}{\partial t}$ represents the "flow of time" and is used to identify points of different hypersurfaces. Thus, given a collection of the surfaces Σ_t, one may imagine constructing a three-dimensional spacetime by not simply stacking one surface atop the other, but by rotating ($N^\psi = -\Omega$) and stacking ($N^\rho = 0$). In terms of a $(2+1)$-decomposition viewpoint, we are simply measuring the discrepancy between $\frac{\partial}{\partial t}$ and the direction normal to each surface. The fact that $N^\rho = 0$ just indicates that $\frac{\partial}{\partial t}$ is orthogonal to ∂_ρ, while the lapse function $N = -1$ since the coordinate z is proper time.

The slicing metric is simply the spacetime metric pulled back to the hypersurfaces Σ_t (see equation (2.4)). Thus $k_{ij} = g_{ij}$ for $i, j \in \{\rho, \psi\}$. Explicitly, one has

$$k_{\rho\rho} = 1 \qquad \text{and} \qquad k_{\psi\psi} = \rho^2$$

as the only non-zero components of the slicing metric.

Finally, one may now calculate the components of the metric dual of the shift vector field. That is, define $N_i = k_{ij}N^j$, yielding

$$\begin{aligned} N_\rho &= k_{\rho j}N^j = 0 \\ N_\psi &= k_{\psi\psi}N^\psi = -\rho^2\Omega. \end{aligned}$$

Note that

$$\begin{aligned} -(N^2 - N_m N^m) &= -1 + (\rho^2\Omega)(-\Omega) \\ &= -1 + \rho^2\Omega^2 \\ &= g_{tt} \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2N_\psi &= -2\rho^2\Omega \\ &= g_{t\psi} \end{aligned}$$

as guaranteed by equation (2.2).

2.5.3 Threading with Rotating Coordinates

We will use $\frac{\partial}{\partial t}$ as our given threading vector field. The threading lapse function is defined such that $-M^2$ is the square of the norm of the threading vector field. Thus,

$$-M^2 = \rho^2\Omega^2 - 1.$$

The threading shift one-form field measures the discrepency between the hypersurfaces Σ_t and the space orthogonal to $\frac{\partial}{\partial t}$. Since $\frac{\partial}{\partial t}$ is orthogonal to $\frac{\partial}{\partial r}$, $M_r = 0$. Now, M_ψ is defined so that $\frac{\partial}{\partial \psi} + M_\psi \frac{\partial}{\partial t}$ is orthogonal to $\frac{\partial}{\partial t}$. This implies that

$$M_\psi = \frac{\rho^2\Omega}{\rho^2\Omega^2 - 1}.$$

According to equation (2.8), the threading metric is defined by $h_{ij} = g_{ij} + M^2 M_i M_j$ yielding the components

$$h_{\psi\psi} = \frac{\rho^2}{1-\rho^2\Omega^2} \qquad \text{and} \qquad h_{\rho\rho} = 1.$$

The importantce of the causality conditions may be noticed at this stage. The threading metric is a Riemannian metric if and only if the threading vector field $(\frac{\partial}{\partial t})$ is timelike whereas the slicing metric is Riemannian if and only if the slices Σ_t are spacelike.

With the above definitions for the threading lapse and shift, the original spacetime metric can be written as

$$ds^2 = -M^2\, dt^2 + M^2 M_\psi\, d\psi\, dt + M^2 M_\rho\, d\rho\, dt + (h_{\psi\psi} - M^2 M_\psi M_\psi) d\psi^2.$$

2.5.4 Parametric Manifolds with Rotating Coordinates

Recall that the theory of parametric manifolds is based upon the threading decomposition of the original spacetime. Therefore, we will define M and M_i for $i \in \{\psi, \rho\}$ as above. Notice now that our parametric coordinate derivative operators are

$$\partial_{*\rho} = \partial_\rho + M_\rho \partial_t = \partial_\rho$$

and

$$\partial_{*\psi} = \partial_\psi + \frac{\rho^2\Omega}{\rho^2\Omega^2 - 1}\partial_t$$

Therefore, the parametric viewpoint is different from either the slicing or threading viewpoints.

Recall that the components of our (parametric) metric are of the form

$$\left(h_{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{-\rho^2}{\rho^2\Omega^2 - 1} \end{pmatrix}$$

with inverse

$$\left(h^{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1-\rho^2\Omega^2}{\rho^2} \end{pmatrix}$$

Since the metric h_{ij} does not depend on the parameter t, all parametric derivatives of the metric components reduce to ordinary partial derivative with respect to the coordinates ρ and ψ. Thus, the parametric Christoffel symbols are just the ordinary Christoffel symbols and the parametric (Zel'manov) curvature tensor reduces to the ordinary Riemann curvature tensor. In terms of our earlier notation these facts can be stated as

$$\gamma^i{}_{jk} = \Gamma^i{}_{jk} \qquad \text{and} \qquad Z^i{}_{jkl} = R^i{}_{jkl} \qquad\qquad i,j \in \{\rho, \psi\}.$$

3. Parametric Manifolds

3.1 Introduction

The ultimate goal of this chapter is to rigorously define a complete set of tools and operators which make up the essence of a parametric structure. As we saw in the last chapter, many of Perjés' definitions can be motivated by the study of spacetimes which admitted a preferred congruence of non-null curves. Such a foliation led to a preferred decomposition of the spacetime which, in turn, led to the notions of parametric tensor fields and parametric derivatives on the manifold of orbits Σ. In the special case where the threading curves are orthogonal to the slicing surfaces, the slicing and threading viewpoints agree. Specifically, the geometry of Σ (given by the threading metric) agrees with the geometry of the immersed surfaces Σ_t.

There already exist standard techniques for relating the geometry of a manifold to the geometry of immersed submanifolds, namely the Gauss-Codazzi equations. What makes the threading-induced parametric structure unique is the absence, in general, of an immersed surface orthogonal to the threading curves. However, even in such a case many of the ideas developed for studying the geometry of immersions are still valid. In fact, I will take these ideas as fundamental in developing a coherent parametric theory.

The Gauss-Codazzi equations are mainly concerned with the relationships between the curvature tensor of an n-dimensional manifold and the induced curvature tensor of an immersed $(n-k)$-dimensional submanifold. While developing the Gauss-Codazzi relations, one also shows that many of the properties of the n-dimensional connection are inherited by the $(n-k)$-dimensional connection. In the general case of threading without orthogonal hypersurfaces, the notions of "induced metric" , "induced connection", and "induced curvature" are perhaps a bit elusive. Indeed, the process of defining a threading-induced curvature tensor on Σ is by no means obvious. A complete understanding of the theory of connections, including the concepts of curvature and torsion, is vital to defining a decent notion of curvature on Σ.

This chapter begins with the definitions of the necessary terminology for the study of connections on manifolds, followed by a discussion of the curvature and torsion tensors. After reviewing the standard Gauss-Codazzi formalism for the case of an $(n-1)$-dimensional immersed submanifold, I then examine the case of a spacetime threading without surfaces orthogonal to the threading curves. By introducing a generalized Gauss-Codazzi formalism I will be able to define terms such as "induced metric" and "induced connection". These terms will then be used to define a metric-compatible connection on Σ. Furthermore, I will treat Gauss' equation as fundamental and use this equation to help define a notion of curvature. As we shall see, this is certainly not the only approach available.

While the generalized Gauss-Codazzi formalism yields notions of a metric and connection for the manifold of orbits Σ via projection operators, the mathematical essence of a parametric manifold is more fully realized by treating Σ as an abstract manifold with additional structure. From this point of view, all tensor fields on Σ are functions of an additional parameter t and the derivative operator $\frac{\partial}{\partial t}$ then becomes a covariant operation (since t is no longer a coordinate). Furthermore, the action of parametric vector fields on parametric functions will depend on an additional one-parameter family of one-forms, ω. As we generalize the notions of connection, Lie bracket, and exterior differentiation to define operators intrinsic to Σ, ω will play a vital role.

In this intrinsic approach, it turns out that the deficiency can no longer be defined by measuring the failure of a distribution to be surface forming. Rather, we will show how the deficiency is related to the failure of the generalized exterior derivative operator to satisfy Poincaré's lemma ($d^2 = 0$). In the presence of deficiency, particular care is taken in defining a Lie bracket, torsion, and parametric connection. As we shall see, all of these definitions are equivalent to those presented in the extrinsic approach.

3.2 Some Definitions

Let us begin by defining the standard notion of a connection on a manifold $\mathcal{M}$, together with some relevant properties of connections. For the following definitions, let $\mathcal{M}$ be a smooth manifold with (Lorentzian or Riemannian) metric g denoted by $\langle\ ,\ \rangle$. Also, let $\mathfrak{X}(\mathcal{M})$ denote the set of all smooth vector fields on $\mathcal{M}$ and $\mathfrak{F}(\mathcal{M})$ the ring of all smooth real-valued functions defined on $\mathcal{M}$.

Definition 3.1 *An (affine) connection* ∇ *on* $\mathcal{M}$ *is a mapping* $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, *usually denoted by* $\nabla(X, Y) = \nabla_X Y$, *which satisfies the following axioms:*

i. Linearity over $\mathfrak{F}(\mathcal{M})$*:* $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

ii. Linearity: $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

iii. Product rule: $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ *for all* $X, Y, Z \in \mathfrak{X}(\mathcal{M})$ *and* $f, g \in \mathfrak{F}(\mathcal{M})$.

The existence of a connection on $\mathcal{M}$ gives one a way of differentiating vector fields along curves. Although traditionally one defines the concept of metric compatibility in terms of parallel vector fields along curves in $\mathcal{M}$, it can be restated (*c.f.*, [6]) as

Definition 3.2 *An affine connection* ∇ *is compatible with the metric of* $\mathcal{M}$ *provided*

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{3.1}$$

for $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

The fact that (3.1) holds is just a statement that covariant differentiation of the metric obeys the familiar Leibnitz rule of derivations.

In general relativity one is usually only concerned with connections which are *torsion-free.*

Definition 3.3 *A connection ∇ is said to be torsion-free when*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y \in \mathcal{X}(\mathcal{M})$.

The action of $[X, Y]$ on functions $f \in \mathfrak{F}(\mathcal{M})$ is defined by the action of the commutator

$$[X, Y]f = XYf - YXf. \tag{3.2}$$

Although it is not *a priori* clear that $[X, Y]$ is a vector field, it can be shown (*c.f.*, [4]) that there exists a unique vector field $[X, Y]$ satisfying (3.2). A torsion free connection is sometimes referred to as a *symmetric connection.*

A fundamental result in the theory of connections is

Theorem 3.4 *There exists a unique connection on $\mathcal{M}$ which is compatible with the metric g and torsion-free.*

Definition 3.5 *This unique connection is called the Levi-Civita connection and may be defined by the equation:*

$$\begin{aligned}\langle X, \nabla_{*Y} Z\rangle =& \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle [X, Z]_*, Y\rangle - \langle [Y, Z]_*, X\rangle - \langle [X, Y]_*, Z\rangle).\end{aligned}$$

3.3 Curvature and Torsion

As we shall see later, the notions of curvature and torsion play an interesting role in the development of a parametric theory. We saw earlier (equation (2.14)) that Perjés' idea of (parametric) torsion appeared to disagree with our standard interpretation. Therefore, a review of torsion is warranted. Also, a clear understanding of the relationships between curvature and torsion will be useful when defining parametric curvature.

Let us begin with the necessary definitions. Using the definitions in [15] rewritten in terms of an affine connection, we have

Definition 3.1 *Define the torsion T and curvature R of ∇ by*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{3.3}$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.4}$$

for $X, Y, Z \in \mathcal{X}(\mathcal{M})$.

The case where $T(X,Y) \equiv 0$ agrees with the earlier notion of torsion-free.

Consider the components of T and R in some patch with coordinates $\{x^\alpha\}$. Defining the symbols $\Gamma^\alpha{}_{\beta\gamma}$ by $\nabla_{\partial_\beta} \partial_\gamma = \Gamma^\alpha{}_{\beta\gamma} \partial_\alpha$ we have

$$\begin{aligned} T(\partial_\beta, \partial_\gamma) &= \nabla_{\partial_\beta} \partial_\gamma - \nabla_{\partial_\gamma} \partial_\beta - 0 \\ &= (\Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}) \partial_\alpha \\ &= \mathrm{T}^\alpha{}_{\beta\gamma} \partial_\alpha . \end{aligned}$$

While it is trivially true that mixed partial derivatives commute, the torsion tensor may be thought of measuring the failure of mixed covariant derivatives to commute. As we see from above

$$\left(\nabla_{\partial_\beta} \partial_\gamma - \nabla_{\partial_\gamma} \partial_\beta \right) (f) = \mathrm{T}^\alpha{}_{\beta\gamma} f_{,\alpha} .$$

For curvature,

$$\begin{aligned} R(\partial_\alpha, \partial_\beta)\partial_\gamma &= (\nabla_{\partial_\alpha}\nabla_{\partial_\beta} - \nabla_{\partial_\beta}\nabla_{\partial_\alpha})\partial_\gamma - 0 \\ &= R^\mu{}_{\gamma\alpha\beta}\partial_\mu. \end{aligned} \tag{3.5}$$

As is often done, $R^\nu{}_{\delta\beta\alpha}$ may be expressed in terms of the connection symbols $\Gamma^\alpha{}_{\beta\gamma}$,

$$R^\nu{}_{\delta\beta\alpha} = \Gamma^\mu{}_{\delta\alpha,\beta} - \Gamma^\mu{}_{\delta\beta,\alpha} + \Gamma^\nu{}_{\mu\beta}\Gamma^\mu{}_{\delta\alpha} - \Gamma^\nu{}_{\mu\alpha}\Gamma^\mu{}_{\delta\beta}.$$

It is worth noting that in the definition of R, (3.4), as well as in the formula for the components $R^\mu{}_{\gamma\alpha\beta}$, (3.5), there is no explicit mention of the torsion. The case is different when using, as Perjés does, the abstract index notation. Since it will become useful in later sections to compare equations written in these different notations, I will briefly outline the definition of curvture in the "index-notation". For a much better description of this notation see [30].

In the index-notation, the vector field $\nabla_X Y$ is represented by $X^a\nabla_a Y^b$. If one is working in a coordinate basis (with coordinates $\{x^\alpha\}$) we have X^a is the vector field $X^\alpha\partial_\alpha$ and the vector field Y^b is $Y^\beta\partial_\beta$. Furthermore, in this notation $\nabla_a Z^b$ would represent the $(1-1)$-tensor (one index up, one index down)

$$\nabla_a Z^b = \partial_a Z^b + \Gamma^b{}_{ca}Z^c$$

where ∂_a is an *ordinary derivative operator.* That is, given any coordinate system, the operator ∂_a is defined to be partial differentiation with respect to the coordinates.

In the absence of torsion, one can define the action of the Riemann curvature tensor by

$$R^d{}_{cba}Z_d = (\nabla_a\nabla_b - \nabla_b\nabla_a)Z_c. \tag{3.6}$$

In terms of the Christoffel symbols $\Gamma^b{}_{ca}$

$$\begin{aligned} \nabla_a\nabla_b Z_d = &\partial_a\left(\partial_b Z_d - \Gamma^c{}_{db}Z_c\right) \\ &- \Gamma^e{}_{ba}\left(\partial_e Z_d - \Gamma^c{}_{eb}Z_c\right) \\ &- \Gamma^e{}_{da}\left(\partial_b Z_e - \Gamma^c{}_{eb}Z_c\right) \end{aligned}$$

yielding

$$
\begin{aligned}
(\nabla_a\nabla_b - \nabla_b\nabla_a)Z_d =&(\Gamma^c{}_{ab} - \Gamma^c{}_{ba})\nabla_c Z_d + (\partial_b\Gamma^c{}_{da} - \partial_a\Gamma^c{}_{db})Z_c \\
&+ (\Gamma^c{}_{db}\Gamma^e{}_{ca} - \Gamma^c{}_{da}\Gamma^e{}_{cb})Z_e \qquad (3.7)\\
=&\mathrm{T}^c{}_{ab}\nabla_c Z_d + R^e{}_{dba}Z_e.
\end{aligned}
$$

Rewriting equation (3.7) yields the curvature tensor in the presence of torsion:

$$
R^e{}_{dba}Z_e = (\nabla_a\nabla_b - \nabla_b\nabla_a)Z_d - \mathrm{T}^c{}_{ab}\nabla_c Z_d. \qquad (3.8)
$$

Thus, there is quite a difference between the treatment of torsion in the two notational schemes. While the first definition of curvature (equation (3.4)) proved to be valid with or without torsion, if one adopts the index notation to describe a theory involving torsion, one must also re-define the curvature tensor to take this into account.

While the index notation is usually used to describe torsion-free theories (*e.g.*, general relativity), the presence of "deficiency" in a parametric theory of space-time has analogous consequences. For example, compare Perjés' definition of the Zel'manov curvature (equation (2.15)) with equation (3.8). The definition of $Z^i{}_{jkl}$ appears to involve torsion. Nevertheless, Perjés claims that the connection ∇_* is torsion free! It can be argued that the extra term in equation (2.15) is actually due to the deficiency of the connection ∇_*. However, since there does not exist a well established theory involving connections with deficiency, it is reasonable to question the appropriateness of the definition of $Z^i{}_{jkl}$. As I have hinted at earlier, determining the "correct" definition for a parametric curvature tensor is not an obvious procedure.

Later in this chapter I will address the specific issue of defining a parametric curvature tensor. For now, I would like to finish the discussion of torsion by stating the symmetries of the curvature tensor when torsion is present.

Consider a manifold $\mathcal{M}$ with an affine connection ∇. The curvature and torsion of ∇ are given by equations (3.3) and (3.4). There exist some obvious and some not so obvious symmetries of T and R (*c.f.*, [29])

Theorem 3.2 *R and T satisfy the following symmetries:*

i. $T(X,Y) = -T(Y,X)$

ii. $R(X,Y)Z = -R(Y,X)Z$

iii. $\langle R(X,Y)Z, W\rangle = \langle R(X,Y)W, Z\rangle$ *if* ∇ *is compatible with* $<\ ,\ >$.

iv. The first Bianchi identity:

$$\begin{aligned} &R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\ &\quad= \nabla_X T(Y,Z) + \nabla_Y T(Z,X) + \nabla_Z T(X,Y) \\ &\qquad + T(X,[Y,Z]) + T(Y,[Z,X]) + T(Z,[X,Y]) \end{aligned} \tag{3.9}$$

Proof: Symmetries *i.* and *ii.* are immediate. To show *iv.* just write out the cyclic sum, use the definition of T, and keep in mind the Jacobi identity for bracket. Explicitly we have,

$$\begin{aligned} R(X,Y)Z + &R(Y,Z)X + R(Z,X)Y \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) \\ &\quad + \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X \\ &\quad - \nabla_{[X,Z]}Y \\ &= \nabla_X(T(Y,Z) + [Y,Z]) + \nabla_Y(T(Z,X) + [X,Z]) \\ &\quad + \nabla_Z(T(X,Y) + [X,Y]) - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X \\ &\quad - \nabla_{[X,Z]}Y \\ &= \nabla_X T(Y,Z) + \nabla_Y T(Z,X) + \nabla_Z T(X,Y) \\ &\quad + T(X,[Y,Z]) + T(Y,[Z,X]) + T(Z,[X,Y]) \\ &\quad + [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \end{aligned}$$

where the last three terms add to zero. To prove *iii.* we need to assume that ∇ is compatible with the metric $\langle\ ,\ \rangle$, thus writing

$$\begin{aligned} \langle \nabla_X \nabla_Y Z, W\rangle &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle \\ &= X\langle \nabla_Y Z, W\rangle - Y\langle Z, \nabla_X W\rangle + \langle Z, \nabla_Y \nabla_X W\rangle \end{aligned}$$

and

$$\left\langle \nabla_{[X,Y]} Z, W \right\rangle = [X,Y] \langle Z, W \rangle - \left\langle Z, \nabla_{[X,Y]} W \right\rangle$$

we have

$$\begin{aligned}
\langle R(X,Y)Z, W \rangle &= \langle \nabla_Y \nabla_X W, Z \rangle - \langle \nabla_X \nabla_Y W, Z \rangle + \left\langle \nabla_{[X,Y]} W, Z \right\rangle \\
&\quad + X \langle \nabla_Y Z, W \rangle - Y \langle Z, \nabla_X W \rangle - Y \langle \nabla_X Z, W \rangle \\
&\quad + X \langle Z, \nabla_Y W \rangle - [X,Y] \langle Z, W \rangle \\
&= - \langle R(X,Y)W, Z \rangle + XY \langle Z, W \rangle - X \langle Z, \nabla_Y W \rangle \\
&\quad - Y \langle Z, \nabla_X W \rangle - YX \langle Z, W \rangle + Y \langle Z, \nabla_X W \rangle \\
&\quad + X \langle Z, \nabla_Y W \rangle - [X,Y] \langle Z, W \rangle \\
&= - \langle R(X,Y)W, Z \rangle .
\end{aligned}$$

♠

3.4 The Standard Gauss–Codazzi Formalism

3.4.1 Introduction

In Chapter 2, a projection operator, $P_\alpha{}^\beta$, was introduced as a way of realizing the correspondence between spacetime tensors and parametric tensors. It was also shown that the parametric derivative operator ∂_{*i} was related to the partial derivative operator ∂_i by

$$P_\alpha{}^\beta(\partial_i)^\alpha = (\partial_{*i})^\beta.$$

Thus, the study of projected spacetime quantities is closely tied to the motivating definitions surrounding parametric manifolds.

In this section I will summarize the standard relationships between spacetime and projected quantities. In particular, I will examine the spacetime metric, Levi-Civita derivative operator, and its curvature and establish the Gauss-Codazzi relations (*c.f.*, [6]) .

3.4.2 The Gauss-Codazzi Relations

The Gauss-Codazzi equations relate the geometry of a manifold to the geometry of an embedded submanifold. Specifically, the higher-dimensional manifold induces a metric on the embedded surface, and thus gives rise to a unique derivative operator (on the surface) and finally a curvature tensor. The Gauss-Codazzi equations relate these induced quantities to the higher-dimensional quantities.

To continue with the original motivating example, let us consider a spacetime which admits a foliation by spacelike hypersurfaces. As we saw in Chapter 2 (equation (2.4)), the spacetime metric g induces a Riemannian metric on the spacelike slice Σ by the projection

$$k = g + n^\flat \otimes n^\flat$$

where $n^\flat$ is the one-form dual (with respect to g) to the future pointing unit vector field n.

Now, for any point $p \in \Sigma$, the tangent space $T_p\mathcal{M}$ may be written as a direct sum

$$\begin{aligned} T_p\mathcal{M} &= T_p\Sigma \oplus (T_p\Sigma)^\perp \\ &= (T_p\mathcal{M})^\perp \oplus (T_p\mathcal{M})^\top \end{aligned}$$

where $(T_p\Sigma)^\perp$ is the orthogonal complement of $T_p\Sigma$ in $T_p\mathcal{M}$ (with respect to the spacetime metric g). For any $v \in T_p\mathcal{M}$, let $v^\top$ and $v^\perp$ be the obvious projections so that

$$v = v^\perp + v^\top$$

where I have used $\perp$ to denote the projection to the *tangent* space of Σ (to agree with the notation of the next section).

Given vector fields X and Y on Σ, one may define a Riemannian connection on Σ by

$$D_X Y = (\nabla_X Y)^\perp. \tag{3.10}$$

Equation (3.10) not only defines an affine connection on Σ, but, as is shown in [6], D is the unique Levi-Civita connection associated with the induced metric k. Being a Riemannian connection, one may define its curvature in the usual manner:

$${}^3R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{3.11}$$

where X, Y, Z are all vectors tangent to Σ. Since Σ is a hypersurface, $[X,Y]$ denotes a vector field tangent to Σ and, hence, $D_{[X,Y]}Z$ is defined. Using $\langle\ ,\ \rangle$ to denote the spacetime metric, one can show, [6], that the curvature ${}^4R$ of $\mathcal{M}$ and the curvature ${}^3R$ of the Cauchy surface Σ are related by Gauss' equation ([6] page 135)

$$\begin{aligned} \langle {}^4R(X,Y)Z, W\rangle &= \langle {}^3R(X,Y)Z, W\rangle \\ &\quad - \langle B(Y,W), B(X,Z)\rangle + \langle B(X,W), B(Y,Z)\rangle \end{aligned} \tag{3.12}$$

where all the vectors X, Y, Z, W are assumed to be tangent to Σ and $B(X,Y)$ is the tensor defined by

$$\begin{aligned} B(X,Y) &= \nabla_X Y - D_X Y \\ &= (\nabla_X Y)^\top. \end{aligned}$$

Theorem 3.1 *Taking ∇, D, and B defined above, if ∇ is torsion-free then*

i. D is torsion-free and

ii. B is symmetric.

Proof: We already claimed (i) above. However, this can easily be shown.

$$\begin{aligned}
T_D(X,Y) &= D_X Y - D_Y X - [X,Y] \\
&= (\nabla_X Y)^{\perp} - (\nabla_Y X)^{\perp} - [X,Y] \\
&= (\nabla_X Y - \nabla_Y X)^{\perp} - [X,Y] \\
&= [X,Y]^{\perp} - [X,Y] \\
&= 0
\end{aligned}$$

since Σ is a hypersurface. The symmetry of B follows from the torsion-free properties of both connections. We have

$$\begin{aligned}
B(X,Y) - B(Y,X) &= \nabla_X Y - \nabla_Y X - (D_X Y - D_Y X) \\
&= [X,Y] - [X,Y] \\
&= 0.
\end{aligned}$$

♠

In the absence of torsion, we see that B is symmetric if and only if the Lie bracket of the spacetime restricted to vector fields on Σ is the 3-dimensional Lie bracket on Σ. While this seems like a trivial statement given the existence of the immersed surface Σ, the relationship between the two bracket operators becomes important in the next section.

B is closely related to the extrinsic curvature K of Σ. The tensor K is defined by

$$K(X,Y) = \langle -\nabla_X Y, n \rangle$$

where $\langle\ ,\ \rangle$ is the metric of the spacetime and n is the unit vector field normal to Σ. If one assumes that ∇ is torsion-free, then K is a symmetric tensor since $[X,Y]$

is tangent to Σ. The relationship between K and B is given by

$$\begin{aligned} K(X,Y) &= \left\langle -\nabla_X Y, n \right\rangle \\ &= \left\langle -B(X,Y) - D_X Y, n \right\rangle \\ &= \langle -B(X,Y), n\rangle - \left\langle D_X Y, n \right\rangle \\ &= \langle -B(X,Y), n\rangle \end{aligned}$$

so that the symmetry of K also follows directly from the symmetry of B. B can be thought of as measuring the difference between the geometries of $\mathcal{M}$ and Σ. In fact, B is identically zero if (and only if) every geodesic of Σ is also a geodesic of $\mathcal{M}$. Equation (3.12) is usually called the *Gauss equation* (one of the Gauss–Codazzi relations). Notice that $B(X,Y)$ is orthogonal to Σ.

It is worth mentioning that the tensor B fails to be symmetric if ∇ possesses torsion. If we let T_∇ and T_D represent the torsion tensors associated with the respective connections ∇ and D, then the above calculation shows that

$$B(X,Y) - B(Y,X) = T_\nabla(X,Y) - T_D(X,Y).$$

Therefore, the failure of B to be symmetric is to be expected in the most general setting.

3.5 A Generalized Gauss-Codazzi Formalism

3.5.1 Connections, Torsion, and Deficiency

The above formalism lends itself nicely to the slicing viewpoint. Both the slicing and Gauss-Codazzi formalisms focus on decomposing the spacetime into a piece tangent to Σ and a piece orthogonal to Σ. While the slicing viewpoint, as presented in Chapter 2, concentrated on splitting the metric, Gauss' equation involves the orthogonal splitting of the Riemann curvature tensor. Given a spacetime which admits a foliation of (spacelike) Cauchy surfaces, one may perform the above decompositions on any of the hypersurfaces. As mentioned earlier, one can view these decompositions as a place to begin an initial-value formulation of spacetime, where the Gauss-Codazzi relations provide initial-value constraints.

The situation more closely connected to the parametric manifold picture of spacetime, however, does not focus on the hypersurfaces but, rather, on a preferred non-null vector field. As we saw in Chapter 2, the fundamental ideas governing a parametric picture of spacetime are related to projections orthogonal to the given vector field. The parametric (and threading) metric was the induced metric on the orthogonal subspace of the tangent space; parametric vectors (and tensors) were identified with vectors orthogonal to the vector field; and the parametric derivative operator was interpreted as a projected derivative operator. However, in the parametric viewpoint it is not assumed there exist (even locally) surfaces orthogonal to the threading vector field.[6] Hence one must be careful when attempting to follow the above formalism leading to equation (3.12).

Given a non-null vector field A (not necessarily unit), at each point p in $\mathcal{M}$ one still has the decomposition

$$T_p\mathcal{M} = (T_p\mathcal{M})^\perp \oplus (T_p\mathcal{M})^\top.$$

---

[6] Barrett O'Neill [23] has studied analogues of the Gauss-Codazzi equations in the case of submersions.

For $v \in T_p\mathcal{M}$, write

$$v = v^{\perp} + v^{\top}$$

with $v^{\perp}$ orthogonal to $A(p)$ and $v^{\top}$ parallel to $A(p)$. As before, the spacetime metric induces a metric h on $(T_p\mathcal{M})^{\perp}$ defined by

$$h = g - \frac{A^{\flat} \otimes A^{\flat}}{\langle A^{\flat}, A^{\flat} \rangle} \tag{3.13}$$

where $A^{\flat}$ is the one-form which is dual (with respect to the metric g) to the vector field A.

Given vector fields X and Y (everywhere) orthogonal to A one may define the operator

$$D_X Y = (\nabla_X Y)^{\perp}.$$

Proposition 3.1 *D satisfies the properties of an affine connection. Specifically:*

1. $D_{fX+gY}Z = fD_X Z + gD_Y Z$
2. $D_X(Y+Z) = D_X Y + D_X Z$
3. $D_X(fY) = fD_X Y + X(f)Y$

for all vector fields $X, Y, Z \in (T\mathcal{M})^{\perp}$.

Proof: This is just a consequence of the linearity of projections.

First,

$$\begin{aligned} D_{fX+gY}Z &= \left(\nabla_{fX+gY}\right)^{\perp} \\ &= (f\nabla_X Z + g\nabla_Y Z)^{\perp} \\ &= fD_X Z + gD_Y Z. \end{aligned}$$

Second,

$$\begin{aligned} D_X(Y+Z) &= (\nabla_X Y + \nabla_X Z)^{\perp} \\ &= D_X Y + D_X Z. \end{aligned}$$

Finally,

$$\begin{aligned} D_X(fY) &= (\nabla_X fY)^{\perp} \\ &= (f\nabla_X Y + X(f)Y)^{\perp} \\ &= fD_X Y + X(f)Y. \end{aligned}$$

Therefore, D is an affine connection. ♠

In the case where $(T\mathcal{M})^{\perp}$ corresponded to the tangent space of some hypersurface, it was stated that D was the Levi-Civita connection of the surface (with respect to the induced metric). Although (in the present scenario) D is not, in general, the Levi-Civita connection on any submanifold, we may still investigate the familiar properties associated with the Levi-Civita connection. Using $\langle\langle\ ,\ \rangle\rangle$ to represent the metric h, we have

Proposition 3.2 *If ∇ is compatible with g, then D is compatible with the metric h. That is,*

$$X\,\langle\langle Y, Z\rangle\rangle = \langle\langle D_X Y, Z\rangle\rangle + \langle\langle Y, D_X Z\rangle\rangle$$

for $X, Y, Z \in (T\mathcal{M})^{\perp}$.

Proof: For $X, Y \in (T\mathcal{M})^{\perp}$, we have $\langle\langle X, Y\rangle\rangle = \langle X, Y\rangle$. Since $D_X Y = \nabla_X Y - (\nabla_X Y)^{\top}$ and $\left\langle (\nabla_X Y)^{\top}, Z\right\rangle = 0$, we have $\left\langle D_X Y, Z\right\rangle = \left\langle \nabla_X Y, Z\right\rangle$. The fact that D is compatible with h is now a consequence of the fact the ∇ is compatible with g. ♠

In the last section we showed that D being torsion-free was an immediate consequence of ∇ being torsion-free. In the present situation, progress is hindered by the fact that while $D_X Y - D_Y X$ represents a vector field orthogonal to A, $[X, Y]$ may not. In fact, $[X, Y] \in (T\mathcal{M})^{\perp}$ for all X and Y in $(T\mathcal{M})^{\perp}$, if and only if $(T\mathcal{M})^{\perp}$ is surface-forming (Frobenius's Theorem). Thus, if one wishes to consider the general threading scenario, it is quite fruitless to compare $D_X Y - D_Y X$ with $[X, Y]$. One may, however, decompose $[X, Y]$ as

$$[X, Y] = [X, Y]^{\top} + [X, Y]^{\perp}.$$

We may now measure the fact that $(T\mathcal{M})^{\perp}$ is not surface forming by the existence of $[X, Y]^{\top}$ and use $[X, Y]^{\perp}$ to measure the torsion of D.

Definition 3.3 *The (generalized) torsion, ${}^{\perp}T_D$, associated with the connection D is defined by*

$$ {}^{\perp}T_D(X,Y) = D_X Y - D_Y X - [X,Y]^{\perp}. $$

Lemma 3.4 *The generalized torsion is precisely the projection of the torsion associated with ∇.*

Proof: We have,

$$ \begin{aligned} {}^{\perp}T_D(X,Y) &= D_X Y - D_Y X - [X,Y]^{\perp} \\ &= \Big(\nabla_X Y - \nabla_Y X - [X,Y]\Big)^{\perp} \\ &= T(X,Y)^{\perp}. \end{aligned} $$

♠

Definition 3.5 *The deficiency, $\mathcal{D}$, of the connection D is defined by*

$$ \mathcal{D}(X,Y) = [X,Y]^{\top}. $$

Theorem 3.6 *The following statements are equivalent:*

i. $(T\mathcal{M})^{\perp}$ *is surface forming.*

ii. *The generalized torsion associated with D, ${}^{\perp}T_D$, is the (standard) torsion T_D as defined by (3.3).*

iii. $\mathcal{D}(X,Y) \equiv 0$ *for all* $X, Y \in (T\mathcal{M})^{\perp}$.

Proof: This theorem is basically the vector field version of Frobenius's theorem rewritten to emphasize the new definitions. By definition, $\mathcal{D}(X,Y) \equiv 0$ if and only if $[X,Y]^{\top} \equiv 0$. Thus $\mathcal{D}(X,Y) \equiv 0$ if and only if $[X,Y] \in (T\mathcal{M})^{\perp}$, yielding $(iii.) \Leftrightarrow (i)$ via Frobenius's theorem. To show $(iii) \Rightarrow (ii)$, we again have $[X,Y]^{\top} \equiv 0$ so $[X,Y]^{\perp} \equiv [X,Y]$, making the two notions of torsion coincide. Since ${}^{\perp}T_D(X,Y) - T_D(X,Y) = [X,Y]^{\top}$, we also easily have $(ii) \Rightarrow (iii)$. ♠

Theorem 3.7 *If ∇ is torsion-free, then ${}^{\perp}T_D(X,Y) \equiv 0$ for all $X, Y \in (T\mathcal{M})^{\perp}$.*

Proof:

$$\begin{aligned} {}^{\perp}T_D(X,Y) &= D_X Y - D_Y X - [X,Y]^{\perp} \\ &= (\nabla_X Y - \nabla_Y X - [X,Y])^{\perp} \\ &= (T(X,Y))^{\perp} \\ &= 0. \end{aligned}$$

♠

Therefore D still inherits its (generalized) torsion only from ∇.

Later in this chapter we will show that, in a coordinate basis, the connection symbols, ${}^{\perp}\Gamma^{i}{}_{jk}$, associated with D obey the symmetry ${}^{\perp}\Gamma^{i}{}_{jk} = {}^{\perp}\Gamma^{i}{}_{kj}$ if and only if ∇ is torsion-free. Thus, the above definition of ${}^{\perp}T_D$ is quite reasonable.

For $X, Y \in (T\mathcal{M})^{\perp}$, define as before

$$B(X,Y) = \nabla_X Y - D_X Y.$$

$B(X,Y)$ is again a vector field orthogonal to the vector fields X and Y and, hence, a vector field tangent to the threading. There is no guarantee, however, that B is symmetric. This is a consequence of the deficiency of D. In general, one has

$$\begin{aligned} B(X,Y) - B(Y,X) &= [X,Y] - [X,Y]^{\perp} + T_{\nabla}(X,Y) - {}^{\perp}T_D(X,Y) \\ &= \mathcal{D}(X,Y) + T_{\nabla}(X,Y) - {}^{\perp}T_D(X,Y). \end{aligned} \tag{3.14}$$

When ∇ is torsion-free, B may still fail to be symmetric.

Theorem 3.8 *If ∇ is torsion-free, then $B(X,Y) = B(Y,X)$ if and only if $\mathcal{D}(X,Y) = 0$.*

Proof: $T_{\nabla}(X,Y) = 0$ implies that ${}^{\perp}T_D(X,Y) = 0$ and, hence, equation (3.14) reduces to

$$B(X,Y) - B(Y,X) = \mathcal{D}(X,Y).$$

♠

We have that the deficiency of the connection D measure the failure of $(T\mathcal{M})^{\perp}$ to be surface-forming and, equivalently, the failure of the extrinsic curvature B to be symmetric in a torsion-free setting.

3.5.2 Curvature

Being an affine connection, D must have an associated "curvature" tensor. However, the existence of the $[X,Y]^{\top}$ component prevents one from proceeding as in equation (3.11). It appears as if this problem may be overcome simply by using the quantity $[X,Y]^{\perp}$ to represent the commutator of two vector fields orthogonal to the original vector field A.

Armed with such a notion of "bracket", the next step would be to define a curvature operator.

Definition 3.9 *Define the operator S by*

$$S(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]_{\perp}} Z. \tag{3.15}$$

Unfortunately, such a definition immediately leads to problems.

Proposition 3.10 *$S(X,Y)Z$ is not function linear. That is*

$$S(X,fY)(gZ) \neq fg\, S(X,Y)Z.$$

Proof:

$$
\begin{aligned}
S(X,fY)(gZ) &= \left(D_X(fD_Y) - fD_YD_X - D_{f[X,Y]^\perp} - D_{X(f)Y}\right)(gZ)\\
&= fS(X,Y)(gZ) + \left(X(f)D_Y - X(f)D_Y\right)(gZ)\\
&= fS(X,Y)(gZ)\\
&= f\Big(D_X\left(Y(g)Z + gD_YZ\right) - D_Y\left(X(g)Z + gD_XZ\right)\\
&\qquad - [X,Y]^\perp(g)Z - gD_{[X,Y]^\perp}Z\Big)\\
&= f\left([X,Y](g)Z - [X,Y]^\perp(g)Z + gS(S,Y)Z\right)\\
&= fgS(X,Y)Z + [X,Y]^\top(g)Z
\end{aligned}
$$

where, in general, $[X,Y]^\top$ is not everywhere zero. ♠

Therefore, in order to define a function linear curvature operator (tensor!), we must keep track of the $[X,Y]^\top$ component (we can not just project it away and forget about it). That is, the $D_{[X,Y]^\perp}Z$ term in equation (3.15) is not complete. We do not want to project the vector field $[X,Y]$ too soon! We will, therefore, consider replacing the last term of (3.15) by the term $(\nabla_{[X,Y]}Z)^\perp$. This term is equivalent to the $D_{[X,Y]}Z$ term in equation (3.11). However, since $[X,Y]$ is not necessarily orthogonal to A we can not write $(\nabla_{[X,Y]}Z)^\perp$ in terms of the connection D.

Definition 3.11 *The (generalized) curvature operator associated with D is defined by*

$$
{}^\perp R(X,Y)Z = D_XD_YZ - D_YD_XZ - (\nabla_{[X,Y]}Z)^\perp.
$$

Proposition 3.12 *${}^\perp R$ is function linear. That is, ${}^\perp R$ is tensorial.*

Proof:

$$
\begin{aligned}
&{}^{\perp}R(X, fY)(gZ) = \\
&\quad \left(D_X(fD_Y) - fD_YD_X - (\nabla_{f[X,Y]} + \nabla_{X(f)}Y)^{\perp}\right)(gZ) \\
&\quad = f\,{}^{\perp}R(X,Y)(gZ) + \left(X(f)D_Y - (\nabla_{X(f)Y})^{\perp}\right)(gZ) \\
&\quad = f\,{}^{\perp}R(X,Y)(gZ) + \left(X(f)D_Y - X(f)D_Y\right)(gZ) \\
&\quad = f\,{}^{\perp}R(X,Y)(gZ) \\
&\quad = f\Big(D_X\left(Y(g)Z + gD_YZ\right) - D_Y\left(X(g)Z + gD_XZ\right) \\
&\qquad - \left([X,Y](g)Z + g\nabla_{[X,Y]}Z\right)^{\perp}\Big) \\
&\quad = f\left(g\,{}^{\perp}R(X,Y)Z + [X,Y](g)Z - ([X,Y](g)Z)^{\perp}\right) \\
&\quad = gf\,{}^{\perp}R(X,Y)Z
\end{aligned}
$$

where the linearity of the projection map was used throughout. ♠

Theorem 3.13 *If* ∇ *is metric compatible, then* ${}^{\perp}R$ *satisfies Gauss' Equation. That is,*

$$
\begin{aligned}
\left\langle {}^{\perp}R(X,Y)Z, W\right\rangle = &\left\langle {}^{4}R(X,Y)Z, W\right\rangle \\
&- \langle B(Y,W), B(X,Z)\rangle + \langle B(X,W), B(Y,Z)\rangle
\end{aligned}
\tag{3.16}
$$

where X, Y, Z *and* W *are orthogonal to* A.

Proof: First, a few computational observations. Since $B(X,Y) = \nabla_XY - D_XY$ is orthogonal to A,

$$
\langle \nabla_XY, Z\rangle = \langle D_XY, Z\rangle \tag{3.17}
$$

for vector fields X, Y, Z orthogonal to A. While we have shown that D is compatible with the metric $\langle\!\langle\ ,\ \rangle\!\rangle$, it is also true that since the four-metric $\langle\ ,\ \rangle$ agrees with the induced metric $\langle\!\langle\ ,\ \rangle\!\rangle$ on $(T\mathcal{M})^{\perp}$, one may write

$$
X\langle Y, Z\rangle = \langle D_XY, Z\rangle + \langle Y, D_XZ\rangle .
$$

That is, D is "compatible" with the metric $\langle\ ,\ \rangle$ when restricted to the subspace $(T\mathcal{M})^{\perp}$. Using the definition of ${}^4R$ and B, we expand the right hand side of equation (3.16)

$$
\begin{aligned}
RHS =& \left\langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \right\rangle \\
&- \langle \nabla_Y W - D_Y W, \nabla_X Z - D_X Z \rangle \\
&+ \langle \nabla_X W - D_X W, \nabla_Y Z - D_Y Z \rangle \\
=& X \langle \nabla_Y Z, W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - Y \langle \nabla_X Z, W \rangle \\
&+ \langle \nabla_X Z, \nabla_Y W \rangle - \left\langle \nabla_{[X,Y]} Z, W \right\rangle - \langle \nabla_Y W, \nabla_X Z \rangle \\
&+ \langle \nabla_Y W, D_X Z \rangle + \langle D_Y W, \nabla_X Z \rangle - \langle D_Y W, D_X Z \rangle \\
&+ \langle \nabla_X W, \nabla_Y Z \rangle - \langle \nabla_X W, D_Y Z \rangle - \langle D_X W, \nabla_Y Z \rangle \\
&+ \langle D_X W, D_Y Z \rangle \\
=& X \langle \nabla_Y Z, W \rangle - Y \langle \nabla_X Z, W \rangle - \left\langle \nabla_{[X,Y]} Z, W \right\rangle \\
&+ \langle D_Y W, D_X Z \rangle - \langle D_X W, D_Y Z \rangle \\
=& X \langle D_Y Z, W \rangle - \langle D_Y Z, D_X W \rangle \\
&- Y \langle D_X Z, W \rangle + \langle D_Y W, D_X Z \rangle - \left\langle \left(\nabla_{[X,Y]} Z \right)^{\perp}, W \right\rangle \\
=& \left\langle D_X D_Y Z - D_Y D_X Z - \left(\nabla_{[X,Y]} Z \right)^{\perp}, W \right\rangle \\
=& \langle {}^{\perp}R(X,Y)Z, W \rangle
\end{aligned}
$$

where the second step involved the symmetry of the metric as well as equation (3.17). ♠

The above proof of Gauss' equation only used the properties of metric compatibility (for both pairs of connections and metrics). In particular, the symmetry (torsion) of either connection was not a concern. Thus, we have further shown that Gauss' equation is valid in the presence of torsion.

Given $\langle\ ,\ \rangle$, ${}^4R$, and B, one may use Gauss' equation to *define* a curvature operator ${}^3R$. In this context, we may view ${}^{\perp}R$ as the unique curvature tensor associated with D which satisfies Gauss' equation.

A word of caution is necessary at this point. If torsion is present in either (or both) of the connections, the tensor $B(X,Y)$ is no longer symmetric. This affects the symmetries of the tensors ${}^{\perp}R$ and ${}^{4}R$. In particular, as we shall see, ${}^{\perp}R$ may not enjoy the familiar cyclic symmetry

$$ {}^{\perp}R(X,Y)Z + {}^{\perp}R(Y,Z)X + {}^{\perp}R(Z,X)Y = 0 $$

even if ${}^{4}R$ does! However, the other symmetries are immediate. More precisely,

Theorem 3.14 *Let ∇ be a torsion-free Riemannian connection associated with the metric $\langle\ ,\ \rangle$, with curvature tensor R. Using $D, \langle\!\langle\ ,\ \rangle\!\rangle$, B, and $\mathcal{D}$ as defined above, if $\hat{R}$ is an induced curvature operator associated with the connection D and $\hat{R}$ and R satisfy Gauss' equation, then $\hat{R}$ has the following symmetries:*

i. $\left\langle \hat{R}(X,Y)Z, W \right\rangle = -\left\langle \hat{R}(Y,X)Z, W \right\rangle$

ii. $\left\langle \hat{R}(X,Y)Z, W \right\rangle = -\left\langle \hat{R}(X,Y)W, Z \right\rangle$

iii. First Bianchi identity:

$$
\begin{aligned}
\langle \hat{R}(X,Y)Z + &\hat{R}(Y,Z)X + \hat{R}(Z,X)Y, W\rangle \\
&= \langle B(X,W), \mathcal{D}(Y,Z)\rangle + \langle B(Y,W), \mathcal{D}(Z,X)\rangle \\
&\quad + \langle B(Z,W), \mathcal{D}(X,Y)\rangle \\
&= -\langle \nabla_X \mathcal{D}(Y,Z), W\rangle - \langle \nabla_Y \mathcal{D}(Z,X), W\rangle \\
&\quad - \langle \nabla_Z \mathcal{D}(X,Y), W\rangle
\end{aligned}
\tag{3.18}
$$

where $\mathcal{D}(X,Y) = B(X,Y) - B(Y,X)$ measures the failure of B to be symmetric. $\mathcal{D}$ may be thought of as the "deficiency" of the connection D.

Proof: The symmetries in (i) and (ii) can be read off directly from equation (3.12), keeping in mind that R satisfies all of the symmetries of the usual Riemann curvature tensor (in the absence of torsion). To prove (iii), just cyclicly permute X, Y, and Z in the terms on the right hand side of equation (3.12) and add, obtaining

$$\begin{aligned}
\langle \hat{R}(X,Y)Z + &\hat{R}(Y,Z)X + \hat{R}(Z,X)Y, W\rangle \\
&= 0 - \langle B(Y,W), B(X,Z)\rangle - \langle B(Z,W), B(Y,X)\rangle \\
&\quad - \langle B(X,W), B(Z,Y)\rangle + \langle B(X,W), B(Y,Z)\rangle \\
&\quad + \langle B(Y,W), B(Z,X)\rangle + \langle B(Z,W), B(X,Y)\rangle \\
&= \langle B(X,W), \mathcal{D}(Y,Z)\rangle + \langle B(Y,W), \mathcal{D}(Z,X)\rangle \\
&\quad + \langle B(Z,W), \mathcal{D}(X,Y)\rangle .
\end{aligned}$$

which is the first line in (*iii*). However, this cyclic sum involving B and $\mathcal{D}$ may be rewritten in terms of ∇ and $\mathcal{D}$. Thus written, claim (*iii*) resembles the standard cyclic symmetry of R (see equation (3.9)). Keep in mind, however, that neither ∇ nor D possess torsion in the theorem. However, deficiency is present. We have

$$\begin{aligned}
\langle B(X,W), \mathcal{D}(Y,Z)\rangle &= \left\langle \nabla_X W - D_X W, \mathcal{D}(Y,Z)\right\rangle \\
&= \left\langle \nabla_X W, \mathcal{D}(Y,Z)\right\rangle \\
&= X\, \langle W, \mathcal{D}(Y,Z)\rangle - \left\langle W, \nabla_X \mathcal{D}(Y,Z)\right\rangle \\
&= -\left\langle \nabla_X \mathcal{D}(Y,Z), W\right\rangle
\end{aligned}$$

since $\mathcal{D}(Y,Z)$ is orthogonal to W. Thus the second equation in (*iii*) is true. ♠

One further comment on the similarities between equations (3.9) and (3.18) is worth making. In equation (3.9) there are three extra terms of the form $T\left(X,[Y,Z]\right)$ (and cyclic permutations). One might expect analogous terms in equation (3.18) involving $\mathcal{D}(X,[Y,Z]^{\perp})$ and cyclic permutations. However, since $\mathcal{D}(X,Y)$ represents a vector field orthogonal to the threading curves,

$$\left\langle \mathcal{D}\left(X,[Y,Z]^{\perp}\right), W\right\rangle = 0.$$

Thus, because of the way I have expressed the first Bianchi identity (in terms of an inner product) equation (3.18) differs slightly from (3.9). We have that the new concept of deficiency resembles the notion of torsion.

3.5.3 Coordinate Expressions

Let us now work in a coordinate patch and investigate the components of the above operators. We will use the adapted coordinate system inherited from the threading decomposition of spacetime. That is, spacetime coordinates $x^\alpha = (x^0, x^i) = (t, x^i)$, $\{i = 1, 2, 3\}$ where the given vector field A can be written $A^\alpha = \frac{1}{M^2}\left(\frac{\partial}{\partial t}\right)^\alpha$ so that $A^0 = \frac{1}{M^2}$ and $A^i = 0$. The coordinates x^i are constant along specific integral curves of $\frac{\partial}{\partial t}$ and can thus be thought of as coordinates on the (local) surfaces $\{t \equiv \text{constant}\}$.

We have the threading lapse function and shift form as before related by

$$dt = -\frac{1}{M}m + M_i dx^i$$

where m is the metric dual of the unit vector tangent to the threading curves. Thus,

$$A_0 = -1 \qquad \text{and} \qquad A_i = M_i.$$

In these coordinates the spacetime metric g has the form (see equation (2.8))

$$(g_{\alpha\beta}) = \begin{pmatrix} -M^2 & M^2 M_j \\ M^2 M_i & h_{ij} - M^2 M_i M_j \end{pmatrix}$$

The functions $h_{ij} = g_{ij} + M^2 M_i M_j$ correspond to the components of the threading metric. That is, the metric on $(T\mathcal{M})^\perp$ induced by g (equation (3.13)). These functions can also be thought of as the nonzero components of the four-dimensional object

$$h_{\alpha\beta} = g_{\alpha\beta} - M^2 A_\alpha A_\beta$$

which is associated with the projection operator

$$P_\alpha{}^\beta = h_\alpha{}^\beta = \delta_\alpha{}^\beta - M^2 A_\alpha A^\beta$$

where $\delta_\alpha{}^\beta$ is the Kronecker delta symbol. Being a projection operator guarantees that $P_\alpha{}^\beta X^\alpha = X^\beta$ for $X \in (T\mathcal{M})^\perp$. It is easy to show that a spacetime vector field $X = X^\alpha \frac{\partial}{\partial x^\alpha} = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i}$ is orthogonal to A if and only if $X^0 = M_i X^i$.

To simplify notation I will introduce the "starry" derivative notation in all coordinate directions. Therefore, at the risk of abusing notation, define

$$\partial_{*\alpha} = \partial_\alpha + A_\alpha \partial_t.$$

Notice that since $A_0 = -1$ and $A_i = M_i$, we have

$$\partial_{*0} = 0$$

and

$$\partial_{*i} = \partial_i + M_i \partial_t.$$

Therefore, this new operator $\partial_{*\alpha}$ agrees with all other previous uses of ∂_*.

Let us work out the action of the connection D in these coordinates. Given X and Y in $(T\mathcal{M})^\perp$, we defined

$$\begin{aligned}
D_X Y &= (\nabla_X Y)^\perp \\
&= P_\gamma{}^\alpha X^\beta \nabla_\beta Y^\gamma \frac{\partial}{\partial x^\alpha} \\
&= P_\gamma{}^\alpha P_\beta{}^\delta X^\beta \nabla_\delta Y^\gamma \frac{\partial}{\partial x^\alpha} \\
&= X^\beta P_\gamma{}^\alpha P_\beta{}^\delta \left(Y^\gamma{}_{,\delta} + \Gamma^\gamma{}_{\mu\delta} Y^\mu \right) \frac{\partial}{\partial x^\alpha} \\
&= X^\beta \left(P_\gamma{}^\alpha \left(Y^\gamma{}_{,\beta} + M^2 A_\beta A^\delta Y^\gamma{}_{,\delta} \right) + P_\gamma{}^\alpha P_\beta{}^\delta \Gamma^\gamma{}_{\mu\delta} Y^\mu \right) \frac{\partial}{\partial x^\alpha} \\
&= X^\beta \left(P_\gamma{}^\alpha Y^\gamma{}_{*\beta} + P_\gamma{}^\alpha P_\beta{}^\delta P_\nu{}^\mu Y^\nu \Gamma^\gamma{}_{\mu\delta} \right) \frac{\partial}{\partial x^\alpha} \\
&= X^\beta \left(Y^\alpha{}_{*\beta} + M^2 A_\gamma A^\alpha Y^\gamma{}_{*\beta} + {}^\perp\Gamma^\alpha{}_{\nu\beta} Y^\nu \right) \frac{\partial}{\partial x^\alpha} \\
&= X^\beta \left(Y^\alpha{}_{*\beta} + \left({}^\perp\Gamma^\alpha{}_{\nu\beta} - M^2 A^\alpha A_{\nu*\beta} \right) Y^\nu \right) \frac{\partial}{\partial x^\alpha}
\end{aligned}$$

where I have defined the symbol ${}^\perp\Gamma^\alpha{}_{\nu\beta}$ by

$${}^\perp\Gamma^\alpha{}_{\nu\beta} = P_\gamma{}^\alpha P_\beta{}^\delta P_\nu{}^\mu \Gamma^\gamma{}_{\mu\delta}.$$

It can be show that the symbol ${}^\perp\Gamma^\alpha{}_{\nu\beta}$ behaves like a projected rank-three tensor. That is,

$${}^\perp\Gamma^0{}_{00} = {}^\perp\Gamma^\alpha{}_{00} = {}^\perp\Gamma^\alpha{}_{0\beta} = {}^\perp\Gamma^\alpha{}_{\beta 0} = {}^\perp\Gamma^0{}_{\beta 0} = 0$$

and

$$^{\perp}\Gamma^{0}{}_{\alpha\beta} = A_i {}^{\perp}\Gamma^{i}{}_{\alpha\beta}.$$

Furthermore, after a long but straightforward calculation, one may show that the terms ${}^{\perp}\Gamma^{i}{}_{jk}$ may be written in a familiar form involving the parametric derivative operator and the components of the induced metric h_{ij} :

$$^{\perp}\Gamma^{i}{}_{jk} = \frac{1}{2} h^{im}(h_{mj*k} + h_{mk*j} - h_{jk*m}). \tag{3.19}$$

Since $D_X Y$ is orthogonal to A, $D_X Y$ is completely determined by its components $(D_X Y)^i$. That is

$$(D_X Y)^{\alpha} \frac{\partial}{\partial x^{\alpha}} = M_i (D_X Y)^i \frac{\partial}{\partial t} + (D_X Y)^i \frac{\partial}{\partial x^i}.$$

We have shown that

$$(D_X Y)^i = X^j \left(Y^i{}_{*j} + {}^{\perp}\Gamma^{i}{}_{kj} Y^k \right)$$

where I have used the facts that $Y^i{}_{*0} \equiv 0$ for all Y and $A^i = 0$ for $i = 1, 2, 3$. The above formula for $(D_X Y)^i$ corresponds exactly to the parametric covariant derivative operator introduced by Perjés. Note that the projected 3-index symbol ${}^{\perp}\Gamma^{i}{}_{jk}$ has the same coordinate representation as the connection symbol used by Perjés, $\gamma^{i}{}_{jk}$ (see equation (2.13)). Thus, we have our first covariant confirmation that Perjés' parametric structure can be induced by a projective geometry of spacetime.

Continuing our coordinate description, let us calculate the components of the three-dimensional curvature tensor ${}^{\perp}R$ defined earlier. Since $\partial_{*i} = \partial_i + M_i \frac{\partial}{\partial t}$ is a basis for $(T\mathcal{M})^{\perp}$, we define the components of ${}^{\perp}R$ by

$$\begin{aligned} {}^{\perp}R(\partial_{*i}, \partial_{*j})\partial_{*k} &= {}^{\perp}R^{l}{}_{kij} \partial_{*l} \\ &= {}^{\perp}R^{l}{}_{kij} (\partial_l + M_l \frac{\partial}{\partial t}). \end{aligned}$$

Calculating the "spatial" components of $D_{\partial_{*i}} D_{\partial_{*j}} \partial_{*k}$, we find:

$$\begin{aligned}\left(D_{\partial_{*i}} D_{\partial_{*j}} \partial_{*k}\right)^l &= \left(D_{\partial_{*i}}\left(D_{\partial_{*j}} \partial_{*k}\right)\right)^l \\ &= \partial_{*i}\left(D_{\partial_{*j}} \partial_{*k}\right)^l + {}^{\perp}\Gamma^l{}_{ni}\left(D_{\partial_{*j}} \partial_{*k}\right)^n \\ &= \partial_{*i}\left(\delta^l{}_{k*j} + {}^{\perp}\Gamma^l{}_{mj}\delta^m_k\right) \\ &\quad + {}^{\perp}\Gamma^l{}_{ni}\left(\delta^n{}_{k*j} + {}^{\perp}\Gamma^n{}_{mj}\delta^m_k\right) \\ &= {}^{\perp}\Gamma^l{}_{jk*i} + {}^{\perp}\Gamma^l{}_{ni}{}^{\perp}\Gamma^n{}_{kj}.\end{aligned} \tag{3.20}$$

Also,

$$\begin{aligned}\left[\partial_{*i}, \partial_{*j}\right] &= (M_{j*i} - M_{i*j})\frac{\partial}{\partial t} \\ &= \mathcal{D}_{ji}\frac{\partial}{\partial t}\end{aligned}$$

where I have introduced the notation $\mathcal{D}_{ji} = M_{j*i} - M_{i*j}$. Therefore,

$$\begin{aligned}\nabla_{[\partial_{*i},\partial_{*j}]}\partial_{*k} &= \mathcal{D}_{ji}\left(\dot{M}_k + \Gamma^0{}_{00}M_k + \Gamma^0{}_{k0}\right)\frac{\partial}{\partial t} \\ &\quad + \mathcal{D}_{ji}\left(\Gamma^l{}_{00}M_k + \Gamma^l{}_{k0}\right)\frac{\partial}{\partial x^l}\end{aligned}$$

thus yielding

$$\left(\nabla_{[\partial_{*i},\partial_{*j}]}\partial_{*k}\right)^{\perp} = \left(\left(M_{j*i} - M_{i*j}\right)\left(\Gamma^l{}_{k0} + M_k\Gamma^l{}_{00}\right)\right)\partial_{*l}.$$

As we see, the components of ${}^{\perp}R$ are not quite as nice as in the case where the ∂_{*i} span a hypersurface. The non-zero contribution of $[\partial_{*i}, \partial_{*j}]$ continues to complicate matters. Writing everything out gives us

$$\begin{aligned}{}^{\perp}R^l{}_{kij} &= {}^{\perp}\Gamma^l{}_{kj*i} - {}^{\perp}\Gamma^l{}_{ki*j} + {}^{\perp}\Gamma^l{}_{ni}{}^{\perp}\Gamma^n{}_{kj} - {}^{\perp}\Gamma^l{}_{nj}{}^{\perp}\Gamma^n{}_{ki} \\ &\quad + 2\left(M_{j*i} - M_{i*j}\right)\left(\Gamma^l{}_{00}M_k + \Gamma^l{}_{k0}\right) \\ &= {}^{\perp}\Gamma^l{}_{kj*i} - {}^{\perp}\Gamma^l{}_{ki*j} + {}^{\perp}\Gamma^l{}_{ni}{}^{\perp}\Gamma^n{}_{kj} - {}^{\perp}\Gamma^l{}_{nj}{}^{\perp}\Gamma^n{}_{ki} \\ &\quad + \left(M_{j*i} - M_{i*j}\right)h^{lm}\left(M^2 M_{m*k} - M^2 M_{k*m} + \partial_t h_{km}\right)\end{aligned} \tag{3.21}$$

where the four-dimensional symbols Γ were replaced by the equivalent expressions involving the threading metric, lapse function, and shift one-form.

Using equation (2.15) in these coordinates, one has

$$Z^l{}_{kij} = {}^{\perp}\Gamma^l{}_{kj*i} - {}^{\perp}\Gamma^l{}_{ki*j} + {}^{\perp}\Gamma^l{}_{ni}{}^{\perp}\Gamma^n{}_{jk} - {}^{\perp}\Gamma^l{}_{nj}{}^{\perp}\Gamma^n{}_{ik}.$$

At this point one notices a deviation from the definitions introduced by Perjés. The Zel'manov curvature does not contain the contribution from $[\partial_{*i}, \partial_{*j}]$.

3.5.4 Zel'manov Curvature

Apparently, if one wants to relate the three-dimensional parametric tensor $Z^i{}_{jkl}$ to a four-dimensional spacetime tensor, one must re-examine the story leading up to the definition of ${}^{\perp}R$.

It seemed most natural to define ${}^{\perp}R$ with the $(\nabla_{[X,Y]})^{\perp}$ term, as this definition closely resembles the definition of the standard curvature tensor. However, consider the definition

$$ {}^{\perp}\bar{R}(X,Y)Z = D_X D_Y Z - D_Y D_X Z - (\pounds_{[X,Y]}Z)^{\perp} \tag{3.22} $$

where $\pounds$ is Lie differentiation.

The difference between the two curvature operators is

$$ {}^{\perp}R(X,Y)Z - {}^{\perp}\bar{R}(X,Y)Z = -(\nabla_Z [X,Y])^{\perp} \tag{3.23} $$

In light of our earlier comments, we know that ${}^{\perp}\bar{R}$ does not satisfy Gauss' equation. However, there do exist the following similarities between ${}^{\perp}R$ and ${}^{\perp}\bar{R}$.

1. ${}^{\perp}R(X,Y)f = {}^{\perp}\bar{R}(X,Y)f \qquad$ for $X, Y \in (T\mathcal{M})^{\perp}$,

and in the case where $(T\mathcal{M})^{\perp}$ is surface forming one has that $[\partial_{*i}, \partial_{*j}] = 0$ which implies

2. ${}^{\perp}R(\partial_{*i}, \partial_{*j}) = {}^{\perp}\bar{R}(\partial_{*i}, \partial_{*j})$

so the coordinate representation of the two tensors agree in this special case.

For the components of ${}^{\perp}\bar{R}$, we must calculate $\left(\pounds_{[\partial_{*i},\partial_{*j}]}\partial_{*k}\right)^{\perp}$. If ∇ is torsion-free, the definition of $\pounds$ yields

$$ \begin{aligned} \left(\pounds_{[\partial_{*i},\partial_{*j}]}\partial_{*k}\right)^{\perp} &= \left[\left[\partial_{*i}, \partial_{*j}\right], \partial_{*k}\right] \\ &= 0 \end{aligned} \tag{3.24} $$

Thus, the nonzero term $[\partial_{*i}, \partial_{*j}]$ does not contribute to the components of ${}^{\perp}\bar{R}$.

To summarize, we have already shown

Theorem 3.15 *${}^{\perp}R$ satisfies Gauss' equation, (3.16).*

Although ${}^{\perp}\bar{R}$ does not satisfy Gauss' equation (${}^{\perp}\bar{R}$ lacks the correct symmetries), we do have

Theorem 3.16 *${}^{\perp}\bar{R}$ is the Zel'manov curvature.*

Proof: Using equations (3.20) and (3.24), we have

$$\begin{aligned} {}^{\perp}\bar{R}^{i}{}_{jkl} &= {}^{\perp}\Gamma^{i}{}_{kl*i} - {}^{\perp}\Gamma^{l}{}_{ki*j} + {}^{\perp}\Gamma^{l}{}_{ni}{}^{\perp}\Gamma^{n}{}_{kj} - {}^{\perp}\Gamma^{l}{}_{nj}{}^{\perp}\Gamma^{n}{}_{ki} \\ &= Z^{l}{}_{ijk}. \end{aligned} \tag{3.25}$$

Since these expressions are invariant under regular coordinate transformation and reparameterizations of Σ, we have that ${}^{\perp}\bar{R} = Z$. ♠

3.5.5 Conclusion

The generalized Gauss-Codazzi approach was successful in defining a parametric structure on Σ. The projected connection D gave us a covariant derivative operator which was also invariant under reparameterizations. Moreover, D was found to be torsion-free if ∇ was torsion-free. Most importantly, the deficiency $\mathcal{D}$ was explicitly defined in such a way as to make its relationship to the torsion tensor clear.

While the generalized Gauss-Codazzi formalism succeeded in providing a curvature operator which satisfied Gauss' equation, the curvature operator did not agree with the Zel'manov curvature. The difference between ${}^{\perp}R$ and Z involved both the deficiency and the lapse function M. The appearance of M is due to the fact that we began with a parameter t whose relationship to proper time was arbitrary. The extra pieces involving the deficiency are a result of using covariant differentiation instead of Lie differentiation.

3.6 The Parametric Structure

3.6.1 Introduction

In the case of a spacetime threading, we were able to recapture much of the geometry of the local rest spaces, even though these spaces did not constitute a surface embedded in the spacetime. As we moved our focus to the manifold of orbits, the threading framework forced some additional structure on Σ. Since the original spacetime may not have been stationary or static, we decided to allow the tensors on Σ to depend on an extra parameter. The parameter on Σ is **not** an extra coordinate. As we saw earlier, while parametric tensors on Σ must (of course) transform properly under a change of coordinates, the components of a parametric tensor remain invariant under a reparameterization. The threading framework also provided us with a natural metric on Σ.

By expanding on the basic threading ideas, we may continue decomposing such objects as the connection or curvature tensor. In the analogous case of slicing, such a procedure would lead quite naturally to the Gauss-Codazzi equations. While we had to generalize a few notions (such as bracket, torsion, and curvature), a similar decomposition may be carried over in the threading scenario. Furthermore, when these objects are thought of as living on the manifold of orbits, they satisfy the necessary reparameterization invariance. The generalized Gauss-Codazzi formalism simply interpreted these new parametric objects as projected spacetime quantities. However, in an attempt to define an abstract parametric theory, introducing an extra dimension (to later project out) is unsatisfying. Fortunately, this approach is not necessary. The essence of a parametric theory is provided by additional structure in the guise of a one-parameter family of one-forms.

As shown below, the projected spacetime quantities obtained from the generalized Gauss-Codazzi techniques may be described in terms of quantities intrinsic to Σ together with the threading shift one-form, $M_i\,dx^i$.

3.6.2 Parametric Functions and Vector Fields

Given any smooth manifold Σ, a parametric structure on Σ is defined by a given one-parameter family of one-forms on Σ, $\omega(t)$, satisfying the *reparameterization property.*

Definition 3.1 *A reparameterization of the parametric structure on Σ is an assignment*

$$s = t + F(p)$$

for $p \in \Sigma$ and $s, t \in \mathbb{R}$.

Definition 3.2 *$\omega(t)$ satisfies the reparameterization property if under a reparameterization*

$$\omega(s) = \omega(t) - dF. \tag{3.26}$$

One may whish to think of ω as the spacetime threading shift one-form. We saw earlier that if one interpreted a reparameterization in terms of a (higher-dimensional) spacetime coordinate transformation (equation (2.18)), then the threading shift one-form satisfied a property identical to the reparameterization property. Under this interpretation, ω keeps track of the tilting of $\frac{\partial}{\partial t}$ with respect to the surfaces of constant time (thought to be diffeomorphic to Σ). As the threading shift form became the fundamental object used earlier to develop a projected spacetime geometry, ω similarly carries all of the parametric structure on Σ. That is, ω carries all of the information necessary to recapture the geometry of the subspace orthogonal to $\frac{\partial}{\partial t}$.

Definition 3.3 *A parametric function on Σ is a mapping $f : \Sigma \times \mathbb{R} \to \mathbb{R}$. Let the collection of such mappings be denoted by $\mathfrak{F}_p(\Sigma)$.*

Given a parametric function $f \in \mathfrak{F}_p(\Sigma)$, for a fixed $t \in \mathbb{R}$ f can be considered as a function from Σ to $\mathbb{R}$. Denote this function by f_t. Thus $f_t \in \mathfrak{F}(\Sigma)$ and can be acted on by tangent vectors of Σ.

Proposition 3.4 *The action of $\frac{\partial}{\partial t}$ on parametric functions is a covariant operation.*

Proof: Under a coordinate transformation of Σ, the operator $\frac{\partial}{\partial t}$ remains unaffected. This is because the parameter t is not a coordinate and, hence, any coordinate transformation of Σ must be independent of t. Therefore $\frac{\partial f}{\partial t}\big|_{(p,t_0)}$ does not depend on the choice of coordinates for $p \in \Sigma$. Furthermore, under a reparameterization $s = t + F(p)$, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial t}$. ♠

Although tangent vector fields do not act uniquely on parametric functions, one-parameter families of tangent vector fields do. These one-parameter families of vector fields, called parametric vector fields, will act on parametric functions in a way reminiscent of the "starry" action of projected spacetime vector fields on spacetime functions.

Definition 3.5 *A parametric vector field is a smooth mapping $X : \Sigma \times \mathbb{R} \to T\Sigma$ such that for each $p \in \Sigma$, $X(p,t) \in T_p\Sigma$ for all $t \in \mathbb{R}$.*

Let $\mathcal{X}_*(\Sigma)$ represent the collection of smooth parametric vector fields defined on Σ.

For a fixed t, let $X_t : \Sigma \to T\Sigma$ denote the obvious tangent vector field. Using the fact that the spacetime derivative operator ∂_* was shown to be invariant under reparameterizations, we define the action of a parametric vector field on a parametric function as follows:

$$Xf(p,t) = X_t f_t(p) + \omega(t)\left(X_t\right)\frac{\partial f}{\partial t}(p).$$

Suppressing the point p, we can write the action as

$$Xf = X_t f_t + \omega(X_t)\dot{f}. \tag{3.27}$$

Theorem 3.6 *Xf is invariant under reparameterizations and coordinate transformations.*

Proof: Consider coordinates $\{x^i\}$ and a parameter t. We have that

$$Xf = X^i\left(\frac{\partial f_t}{\partial x^i} + M_i\frac{\partial f}{\partial t}\right).$$

Under a reparameterization $s = t+F(p)$, the components of ω transform according to equation (3.26). Denote the parametric structure ω under this new parameterization by $\hat{\omega}$. Thus,

$$\begin{aligned}\hat{\omega} &= \hat{M}_i\,dx^i\\ &= \left(M_i - \frac{\partial F}{\partial x^i}\right)dx^i\\ &= \omega - dF.\end{aligned}$$

Although $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial t}$, we must be careful computing $\frac{\partial f}{\partial x^i}$. Using the notation introduced above, let $f_t : \Sigma \to \mathbb{R}$ and $\hat{f}_s : \Sigma \to \mathbb{R}$. Clearly $\frac{\partial f_t}{\partial x^i} = \frac{\partial f}{\partial x^i}\Big|_t$. Since $\hat{f}_s(p) = f(p,s) = f(p, t+F(p))$,

$$\begin{aligned}\frac{\partial \hat{f}_s}{\partial x^i} &= \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x^i} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x^i}\\ &= \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial s}\frac{\partial F}{\partial x^i}\\ &= \frac{\partial f_t}{\partial x^i} + \frac{\partial F}{\partial x^i}\dot{f}.\end{aligned}$$

Therefore,

$$\begin{aligned}Xf &= X^i\left(\frac{\partial f_t}{\partial x^i} + M_i\frac{\partial f}{\partial t}\right)\\ &= X^i\left(\frac{\partial f_t}{\partial x^i} + \left(\hat{M}_i + \frac{\partial F}{\partial x^i}\right)\frac{\partial f}{\partial t}\right)\\ &= X^i\left(\left(\frac{\partial \hat{f}_s}{\partial x^i} - \frac{\partial F}{\partial x^i}\frac{\partial f}{\partial t}\right) + \left(\hat{M}_i + \frac{\partial F}{\partial x^i}\right)\frac{\partial f}{\partial s}\right)\\ &= X^i\left(\frac{\partial \hat{f}_s}{\partial x^i} + \hat{M}_i\frac{\partial f}{\partial s}\right)\end{aligned}$$

which is the expression for Xf with respect to the parameter s, showing that Xf is invariant under a reparameterization. If we consider a coordinate transformation of Σ, X_t and $\frac{\partial}{\partial x^i}$ will transform as usual guaranteeing that $X_t(f_t)$ is independent of the choice of coordinates. Since ω and $\frac{\partial}{\partial t}$ are unaffected, Xf remains invariant under a coordinate transformation of Σ. ♠

Theorem 3.7 *Parametric vector fields are derivations on the ring $\mathfrak{F}(\Sigma \times \mathbb{R})$. That is,*

i. $X(rf + sg) = rX(f) + sX(g)$ *and*

ii. $X(fg) = fX(g) + gX(f)$ *for all* $r, s \in \mathbb{R}$ *and* $f, g \in \mathfrak{F}(\Sigma \times \mathbb{R})$.

Proof: This follows directly from the derivational properties of X_t and $\frac{\partial}{\partial t}$. Written out we have

$$\begin{aligned}
X(rf + sg)(p, t) \\
&= X_t(rf + sg)(p) + \omega(t)(X_t)\frac{\partial}{\partial t}(rf + sg)(p) \\
&= rX_t(f)(p) + sX_t(g)(p) + \omega(t)(X_t)(r\dot{f} + s\dot{g})(p) \\
&= r\left(X_t(f)(p) + \omega_r(X_t)\dot{f}(p)\right) \\
&= rX(f) + sX(g),
\end{aligned}$$

and

$$\begin{aligned}
X(fg)(p, t) &= X_t(fg)(p) + \omega(t)(X_t)\frac{\partial}{\partial t}(fg) \\
&= fX(g)(p, t) + gX(f)(p, t).
\end{aligned}$$

♠

Parametric vector fields have a very nice representation in terms of a local coordinate system, $\{x^i\}$. Since a parametric vector field is just a family of tangent vector fields, we may write

$$X = X^i \frac{\partial}{\partial x^i} = X^i \partial_i$$

as usual, where we let the functions X^i depend on the parameter. That is, the X^i are parametric functions on Σ. In terms of this representation we may write out the action of parametric vector fields on parametric functions

$$\begin{aligned}
X(f) &= X_t(f_t) + \omega(t)(X_t)\dot{f} \\
&= X^i f_{,i} + \omega_i \dot{f} \\
&= X^i f_{*i}.
\end{aligned}$$

The use of * in the above equation agrees with the earlier uses. That is, the action of parametric vector fields on parametric functions mimics the action of spacetime vector fields which are orthogonal to $\frac{\partial}{\partial t}$.

We can similarly define parametric tensors of higher rank.

Definition 3.8 *A parametric (p,q)-tensor, $T \in \mathcal{T}^p_q(\Sigma)$, on Σ is a one parameter family of (p,q)-tensors on Σ. That is,*

$$T : T\Sigma \times \ldots \times T\Sigma \times T^*\Sigma \times \ldots \times T^*\Sigma \times \mathbb{R} \to \mathbb{R}$$

such that $T(\ ,\ldots,\ ,t) \in \mathcal{T}^p_q(\Sigma)$.

As with parametric vector fields, parametric tensors can easily be expressed in a coordinate basis

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_p}} dx^{j_1} \ldots dx^{j_q}$$

where the $T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ are parametric functions. We can also talk about one-parameter families of metrics on Σ, that is a *parametric metric.*

We saw earlier that in a threaded spacetime the Lie bracket of two vector fields orthogonal to the threading need not be a vector field orthogonal to the curves. This "deficiency" is carried over to the parametric theory. This can be seen explicitly by calculating the action of the commutator $(XY - YX)$ on a parametric function.

$$\begin{aligned} X\left(Y(f)\right) &= X^i \left(Y^j f_{*j}\right)_{*i} \\ &= X^i \left(Y^j{}_{*i} f_{*j} + Y^j f_{*ji}\right) \end{aligned}$$

so

$$(XY - YX)(f) = \left(X^i Y^j{}_{*i} - Y^i X^j{}_{*i}\right) f_{*j} + X^i Y^j \left(f_{*ji} - f_{*ij}\right) \tag{3.28}$$

where, in general, $f_{*ji} - f_{*ij} \neq 0$.

In the earlier language of projections, the first term on the right hand side of equation (3.28) was called $[X,Y]^{\perp}(f)$ and can be identified with a parametric vector field, while the second term refers to the earlier object $[X,Y]^{\top}(f)$, which is not a parametric vector field.

We would like to define a notion of "bracket" of parametric vector fields. The non-commutivity of the mixed parametric derivative makes this non-trivial. Without the use of a projection operator, it is hard to describe the quantity we earlier called $[X,Y]^{\perp}$ (at least in a coordinate-free way). However, there is an intrinsic calculation that yields the $[X,Y]^{\top}$ term, or the deficiency. In order to define the deficiency intrinsically we will turn our attention to exterior differentiation of parametric forms.

3.6.3 Parametric Exterior Differentiation

In Chapter 2, it was pointed out that Perjés introduced a notion of exterior differentiation of parametric functions; namely

$$d_* f = df + \omega \dot{f}$$

where d is the usual exterior differentiation on differential forms. Parametric functions may be considered as parametric differential 0-forms. Parametric differential p-forms are just one-parameter families of differential p-forms defined on Σ. Thus, in a coordinate basis, a parametric differential p-form may be written as

$$\theta = \theta_{(i_1 \ldots i_p)}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

where the $\theta_{i_1 \ldots i_p}$ are functions of x^i and t and I have adopted the notation in [4] where $(i_1 \ldots i_p)$ gives us sums running through increasing sets of indices.

There are four axioms needed to completely determine the exterior derivative d (see [4]), namely

i. $df(X) = X(f)$ for functions f and vector fields X,

ii. wedge-product rule: $d(\theta \wedge \tau) = d\theta \wedge \tau + (-1)^p \theta \wedge d\tau$ where θ is a p-form, *iii.* $d(df) = 0$, and

iv. d is linear: $d(\theta + \tau) = d\theta + d\tau$.

We already have that $d_* f(X) = X(f)$ for parametric vector fields X and parametric functions f. Properties (*ii*) and (*iv*) also carry over easily. However, it is not clear that we wish $d_*(d_* f) = 0$. For the parametric case, consider replacing axiom (*iii*) by

iii′. $d_*(d_* f) = 0$ for parameter independent functions f.

Consider an exterior derivative operator, d_*, on parametric differential forms satisfying (*i*), (*ii*), (*iii′*), and (*iv*) for parametric forms, vector fields, and functions. We have the following familiar coordinate expressions:

1. since the coordinate functions do not depend on the parameter, we have, by (*ii*) and (*iii*)

$$\begin{aligned} d_*(dx^{i_1} \dots dx^{i_p}) =& (d_*^2 x^{i_1}) \wedge dx^{i_2}\, dx^{i_3} \dots dx^{i_p} \\ & - dx^{i_1} \wedge (d_*^2 x^{i_2}) \wedge dx^{i_3} \dots dx^{i_p} + \dots \\ & \dots + (-1)^{p-1} dx^{i_1} \dots dx^{i_{p-1}} \wedge (d_*^2 x^{i_p}) \\ =& 0, \end{aligned}$$

2.

$$\begin{aligned} d_*(f dx^{i_1} \dots dx^{i_p}) =& d_* f \wedge dx^{i_i} \dots dx^{i_p} + f d_*(dx^{i_1} \dots dx^{i_p}) \\ =& d_* f \wedge dx^{i_1} \dots dx^{i_p}, \text{ and} \end{aligned}$$

3. using (*iv*), d_* on any parametric p-form has the coordinate expression

$$d_*(\theta) = d_* \left(\theta_{(i_1 \dots i_p)} \right) \wedge dx^{i_1} \dots dx^{i_p}.$$

What about $d_*(d_* f)$ on arbitrary parametric functions? According to this set of axioms we have

$$\begin{aligned} d_*^2 f &= d_*(f_{*i} dx^i) \\ &= f_{*ij} dx^j \wedge dx^i \\ &= -f_{*ji} dx^j \wedge dx^i. \end{aligned}$$

Therefore $2d_*^2 f = (f_{*ij} - f_{*ji}) dx^j \wedge dx^i$, and we have seen earlier that this term is generally non-zero. In fact, this term reproduces the $\left[\partial_{*i}, \partial_{*j}\right]^\top f$ term, which measures the deficiency.

Extrinsically, we related the deficiency to the fact that $(T\mathcal{M})^\perp$ was not surface forming. Intrinsically we can define the deficiency as the failure of d_*^2 to be identically zero. In either interpretation, it merits its name.

Definition 3.9 *The deficiency, $\mathcal{D}$, is a derivative operator defined by*

$$\mathcal{D}(X,Y)f = 2d_*^2 f(X,Y),$$

for $X, Y \in \mathcal{X}_*(\Sigma)$ and $f \in \mathfrak{F}_p(\Sigma)$.

In terms of a coordinate basis we have

$$\begin{aligned} \mathcal{D}(X,Y)f &= 2d_*^2 f(X^i \partial_i, Y^j \partial_j) \\ &= X^i Y^j (f_{*ji} - f_{*ij}) \\ &= X^i Y^j (\omega_{j*i} - \omega_{i*j}) \dot{f} \\ &= X^i Y^j \mathcal{D}_{ji} \dot{f} \end{aligned}$$

which is the same quantity that we defined extrinsically by $[X,Y]^\top (f)$.

3.6.4 A Bracket Operator

We can now easily define the bracket of two parametric vector fields intrinsically. We want our intrinsic definition to agree with the projected spacetime quantity $[X, Y]^{\perp}$. For two parametric vector fields X and Y, define

$$[X, Y]_* f = X\left(Y(f)\right) - Y\left(X(f)\right) - \mathcal{D}(X, Y)f.$$

We have already worked out these terms in a coordinate basis. Putting old facts together we have

$$\begin{aligned}[X, Y]_* f &= \left(X^i Y^j{}_{*i} - Y^i X^j{}_{*i}\right) f_{*j} \\ &\quad + X^i Y^j (f_{*ji} - f_{*ij}) - X^i Y^j (f_{*ji} - f_{*ij}) \\ &= \left(X^i Y^j{}_{*i} - Y^i X^j{}_{*i}\right) f_{*j}\end{aligned}$$

which reproduces the correct vector field. If $\{x^i\}$ are coordinates on Σ, then $\left[\partial_i, \partial_j\right]_* = 0$ as one would like.

Given a parametric vector field X, we can define an $\mathbb{R}$-linear mapping $\mathcal{L}_{*_X}$: $\mathcal{X}_*(\Sigma) \to \mathcal{X}_*(\Sigma)$ by $\mathcal{L}_{*_X} Y = [X, Y]_*$. Since

$$\begin{aligned}\mathcal{L}_{*_X}(fY)e &= [X, fY]_* e \\ &= Xf\left(Y(e)\right) - fY\left(X(f)\right) - \mathcal{D}(X, fY)e \\ &= X(f)Y(e) + fXY(e) - fY\left(X(e)\right) - f2d_*^2 e(X, Y) \\ &= \left(X(f)Y + f\mathcal{L}_{*_X} Y\right) e\end{aligned}$$

for all $e, f \in \mathfrak{F}_p(\Sigma)$ and $X, Y \in \mathcal{X}_*(\Sigma)$, $\mathcal{L}_{*_X}$ may be extended uniquely to a parametric tensor derivation on Σ, the *parametric Lie derivative.* (See theorem 15 in Chapter 2 of [22].)

3.6.5 Parametric Connections

I will now introduce the notion of a connection on a parametric manifold. Although the following definition looks identical to the definition of a standard affine connection on a manifold, this is an illusion created by the choice of notation. Specifically, I have been using Xf to denote the action of a parametric vector field on a parametric function. The underlying operator for such an action is **not** partial differentiation, but parametric differentiation via the operator ∂_{*i}. In this sense, one can view a parametric connection as a *generalized* connection on a manifold. [7] That is, we generalize the notion of a vector field acting on a function.

Definition 3.10 *An (affine) parametric connection, ∇_*, on Σ is a mapping ∇_* : $\mathcal{X}_*(\Sigma) \times \mathcal{X}_*(\Sigma) \to \mathcal{X}_*(\Sigma)$ denoted by $\nabla_*(X, Y) = \nabla_{*X} Y$, which satisfies the following properties:*

i. Linearity over $\mathfrak{F}_p(\Sigma)$: $\nabla_{*(fX+gY)} Z = f\nabla_{*X} Y + g\nabla_{*Y} Z$

ii. Linearity: $\nabla_{*X}(Y + Z) = \nabla_{*X} Y + \nabla_{*X} Z$

iii. Derivation: $\nabla_{*X}(fY) = X(f)Y + f\nabla_{*X} Y$ *for all* $X, Y, Z \in \mathcal{X}_*(\Sigma)$, $f, g \in \mathfrak{F}_p(\Sigma)$, *and $X(f)$ refers to the parametric action of X of f.*

As before, given $X \in \mathcal{X}_*(\Sigma)$ one can consider the $\mathbb{R}$-linear mapping ∇_{*X} : $\mathcal{X}_*(\Sigma) \to \mathcal{X}_*(\Sigma)$. Condition (*iii*) above and [22] guarantee that ∇_{*X} may be extended uniquely to a parametric tensor derivation on Σ. Thus, we may treat ∇_{*X} as a covariant derivative operator on any parametric tensor.

We next wish to show that given a parametric metric h on Σ, then there exists a unique parametric connection on Σ which is compatible with h and torsion-free. Hence, we need to define these last two properties.

Let h be a parametric metric on Σ, denoted by $\langle\ ,\ \rangle$. Metric compatibility is defined in the usual way.

Definition 3.11 *A parametric connection is said to be compatible with the parametric metric h provided*

$$X \langle Y, Z\rangle = \langle \nabla_{*X} Y, Z\rangle + \langle Y, \nabla_{*X} Z\rangle .$$

---

[7] In [24], Otsuki describes *generalized connections* which do not always reduce to partial differentiation on functions.

Definition 3.12 *The parametric torsion, T_*, of ∇_* is defined by*

$$T_*(X,Y) = \nabla_{*X}Y - \nabla_{*Y}X - [X,Y]_* .$$

If $T_*(X,Y) = 0$ for all $X, Y \in \mathfrak{X}_*(\Sigma)$, then ∇_* is said to be torsion free.

Theorem 3.13 *There exists a unique torsion-free parametric connection compatible with h.*

Proof: The proof is exactly the same as the proof for the existence and uniqueness of the Levi-Civita connection. The following proof is taken from [6]. Suppose that such a ∇_* exists. Then we have

$$\begin{aligned} X\langle Y,Z\rangle &= \left\langle \nabla_{*X}Y, Z\right\rangle + \left\langle Y, \nabla_{*X}Z\right\rangle, \\ Y\langle Z,X\rangle &= \left\langle \nabla_{*Y}Z, X\right\rangle + \left\langle Z, \nabla_{*Y}X\right\rangle, \\ -Z\langle X,Y\rangle &= -\left\langle \nabla_{*Z}X, Y\right\rangle - \left\langle X, \nabla_{*Z}Y\right\rangle. \end{aligned}$$

Adding the above equations yields

$$\begin{aligned} X\langle Y,Z\rangle &+ Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &= -\langle [Z,X]_*, Y\rangle + \langle [Y,Z]_*, X\rangle \\ &\quad + \langle [X,Y]_*, Z\rangle + 2\left\langle Z, \nabla_{*Y}X\right\rangle. \end{aligned}$$

Therefore, $\nabla_{*Y}X$ is uniquely determined by

$$\begin{aligned} \left\langle Z, \nabla_{*Y}X\right\rangle = &\frac{1}{2}(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &+ \langle [Z,X]_*, Y\rangle - \langle [Y,Z]_*, X\rangle - \langle [X,Y]_*, Z\rangle). \end{aligned} \tag{3.29}$$

One may also use this equation to define ∇_*, thus proving existence. ♠

We can use equation (3.29) to write out the unique parametric connection ∇_* in a coordinate basis. If we let $h_{ij} = \left\langle \partial_i, \partial_j \right\rangle$, we can define the connection symbols by $\nabla_{*\partial_i}\partial_{*j} = \gamma^k{}_{ij}\partial_k$. Equation (3.29) now gives us

$$\gamma^l{}_{ij}h_{lk} = \frac{1}{2}\left(h_{jk*i} + h_{ki*j} - h_{ij*k}\right)$$

or

$$\gamma^{k}_{\ ij} = \frac{1}{2} h^{km} \left(h_{jm*i} + h_{mi*j} - h_{ij*m} \right).$$

Therefore the connection symbols associated with ∇_* agree with the connection symbols associated with the earlier projected covariant derivative D.

3.6.6 Curvature

While introducing a generalized Gauss-Codazzi formalism, the definition of curvature presented the greatest problem. In such a general setting, it was pointed out that the most "natural", and naive, definition of a curvature operator (see the definition of S in equation (3.15)) failed to be a tensor! The exact same problems are reproduced in the definition of the parametric objects, $\nabla_{*X} Y$ and $[X,Y]_*$. That is, the operator

$$S(X,Y)Z = \nabla_{*X} \nabla_{*Y} Z - \nabla_{*Y} \nabla_{*X} Z - \nabla_{*[X,Y]_*} Z$$

is not function-linear. This is due to the fact that $[X,Y]_* f \neq XY(f) - YX(f)$. Rather, it is the case that

$$[X,Y]_* f = XY(f) - YX(f) - \mathcal{D}(X,Y) f.$$

Two alternate definitions of curvature were proposed earlier that overcame this problem. While working in a spacetime setting with projection operators, it seemed that the easiest way to make the operator S a tensor was to add the necessary terms "by hand". Thus, we traded the term $(\nabla_{[X,Y]^\perp} Z)^\perp$ for $(\nabla_{[X,Y]^\perp} Z + \nabla_{[X,Y]^\top} Z)^\perp = (\nabla_{[X,Y]} Z)^\perp$, leading to the definition of ${}^\perp R$. Not only did this approach seem the "easiest" way to make S function-linear, it also had the favorable consequence of satisfying a generalized Gauss equation (equation (3.16)).

We also introduced, without much motivation, a second candidate for a curvature operator; namely ${}^\perp \bar{R}$. In defining ${}^\perp \bar{R}$ the troublesome term $(\nabla_{[X,Y]^\perp} Z)^\perp$ was replaced by $\left(\pounds_{[X,Y]} Z\right)^\perp$. With such a definition, ${}^\perp \bar{R}$ also possessed several advantageous properties. First, the components of ${}^\perp \bar{R}$ could be written in terms of the connection symbols $\gamma^i{}_{jk}$ in a way analogous to the components of the classical Riemann tensor (see equation (3.25)). Second, it was pointed out that the components of ${}^{\perp}\bar{R}$ agree with the Zel'manov curvature tensor. Again, ${}^{\perp}R$ and ${}^{\perp}\bar{R}$ are both generalizations of the projected curvature tensor ${}^{3}R$ in the sense that all three tensors agree when the subspace orthogonal to $\frac{\partial}{\partial t}$ forms a hypersurface.

One must now decide how to proceed to define a parametric curvature tensor on our abstract parametric manifold Σ. One approach would be to try to define the tensors ${}^{\perp}R$ and ${}^{\perp}\bar{R}$ in terms of ∇_*, $[\ ,\ ]_*$, and $\mathcal{D}$. It is not obvious that the terms $(\nabla_{[X,Y]})^{\perp}$ and $\left(\pounds_{[X,Y]}Z\right)^{\perp}$ can be defined in such a manner. Another more straight-forward approach would be analogous to the definition of ${}^{\perp}\bar{R}$. That is, since we know why S is not function linear (the presence of deficiency) we can easily correct the problem. First, one must extend the action of $\mathcal{D}(X,Y)$ to tensors of rank $(p$-$q)$ by differentiating the components of an arbitrary tensor with respect to the parameter t. Since the action of ∂_t on p-forms is covariant, the result is a $(p$-$q)$ tensor. Now, define

$$Z(X,Y)W = \nabla_{*X}\nabla_{*Y}W - \nabla_{*Y}\nabla_{*X}W - \nabla_{*[X,Y]_*}W - \mathcal{D}(X,Y)W.$$

Such a definition makes use of the various derivative operators present in a parametric theory. Not only does the parametric manifold Σ have the natural parametric derivative operator ∇_*, but the covariant operation of differentiation with respect to the parameter is also present. The deficiency operator is built out of this parametric derivative.

Given coordinates, x^i, the components of Z may be computed

$$\begin{aligned} Z(\partial_i,\partial_j)\partial_k &= \nabla_{*\partial_i}\nabla_{*\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{*\partial_i}\partial_k - 0 - 0 \\ &= \left(\gamma^l{}_{jk*i} - \gamma^l{}_{ik*j} + \gamma^l{}_{mi}\gamma^m_{jk} - \gamma^l{}_{mn}\gamma^m_{ik}\right)\partial_l \\ &= Z^l{}_{kij}\partial_l. \end{aligned}$$

Thus, the components of Z are exactly the components of the Zel'manov curvature.

So far I have begged the question of how one could reproduce ${}^{\perp}R$ intrinsically. In terms of a coordinate basis, we saw before (equation (3.21)) that the difference between ${}^{\perp}R$ and Z is

$$
\begin{aligned}
{}^{\perp}R^{l}{}_{kij} - Z^{l}{}_{kij} &= \left(M_{j*i} - M_{i*j}\right) h^{lm} \left(M^2 M_{m*k} - M^2 M_{k*m} + \partial_t h_{km}\right) \\
&= \mathcal{D}_{ji} h^{lm} \left(M^2 \mathcal{D}_{mk} + \partial_t h_{km}\right),
\end{aligned}
$$

which involves the deficiency $\mathcal{D}$ and the threading lapse function M. As we mentioned in the last section, the appearance of M is due to the fact that we began with a parameter t whose relationship to proper time was arbitrary. While we have an intrinsic definition for the deficiency, we can not recover the lapse function without explicitly introducing it.

Abandoning ${}^{\perp}R$ for Z results in a curvature operator that can be defined entirely in terms of Σ and the parametric structure ω. However, we know in advance that Z will not possess all of the symmetries of the Riemann curvature tensor. Earlier it was shown that ${}^{\perp}R$ was the unique curvature satisfying Gauss' equation and, hence, enjoying all of the inherited symmetries of the Riemann tensor (where the first Bianchi identity for ${}^{\perp}R$ resembled the identity in the presence of torsion). As we saw in Chapter 2, the symmetries of Z may be written

i. $Z(X,Y)W = -Z(Y,X)W$ and

ii. $Z(X,Y)W + Z(Y,W)X + Z(W,X)Y = 0$.

3.6.7 Conclusion

The above intrinsic approach to parametric manifolds is more mathematically satisfying than the earlier extrinsic approach. It proved to be quite interesting to develop a generalized Gauss-Codazzi formalism and, perhaps, such a projective approach is closer to the historical roots of parametric manifolds. However, the exciting field of differential geometry is elegant precisely because of its ability to describe geometric objects intrinsically. The ability to work on manifolds instead of surfaces and define "tangent" vectors without making use of a higher dimensional space makes the field of differential geometry very appealing. In such a way, the intrinsic approach to parametric manifolds is also very appealing. I have shown how to recapture the projective flavor of the Gauss-Codazzi formalism without introducing any projection operators. After defining the correct action of parametric vector fields on parametric functions, equation (3.27), and recapturing this action in the guise of an exterior derivative operator, the correct generalizations of Lie bracket, torsion, and affine connection naturally followed. Furthermore, in such an intrinsic setting the Zel'manov curvature tensor (used by Einstein, Bergmann, Zel'manov, and Perjés) is the most natural generalization of the Riemann curvature tensor.

With such a firm foundation, the theory of parametric manifolds may now be easily explored. It should now be straightforward, for example, to develop the analytical theory stemming from the parametric exterior derivative or, perhaps, to continue nd expand the field of parametric manifolds by studying the behavior of (parametric) geodesics and answering question about completeness and other fundamental geometrical properties.

4. Fibre Bundles and Foliations

4.1 Introduction

Using the language of fibre bundles and foliations, this chapter will discuss some of the central concepts presented in the previous two chapters. This dissertation began with the slicing and threading decompositions of spacetime which were induced by the existence of two different, but very well behaved, foliations. The slicing viewpoint assumed a foliation by spacelike hypersurfaces, while the threading viewpoint depended on a foliation by timelike curves.

We will begin this chapter with precise definitions of foliations in order to place the slicing and threading viewpoints in the proper mathematical context. Furthermore, when the leaves of these foliations coincide with the fibres of a fibre bundle, we will see how both the slicing and threading decompositions of spacetime can be described by the same mathematical structure. By studying how Riemannian metrics on fibre bundles are related to metrics on the base space and typical fibre, we will recover both the slicing and threading decompositions. Both Reinhart and Hermann placed certain conditions (discussed below) on metrics in order to guarantee when a given foliation can be thought of as a fibre bundle.

In the setting of general relativity these conditions are very similar to those Einstein and Bergmann imposed on the spacetime metric in an attempt to generalize Kaluza's ideas. Hence, we will include a review of the work of Einstein and Bergmann leading up to a parametric theory of spacetime.

This chapter will conclude by exploring the definitions of ∂_* and d_* in the context of fibre bundles. We will see that if $\mathcal{M} = \Sigma \times \mathbb{R}$ is thought of as a fibre bundle over Σ, then the horizontal subspaces defined by $\partial_{*i} = \partial_i + M_i \frac{\partial}{\partial t}$ are "almost" a connection in a principal bundle, and that d_* is the induced covariant exterior derivative on Σ. We will also see how the operator d_* can also be thought of as a *total derivative* in the context of jet bundles.

4.2 Fibre Bundles and Foliations

4.2.1 Quotient Manifolds Defined by Foliations

There exist many sources, and many definitions, for foliations of manifolds. Both [25] and [17] offer nice introductions to the geometric and topological properties of foliations and quotient manifolds defined by foliations. Generally speaking, foliations are a generalized differentiable structure on a manifold. That is, one may think of a differentiable structure on a manifold as a zero-dimensional foliation.

There are two standard approaches to the study of foliations. One of these defines foliations in terms of a decomposition of the manifold (the study of submersions), while the other defines foliations in terms of a decomposition of the tangent space of the manifold (the study of distributions). Both approaches are appealing, but for the sake of simplicity, I will only discuss the former. Other definitions may be found in [25] and [17].

For the following, let $\mathcal{M}$ be a smooth n-dimensional manifold. I will be concerned only with smooth foliations and assume all differentiable structures are of class C^∞.

Definition 4.1 *A p-dimensional foliation is a decomposition of $\mathcal{M}$ into distinct subsets $\{\mathcal{L}_a\}$, for $a \in A$ some index set, such that each point of $\mathcal{M}$ has a neighborhood U and a coordinate system $(x, y) : U \to \mathbb{R}^p \times \mathbb{R}^q$ $(q = n - p)$ such that for all $\mathcal{L}_a$, the components $U \cap \mathcal{L}_a$ are described by equations of the form*

$$\begin{aligned} y_1 &= c_1 \\ y_2 &= c_2 \\ \vdots\ & \quad \vdots \\ y_q &= c_q \end{aligned}$$

where $c_1, \ldots, c_q$ are constants. Denote the foliation by $\mathcal{F} = \{\mathcal{L}_a\}$.

Definition 4.2 *Such a coordinate system is said to be distinguished by the foliation* $\mathcal{F}$.

Definition 4.3 *The subsets* $\mathcal{L}_a$ *are called the leaves of the foliation* $\mathcal{F}$.

Example 4.4. *Slicing and Threading.*

Earlier we were concerned with spacetimes which were foliated by, for example, spacelike hypersurfaces or timelike curves. The slicing viewpoint concerns itself with the foliation by hypersurfaces. Thus, the leaves in this case would correspond to the surfaces Σ_t. The adapted coordinate system used throughout this paper made use of the fact that there existed a coordinate system in $\mathcal{M}$ such that the surfaces (leaves) Σ_t were given by $\{t \equiv \text{constant}\}$.

Similarly, for threading we assumed the spacetime $\mathcal{M}$ could be decomposed into leaves which were timelike curves. Again,we made use of the fact that these curves would be defined (locally) by $\{x^i \equiv \text{constant}\}$.

In the above example (and hence throughout this entire paper), the foliations were especially well-behaved. For instance, we intuitively assumed that each timelike curve intersected some fixed hypersurface at most one time. We did not concern ourselves with the pathological case of a curve passing through a neighborhood of $\mathcal{M}$ infinitely often. Nor were we interested in the case where a surface Σ_t intersected the same family of threading curves more than once.

Definition 4.5 *A leaf is said to be regular if it intersects a distinguished coordinate neighborhood in at most one p-dimensional slice.*

Definition 4.6 $\mathcal{F}$ *is a regular foliation if every leaf of* $\mathcal{F}$ *is regular.*

This regularity condition has been assumed in previous sections of this paper. Regularity assures that the manifold topology of the leaf is the same as the topology induced by the manifold $\mathcal{M}$. Stated another way, the manifold of leaves, $\mathcal{M}/\mathcal{F}$, is a manifold (with the quotient topology) only if $\mathcal{F}$ is a regular foliation. Stated formally,

Theorem 4.7 *If $\mathcal{F}$ is a regular foliation of $\mathcal{M}$, then*

i. $\mathcal{M}/\mathcal{F}$ is a (not necessarily Hausdorff) manifold with the quotient topology and

ii. $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{F}$ is differentiable and surjective.

For a proof see [25].

4.2.2 Fibre Bundles

Fibre bundles come with many different structures; principal (fibre) bundles, vector bundles, and (topological) bundles. The fibre bundle with the least amount of structure is the topological fibre bundle.

Definition 4.8 *A (topological) fibre bundle is a collection (E, π, F, B) satisfying*

i. E, F, and B are topological spaces typically referred to as the total space (or bundle), typical fibre, and base space respectively,

ii. $\pi : E \to B$ is a continuous surjective map, and

iii. (local triviality): for any point $b \in B$ there exists an open coordinate neighborhood $U \subset B$ containing b and a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ satisfying $\pi \circ \phi^{-1}(x, f) = x$ for $x \in U$ and $f \in F$. That is, ϕ preserves fibres.

Property (*iii*) may be interpreted as meaning that the total space E is locally a product space. If $E = B \times F$, one would call E a trivial bundle. The local triviality of any bundle E guarantees that the set $\pi^{-1}(b)$, $b \in B$, is homeomorphic to the typical fibre F. The collection of sets $\{\pi^{-1}(b)\}$ are generally referred to as the fibres of E. While fibre bundles are fundamental to the study of differential geometry (see [15]), this dissertation concerns itself with fibre bundles for two reasons.

First, a complete description of both the slicing and threading decompositions of spacetime can be achieved by the use of a single mathematical structure, namely a fibre bundle. In fact, the study of the slicing, threading, and parametric pictures of spacetime is very interesting even in the most trivial setting where $\mathcal{M} = \Sigma \times \mathbb{R}$. Given that the spacetime $\mathcal{M}$ is a global product, it is clearly a fibre bundle. The slicing and threading viewpoints simply correspond to different choices of fibre and base space. We will examine this relationship later.

Second, it seems most natural to discuss the parametric connection, ∇_*, in the same terms one would discuss a standard connection of a manifold. Since connections on manifolds fundamentally arise from studying connections on principal bundles, one must certainly be interested in tracing the origins of ∇_* back to such a setting. I will address this issue towards the end of the chapter.

There is also a direct relationship between foliated manifolds and fibre bundles. I already mentioned that while the slicing and threading viewpoints depend upon foliations of the spacetime, these two viewpoints can also be easily discussed in terms of fibre bundles. In general, one may use the fibres $\pi^{-1}(b)$ of a fibre bundle to define a foliation of the total space E. Although I have been considering only the nicest foliated spacetimes in this dissertation, it is still an interesting problem to study when a foliated manifold may be considered to be a fibre bundle. Since the major motivation for this paper came from studying manifolds with metrics (e.g. spacetimes), let us turn our attention to studying foliated manifolds with metrics and their relationships to fibre bundles.

4.3 Bundle-Like Metrics

4.3.1 Introduction and Definitions

The study of foliated manifolds which possess bundle-like metrics has allowed one to give certain conditions which guarantee that such a space is a fibre bundle ([27] and [11]) as well as offer insight into how the differential-geometric structure of a foliation affects the global properties of the foliation ([12]). As Reinhart explains in [27], the motivating example for the concept of a bundle-like metric comes from trying to construct a Riemannian metric on the total space of a fibre bundle out of Riemannian metrics on the base space and the typical fibre.

Consider a smooth fibre bundle $\pi : \mathcal{M} \to B$ with typical fibre F. Now the fibres $\pi^{-1}(b)$ above each point $b \in B$ are the *leaves* of a foliation of $\mathcal{M}$. Let $V \subset B$ be such that $\pi^{-1}(V) \approx V \times F$. We will first define a metric on $V \times U$ where U is a coordinate neighborhood of F. If $h_{ij}dx^i dx^j$ and $k_{\alpha\beta}dy^\alpha dy^\beta$ are metrics on V and F respectively, on $V \times U$ we take the metric to be $h_{ij}dx^i dx^j + k_{\alpha\beta}dy^\alpha dy^\beta$. By using a partition of unity, this metric can be extended to a metric on $\mathcal{M}$. This is an example of a bundle-like metric. While the functions h_{ij} depend only on the position in B and the $k_{\alpha\beta}$ depend only on the fibre coordinate, this will not be the case in a typical example of a foliated manifold with bundle-like metric. In the general case the fibre-dependent functions $k_{\alpha\beta}$ will also be allowed to depend on the corresponding base point in B.

Proceeding as in [27], let us define what is meant by *bundle-like* in the most general setting. Let $\mathcal{M}$ be an n-dimensional manifold together with a r-dimensional *foliation* F. F is defined by a smooth mapping $p \mapsto F_p \subset T_p\mathcal{M}$ to subspaces of dimension r satisfying the complete integrability conditions. Complete integrability guarantees that through each point $p \in \mathcal{M}$ there passes a submanifold N such that $T_qN \subseteq F_q$ for all $q \in N$. N is called an *integrable submanifold* of $\mathcal{M}$. Maximally connected integrable submanifolds are referred to as *leaves* of the foliation F. In a neighborhood of each point of $\mathcal{M}$ we can define local coordinates $(x^1, \ldots, x^s, y^1, \ldots, y^r)$

such that the leaves of F are defined (locally) by $x^i \equiv c^i$ where the c^i are constants. The following index conventions will be adopted: $i, j = 1, \ldots, s$ and $\alpha, \beta = 1, \ldots, r$. The form $dx^1 \wedge \ldots \wedge dx^s$ corresponds to the subspace of forms which are zero on all vectors belonging to the foliation (i.e. those vectors in F_p). When working with foliations it is usually assumed that such a coordinate neighborhood is *flat* (*i.e.* it is the product of cubical neighborhoods of Euclidean s and r space).

Now, choose 1-forms $\omega^1, \ldots, \omega^r$ and vectors $v_1, \ldots, v_s$ such that (dx^i, ω^α) form a basis for the cotangent space and $(v_i, \frac{\partial}{\partial y^\alpha})$ is the dual basis. If $\mathcal{M}$ possesses a metric, one usually chooses the ω^α so that they are zero on the orthogonal space to the tangent space of a leaf through a given point.

Definition 4.1. *A metric on $\mathcal{M}$ is said to be bundle-like if it has the following form in such a flat coordinate system:*

$$ds^2 = g_{ij}(x)dx^i dx^j + g_{\alpha\beta}(x, y)\omega^\alpha \omega^\beta$$

Definition 4.2. *A differential form σ is said to be base-like if*

$$\sigma = \sigma_{i_1 \ldots i_s}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_s}.$$

These coordinate dependent definitions are indeed well defined as Reinhart shows in [27]. In the above case where the foliation corresponds to the fibres of a fibre bundle $\pi : \mathcal{M} \to B$, the base like forms are just those forms on $\mathcal{M}$ induced from forms on B (via π). That is, dx^i on $\mathcal{M}$ actually corresponds to $\pi^*(dx^i)$ in any local trivialisation. Reinhart also mentions that the collection of the ω^α defines "a sort of connection in this fibre space." That is, one lets the ω^α define a notion of *horizontal* in the fibre space. As was mentioned earlier, if there is a metric on $\mathcal{M}$, this notion of horizontal is chosen to be the direction orthogonal to the leaves (fibres) of the foliation.

While these definitions have no intrinsic geometric meaning, Reinhart addresses this deficiency and offers some geometric interpretation of bundle-like metrics in [27].

4.3.2 Foliations With Bundle-Like Metrics

In the Riemannian case, there are some known results about foliations with bundle-like metrics and the resulting quotient spaces. The following theorem appeared as Corollary 3 in [27]:

Theorem 4.3. *Let $\mathcal{M}$ be a foliated manifold complete in a bundle-like metric. If the foliation is regular, then $\mathcal{M}$ is a fibre space over a complete (Hausdorff) manifold B.*

Here B is the *manifold of leaves* whose space is defined by identifying each leaf to a point. B is given the quotient topology with respect to this identification, so that a set in B is open if and only if its inverse image under π is open in $\mathcal{M}$. Regularity of the foliation insures that B is indeed a (Hausdorff) manifold.

Theorem 4.4. *Let $\mathcal{M}$ be a foliated manifold complete in a Riemannian metric that is bundle-like with respect to the foliation. If all of the leaves are closed and the holonomy group of each leaf with respect to the foliation is trivial, then B can be made into a smooth manifold so that π is a smooth map of maximal rank.*

This theorem originally appeared as Theorem 4.4 in [12].The fact that $\mathcal{M}$ is complete and each leaf is closed implies that there is a well-defined distance function between leaves. Thus, one has a metric on the space of leaves B. It turns out that if the holonomy group of each leaf is trivial, then if one leaf is closed it must also be regular and hence all leaves are closed and regular guaranteeing that B is a manifold.

4.3.3 Two Examples (Slicing and Threading)

Let us state what is meant by a bundle-like metric with respect to the two familiar foliations which yield the slicing and threading decompositions of spacetime.

Example 4.5. *Slicing*

Consider a spacetime $\mathcal{M}$ together with a foliation by spacelike hypersurfaces (in terms of the above terminology, the hypersurfaces are the *leaves* of the foliation). Here $p = 3$ and $q = 1$. We'll adopt the earlier notation so that local coordinates are given by (t, x^i) and the hypersurfaces (leaves) are defined locally by $\{t \equiv constant\} = \Sigma_t$. This notation agrees with the earlier discussion of slicing spacetime, but disagrees with the above section. We will choose one-forms ω^i $(i = 1, 2, 3)$ such that the ω^i are zero on vectors orthogonal to each hypersurface Σ_t, thus yielding (dt, ω^i) as a basis for the cotangent space. Let

$$\omega^i = N^i dt + dx^i.$$

We had earlier ((2.3)) that the unit normal to each slice Σ_t had the form

$$n = \frac{1}{N}\frac{\partial}{\partial t} - \frac{1}{N}N^i\frac{\partial}{\partial x^i}$$

thus, the ω^i as defined have the desired action. The dual basis is given by $(v, \frac{\partial}{\partial x^i})$ where

$$v = \frac{\partial}{\partial t} - N^i\frac{\partial}{\partial x^i}.$$

In the slicing notation, the spacetime metric takes the following form:

$$\begin{aligned} ds^2 &= -(N^2 - N^m N_m)dt^2 + 2N_i dt dx^i + k_{ij}dx^i dx^j \\ &= -N^2 dt^2 + k_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \\ &= -N^2 dt^2 + k_{ij}\omega^i\omega^j \end{aligned}$$

Therefore, for the spacetime metric to be bundle-like, one must require that the slicing lapse function $-N^2$ be a function of t only, while the slicing metric k_{ij} is allowed to be a function of both t and the x^i.

Requiring the slicing lapse function to be independent of the spatial coordinates x^i is quite restrictive. However, this condition arises as a result of trying to define a metric on the base space $\mathbb{R}$. Clearly, any metric on $\mathbb{R}$ should depend only on the coordinate t. Such a situation would arise in spatially homogeneous model in which ∂_x is Killing.

Example 4.6. *Threading*

In the threading scenario we concentrate on the foliation of spacetime with timelike curves. The leaves of the foliation correspond to the integral curves of $\frac{\partial}{\partial t}$. We will again work in local coordinates (x^i, t) where the leaves are defined by $\{x^i \equiv constants\}$. We now choose a one-form $\bar{\omega}$ such that $\bar{\omega}$ is zero on vectors orthogonal to $\frac{\partial}{\partial t}$. That is, if we consider each leaf as the world line of an observer, $\bar{\omega}$ is zero on all vectors in the local rest space of that observer. Let

$$\bar{\omega} = dt - M_i dx^i.$$

Comparing with equation (2.7), one sees that $\bar{\omega}$, being a multiple of the metric dual of $\frac{\partial}{\partial t}$, is indeed zero on the local rest space of each observer. Dual to the basis $(dx^i, \bar{\omega})$ is $(v_i, \frac{\partial}{\partial t})$ where

$$v_i = M_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}.$$

Using the threading notation, the spacetime metric is

$$\begin{aligned} ds^2 &= -M^2 dt^2 + 2M^2 M_i dt dx^i + (h_{ij} - M^2 M_i M_j) dx^i dx^j \\ &= h_{ij} dx^i dx^j - M^2 (dt - M_i dx^i)^2 \qquad (4.1) \\ &= h_{ij} dx^i dx^j + g_{00} \omega^2 \end{aligned}$$

For the metric to be considered bundle-like, the threading metric h_{ij} must be independent of t, while the threading lapse function $g_{00} = -M^2$ is allowed to depend on both t and the x^i.

The conditions for a bundle-like metric seem more reasonable in the threading interpretation. In fact, since we only require the three-dimensional metric components, h_{ij}, to be independent of t, the property of bundle-like is less restrictive then requiring $\frac{\partial}{\partial t}$ to be a Killing vector field. The condition of bundle-like allows the relationship between an observers clock and proper time (the threading lapse functions) to be time dependent, whereas the condition of $\frac{\partial}{\partial t}$ being Killing would not allow for such a possibility.

4.4 Theodor Kaluza and the Seed of Parametric Manifolds

4.4.1 Introduction

Anyone familiar with the Kaluza-Klein theories of spacetime will notice a similarity between the threading $(1+3)$ formalism and the standard Kaluza-Klein $(1+4)$ framework. While it appears that Gunnar Nördstrom made the first attempt to unify gravity with Maxwell's theory of electromagnetism via the introduction of a higher-dimensional theory, Nördstrom's theory of gravity differed from Einstein's general theory of relativity (see [1]). Kaluza first attempted such a higher-dimensional unification of Einstein's theory of gravity with Maxwell's theory of electromagnetism in [14]. Kaluza attempted to describe ordinary four-dimensional Einstein gravity and Maxwell electromagnetism by working in a five-dimensional space. Gravity and electromagnetism were then obtained by imposing a "cylindrical" condition on the fifth dimension. As Kaluza is basically beginning with a $(1+4)$ decomposition of a five-dimensional space, many of his calculation are reminiscent of the threading viewpoint. In place of the threading metric, Kaluza has the ordinary Einstein metric of spacetime, and taking the place of the threading shift is the electromagnetic vector potential.

Later, Einstein and Bergmann generalized Kaluza's theory. In [8] Einstein and Bergmann reformulated Kaluza's ideas and then proceeded to replace the "cylindrical" condition imposed by Kaluza by a "periodic" assumption, thus ascribing physical reality to Kaluza's fifth dimension. In their resulting ansatz lies the beginning of a true parametric picture of spacetime (although still nestled in the comforts of a five-dimensional space).

The collection [1] is a valuable source of many of the early papers on the Kaluza-Klein theories of unification. As many of the original papers are hard to find, I will reference this collection in most instances.

4.4.2 Threading and Kaluza's Theory

Einstein presented Theodor Kaluza's paper *On the Unity Problem of Physics*, [14], on December 8, 1921. Conjecturing that the components of the electromagnetic tensor $\frac{1}{2}F_{ab} = \frac{1}{2}(A_{a,b} - A_{b,a})$ could somehow be truncated versions of a Christoffel symbol $\Gamma^{a}{}_{bc}$, Kaluza turned to the freedom of a fifth dimension to carry out his theory. By considering physical spacetime as a subspace of a five-dimensional world, Kaluza had to introduce his "cylindrical condition" to account for the fact that we are only aware of the four-dimensional spacetime around us.

In [8], Einstein and Bergmann state Kaluza's ansatz as follows:

1. One has a five-dimensional metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (\alpha, \beta = 0, \ldots, 4).$$

2. *Cylindrical condition:* There exists a Killing vector field A^α. That is, setting $A_\alpha = g_{\alpha\beta} A^\beta$, one has

$$\nabla_\alpha A_\beta + \nabla_\beta A_\alpha = 0.$$

   ∇_α is the derivative operator associated with the five-metric $g_{\alpha\beta}$.

3. The integral curves of the vector field A^α are assumed to be geodesics. Call these integral curves *A-curves*

It follows from the cylindrical condition (Killing's equation) that the norm of A^α is constant along the A-curves. However, it can also be shown that the norm of A^α is constant throughout the entire space. As one should guess, the antisymmetrical derivatives of A^α, $A_{\alpha,\beta} - A_{\beta,\alpha}$, are supposed to make up the electromagnetic field.

Reminicent of threading, one introduces an adapted coordinate system. Consider an arbitrary four-dimensional surface which meets each A-curve exactly once. On this surface one introduces coordinates x^i $(i = 1, \ldots, 4)$ and assumes that the fifth coordinate x^0 is identically zero. Picking an orientation, one can now define the coordinate x^0 as the distance from the surface along one of the A-curves. Then, since

$$x^0 = \int_0^{x_0} \sqrt{g_{00}}\, dx^0,$$

in these coordinates $g_{00} = 1$ on each A-curve and hence in the entire space. By choice of the coordinate x^0, $A^i = 0$. If the constant norm of A is considered to be 1, then one has $A^0 = 1$. One should note that the coordinate vector field ∂_0 is tangent to the A-curves. Therefore

$$\begin{aligned} 0 &= A_{\alpha;\beta} + A_{\beta;\alpha} \\ &= A_{\alpha,\beta} + A_{\beta,\alpha} - 2A_\gamma \Gamma^\gamma{}_{\alpha\beta} \\ &= (g_{\alpha\gamma}A^\gamma)_{,\beta} + (g_{\beta\gamma}A^\gamma)_{,\alpha} - A^\gamma(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}) \\ &= A^\gamma g_{\alpha\beta,\gamma} + g_{\alpha\gamma}A^\gamma{}_{,\beta} + g_{\beta\gamma}A^\gamma{}_{,\alpha} \\ &= g_{\alpha\beta,0} \end{aligned}$$

and we see that the cylindrical condition can be restated as the condition that the components of the metric do not depend on the extra parameter x^0. Since $A_\alpha = g_{\alpha\beta}A^\beta$,

$$A_i = g_{i0} \qquad \text{and} \qquad A_0 = g_{00} = 1.$$

It should be noted that since the components of the metric are independent of x^0, so are the components A_α.

At this point one could argue that we have expressed our five-dimensional space in terms of the components of a four-dimensional metric tensor, g_{ij}, and the components of a four-vector A_i. However, this description is not invariant under reasonable change of coordinates. As Einstein and Bergmann mention, there are two families of coordinate transformations which preserve the nature of the adapted coordinate system. The coordinates on the original four-dimensional surface were chosen arbitrarily and hence the following coordinate changes should be allowed:

$$\begin{aligned} \bar{x}^i &= \bar{x}^i(x^i) \\ \bar{x}^0 &= x^0 \end{aligned} \tag{4.2}$$

In keeping with their terminology, we will refer to such a transformation as a *four-transformation.* Furthermore, since the original four-surface was chosen arbitrarily (and hence the origin of the extra parameter x^0), one can pick another four-dimensional surface (with the same coordinates x^i and the same A-curves), in effect re-parameterizing the five-dimensional space. Such a transformation will be called a *cut transformation* and has the following form:

$$\begin{aligned} \bar{x}^i &= x^i \\ \bar{x}^0 &= x^0 + f(x^i). \end{aligned} \tag{4.3}$$

Under a four-transformation it can be shown that the A_i and g_{ij} do indeed transform like four-dimensional tensors. However, under a cut transformation (reparameterization) the components transform according to

$$\begin{aligned} \bar{A}_i &= A_i - \frac{\partial f}{\partial x^i} \\ \bar{g}_{ij} &= g_{ij} - g_{0i}\frac{\partial f}{\partial x^j} - g_{0j}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \end{aligned}$$

Even though the components A_i do not transform correctly, the antisymmetrical derivatives of A_i are invariant under such a transformation.[8] Thus one considers the quantities $A_{i,j} - A_{j,i}$ rather then simply the A_i. In an attempt to overcome the non-covariance of the g_{ij}, Einstein and Bergmann are lead to consider the five dimensional tensor

$$h_{\alpha\beta} = g_{\alpha\beta} - A_\alpha A_\beta.$$

In our adapted coordinate system the only nonzero components are h_{ij}. Furthermore, the h_{ij} transform like the components of a tensor under both four-transformations and cut transformations. As we saw in the case of threading, the h_{ij} arise from calculating the distance of two infinitesimally close A-curves. Therefore, we have in this coordinate system the standard Kaluza ansatz in which the five-dimensional metric takes the form

$$g_{\alpha\beta} = \begin{pmatrix} 1 & A_j \\ A_i & h_{ij} + A_i A_j \end{pmatrix}$$

[8] As the authors mention in [8], this corresponds to the fact that the electromagnetic potentials are defined only up to additive terms which are gradients of an arbitrary function.

where h_{ij} represents the four-dimensional metric of spacetime and A_i are the components of a co-vector field meant to be interpreted as the electromagnetic potential.

4.4.3 Einstein and Bergmann's Generalization

As a result of Kaluza's "cylindrical condition", the components of the higher-dimensional metric are assumed to be independent of the extra parameter x^0. In light of the above discussion, it is clear that this is analogous to the case of a $(1+3)$-decomposition of spacetime in which the threading curves represent an isometry of spacetime (i.e. $\frac{\partial}{\partial t}$ is a Killing vector). In an attempt to give Kaluza's fifth dimension some physical meaning, Einstein and Bergmann generalized Kaluza's original theory by replacing the "cylindrical condition" with a "periodic condition". As we shall see, the net result is the fact that the components of the resulting four-dimensional metric tensor h_{ij} are, in general, functions of the parameter x^0. Thus, the seeds of a parametric structure of spacetime are planted. The following paragraphs will outline Einstein's and Bergmann's generalization.

Again working in a five-dimensional space, Einstein and Bergmann made the following assumptions:

1′. One has a five-dimensional metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \quad (\alpha, \beta = 0, \ldots, 4).$$

2′. *Periodic postulate:* The five-dimensional space will be considered to be closed with respect to one dimension. However, this fact will be represented by an open space that is periodic with respect to this dimension ("unroll" the cylinder). Therefore, as one moves about in this dimension one will repeatedly encounter points $p, p', p'', \ldots$ that represent a single point in the five dimensional space.

3′. There also exists a family of closed geodesics such that each point in space lies on exactly one geodesic. In terms of the open space, each equivalence class of points $\{p, p', p'', \ldots\}$ lie on the same geodesic. These geodesics will again be called A-curves.

The adapted coordinate system is introduced as before. On a four-dimensional surface which meets each A-curve exactly once, choose coordinates x^i. Picking a positive direction, one can define a fifth coordinate by the metric distance from the surface (considered to the the set $x^0 \equiv 0$) measured along an A-curve. Letting b represent the length between two consecutive periodic points,

$$b = \int_p^{p'} ds,$$

one has the x^0 coordinate of a point p

$$x^0 = \frac{1}{b} \int_{p_0}^{p} ds$$

where p_0 is on the initial surface.

As before the vector field A^α is given by

$$A^0 = 1 \qquad A^i = 0$$

and thus

$$A_i = g_{0i}.$$

One also introduces the tensor with components

$$h_{\alpha\beta} = g_{\alpha\beta} - A_\alpha A_\beta$$

of which only the h_{ij} are non zero. Thus, the h_{ij} are interpreted as a four-dimensional metric tensor (the actual spacetime metric).

It turns out that, as with Kaluza's original theory, the components A_i are independent of the parameter x^0. However, the difference in this generalization lies in the fact the the metric tensor *is* allowed to be a function of x^0. Thus, we see that Einstein and Bergmann are requiring precisely that $g_{\alpha\beta}$ be bundle-like.

Einstein and Bergmann continued by defining all of the covariant operations necessary to be able to analyze four-dimensional (spacetime) tensors in this five-dimensional framework. In other words, they are treating spacetime as a parametric manifold! For Einstein and Bergmann, the components of a *four-tensor* are allowed to depend on all five coordinates x^α, but must transform like an ordinary four-dimensional tensor under a four-transformation ((4.2)) and remain invariant under a cut transformation ((4.3)). The fundamentals of their tensor analysis can be summarized by

1. Since $\frac{\partial}{\partial \bar{x}^0} = \frac{\partial}{\partial x^0}$ under either of the transformations, differentiation with respect to the parameter x^0 is a covariant operation.

2. The action of the operator

$$\frac{\partial}{\partial x^i} - A_i \frac{\partial}{\partial x^0} \tag{4.4}$$

   leave the components of a tensor invariant under a cut transformation. Thus, for a covariant derivative operator one replaces ordinary partial derivatives, $\frac{\partial}{\partial x^i}$, by (4.4), yielding (for example)

$$\begin{aligned} \nabla_{*i} B_j &= (\frac{\partial}{\partial x^i} - A_i \frac{\partial}{\partial x^0}) B_j - \gamma^m_{ji} B_m \\ &= B_{j,i} - A_i B_{j,0} - \gamma^m_{ji} B_m \\ &= B_{j*i} - \gamma^m_{ji} B_m \end{aligned}$$

   where

$$\gamma^i{}_{jk} = \frac{1}{2} h^{im} (h_{mj*k} + h_{mk*j} - h_{jk*m}).$$

   These are the main analysis rules for parametric tensors.

Einstein and Bergmann also used the $\gamma^i{}_{jk}$'s to define a notion of curvature which agrees with Zel'manov's later definition.

4.5 Decomposition of Metrics on Fibre Bundles

4.5.1 Introduction

The slicing and threading frameworks can be described as part of a single mathematical structure; a *fibre bundle.* The slicing and threading metrics, shifts, and lapse functions naturally arise when one examines the decomposition of a bundle metric in terms of metrics on the base space and the fibre space. By choosing the base space and fibre space correctly, one recovers either the threading or slicing framework.

As the presentations of the slicing and threading viewpoints in chapter one mainly focused on the decomposition of the spacetime metric, that will be my main concern in this section as well. The motivation behind Reinhart's notion of bundle-like metric came from an attempt to construct a metric on the total space of a fibre space out of metrics on the base space and typical fibre. Such a construction was very reminiscent of the construction process of slicing and threading. In this section, I would like to address the converse. That is, given a metric on the total space, what conditions are necessary in order to construct metrics on the base space as well as the typical fibre. Since I am interested in this problem in order to better describe the slicing and threading decompositions, I am really only interested in local decompositions of the metric. Thus, I will be working in a single local trivialisation of the bundle.

Let $\mathcal{M}$ be a fibre bundle with base space B, fibre F, and continuous, surjective projection $\pi : \mathcal{M} \to B$. For the purposes of slicing and threading one should take $\mathcal{M}$ to be a spacetime and the collection of fibres $\{\pi^{-1}(x) : x \in B\}$ to represent the appropriate foliation (spacelike hypersurfaces for slicing or timelike curves for threading). Furthermore, we have $B = \mathcal{M}/F$ under the equivalence induced by the fibres (i.e. for $p, q \in \mathcal{M}$, $p \sim q \Leftrightarrow \pi(p) = \pi(q)$). We may call B the *manifold of* *leaves.* Using the terminology of the preceding chapters, we may call B the *manifold of orbits* also.

For the following I will work within a single local trivialisation with adapted coordinates. Let $U \subset B$ be a coordinate neighborhood with coordinates x^i such that $\pi^{-1}(U) \approx U \times F$. Furthermore, within a coordinate neighborhood of $\pi^{-1}(U)$ we may use coordinates (x^i, y^α) where y^α are coordinates on F.

For any $p \in \pi^{-1}(U)$, there exists a natural subspace $V_p \subset T_p\mathcal{M}$ called the *vertical subspace.* V_p is defined by

$$V_p = \{X \in T_p\mathcal{M} : \pi_*(X) = 0\}.$$

Complementing the notion of vertical, define a subspace $H_p \subset T_p\mathcal{M}$ so that $T_p\mathcal{M} = V_p \oplus H_p$ and call H_p the *horizontal subspace.* Certainly there are many smooth choices for H_p. If $\mathcal{M}$ has a metric, H_p may be chosen quite naturally to be the orthogonal complement to V_p.

Now, given a metric g on $\mathcal{M}$ is there a natural choice for metrics h and k on B and F respectively? Not unless additional structure on $\mathcal{M}$ is given or we allow for the additional freedom of a parametric metric. Let us mention a few of these possibilities in greater detail.

4.5.2 Metric on the Base Space

For $X,\ Y \in T_xB$ there exist unique horizontal lifts of X and Y at each point $p \in \pi^{-1}(x)$. Call these lifted vectors $\hat{X}_p$ and $\hat{Y}_p$. It would seem natural to define $h(X, Y)$ in terms of these lifts. In order for h to be well-defined there are various options depending on the additional structure one is willing to assume.

1. If g were constant along each fibre, then $g(\hat{X}_p, \hat{Y}_p) = g(\hat{X}_q, \hat{Y}_q)$ for all $p, q \in \pi^{-1}(x)$. One could then define

$$h(X, Y) = g(\hat{X}_p, \hat{Y}_p)$$

for any $p \in \pi^{-1}(x)$. Actually, we can loosen this restriction somewhat. We only need that g restricted to the horizontal subspace H_p is constant along each fibre. This is essentially Reinhart's condition that g be bundle-like with respect to the given foliation.

2. If there were some preferred global section $\sigma : B \to \mathcal{M}$ (*e.g.* F is a vector space) one could define

$$h(X,Y) = g(\hat{X}_{\sigma(x)}, \hat{Y}_{\sigma(x)}).$$

σ may refer to some initial hypersurface in an initial value formulation.

3. One could allow the metric on B to carry an extra parameter, namely y^α, and define

$$h(X,Y)\big|_{y^\alpha} = g(\hat{X}_{(x^i,y^\alpha)}, \hat{Y}_{(x^i,y^\alpha)})$$

and, hence, begin to consider B as a *parametric manifold.*

If h has been defined in one of the above situations, the component functions h_{ij} can be defined and computed. Suppose the horizontal direction is defined by the basis

$$H_i = \frac{\partial}{\partial x^i} + \Gamma^\alpha_i \frac{\partial}{\partial y^\alpha}.$$

We define the *horizontal lift* of $\frac{\partial}{\partial x^i}$ to be $\frac{\hat{\partial}}{\partial x^i} = H_i$.

One may now define the components of h by

$$\begin{aligned} h_{ij} &= h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= g(\frac{\hat{\partial}}{\partial x^i}, \frac{\hat{\partial}}{\partial x^j}) \\ &= g(\frac{\partial}{\partial x^i} + \Gamma^\alpha_i \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial x^j} + \Gamma^\beta_i \frac{\partial}{\partial y^\beta}) \\ &= g_{ij} + 2\Gamma^\alpha_i g_{j\alpha} + \Gamma^\alpha_i \Gamma^\beta_j g_{\alpha\beta} \end{aligned}$$

We are assuming $h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ is well defined, but the functions h_{ij} may be functions of y^α as well as x^i (as in 3.).

4.5.3 Metric on the Fibre

There are similar obstructions to defining a metric k on F. Vectors $Z, W \in T_yF$ can be naturally identified with vertical vectors (tangent to the fibres) of $\mathcal{M}$. We have many natural embeddings of F into $\mathcal{M}$. The problem is which fibre? As with h, we need some additional structure which allows us a definition in the following sense:

$$k(Z, W) = g(\hat{Z}, \hat{W})$$

where $\hat{Z}$ and $\hat{W}$ represent some mapping of Z and W into vertical vectors of $\mathcal{M}$. That is, we define k to be a pullback of g. Since k depends on which imbedding of F we use, we can think of k as being parameterized by the coordinates (x^i) of B. We therefore define the components of k by $k_{\alpha\beta}\big|_{(x^i)} = g_{\alpha\beta}\big|_{(x^i)}$. That is, one can use the local trivialisations to pull back the metric g to a parametric metric on F.

4.5.4 The Decomposition

We can now write the original metric g of $\mathcal{M}$ in terms of h and k. We have:

$$(g_{ab}) = \begin{pmatrix} k_{\alpha\beta} & g_{\alpha j} \\ g_{i\beta} & h_{ij} - 2\Gamma_i^\alpha g_{\alpha j} - \Gamma_i^\alpha \Gamma_j^\beta k_{\alpha\beta} \end{pmatrix}$$

where

$$\begin{aligned} g_{\alpha j} &= g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^\alpha}\right) \\ &= g\left(\frac{\partial}{\partial x^j} + \Gamma_j^\beta \frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial y^\alpha}\right) - \Gamma_j^\beta g_{\alpha\beta} \\ &= g\left(H_j, \frac{\partial}{\partial y^\alpha}\right) - \Gamma_j^\beta g_{\alpha\beta} \end{aligned}$$

.

Since $\mathcal{M}$ has a metric, we may choose our notion of horizontal so that H_i is orthogonal to $\frac{\partial}{\partial y^\alpha}$. In which case we have $g(H_j, \frac{\partial}{\partial y^\alpha}) = 0$ and $g_{\alpha j} = -\Gamma_j^\beta g_{\alpha\beta} = -\Gamma_j^\beta k_{\alpha\beta}$.

In this special case g takes the following form:

$$(g_{ab}) = \begin{pmatrix} k_{\alpha\beta} & -\Gamma^{\beta}_{j} k_{\alpha\beta} \\ -\Gamma^{\alpha}_{i} k_{\alpha\beta} & h_{ij} + \Gamma^{\alpha}_{i}\Gamma^{\beta}_{j} g_{\alpha\beta} \end{pmatrix}.$$

Example 4.1. *Threading.*

Take $F = \mathbb{R}$ with coordinate $y^0 = t$ and $B = \Sigma$ to be a three-dimensional manifold with coordinate x^i $(i = 1, 2, 3)$ as before. In terms of the above decompositions, the spacetime metric is of the following form

$$(g_{ab}) = \begin{pmatrix} k_{00} & -\Gamma_j k_{00} \\ -\Gamma_i k_{00} & h_{ij} + \Gamma_i \Gamma_j k_{00} \end{pmatrix}$$

Since k_{00} represents the squared norm of $\frac{\partial}{\partial t}$, according to previous notation $k_{00} = -M^2$. This decomposition is then precisely the same as the threading decomposition with $\Gamma_i = M_i$ and where h_{ij} is the threading metric on the manifold of orbits Σ. Thus, the notion of horizontal is given by $H_i = \frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t}$ which corresponds to the orthogonal subspace to $\frac{\partial}{\partial t}$.

Example 4.2. *Slicing*

By switching the roles of F and B in the above example, one has the original slicing story. Let $F = \Sigma$ with coordinates $y^\alpha = X^i$ $(i = 1, 2, 3)$ and $B = \mathbb{R}$ with a single coordinate $x^0 = t$. One has:

$$(g_{ab}) = \begin{pmatrix} k_{ij} & -\Gamma^j k_{ij} \\ -\Gamma^i k_{ij} & h_{00} + \Gamma^i \Gamma^j k_{ij} \end{pmatrix}$$

As before, k_{ij} is the slicing metric, $\Gamma^i = -N^i$, and $h_{00} = -N^2$. Here the horizontal subspace is given by $H_0 = \frac{\partial}{\partial t} + N^i \frac{\partial}{\partial x^i}$ which is orthogonal to the hypersurfaces Σ_t.

Now, in the slicing and threading pictures of spacetime, one always has both sets of foliations (foliations by hypersurfaces as well as by curves), thus the slicing and threading pictures of spacetime can be described nicely as part of a single mathematical structure; a *fibre bundle.* By choosing which foliation corresponds to the fibres, one is handed either the threading or slicing framework.

4.6 Parametric Manifolds and Fibre Bundles

Consider the fibre bundle picture that leads to the threading decomposition so that $F = \mathbb{R}$ and $B = \Sigma$. For simplicity, assume $\mathcal{M} = \Sigma \times \mathbb{R}$, so we have a trivial bundle $\pi : \Sigma \times \mathbb{R} \to \Sigma$. We just saw how the spacetime metric naturally decomposes into the threading metric on Σ. More precisely, if there were no special conditions imposed on the spacetime, it decomposes into a one-parameter family of threading metrics.

We also used the spacetime metric to designate a preferred notion of horizontal in $T_p(\Sigma \times \mathbb{R}) = T_p\mathcal{M}$. We had $H_i = \frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t}$ where $M_i dx^i$ was the threading shift one-form. This notion of horizontal depends upon the t coordinate of $\mathcal{M}$, as, in general, the functions M_i depend on t. Thus, without imposing additional restrictions on the spacetime $\mathcal{M}$ there is no natural relationship between the horizontal spaces $H_i(p,t)$ and $H_i(p,t+s)$. That is, the choice of H_i does not constitute a *connection* in the usual sense of a connection on a vector bundle (since $F = \mathbb{R}$, $\pi : \mathcal{M} \to \Sigma$ is a vector bundle). Stated more precisely, if $\alpha_s : \pi^{-1}(b) \to \pi^{-1}(b)$, $s \in \mathbb{R}$ induces some action on the vector space $\mathbb{R}$ (say $\alpha_s(b,t) = (b, s+t)$), the induced map on $T\mathcal{M}$ given by α_{s*} may not take horizontal vectors into horizontal vectors. As Reinhart mentions in [27], these choices for H_i nevertheless define a "sort of connection". We still have a smooth decomposition of $T_p\mathcal{M} = V_p \oplus H_p$, and we can still talk about horizontal lifts. However, we will not require the horizontal subspaces to transform in any particular way as one travels up and down a specific fibre.

For $X = X^i \frac{\partial}{\partial x^i} \in T_x\Sigma$, we can define its horizontal lift at $(x,t) \in \pi^{-1}(x)$ to be

$$\hat{X} = X^i \left(\frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t} \right) \Big|_{(x,t)}.$$

Now, a parametric function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ is just a function $f : \mathcal{M} \to \mathbb{R}$, and thus there exists a natural action of $X \in T_p\Sigma$ on f given by the horizontal lift of X.

Therefore we have

$$\begin{aligned}\hat{X}(f) &= X^i\left(\frac{\partial}{\partial x^i} + M_i\frac{\partial}{\partial t}\right) f \\ &= X^i f_{*i}.\end{aligned}$$

Again, we see the parametric derivative operator ∂_{*i} as the most natural derivative operator on parametric functions.

If Σ is thought of as a parametric manifold, a reparameterization of Σ given by $s = t + F(x)$ is just a coordinate transformation on the fibre $\pi^{-1}(x)$. We have studied the effect of such a coordinate transformation on the horizontal bases H_i earlier. We had

$$\begin{aligned}H_i' &= \frac{\partial}{\partial x^i} - \frac{\partial F}{\partial x^i}\frac{\partial}{\partial t} + M_i\frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial x^i} + \left(M_i - \frac{\partial F}{\partial x^i}\right)\frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial x^i} + M_i'\frac{\partial}{\partial t}\end{aligned}$$

where M_i' are the components of the threading shift one-form after a coordinate transformation of the fibres. Therefore, when defining a parametric structure on Σ we require that the one-parameter family of one-form field obey the reparameterization property (equation (3.26)).

Not only does this fibre bundle setting easily reproduce many of the results we obtained through projection techniques, it also offers a nice description of the parametric exterior derivative operator, d_*. When studying the curvature in a principal bundle (since $F = \mathbb{R}$ is a Lie group, $\pi : \mathcal{M} \to \Sigma$ is also a principal bundle) one uses the exterior derivative on $\mathcal{M}$ and the notion of horizontal to define an exterior covariant derivative operator. For any k-form θ on $\mathcal{M}$ and letting d denote the usual exterior derivative on $\mathcal{M}$, define D by

$$D\theta\left(X_i, \ldots, X_{k+1}\right) = d\theta\left(H(X_1), \ldots, H(X_{k+1})\right).$$

For a 0-form (or parametric function) f we have

$$\begin{aligned} Df(X) &= df\,(H(X)) \\ &= df\left(X^i M_i \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i}\right) \\ &= M_i X^i \frac{\partial f}{\partial t} + X^i \frac{\partial f}{\partial x^i} \\ &= X^i f_{*i} \\ &= d_* f(X), \end{aligned}$$

where the last line emphasizes that Df agrees with the earlier notion of parametric exterior derivative. Using the generalized axioms for a parametric exterior derivative, we can extend $Df = d_* f$ to a parametric exterior derivative operator on Σ. Thus the exterior covariant derivative operator in the bundle gives rise to a parametric exterior derivative operator on the parametric manifold Σ. It should be emphasized again that the subspaces H_i do not constitute a connection in a principal bundle since they are not preserved under the group operation (addition).

The next, and final, step would be to write ∇_* as a connection in a principal bundle. That is, use D to define a covariant derivative on the parametric manifold Σ. There are, however, barriers for such an interpretation of ∇_*. Difficulties arise because the choices of H_i do not constitute a true connection on the principal bundle $\pi : \Sigma \times \mathbb{R} \to \Sigma$. This is due to the fact that any group action α on a fibre will not necessarily induce a map $\alpha_* : T_p(\Sigma \times \mathbb{R}) \to T_{\alpha(p)}(\Sigma \times \mathbb{R})$ which preserves the choice of H_i (at lease without introducing restriction on $\mathcal{M}$).

As the notion of horizontal corresponds to the subspace of vectors orthogonal to $\frac{\partial}{\partial t}$, one may wish to consider the notion of Fermi-Walker transport. Fermi-Walker transport arises when one considers an orthonormal tetrad for an arbitrarily accelerated observer (see [19]p170). If m represents an observer, a vector field X is said to have been *Fermi-Walker transported* along m if

$$m^\alpha \nabla_\alpha X^\beta + m_\alpha X^\alpha A^\beta - A_\alpha X^\alpha m^\beta = 0 \tag{4.5}$$

where $A^\beta = m^\alpha \nabla_\alpha m^\beta$ is the acceleration of the observer. It is true that Fermi-Walker transport preserves norms (and hence angles) of arbitrary vectors. That is, vectors which are perpendicular to the observer will remain perpendicular to the observer after Fermi-Walker transport. Thus, horizontal vector fields which have been Fermi-Walker transported along integral curves of $\frac{\partial}{\partial t}$ remain horizontal. There is, however, no group action of $\mathbb{R}$ which induces Fermi-Walker transport on $T\mathcal{M}$. As with all other parametric object, the best one can achieve is by interpreting ∇_* as a one-parameter family of (standard) connections.

It should be emphasized that in the theories of Kaluza as well as Einstein and Bergmann additional structure (Killing and bundle-like respectively) was imposed on the spacetime. This additional structure provided the much needed symmetry in the vertical direction. Furthermore, in more general settings such as Yang-Mills theories, the group (fibre) symmetry is still present. Thus, we have that ∇_* is a notion of connection in a more general setting. Perjés mentions how the parametric derivative ∂_* may be interpreted as a generalization of an invariant derivative in gauge theories. Thus, one may anticipate the usefulness of such derivative operators in a generalized Yang-Mills setting.

4.7 Parametric Manifolds and Jet Bundles

4.7.1 Introduction to Jet Bundles

Jet bundles may be thought of as generalized tangent spaces to manifolds. As Saunders mentions in [28], a first-order jet generalizes the notion of a tangent vector by considering equivalence classes of higher-dimensional manifolds passing though a point (rather then curves). The relationship between jet bundles and parametric manifolds begins with considering the trivial bundle associated with the threading viewpoint:

$$\begin{array}{c} \Sigma \times \mathbb{R} \\ \pi \big\downarrow \\ \Sigma \end{array}$$

It turns out that jet fields associated with the first jet bundle J^1_π correspond to one-parameter families of one-form fields on Σ. Also, for each jet field there corresponds a notion of a total derivative. If one chooses the preferred jet field associated with the threading (i.e. parametric) decomposition of $\Sigma \times \mathbb{R} \approx \mathcal{M}$, then we can interpret the corresponding derivative as a parametric exterior derivative d_*. This interpretation of d_* agrees with our earlier definition.

Let (E, π, F, B) be any fibre bundle. We will refer to this bundle by the projection map π. For $p \in B$ let $\phi, \psi \in \Gamma_p(\pi)$ be sections of the bundle π. The definitions contained in this section can be found in [28].

Definition 4.1 *We say that ϕ is one-equivalent to ψ at p if*

i. $\phi(p) = \psi(p)$ *and*

ii. $\phi_*\big|_p = \psi_*\big|_p$.

Definition 4.2 *The equivalence class* $[\phi]$ *is called the first-jet of* ϕ *at* p *and is written* $j_p^1\phi$.

Consider the set

$$J_\pi^1 = \{j_p^1\phi : p \in B, \phi \in \Gamma_p(\pi)\}$$

with maps $\pi_1 : J_\pi^1 \to B$ and $\pi_{1,0} : J_\pi^1 \to E$ defined by

$$\pi_1(j_p^1\phi) = p$$
$$\pi_{1,0}(j_p^1\phi) = \phi(p).$$

Claim 4.3 J_π^1 *is manifold.*

Proof: I will define a set of coordinates for J_π^1. For a complete proof see [28]. Consider a locally trivial neighborhood U in E so that we can use adapted coordinates (x^i, u^α) in U. That is, the coordinate functions x^i are pulled back from B via π. We define an induced coordinate system (U', u') on J_π^1 by

$$U' = \{j_p^1\phi : \phi(p) \in U\}$$
$$u' = \left(x^i, u^\alpha, u_i^\alpha\right)$$

with

$$x^i(j_p^1\phi) = x^i(p)$$
$$u^\alpha(j_p^1\phi) = u^\alpha(\phi(p))$$
$$u_i^\alpha(j_p^1\phi) = \frac{\partial\phi^\alpha}{\partial x^i}\Big|_p$$

Thus, one uses derivatives of ϕ to define coordinates on J_π^1. Using these coordinates, one can show that J_π^1 is indeed a manifold. ♠

Furthermore, it is shown in [28] that

Claim 4.4 $\pi_{1,0} : J_\pi^1 \to E$ *and* $\pi_1 : J_\pi^1 \to B$ *are fibre bundles.*

$$
\begin{array}{ccc}
J^1_\pi & \xrightarrow{\pi_{1,0}} & E \\
 & \searrow^{\pi_1} & \downarrow \pi \\
 & & \Sigma
\end{array}
$$

Although I will not take the space for a complete description of jet bundles, it is worth noting that one of the most important features of the bundle $(J^1_\pi, \pi_{1,0}, E)$ is its affine structure. This gives sections of the bundle many of the features of a vector field on a manifold. While a vector field has a "flow" parameterized by a one-dimensional manifold, the "flow" of a jet field is parameterized by the base space B. Furthermore, in some cases (soon to be shown) the jet field may act as a derivation. We will see how the action of a jet field corresponds to the action of a parametric vector field on parametric functions.

The map π^* may be used to pull back forms on B to forms on E. These forms are called horizontal forms. More precisely,

Definition 4.5 *A horizontal one-form on E is a section of $\pi^*(T^*B) \to E$. Denote the collection of such forms by $\bigwedge^1_0 \pi$.*

Definition 4.6 *Given a jet $j^1_p\phi \in J^1_\pi$, the action of the jet on functions on E is the mapping $j^1_p\phi : C^\infty(E) \to T^*_{\phi(p)}$ defined by*

$$
j^1_p\phi[f] = \pi^* \left(d\left(\phi^*(f)\right)\big|_p \right).
$$

In coordinates this action takes the form

$$
j^1_p\phi[f] = \left(\frac{\partial f}{\partial x^i}\Big|_{\phi(p)} + u^\alpha_i(j^1_p\phi)\frac{\partial f}{\partial u^\alpha}\Big|_{\phi(p)} \right) dx^i\Big|_{phi(p)}
$$

Definition 4.7 *A section $\Gamma : E \to J^1_\pi$ of the bundle $\pi_{1,0}$ will be called a jet field. The action of Γ on functions is the mapping $C^\infty(E) \to \bigwedge^1_0 \pi$ given by*

$$
(\Gamma f)\big|_{\phi(p)} = \Gamma(\phi(p))[f]. \tag{4.6}
$$

In terms of the coordinates introduced above, we have

$$\Gamma f = \left(\frac{\partial f}{\partial x^i} + \Gamma_i^\alpha \frac{\partial f}{\partial u^\alpha} \right) dx^i$$

where Γ_i^α is the coordinate representation of Γ given by $\Gamma_i^\alpha = u_i^\alpha \circ \Gamma$.

Claim 4.8 *Starting with the trivial bundle* $(\Sigma \times \mathbb{R}, \pi, \Sigma)$ *associated with the threading viewpoint there exists a canonical diffeomorphism between the first jet manifold* J_π^1 *and* $T^*\Sigma \times \mathbb{R}$.

Proof: The diffeomorphism is given in [28]. Locally (for some neighborhood $W \subseteq \Sigma$), we have that for any section $\phi \in \Gamma_W(\pi)$ we use the standard projection $\pi_2 : \Sigma \times \mathbb{R} \to \mathbb{R}$ to define $\bar{\phi} = \pi_2 \circ \phi$ in some neighborhood $W \subset B$. Now, $\bar{\phi} \in C^\infty(W)$ and the diffeomorphism is given by

$$j_p^1 \phi \mapsto \left(d\bar{\phi}\big|_p, \bar{\phi}(p) \right).$$

Saunders proves that this map is indeed a diffeomorphism. ♠

4.7.2 Example

Let us suppose that on is given a spacetime (M, g) with a timelike congruence given by the vector field $\frac{\partial}{\partial t}$. One may now work on the three-dimensional manifold of orbits, Σ, and consider a $(1+3)$ decomposition of M.

Consider a decomposition of g in terms of the threading lapse function, M, the threading shift one-form $M_i dx^i$, and the threading metric h_{ij}. Recall that while the functions M, M_i, and h_{ij} are functions of (t, x^i), they do define three-dimensional tensors on Σ which carry an extra parameter. Furthermore, assume that the coordinates (t, x^i) on M respect the timelike congruence. That is, an integral curve of $\frac{\partial}{\partial t}$ is given (locally) by the set $\{x^i \equiv constant\}$. Therefore the x^i can be used as coordinates on Σ.

Consider the bundle $\pi : M \approx \Sigma \times \mathbb{R} \to \Sigma$, where $\pi(p)$ is the unique integral curve of $\frac{\partial}{\partial t}$ which passes through $p \in M$. Now construct the first jet bundle, $\pi_{1,0} : J^i_\pi \to \Sigma \times \mathbb{R}$. Let $\Gamma : \Sigma \times \mathbb{R} \to J^1_\pi$ be a jet field. Since $J^1_\pi \approx T^*\Sigma \times \mathbb{R}$, Γ can be thought of as a map

$$\Gamma : \Sigma \times \mathbb{R} \to T^*\Sigma \times \mathbb{R} \tag{4.7}$$

We will call such a map (section) a *parametric one-form field* and denote the collection of such maps by $\mathcal{PT}_1(\Sigma)$.

In particular, the smooth functions M_i yield a preferred jet field Γ given by

$$\Gamma(x^i, t) = (x^i, t, M_i(x^i, t)).$$

Furthermore, under the diffeomorphism $J^1_\pi \equiv T^*\Sigma \times \mathbb{R}$, the point $(x^i, t, M_i(x^i, t))$ is identified with the one form $M_i dx^i$. Now, $M_i dx^i$ is a one-parameter family of one forms on Σ.

Given any jet field Γ, there exists an action on $C^\infty(\Sigma \times \mathbb{R})$. We will call $f \in C^\infty(\Sigma \times \mathbb{R})$ a *parametric function.*

For each $j^1_p\phi \in J^1_\pi$, define a map $j^1_p\phi : C^\infty(\Sigma \times \mathbb{R}) \to T^*_{\phi(p)}(\Sigma \times \mathbb{R})$ by

$$j^1_p\phi[f] = \pi^*(d(\phi^* f)|_p).$$

In coordinates,

$$j^1_p\phi[f] = \left(\frac{\partial f}{\partial x^i}\Big|_{\phi(p)} + M_i(j^1_p\phi)\frac{\partial f}{\partial t}\Big|_{\phi(p)} \right) dx^i\Big|_{\phi(p)}$$

We will write $j^1_p\phi[f] = \frac{df}{dx^i}\big|_{j^1_p\phi} dx^i\big|_{\phi(p)}$.

Now, the action of jet field Γ is given by equation (4.6)and from the definition of $j^1_p\phi[f]$, one readily sees that $j^1_p\phi[f]$ is a horizontal one-form on $\Sigma \times \mathbb{R}$.

Therefore, we have for each jet field Γ a notion of a *parametric exterior derivative*, $d : C^\infty(\Sigma \times \mathbb{R}) \to \mathcal{PT}_1$ given by

$$df\big|_{\phi(p)} = \Gamma(\phi(p))[f] \tag{4.8}$$

We explicitly note that the parametric structure depends on a choice of a preferred jet field. However, we have a preferred choice of jet field given by our (1+3) decomposition of spacetime. Namely, we have a jet field Γ taking $(x^i, t) \mapsto (x^i, t, M_i|_{(x^i,t)}) \in J^1_\pi$.

Thus, our preferred notion of parametric exterior derivative takes the following form:

$$\begin{aligned}
df|_{\phi(p)} &= \frac{df}{dx^i}|_{\Gamma(\phi(p))} dx^i|_{\phi(p)} \\
&= (\frac{\partial f}{\partial x^i}|_{\phi(p)} + v_i(\Gamma(\phi(p)))\frac{\partial f}{\partial t}|_{\phi(p)})dx^i|_{\phi(p)} \\
&= (\frac{\partial f}{\partial x^i}|_{\phi(p)} + M_i|_{\phi(p)}\frac{\partial f}{\partial t}|_{\phi(p)})dx^i|_{\phi(p)} \\
&= \frac{\partial_* f}{\partial x^i}|_{\phi(p)} dx^i|_{\phi(p)} \\
&= f_{*i} dx^i
\end{aligned}$$

Here we see that the above "starry" notation is the same as before! We have indeed reproduced the earlier parametric exterior derivative!

This notation agrees with the anticipated action of a parametric vector field on a parametric function.

We thus define a *parametric vector field* of Σ to be a section $X : \Sigma \times \mathbb{R} \to T\Sigma \times \mathbb{R}$. Now, let f be a parametric function (i.e. $f \in C^\infty(\Sigma \times \mathbb{R})$). If $X(x^i, t) = (X^i \frac{\partial}{\partial x^i}, t)$, then let $\hat{X}$ be the horizontal lift of X at time t and define

$$X(f) = \hat{X}(f)$$

Since $\hat{X} = X^i(\frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t}) \in T(\Sigma \times \mathbb{R})$, he have

$$X(f) = X^i f_{*i}.$$

5. Avenues of Future Research

5.1 Introduction

In this chapter I will discuss some possible applications of parametric manifolds to general relativity. As the theory of parametric manifolds gives one a way of analyzing time-dependent fields on fixed manifolds, it is particularly useful in the study of initial-value problems.

First, I will address the issue of the quantized scalar field. In particular, I will show how the decomposition of the spacetime Laplacian is simplest in terms of a parametric theory.

Second, I will address the initial-value problem of general relativity itself. As the standard initial-value formula-ton of general relativity relies on the Gauss-Codazzi equations, the generalized Gauss-Codazzi equations can be used with less restrictive sets of initial-data.

5.2 Quantization of the Scalar Field in Curved Spacetime

5.2.1 The Problem

For spacetimes which admit an everywhere timelike, hypersurface-orthogonal Killing vector field, there exists a standard procedure for the quantization of the Klein-Gordon equation. Such spacetimes are called *static* and, if we take the integral curves of the Killing vector field to be our threading curves, then the slicing and threading viewpoints agree. Moreover, since the threading vector field is Killing, all "time" derivatives of the metric are zero. Hence many aspects of the parametric viewpoint will agree with the usual threading (or slicing) decomposition. That is, "starry" derivatives of the metric are ordinary partial derivatives and the parametric Christoffel symbols agree with the standard Christoffel symbols associated with the threading metric.

In the case of spacetimes which are stationary but non-static, the slicing and threading viewpoints are different (stationary but non-static implies non-zero shift). In such a situation one has (at least) two options. One could follow the procedure of Ashtekar and Magnon [2] and regard the timelike Killing vector field as fundamental, or one could follow the procedure of Dray, Kulkarni, and Manogue [7] and regard the surfaces of constant Killing time as fundamental. As in shown in [7], these two procedures differ. Consider this problem from the slicing, threading, and parametric points of view.

The massless Klein-Gordon equation is

$$\Box\phi = 0$$

where $\Box\phi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$ for a scalar field ϕ. At the heart of any quantization procedure is an initial-value formulation of this equation, *i.e.*, a decomposition of spacetime into "space" and "time". As shown below, in a static spacetime the decompositions are identical. However, when one considers a stationary but non-static spacetime, one has a simpler decomposition in the parametric setting. Further work on applying the parametric viewpoint to the quantization of the scalar field in a stationary but non-static spacetime, as well as on an arbitrary spacetime background, is in progress.

5.2.2 Hypersurface-Orthogonal Decompositions

For the sake of generality, let us assume that $\frac{\partial}{\partial t}$ is not necessarily Killing. In terms of the spacetime metric $g_{\alpha\beta}$, one may write

$$\begin{aligned}\Box\phi &= g^{\alpha\beta}\nabla_\alpha\nabla_\beta\,\phi \\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}g^{\alpha\beta}\phi_{,\alpha}\right)_{,\beta}.\end{aligned} \tag{5.1}$$

I will introduce the notation $\triangle_k$, $\triangle_h$, and $\triangle_*$ for the three-dimensional Laplace operators in the slicing, threading, and parametric viewpoints respectively. As in (5.1), one may show tha these operators may be written

$$\triangle_k\phi \doteq \frac{1}{\sqrt{|k|}}\left(\sqrt{|k|}k^{ij}\phi_{,i}\right)_{,j}$$
$$\triangle_h\phi = \frac{1}{\sqrt{|h|}}\left(\sqrt{|h|}h^{ij}\phi_{,i}\right)_{,j}$$
$$\triangle_*\phi = \frac{1}{\sqrt{|h|}}\left(\sqrt{|h|}h^{ij}\phi_{*i}\right)_{*j}$$

where h_{ij} and k_{ij} are the threading and slicing metrics respectively. As usual, Greek indices run over all four spacetime coordinates, Latin indices run over the three "spatial" coordinates, and the threading vector field is assumed to be $\frac{\partial}{\partial t}$. Also, let m^α be the unit vector tangent to the threading curves and n^α be the unit vector normal to the slicing surfaces. Continuing with the earlier notation, let M and N be the lapse functions and M_i and N^i the components of the shift one-form and vector fields respectively. If $M_i \equiv 0 \equiv N^i$, then direct computation shows that

$$\begin{aligned}
\Box\phi &= \triangle_k\phi + \frac{1}{N}k^{ij}N_{,i}\phi_{,j} - n^\beta\nabla_\beta(n^\alpha\nabla_\alpha\phi) - \frac{\dot{\sqrt{|k|}}}{N\sqrt{|k|}}n^\alpha\nabla_\alpha\phi \\
\Box\phi &= \triangle_h\phi + \frac{1}{M}h^{ij}M_{,i}\phi_{,j} - m^\beta\nabla_\beta(m^\alpha\nabla_\alpha\phi) - \frac{\dot{\sqrt{|h|}}}{M\sqrt{|h|}}m^\alpha\nabla_\alpha\phi \\
\Box\phi &= \triangle_*\phi + \frac{1}{M}h^{ij}\phi_{*j}M_{*i} - m^\beta\nabla_\beta(m^\alpha\nabla_\alpha\phi) - \frac{\dot{\sqrt{|h|}}}{M\sqrt{|h|}}m^\alpha\nabla_\alpha\phi
\end{aligned} \tag{5.2}$$

where ∇ is the spacetime Levi-Civita connection, and a dot denotes differentiation with respect to t. These equations are valid regardless of the Killing condition on the vector field $\frac{\partial}{\partial t}$. In the case of a static spacetime, the last term in each of the above expressions will vanish.

One should note that the above equation (5.2) are all identical, since in the hypersurface-orthogonal case we have $M \equiv N$, $m^\alpha \equiv n^\alpha$, $h_{ij} \equiv k_{ij}$, and $\partial_{*i} \equiv \partial_i$, thus making all three decompositions coincide.

5.2.3 Non-Hypersurface-Orthogonal Decompositions

When the threading curves are no longer orthogonal to the slicing hypersurfaces, shift is present. The presence of shift implies that the lapse functions M and N are different, $h_{ij} \neq k_{ij}$, $m^\alpha \neq n^\alpha$, and $\partial_{*i} \neq \partial_i$. In such a case we have

$$\begin{aligned}
\Box\phi =& \triangle_k \phi + \frac{1}{N} k^{ij} N_{,i} \phi_{,j} - n^b \nabla_b (n^a \nabla_a \phi) - \frac{\dot{\sqrt{|k|}}}{N\sqrt{|k|}} n^a \nabla_a \phi \\
& + \frac{1}{N} n^a \nabla_a \phi D_i N^i \\
\Box\phi =& \triangle_h \phi - m^b \nabla_b (m^a \nabla_a \phi) + \frac{1}{M} h^{ij} \phi_{,j} M_{,i} - \frac{\dot{\sqrt{|h|}}}{M\sqrt{|h|}} m^a \nabla_a \phi \\
& + M m^\alpha \nabla_\alpha \phi \hat{D}_i M^i \\
\Box\phi =& \triangle_* \phi + \frac{1}{M} h^{ij} \phi_{*j} (M_{*i} + M \dot{M}_i) - m^b \nabla_b (m^a \nabla_a \phi) \\
& - \frac{\dot{\sqrt{|h|}}}{M\sqrt{|h|}} m^a \nabla_a \phi
\end{aligned}$$

where D and $\hat{D}$ are the Levi-Civita connections associated with the slicing and threading metrics respectively. Again, a dot denotes differentiation with respect to t, so if $\frac{\partial}{\partial t}$ is assumed to be a Killing vector, some of the above terms are identically zero. Note that only the parametric decomposition contains no divergence term, which should simplify the quantization procedure.

5.3 Initial-Value Formulation in General Relativity

The standard initial-value formulation of general relativity begins with a set of initial data consisting of a three-dimensional Riemannian manifold Σ (with metric k) together with a symmetric rank-two tensor field K on Σ. It is possible (*c.f.* [30]) to use the Gauss-Codazzi equations to obtain the initial-value constraints

1. $D_i K^i{}_j - D_j K^i{}_i = 0$ and
2. $\frac{1}{2}\left({}^3R + \left(K^i{}_i\right)^2 - K_{ij}K^{ij}\right) = 0$

where D is the Levi-Civita connection associated with k and ${}^3R$ is the Ricci scalar. If k and K satisfy the initial value constraints, it can be shown that there exists a globally hyperbolic spacetime satisfying Einstein's equation which admits a Cauchy surface diffeomorphic to Σ. That is, there exists a spacetime with an embedded spacelike surface diffeomorphic to Σ possessing the property that the past and future of this *Cauchy surface* is the entire spacetime $\mathcal{M}$. Furthermore, the induced metric on Σ is k and the induced extrinsic curvature of the Cauchy surface is K (see [30]). Recall that if n is the unit vector normal to Σ and if X and Y are tangent vector fields on Σ, the extrinsic curvature K is defined by

$$K(X,Y) = \langle -\nabla_X Y, n \rangle$$

where $\langle\ ,\ \rangle$ is the metric of the spacetime. If one assumes that ∇ is torsion-free, then K is a symmetric tensor.

In the generalized Gauss-Codazzi formalism, the generalized extrinsic curvature operator was no longer symmetric. However, the anti-symmetric part of K was measured by the deficiency, a well-defined quantity on a parametric manifold. Furthermore, we saw that the non-symmetric nature of K did not affect a re-formulation of Gauss' equation, the first of the initial-value constraints. The theory of parametric manifolds should allow one to first treat the generalized Gauss-Codazzi equations as initial-value constraints and second, to formulate a theorem similar to the standard case.

Such an approach to an initial-value formulation requires one to use the curvature operator $^{\perp}R$, rather than the Zel'manov curvature Z, as only $^{\perp}R$ satisfied Gauss' equation.

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