With the development of converters and inverters, it became possible to control the stator current of an induction motor. The differential equations which describe transient behavior of the induction motor are conveniently expressed by transforming the stator and rotor variables to d-q axes rotating at synchronous speed. The transient behavior of a current-controlled induction motor is improved by optimizing the state equations of the system with respect to a cost function. A first-order gradient method is used to obtain numerical solutions to this optimization problem.
OPTIMAL CONTROL OF AN INDUCTION MOTOR

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Meaning of Subscript

d, q \quad \text{Equivalent two-phase transformed variable as in } i_{qs}
s \quad \text{Stator quantity as in } L_s
r \quad \text{Rotor quantity as in } L_r

Variables

\begin{align*}
\begin{array}{lcl}
i_{\alpha s}, i_{\beta s} & \text{Three-phase stator current variables} \\
i_{\alpha r}, i_{\beta r} & \text{Three-phase rotor current variables} \\
i_{as}, i_{bs} & \text{Two-phase } a, b \text{ axis motor stator current variables} \\
i_{ar}, i_{br} & \text{Two-phase } a, b \text{ axis motor rotor current variables} \\
i_{ds}, i_{qs} & \text{Two-phase } d, q \text{ axis motor stator current variables} \\
i_{dr}, i_{qr} & \text{Two-phase } d, q \text{ axis motor rotor current variables} \\
v_{\alpha s}, v_{\beta s} & \text{Three-phase stator voltage variables} \\
v_{\alpha r}, v_{\beta r} & \text{Three-phase rotor voltage variables} \\
v_{as}, v_{bs} & \text{Two-phase } a, b \text{ axis motor stator voltage variables} \\
v_{ar}, v_{br} & \text{Two-phase } a, b \text{ axis motor rotor voltage variables} \\
v_{ds}, v_{qr} & \text{Two-phase } d, q \text{ axis motor stator voltage variables} \\
v_{dr}, v_{qr} & \text{Two-phase } d, q \text{ axis motor rotor voltage variables} \\
L_s, L_r & \text{Self inductance of stator and rotor} \\
R_s, R_r & \text{Phase resistances of stator and rotor} \\
M & \text{Mutual inductance between stator and rotor}
\end{array}
\end{align*}
\( n_p \) Number of pole pairs
\( p = \frac{d}{dt} \) Differential operator
\( J \) Machine and load inertia
\( B \) Friction coefficient
\( w_r \) Rotor speed
\( w_s \) Stator current angular speed
\( T_r \) Load torque
\( H \) Hamiltonian
\( t_0, t_f \) Initial time and terminal time
\( \lambda_1, \lambda_2, \lambda_3 \) Lagrange multipliers
OPTIMAL CONTROL OF AN INDUCTION MOTOR

SECTION 1. INTRODUCTION

1.1 General

The induction motor represents one of the most useful design forms of a-c powered electromechanical rotating machines. This device can be constructed with no physical connections between the stator and rotor circuits. The currents in the rotor are generated because of the magnetic coupling between the stator and rotor, and the motor becomes extremely rugged, dependable, and almost maintenance free.

Variable-speed drives built using AC motors and static converters have been developed considerably during the last two decades. With the development of inverters, it became possible to control the stator current and frequency in induction motors. There is an extensive literature on the performance of inverters.3,4,5,10,16 After the development of inverters, an extensive study on induction motors controlled with stator voltage, frequency and current has been made. These studies are roughly classified into the following categories: (i) stability performance2,5,6,7; (ii) computer simulation of motor equations8,10; (iii) optimal control9,11; (iv) speed control.4

In the papers about stability performance, a dq model is developed and nonlinear equations are linearized around
the operating points. Transfer functions related to these linearized equations are found and stability performance is examined for various control strategies such as current control, frequency control, and voltage control.\(^2\) The boundness of the solutions of the bilinear and nonlinear differential equations describing the dynamic behavior of an ideal three-phase induction motor is shown by Dote\(^6\) using a Lyapunov function. It is then shown with a computer simulation that the machine has a limit cycle. Utilizing these results, a bilinear and nonlinear state observer is constructed to estimate the unmeasurable state variables.

Computer simulation of induction motor is important to find numerical solutions to the nonlinear motor equations. Jordan\(^8\) found the transient and steady-state performance of the motor from its equivalent circuit using a predictor-corrector type of differential equation solving algorithm.

The optimal control of a voltage driven variable reluctance stepping motor, with an inertia load, was studied by Jones.\(^9\) A solution for the optimal control policy is achieved by a conjugate gradient method in function state, optimizing the response of an established nonlinear motor model.

The speed control of the induction motor is possible by using a variable stator frequency. Wang et al\(^4\) studied the speed control of an induction motor fed by a current source inverter using linearized state equations with the control functions chosen as frequency and load torque.
1.2 Purpose of Study

The motion of the induction motor at any speed requires a three-phase current or voltage source with variable frequency and amplitude. An important operating mode of induction motors is accompanied with controlled stator currents. The stator currents are directed on the motor either by feedback or a current source inverter. Assuming the stator currents as input variables of the motor, only the dynamics of the rotor motion are required. For the analysis this system is nonlinear due to the several products of system variables. Therefore it is not possible to obtain a closed solution of these equations.

The transient response of the system is improved by optimizing the time-invariant differential equations of current controlled induction motor respect to a performance index. An equivalent motor called a-b machine is derived by using the three-phase to two phase transformation. This motor is a straightforward model of the two-phase servomotor which is commonly used in positioning control systems and servomechanisms. The two phase a-b model can most easily be analyzed in terms of a d-q machine obtained by using the a-b to d-q transformation. The resulting third order state equations are solved with different parameters using the fourth order Runge-Kutta method to examine the parameter sensitivity of the system. These equations are also solved to study the effect of stator current, stator frequency and...
load torque on dynamic of induction motor. An optimization method using the first order gradient technique is applied to the induction motor equations to obtain the optimum transient response of induction motor.
SECTION 2. STATE EQUATIONS OF THE THREE-PHASE INDUCTION MACHINE

2.1 Primitive Machine Model of The Three-Phase Induction Machine

The induction machine is usually constructed with a uniform air gap; thus both rotor and stator are nonsalient structures. The three stator windings and the three rotor windings are sinusoidally distributed with equal distribution factors \( K_s \) and \( K_r \) turns/m. The axes of the stator windings are separated by \( 2\pi/3n \) radians, where \( n_p \) is the number of pole pairs. The axes of the rotor windings are also separated by \( 2\pi/3n \) radians.

The three-phase stator is usually connected to a balanced three-phase sinusoidal voltage supply known as a three-phase infinite bus. The rotor windings are either externally short-circuited through the slip rings or connected to a balanced three-phase load. In this paper we will assume that the rotor windings are short-circuited.

The three-phase machine pictured in Fig.2-1 can most easily be analyzed by transforming it to an equivalent two phase a-b model machine, as shown in Fig.2-2. The a-b machine has two identical stator windings and two identical rotor windings which are sinusoidally distributed. Notice that the stator windings on the a-b model are identical with the three stator windings on the original three-phase
Figure 2-1. The three-phase wound rotor induction machine
Figure 2-2. Equivalent two-phase a-b model of the induction machine
machine. Similarly, the two rotor windings have exactly the same construction as the rotor windings on the three-phase machine.

The three-phase variables can be transformed to an equivalent set of two-phase variables by using the $\alpha-\beta-\gamma$ to $a-b$ transformation given the equation which is known as transformation matrix

$$[Q_{ab}] = \begin{bmatrix}
1 & -1/2 & -1/2 & 0 & 0 & 0 \\
0 & -\sqrt{3}/2 & \sqrt{3}/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1/2 & -1/2 \\
0 & 0 & 0 & 0 & -\sqrt{3}/2 & \sqrt{3}/2
\end{bmatrix}$$

(2.1)

and the three-phase currents are transformed according to the matrix equation

$$[i_{ab}] = [Q_{ab}][i_{\alpha\beta\gamma}]$$

(2.2)

where

$$[i_{ab}] = \begin{bmatrix} i_{as} \\
i_{bs} \\
i_{ar} \\
i_{br} \end{bmatrix} \quad [i_{\alpha\beta\gamma}] = \begin{bmatrix} i_{\alpha s} \\
i_{\beta s} \\
i_{\gamma s} \\
i_{\alpha r} \\
i_{\beta r} \\
i_{\gamma r} \end{bmatrix}$$

(2.3)
The three-phase voltage variables can also be transformed similarly

\[ [v_{ab}] = [Q_{ab}][v_{\alpha\beta}] \]  \hspace{1cm} (2.4)

where

\[
[v_{ab}] =
\begin{bmatrix}
v_{as} \\
v_{bs} \\
v_{ar} \\
v_{br}
\end{bmatrix} \hspace{1cm} [v_{\alpha\beta}] =
\begin{bmatrix}
v_{\alpha s} \\
v_{\beta s} \\
v_{r s} \\
v_{\alpha r} \\
v_{\beta r} \\
v_{r r}
\end{bmatrix}
\]  \hspace{1cm} (2.5)

The analysis of the induction machine is greatly simplified if we transform the a-b model to an equivalent d-q model, as shown in Fig.2-3. The equivalent d-q variables are obtained from the a-b variables according to the matrix equations

\[ [i_{dq}] = [Q_{dq}][i_{ab}] \]  \hspace{1cm} (2.6)

\[ [v_{dq}] = [Q_{dq}][v_{ab}] \]  \hspace{1cm} (2.7)

where

\[
[i_{dq}] =
\begin{bmatrix}
i_{ds} \\
i_{qs} \\
i_{dr} \\
i_{qr}
\end{bmatrix} \hspace{1cm} [v_{dq}] =
\begin{bmatrix}
v_{ds} \\
v_{qs} \\
v_{dr} \\
v_{qr}
\end{bmatrix}
\]  \hspace{1cm} (2.8)
Figure 2-3. Equivalent d-q model of the induction machine
The a-b to d-q transformation is defined as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \alpha & -\sin \alpha \\
0 & 0 & \sin \alpha & \cos \alpha
\end{bmatrix}
\]  

(2.9)

It can be seen from the matrix equation above that the transformation does not change stator variables. However, the rotor variables are transformed in such a fashion that the d-q machine using a commutator and brush mechanism has a total rotor surface current density that is identical with the total rotor surface current density of the a-b machine using slip ring connections.

In construction the stator and the rotor windings of the d-q machine are respectively identical with the stator and rotor windings of the a-b machine. The differential equations of the d-q primitive-machine model are given in general by

\[
\begin{bmatrix}
v_{ds} \\
v_{qs} \\
v_{dr} \\
v_{qr}
\end{bmatrix}
= 
\begin{bmatrix}
R_s + L_s p & 0 & M_p & 0 \\
0 & R_s + L_s p & 0 & M_p \\
M_p & G_{rs} w_r & R_r + L_r p & G_{rr} w_r \\
-G_{sr} w_r & M_p & -G_{rr} w_r & R_r + L_r p
\end{bmatrix}
\begin{bmatrix}
i_{ds} \\
i_{qs} \\
i_{dr} \\
i_{qr}
\end{bmatrix}
\]  

(2.10)
The mechanical port equation is given by

\[ T_r = (Jp + D)w_r + T_e \] (2.11)

where

- \( T_r \) = externally applied torque
- \( J \) = total polar moment of inertia of the rotor
- \( D \) = angular viscous friction coefficient
- \( T_e \) = torque of electrical origin
- \( R_s \) = stator phase resistance
- \( R_r \) = rotor phase resistance
- \( L_s = (3/2)L_1 \) (\( L_1 \) is the stator phase inductance)
- \( L_r = (3/2)L_2 \) (\( L_2 \) is the rotor phase inductance)
- \( M = (3/2)M_0 \) (\( M_0 \) is the mutual inductance between stator and rotor)
- \( G_{rs} = n_p M \) (rotational inductance between rotor and stator) (2.12)
- \( G_{rr} = n_p L_r \) (rotational inductance in rotor) (2.13)

\( T_e \) is given by

\[ T_e = G_{rs}i_ds^iqr - G_{rs}i_qsi_dr \] (2.14)

Substituting the simplifications of Eqs. (2.12), (2.13) and (2.14) into (2.10) and (2.11) results in
\[
\begin{bmatrix}
  v_{ds} \\
  v_{qs} \\
  v_{dr} \\
  v_{qr}
\end{bmatrix}
= 
\begin{bmatrix}
  R_s + L_s p & 0 & M_p & 0 \\
  0 & R_s + L_s p & 0 & M_p \\
  M_p & n_p M w_r & R_r + L_r p & n_p L_r w_r \\
  -n_p M w_r & M_p & -n_p L_r w_r & R_r + L_r p
\end{bmatrix}
\begin{bmatrix}
  i_{ds} \\
  i_{qs} \\
  i_{dr} \\
  i_{qr}
\end{bmatrix}
\] (2.15)

\[T_r = (J_p + D) w_r + M (i_{ds} i_{qr} - i_{qs} i_{dr})\] (2.16)

Equations (2.15) and (2.16) represent a set of nonlinear coupled differential equations describing the operation of the d-q model of the induction machine.

2.2 State Equations of the Current-Controlled Induction Machine

Assuming that the rotor circuit of the induction machine is short circuited, the port voltages can be set equal to zero:

\[v_{\omega r} = v_{\beta r} = v_{rr} = 0\] (2.17)

By substituting (2.17) into (2.4), we obtain for the equivalent a-b rotor voltages

\[v_{ar} = v_{br} = 0\] (2.18)
Similarly by use of the a-b to d-q transformation equation (2.7), the equivalent d-q rotor voltages

\[ v_{dr} = v_{qr} = 0 \]  

(2.19)

for normal induction machine operation. In the analysis of the stator-current-controlled motor, it is assumed that the stator current is sinusoidal and given by

\[ i_{aS} = I_m \cos \omega_s t \]
\[ i_{bS} = I_m \cos (\omega_s t + 2\pi/3) \]
\[ i_{rS} = I_m \cos (\omega_s t - 2\pi/3) \]  

(2.20)

where \( I_m \) is the peak value of stator phase current and \( \omega_s \) is the angular frequency of stator phase current.

By using the stator portion of the \( \alpha-\beta-\gamma \) to a-b transformation given by equation (2.2), the equivalent two-phase stator currents are found to be

\[ i_{aS} = I_m \cos \omega_s t \]
\[ i_{bS} = I_m \sin \omega_s t \]  

(2.21)

The stator portion of the a-b to d-q transformation, given by (2.6), shows that
It is assumed that stator currents are sinusoidal. As it can be seen from the equation (2.22), the d-q currents depend on the peak value of stator current \( I_m \) and the stator current frequency \( w_s \). It is very difficult to analyze the state equations (2.15) and (2.16) with the control variables (2.22) in this form. For this reason we need to introduce another transformation which eliminates the sinusoidal part of control currents. This is accomplished by choosing a transformation matrix as

\[
[Q_{DQ}] = \begin{bmatrix}
  \cos w_s & \sin w_s & 0 & 0 \\
  -\sin w_s & \cos w_s & 0 & 0 \\
  0 & 0 & \cos w_s & \sin w_s \\
  0 & 0 & -\sin w_s & \cos w_s
\end{bmatrix}
\]  

(2.23)

D-q currents and voltages are transformed according to the matrix equations

\[
[i_{DQ}] = [Q_{DQ}][i_{dq}]
\]  

(2.24)

\[
[v_{DQ}] = [Q_{DQ}][v_{dq}]
\]  

(2.25)

where
\[ [i_{DQ}] = \begin{bmatrix} i_{Ds} \\
Qs \\
i_{Dr} \\
i_{Qr} \end{bmatrix} \quad [v_{DQ}] = \begin{bmatrix} v_{Ds} \\
Qs \\
v_{Dr} \\
v_{Qr} \end{bmatrix} \quad (2.26) \]

By substituting (2.23), (2.24) and (2.25) into equations (2.15) and (2.16), the differential equations of induction motor are found in matrix form to be:

\[
\begin{bmatrix} v_{Ds} \\
Qs \\
v_{Dr} \\
v_{Qr} \end{bmatrix} = \begin{bmatrix} R_r & -w_sL_s & 0 & -w_sM \\
w_sL_s & R_s & w_sM & 0 \\
0 & M(n_pw_r - w_s) & R_r & L_r(n_pw_r - w_s) \\
M(w_s - n_pw_r) & 0 & L_r(w_s - n_pw_r) & R_r \end{bmatrix} \begin{bmatrix} i_{Ds} \\
Qs \\
i_{Dr} \\
i_{Qr} \end{bmatrix} \quad (2.27)
\]

and the mechanical port equation

\[ T_r = Jp w_r + D w_r + n_pM(i_{Ds} i_{Qr} - i_{Qs} i_{Dr}) \quad (2.28) \]

where

\[ i_{Ds} = I_m \quad (2.29) \]
\[ i_{Qs} = 0 \]
Assuming that the stator current is supplied with an ideally controllable current source, only the dynamics in rotor circuit and rotor motion are required. Disregarding the stator equations in (2.27), the differential equations are

\[
\begin{align*}
\frac{di_{Dr}}{dt} &= -\left(\frac{M}{L_r}\right) \frac{di_{Ds}}{dt} + \left(\frac{M}{L_r}\right) (w_s - n_p w_r) i_{Qs} - \left(\frac{R}{L_r}\right) i_{Dr} + \left(\frac{w_s - n_p w_r}{L_r}\right) i_{Qr} \\
\frac{di_{Qr}}{dt} &= -\left(\frac{M}{L_r}\right) (w_s - n_p w_r) i_{Ds} - \left(\frac{M}{L_r}\right) \frac{di_{Qs}}{dt} - \left(\frac{w_s - n_p w_r}{L_r}\right) i_{Dr} - \left(\frac{R}{L_r}\right) i_{Qr} \\
\frac{dw_r}{dt} &= -\left(\frac{n_p M}{J}\right) i_{Qr} i_{Ds} + \left(\frac{n_p M}{J}\right) i_{Qs} i_{Dr} - \left(\frac{D}{J}\right) w_r + \left(\frac{1}{J}\right) T_r
\end{align*}
\]

2.2.1 State-Space Representation of the Current-Controlled Induction Motor

In equations (2.30) define

\[
\begin{align*}
i_{Dr} &= Y_1 \\
i_{Qr} &= Y_2 \\
w_r &= Y_3 \\
i_{Ds} &= u_1 = I_m \\
w_s &= u_2 = \omega
\end{align*}
\]

Using the above in equation (2.30), we find
\[ y_1 = -\frac{R_r}{L_r} y_1 + u_2 y_2 - n_p y_2 y_3 - \frac{M}{L_r} u_1 \]  
\[ y_2 = -u_2 y_1 + n_p y_1 y_3 - \frac{R_r}{L_r} y_2 + \frac{n_p M}{L_r} y_1 y_3 - \frac{M}{L_r} u_1 u_2 \]  
\[ y_3 = -\frac{n_p M}{J} y_1 y_2 - \frac{D}{J} y_3 + \frac{1}{J} T_r \]  

(2.32)

To eliminate \( u_1 \) and \( u_1 u_2 \) from equations above, new variables defined as

\[ x_1 = y_1 + \frac{M}{L_r} u_1 \]
\[ x_2 = y_2 \]
\[ x_3 = y_3 \]  

(2.33)

It should be noted that this change of variables is equivalent to specifying rotor flux linkages rather than currents as the state variables. Finally the state equations of induction motor are

\[ x_1 = -\frac{R_r}{L_r} x_1 + u_2 x_2 - n_p x_2 x_3 + \left( \frac{R_r M}{L_r^2} \right) u_1 \]  
\[ x_2 = -u_2 x_1 + n_p x_1 x_3 - \frac{R_r}{L_r} x_2 \]  
\[ x_3 = -\frac{n_p M}{J} u_1 x_2 - \frac{D}{J} x_3 + \frac{1}{J} T_r \]  

(2.34)
2.2.2 Parameter Sensitivity

The performance of an induction motor depends upon its parameters. As the motor ratings go up, saturation, temperature, and harmonics start affecting the parameters significantly. As an example of parameter sensitivity, the effect of rotor resistance variation on motor speed is examined in this section.

Consider the system equations
\[ x = f[x(t), u(t), p, t] \] (2.35)
where \( x(t) \) is an \( n \)-dimensional state vector, \( u(t) \) is an \( m \)-dimensional control vector and \( p \) is an \( r \)-dimensional parameter vector. We define the dynamic sensitivity coefficients as
\[ s = \frac{\partial x}{\partial p} \] (2.36)
where \( s \) is an \((nxr)\)-dimensional parameter sensitivity matrix. By partial differentiation of equation (2.35) with regard of one parameter \( p \), we obtain the sensitivity equation of the system
\[ \dot{s} = \frac{\partial f}{\partial x} s + \frac{\partial f}{\partial p} \] (2.37)
The equation (2.36) can be approximated as
\[ s = \frac{\Delta x}{\Delta p} \] (2.38)
In the analysis to see the effect of resistance variations on the motor speed, it is assumed that the rotor resistance is increased by 8% as a result of an increase in the operating temperature of the motor. Equations (2.34) are solved for this case and results are given in figure 2-4.
using approximated sensitivity equation (2.38). In figure 2-4(a), the speed of the motor is shown before and after resistance change. Figure 2-4(b) shows the effect of resistance change on speed in the time domain. These results show that a variation in the rotor circuit changes the transient response significantly. For this reason, it is essential to provide a feedback type controller to compensate the effect of the resistance variation.

### 2.2.3 Parameter Transformation

In order to decrease the computation time in computer, the differential equations of induction motor are transformed to another state with one parameter. In the motor equations, the friction coefficient is assumed to be close enough to zero so that we can neglect it and following new variables are introduced:

\[
\begin{align*}
    x_1^* &= \frac{R_r}{L_r} x_1 \\
    u_1^* &= \frac{R_r M}{L_r^2} u_1 \\
    x_2^* &= \frac{R_r}{L_r} x_2 \\
    u_2^* &= \frac{L_r}{R_r} u_2 \\
    x_3^* &= \frac{n_p L_r}{R_r} x_3 \\
    T_r^* &= \frac{J R_r^2}{n_p L_r^3} T_r \\
    t^* &= \frac{R_r}{L_r} t
\end{align*}
\] (2.39)
Plot of speed $(x3)$ vs time $(t)$ for rotor resistances $0.176(a)$ and $0.22(b)$

Figure 2-4(a). Transient rotor speed for variable rotor resistance

Plot of speed sensitivity function vs time

Figure 2-4(b). Speed sensitivity function for variable resistance
These new variables are substituted into the equations (2.34):

\[
\begin{align*}
\frac{dx_1^*}{dt^*} &= -x_1^* + u_2^*x_2^* - x_2^*x_3^* + u_1^* \\
\frac{dx_2^*}{dt^*} &= -u_2^*x_1^* + x_1^*x_3^* - x_2^* \\
\frac{dx_3^*}{dt^*} &= A(-u_1^*x_2^* + T_r^*)
\end{align*}
\] (2.40)

The parameter A in the new system is given as

\[
A = \frac{n_p^2 L_r^5}{JR_r^4}
\] (2.41)

Figure 2-5 shows the dynamic structure of the system in a block diagram.

Figure 2-6, 2-7 and 2-8 show the graphical output of normalized rotor speed for various values of normalized control current, control frequency and load torque. In this simulation, the system parameter A is kept constant. These figures show that the transient time decreases while control current increases; on the other hand oscillations become larger. The effect of the stator frequency can be observed from the simulation results. The speed control is possible with the variations in the stator angular frequency. The load torque also has an effect on the transient time of induction motor. An increase on the load torque causes an
Figure 2-5. Block diagram of the current-controlled induction motor
Figure 2-6. The transient rotor speed for variable control current
Plot of rotor speed (x3) vs time (t)

$u_1=300$, $u_2=0.67$ (a), 1 (b), 1.33 (c)

Figure 2-7. The transient rotor speed for variable control frequency
Plot of speed (x3) vs time (t)

\[ u_1 = 1, \; u_2 = 1, \; T_r = -0.13 \; (a), \; 0 \; (b), \; 0.13 \; (c) \]

Figure 2-8. The transient rotor speed for variable load torque
increase on the transient time. In the next section, optimal control functions that minimize the oscillations will be found using the first order gradient technique.
SECTION 3. OPTIMAL CONTROL OF THE INDUCTION MOTOR
TO IMPROVE OPTIMUM TRANSIENT BEHAVIOR

A strategy for computation of optimal control functions is presented in this section. Then the principle of the iteration process based on the gradient technique is described. Next, first order gradient technique is applied to the differential equations of the current-controlled induction motor. The optimum control functions are found for starting, acceleration and deceleration process.

3.1 The Structure of The System Analyzed

In order to assess or design control loops, it is necessary to have a simple but reliable model for the inverter, induction motor, and any inner loops which might be considered as basic to drive. Figure 3-1 shows an example of a current source inverter which might be used to control the stator current and frequency of an induction motor with some modifications for fast current response. This drive is capable of recovery from short circuits or commutation faults. It offers inherent overcurrent protection when current feedback is used.

The idealized model used for this analysis contains the conventional assumptions such as sinusoidal distribution of magnetomotor force in the air gap, negligible saturation, negligible iron losses and sinusoidal stator currents.
Figure 3-1. Current source inverter
3.2 The Strategy of Optimization

Optimal control problems for continuous system may be considered as limiting cases of optimal programming problems for multistage systems in which the time increment between steps becomes small compared to times of interest. Actually, the reverse procedure is more common today: Continuous systems are approximated by multistage systems for solutions on digital computers. Consider the system described by the following nonlinear differential equations:

\[
\dot{x} = f[x(t), u(t), t]; \quad x(t_0) \text{ given, } t_0 \leq t \leq t_f \quad (3.1)
\]

where \(x(t)\), an \(n\)-vector function is determined by \(u(t)\), an \(m\)-vector function, constrained by

\[
C[u(t), t] \leq 0 \quad (3.2)
\]

Consider a performance index that is minimized or maximized of the form

\[
J = \Phi_1[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t]dt \quad (3.3)
\]

subject to the constraints on terminal state

\[
\Phi_2[x(t_f), t_f] = 0 \quad (q\text{-equations}) \quad (3.4)
\]

The problem is to find the control functions \(u(t)\) that minimize or maximize the performance index \(J\). One of the approaches to solve this problem is the maximum principle of
Pontryagin. Let us define the Hamiltonian using equations (3.1), (3.2), (3.3) and (3.4), introducing the n-undetermined multipliers, $\psi^t = [\psi_1, \psi_2, \ldots, \psi_n]$, and $\alpha$ as follows

$$H[x(t), u(t), y(t), t] = L[x(t), u(t), t] + \psi^t f[x(t), u(t), t] + \alpha C[u(t), t]$$  \hspace{1cm} (3.5)$$

with the requirement that

$$\alpha \begin{cases} > 0, & C = 0 \\ = 0, & C < 0 \end{cases}$$ \hspace{1cm} (3.6)$$

A set of necessary conditions for $J$ to have a stationary value is (derivation of these conditions is given in Appendix A)

$$\dot{x}(t) = f[x, u, t]$$ \hspace{1cm} (3.7)$$

$$\dot{\psi}(t) = -(\partial H/\partial x)^t - (\partial f/\partial x)^t - (\partial L/\partial x)^t$$ \hspace{1cm} (3.8)$$

$$0 = (\partial H/\partial u)^t = (\partial f/\partial u)^t + (\partial L/\partial u)^t + (\partial C/\partial u)\alpha$$ \hspace{1cm} (3.9)$$

$$C(u, t) \leq 0$$ \hspace{1cm} (3.10)$$

$x_k(t_0)$ is given or $\psi_k(t_0) = 0$ \hspace{1cm} (3.11)$$

$$\psi(t_f) = [\partial \phi_1/\partial x + v(\partial \phi_2/\partial x)]^t$$ \hspace{1cm} (3.12)$$

$$\Omega = (\partial \phi_1/\partial t + v^t(\partial \phi_2/\partial t) + [(\partial \phi_1/\partial x) + v^t(\partial \phi_2/\partial x)]f + L)|_{t=t_f} = 0$$ \hspace{1cm} (3.13)$$

$$\phi_2[x(t_f), t_f] = 0$$ \hspace{1cm} (3.14)$$

For $C<0$, we have $\alpha = 0$ and the optimality condition (3.9) determines the m-vector $u(t)$. For $C=0$, equation (3.10) and
(3.9) determines \( u \) and \( \alpha \) respectively. The solution to the 
2n differential equations (3.7) and (3.8) and the choice of 
q parameters \( v \) are determined by the 2n+q boundary 
conditions (3.11), (3.12) and (3.14). This two-point 
boundary problem is, in general, not easy to solve. However 
if we specify \( v \) instead of \( \phi_2 \), (3.11) and (3.12) provide 2n 
boundary conditions for a fixed terminal time, it becomes 
two-point boundary-value problem of order 2n. If we change 
the values of \( v \), it may be possible to bring \( \phi_2 \) to zero at 
time \( t=t_f \).\(^{12} \)

If \( C[u(t),t] \leq 0 \) had been a vector function with \( s \) 
components, equations (3.5) and (3.9) would remain 
applicable if we replace \( \alpha C \) and \( (dC/du)\alpha \) by \( \alpha^t C \) and 
\( (dC/du)\alpha^t \) respectively.

3.2.1 Numerical Solutions for Optimization Problems

Using a First-Order Gradient Technique

Mathematically, the nonlinear programming methods 
belong to one of two categories. The first category of 
methods utilizes the derivative information of the objective 
function, (known as the gradient) to search for descent 
direction in order to decrease the value of the objective 
function. The second category of methods calculates the 
values of the objective function without computing the 
derivatives, followed by comparison of each of the proposed 
values of the objective function to find a better value of
Because of the complexity of the objective function both categories of methods may require many iterations, and time consumption to solve the problem is very high depending on the number of state variables.

Gradient methods are divided into two categories, called first and second order Gradient techniques. First order gradient methods usually show great improvements in the first few iterations but have poor convergence characteristics as the optimal solution is approached. Second order gradient methods use information on the curvature as well as the slope at the nominal point in the u-space. They have excellent convergence characteristics as the optimal solution is approached but may have starting difficulties. In this thesis, first order gradient technique is used to obtain the optimum transient behavior of induction motor.

The steps in using the first order gradient method are as follows:

(a) Guess a set of values for input functions \( u(t) \), \( t \in [t_0, t_f] \)

(b) Determine the values of \( x \) from \( x - f(x, u, t) = 0 \)

(c) Determine the values of \( \dot{\psi}(t) \) from \( \dot{\psi}(t) = -(\partial H/\partial x) \)

(d) Determine the values of \( (\partial H/\partial u) \) which in general will not be zero.

(e) Interpreting \( (\partial H/\partial u) \) as a gradient vector, change the estimates of \( u \) by amounts \( \delta u = -K(\partial H/\partial u)^T \), where \( K \) is a positive scalar
constant. (If maximum is being sought, "-" is replaced by a "+".)

(f) Assuming $u$ is constrained as $u_{\text{min}} \leq u \leq u_{\text{max}}$, choose $u = u_{\text{min}}$ if $u$ becomes smaller than $u_{\text{min}}$ or choose $u = u_{\text{max}}$ if $u$ exceeds $u_{\text{max}}$.

(g) Repeat steps (a) through (f), using the revised estimates of $u$, until

$$\|{(\Delta H/\Delta u)}\| = \int_{t_0}^{t_f} (\Delta H/\Delta u)(\Delta H/\Delta u)^t dt$$

is very small.

The determination of values of $x$ usually requires numerical solutions. There are many numerical solution methods to solve nonlinear differential equations such as Euler's methods, high order Taylor methods, Runge-Kutta methods, Predictor-Corrector methods. Procedures for the solutions of ordinary differential equations can be found in very extensive literature.$^{14,15}$ In this thesis, a fourth order Runge-Kutta method is used to solve the nonlinear differential equations for the current controlled induction motor. A procedure for solving a system of ordinary differential equations is given in Appendix B.

The choice of $K$ determines the magnitude of $u$. If we take $K$ larger, we do not need so many steps to reach maximum (or minimum.) But if $K$ is too large, the iteration process may become unstable and the solutions may not converge.
3.3 Optimum Control of the Speed of Induction Motor with Controlled Stator Current

We present a method for the computation of the optimum control functions \( u_1^*(t) \) and \( u_2^*(t) \) to find an optimum transient behavior. The following control process constraints are used:

We will obtain a time optimum transient process of the speed \( x_3^*(t) \) from the initial speed \( x_{30} \) to the terminal speed \( x_{3n} \). When the terminal speed \( x_{3n} \) is reached, we demand \( (d/dt)x_3(t)|_{t=t_f} = 0 \). We can write a cost function which satisfies this requirement approximately:

\[
J = \int_{t_0}^{t_f} \xi [x_3^*(t) - x_{3n}]^2 dt
\]  

This cost function also tends to minimize the oscillations of speed in the transient process, so a smooth speed profile can be obtained.

The magnitude of control functions, stator current and stator frequency, are limited because of the design of induction motor. Consequently, these control functions shall be bounded by the values of minimum and maximum stator current and frequency:

\[
I_{\text{min}} \leq u_1^*(t) \leq I_{\text{max}}, \quad \omega_{\text{min}} \leq u_2^*(t) \leq \omega_{\text{max}} \tag{3.16}
\]
The influence factor will be constant during the iteration process. We need some test computations to find the right relative size of this factor.

With the cost function (3.15), and the state equations of induction motor (2.40) with no load torque, the Hamiltonian is:

\[
H[x(t), u(t), \psi(t), t] = 5(x_3^* - x_{3n})^2
+ \psi_1(-x_1^* + u_2^* x_2 - x_2^* x_3 + u_1^*) + \psi_2(-u_2^* x_1 + x_1^* x_3 - x_2^*)
+ \psi_3 A(-u_1^* x_2) + \alpha_1(-u_1^* + I_{\text{min}}) + \alpha_2(u_1^* - I_{\text{max}})
+ \alpha_3(-u_2^* + W_{\text{min}}) + \alpha_4(u_2^* - W_{\text{max}})
\]  

subject to requirements that

\[
\begin{align*}
\alpha_1 &= 0, & \text{if } u_1^* > I_{\text{min}} \\
&> 0, & \text{if } u_1^* = I_{\text{min}} \\
\alpha_2 &= 0, & \text{if } u_1^* < I_{\text{max}} \\
&> 0, & \text{if } u_1^* = I_{\text{max}} \\
\alpha_3 &= 0, & \text{if } u_2^* > W_{\text{min}} \\
&> 0, & \text{if } u_2^* = W_{\text{min}} \\
\alpha_4 &= 0, & \text{if } u_2^* < W_{\text{max}} \\
&> 0, & \text{if } u_2^* = W_{\text{max}}
\end{align*}
\]
The co-state equations of (2.40) are:

\[ \dot{\psi}_1 = -\left(\frac{\partial H}{\partial x_1}\right)^* = \psi_1 + \psi_2 (u_2^* - x_3^*) \]

\[ \dot{\psi}_2 = -\left(\frac{\partial H}{\partial x_2}\right)^* = \psi_1 (x_3^* - u_2^*) + \psi_2 + \psi_3^* u_1 \]

\[ \dot{\psi}_3 = -\left(\frac{\partial H}{\partial x_3}\right)^* = \psi_1 x_2^* - \psi_2 x_1^* + 25(x_3^* - x_{3n}) \]

(3.19)

Optimum control functions \( u_1^*(t) \) and \( u_2^*(t) \) can be found as follows if \( I_{\text{min}} < u_1^* < I_{\text{max}} \) and \( W_{\text{min}} < u_2^* < W_{\text{max}} \)

\[ (\frac{\partial H}{\partial u_1})^* = \psi_1 - A\psi_3 x_2^* \]

\[ (\frac{\partial H}{\partial u_2})^* = \psi_1 x_2^* - \psi_2 x_1^* \]

(3.19)

and following equations are found for the iteration

\[ \delta u_1^* = -K_1 (\psi_1 - A\psi_3 x_2^*) \]

\[ \delta u_2^* = -K_2 (\psi_1 x_2^* - \psi_2 x_1^*) \]

(3.20)

With \( K_1, K_2 = \text{const} > 0 \), the value of the iteration step can be controlled. Initial conditions for state equations and terminal conditions for co-state equations are given as
\[ x_1(t_0) = x_{10} \quad \dot{\psi}_1(t_f) = \left. \frac{d\mathcal{J}}{dx_1^*} \right|_{t=t_f} = 0 \]

\[ x_2(t_0) = x_{20} \quad \dot{\psi}_2(t_f) = \left. \frac{d\mathcal{J}}{dx_2^*} \right|_{t=t_f} = 0 \quad (3.21) \]

\[ x_3(t_0) = x_{30} \quad \dot{\psi}_3(t_f) = \left. \frac{d\mathcal{J}}{dx_3^*} \right|_{t=t_f} = 0 \]

The solution of this optimization problem is computed using a program whose flow chart is given in figure 3-2. The computation process starts with constant control functions. Then system equations of induction motor (2.40) are integrated using fourth order Runge-Kutta method in time direction from \( t_0 \) to \( t_f \). In this simulation, the terminal time is chosen as constant. Next, the co-state equations (3.19) are solved in time direction from \( t_f \) to \( t_0 \) starting with zero terminal conditions. With these results, the value of control functions can be improved using equations (3.20). The computation stops when \( \delta u_1^* \) and \( \delta u_2^* \) become so small that there is no appreciable change from step to step of the iteration.
Data-Input

Starting Control Functions

System Computation

\[ x_1(0), x_2(0), x_3(0) \]

\[ u_1(t) \]

System:

Induction Motor

Time direction 0 \( \rightarrow \) \( t_f \) (forward)

Adjoint System Computation

\[ u_1(t_f), u_2(t_f), u_3(t_f) \]

Time direction \( t_f \) \( \rightarrow \) 0 (backward)

Computation of the iteration process

\[ u_1^{N+1}(t) = u_1^N(t) + \frac{\partial H}{\partial u_1} \cdot K_1 \]

\[ u_2^{N+1}(t) = u_2^N(t) + \frac{\partial H}{\partial u_2} \cdot K_2 \]

No

\[ \left| \frac{\partial H}{\partial u} \right| \leq \varepsilon ? \]

Yes

Optimum is reached

Data-Output

Stop

Figure 3-2. Flowchart of the iteration process
3.3.1 Results of Computations

3.3.1.1 Optimization of the Starting Process

Figure 3-3 shows the computer simulation results for the starting process. The normalized starting input functions are chosen as \( u_1^* = u_2^* = 1 \) in the simulation. Figure 3-3(a) shows large oscillations in the transient speed of induction motor before optimization. Figure 3-3(b) shows that the transient time and oscillations becomes very small after optimization, and we find very smooth function of speed. Figure 3-3(c) and (d) shows the control functions that drive the speed of motor from \( x_3^*(0) = 0 \) to steady state in minimum time with minimum oscillations. The iteration process starts with the input functions \( u_1^*(t) = u_2^*(t) = 1 \). The factors are chosen as \( \delta = 20, K_1 = 0.01 \) and \( K_2 = 0.001 \). The iteration is terminated when \( \delta u_1^* < 0.01 \) and \( \delta u_2^* < 0.05 \). The number of iterations was 150 when the iteration stopped.

3.3.1.2 Optimization of an Acceleration and Deceleration Process

Figure 3-4 shows the optimization of an acceleration process of speed from \( x_3^*(0) = 0.2 \) to \( x_3^*(t_f) = 1 \) without load torque. The motor is in the steady state before and after the acceleration process. Figure 3-4 (a) shows the
speed of motor before optimization with \( u_1 = u_2 = 1 \). Figure 3-4 (b), (c) and (d) show the speed, stator current and frequency (control functions) of the induction motor after optimization. Iteration starts with \( u_1(t) = u_2(t) = 1 \). From these results we can see that the transient behavior of speed improves significantly after optimization. This can also be seen for deceleration process from \( x_3(0) = 1 \) to \( x_3(0) = 0.5 \) as in figure 3-5. Figure 3-5(a) shows the speed of motor before optimization. Figures 3-5(b), (c), and (d) show the transient speed, stator current and frequency after optimizations. Starting functions for iteration were \( u_1(t) = 1, u_2(t) = 0.50 \).

### 3.3.2 Closed-Loop Control Forms for Current Controlled Induction Motor

Optimization results were found for the open loop-control of the current controlled induction motor in the previous section. In practice, however, a closed-loop control is needed to improve the transient behavior of the induction motor. The optimization technique used in the previous section can be applied to the induction motor for closed-loop control. There are many interesting examples of different types of feedback control reported in the literature. The feedback form is usually a function of slip, \( (u_2 - x_3) \), with a variable multiplier. For simplicity let’s choose a feedback form as follows:
Figure 3-3(a). The rotor speed before optimization (Switch-on process)

Figure 3-3(b). The rotor speed after optimization (Switch-on process)
Figure 3-3(c). Optimizing control current for switch-on process

Figure 3-3(d). Optimizing control frequency for switch-on process
Figure 3-4(a). The rotor speed before optimization (Acceleration process)

Figure 3-4(b). The rotor speed after optimization (Acceleration process)
Figure 3-4(c). Optimizing control current for acceleration process

Figure 3-4(d). Optimizing control frequency for acceleration process
Figure 3-5(a). The rotor speed before optimization
(Deceleration process)

Figure 3-5(b). The rotor speed after optimization
(Deceleration process)
Figure 3-5(c). Optimizing control current for deceleration process

Figure 3-5(d). Optimizing control frequency for deceleration process
where \( C_1(t) \) and \( C_2(t) \) are chosen so that the oscillations become minimum in the transient process. This control can be substituted into the system equations of induction motor and optimization results can be found similarly to the previous section. Notice that stator frequency can be chosen to control the speed of the induction motor. The slip of the motor goes to zero as the motor approaches the steady state, and first term on the right hand side of the equation (3.19) vanishes. For this reason, \( C_2 \) can be referred to as no-load current. The results of optimization are shown in figure (3-7). Although these results are inferior to the open-loop case, an improvement in the transient respond of the motor can be observed. The results may improve with the different types of control functions.
Figure 3-6(a). The rotor speed before and after optimization with the feedback control

Figure 3-6(b). The plot of feedback multiplier
SECTION 4. CONCLUSIONS

The differential equations of the induction motor are very difficult to analyze due to variable inductances, sinusoidal input voltages and currents. Solving these equations numerically has two main disadvantages: It is very time-consuming, and it may affect the precision of the computation. It is shown that motor equations with variable inductances and sinusoidal currents can be transformed from the $\alpha-\beta-\gamma$ axes to the a-b axes to reduce the number of state variables under certain conditions, such as balanced sinusoidal excitations. The transformation from a-b to d-q axes converts the variable inductance matrices to constant values. Finally, the last transformation from d-q to D-Q results in time invariant equations. The final equations can be solved easily with the help of a computer.

The controlled variable in an induction motor is usually the speed; and control may involve the maintaining of a pre-set constant speed, or change over to some other predetermined speed. A parameter change, such as a variation in the rotor resistance, may affect the dynamics of induction motor. Simulation results show that the rate of acceleration in the transient process is proportional to the resistance variation in the rotor circuit. The resistance variation, however, does not have any effect on the speed in the steady state.
The current-controlled induction motor with constant current and frequency has large oscillations and large settling time that may result in an unstable operation. For this reason, it is necessary to use proper control functions to improve the transient response of the system. The results in Section 3 show that the oscillations and settling time became significantly smaller after optimization using a first order gradient technique. It is also observed from these results that the constraints for the control functions were not active.

However, open-loop control solutions are not necessarily useful in practical applications. For this reason, a feedback solution is needed to apply the optimization results to the motor. The results in the open-loop control case can be used to relate the control functions to the state variables. The optimized control current, for example, can be related to the speed of the motor by eliminating the independent time variable from these two functions. The results in the "semi closed-loop" control case show improvements in the transient process even though they are inferior to the open-loop case. It might be possible to find different control loops with better results.

It is possible to find similar results using the stator voltage and frequency as control variables. The number of state variables, however, will be five in this case. The transformation technique from $\alpha-\beta-f$ to d-q can be
used without any modifications. As a continuation, different optimization problems can be examined. Energy input of motor, for example, can be minimized using proper cost function.


APPENDICES
APPENDIX A

Optimization of Continuous Systems; Function of State Variables Specified at a Fixed Terminal Time With the Constrained Control

Consider the system described by the following nonlinear differential equations:

\[ \dot{x} = f[x(t),u(t),t]; \quad x(t_0) \text{ given, } t_0 \leq t \leq t_f \] (A.1)

where \( x(t) \), an \( n \)-vector function is determined by \( u(t) \), an \( m \)-vector function, constrained by

\[ C[u(t),t] \leq 0 \] (A.2)

Consider a performance index that is minimized or maximized of the form

\[ J = \phi_1[x(t_f),t_f] + \int_{t_0}^{t_f} L[x(t),u(t),t]dt \] (A.3)

subject to the constraints on terminal state

\[ \phi_2[x(t_f),t_f] = 0 \quad (\text{q-equation}) \] (A.4)

where \( \phi_2 \) is a \( q \)-vector terminal constraints, \( v \) is lagrange multiplier, and \( L[x(t),u(t),t] \) is a performance index function to be minimized.
The problem is to find the control functions \( u(t) \) that minimize or maximize the performance index \( J \). One of the approaches to solve this problem is the maximum principle of Pontryagin. Let us define the Hamiltonian using equations (A.1), (A.2), (A.3) and (A.4), introducing the \( n \)-undetermined multipliers, \( \mathcal{V}^t = [\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n] \), and \( \alpha \) as follows:

\[
\mathcal{H}[x(t), u(t), \mathcal{V}(t), t] = \mathcal{L}[x(t), u(t), t] + \mathcal{V}^t [f(x(t), u(t), t)] + C[u(t), t]
\]  
(A.5)

with the requirement that

\[
\alpha \begin{cases} 
\geq 0, C=0 \\
=0, C<0
\end{cases} \quad (A.6)
\]

Adjoining the system differential equations (A.1) to \( J \) with multiplier function \( \mathcal{V}(t) \)

\[
\dot{J} = \hat{\mathcal{V}}[x(t_f), t_f] + \int_{t_0}^{t_f} \mathcal{L}[x(t), u(t), t] + \mathcal{V}^t(t) [f(x(t), u(t), t) - \dot{x}(t)] dt
\]
(A.7)

where

\[
\hat{\mathcal{V}}[x(t_f), t_f] = \hat{\mathcal{V}}_1[x(t_f), t_f] + \mathcal{V}^t \hat{\mathcal{V}}_2[x(t_f), t_f]
\]
(A.8)
Now consider the variation in $J$ due to variations in the control vector $u(t)$ for fixed times $t_0$ and $t_f$, and integrate the last term on the right hand side of (A.7) by parts,

$$
\delta J = \left\{ \frac{\partial \phi}{\partial x} - \dot{\psi}^t \delta x \right\} \left[ \left( \frac{\partial x}{\partial t} + \dot{\psi}^t \delta x \right) \right]_t=t_f \quad t=t_0
$$

$$
+ \int_{t_0}^{t_f} \left[ \left( \frac{\partial H}{\partial x} + \dot{\psi}^t \delta x + \frac{\partial H}{\partial u} \right) \right] dt
$$

(A.9)

Now, we choose the lagrange multipliers $\psi(t)$ to make the coefficients of $x(t)$ in (A.9) vanish:

$$
\dot{\psi} = -\frac{\partial H}{\partial x} = -\dot{\psi} \frac{\partial f}{\partial x} - \frac{\partial L}{\partial x}
$$

$$
\psi^t(t_f) = \left( \frac{\partial \phi_1}{\partial x} + \dot{v}^t \frac{\partial \phi_2}{\partial x} \right) \bigg|_{t=t_f}
$$

(A.10)

After we choose $(t)$ as in equations (A.10), (A.11) becomes

$$
\delta J = \int_{t_0}^{t_f} \frac{\partial H}{\partial u} \delta u dt + \dot{\psi}^t(t_0) \delta x(t_0)
$$

(A.11)

For a stationary value of $J$, we have

$$
\frac{\partial H}{\partial u} + \frac{\partial f}{\partial u} + \frac{\partial C}{\partial u} = 0 , \quad t_0 \leq t \leq t_f, \quad (A.12)
$$

and if a component $x_k(t_0)$ is not specified, we have

$$
\psi_k(t_0) = 0.
$$
For $C < 0$, we have $\alpha = 0$ and the optimality condition (A.12) determines the m-vector $u(t)$. For $C = 0$, equation (A.2) and (A.12) determines $u$ and $\alpha$ respectively.

If $C[u(t), t] \leq 0$ had been a vector function with s-components, equations (A.5) and (A.12) would remain applicable if we replace $\alpha C$ and $\alpha(\partial C / \partial u)$ by $\alpha^t C$ and $\alpha^t (\partial C / \partial u)$ respectively.
APPENDIX B

Runge Kutta (Order Four) Algorithm

To approximate the solution of the initial-value problem

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, \ldots, x_m, t), & x_1(a) &= x_{1a} \\
\dot{x}_2 &= f_2(x_1, \ldots, x_m, t), & x_2(a) &= x_{2a} \\
&\quad \vdots \\
\dot{x}_m &= f_m(x_1, \ldots, x_m, t), & x_m(a) &= x_{ma} \\
\end{align*}
\]

\[a \leq t \leq b\]

at \((N+1)\) equally spaced numbers in the interval \([a, b]\)

INPUT endpoints \(a, b\); integer \(N\); initial conditions

\[x_{1a}, x_{2a}, \ldots, x_{ma}\]

OUTPUT approximation \(w_1, \ldots, w_m\) to \(x_1, \ldots, x_m\) at the \((N+1)\) values of \(t\).

Step 1 Set \(h = (b-a)/N\);

\[t_1 = a;\]

\[w_{1,1} = x_{1a}, \ldots, w_{m,1} = x_{ma};\]

OUTPUT \((w_{1,1}, \ldots, w_{m,1}, t_1)\).

Step 2 For \(j = 1, 2, \ldots, N\) do steps 3-5.
Step 3 Set $K_{1,i} = h_{f_i}(w_{1,j}, \ldots, w_{m,j})$ for each $i=1,\ldots,m$

$K_{2,i} = h_{f_i}(w_{1,j} + K_{1,1}/2, \ldots, w_{m,j} + K_{1,m}/2, t_{j+h/2})$

for each $i=1,\ldots,m$

$K_{3,i} = h_{f_i}(w_{1,j} + K_{2,1}/2, \ldots, w_{m,j} + K_{2,m}/2, t_{j+h/2})$

for each $i=1,\ldots,m$

$K_{2,i} = h_{f_i}(w_{1,j} + K_{3,1}, \ldots, w_{m,j} + K_{3,m}, t_{j+h})$

for each $i=1,\ldots,m$

Step 4 Set $w_{i,j+1} = w_{i,j} + [K_{1,i} + 2K_{2,i} + 2K_{3,i} + K_{4,i}]/6$

for each $i=1,\ldots,m$

$t_{j} = t_{j} + jh$

Step 5 OUTPUT $(w_{1,j}, w_{2,j}, \ldots, w_{m,j}, t_{j})$

Step 6 STOP
APPENDIX C

Motor Data

Rotor resistance per phase : \( R_r = 0.22 \, \text{ohm} \)
Rotor self inductance per phase: \( L_r = 0.025 \, \text{H} \)
Number of pole pairs : \( n_p = 4 \)
Mutual inductance : \( M = 0.0246 \, \text{H} \)
Voltage : \( v_S = 208 \, \text{V} \)
Current : \( i_S = 30 \, \text{A} \)
Moment of inertia : \( J = 0.01 \, \text{kg.m}^2 \)
Frequency : \( f_S = 60 \, \text{Hz} \)