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The topological structure of the Royden compactification U^* of the open unit disk U is investigated in detail. The space U^* is a fiber space (bundle) over $\hat{U}$ (the closed unit disk) with projection π which maps U^* onto $\hat{U}$ and is identity on U. Let γ be the unit circumference. Then by virtue of Nakai's function

$$g_\zeta(z) = \sin(\log(\log \frac{e}{|z - \zeta|}))$$

on U $(\zeta \in \gamma)$ we can show that $\Gamma = U^* - U$ is a non-trivial fiber space (bundle) over γ. Furthermore, we obtain the existence of points in each fiber of Γ which are approachable from only one side, and points in each fiber which are not approachable: i.e., do not belong to the closure of the complement of the fiber in Γ.

As an application of our study of Royden's boundary

we obtain a boundary identity theorem of the Fatou-Lusin-Privaloff type for meromorphic functions in U with entirely different hypothesis on the manner in which the boundary values are approached.

STUDY OF THE TOPOLOGICAL STRUCTURE
OF THE ROYDEN COMPACTIFICATION
OF THE UNIT DISK

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TABLE OF CONTENTS

STUDY OF THE TOPOLOGICAL STRUCTURE OF THE ROYDEN COMPACTIFICATION OF THE UNIT DISK

INTRODUCTION

In the study of Dirichlet-finite harmonic functions on open Riemann surfaces Royden in his thesis [13] made systematic use of the method of Orthogonal Projection and also suggested a compactification related to this method [14]. The name, the Royden compactification, was coined by M. Nakai [6] and further studied in detail by himself [7], [8], [9], [10], [11], [12] and also by Kusunoki-Mori [3] and [4]. These studies were instigated to provide tools for the investigation of the classification theory of Riemann surfaces. Many theorems in function theory on the plane are carried over to Riemann surfaces formally but not as genuine generalizations since the Royden topology is different from the relative plane topology for plane regions. Although the primary object of these extensions was not to extend theorems in function theory but to provide tools for classification theory, it seems nevertheless important and interesting to examine those generalizations for plane regions. With this motivation we take up the simplest plane regions, the open unit disk, and study in detail the topological structure of the Royden compactification.

Let U be the open unit disk $[|z| < 1]$, and $M(U)$ the real Royden algebra, the totality of real-valued functions f on U satisfying the following three conditions:

(M1) f is bounded on U,

(M2) f is a Tonelli function on U,

(M3) the Dirichlet integral $D_U(f)$ is finite.

The Royden compactification U^* of U is the topological space uniquely determined up to homeomorphism by the following properties:

(R1) U^* is a compact Hausdorff space,

(R2) U^* contains U as an open and dense subspace,

(R3) every function f in $M(U)$ can be extended to a continuous function on U^*,

(R4) $M(U)$ separates points in U^*.

We define a character χ on $M(U)$ as a multiplicative linear functional on $M(U)$ such that $\chi(1) = 1$ and denote by X the set of all characters on $M(U)$. We topologize X by the weak* topology. Then X is the Royden compactification of U.

Let γ by the unit circumference $[|z| = 1]$. The Nakai function $g_\zeta(z)(\zeta \in \gamma)$ is defined as follows:

$$g_\zeta(z) = \sin(\log(\log \frac{e}{|z - \zeta|})).$$

Then g_ζ has the properties:

(1) for each $\zeta \in \gamma$ $|g_\zeta| \leq 1$ on U, (2) $g_\zeta \in C^\infty(U)$,

(3) $D_U(g_\zeta) < \infty$.

It follows that $g_\zeta \in M(U)$ and hence g_ζ has a continuous extension to U^*.

A rotation $f_\theta(z) = e^{i\theta}z$, $\theta \in [0, 2\pi)$, can be extended to a homeomorphism, also denoted by f_θ, on U^*.

Let A be the set $\{f \in M(U) : f$ is continuous on $\hat{U}\}$ where $\hat{U}$ is the closed unit disk and S the totality of characters on A provided with the weak* topology. Then $S = \hat{U}$ and for any point $p \in U^*$ the map $\pi(p) : f \to f(p)$ defines a character on A and hence a point of $\hat{U}$. Then π is a projection from U^* onto $\hat{U}$ such that $\pi(\Gamma) = \gamma$ and π is the identity on U where $\Gamma = U^* - U$. Thus U^* is a fiber space (bundle) over $\hat{U}$ with projection π. From $\pi \circ f_\theta = f_\theta \circ \pi$, $\theta \in [0, 2\pi)$, it follows that f_θ maps the fiber $\pi^{-1}(\zeta)$ homeomorphically onto the fiber $\pi^{-1}(e^{i\theta}\zeta)$.

We denote the extension of the Nakai function g_ζ also by g_ζ. The range of g_ζ $(\zeta \in \gamma)$ is $[-1, +1]$ and $g_\zeta(q) = g_\zeta(\zeta')$ for all $q \in \pi^{-1}(\zeta')$ if $\zeta' \neq \zeta$, i.e. the value of g_ζ on $\pi^{-1}(\zeta')$ $(\zeta' \neq \zeta)$ is constant.

We can prove that the fiber space (bundle) (Γ, π, γ) is not locally trivial by showing that π is not an open mapping. Moreover this fiber space (bundle) does not even have any local (continuous) sections.

We show the existence of three types of points on each fiber. One type consists of the points of the fiber which do not belong to the closure of the complement of the fiber in Γ. The other two types are those which do belong to the closure. This latter class subdivides into those approachable from only one side and those approachable from either side.

Take an arbitrary pair of real numbers a and b with $-1 < a < b < 1$ and for each $\zeta \in \gamma$ let $A(\zeta : a, b)$ be the set $\{z \in U : a < g_\zeta(z) < b\}$. Then

$$A(\zeta : a, b) = \bigcup_{n=1}^{\infty} A_n(\zeta : a, b) \quad \text{where}$$

$A_n(\zeta : a, b) = \{z \in U : \alpha_n < |z - \zeta| < \beta_n\}$ with $\beta_n - \alpha_n$ converging to zero as n converging to ∞. For each $\zeta \in \gamma$ a stairway V at ζ is defined as follows:

(S1) V is a subset of U,

(S2) there exists a pair of real numbers a and b with $-1 < a < b < 1$ and a positive integer n such that $\overline{A^n(\zeta : a, b)} \cap U \subset V$ where $A^n(\zeta : a, b) = \bigcup_{k=n}^{\infty} A_k(\zeta : a, b)$. We say that a complex-valued function function f on U has a limit α along V at ζ if

$$\lim_{n} \sup_{z\varepsilon V, |z-\zeta|<\frac{1}{n}} |f(z) - \alpha| = 0.$$

As an application of the Royden compactification of U, by using the Nakai Unicity Theorem we obtain a boundary identity theorem: Let $f(z)$ be a meromorphic function on the open unit disk U. Suppose that there exists a set E in the unit circumference with positive linear measure and a stairway V at 1 such that f has limit zero along ζV at each point $\zeta \varepsilon E$. Then f is identically zero.

The results of Chapters I and II are known background material. The results of Chapter III and IV with the exception of 3.2A, 3.2B, 4.2A, 4.2B, are original.

The term fiber space or bundle is used throughout to mean a surjective map with no additional structure.

I. ROYDEN COMPACTIFICATION

In this chapter we shall present an introduction to the Royden compactification of an open Riemann surface R. In §1, we define Royden's algebra M(R) and discuss various topologies on M(R). For convenience we state Green's formula, the Dichlet principle and explain potential subalgebras of M(R). §2 treats the theory of Royden compactification R^* of R leading up to the existence and uniqueness of R^*. The Royden and harmonic boundaries are discussed in §2. §3 contains the decomposition theorem, one of the fundamental theorems in the study of HD-functions, and the important concept of minimal functions.

§1. Royden's Algebra.

1A. Tonelli Functions

Let R be a plane region and f(x, y) a real-valued function defined on R. f is called a Tonelli function on R if it satisfies the following conditions:

(T.1) f is continuous on R,

(T.2) $x \to f(x, y)$ is absolutely continuous on x for almost all y and the same holds if x and y are interchanged.

(T.3) $(\partial f/\partial x)(x, y)$ and $(\partial f/\partial y)(x, y)$ are square integrable on any compact subset of R.

A Riemann surface R is a real two dimensional manifold with conformal structure, or equivalently, a one dimensional complex analytic manifold.

A real-valued function f on a Riemann surface R is a Tonelli function if it is Tonelli on every parametric disk of R.

It is clear that if f and g are Tonelli functions on a Riemann surface R, then so are $f \vee g$ and $f \wedge g$ where $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$.

1B. Definition of Royden's Algebra

The real (or complex) Royden algebra $M(R)$ associated with a Riemann surface R is the totality of real (or complex) functions f on R satisfying the following three conditions:

(M.1) f is bounded on R,

(M.2) f is a Tonelli function on R,

(M.3) the Dirichlet integral $D_R(f)$ of f on R is finite, i.e. $D_R(f) = \int_R |f'(z)|^2 dxdy < \infty$ where $z = x + iy$.

Royden's algebra $M(R)$ actually forms an algebra over the real (or the complex) number field with the usual addition and multiplication of functions and scalar multiplication defined pointwise. Moreover, it is a vector lattice by the observation in 1A.

1C. Completeness

Here we consider various topologies on $M(R)$. Let $\{f_n\}$ be a sequence of functions on R and f a function on R. We use the following terminology for convenience:

(1) $\{f_n\}$ converges to f on R in C-topology,

$f = \text{C-}\lim_{n\to\infty} f_n$, if $\lim_{n\to\infty} \sup_K |f_n - f| = 0$ for each compact subset K in R,

(2) $\{f_n\}$ converges to f on R in B-topology,

$f = \text{B-}\lim_{n\to\infty} f_n$, if $\{f_n\}$ is uniformly bounded on R and $f = \text{C-}\lim_{n\to\infty} f_n$ on R,

(3) $\{f_n\}$ converges to f on R in U-topology,

$f = \text{U-}\lim_{n\to\infty} f_n$, if $\lim_{n\to\infty} \sup_R |f_n - f| = 0$,

(4) $\{f_n\}$ converges to f on R in D-topology,

$f = \text{D-}\lim_{n\to\infty} f_n$, if $\lim_{n\to\infty} D_R(f_n - f) = 0$,

(5) $\{f_n\}$ converges to f on R in QD-topology

$(Q = C, B, U)$, $f = \text{QD-}\lim_{n\to\infty} f_n$, if $f = \text{Q-}\lim_{n\to\infty} f_n$ on R and $f = \text{D-}\lim_{n\to\infty} f_n$ on R.

In particular, the UD-topology is given by the norm

$$\|f\| = \|f\|_R = \sup_R |f| + \sqrt{D_R(f)} \quad . \tag{1}$$

Then we can easily see that $M(R)$ is a normed algebra in the UD-topology.

Regarding the completeness of $M(R)$, we have the following theorem due to Kawamura [5]:

<u>Theorem</u> 1.1C. Let $\{f_n\}$ be a sequence of functions in $M(R)$ and f a bounded function on R satisfying $f = C\text{-}\lim_n f_n$ on R and such that $D_R(f_n) \leq K < \infty$ for every n. Then $f \in M(R)$ and for any $g \in M(R)$

$$D_R(f, g) = \lim_n D_R(f_n, g)$$

where
$$D_R(f, g) = \int_R df \wedge * \overline{dg}.$$

<u>Proof</u>. Let $\Gamma^2(R)$ be the set of all first order differentials α with the local representation $\alpha = a(x, y)dx + b(x, y)dy$ in terms of a local parameter $z = x + iy$ such that $a(x, y)$ and $b(x, y)$ are measurable and

$$\int_R \alpha \wedge * \overline{\alpha} = \int_R (|a(x, y)|^2 + |b(x, y)|^2)dxdy < \infty.$$

We set $(\alpha, \beta) = \int_R \alpha \wedge * \overline{\beta}$ for α and β in $\Gamma^2(R)$.

Clearly $\Gamma^2(R)$ is a Hilbert space. Since $(df, df) = D_R(f) < \infty$ for every $f \in M(R)$, $\{df : f \in M(R)\} \subset \Gamma^2(R)$.

By the hypothesis $\{df_n\}$ is a uniformly bounded sequence in $\Gamma^2(R)$. Hence it contains a weakly convergent subsequence $\{df_{n_k}\}$ by a well-known fact from Hilbert space theory.

Let $z = x + iy$ be a local parameter on R in $T = (S, t) \times (u, v)$ and let $C_0^\infty(T)$ be the class of C^∞-functions on R with supports in T. Then we have

$$\int_T f_n(x, y) \frac{\partial}{\partial x} \varphi(x, y) dxy = -\int_T (\frac{\partial}{\partial x} f_n(x, y)) \varphi(x, y) dxdy$$

for every $\varphi \in C_0^\infty(T)$. Let $\alpha = a(x, y)dx + b(x, y)dy$ be the weak limit of $\{df_{n_k}\}$. Since $\varphi(x, y)dx \in \Gamma^2(T)$ and

$$\int_T \left(\frac{\partial}{\partial x} f_{n_k}(x, y)\right) \varphi(x, y) dxdy = (df_{n_k}, \varphi dx)$$

$$\to (\alpha , \varphi dx) = \int_T a(x, y)\varphi(x, y) dxdy$$

as $k \to \infty$, we obtain

$$\int_T f_{n_k}(x, y) \left(\frac{\partial}{\partial x} \varphi(x, y)\right) dxdy \to$$

$$\int_T a(x, y)\varphi(x, y) dxdy \quad \text{as} \quad k \to \infty.$$

Moreover, from $f = \underset{n}{C\text{-}\lim}\, f_n$

$$\int_T f_{n_k}(x, y)\left(\frac{\partial}{\partial x}\varphi(x, y)\right) dxdy \to$$

$$\int_T f(x, y)\left(\frac{\partial}{\partial x}\varphi(x, y)\right) dxdy \quad \text{as} \quad k \to \infty$$

and hence we have

$$\int_T f(x, y)\left(\frac{\partial}{\partial x}\varphi(x, y)\right) dxdy = -\int_T a(x, y)\varphi(x, y)dxdy. \quad (2)$$

Similarly

$$\int_T f(x, y)\left(\frac{\partial}{\partial y}\varphi(x, y)\right) dxdy = -\int_T b(x, y)\varphi(x, y)dxdy. \quad (3)$$

From (2) and (3) it follows that the distributional derivatives $[\partial f/\partial x]_{dis}$ and $[\partial f/\partial y]_{dis}$ of $f(x, y)$ on T are $a(x, y)$ and $b(x, y)$, respectively. By the theorem of Nikodym f satisfies (T.2) and the usual derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of f coincide with a and b on T, respectively (cf. L. Schwartz, Théorie des distributions. p. 57). Since T is arbitrary $df = \alpha$ on T and thus $D_R(f) < \infty$, i.e. $f \in M(R)$.

q.e.d.

As an immediate consequence of this theorem, we obtain two corollaries.

<u>Corollary</u> 1.1C.1. $M(R)$ is BD-complete.

<u>Corollary</u> 1.1C.2. $M(R)$ is a Banach algebra with respect to the norm (1).

1D. <u>Approximation</u>.

Here we state an important approximation theorem as a computational rule for functions in $M(R)$ analogous to Green's formula for C^∞-functions.

<u>Theorem</u> 1.1D. For any $f \in M(R)$ and positive number ε there exists a function $f_\varepsilon \in C^\infty(R) \cap M(R)$ such that $\|f - f_\varepsilon\| < \varepsilon$. Furthermore, if f has compact support in an open set $G \subset R$, then f_ε can be chosen so as to have compact support in G.

<u>Proof</u>. Let $z = x + iy$ be a local parameter of R valid in $|z| < 2$. We assume that the support of f is contained in $|z| < 1$. Let ρ_n be the function on R defined by

$$\left(\pi \int_0^{n^{-2}} e^{-t^{-1}} dt\right) \rho_n(z) = \begin{cases} \exp(-(n^{-2} - x^2 - y^2)^{-1}) & \text{on } |z| < \frac{1}{n}, \\ 0 \quad \text{on } R - \{|z| < \frac{1}{n}\}. \end{cases}$$

It is a nonnegative C^∞- function on R with

$\int_R \rho_n(z)\,dxdy = 1.$ Let

$$f_n(w) = \int_R \rho_n(w - z)f(z)\,dxdy = \int_R \rho_n(w)f(w - z)\,dxdy.$$

It is easy to see that $f_n \in C^\infty(R)$, $f_n = 0$ on R outside of $|z| < 2$ and $f = \underset{n}{\text{U-lim}}\, f_n$ on R. Clearly

$$\partial f_n(w) = \int_R \rho_n(z)\,\partial f(w - z)\,dxdy$$

almost everywhere on $|z| < 2$, where $w = u + iv$ and ∂ stands for $\partial/\partial u$ or $\partial/\partial v$. We obtain by Schwarz's inequality

$$|\underset{w}{\text{grad}}\ (f_n(w) - f(w)|^2$$

$$\leq \left(\int_R \rho_n(z)\,dxdy\right)\left(\int_R \rho_n(z)\,|\underset{w}{\text{grad}}(f(w - z) - f(w))|^2 dxdy\right)$$

$$= \int_R \rho_n(z)\ |\ \underset{w}{\text{grad}}\ (f(w - z) - f(w))|^2 dxdy.$$

Hence we have that

$$D_R(f_n - f) \leq \int_{|z|<\frac{1}{n}} \rho_n(z)\left(\int_{|w|<2} |\text{grad}(f(w - z) - f(w))|^2 dudv\right) dxdy$$

Since $|\underset{w}{\text{grad}}\ f(w)|$ is square integrable on $|w| < 2$, Lebesgue's theorem gives

$$\lim_{|z|\to 0} \int_{|w|<2} |\operatorname{grad}_w (f(w - z) - f(w)|^2 du dv = 0.$$

Thus $f = D\text{-}\lim_n f_n$ on R and $\lim_n \|f - f_n\| = 0$.

Let $\{\varphi_j\}_1^\infty$ be a sequence of C^∞-functions on R such that for some parametric disk $U_j : |z| < 2$, the support of φ_j is contained in $|z| < 1$, $\{U_j\}_1^\infty$ is locally finite covering of R, and $\sum_{j=1}^\infty \varphi_j = 1$ on R. Since $f\varphi_j$ satisfies the requirement of the previous step, we can find an $f_j \in C^\infty(R) \cap M(R)$ such that the support of f_j is compact in U_j and $\|f\varphi_j - f\| < \varepsilon/2^j$. Moreover if f has compact support in an open subset G of R then we may assume that the support of f_j is contained in G. Set $f_\varepsilon = \sum_{j=1}^\infty f_j$. Since almost every f_j vanishes at any given point of R we have $f_\varepsilon \in C^\infty(R)$. Furthermore

$$\|f - f_\varepsilon\| \leqq \sum_{j=1}^\infty \|f\varphi_j - f_j\| < \varepsilon.$$

Hence $f_\varepsilon \in C^\infty(R) \cap M(R)$ and if the support of f is contained in an open set G then the support of f_ε is contained in G.

q.e.d.

1E. <u>Green's Formula</u>.

As an example of an application of the approximation theorem we give the following form of Green's formula. Let G be an open set with compact closure in R and let ∂G consist of a finite number of disjoint analytic Jordan curves.

Let $f \in M(G)$ and $u \in H(\overline{G})$, where $H(\overline{G})$ is the family of all harmonic functions on $\overline{G}$. Then

$$D_G(f, u) = \int_{\partial G} f{*}du. \tag{4}$$

Another form of Green's formula which we shall often use is the following:

Let $f \in M(R)$, $u \in HD(G)$, choose a union γ_1 of some components of ∂G and set $\gamma_2 = \partial G - \gamma_1$. Assume that $f = 0$ on γ_1 and u is harmonic on γ_2. Then

$$D_G(f, u) = \int_{\partial G} f{*}du = \int_{\gamma_2} f{*}du. \tag{5}$$

Here $*du$ is meaningless in general on γ_1 but since $f/\gamma_1 = 0$ we may write $f{*}du$ on γ_1 with the agreement that $f{*}du = 0$ on γ_1 (cf. [16] p. 152).

1F. Dirichlet Principle.

Let G be as in 1E. For $f \in M(R)$ and $u \in HD(G)$ with boundary values $u = f$ on ∂G

$$D_G(f) = D_G(u) + D_G(f - u), \tag{6}$$

i.e. the Dirichlet principle is valid for f and G.

In fact by (5)

$$D_G(f - u, u) = \int_{\partial G} (f - u) * du = 0$$

and (6) follows immediately.

1G. Potential Subalgebra.

We denote by $M_0(R)$ the class of all functions in $M(R)$ with compact supports and by $M_\Delta(R)$ the BD-closure of $M_0(R)$ in $M(R)$. Hence $M_0(R) \subset M_\Delta(R) \subset M(R)$. Here M_0 is a subalgebra of M_Δ and also of M, and M_Δ is a subalgebra of M. In particular, we call $M_\Delta(R)$ the potential subalgebra of $M(R)$.

The following theorem corresponds to Theorem 1.1C. We state it without proof.

Theorem 1.1G. Let $\{f_n\}$ be a sequence of functions in $M_\Delta(R)$ and f a bounded function on R such that $f = C\text{-}\lim_n f_n$ and $D_R(f_n) \leq K < \infty$ for every n. Then

$f \in M_\Delta(R)$ (cf. [16] p, 153).

Clearly $M_0(R)$ is an ideal of $M_\Delta(R)$ and of $M(R)$ as well. Similarly we may prove that $M_\Delta(R)$ is an ideal of $M(R)$ (cf. [16] p. 154).

§2. Royden's Compactification.

2A. Definition of Royden's Compactification.

Let R be a Riemann surface and $M(R)$ Royden's algebra associated with R.

A topological space R^* is called Royden's compactification of R if R^* satisfies the following conditions:

(R*.1) R^* is a compact Hausdorff space,

(R*.2) R^* contains R as an open and dense subspace,

(R*.3) every function in $M(R)$ can be continuously extended to R^*,

(R*.4) $M(R)$ separates points in R^*.

Since R is dense in R^*, the continuous extension of any $f \in M(R)$ to R^* is unique. Hence we may use the same notation f for the extension. Let $B(R^*)$ be the totality of real-valued bounded continuous functions on R^*. Then $M(R) \subset B(R^*)$.

2B. Characters.

Let χ be a multiplicative linear functional on $M(R)$ such that $\chi(1) = 1$. We call such a χ a character on $M(R)$ and denote by X the set of all characters on $M(R)$. We topologize X by the weak* topology. Then we have the following lemma:

Lemma 1.2B. X is a compact Hausdorff space.

Proof. X is a weak* closed subset of the unit ball in the dual space of $M(R)$ and hence it is a compact Hausdorff space when provided with the weak* topology.

q.e.d.

2C. Existence and Uniqueness of R*.

We are ready to show the existence and uniqueness of the Royden compactification R^* of R.

Theorem 1.2C. For any open Riemann surface R the Royden compactification R^* of R exists and is uniquely determined up to a homeomorphism fixing R pointwise.

Proof. For a point $z \in R$, $\chi_z(f) = f(z)$ defines a character on $M(R)$. Consider the mapping $\psi : R \to X$ by $z \to \chi_z$. Then ψ is one-to-one because $M(R)$ separates points in R. Clearly ψ is continuous. Each $f \in M(R)$ may be considered as defined on X by $f(\chi) = \chi(f)$. Then we can easily see that $M(R)$ is a subalgebra of $B(X)$, the set of all bounded continuous functions on X, and $M(R)$ separates points in X. Hence by the Stone-Weierstrass theorem $M(R)$ is dense in $B(X)$ with respect to the uniform topology. Now we show that ψ is a homeomorphism (into). Let z_0 be an arbitrary point in R and U any open neighborhood of z_0 in R. Choose $g \in B(R)$ so that $g(z_0) = 0$, $g = 3$ on $R - U$ and $0 \le g < 3$ on U, and $h \in M(R)$ so that $|h - g| < 1$. Let $G = \{\chi \in X: |\chi(h) - \chi_{z_0}(h)| < 1\}$. Then $G \cap R = \{z \in R: |h(z) - h(z_0)| < 1\}$

$$\subseteq \{z \in R: |h(z)| < 2\} \subseteq U.$$

Therefore ψ is a relatively open map and hence ψ is a homeomorphism (into). If we identify z and χ_z we may view $R \subset X$ as a topological subspace. Thus X satisfies (R*.1) - (R*.4) except possibly (R*.2). We shall prove that R is dense in X. It will be convenient to denote the extension of $f \in M(R)$ to X defined above by $\tilde{f}$, i.e. $\tilde{f}(\chi) = \chi(f)$. Suppose that $f \ge c > 0$ and $f \in M(R)$. Clearly $f^{\frac{1}{2}} \in M(R)$ and hence

$\chi(f) = \chi(f^{\frac{1}{2}})\chi(f^{\frac{1}{2}}) \geq 0$, i.e. $\tilde{f} \geq 0$. Now if $f \geq 0$, then $f + \frac{1}{n} \geq \frac{1}{n}$ for all n. It implies that $(f + \frac{1}{n})^{\sim} \geq 0$, i.e. $\tilde{f} \geq -\frac{1}{n}$ for all n. Hence $\tilde{f} \geq 0$. Suppose that R is not dense in X. Let U be an open subset of X which is disjoint from R and choose V so that V is open in X and $\bar{V} \subset U$. Then $\bar{V}$ and $\bar{R}$ are mutually disjoint closed sets and hence there exists $g \in B(x)$ such that $g = 1$ on $\bar{R}$ and $g = -1$ on $\bar{V}$. Choose a sequence $\{f_n\}_1^\infty$ in $M(R)$ so that $\tilde{f}_n \to g$ uniformly on X as $n \to \infty$. By uniform convergence and the fact that $g = 1$ on R, we have $\tilde{f}_n \geq 0$ on R for sufficiently large n, i.e. $\tilde{f}_n \geq 0$. From the above observation $\tilde{f}_n \geq 0$ on X for sufficiently large n and so $g \geq 0$ on X which is contradiction. In particular, we may now drop the tildas. The fact that R is open in X is a consequence of the local compactness of R and $\bar{R} = X$. In fact, let z_0 be any point in R and F a compact neighborhood of z_0 in R. Since R is dense in X $\bar{F}$ in X is a neighborhood of z_0 in X. But F is compact in R and so in X, i.e. $\bar{F} = F$. Consequently R is open in X. Thus X also satisfies (R*.2).

Let R^* be a Royden compactification of R. For each $p \in R^*$ define $\chi_p \in X$ by $\chi_p(f) = f(p)$ for every $f \in M(R)$. Let the map $\theta : R^* \to X$ be defined by $\theta(p) = \chi_p$. Then θ is one-to-one since $M(R)$ separates points of R^* and θ is clearly continuous. By compactness of R^* and X, θ is a homeomorphism of R^* into X with closed image. But θ is the identity on $R \subset R^*$ and R is dense in X. Hence θ is onto.

q.e.d.

Corollary 1.2C.1. $M(R)$ is dense in $B(R^*)$ in U-topology.

Corollary 1.2C.2. For any character χ on $M(R)$ there exists a unique point $p \in R^*$ such that $\chi(f) = f(p)$ for every $f \in M(R)$, and conversely.

2D. Urysohn's Property.

$M(R)$ has the following Urysohn's property:

Theorem 1.2D. For any two nonempty disjoint compact sets K_1 and K_2 in R^* and real numbers a_1 and a_2 there exists a real-valued function f in $M(R)$ such that $f = a_j$ on K_j $(j = 1, 2)$ and $\min(a_1, a_2) \leq f \leq \max(a_1, a_2)$.

Proof. We may assume that $a_1 > a_2$. By Urysohn's lemma there exists a real-valued function g in $B(R^*)$

such that $g = a_1 + 2$ on K_1 and $g = a_2 - 2$ on K_2. Since $M(R)$ is dense in $B(R^*)$ in the uniform topology by corollary 1.2C.1, there exists a h in $M(R)$ with $|g - h| < 1$ on R^*. Then $M(R)$ is a vector lattice with respect to $\vee$(max.) and $\wedge$(min.) and $f = (h \wedge a_1) \vee a_2$ which is in $M(R)$. Thus this f satisfies the required property.

q.e.d.

2E. Royden's Boundary.

The set $\Gamma = R^* - R$ is a compact subset of R^*. We call Γ Royden's boundary of R. It is easily seen that

$$\Gamma = \{p \in R^* : f(p) = 0 \quad \text{for every} \quad f \in M_0(R)\}$$

and hence $\Gamma = \emptyset$ if and only if R is compact. Next we give a topological condition for distinguishing points in Γ from those in R:

Theorem 1.2E. A point $p \in R^*$ belongs to Γ if and only if p is not a G_δ-set in R^*.

Proof. Each point p in R is evidently a G_δ-set. Hence we must only show that $p \in \Gamma$ is not a G_δ-set. Suppose that $p \in \Gamma$ and p is a G_δ-set. Then we can find a sequence $\{U_n\}_1^\infty$ of open neighborhoods U_n of p such that $\overline{U}_{n+1} \subset U_n$ and $p = \bigcap_1^\infty U_n$. For each n we choose two open disks B_n and B_n' in $R \cap (U_n - \overline{U}_{n+1})$ with $B_n \supset \overline{B}_n'$ such that the annulus $A_n = B_n - \overline{B}_n'$ satisfies $\log(\text{mod } A_n) = 2^n$ (cf. [16] p.11). Let w be continuous on R with $w|(\bigcup_1^\infty \overline{B}_n') = 1$, $w|(R - \bigcup_1^\infty B_n) = 0$, and $w \in H(\bigcup_1^\infty A_n)$. Clearly w is a bounded Tonelli function on R, and

$$D_R(w) = \sum_1^\infty D_{A_n}(w) = \sum_1^\infty \frac{2\pi}{\log \text{mod } A_n} = 2\pi.$$

Hence w is in $M(R)$ and hence w is continuous on R^*. Let $z_n \in \partial B_n$ and $z_n' \in \partial B_n'$. Then $\lim_n z_n = \lim_n z_n' = p$ on R^* but $w(z_n) = 0$ and $w(z_n') = 1$ violate the continuity of w at p.

q.e.d.

2F. Harmonic Boundary.

We distinguish the following important part of the Royden boundary Γ of R:

$$\Delta = \{p \in R^* : f(p) = 0 \text{ for every } f \in M_\Delta(R)\}.$$

This is a compact set in R^*, and since $M_0(R) \subset M_\Delta(R)$ Δ is a compact subset of Γ. We call Δ Royden's <u>harmonic boundary</u> of R. First we shall show how Δ is distributed in Γ from the topological viewpoint:

<u>Theorem</u> 1.2F. $\overline{\Gamma - \Delta} = \Gamma$.

<u>Proof</u>. We must show that for any p_0 in Γ and for any open neighborhood U of p_0, $U \cap (\Gamma - \Delta) \neq \emptyset$. Let V be an open neighborhood of p_0 with $\overline{V} \subset U$ and let $\{R_n\}_1^\infty$ be an exhaustion of R (cf. [16] p. 2) with $V \cap (R_{n+1} - \overline{R}_n) \neq \emptyset$. For each n we choose two open disks B_n and B_n' in $V \cap (R_{n+1} - \overline{R}_n)$ such that the annulus $A_n = B_n - \overline{B}_n'$ satisfies $\log \operatorname{mod} A_n = 2^n$. Let w_n be continuous on R with $w_n | (R_n - B_n) = 0$,

$w_n | \overline{B}_n' = 1$, and $w_n \in H(A_n)$. Then $w = \sum_1^\infty w_n$ is obviously B-convergent and

$$D_R(w - \sum_1^m w_n) = \sum_{m+1}^\infty D_{A_n}(w_n) = \sum_{m+1}^\infty \frac{2\pi}{\log \operatorname{mod} A_n} = \frac{\pi}{2^{m-1}} .$$

Thus $w = \text{BD-}\lim_m \sum_1^m w_n$ on R. Since $\sum_1^m w_n$ is in $M_0(R)$, $w \in M_\Delta(R)$. Let $z_n \in \overline{B}_n'$ and let p be an accumulation point of $\{z_n\}_1^\infty$. Then $p \in \Gamma \cap \overline{V} \subset U$. On the other hand from $w(z_n) = 1$ we have $w(p) = 1$

and hence $p \in \Gamma - \Delta$. A fortiori $p \in (\Gamma - \Delta) \cap U$.

q.e.d.

This theorem shows that Δ is a small set in Γ from a topological viewpoint. However if Δ is not empty then it is large in Γ in a function-theoretic sense.

2G. Parabolic Surfaces

For an open Riemann surface R let $\{R_n\}_0^\infty$ be an exhaustion with connected $R - \overline{R}_0$. Let $\omega_n(z; R_n, R_0)$ be a continuous function on R such that $\omega_n | \overline{R}_0 = 0$, $\omega_n | (R - R_n) = 1$, and $\omega_n \in H(R_n - \overline{R}_0)$. The function ω_n is called the harmonic measure of ∂R_n with respect to $R_n - \overline{R}_0$. By the maximum principle we see that $\omega_{n+p} \leq \omega_n$ and thus $\lim_n \omega_n$ exists on R, vanishes on $\overline{R}_0$, and harmonic on $R - \overline{R}_0$. We denote it by $\omega(z) = \omega(z; R, \overline{R}_0)$. This $\omega(z)$ is referred to as the harmonic measure of the ideal boundary of R with respect to $R - \overline{R}_0$.

A surface R which is either closed or open with $\omega \equiv 0$, is called parabolic. Otherwise it is said to be hyperbolic. The class of parabolic surfaces is denoted by O_G. We know that $R \in O_G$ is characterized by the nonexistence of Green's functions of R (cf. [1] p. 204). Furthermore, we have the following characterization of the class O_G in terms of the harmonic boundary (cf. [14]):

Theorem 1.2G. The following four conditions for Riemann surfaces R are equivalent:

$$R \in O_G \tag{7}$$

$$1 \in M_\Delta(R) \tag{8}$$

$$M_\Delta(R) = M(R) \tag{9}$$

$$\Delta = \emptyset \tag{10}$$

Proof. The assertion being trivial for closed surfaces we assume that R is open. Since $M_\Delta(R)$ is an ideal of $M(R)$ the equivalence of (8) and (9) is obvious. It is also trivial that (8) implies (10). Next assume that $\Delta = \emptyset$. For each $p \in R^*$ there exists a function $f_p \in M_\Delta(R)$ such that $f_p \geq 0$ on R^* and $f_p(p) > 1$. Since R^* is compact we can find a finite number of points $p_1, \ldots, p_n$ in R^* such that $\bigcup_{j=1}^n \{p \in R^*: f_{p_j}(p) > 1\} = R^*$. Let $f = \sum_1^n f_{p_j}$. Then $f \in M_\Delta(R)$ and $f > 1$ on R. Hence $1/f \in M(R)$ and $1 = (1/f)f \in M_\Delta(R)$ i.e. (8) and (10) are equivalent.

All that remains to be proved is the equivalence of (7) and (8). Let $u_n = 1 - \omega_n$ and $u = 1 - \omega$. Then $R \in O_G$ if and only if $u \equiv 1$ on R. By Green's formula

$$D_R(u_{n+p} - u_n, u_{n+p}) = \int_{\partial(R_{n+p}-R_0)} (u_{n+p} - u_n) * du_{n+p} = 0$$

and hence $D_R(u_{n+p} - u_n) = D_R(u_n) - D_R(u_{n+p})$, i.e. $\{u_n\}_1^\infty$ is D-Cauchy. But $u_n \in M_0(R)$ and $u = \text{BD-}\lim_n u_n$ mean that $u \in M_\Delta(R)$. For $R \in O_G$ we therfore have $1 = u \in M_\Delta(R)$.

Conversely assume that $1 \in M_\Delta(R)$. Take a sequence $\{f_n\} \subset M_0(R)$ such that $1 = \text{BD-}\lim_n f_n$. Then $u = \text{BD-}\lim_n uf_n$. Let K be an arbitrary compact set in R. Then

$$D_R(uf_n - u) \leq 2\int_R |u|^2 |\text{grad}(f_n - 1)|^2 dxdy$$

$$+ 2\int_R |f_n - 1|^2 |\text{grad } u|^2 dxdy$$

$$\leqq a\, D_R(f_n) + 2 \sup_K |f_n - 1|^2 D_R(u) + b_n\, D_{R-K}(u)$$

where $a = 2 \sup_R |u|^2$ and $b_n = 2 \sup_R |f_n - 1|^2$.

We set $b = \lim_n \sup b_n$ and obtain

$$\lim_n \sup\ D_R(uf_n - u) \leq b\, D_{R-K}(u).$$

On letting $K \to R$ we conclude that

$u = \text{D-}\lim_n uf_n$ and $1 - u = \text{BD-}\lim_n (1 - u) f_n$ on R.

By Green's formula

$$D_R((1 - u)f_n, (1 - u)) = \int_{\partial(R_m - R_0)} (1 - u)f_n * d(1 - u) = 0$$

where $R_m \supset \text{supp}\ f_n$. It follows that

$$D_R(u) = D_R(1 - u) = \lim_n D_R((1 - u)f_n, (1 - u)) = 0.$$

Therefore u is a constant on R and hence $u \equiv 1$, which implies $R \in O_G$.

q.e.d.

As a modification of the above theorem for a relative case we state the following theorem without proof (cf. [3], [16] p. 159): Let G be a subregion of R with analytic relative boundary. If $\overline{G} \cap \Delta = \emptyset$, then the double $\hat{G}$ of G about ∂G belongs to O_G.

2H. Maximum Principles.

Here we shall prove two forms of the maximum principle which are often used.

Let $HB = HB(R)$ be the set of all bounded harmonic functions on a Riemann surface R and O_{HB} the class of surfaces R such that $HB(R)$ consists of only constants. We observe that $O_G \subset O_{HB}$.

The first maximum principle is

<u>Theorem</u> 1.2H.1. Let G be a subregion of R (G may be R) and u a real-valued harmonic function on G bounded from above (resp. below). Suppose

$$\limsup_{z \in G,\ z \to p} u(z) \le m$$

$$(\text{resp. } \liminf_{z \in G,\ z \to p} u(z) \ge m) \qquad (11)$$

for every $p \in (\overline{G} \cap \Delta) \cup \partial G$. Then $u \le m$ (resp. $u \ge m$) on G.

<u>Proof</u>. Suppose that $u(z_0) > m$ at a point $z_0 \in G$. Choose a number c such that $u(z_0) > c > m$ and $du \neq 0$ on $\{z \in G : u(z) = c\}$. Let G_0 be the component of $\{z \in G : u(z) > c\}$ which contains z_0. Then G_0 is a subregion of G and $\overline{G}_0 \wedge \Delta = \emptyset$ and hence $\hat{G}_0 \in O_G$ by the observation in 2G. On the other hand $u - c$ is in $HB(G_0)$ and $u - c = 0$ on ∂G_0. Therefore $u - c$ is continued to $\hat{G}_0$ so that $u - c \in HB(\hat{G}_0)$. This contradicts $O_G \subset O_{HB}$.

q.e.d.

The second maximum principle is

<u>Theorem</u> 1.2H.2. Let G be a subregion of R (G may be R) and $u \in HD(G)$ where $HD(G)$ is the class of all

real-valued harmonic functions on G whose Dirichlet integrals are finite. If

$$-\infty \le a \le \liminf_{z\in G,\ z\to p} u(z) \le \limsup_{z\in G,\ z\to p} u(z) \le b \le \infty \qquad (12)$$

for every point $p \in (\overline{G} \cap \Delta) \cup \partial G$. Then $a \le u \le b$ on G.

Proof. It suffices to show that $b \ge u$ on G. We may assume that $b < \infty$. Suppose that the assurtion is not true, i.e. there exists a point $z_0 \in G$ with $b < u(z_0)$. Let c be a real number such that $b < c < u(z_0)$ and $du \ne 0$ on $\{z \in G : u(z) = c\}$. Let G_0 be the component of the open set $\{z \in G : u(z) > c\}$ which contains z_0. Then $\overline{G}_0 \cap \Delta = \emptyset$ and thus $1 \in M_\Delta(\hat{G}_0)$ by Theorem 1.2G and the observation in 2G. Hence we can find a sequence $\{\varphi_n\}_1^\infty \subset M_0(\hat{G}_0)$ with $1 = \text{BD-}\lim_n \varphi_n$ on $\hat{G}_0$. Clearly $u = \text{CD-}\lim_n u\varphi_n$ on G_0. Let $v = u - c$ and let φ_n vanish on $\hat{G}_0$ outside of a regular region S. Then the relative boundary $\partial(G_0 \cap S)$ with respect to R consists of a finite number of piece-wise analytic simple curves on which $v\varphi_n = 0$. By Green's formula

$$D_{G_0}(v\varphi_n, u) = \int_{\partial(G_0 \cap S)} v\varphi_n {*du} = 0.$$

It follows that

$$D_{G_0}(u) = D_{G_0}(v, u) = \lim_n D_{G_0}(v\varphi_n, u) = 0.$$

As a consequence $u \equiv c$ on G_0, which contradicts $u(z_0) > c$.

q.e.d.

2K. Duality.

Recall that Δ is defined in terms of $M_\Delta(R)$ by

$\Delta = \{p \in R^* : f(p) = 0$ for every $f \in M_\Delta(R)\}$.

But the following duality relation is valid between $M_\Delta(R)$ and the harmonic boundary Δ:

Theorem 1.2K. $M_\Delta(R) = \{f \in M(R) : f = 0$ on $\Delta\}$

Proof. We may assume that $\Delta \neq \emptyset$ since if $\Delta = \emptyset$ then the assertion is trivial. Let $f \in M(R)$ and $f = 0$ on Δ. For a regular exhaustion $\{R_n\}_1^\infty$ of R we construct continuous functions u_n on R with $u_n|R_n \in H(R_n)$ and $u_n|(R - R_n) = f$. Then $u_n \in M(R)$ and $|u_n| \leq \sup_R |f|$ on R. By Green's formula

$$D_R(u_{n+p} - u_n, u_{n+p}) = \int_{\partial R_{n+p}} (u_{n+p} - u) * du_{n+p} = 0$$

and hence $D_R(u_{n+p} - u_n) = D_R(u_n) - D_R(u_{n+p})$.

Therefore $\{u_n\}_1^\infty$ is D-Cauchy. By choosing a suitable subsequence we may assume that $u = \text{BD-}\lim_n u_n$ exists. Clearly $u \in HD(R)$. Since $u_n - f \in M_0(R)$, $u - f \in M_\Delta(R)$ or $u = f = 0$ on Δ. By Theorem 1.2H.2, $u \equiv 0$ on R and $f \in M_\Delta(R)$.

q.e.d.

§3. Orthogonal Projection

3A. Quasi-Dirichlet finiteness.

A real-valued function f on R is called *quasi-Dirichlet finite* if $D_R((f \wedge c) \vee (-d))$ exists and is finite for all nonnegative numbers c and d.

As an immediate consequence of the above definition we have the following observations:

(a) A Tonelli function f on R is quasi-Dirichlet finite if and only if $(f \wedge c) \vee (-d) \in M(R)$ for every pair of nonnegative numbers c and d.

(b) If f is quasi-Dirichlet finite then so is αf for any real number α.

Theorem 1.3A.1. If f_1 and f_2 are nonnegative and quasi-Dirichlet finite, then the same is true for $f_1 + f_2$.

<u>Proof</u>. We must show that $D_R((f_1 + f_2) \wedge c) < \infty$. To do this let $A_j = \{z \in R : f_j(z) > c\}$ $(j = 1, 2)$ and $B = \{z \in R : f_1(z) + f_2(z) > c\}$. Then $B \supset A_j$ $(j = 1, 2)$ and hence

$$\sqrt{D_R((f_1 + f_2) \wedge c)} = \sqrt{D_{R-B}(f_1 + f_2)} \leqq \sqrt{D_{R-B}(f_1)}$$

$$+ \sqrt{D_{R-B}(f_2)} \leqq \sqrt{D_{R-A_1}(f_1)} + \sqrt{D_{R-A_2}(f_2)}$$

$$= \sqrt{D_R(f_1 \wedge c)} + \sqrt{D_R(f_2 \wedge c)} < \infty .$$

q.e.d.

<u>Theorem</u> 1.3A.2. If f is continuous on R, a Tonelli function on $R - \{|f| = \infty\}$, and quasi-Dirichlet finite on R. Then f has a continuous extension to R^*.

<u>Proof</u>. First assume that $f \geq 0$ on R. For each $n = 1, 2, \ldots$ we have $f \wedge n \in M(R)$ by the quasi-Dirichlet finiteness and so $f \wedge n$ can be continuously extended to R^* in a unique manner. We denote by f_n this continuous extension. Since $\{f_n\}_1^\infty$ is nondecreasing on R^* we can define $h(p) = \lim_n f_n(p)$ on R^*. For $m > n$ we have $(f \wedge m) \wedge n = f \wedge n$ on R and thus $f_m \wedge n = f_n$ on R. Assume that $h(p_0) < \infty$ for some $p_0 \in R^*$ and take an integer n such that $h(p_0) < n$. Then $f_n(p_0) < n$.

Choose a neighborhood U of p_0 such that $f_n(p) < n$ on U. Then $f_m(p) \wedge n = f_n(p) < n$ on U implies that $f_m(p) = f_n(p)$ on U for all $m > n$. Therefore $h|U = f_n|U$ is continuous on U. Next assume $h(p_0) = \infty$. Then for any $c > 0$ there exists an f_n such that $f_n(p_0) > c$. Let U be a neighborhood of p_0 with $f_n(p) > c$ on U. Then $h(p) > c$ on U, i.e. h is continuous at p_0. Clearly $f_n = f \wedge n$ on R and thus $h|R = f$.

In the general case of real-valued f, $g = f \vee 0$ and $h = -(f \wedge 0)$ satisfy the conditions of the theorem if f does, and g and h can be continuously extended to R^*. We use the same notation g and h for these extensions. It is easily seen that $g - h$ has a definite meaning on R^* and is continuous on R^*. Since $(g - h)|R = f$ the assertion is true for f.

q.e.d.

<u>Corollary</u> 1.3A. Dirichlet finite Tonelli functions on R have continuous extensions to R^*.

3B. <u>Orthogonal Decomposition.</u>

Let K be a compact set in R^* such that $\overline{K \cap R} = K$ and for any $z \in \partial(K \cap R)$ there is a disk U such that

$U \cap \partial(K \cap R)$ is a piecewise analytic arc joining two different points in ∂U. We call such a compact set K in R^* a <u>distinguished compact set</u>.

We make the following convention: if $R \in O_G$, then $HD(R) = \{0\}$. This convention which may seem somewhat artificial will turn out to be in a sense quite natural. At this point we remark that $O_G \subset O_{HD}$. Thus $HD(R) = \{0\}$ for $R \in O_G$ means that we shall not consider nonzero constants as harmonic functions.

Now we establish the following decomposition theorem, one of the fundamental theorems in the study of HD-functions.

<u>Theorem</u> 1.3B. Let f be a Dirichlet finite Tonelli function on R and $K \subset R^*$ a distinguished compact set which may be empty. Then

(a) f extended to R^* can be uniquely decomposed into the form $f = u + g$ where u and g are Dirichlet finite Tonelli functions on R with $u \in HD(R - K)$ and $g = 0$ on $K \cup \Delta$,

(b) $D_R(f) = D_R(u) + D_R(g)$,

(c) $|u| \leq \sup_{(\partial(K \cap R)) \cup \Delta} |f|$ on $R - K$,

(d) $D_R(u, \varphi) = 0$ for any Dirichlet finite Tonelli function φ on R with $\varphi = 0$ on $K \cup \Delta$,

(e) if $R \notin O_G$ or $K \neq \emptyset$ and v is super-harmonic (resp.) subharmonic) on $R - K$ with $v \geq f$ (resp. $v \leq f$) on $R - K$ then $v \geq u$ (resp. $v \leq u$) on $R - K$.

We shall prove this theorem in 3K.

3C. Reformulation.

We denote by $\hat{M}(R)$ the class of Dirichlet finite Tonelli functions, i.e. function f on R satisfying (M.2) and (M.3) of 1B. Thus $\hat{M}(R) \supset M(R)$ and $\hat{M}(R) \subset C(R^*)$. Let $\hat{M}_{\Delta \cup K}(R)$ be the set of functions $f \in \hat{M}(R)$ such that $f = 0$ on $K \cup \Delta$ where K is a distinguished compact set in R^*. If $K = \emptyset$, then we write $\hat{M}_\Delta(R)$ instead of $\hat{M}_{\Delta \cup \emptyset}(R)$. From Theorem 1.3B. we obtain:

Corollary 1.3C.1. The CD-closure of $\{f : f \in M_0(R)$, $f = 0$ on $K\}$ is $\hat{M}_{\Delta \cup K}(R)$ where K is a distinguished compact set in R^*.

Corollary 1.3C.2. The orthogonal decomposition $\hat{M}(R) = HD(R - K) + \hat{M}_{\Delta \cup K}(R)$ holds for any distinguished compact set K in R^* including $K = \emptyset$.

Corollary 1.3C.3. $M(R) = HBD(R) + M_\Delta(R)$.

Furthermore, we have the following observation:

$R \in O_G$ if and only if $\hat{M}(R) = \hat{M}_\Delta(R)$.

Hence if $R \notin O_G$, we have $\hat{M}(R) = HD(R) + \hat{M}_\Delta(R)$, and in order that this formula be valid even for $R \in O_G$ it is necessary and sufficient to assume $HD(R) = \{0\}$ for $R \in O_G$. This is the meaning of our convention in 3B.

3D. Harmonic Projection.

Let K be a distinguished compact set in R^*. For $f \in \hat{M}(R)$ we denote by $\pi_K f$ the function u of Theorem 1.3B. $\pi_K f$ is in $HD(R - K)$, coincide with f on $K \cup \Delta$, and is continuous on R^*. For $K = \emptyset$ we simply write $\pi = \pi_\emptyset$. We call $\pi_K f$ the harmonic projection of f on $R - K$, and the operator π_K the harmonizer on $R - K$.

3E. HD-Minimal Functions.

A positive HD-function u on R which is not identically zero is called HD-minimal if for any $v \in HD(R)$, $u \geq v \geq 0$ on R implies the existence of a constant $c(v)$ such that $v = c(v)u$ on R.

If $R \in O_G$ then $HD(R) = \{0\}$ by our convention and therefore there is no HD-minimal function on R. If

$R \in O_{HD} - O_G$ then each $f \in HD(R)$ is constant, and HD-minimal unless $f \leq 0$.

The HD-minimality is characterized in terms of Δ as follows:

Theorem 1.3D. An HD-function u on R is HD-minimal if and only if there exists an isolated point $p \in \Delta$ such that $0 < u(p) < \infty$ and $u = 0$ on $\Delta - \{p\}$. In particular an HD-minimal function is automatically strictly positive and bounded.

Proof. Assume that $u \in HD(R)$, $0 < u(p) < \infty$, and $u = 0$ on $\Delta - \{p\}$ for an isolated point p in Δ. By Theorem 1.2H.2 $u \geq 0$ on R. By Theorem 1.2D there exists an $f \in M(R)$ such that $f(p) = 1$ and $f = 0$ on $\Delta - \{p\}$. The function $u_0 = \pi f$ is in $HBD(R)$ with $u_0(p) = 1$ and $u_0 = 0$ on $\Delta - \{p\}$. For any c in $(0, u(p))$ the difference $u - cu_0$ is in $HD(R)$ and is nonnegative on Δ. Hence by Theorem 1.2H.2 $u \geq cu_0$ on R. Fix a point $z_0 \in R$, then $c \leq u(z_0)/u_0(z_0)$. This means that $0 < u(p) < \infty$, i.e. $u \in HBD(R)$. For any $v \in HD(R)$ with $u \geq v \geq 0$ on R we have $v = 0$ on $\Delta - \{p\}$ and $0 \leq v(p) < \infty$. As a consequence $c(v)u - v$ with $c(v) = v(p)/u(p)$ vanishes identically on Δ, i.e. $v = c(v)u$ and u is an HD-minimal function.

Conversely assume that u is HD-minimal on R. Of course $u \geq 0$ and $u \not\equiv 0$ on Δ. Thus we can find a point $p \in \Delta$ with $u(p) > 0$. Suppose that there exists a point $q \in \Delta$, $q \neq p$ such that $u(q) > 0$. Take $f \in M(R)$ such that $f(p) = 1$, $f(q) = 0$, and $0 \leq f \leq 1$ on R^*. Consider $v = \pi(fu)$. Clearly $0 \leq v = fu \leq u$ on Δ. Hence $0 \leq v \leq u$ on R and so we can find a constant $c(v)$ such that $v = c(v)u$. We now encounter the contradiction $0 = v(q) = c(v)u(q) > 0$. If follows that u vanishes on Δ except at p. Continuity of u on Δ implies that p is isolated in Δ, $u(p) > 0$, and $u = 0$ on $\Delta - \{p\}$. From the first part of the proof we have $u \in HBD(R)$.

q.e.d.

Corollary 1.3E. Let $p \in \Delta$ be isolated in Δ. Then there always exists a $u \in HBD(R)$ such that $u(p) = 1$ and $u = 0$ on $\Delta - \{p\}$. Moreover any HD-function v on R has a finite value at p.

Proof. Only the second part requires proof. The functions $v_1 = \pi(v \vee 0)$ and $v_2 = -\pi(v \wedge 0)$ are in $HD(R)$ and are nonnegative. Since $v_1 - v_2 = v \vee 0 + v \wedge 0 = v$ on Δ, $v = v_1 - v_2$ on R. Thus we may assume that $v > 0$ on R. Suppose $v(p) = \infty$. Then $v - nu \geq 0$ on Δ and therefore $v \geq nu$ on R for any $n = 1, 2, \ldots$. This implies that $v \equiv \infty$, a contradiction.

q.e.d.

3F. Evans' Superharmonic Functions.

We have seen thus far that the set $\Gamma - \Delta$ is in a sense harmonically negligible. This is best illustrated by the following property of $\Gamma - \Delta$:

Theorem 1.3F. Let F be an arbitrary nonempty compact set in $\Gamma - \Delta$. There exists a continuous positive superharmonic function $v \in \hat{M}(R)$ with $v = \infty$ on F and $v = 0$ on Δ.

Proof. Let K be a distinguished compact set in R^* such that $K \cap \Delta = \emptyset$ and the interior K^0 of K contains F. For a regular exhaustion $\{R_n\}_1^\infty$ of R set $K_n = K - R_n$. Take an $f \in M(R)$ such that $f|K = 1$, $f|\Delta = 0$, and $0 \leq f \leq 1$ on R^*. Let $v_n = \pi_{K_n} f \in M(R)$. Since $\pi_{K_{n+p}}(\pi_{K_n} f) = \pi_{K_{n+p}} f$, by Theorem 1.3B and Theorem 1.2H.2 we obtain $D_R(v_n - v_{n+p}) = D_R(v_n) - D_R(v_{n+p})$ and

$0 \leq v_{n+p} \leq v_n \leq 1$ on R^*. Hence $\{v_n\}_1^\infty$ is BD-Cauchy. Let $v_0 = \text{BD-}\lim_n v_n$ on R. Then $v_0 \in HBD(R)$ and since $0 \leq v_0 \leq v_n$ on R, $v_0 = 0$ on Δ, i.e. $\text{BD-}\lim_n v_n \equiv 0$ on R.

Let z_0 be a fixed point in R_1. We can choose a subsequence $\{n_k\}_1^\infty$ of positive integers such that $v_{n_k}(Z_0) < 2^{-k}$ and $\sqrt{D_R(v_{n_k})} < 2^{-k}$. Let $v'_m = \sum_1^m v_{n_k}$ and $v = \sum_1^\infty v_{n_k}$ on R. Clearly $v = \text{CD-}\lim_m v'_m$ on R and thus $v \in \hat{M}(R)$. It is also easy to see that v is positive and superharmonic on R. Observe that

$$\frac{v}{1+v} = \text{BD-}\lim_m \frac{v'_m}{1+v'_m} \quad \text{on } R.$$

Since v'_m is in $M_\Delta(R)$ so is $v'_m/\ 1 + v'_m$ and hence by Theorem 1.1G $v/1 + v \in M_\Delta(R)$, i.e. $v = 0$ on Δ. On the other hand $v > v'_m = m$ on $K_{n_m} \supset F$. Therefore $v = \infty$ on F.

q.e.d.

3G. Another Maximum Principle.

Theorem 1.3G. Let G be a subregion of R (G may be R) and u a superharmonic function on G bounded from below. Suppose that

$$\liminf_{z \in G,\ z \to p} u(z) \geq m$$

for any $p \in (\overline{G} \cap \Delta) \cup \partial G$. Then $u \geq m$ on G.

The analogue, mutatis mutandis, is true for subharmonic functions.

Proof. It suffices to consider the case where u is superharmonic. We define a function u' on $\gamma = \overline{G} - G$ by $u'(p) = \liminf_{z \in G,\ z \to p} u(z)$. Then u' is lower semicontinuous on γ. Let c be an arbitrary real number with $m > c$ and consider the open set

$$U = \{p \in \gamma : u'(p) > c\}$$

in γ. Since $u'(p) \geq m > c$ for every $p \in (\overline{G} \cap \Delta) \cup \partial G$, $U \supset (\overline{G} \cap \Delta) \cup \partial G$. Hence $F = \gamma - U$ is a compact subset of $\Gamma - \Delta$.

Let v be as in Theorem 1.3F. For each $n = 1, 2, \ldots$ set $w = u + v/n$ on G. Then w is superharmonic, bounded from below on G, and

$$\liminf_{z \in G,\ z \to p} w(z) > c$$

for every $p \in \gamma$. Thus $w(z) > c$ on G. On letting $n \to \infty$ we obtain the desire conclusion.

q.e.d.

3H. Dirichlet Integral of the Harmonic Measure.

Let K be a distinguished compact set in R, and w_K the lower envelope of the family of nonnegative superharmonic functions on R which dominate 1 on K. Then w_K is called the harmonic measure of K relative to R. Immediately we have the observation:

$w_K \in M_\Delta(R)$, $w_K|K = 1$, $w_K|\Delta = 0$, and

$$D_R(w_K) = \int_{-\partial K} {*dw_K} = \int_{-\partial R_0} {*dw_K}$$

where R_0 is a regular region containing K.

Theorem 1.3H. Let K be a nonempty distinguished compact set in R^* with $K \cap \Delta = \emptyset$ and set $u(z) = \inf\{v(z) : v \in \mathfrak{F}_K\}$ on R where $\mathfrak{F}_K$ is the family of nonnegative superharmonic functions v on R such that $v|K \cap R \geq 1$. The function u has the following properties:

(a) $u \in M(R)$, $u|K = 1$, $u|\Delta = 0$, and $u \in HD(R - K)$,

(b) $D_R(u) = \int_{-\partial K} {*du}$,

(c) if K' is a distinguished compact set in R^* such that the interior K'^0 of K' contains K, $K' \cap \Delta = \emptyset$ and $\int_{\partial K'} |{*du}| < \infty$

then $\displaystyle\int_{\partial K'} *du = \int_{\partial K} *du.$

Proof. Since $\mathfrak{F}_K$ is a Perron family (cf. II. §2, [2] p. 14) on $R - K$ and each point of $\partial(R - K)$ is regular for the Dirichlet problem (cf. [2] p. 25) u is nonnegative and harmonic on $R - K$, $u = 1$ on K, and u is continuous on R. Let $f \in M(R)$ such that $f|K = 1$, $f|\Delta = 0$, and $0 \leq f \leq 1$ on R^*. Then

$$\liminf_{z \in R,\ z \to p} (u(z) - (\pi_K f)(z)) \geq 0$$

for every $p \in \Delta \cup \partial K$. Thus by Theorem 1.3G, $u \geq \pi_K f$ on R and since $\pi_K f \in \mathfrak{F}_K$, $u = \pi_K f$ on R. This proves (a).

Let $\{R_n\}_1^\infty$ be a regular exhaustion of R and set $K_n = \overline{R}_n \cap K$ $(n = 1, 2, \ldots)$, $K_\infty = K$. Then $\pi_{K_n}(\pi_{K_{n+p}} f) = \pi_{K_n} f$ $(p = 1, 2, \ldots, \infty)$. Hence by Theorem 1.3B we obtain $D(\pi_{K_{n+p}} f - \pi_{K_n} f) = D(\pi_{K_{n+p}} f) - D(\pi_{K_n} f)$ and $D(\pi_{K_n} f) \leq D(\pi_K f) = D(u)$. Thus $\{\pi_{K_n} f\}$ is D-Cauchy. Since $\{\pi_{K_n} f\}$ is nondecreasing and $\pi_{K_n} f = u$ on K_n it is easy to see that

$$u = \pi_K f = \text{BD-}\lim_n \pi_{K_n} f \quad \text{on} \quad R.$$

For a fixed n and for $m > n$ let $v_m \in M_0(R)$

such that $v_m|K_n = 1$, $v_m|(R - R_m) = 0$, and $v_m \in H(R_m - K_n)$. As in the proof of Theorem 1.3B we can readily deduce that $\pi_{K_n} f = \text{BD-}\lim_m v_m$. Since $u/K = 1$ and $v_m|(R - R_m) = 0$ we obtain by Green's formula

$$D_R(v_m, u) = D_{R_m-K}(v_m, u) = \int_{\partial(R_m-K)} v_m{*du} = \int_{-\partial K} v_m{*du}.$$

From $du > 0$ along $-\partial K$ and the fact that $\{v_m\}_1^\infty$ is nondecreasing on ∂K we conclude by Lebesgue's convergence theorem:

$$D_R(\pi_{K_n} f, u) = \lim_m D_R(v_m, u) = \lim_m \int_{-\partial K} v_m{*du} = \int_{-\partial K} \pi_{K_n} f{*du}.$$

Again since $\pi_{K_n} f$ increases to 1 on ∂K it follows that

$$D_R(u) = \lim_n D_R(\pi_{K_n} f, u) = \lim_n \int_{-\partial K} \pi_{K_n} f{*du} = \int_{-\partial K} {*du}.$$

Thus we have proved (b).

We turn to (c). Let $(K'^0 - K) \cap R = \cup_{n=1}^\infty S_n$ be the decomposition into components. If we can prove that

$$\int_{(\partial S_n)\cap(\partial(K'^0-K))} {*du} = 0$$ for each S_n then we shall have

$$\int_{\partial(K'^0-K)} {*du} = \sum_1^\infty \int_{(\partial S_n)\cap(\partial(K'^0-K))} {*du} = 0$$

since $\int_{\partial(K'^0-K)} |*du| < \infty$. Therefore we may assume without loss of generality that $S = (K'^0 - K) \cap R$ is connected. In view of $\overline{S} \cap \Delta = \emptyset$ by the remark in 2G $\hat{S} \in O_G$. Let $\{R_n\}_0^\infty$ be an exhaustion of R such that $\overline{R}_0 \subset S$. Take a function w_n continuous on $S \cup \partial S$ and such that $w_n|\overline{R}_0 = 1$, $w_n|(S - R_n) = 0$, $w_n \in H(R_n \cap S - \overline{R}_0)$, and $*dw_n = 0$ on ∂S.

The symmetric extension $\hat{w}_n$ of w_n to $\hat{S}$ is the harmonic measure of $\overline{R}_0 \cup \overline{G}_0$ with respect to $(S \cap R_n) \cup H_n$, where G_0 and H_n are the reflections about ∂S of R_0 and $(S \cap R_n) \cup ((\partial S) \cap R_n)$ in $\hat{S}$. From $\hat{S} \in O_g$ it follows that $\text{BD-}\lim_n w_n = 1$ on S.

By Green's formula $D_S(w_n, u) = \int_{\partial S} w_n *du - \int_{\partial R_0} *du.$

Since u is harmonic on $\overline{R}_0$

$$D_S(w_n, u) = \int_{\partial S} w_n *du.$$

From $|w_n *du| \leqq |*du|$ on ∂S, $\int_{\partial S} |*du| < \infty$, and $\lim_n w_n = 1$ on ∂S we conclude by Lebesgue's convergence theorem that

$$\int_{\partial S} *du = \lim_n \int_{\partial S} w_n *du = \lim_n D_S(w_n, u) = D_S(1, u) = 0.$$

q.e.d.

3K. Proof of Theorem 1.3B.

If $K = \emptyset$ and $R \in O_G$ then our assertion is trivial. The same is true if R is closed and $K \neq \emptyset$. We therefore exclude these cases. Moreover we may assume that f is real-valued.

Let $\{R_n\}_0^\infty$ be a regular exhaustion of R with $\overline{R}_0 \subset R - K$, and for $n \geq 1$ let u_n, u_n', and u_n'' be harmonic on $R_n - K$ and continuous on R with

$u_n (u_n'$ or $u_n'') = f$ $(f \vee 0$ or $-(f \wedge 0))$ on $R - (R_n - K)$.

Since u_n', $u_{n+p}' \in M(R_{n+p+1})$ we may use Green's formula:

$$D_R(u_{n+p}' - u_n', u_{n+p}') = \int_{\partial(R_{n+p}-K)} (u_{n+p}' - u_n') *du_{n+p}' = 0.$$

Thus $D_R(u_{n+p}' - u_n') = D_R(u_n') - D_R(u_{n+p}')$ and $\{u_n'\}_1^\infty$ is D-Cauchy. For $g_n' = f \vee 0 - u_n'$ we have similarly

$$D_R(u_n') + D_R(g_n') = D_R(f \vee 0) \leq D_R(f). \qquad (*)$$

Let w_n be continuous on R with $w_n | \overline{R}_0 = 1$, $w_n | (R - (R_n - K)) = 0$, and $w_n \in H(R_n - K - \overline{R}_0)$. By Green's formula

$$D_R(g_n', w_n) = \int_{\partial(R_n - R_0)} g_n' * dw_n = \int_{-\partial R_0} g_n' * dw_n$$

$$= \int_{-\partial R_0} (f \vee 0) * dw_n - \int_{-\partial R_0} u_n' * dw_n .$$

On setting $a_n = \inf_{\partial R_0} u_n'$ and $b = \sup_{\partial R_0} (f \vee 0)$ we see that

$$a_n D_R(w_n) = a_n \int_{-\partial R_0} * dw_n \leq \int_{-\partial R_0} u_n' * dw_n$$

$$\leq b \int_{-\partial R_0} * dw_n - D_R(g_n', w_n) \leq b D_R(w_n) + \sqrt{D_R(f) \cdot D_R(w_n)} .$$

Hence we obtain $a_n \leq b + \sqrt{D_R(f)/D_R(w_n)}$. By our assumption R is not in O_G if $K = \emptyset$, and thus $\limsup_n a_n < \infty$. In view of Harnack's inequality $\{u_n'\}$ is bounded on R_0. On choosing a suitable subsequence we may therefore assume that the sequence $\{u_n'\}_1^\infty$ is C-Cauchy on R. If $K \neq \emptyset$ and $R \in O_G$, then again $\{u_n'\}$ is C-Cauchy because of (*) and the fact that $u_{n+p}' - u_n'$ vanishes along some subarc of $\partial(K \cap R)$. It is thus legitimate to suppose that $\{u_n'\}$ is CD-Cauchy on R.

Similarly $\{u_n''\}$ may be assumed to be CD-Cauchy on R and the same is true of $\{u_n\}$. Let

$$u = \text{CD-}\lim_n u_n$$

on R. Then u is a Dirichlet finite Tonelli function on R and harmonic on $R - K$, with $u = f$ on $R \cap K$ and hence on $K = \overline{R \cap K}$.

Let $g_n = f - u_n$ and $g = f - u$. Then $g_n \in M_0(R)$, $g_n = 0$ on K, and $g = \text{CD-}\lim_n g_n$ on R. Thus $g = 0$ on K. It is also easy to see that

$$\frac{g}{1 + |g|} = \text{BD-}\lim_n \frac{g_n}{1 + |g_n|}$$

on R, and since $g_n/(1 + |g_n|) \in M_0(R)$, $g/(1 + |g|) \in M_\Delta(R)$. Hence $g = 0$ on Δ, and $f = u + g$ is a required decomposition.

Let $f = u' + g'$ be another decomposition. Then $u - u' = 0$ on $\Delta \cup K$ and therefore $u \equiv u'$ on $R - K$ by Theorem 1.2H.2. Hence $u \equiv u'$ on R and the decomposition is unique. This completes the proof of (a).

By the same theorem, (c) is trivially valid. Assertion (e) is clear if we observe that $v \geq u_n$ (resp. $v \leq u_n$) on $R_n - K$.

Finally we prove (d), from which (b) will follow. Let $\varphi = 0$ on $K \cup \Delta$. By the above proof φ is the CD-limit of a sequence $\{\varphi_n\}_1^\infty \subset M_0(R)$ such that $\varphi_n = 0$ on K. By Green's formula $D_R(\varphi_n, u) = 0$ and thus $D_R(\varphi, u) = 0$ in the limit.

II. HAMONIC MEASURE.

One of the benefits of the existence of a boundary of a Riemann surface lies in the possibility of the integral representation. In the case of the unit disk the Poisson formula gives explicitly the behavior of harmonic functions. For the integral representation we shall need the existence of a measure which we call the harmonic measure on Royden's boundary.

The connection between the Dirichlet principle and the Perron method is of considerable interest. We shall discuss the Dirichlet problem from the Perron viewpoint. This is used to study ideal boundary points of subregions.

§1 introduces the harmonic measure and the kernel which leads to the integral representation of harmonic functions on R. §2 treats the Dirichlet problem by using the Perron method. Also in §2 we state the relation between the closure of a subregion and its Royden's compactification. Moreover we discuss the relation between the harmonic measures on Royden's boundaries of R and its subregion G in §2.

§1. Harmonic Measure and Kernel.

1A. Harmonic Measure on Γ.

We shall assume throughout this section that R is an open Riemann surface. The harmonic measure μ on Royden's boundary Γ of R with respect to a fixed point $z_0 \in R$ is the measure on Γ such that

(μ.1) μ is a positive regular Borel measure on Γ,

(μ.2) every superharmonic function $v \in M(R)$ has the Gauss property $v(z_0) \geq \int_\Gamma v(p)d\mu(p)$.

The point z_0 will be referred to as the center of of the measure μ. We denote by S_μ the support of μ in Γ. As an immediate consequence of (μ.2) we have

$$u(z_0) = \int_\Gamma u(p)d\mu(p) \tag{13}$$

for every $u \in HBD(R)$. From Theorem 1.3B and property (μ.3) proved below we have that

$$v(z_0) \geq \pi v(z_0) = \int_\Gamma \pi v(p)d\mu(p) = \int_\Gamma v(p)d\mu(p),$$

and thus (μ.2) and (13) are equivalent.

Theorem 2.1A. Suppose that $R \notin O_G$. There exists a unique harmonic measure μ on Γ with respect to an arbitrary fixed center $z_0 \in R$. It satisfies the following conditions:

(μ.3) $S_\mu = \Delta$,

(μ.4) $\mu(\Delta) = 1$,

(μ.5) for any open set U in Γ with $U \cap \Delta \neq \emptyset$, $\mu(U) > 0$.

Proof. First assume the existence of a μ with (μ.1) and (μ.2). Take an arbitrary nonempty compact set K in $\Gamma - \Delta$ and let $v_n \in M(R)$ be the function $v \wedge n$ obtained from v of Theorem 1.3F for K. By (μ.2)

$$\frac{1}{n} v(z_0) \geq \frac{1}{n} v_n(z_0) \geq \int_\Gamma \frac{v(p) \wedge n}{n} d\mu(p) \geqq \mu(K).$$

Thus on letting $n \to \infty$ we obtain $\mu(K) = 0$. By the regularity of μ we have that $S_\mu \subset \Delta$. Let U be open in Γ with $U \cap \Delta \neq \emptyset$, and F an arbitrary compact set in $U \cap \Delta$. Take an $f \in M(R)$ with $f|F = 1$, $f|(\Delta - U \cap \Delta) = 0$, and $0 \leq f \leq 1$ on R^*. Set $u = \pi f \in HBD(R)$. By (13) and $S_\mu \subset \Delta$

$$u(z_0) = \int_\Gamma u(p) d\mu(p) = \int_\Delta u(p) d\mu(p) \leq \int_U d\mu(p) = \mu(U).$$

Since $u(z_0) > 0$ we have (μ.5) and further (μ.3). By setting $u \equiv 1$ in (13) we obtain (μ.4).

If μ' is another measure with (μ.1) and (μ.2), then as we have seen above, μ' satisfies (μ.3), (μ.4), and (μ.5). Let $f \in M(R)$. By (13) and (μ.3)

$$(\pi f)(z_0) = \int_\Delta f(p)\,d\mu(p) = \int_\Delta f(p)\,d\mu'(p).$$

Let $B(\Delta)$ be the space of bounded continuous functions on Δ. Since $M(R)|\Delta$ is dense in $B(\Delta)$ with respect to the U-topology we conclude that $\mu = \mu'$.

Finally we shall prove the existence of a μ with (μ.1) and (μ.2) and a fortiori (μ.3). Observe that

$$\sup_{z\in R} |(\pi f)(Z)| = \sup_{p\in\Delta} |f(p)| \tag{14}$$

and that $M(R)$ is dense in $B(R^*)$ with respect to the U-topology, i.e. the topology induced by $\sup_{z\in R} |f(z)|$.

Thus π can be extended to $B(R^*)$, (14) is also satisfied and $\pi f \in HB(R) \cap B(R^*)$. Let $f \in B(\Delta)$ and $f' \in B(R^*)$ with $f = f'|\Delta$. We set $\rho f = \pi f'$. By (14), ρf is independent of the choice of f' and $\rho f \in HB(R) \cap B(R^*)$ with

$$\sup_{z\in R} |(\rho f)(z)| = \sup_{p\in\Delta} |f(p)|.$$

Thus $f \to (\rho f)(z_0)$ is a bounded linear functional on $B(\Delta)$ and hence there exists a measure μ on Δ with (μ.1) and (μ.3) such that

$$(\rho f)(z_0) = \int_\Delta f(p)\,d\mu(p).$$

If $u \in HBD(R)$ then $\rho u = u$ on R^*. On setting

$\mu(\Gamma - \Delta) = 0$ we obtain (13) and conclude that μ satisfies (μ.2).

q.e.d.

1B. Harmonic Kernel.

Suppose that $R \notin O_G$ and the center of the harmonic measure μ is $z_0 \in R$. The harmonic Kernel $P(z, p)$ associated with (R, μ, z_0) is the real-valued function on $R \times \Gamma$ such that

(P.1) $P(z_0, p) = 1$ on Γ,

(P.2) $P(z, p) = 0$ on $R \times (\Gamma - \Delta)$,

(P.3) $P(z, p) \in HP(R)$ for every fixed $p \in \Gamma$,

(P.4) for every fixed $z \in R$, $P(z, p)$ is a nonnegative bounded Borel function on Γ,

(P.5) for every superharmonic function $v \in M(R)$ and every point $z \in R$ the inequality

$$v(z) \geq \int_{\Gamma} P(z, p) v(p) d\mu(p) \quad \text{is valid.}$$

In particular (P.5) implies that

$$u(z) = \int_{\Gamma} P(z, p) u(p) d\mu(p) \qquad (15)$$

for every $z \in R$ and $u \in HBD(R)$. By considering πv

we can deduce (P.5) from (15), i.e. (P.5) and (15) are equivalent.

<u>Theorem</u> 2.1B. There exists a unique harmonic kernel $P(z, p)$ on $R \times \Gamma$ with respect to an arbitrary fixed point $z_0 \in R$.

Here the uniqueness means that for two harmonic kernels P and P' there exists a set E of μ-measure zero in Δ with $P \equiv P'$ on $R \times (\Gamma - E)$.

In fact, let $\{z_n\}_1^\infty$ be dense in R and $Q(z, p) = P(z, p) - P'(z, p)$. Since $Q(z, p) \equiv 0$ on $R \times (\Gamma - \Delta)$ it follows from (15) and the denseness of $M(R)|\Delta$ in $B(\Delta)$ that

$$\int_\Delta Q(z, p) f(p) d\mu(p) = 0 \tag{16}$$

for any $f \in B(\Delta)$. Thus for fixed $z \in R$, $Q(z, p) = 0$ on Δ except a set $E(Z)$ of μ-measure zero. Let

$E = \bigcup_{n=1}^\infty E(z_n)$. Then $\mu(E) = 0$ and $Q(z_n, p) = 0$ for $p \in \Gamma - E$ and for every $n = 1, 2, \ldots$. For a fixed $p \in \Gamma - E$, $Q(z, p) \in H(R)$. Thus the denseness of $\{z_n\}_1^\infty$ in R implies $Q(z, p) = 0$ for every $z \in R$ and a fixed $p \in \Gamma - E$. Since p is arbitrary we have established the assertion.

The proof of existence will be given in 1C and 1D.

1C. Harneck's Function.

Let k be a Harnack's function on $R \times R$, i.e. the function k defined by

$$k(z', z) = \inf\{c : c^{-1}u(z) \leq u(z') \leq cu(z) \text{ for every } u \in HP(R)\} \tag{16}$$

for every $(z', z) \in R \times R$ where HP is the set of positive harmonic functions on R.

Theorem 2.1C. $\log k(z', z)$ is finitely continuous on $R \times R$ and a semimetric on R.

Proof. Let $z' - z = re^{i\theta}$ in a closed parametric disk $|z' - z| < 1$. For any $u \in HP(R)$ we have

$$u(z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} u(e^{it} + z)dt$$

and thus

$$\frac{1 - r}{1 + r} u(z) \leq u(z') \leq \frac{1 + r}{1 - r} u(z).$$

Therefore

$$1 \leq k(z', z) \leq (1 + |z' - z|)/(1 - |z' - z|) \to 1$$

as $z' \to z$ and consequently $\log k(z', z) \to 0$ as $z' \to z$. It follows easily from the definition that $\log k(z', z)$ is a semimetric on R. By the triangle inequality $\log k(z', z)$ is finitely continuous on $R \times R$ since $\log K(z', z) \to 0$ as $z' \to z$. q.e.d.

Let μ_z be the harmonic measure on Γ with center z; $\mu_{z_0} = \mu$. For any nonnegative function $f \in B(\Delta)$,

$$(\pi f)(z) = \int_\Delta f(p)d\mu(p) \in HP(R).$$

Therefore

$$k(z, z_0)^{-1} \int_\Delta f(p)d\mu(p) \leq \int_\Delta f(p)d\mu_z(p) \leq k(z, z_0) \int_\Delta f(p)d\mu(p).$$

It follows that

$$k(z, z_0)^{-1}\mu(X) \leq \mu_z(X) \leq k(z, z_0)\mu(X) \tag{17}$$

for every Borel set X in Δ and hence in Γ. Let $\tilde{P}(z, p)$ be a function on $R \times \Gamma$ which can be used as the Radon-Nikodym derivative of μ_z with respect to μ. It is nonnegative and Borel measurable on Γ. Moreover

$$k(z, z_0)^{-1} \leq \tilde{P}(z, p) \leq k(z, z_0) \tag{18}$$

μ-a.e. on Γ for fixed $z \in R$. We may assume that $\tilde{P}(z_0, p) \equiv 1$ on Δ and $\tilde{P}(z, p) \equiv 0$ on $\Gamma - \Delta$ for every $Z \in R$. From (17) and that definition of k we can easily see that

$$k(z, z')^{-1}\tilde{P}(z, p) \leq \tilde{P}(z', p) \leq k(z', z)\tilde{P}(z, p) \tag{19}$$

μ-a.e. on Δ for fixed z, $z' \in R$.

Let T be a countable dense subset of R. Take a Borel subset $E(z, z')$ of Δ such that (19) is true for $p \in \Gamma - E(z, z')$ and $\mu(E(z, z')) = 0$. Set

$E = \bigcup_{z,z' \in T} E(z, z')$. Again $\mu(E) = 0$ and (19) is valid on $\Gamma - E$ for every z, $z' \in T$. In particular (19) holds on $\Gamma - E$ for all $z \in T$. Thus for z', $z \in T$ and $p \in \Gamma - E$

$$|\tilde{P}(z, p) - \tilde{P}(z', p)| \leq k(z, z_0) \max(k(z, z') - 1,$$
$$1 - k(z, z')^{-1}). \qquad (20)$$

Since $\log k(z, z')$ is a continuous semimetric on R, (20) implies

$$\lim_{z,z' \in T, z,z' \to z''} |\tilde{P}(z, p) - \tilde{P}(z', p)| = 0 \qquad (21)$$

for every $p \in \Gamma - E$. This means that $\lim_{z \in T, z \to z'} \tilde{P}(z, p)$ exists for every $z' \in R$ and $p \in \Gamma - E$, and if $z' \in T$ then the limit is $\tilde{P}(z', p)$.

Therefore it is possible to define

$$P(z, p) = \lim_{z' \in T, z' \to z} \tilde{P}(z', p) \qquad (21)$$

for $z \in R$ and $p \in \Gamma - E$. By (20), $P(z, p)$ is finitely continuous on R for each fixed $p \in \Gamma - E$.

1D. Harmonicity of $P(z, p)$.

For $z \in R$ take a sequence $\{z_n\}_1^\infty \subset T$ converging to z. For any $u \in HBD(R)$ we see by (18), Lebesgue's convergence theorem, and the definition of $P(z, p)$ that

$$u(z) = \lim_n u(z_n) = \lim_n \int_{\Delta - E} \tilde{P}(z_n, p) u(p) d\mu(p)$$

$$= \int_{\Delta - E} \lim_n \tilde{P}(z_n, p) u(p) d\mu(p)$$

$$= \int_{\Gamma} P(z, p) u(p) d\mu(p). \qquad (22)$$

Let z be a local parameter in $|z| < 1$ and $\{z_n\}_1^\infty$ a dense sequence in it. Then for any $f \in B(\Delta)$, since $\int_\Delta P(z, p) f(p) d\mu(p) \in H(|z| < 1)$, we have by Fubini's theorem

$$\int_\Delta P(z_n, p) f(p) d\mu(p)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\int_\Delta P(z_n + r_m e^{i\theta}, p) f(p) d\mu(p) \right] d\theta$$

$$= \int_\Delta \left[\frac{1}{2\pi} \int_0^{2\pi} P(z_n + r_m e^{i\theta}, p) d\theta \right] f(\theta) d\mu(\theta)$$

where $\{r_m\}_1^\infty$ is a dense sequence in $(0, 1 - |z_n|)$. Hence there exists a set $F_{n,m}$ in Δ with $\mu(F_{n,m}) = 0$ such that for any $p \in \Delta - F_{n,m}$

$$P(z_n, p) = \frac{1}{2\pi} \int_0^{2\pi} P(z_n + re^{i\theta}, p)\,d\theta \qquad (23)$$

with $r = r_m$. Let $F_n = (\cup_{m=1}^{\infty} F_{n,m}) \cup E$. Then $\mu(F_n) = 0$ and (23) holds for every $r = r_m$ $(m = 1, 2, \ldots)$ and p in $\Delta - F_n$. By the continuity of $P(z, p)$ in z for a fixed $p \in \Delta - E$ we obtain (23) for every $r \in (0, 1 - |z_n|)$ and $p \in \Gamma - F_n$. Finally let $F = \cup_{n=1}^{\infty} F_n$. Again $\mu(F) = 0$ and (23) holds for all $n = 1, 2, \ldots$ and for all $r \in (0, 1 - |z_n|)$ if z_n is fixed. By the continuity of $P(z, p)$ in z we conclude that

$$P(z, p) = \frac{1}{2\pi} \int_0^{2\pi} P(z + re^{i\theta}, p)\,d\theta \qquad (24)$$

for every $z \in \{|z| < 1\}$ and $r \in (0, 1 - |z|)$ with an arbitrarily fixed $p \in \Gamma - F$.

We set $P(z, p) \equiv 1$ on R for $p \in F$ and leave P unchanged for $p \in \Gamma - F$. Then (P.1) and (P.2) are satisfied in view of the definition of P, and (P.3) follows from (24). By (18) and the definition of P, (P.4) is seen to be valid. Finally (22) gives (15) and thus (P.5). Therefore we proved the existence of $P(z, p)$ satisfying the conditions (P.1) - (P.5) in 1B.

1E. Integral Representation.

By using the harmonic kernel we can prove a Schwartz-type theorem:

Theorem 1.1E. If f is μ-integrable on Γ, then

$$u(z) = \int_{\Gamma} P(z, p) f(p) d\mu(p) \tag{25}$$

is on R a harmonic function which is a C-limit of HBD-functions on R.

If in addition f is bounded on Δ and continuous at $q \in \Delta$ as a function on Δ then

$$\lim_{z \in R, z \to q} u(z) = f(q).$$

Proof. Since the restrictions of HBD-functions to Δ are dense in $B(\Delta)$ with respect to the U-topology and $B(\Delta)$ is dense in $L^1(\Delta, \mu)$ with respect to the L^1-norm there exists a sequence $\{u_n\}_1^\infty$ in $HBD(R)$ such that

$$\lim_{n \to \infty} \int_{\Gamma} |u_n(p) - f(p)| d\mu(p) = 0.$$

Since $u_n(z) = \int_{\Delta} P(z, p) u_n(p) d\mu(p)$ is in $HBD(R)$ and $u = C\text{-}\lim_{n \to \infty} u_n$ on R we have $u \in H(R)$.

Let $\varepsilon > 0$. If f is continuous at $q \in \Delta$ as a function on Δ then we can find a neighborhood U of q in Δ such that $|f(p) - f(q)| < \varepsilon$ for every $p \in U$. If f is bounded then there exists a $w \in HBD(R)$ with $w(q) = 0$ such that $|f(p) - f(q)| \leq w(p) + \varepsilon$ on Δ. If follows that

$$\left| \int_\Delta P(z, p) f(p) d\mu(p) - f(q) \right|$$

$$\leq \int_\Delta P(z, p) |f(p) - f(q)| d\mu(p)$$

$$\leq \int_\Delta P(z, p)(w(p) + \varepsilon) d\mu(p) = w(z) + \varepsilon.$$

Hence $\lim_{z \in R, z \to q} |u(z) - f(q)| \leq w(q) + \varepsilon = \varepsilon.$

q.e.d.

1F. <u>The Class $\widetilde{HD}$</u>.

We denote by $\widetilde{HD}(R)$ the class of harmonic functions $u \geq 0$ on R such that $u = \inf F$ where F is a lower directed subfamily of $HD(R)$, i.e. $u_1, u_2 \in F$ implies $u_1 \wedge u_2 \in F$. We have the following observation: $u \in \widetilde{HD}(R)$ if and only if there exists a nonincreasing sequence $\{u_n\}_1^\infty$ of nonnegative functions in $HD(R)$ such that $u = C\text{-}\lim_{n} u_n$ on R (cf. [2] p.18).

An <u>$\widetilde{HD}$-minimal function</u> u is a positive function in $\widetilde{HD}(R)$ such that for any $v \in \widetilde{HD}(R)$ with $0 \leq v \leq u$ there exists a constant $c(v)$ with $v = c(v)u$ on R. From the definition it follows that HD-minimal functions are $\widetilde{HD}$-minimal.

1G. <u>Upper Semicontinuous Functions on Δ.</u>

We denote by $U(\Delta)$ the set of nonnegative functions f on Δ such that

$$f(p) = \inf_{v \in F_f} v(p) \qquad (26)$$

for every $p \in \Delta$, where

$$F_f = \{v \in HD(R) : v \geq f \text{ on } \Delta\}.$$

We can see at once that F_f is a lower directed subfamily of $HE(R)$ and hence $f \in U(\Delta)$ is upper semicontinuous and μ-integrable on Δ.

It is also easily seen that $U(\Delta)$ is a half vector space and forms a lattice. Moreover we have the observation that if $f \geq 0$ is bounded and upper semicontinuous on Δ then $f \in U(\Delta)$.

We can see the significance of $U(\Delta)$ in the following theorem:

<u>Theorem</u> 2.1G. A function u belongs to $\widetilde{HD}$ if and only if there exists an $f \in U(\Delta)$ such that the integral representation

$$u(z) = \int_{\Delta} P(z, p) f(p) d\mu(p) \tag{27}$$

with the boundary function f is valid on R.

<u>Proof</u>. First assume that $u \in \widetilde{HD}(R)$ and set $F = \{v \in HD(R) : v \geq u \text{ on } R\}$. Then F is lower directed and $u(z) = \inf_{v \in F} v(z)$ on R. Set $f(p) = \inf_{v \in F} v(p)$ on Δ. Clearly $F_f = F$ and $f \in U(\Delta)$. By interchanging the directed infimum and the integration we obtain

$$u(z) = \inf_{v \in F} v(z) = \inf_{v \in F} \int_{\Delta} P(z, p) v(p) d\mu(p)$$

$$= \int_{\Delta} P(z, p) (\inf_{v \in F} v(p)) d\mu(p) = \int_{\Delta} P(z, p) f(p) d\mu(p).$$

Conversely assume that u is given by (27). Again we interchange the directed infimum and the integration:

$$u(z) = \int_{\Delta} P(z, p) (\inf_{v \in F_f} v(p)) d\mu(p)$$

$$= \inf_{v \in F_f} \int_{\Delta} P(z, p) v(p) d\mu(p) = \inf_{v \in F_f} v(z).$$

Let $F = \{v \in HD(R) : v \geq u$ on $R\}$. Then $F \supset F_f$ and $u(z) = \inf_{v \in F} v(z)$, i.e. $u \in \widetilde{HD}(R)$.

q.e.d.

1H. Boundary Functions.

For a real-valued function f on R we write

$$\overline{f}(p) = \lim_{z \in R,} \sup_{z \to p} f(z) \tag{28}$$

with $p \in \Delta$. Clearly $\overline{f}$ is upper semicontinuous on Δ. In the representation (27) the function f is not uniquely determined since we may change an f on a set in Δ of measure zero so that the resulting function is still upper semicontinuous. However we can find the least possible f as follows:

Theorem 2.1H. Assume that $u \in \widetilde{HD}(R)$ is expressed as (27) with an $f \in U(\Delta)$. Then

$$\overline{u}(p) \leq f(p) \tag{29}$$

everywhere on Δ and

$$\overline{u}(p) = f(p) \tag{30}$$

μ-almost everywhere on Δ. In particular

$$u(z) = \int_{\Delta} P(z, p)\overline{u}(p)\,d\mu(p). \tag{31}$$

<u>Proof</u>. $u(z) = \int_\Delta P(z, p) f(p) d\mu(p)$

$$= \inf_{v \in F_f} \int_\Delta P(z, p) v(p) d\mu(p)$$

$$= \inf_{v \in F_f} v(z)$$

on R. For any $v \in F_f$, $u \leq v$ on R implies $\bar{u}(p) \leq v(p)$ on Δ. Thus by $f(p) = \inf_{v \in F_f} v(p)$ we obtain (29).

To prove (30) we first assume that f and consequently u is bounded. Suppose $\bar{u}(p) < f(p) - \varepsilon (\varepsilon > 0)$ on a compact set $K \subset \Delta$. We wish to conclude that $\mu(K) = 0$. Suppose $\mu(K) > 0$. Then the function

$$w(z) = \varepsilon \int_K P(z, p) d\mu(p)$$

is in $HP(R)$ and $0 < w(z) \leq v$ on R. By

Theorem 2.1E. $\lim_{z \in R,\ z \to q} w(z) = 0$ for every $q \in \Delta - K$.

Thus

$$\limsup_{z \in R, z \to q} (u(z) + w(z)) = \bar{u}(q) \leq f(q)$$

for $q \in \Delta - K$, and

$$\limsup_{z \in R, z \to q} (u(z) + w(z)) \leq \bar{u}(q) + \varepsilon < f(q)$$

for $q \in K$. Therefore for every $v \in F_f$ and all $q \in \Delta$

$$\lim_{z \in R, z \to q} \inf(v(z)-(u(z) + w(z))) \geq 0$$

which by Theorem 1.2H.2

$$v \geq u + w \quad \text{on} \quad R.$$

We conclude that $u(z) \geq u(z) + w(z)$ and in particular $u(z_0) \geq u(z_0) + w(z_0) = u(z_0) + \varepsilon\mu(K)$; this contradicts $\mu(K) > 0$.

We now drop the assumption that f is bounded and set

$$u_n(z) = \int_\Delta P(z, p)(f(p) \wedge n)d\mu(p)$$

for $n = 1, 2, \ldots$. Since $u_n \in \widetilde{HD}(R)$ by virtue of $f \wedge n \in U(\Delta)$, we have $\overline{u}(p) \geq \overline{u}_n(p) = f(p) \wedge n$ on Δ μ-almost everywhere. If follows that $\overline{u}(p) \geq f(p)$ on Δ μ-almost everywhere. By (29) we obtain (30), and assertion (31) is trivial.

q.e.d.

§2. Perron's Method.

2A. Perron Family.

Let $R \notin O_G$ and consider a bounded real-valued function f on the Royden boundary Γ of R. We denote by $\mathfrak{G}(R^*, f)$ the class of superharmonic functions v on R such that

$$\liminf_{z \in R,\ z \to p} v(z) \geq f(p) \qquad (32)$$

for any point $p \in \Gamma$. We call the class $\mathfrak{G}(R^*, f)$ the <u>Perron family</u> with respect to R^* and f. Set

$$\overline{H}(z; R^*, f) = \overline{H}(z; f) = \inf \{v(z) : v \in \mathfrak{G}(R^*, f)\} \qquad (33)$$

and

$$\underline{H}(z; R^*, f) = \underline{H}(z; f) = -\overline{H}(z; -f) \qquad (34)$$

for $z \in R$. It is easy to see that $\underline{H}(z; f)$ and $\overline{H}(z; f)$ are harmonic on R and $\overline{H}(z; f) \geq \underline{H}(z; f)$ on R.

If $\overline{H}(z; f) = \underline{H}(z; f)$, then we denote this common function by $H(z; R^*, f) = H(z; f)$. In this case we say that f is <u>resolutive</u> with respect to R^*.

A point $p \in \Gamma$ is said to be a <u>regular point</u> for the Dirichlet problem with respect to R^* if

$$\lim_{z \in R,\, z \to p} H(z; f) = f(p) \qquad (35)$$

for any resolutive function f on Γ which is continuous at p. We shall show the relation between regular points and Δ in the following theorem:

<u>Theorem</u> 2.2A. A bounded real Borel function f on Γ is resolutive and

$$H(z;\ f) = \int_{\Gamma} P(z,\ p)f(p)d\mu(p) \tag{36}$$

on R. The set of all regular points in Γ coincides with Δ.

Proof. First we assume that f is continuous on Γ. Then

$$v(z) = \int_{\Gamma} P(z,\ p)f(p)d\mu(p) \in HB(R)$$

is continuous on R^* and $v(p) = f(p)$ on Δ by Theorem 2.1E. For any positive constant $\varepsilon > 0$ there exists an open set W in Γ such that $W \supset \Delta$ and

$$-\varepsilon < v(p) - f(p) < \varepsilon$$

for every $p \in W$. Let $K = \Gamma - W$. Then K is compact with $K \subset \Gamma - \Delta$ and there exists on R a positive superharmonic function u which is continuous on R^* with $u = \infty$ on K and $u = 0$ on Δ by Theorem I.3F. Clearly $v + \varepsilon + \varepsilon\mu \in \mathfrak{G}(R^*,\ f)$ and hence $\overline{H}(z,\ f) \leq v(z) + \varepsilon + \varepsilon\mu(z)$ on R. On letting $\varepsilon \to 0$ we obtain

$$\overline{H}(z,\ f) \leq v(z) \tag{37}$$

on R. Since $-f$ is continuous and

$$-v(z) = \int_{\Gamma} P(z,\ p)(-f(p))d\mu(p)$$

we obtain as above $\overline{H}(z;\ -f) \leq -v(z)$. From

$\underline{H}(z; f) = -\overline{H}(z; -f)$ we see that $\underline{H}(z; f) \geq v(z)$. This with (37) gives $\overline{H}(z; f) = \underline{H}(z; f) = v(z)$, and (36) has been proved for a continuous function on Γ.

It is easy to see that $H(z; f)$ is for each fixed $z \in R$ a continuous linear functional on the family of resolutive function f. Hence (36) remains valid for bounded Borel function f.

Next assume that $q \in \Gamma$ is regular. We shall show that $q \in \Delta$. If $q \in \Gamma - \Delta$ then we can find a positive superharmonic function $\mathcal{S}$ on R such that $\mathcal{S}$ is continuous on R^* and $\mathcal{S}(q) = 1$, $\mathcal{S}|\Delta = 0$ by Theorem 1.3F. Clearly $f = \mathcal{S}|\Gamma$ is resolutive and

$$H(z; f) = \int_{\Gamma} P(z, p)\mathcal{S}(p)d\mu(p)$$

$$= \int_{\Delta} P(z, p)\mathcal{S}(p)d\mu(p) = 0.$$

Thus $\lim_{z \in R, z \to q} H(z; f) = 0 \neq f(q) = 1$, which contradicts the regularity of q.

Conversely assume that $q \in \Delta$. Let f be an arbitrary resolutive function on Γ, continuous at q on Γ and a fortiori on Δ. We may suppose that $f(q) = 0$. For an arbitrary $\varepsilon > 0$ there exist functions $g_1, g_2 \in B(\Gamma)$ such that

$$g_1 - \varepsilon < f < g_2 + \varepsilon$$

on Γ and $g_1(q) = g_2(q) = 0$. Clearly

$$\int_\Gamma P(z, p) g_1(p)\, d\mu(p) - \varepsilon \leq H(z; f)$$
$$\leq \int_\Gamma P(z, p) g_2(p)\, d\mu(p) + \varepsilon$$

and we conclude that

$$-\varepsilon \leq \liminf_{z \in R,\ z \to q} H(z; f) \leq \limsup_{z \in R,\ z \to q} H(z; f) \leq \varepsilon.$$

Thus $\lim_{z \in R, z \to q} H(z; f) = 0 = f(q)$, i.e. q is regular.

q.e.d.

2B. Compactification of Subregions.

Let R be an arbitrary Riemann surface and G a subregion of R. Here the case $G = R$ is admitted. We denote by $\overline{G}$ the closure of G in R^* and by G^* the Royden compactification of the Riemann surface G. We give a relation between G^* and $\overline{G}$:

Theorem 2.2B. There exists a unique continuous mapping $j = j(G^*, \overline{G})$ of G^* onto $\overline{G}$ fixing G pointwise.

The mapping $j(G^*, \overline{G})$ shall be referred to as the projection of G^* onto $\overline{G}$. A set $\overline{E} \subset \overline{G}$ such that $\overline{E} = j(E^*)$ with $E^* \subset G^*$ is called the projection of E^*. The set $j^{-1}(\overline{p})$, with $\overline{p}$ in $\overline{G}$, is the fiber over $\overline{p}$.

<u>Proof</u>. The uniqueness of such a j is clear since $j(p) = p$ for every $p \in G$ and G is dense in both G^* and $\overline{G}$. Hence we have only to show the existence. Let $M(G; R) = M(R)|G$. Observe that $f \in M(G; R)$ can be extended to $\overline{G}$ and to G^* as continuous function $\overline{f}$ and f^*. Let $p^* \in G^*$. Then $f^* \to f^*(p^*)$ is a character χ^* on $M(G)$, and $\overline{\chi} = \chi^*|M(G; R)$ is a character on $M(G; R)$. We prove the existence of a unique $\overline{p} \in \overline{G}$ such that

$$\overline{\chi}(\overline{f}) = \overline{f}(\overline{p}) \tag{38}$$

for all $\overline{f} \in M(G; R)$. Consider $\mathcal{J} = \{\overline{f} \in M(G; R): \overline{\chi}(\overline{f}) = 0\}$. We shall deduce a contradiction from the assumption that there exists an $\overline{f}_q \in \mathcal{J}$ for every $q \in \overline{G}$ such that $\overline{f}_q(q) \neq 0$. Since $\mathcal{J}$ is an ideal of $M(G; R)$ we may assume that $\overline{f}_q \geq 0$ on $\overline{G}$ by taking $\overline{f}_q^2$ instead of $\overline{f}_q$. Moreover we may suppose that $\overline{f}_q(q) > 1$. By the compactness of G we can find points $q_1, \ldots, q_n$ such that $\overline{f} = \sum_{j=1}^{n} \overline{f}_{q_j} \geq 1$ on $\overline{G}$. Then $1 = \overline{f}\cdot(1/\overline{f}) \in \mathcal{J}$ and $\overline{\chi}(1) = 0$, a contradiction. Thus there exists a $\overline{p} \in \overline{G}$ with $\overline{f}(\overline{p}) = 0$ for all $\overline{f} \in \mathcal{J} \subset M(G; R)$. Since $\overline{f} - \overline{f}(\overline{p}) \in \mathcal{J}$, (38) has been proved. The uniqueness follows from the fact that $M(G; R)$ separates points in $\overline{G}$.

We denote by $j(p^*)$ the point $\overline{p}$ in (38). We have shown that

$$f^*(p^*) = \overline{f}(j(p^*)) \qquad (39)$$

for all $f \in M(G; R)$. From this the continuity of f follows.

Finally we have to prove that f is onto. Let $\overline{p} \in \overline{G}$ and set

$$\mathcal{J} = \{f \in M(G); \lim_{z \in G, z \to \overline{p}} f(z) = 0\}.$$

$\mathcal{J}$ contains $\{f \in M(G; R); \overline{f}(\overline{p}) = 0\}$ and $1 \notin \mathcal{J}$. Thus $\mathcal{J}$ is nontrivial ideal of the Banach algebra $M(G)$. By virtue of Gelfand's theorem (cf. [14]) there exists a character χ^* on $M(G)$ such that $\chi^*|\mathcal{J} = 0$. Corollary 1.2B.2 gives the existence of a point $p^* \in G^*$ such that $\chi^*(f) = f^*(p^*)$. Take an arbitrary $f \in M(G; R)$. Then $f - \overline{f}(\overline{p}) \in \mathcal{J}$ and hence $\overline{f}(\overline{p}) = f^*(p^*)$. We conclude by (39) that $\overline{p} = j(p^*)$.

Again by (39) it is clear that $j(z) = j^{-1}(z) = z$ for G.

q.e.d.

2C. Coincidence of Boundary Points.

The boundary $\overline{G} - G$ of G consists of two parts: the "relative" boundary $\overline{\partial G}$ and the "ideal" boundary

$$b_G = (\overline{G} - \overline{\partial G}) \cap \Gamma. \tag{40}$$

Correspondingly the boundary $G^* - G$ of G can be divided into two parts: $j^{-1}(\overline{\partial G})$ and $j^{-1}(b_G)$. The structures $\overline{\partial G}$ and $j^{-1}(\overline{\partial G})$ are quite different since every $f \in M(G)$ is continuous at each point of $j^{-1}(\overline{\partial G})$ but not at $\overline{\partial G}$. In contrast we can prove:

<u>Theorem</u> 2.2C. The projection $j = j(G^*, \overline{G})$ is a homeomorphism of $G \cup j^{-1}(b_G)$ onto $G \cup b_G$.

<u>Proof</u>. Since $G \cup j^{-1}(b_G)$ and $G \cup b_G$ are locally compact and since j is a continuous map of $G \cup j^{-1}(b_G)$ onto $G \cup b_G$. We must show that the fiber $j^{-1}(p)$ consists of one point for every $\overline{p} \in b_G$. Assume that $j^{-1}(\overline{p})$ contained two distinct points p_1^* and p_2^*. Then there would exist an $f \in M(G)$ such that $f(p_i^*) = i$ $(i = 1, 2)$. Let U be an open neighborhood of $\overline{p}$ in R^* such that $U \subset G \cup b_G$. Let $\varphi \in M(R)$ such that $\varphi|U = 1$ and $\varphi|V = 0$ where V is an open set containing $(R - \overline{G}) \cup \overline{\partial G}$, with $\overline{U} \cap \overline{V} = \emptyset$ (see I.2D). Then clearly $f\circ\varphi \in M(R)$, it being understood that $f\circ\varphi = 0$ on $R - \overline{G}$. Thus we may consider that $f\circ\varphi$ is in $M(G; R)$. By (39)

$$i = f^*(p_i^*) = f^*(p_i^*)\varphi^*(p_i^*) = (f\circ\varphi)^*(p_i^*)$$

$$= (\overline{f\circ\varphi})(j(p_i^*)) = (\overline{f\circ\varphi})(\overline{p})$$

for $i = 1, 2$. This is a contradiction.

q.e.d.

2D. Correspondence of Harmonic Measures I.

We denote by μ_R (resp. μ_G) the harmonic measure on Royden's boundary of R (resp. G) and fix the point $z_0 \in G \subset R$ as the center of μ_R and μ_G (see II.1A).

The following result shows that $G \cup j^{-1}(b_G)$ and $G \cup b_G$ are the same not only topologically but also harmonically.

Theorem 2.2D. Let E be a Borel set in b_G. Then $\mu_R(E) > 0$ if and only if $\mu_G(j^{-1}(E)) > 0$.

Proof. Since μ_R and μ_G are regular we may assume that E is a compact set. Note that $E \cap \overline{\partial G} = \emptyset$ in $\overline{G} - G$. Thus $(j^{-1}(E)) \cap (j^{-1}(\overline{\partial G})) = \emptyset$ in Γ_G. From the regularity of μ_R and μ_G and the fact that f is a homeomorphism from $G \cup j^{-1}(b_G)$ onto $G \cup b_G$, it follows that there exists a sequence $\{V_n\}_1^\infty$ of open sets V_n in R^* such that $G \cup b_G \supset V_n \supset \overline{V}_{n+1} \supset E$ and

$$\mu_R(E) = \lim_n \mu_R(V_n \cap \Gamma_R), \qquad (41)$$

$$\mu_G(j^{-1}(E)) = \lim_n \mu_G(j^{-1}(V_n) \cap \Gamma_G).$$

Let $\{f_n\}_1^\infty \subset M(R)$ satisfy $f_n|\overline{V}_{n+1} = 1$, $f_n|(R^* - V_n) = 0$, $0 \le f_n \le 1$ and $f_n \ge f_{n+1}$ (see I.2D). Clearly f_n may be viewed as an element of $M(G)$, and $f_n|j^{-1}(\overline{V}_{n+1}) = 1$, $f_n|(G^* - j^{-1}(V_n)) = 0$. Let

$$u_n(z) = \int_{\Gamma_R} P_R(z, p) f_n(p) d\mu_R(p) \tag{42}$$

for $z \in R$ and

$$v_n(z) = \int_{\Gamma_G} P_G(z, p) f_n(p) d\mu_G(p) \tag{43}$$

for $z \in G$. Take a regular exhaustion $\{R_m\}_1^\infty$ of R and set

$$v_{nm} = \pi_{R^*-R_m \cap G}\, f_n$$

on R. By Theorem 1.3B we see that $\{v_{nm}\}_{m=1}^\infty$ is BD-Cauchy on R and a fortiori on G. By the property of j, $v_{nm} - f_n$ vanishes on Γ_G and hence $v_{nm} - f_n \in M_0(G)$. Thus $\text{BD-}\lim_m (v_{nm} - f_n) \in M_\Delta(G)$ and it follows that $\text{BD-}\lim_m v_{nm} = f_n$ on Δ_G. In view of this and (43) we obtain

$$v_n = \text{BD-}\lim_m v_{nm} \quad \text{on } G. \tag{44}$$

Since $v_{nm} - f_n = 0$ on $R^* - R_m \cap G$ we have $v_n \in M(R)$ if we define $v_n = 0$ on R outside of G.

On the other hand $v_{nm} - f_n \in M_0(R)$ and, since

$$v_n = \text{BD-}\lim_m v_{nm} \quad \text{on} \quad R \tag{45}$$

$v_n - f_n \in M_\Delta(R)$. Therefore

$$v_n = f_n \tag{46}$$

on Δ_R.

As above we can show that

$$u_n = \text{BD-}\lim_m u_{nm} \tag{47}$$

on R where

$$u_{nm} = \pi_{R^*-R_m} f_n = \pi_{R^*-R_m} v_n . \tag{48}$$

By the construction of $\{f_n\}_1^\infty$ and by the maximum principle we conclude that $u_n \geq u_{n+1}$ on R, $v_n \geq v_{n+1}$ on G, and $u_n \geq v_n$ on G. In particular $u_n(z_0) \geq v_n(z_0)$. This means that

$$\mu_R(V_n \cap \Gamma_R) \geq \mu_G(j^{-1}(V_{n+1}) \cap \Gamma_G)$$

and by the regularity of the measures μ_R and μ_G

$$\mu_R(E) \geq \mu_G(j^{-1}(E)). \tag{49}$$

In particular $\mu_G(j^{-1}(E)) > 0$ implies $\mu_R(E) > 0$.

Next assume that $\mu_G(j^{-1}(E)) = 0$. Then $\lim_n v_n(z_0) = \mu_G(j^{-1}(E)) = 0$. Since the continuous function on R^* which is harmonic on R_m and equals $v_1 - v_n$ on $R^* - R_m$ must coincide with $u_{1m} - u_{nm}$ and since $u_{1m} - u_{nm} \geq v_1 - v_n$ we conclude by the maximum principle that $u_{1m} - u_{nm} \leq u_{1m+1} - u_{nm+1}$ and $\lim_m (u_{1m} - u_{nm}) = u_1 - u_n$. Hence $u_1 - u_n \geq u_{1m} - u_{nm}$. From $U\text{-}\lim_n v_n = 0$ on ∂R_m and $u_{nm} = v_n$ on ∂R_m it follows that $U\text{-}\lim_n u_{nm} = 0$ on R_m. Therefore $\lim_n (u_1 - u_n) \geq u_{1m}$ on R_m. On letting $m \to \infty$ we infer that $\lim_n (u_1 - u_n) \geq u_1$ on R. Since $u_n \geq 0$ we finally conclude that $\lim_n u_n(z_0) = 0$, i.e. $\mu_R(E) = 0$.

q.e.d.

2E. Correspondence of Harmonic Measures II.

Let $\omega_{\partial G}$ be the harmonic measure on ∂G with respect to G for the center $z_0 \in G$, i.e. for each Borel set X in ∂G

$$\omega_{\partial G}(X) = \inf\{\mathcal{S}(z_0);\ \ \mathcal{S} \in \mathfrak{S}(\overline{G}, f_X)\} \tag{50}$$

where f_X is the characteristic function of X on $\overline{G} - G$ and $\mathfrak{S}(\overline{G}, f_X)$ is the set of superharmonic function $\mathcal{S}$ on G with

$$\liminf_{z \in G,\ z \to p} \mathcal{S}(z) \geq f_X(p) \tag{51}$$

for every $p \in \overline{G} - G$. It is easy to see that $\omega_{\partial G}$ is a regular Borel measure for Borel subset in ∂G.

Theorem 2.2E. For every Borel set $E \subset \partial G$,

$$\mu_G(j^{-1}(E)) = \omega_{\partial G}(E).$$

Proof. It is easy to see that

$$\mathfrak{a}(G^*, f_E \circ j) = \mathfrak{a}(\overline{G}, f_E)$$

and the assertion follows from Theorem 2.2A.

q.e.d.

III. TOPOLOGICAL STRUCTURE OF THE ROYDEN COMPACTIFICATION OF THE UNIT DISK.

§1 introduces Nakai's function and its properties. This function plays an important role in the investigation of the topological structure of the Royden compactification U^* of the unit disk U. In §2 we obtain the projection $\pi : U^* \to \hat{U}$ where $\hat{U}$ is the closed unit disk, and non-triviality of the fiber space (bundle) $\Gamma = U^* - U$ by virtue of Nakai's function. Next we show that a rotation can be extended to a homeomorphism on U^*. Finally in §2 we show the existence of points in each fiber of Γ which are approachable from only one side, and points in each fiber which are not approachable.

§1. An Auxiliary Function.

1A. Nakai's Function.

Let U be the open unit disk $\{|z| < 1\}$ and γ the unit circumference $\{|z| = 1\}$. For each $\zeta \in \gamma$ we define a real-valued function on U as follows:

$$g_\zeta(z) = \sin(\log(\log \frac{e}{|z - \zeta|})), \qquad z \in U.$$

We call this function Nakai's function (cf. [10]).

1B. Properties of g_ζ.

The Nakai function g_ζ has the following properties:

Theorem 3.1B.

(1) $|g_\zeta(z)| \leq 1$,

(2) $g_\zeta \in C^\infty(U)$,

(3) $D_U(g_\zeta) < \infty$.

Proof. Properties (1) and (2) are obvious. Hence we only have to prove (3). Let $z - \zeta = re^{i\theta}$ and $g_\zeta(z) = g_\zeta(\zeta + re^{i\theta}) = \varphi(r, \theta)$. Then

$\varphi(r, \theta) = \sin(\log(\log \frac{e}{r}))$,

$\frac{\partial\varphi}{\partial r} = (\cos(\log(\log (\frac{e}{r})))(\frac{1}{1 - \log r}) \times (\frac{-1}{r})$, and $\frac{\partial\varphi}{\partial\theta} = 0$.

Hence $|\mathrm{grad}_z\, \varphi|^2 = \left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2 = \left(\frac{\partial\varphi}{\partial r}\right)^2$ and thus we obtain

$$D_U(g_\zeta) = \int_U |\mathrm{grad}_z\varphi|^2 dxdy = \int_U \left(\frac{\partial\varphi}{\partial r}\right)^2 rdrd\theta$$

$$\leq \int_0^{2\pi}\int_0^2 \frac{dr}{r(1 - \log r)^2} = 2\pi \int_0^2 \frac{dr}{r(1 - \log r)^2} .$$

Setting $t = 1 - \log r = \log (\frac{e}{r})$, we have $dt = -\frac{dr}{r}$

and $r = e^{1-t}$. Let $A = 1 - \log 2$.

then

$$\int_0^{2\pi} \frac{dr}{r(1 - \log r)^2} = \int_\infty^A \left(-\frac{1}{t^2}\right) dt = \int_A^\infty \frac{dt}{t^2} = \frac{1}{A}$$

$$= \frac{2\pi}{1 - \log 2} < \infty,$$

i.e. $D_U(g_\zeta) < \infty$.

q.e.d.

From this theorem we have $g_\zeta \in M(U)$ for every $\zeta \in \gamma$. Hence g_ζ can be extended continuously to U^*.

§2. Fiber Space.

2A. Projection.

Theorem 3.2A. There exists a unique continuous mapping π of U^* onto $\hat{U}$ fixing U elementwise such that $\pi^{-1}(U) = U$ where $\hat{U}$ is the closed unit disk (cf. [10]).

Proof. The uniqueness of such a π is obvious. Hence only we must show the existence. Let A be the set $A = \{f \in M(U) : f$ is continuous on $\hat{U}\}$. Then A contains sufficiently many functions on $\hat{U}$ because $C^\infty(U) \subset M(U)$ and $C^\infty(\hat{U}) \subset A$. Let S be the space of all characters on A (cf. I.2B). The topology in S is defined by the weak* topology, i.e. a directed net $\{q_\lambda\}$ in S converges to q in S if $\{f(q_\lambda)\}$ converges to $f(q)$ for every $f \in A$. Then S is a compact Hausdorff space containing U as its open and dense subset (cf. I.2C).

First we shall show that $S = \hat{U}$. Clearly $\hat{U} \subset S$. Take an arbitrary q in S and set

$$A_q = \{f \in A : f(q) = 0\}.$$

Then for some $z_0 \in U$ $f(z_0) = 0$ for all $f \in A_q$. If it is not so, then we can find a $f_z \in A_q$ such that $f_z \geq 0$ on $\hat{U}$ and $f_z(z) = 1$ for each $z \in \hat{U}$ since A_q is an ideal in A. By the compactness of $\hat{U}$ and the continuity of each f_z on $\hat{U}$, we can find a system of points $z_1, z_2, \ldots, z_n$ in $\hat{U}$ such that

$$g(z) = \sum_{k=1}^{n} f_{z_k}(z) > \frac{1}{2}$$

on $\hat{U}$. As $g \in A_q$ and $\frac{1}{g} \in A$, $1 = (\frac{1}{g})g \in A_q$ and hence $A_q = A$. It is a contradiction. Therefore we have proved the existence of a point $z_0 \in \hat{U}$ with the property mentioned above. Since $f - f(q) \in A_q$ we get $f(z_0) = f(q)$ for any function $f \in A$. This proves that $S = \hat{U}$.

Take a point p in U^*. Then $f \to f(p)$ defines a character $\pi(p)$ on A. This gives a mapping of U^* into $S = \hat{U}$. Moreover π is onto. In fact, for any point $z_0 \in \hat{U}$ consider the set

$$M_{z_0} = \{f \in M(U) : \lim_{U \ni z \to z_0} f(z) = 0 \text{ in } U\}.$$

This is a proper ideal of $M(U)$. Since $M(U)$ is normed so as to be a Banach algebra, there exists a character p in $M(U)$ vanishing on M_{z_0} by Mazur-Gelfand's theorem that a normed field is the complex number field. This p can be considered to be a point in U^*. If f belongs to A, $f - f(z_0)$ is contained in M_{z_0} and a fortiori

$$f(p) = f(z_0) \quad \text{for any} \quad f \in A.$$

This shows that $\pi(p) = z_0$ or π is onto. Again by

$$f(p) = f(\pi(p)) \quad \text{for any} \quad f \in A$$

and for any p in U^* we can conclude that π is a continuous mapping of U^* onto $\hat{U}$ fixing U elementwise and $\pi^{-1}(U) = U$.

q.e.d.

We shall quote π as a <u>projection</u> of U^* onto $\hat{U}$.

2B. <u>Fibers</u>.

We call the set $\pi^{-1}(z)$ the <u>fiber</u> in U^* over a point z in $\hat{U}$. The fiber $\pi^{-1}(z)$ is one point z if z is in U but $\pi^{-1}(z)$ contains infinitely many points if z is in γ. This is shown by using the following theorem:

Theorem 3.2B (cf. [10]). For each function f in $M(U)$ and ζ in γ

$$\{f(p) : p \in \pi^{-1}(\zeta)\} = C_U(f, \zeta)$$

where $C_U(f, \zeta)$ is the totality of a such that there exists a sequence $\{z_n\}$ in U with $\lim_{n\to\infty} z_n = \zeta$ and $\lim_{n\to\infty} f(z_n) = a$.

Proof. First we shall show that $f(p)$ is in $C_U(f, \zeta)$ for any $p \in \pi^{-1}(\zeta)$. Considering $f - f(p)$ instead of f, we may assume $f(p) = 0$. Contrary to the assertion, we assume that 0 is not in $C_U(f, \zeta)$. Then we can find a neighborhood G of ζ in $\hat{U}$ such that

$$|f(z)| > d > 0 \quad \text{on} \quad G \cap U.$$

Clearly we can find a function $g \in M(U)$ such that $g(z) = \frac{1}{f(z)}$ on $G \cap U$ and hence $g(z)f(z) = 1$ on $G \cap U$. By the continuity of π, $\pi^{-1}(G \cap \hat{U})$ is a neighborhood of p in U^*. Since U is dense in both U^* and $\hat{U}$, $\pi^{-1}(G \cap \hat{U}) \cap U = \pi^{-1}(G \cap U) = G \cap U$ is dense in $\pi^{-1}(G \cap \hat{U}) \cap U^* = \pi^{-1}(G \cap \hat{U})$. Hence $f(z)g(z) = 1$ on $G \cap U$ implies $f(p)g(p) = 1$ on $\pi^{-1}(G \cap \hat{U})$. In particular, $f(p)g(p) = 1$. This is a contradiction, because $f(p) = 0$.

Conversely we assume that a is in $C_U(f, \zeta)$. We must show the existence of a point p in $\pi^{-1}(\zeta)$ such that $f(p) = a$. To this end, we may assume $a = 0$ by considering $f - a$ instead of f. Also we assume that f does not vanish on $\pi^{-1}(\zeta)$. As $\pi^{-1}(\zeta)$ is compact and f is continuous on this set, there exists a positive number d such that

$$|f(q)| > d \quad \text{on} \quad \pi^{-1}(\zeta).$$

On the other hand, from $0 \in C_U(f, \zeta)$ we can find a sequence $\{z_n\}$ in U such that $\lim_{n\to\infty} z_n = \zeta$ in $\hat{U}$ and

$$|f(z_n)| < d.$$

Let r be an accumlation point of the set $\{z_n\}$ in U^*. Clearly r is in Γ. Let $\{z_\lambda\}$ be a directed net converging to r in U^* whose terms are choosen from the set $\{z_n\}$. Then by the continuity of the projection π and the fact that r is in Γ and $\pi^{-1}(U) = U$, $\{z_\lambda\}$ must converge to ζ in $\hat{U}$ and $\pi(r) = \zeta$. Since $|f(z_\lambda)| < d$,

$$|f(r)| = \lim_{\lambda} |f(z_\lambda)| \leq d.$$

This shows that r is not in $\pi^{-1}(\zeta)$. This is a contradiction, because $\pi(r) = \zeta$.

q.e.d.

We call the set $C_U(f, \zeta)$ the <u>interior cluster set of f at ζ in γ</u>.

2C. Behavior of g_ζ on $\pi^{-1}(\zeta)$.

Theorem 3.2C.1. $C_U(g_\zeta, \zeta) = [-1, 1]$.

Proof. It suffices to show that

$$g_\zeta(\pi^{-1}(\zeta)) = [-1, 1].$$

Take a sequence of neighborhoods

$$U_n = \{z \in U : |z - \zeta| < \frac{1}{n}\} \quad (n = 1, 2, \ldots)$$

of ζ in U. Then clearly $\bigcap_{n=1}^{\infty} \overline{U}_n = \pi^{-1}(\zeta)$. Since $\{\overline{U}_n\}$ is a decreasing sequence, $\bigcap_{k=1}^{n} g_\zeta(\overline{U}_k) = g_\zeta(\overline{U}_n) = g_\zeta\left(\bigcap_{k=1}^{n} \overline{U}_k\right)$ for all n and hence $g_\zeta(\pi^{-1}(\zeta)) = \bigcap_{n=1}^{\infty} g_\zeta(\overline{U}_n)$. It is easy to see that $g_\zeta(\overline{U}_n) \supset [-1, 1]$ for all n and thus $g_\zeta(\pi^{-1}(\zeta)) \supset [-1, 1]$. Therefore $g_\zeta(\pi^{-1}(\zeta)) = [-1, 1]$ for every $\zeta \in \gamma$.

q.e.d.

Thus every fiber $\pi^{-1}(\zeta)(\zeta \in \gamma)$ contains a point set whose cardinal number is at least the cardinal number of continum.

Theorem 3.2C.2. Let ζ and ζ' be arbitrary distinct points in γ. Then

$$g_\zeta(q) = g_\zeta(\zeta') \quad \text{for all} \quad q \in \pi^{-1}(\zeta').$$

Proof. Let p be an arbitrary point in $\pi^{-1}(\zeta')$. Take a directed net $\{z_\lambda\}$ in U which converges to p. Then clearly $\{z_\lambda\}$ converges to $\pi(p) = \zeta'$ in $\hat{U}$. By the continuity of g_ζ on U^* $g_\zeta(z_\lambda)$ converges to $g_\zeta(p)$. On the other hand $g_\zeta(z)$ is continuous on $\hat{U} - \{\zeta\}$ and hence $g_\zeta(z_\lambda)$ converges to $g_\zeta(\zeta')$. Thus $g_\zeta(p) = g_\zeta(\zeta')$.

q.e.d.

This shows that g_ζ is a constant on $\pi^{-1}(\zeta')$ for $\zeta' \neq \zeta$.

2D. Rotation.

Theorem 3.2D.1. A rotation $f_\theta(z) = e^{i\theta}z$ $(\theta \in [0, 2\pi))$ has the unique continuous extension f^* to U^* which is a homeomorphism.

Proof. Let X be the set of all characters on $M(U)$. Then $X = U^*$ by indentifying $z \in U$ with the corresponding character χ_z (cf. I.2C). We define a map $f_\theta^* : X \to X$ by $f_\theta^*(\chi)(g) = \chi(g \circ f_\theta)$ for each $\chi \in X$ and for each $g \in M(U)$. Then it is well-defined and

$f_\theta^* = f_\theta$ on U, i.e. $f_\theta^*(\chi_z) = \chi_{f_\theta(z)}$ for each $z \in U$. In fact, for each $z \in U$ and for each $g \in M(U)$

$$f_\theta^*(\chi_z)(g) = \chi_z(g \circ f_\theta) = g \circ f_\theta(z) = \chi_{f_\theta(z)}(g).$$

Hence f_θ^* is a continuous extension of f_θ to X for each $\theta \in [0, 2\pi)$. Since U is open and dense in X this extension is unique.

$f_\theta \circ f_{-\theta} = f_{-\theta} \circ f_\theta$ $(\theta \in [0, 2\pi))$ is the identity map on U. It implies that $f_\theta^* \circ f_{-\theta}^* = f_{-\theta}^* \circ f_\theta^*$ is the identity on X because U is dense in X. Thus f is clearly a homeomorphism.

q.e.d.

By the same proof we see that any homeomorphism of U which maps $M(U)$ continuously into itself (by composition) has a unique extension to a homeomorphism of U^*.

<u>Theorem</u> 3.2D.2. $\pi \circ f_\theta^* = f_\theta \circ \pi$ $(\theta \in [0, 2\pi))$.

<u>Proof</u>. It is trivial on U. Let p be an arbitrary point in Γ and $\{z_\lambda\}$ a directed net in U which converges to p. Then by the continuity of f_θ^* $f_\theta(z_\lambda)$ converges to $f_\theta^*(p)$ and hence $\pi \circ f_\theta(z_\lambda)$ converges to $\pi \circ f_\theta^*(p)$. But $\pi \circ f_\theta(z_\lambda) = f_\theta(z_\lambda) = f_\theta \circ \pi(z_\lambda)$ converges to $f_\theta \circ \pi(p)$ since $\pi(z_\lambda)$ converges to $\pi(p)$. Hence $\pi \circ f_\theta^*(p) = f_\theta \circ \pi(p)$. By the arbitrariness of p in Γ, $\pi \circ f_\theta^* = f_\theta \circ \pi$ on Γ.

q.e.d.

From this theorem we understand that

$$f_\theta^*(\pi^{-1}(\zeta)) = \pi^{-1}(f_\theta(\zeta)), \quad \text{i.e.} \quad f_\theta^* \text{ carries}$$

a fiber $\pi^{-1}(\zeta)$ to a fiber $\pi^{-1}(e^{i\theta}\zeta)$. Therefore f_θ^* is a rotation on U^*.

2E. Non-triviality.

"A fiber space (bundle) (X, π, B) is locally trivial" means that each point in B has an open neighborhood G such that $\pi^{-1}(G)$ is homeomorphic to a product space $G \times F$ by a homeomorphism which carries the projection π to the canonical projection of $G \times F$ onto G.

Lemma 3.2E. If a fiber space (X, π, B) is locally trivial, then the projection π is an open mapping.

We omit the proof.

Let (X, π, B) be a fiber space (bundle) and let G be an open subset of B. Then $\mathcal{S}: G \to X$ is called a local cross section if $\pi \circ \mathcal{S}$ is the identity on G.

Now we are ready to determine the topological structure of the fiber space (Γ, π, γ) as follows:

Theorem 3.2E. The fiber space (Γ, π, γ) is not locally trivial. Moreover this space does not have a local cross section.

Proof. Fix an arbitrary point ζ in γ and take an open subset of Γ

$$G = \{p \in \Gamma \; : a < g_\zeta(p) < b\}$$

where $-1 < a < b < 1$. Then by setting

$$G_1 = \bigcup_{\substack{\zeta' \in \gamma \\ a < g_\zeta(\zeta') < b}} \pi^{-1}(\zeta')$$

$$G = G_1 \cup (G \cap \pi^{-1}(\zeta)).$$

Therefore

$$\pi(G) = \{\zeta' \in \gamma : a < g_\zeta(\zeta') < b\} \cup \{\zeta\}.$$

We denote $\hat{G} = \{\zeta' \in \gamma : a < g_\zeta(\zeta') < b\}$. Recall g_ζ is defined in U by

$$g_\zeta = \sin(\log(\log \frac{e}{|z - \zeta|})) \quad (\zeta \in \gamma)$$

and the extended to U^*. Clearly we can also regard g_ζ as a continuous function on $\hat{U} - \{\zeta\}$. Hence $\hat{G}$ is an open subset of γ. It follows that $\pi(G) = \hat{G} \cup \{\zeta\}$ is not open, i.e. π is not an open map. By the above lemma (Γ, π, γ) is not locally trivial.

Let ζ be an arbitrary point in γ and $\hat{G}$ be an open subset of γ which contains ζ. Suppose that there exists a local cross section $\mathfrak{S} : \hat{G} \to \Gamma$. By the definition $\pi \circ \mathfrak{S}$ is the identity map on $\hat{G}$.

If $\zeta' \in \gamma$ and $\zeta' \neq \zeta$, then $g_\zeta(\zeta')$ is the constant value of g_ζ in the fiber over ζ'. Therefore $g_\zeta \circ \mathfrak{S} = g_\zeta$ on $\hat{G} - \{\zeta\}$. $g_\zeta \circ \mathfrak{S}$ is continuous on G, but g_ζ has no continuous extension over all $\hat{G}$. This is a contradiction. Thus there are no local sections.

q.e.d.

2F. Accessible Point From One Side.

A point p in Γ is called a point which is <u>accessible from the right (resp. left)</u> if there exists a directed net $\{p_\lambda\}$ in Γ such that $\arg(\pi(p_\lambda)) < \arg(\pi(p))$ (resp. $\arg(\pi(p)) < \arg(\pi(p_\lambda))$) for all λ and $\{p_\lambda\}$ converges to p.

We shall show the existence of points on every fiber $\pi^{-1}(\zeta)$ which are accessible only from the right. Take a sequence $\{\zeta_n\}$ on γ such that ζ_n converges to ζ from the right, i.e. $\arg \zeta_n < \arg \zeta_{n+1} < \arg \zeta$ $(n = 1, 2, \ldots)$ and $|\zeta_{n+1} - \zeta_{n+2}| < \frac{1}{2} |\zeta_n - \zeta_{n+1}|$ $(n = 1, 2, \ldots)$. Let

$$A_n = \{z \in \mathbb{C} : r_n < |z - \zeta_n| < \rho_n\}$$

with $\rho_n < \frac{1}{2} |\zeta_n - \zeta_{n+1}|$ and

$$B_n = \{z \in \mathbb{C} : |z - \zeta_n| < r_n\} \quad (n = 1, 2, \ldots)$$

where $\mathbb{C}$ is the complex plane. Here we choose

$r_n = e^{-2^n} \rho_n$ $(n = 1, 2, \ldots)$. For each n we define a continuous function w_n on U such that $0 \leq w_n \leq 1$, $w_n = 1$ on $\overline{B}_n \cap U$, $w_n = 0$ on $U - (A \cup \overline{B}_n)$, and $w_n - (A_n - \overline{B}_n) \in H(A - \overline{B}_n)$. Since

$$D_U(w_n) \leqq \frac{2\pi}{\operatorname{mod} A_n} = \frac{2\pi}{\log(\rho_n/r_n)} = 2\pi \cdot 2^{-n},$$

every w_n $(n = 1, 2, \ldots)$ is a bounded Tonelli function on U and hence $w_n \in M(U)$ $(n = 1, 2, \ldots)$. Take $\sum_{n=1}^{\infty} w_n$. Clearly w is a bounded Tonelli function on U and $D_U(w) = \sum_{n=1}^{\infty} D_U(w_n) = 2\pi \sum_{n=1}^{\infty} 2^{-n} = 2\pi$. It follows that $w \in M(U)$.

Let $\{p_n\}$ be a sequence in Γ such that $p_n \in \pi^{-1}(\zeta_n)$ $(n = 1, 2, \ldots)$. Then $\{p_n\}$ has an accumlation point p in $\pi^{-1}(\zeta)$. Since $w(p_n) = 1$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} w(p_n) = w(p)$, $w(p) = 1$. Suppose that there exists a directed net $\{q_\lambda\}$ converging to p from the left, i.e. $\arg(\pi(q_\lambda)) > \arg(\pi(p))$ for all λ. From $w_n(q_\lambda)) = 0$ for all λ and for all n, $w(q_\lambda) = 0$ for all λ and hence $w(p) = 0$. This is a contradiction. Therefore there exists no directed net converging to p from the left.

Similarly, we may show that the existence of points in each fiber $\pi^{-1}(\zeta)$ which are accessible only from the left.

2G. Inaccessible Points.

We say that a point p in Γ is inaccessible if p is not accessible from any side, i.e. $p \notin \overline{\Gamma - \pi^{-1}(\zeta)} \cap \Gamma$, for $\zeta = \pi(p)$.

We can also show the existence of such points in every fiber $\pi^{-1}(\zeta)$. Take an arbitrary point ζ in γ and a sequence of points $\{z_n\}$ in U which converges to ζ. We define sets A_n and B_n as follows:

$$A_n = \{z \in U : |z - z_n| < \rho_n\}$$

and

$$B_n = \{z \in U : |z - z_n| < r_n\}$$

with $0 < \rho_n < \frac{1}{2}|z_n - z_{n+1}|$ and $r_n = e^{-2^n}\rho_n$ $(n = 1, 2, \ldots)$. Then $\bar{A}_n \cap \bar{A}_m = \emptyset$ if $m \neq n$. For each n we define a continuous function u_n on U such that $0 \leq u_n \leq 1$, $u_n = 0$ on $U - A_n$, $u_n = 1$ on $\bar{B}_n$, and $u_n|(A_n - \bar{B}_n) \in H(A_n - \bar{B}_n)$. Clearly u_n $(n = 1, 2, \ldots)$ are bounded Tonelli functions. Since

$$D_U(u_n) = \frac{2\pi}{\log \operatorname{mod}(A_n - \bar{B}_n)} = \frac{2\pi}{\log(\rho_n/r_n)} = 2\pi 2^{-n},$$

$u_n \in M(U)$ $(n = 1, 2, \dots)$. Take $u = \sum_{n=1}^{\infty} u_n$. Then by

$$D_U(u) = \sum_{n=1}^{\infty} D_U(u_n) = \sum_{n=1}^{\infty} 2\pi 2^{-n} = 2\pi \quad u \in M(U) \quad \text{and}$$

hence u has the continuous extension on U^*. On the other hand u is continuous on $U \cup (\gamma - \{\zeta\})$ and $u = 0$ on $\gamma - \{\zeta\}$ by the construction of u. Hence $u = 0$ on $\Gamma - \pi^{-1}(\zeta)$.

Let p be an accumulation point in Γ of a sequence $\{z_n\}$. We can see that $p \in \pi^{-1}(\zeta)$ and $u(p) = 1$ since $u(z_n) = 1$ $(n = 1, 2, \dots)$. From the above fact

$p \notin \overline{\Gamma - \pi^{-1}(\zeta)} \cap \Gamma$, i.e. p is not an accumulation point from either side, the left or the right.

IV. AN APPLICATION TO FUNCTION THEORY.

In §1 we define a stairway and the limit of a function along a stairway and in §2 we prove a boundary theorem for meromorphic functions by Nakai's unicity theorem.

§1. Stairway.

1A. Preliminary.

Let U be the open unit disk $\{|z| < 1\}$ and γ the unit circumference $\{|z| = 1\}$. Take an arbitrary real numbers a and b so that $-1 < a < b < 1$. Consider a set $A(\zeta; a, b) = \{z \in U : a < g_\zeta(z) < b\}$. This set can be written as a union of disjoint subsets of U such that

$$A(\zeta; a, b) = A_0(\zeta; a, b) \cup \left(\bigcup_{n=1}^{\infty} A_n(\zeta; a, b) \right)$$

with the conditions:

(1) each A_j $(j = 1, 2, \ldots)$ is a region bounded by two arcs in γ and two cross-cuts contained in $\partial A(\zeta; a, b)$,

(2) A_0 is either empty or a region bounded by an arc in γ and a cross-cut,

(3) A_{j-1} and A_{j+1} are divided by A_j $(j = 1, 2, \ldots)$.

In fact, $A_n = \{z \quad U : \alpha_n < |z - \zeta| < \beta_n\}$ for $n = 1, 2, \ldots$ and we may compute α_n and β_n as follows: let $X_0 = \log(1 - \log 2)$, $t_0 = -(\sin^{-1} a)$ and $S_0 = \sin^{-1} b$.

If $e^{1-e^{(2m+1)\pi - S_0}} < X_0 < e^{1-e^{2(m+1)\pi + S_0}}$ or

$e^{i-e^{2m\pi - t_0}} < X_0 < e^{1-e^{(2m+1)\pi + t_0}}$

for some integer m, $A_0 = \emptyset$. Otherwise, A_0 is a region bounded by an arc in γ and a cross-cut. Now we compute α_n and β_n $(n = 1, 2, \ldots)$

for $e^{1-e^{(2m+1)\pi - S_0}} < X_0 < e^{1-e^{2(m+1)\pi + S_0}}$, i.e.

$$\alpha_n = \begin{cases} e^{1-e^{2(m+n)\pi + S_0}} & \text{if } n = \text{odd}, \\ e^{1-e^{(2(m+n)+1)\pi + t_0}} & \text{if } n = \text{even} \end{cases}$$

and

$$\beta_n = \begin{cases} e^{1-e^{2(m+n)\pi - t_0}} & \text{if } n = \text{odd}, \\ e^{1-e^{(2(m+n)+1)\pi - S_0}} & \text{if } n = \text{even}. \end{cases}$$

For other cases we omit the computation of α_n and β_n.

We set $A^{(n)}(\zeta;\ a,\ b) = \bigcup_{k=n}^{\infty} A_n(\zeta;\ a,\ b)$.

1B. Stairway.

For each point ζ in γ we define a Stairway V at ζ if V satisfies the following conditions:

(S.1) V is a subset of U,

(S.2) there exists a pair of real numbers a and b with $-1 < a < b < 1$, and a positive integer n such that $\overline{A^{(n)}(\zeta;\ a,\ b)}^0 \subset V$ where $\bar{A}^0 = \bar{A} \cap U$ and $\bar{A}$ is the closure of A in U^*.

By the definition $A^{(n)}(\zeta;\ a,\ b)$ and $A(\zeta;\ a,\ b)$ are stairways at ζ.

1C. Limit of a Function Along a Stairway.

Let f be a function on U and α be a complex number. We say that f has a limit α along V at ζ if

$$\lim_{n\to\infty} \sup_{\substack{z\in V \\ |z-\zeta|<\frac{1}{n}}} |f(z) - \alpha| = 0.$$

§2. A Boundary Theorem of Meromorphic Function.

2A. Nakai's Unicity Theorem.

We shall introduce Nakai's unicity theorem which Nakai called the boundary theorem of Riesz-Lusin-Privaloff Type in [10] and [16].

Theorem 4.2A. Let E be a Borel subset of Royden's boundary Γ of a Riemann surface R with positive harmonic measure, i.e. $\mu_R(E) > 0$, and G a subregion of R such that its closure $\overline{G}$ in U^* is a neighborhood of E. Then every meromorphic function f on G with boundary value zero at each point of E vanishes identically on G. (cf. [10], [16]).

Proof. Suppose $f \not\equiv 0$ on G. We may assume that E is a compact subset of the harmonic boundary Δ by the regularity of μ_R. We may also take $G = R$ by Theorem 2.2D. Consider

$$F = \{z \in R : |f(z)| < 1\}$$

and let $F = \bigcup_{k=1}^{N} F_k$ $(N \leq \infty)$ be the decomposition into components. Set $E_k = E \cap \overline{F}_k$. Since $\overline{F} - \overline{\partial F}$ is a neighborhood of E and $\partial F_k \subset \partial F$,

$$E_k \subset \overline{F}_k \cap \Gamma - \overline{\partial F}_k.$$

Let $j_k = j(F_k^*, \overline{F}_k)$. By Theorem 2.2C j_k is a homeomorphism on $j_k^{-1}(\overline{F}_k \cap \Gamma - \overline{\partial F}_k)$. We write $j_k^{-1}(E_k) = E_k^*$.

Now we shall show that

$$\mu_{F_k}(E_k^*) > 0 \qquad (52)$$

for at least one k.

If this were not so then by Theorem 2.2D $\mu_R(E_k) = 0$ for every $k = 1, \ldots, N$. In this case $N = \infty$ since if $N < \infty$ then $E = \bigcup_{k=1}^{N} E_k$ and hence $\mu_R(E) = 0$ which is a contradiction.

Let U be an open set in R^* such that $E \subset U \subset \overline{U} \subset \overline{F} - \overline{\partial F}$ and $\mu_R(U \cap \Gamma - E) < \frac{1}{2}\mu_R(E)$.

Take $f_\infty \in M(R)$ such that $0 \leq f_\infty \leq 1$ on R^*, $f_\infty|E = 1$, and $f_\infty|(R^* - U) = 0$. Let $f_n (1 \leq n < \infty)$ be defined by $f_n|\bigcup_{k=1}^{n} F_k = f_\infty$ and $f_n|(R - \bigcup_{k=1}^{n} F_k) = 0$. Clearly $f_n \in M(R)$, $f_n|\left(\bigcup_{k=1}^{N} E_K\right) = 1$, and $f_n|((R^* - U) \cup (U - \bigcup_{k=1}^{n} \overline{F}_k)) = 0$. Observe that $\{f_n\}_1^\infty$ is increasing and dominated by f_∞. Thus the same is true of $\{f_n\}_1^\infty$ and πf_∞. Since

$$D_R(\pi f_\infty - \pi f_n) \leq D_R(f_\infty - f_n) = \sum_{k=n+1}^{\infty} D_{F_R}(f_\infty) \to 0 \quad \text{as} \quad n \to \infty,$$

we conclude that $\text{BD-}\lim_{n\to\infty} (\pi f_\infty - \pi f_n) = c$ exists and is a nonnegative constant. Again $\pi f_\infty - \pi f_n = f_n = 0$ on $E_{k_0} \subset \Delta$ for some E_{k_0}. Thus $c = 0$, i.e.

$\pi f_\infty = \text{BD-}\lim_{n\to\infty} \pi f_n$ on R. Therefore

$$\mu_R(E) = \int_E d\mu_R(p) \leq \int_\Gamma (\pi f_\infty)(p)\, d\mu_R(p)$$

$$= (\pi f_\infty)(z_0) \quad \text{(by Theorem 2.1E.)}$$

$$= \lim_{n\to\infty} (\pi f_n)(z_0)$$

$$= \lim_{n\to\infty} \int_\Gamma (\pi f_n)(p)\, d\mu_R(p)$$

$$\leq \lim_{n\to\infty} \int_{H_n} d\mu_R(p)$$

where $H_n = (\bigcup_{k=1}^{n} \overline{F}_k) \cap \Gamma \cap U$. Hence

$$\mu_R(E) \leq \lim_{n\to\infty} \mu_R(H_n) = \lim_{n\to\infty} (\mu_R(H_n \cap E) + \mu_R(H_n - E))$$

$$\leq \lim_{n\to\infty} \left(\sum_1^n \mu_R(E_k) + \mu_R(U \cap \Gamma - E) \right) \leq \frac{1}{2}\, \mu_R(E)$$

because $\mu_R(U \cap \Gamma - E) < \frac{1}{2}\, \mu_R(E)$ and $\mu_R(E_k) = 0$.

This is a contradiction. Therefore we have proved (52).

Let $\mu_{F_k}(E_k^*) > 0$ and $w(z) = -\log|f(z)|$. Then $w(z)$ is a positive superharmonic function on F_k with (52), and $w = \infty$ at each point of E_k^*. Clearly $w/n \in \mathfrak{a}(F_k^*, f_{E_k^*})$ for every $n = 1, 2, \ldots,$ where $f_{E_k^*}$ is the characteristic function on E_k^* on Γ_{F_k}. Thus

$$\mu_{F_k}(E_k^*) = H(z_0; f_{F_k^*}) \leq w(z_0)/n$$

for every $n = 1, 2, \ldots,$ by Theorem 2.2A, i.e. $\mu_{F_k}(E_k^*) = 0$ which is a contradiction.

q.e.d.

2B. <u>Traces on the Boundary</u>.

Let G be a non-compact subdomain of a arbitrary open Riemann surface R.

<u>Theorem</u> 4.2B. The set $\overline{G} - \overline{\partial G}$ is an open set in R^*.

<u>Proof</u>. We must show that for any $p \in \overline{G} - \overline{\partial G}$ we can find an open set U in R^* such that $p \in U \subset \overline{G} - \overline{\partial G}$. This is trivial for p in G. Hence we suppose that p is contained in $(\overline{G} - \overline{\partial G}) \cap \Gamma$. There exists a continuous real-valued function $f(q)$ on R^* such that $f(p) = 2$ and $f(q) = -1$ on V, where V is an open

neighborhood of $\overline{\partial G}$ in R^* which does not contain p. Since $M(R)$ is uniformly dense in the totality of continuous functions on R^*, we can find a real-valued function $f' \in M(R)$ such that

$$|f'(q) - f(q)| < \frac{1}{2} \text{ on } R^*.$$

We define a function $g(q)$ by

$$g(q) = \max(\min(1, f'(q)), 0).$$

Then $g \in M(R)$ since $M(R)$ forms a vector lattice and $g(p) = 1$, $g(q) = 0$ on V. As the point p is an accumulation point of G, we can find a directed net $\{p_\lambda\}$ in G such that $\lim_\lambda p_\lambda = p$ and hence

$$\lim_\lambda g(p_\lambda) = g(p) = 1.$$

Now a function $h(z)$ is defined on R as follows:

$$h(z) = \begin{cases} g(z) & \text{on } G \\ 0 & \text{on } R - G \end{cases}$$

Since $g(z)$ vanishes on a neighborhood of ∂G, $h(z)$ is a bounded Tonelli function. Moreover,

$$D_R(h) = D_G(g) \leq D_R(g) < \infty$$

which shows that $h \in M(R)$ and thus it is extended continuously to R^*.

Consider the set

$$U = \{q \in R^* : |h(q)| > 0\}.$$

This is an open set in R^*. Let r be a point in

$R^* - (\overline{G} - \overline{\partial G})$. If $r \in R$, then $r \in R - G$ and $h(r) = 0$. If $r \in \Gamma$, then $r \in \overline{R - G}$ or $r \in \overline{G}$. In the former case, there exists a directed net $\{r_\lambda\}$ in $R - G$ with $\lim_\lambda r_\lambda = r$. Then $h(r) = \lim_\lambda h(r_\lambda) = 0$. In the latter case, $r \in \overline{\partial G}$. Hence there exists a directed net $\{q_\lambda\}$ in ∂G with $\lim_\lambda q_\lambda = r$ and hence $h(r) = \lim_\lambda h(q_\lambda) = 0$. Thus $h(r) = 0$ for any $r \in R^* - (\overline{G} - \overline{\partial G})$. This shows $U \subset \overline{G} - \overline{\partial G}$. Furthermore, as $\{p_\lambda\}$ is in G, we have

$$h(p) = \lim_\lambda h(p_\lambda) = \lim_\lambda g(p_\lambda) = 1.$$

This tells $p \in U$.

q.e.d.

2C. <u>A Unicity Principle</u>.

As an application of the Royden compactification of the open unit disk we obtain the following interesting theorem. The Royden compactification seems to be an essential tool in the proof of this unicity principle.

<u>Theorem</u> 4.2C. Let $f(z)$ be a meromorphic function in the open unit disk U. Suppose that there exists a set E in the unit circumference γ with positive linear measure and a stairway V at 1 such that f has limit zero along ζV at each point ζ in E. Then $f(z)$ is

identically zero.

Proof. We may assume that E is a compact subset of γ by the regularity of linear measure. By the definition of stairway, there exists a pair of real numbers a_0 and b_0 such that $-1 < a_0 < b_0 < 1$ and $V \supset \overline{A(1;\ a_0,\ b_0)}^0$ where $\overline{A(1;\ a_0,\ b_0)}^0 = \overline{A(1;\ a_0,\ b_0)} \cap U$. Since f has limit zero along ζV at each point ζ in E, for every pair of real numbers a and b with $-1 < a_0 < a < b < b_0 < 1$, and for each point ζ in E f has limit zero along $\overline{A(\zeta;\ a,\ b)}^0$. Now we fix pairs $(a_1,\ b_1)$ and $(a,\ b)$ of real numbers with $-1 < a_0 < a_1 < a < b < b_1 < b_0 < 1$ and define sets $X_\zeta^{(n)}$ $(n = 1,\ 2,\ \ldots)$ by

$$X_\zeta^{(n)} = \Gamma \cap [\overline{A^{(n)}(\zeta;\ a,\ b)} - \overline{\partial A^{(n)}(\zeta;\ a,\ b)}].$$

By Theorem 4.2B these are open sets in Γ. Set

$$X^{(n)} = \bigcup_{\zeta \in E} X_\zeta^{(n)} \quad (n = 1,\ 2,\ \ldots) \quad \text{and} \quad X = \bigcap_1^\infty X^{(n)}.$$

Clearly for each $n = 1,\ 2,\ \ldots$ $X^{(n)}$ is an open set in Γ and hence X is a G_δ-set in Γ. Moreover, for each $\zeta \in E$ we take a set $X_\zeta = X \cap \pi^{-1}(\zeta)$.

Let q_0 be an arbitrary point in X. Then $q_0 \in X_\zeta$ for some $\zeta \in E$ and $a \leq g_\zeta(q_0) \leq b$. First we shall show that $\overline{A(\zeta;\ a_1,\ b_1)}$ is a neighborhood of q_0. To see this observe that $q_0 \in \{q \in \Gamma : a_1 < g_\zeta(q) < b_1\}$. Take an arbitrary point q in Γ with $a_1 < g_\zeta(q) < b_1$ and a directed net $\{z_\lambda\}$ in U converging to q. We may assume $a_1 < g_\zeta(z_\lambda) < b_1$ for all λ since $\lim_\lambda g_\zeta(z_\lambda) = g_\zeta(q) \in (a_1, b_1)$. It follows that $\{z_\lambda\} \subset A(\zeta;\ a_1, b_1)$. Consequently, $q \in \overline{A(\zeta;\ a_1, b_1)}$. It implies that $\overline{A(\zeta;\ a_1, b_1)}$ contains an open set $\{q \in \Gamma : a_1 < g_\zeta(q) < b_1\}$, i.e. $\overline{A(\zeta;\ a_1, b_1)}$ is a neighborhood of q_0.

From the hypothesis

$$\lim_{n\to\infty} \sup_{\substack{z\in\zeta V \\ |z-\zeta|<\frac{1}{n}}} |f(z) - 0| = 0$$

together with the fact that for some $m \geq n$

$$\sup_{z\in A^{(m)}(\zeta;a_0,b_0)} |f(z) - 0| \leq \sup_{\substack{z\in\zeta V \\ |z-\zeta|<\frac{1}{n}}} |f(z) - 0|$$

it follows that

$$\lim_{n\to\infty} \sup_{z\in A^{(m)}(\zeta;a_0,b_0)} |f(z) - 0| = 0. \qquad (1)$$

For any $\varepsilon > 0$ there exists a positive integer N such that

$$|f(z)| < \varepsilon \quad \text{for all} \quad z \in A^{(N)}(\zeta;\ a_0, b_0).$$

By the construction $\overline{A^{(N)}(\zeta;\ a_1, b_1)} \cap U$ is contained in $A^{(N)}(\zeta;\ a_0, b_0)$. Then we may take $A(\zeta;\ a_1, b_1)$ as $A^{(N)}(\zeta;\ a_1, b_1)$. Now we showed that for each $\varepsilon > 0$ there exists a neighborhood $\overline{A(\zeta;\ a_1, b_1)}$ of q_0 such that $|f(z)| < \varepsilon$ for all $z \in \overline{A(\zeta;\ a_1, b_1)}^0$. This shows that f has limit zero at q_0 and hence f has limit zero at each point p in X by the arbitraryness of q_0 in X. Therefore from Theorem 4.2A $f \equiv 0$ if $\mu_U(X) > 0$.

The sequence of subsets $\{X^{(n)}\}$ decreases to X, so does $\{\mu_U(X^{(n)})\}$ to $\mu_U(X)$. Set $\gamma_k(\zeta) = \{\eta \in \gamma : \alpha_k < |\eta - \zeta| < \beta_k\}$ where $\{(\alpha_k, \beta_k)\}_1^\infty$ is associated with (a,b).

$$X_\zeta^{(n)} = X_\zeta \cup \{\bigcup_{k=1}^{\infty} \pi^{-1}(\gamma_k(\zeta))\}.$$

We define a real-valued function $w(z)$ on U by

$$w(z) = \int_{X^{(n)}} P(z, p)\,d\mu_U(p)$$

where $P(z, p)$ is the harmonic kernel associated with (U, μ_U, z). By Theorem 2.1E $w(z)$ is a harmonic function on U and for each $q \in \Delta \cap X^{(n)}$ $\lim_{U \ni z \to q} w(z) = 1$. It follows that $\lim_{U \ni z \to \xi} w(z) = 1$ for each $\xi \in \gamma^{(n)}$ where $\gamma^{(n)} = \bigcup_{\zeta \in E} \gamma^{(n)}(\zeta) = \bigcup_{\zeta \in E}\left(\bigcup_{k=n}^{\infty} \gamma_k(\zeta)\right)$.

In fact, let ξ be an arbitrary point in $\gamma^{(n)}$, then ξ is in some $\gamma_k(\zeta)$ for $k \geq n$ and $\zeta \in E$ which are open arcs in γ. Take the intersection G of U with the open disk with center at the middle point ζ_0 of the component Λ of $\gamma_k(\zeta)$ containing ξ and radius the chord from ζ_0 to an endpoint of Λ. We define a real-valued function $\hat{w}$ on $G \cup (\partial G) \cup \Lambda$ such that $\hat{w}$ is harmonic in G, $\hat{w}|\partial G = w$, and $\hat{w}|\Lambda = 1$. Then $\lim_{G \ni z \to p} \hat{w}(z) = 1$ for every $p \in \pi^{-1}(\Lambda) \cap \Delta - \overline{\partial G}$ because $\hat{w} \circ \pi$ is continuous on $\overline{G} - \overline{\partial G}$ in U^*. Consider $\tilde{w} = \hat{w} - w$ on G. Then $\tilde{w} \geq 0$ since $0 \leq w \leq 1$ on U (cf. II. 1B). Moreover $\lim_{G \ni z \to p} \tilde{w}(z) = 0$ for every $p \in (\partial G) \cup (\overline{G} \cap \Delta - \overline{\partial G})$. Let G^* be the Royden compactification of G and Δ_G its harmonic boundary. Since $\tilde{w} \in HB(G)$, by Theorem 2.2B and Theorem 2.2C the above implies that $\lim_{G \ni z \to p} \tilde{w}(z) = 0$ for every $p \in \Delta_G$ except a set of μ_G measure zero where μ_G is the harmonic measure relative to G. Hence we have that $\tilde{w} \equiv 0$ on G by Theorem 1.2H.1. Consequently $\lim_{G \ni z \to \xi} w(z) = 1$ for every $\xi \in \gamma^{(n)}$.

By the Fatou theorem (cf. [15] p. 235)

$$w(z) = \frac{1}{2\pi} \int_0^{2\pi} \left[\mathrm{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \right] w(e^{i\theta})\, d\theta \quad \text{on} \quad U.$$

Therefore $w(z) \geq \frac{1}{2\pi} \int_{\gamma^{(n)}} \mathrm{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) d\theta$ on U and hence

$$w(0) \geq \frac{1}{2\pi} \int_{\gamma^{(n)}} d\theta = \frac{1}{2\pi}\, \ell(\gamma^{(n)})$$

where ℓ is the linear measure.

On the other hand from Theorem 1.3B

$$P(0, q) = \begin{cases} 1 & \text{on } \Delta \\ 0 & \text{on } \Gamma - \Delta \end{cases}$$

It follows that

$$w(0) = \int_{X^{(n)}} P(0, q) d\, \mu_U\, (q) = \mu_U(X^{(n)} \cap \Delta)$$

$$= \mu_U(X^{(n)}).$$

Combining this with the previous inequality, we obtain $\mu_U(X^{(n)}) \geq \ell(\gamma^{(n)})$. Next we rotate some $\gamma_n(\zeta')$ by a rotation f_θ so that $f_\theta \gamma_n(\zeta')$ covers ζ. It is easily seen that $f_\theta \gamma^{(n)} \supset E$. Thus

$$\mu_U(X^{(n)}) \geq \ell(\gamma^{(n)}) = \ell(f_\theta \gamma^{(n)}) > \ell(E) > 0$$

for all n and hence

$$\mu_U(X) = \lim_{n \to \infty} \mu_U\, (X^{(n)} \geq \ell(E) > 0.$$

q.e.d.

2D. <u>Remark</u>.

Let U and γ be the open unit disk and the unit circumference, respectively and $\{\zeta_n\}_1^\infty$ a sequence of points on γ converging ζ on γ so that ζ_n is in the left of ζ_{n+1} $(n = 1, 2, \ldots)$, i.e.

$\arg \zeta_{n+1} < \arg \zeta_n$, and $\{|\zeta_n - \zeta_{n+1}|\}$ is decreasing.

Let

$$B_n(\zeta) = \{z \in U : |z - \zeta_n| < \frac{|\zeta_n - \zeta_{n+1}|}{2}\},$$

$$B^{(n)}(\zeta) = \bigcup_{k=n}^{\infty} B_k(\zeta), \quad \text{and} \quad B(\zeta) = \bigcup_{1}^{\infty} B^{(n)}(\zeta).$$

We say a stepping stone V at ζ if V has the following conditions:

(1) V is a subset of U,

(2) there exists a positive integer n such that $V \supset \overline{B^{(n)}(\zeta)}^0$, where $\overline{B^{(n)}(\zeta)}^0 = \overline{B^{(n)}(\zeta)} \cap U$.

We may replace a stairway in the Unicity theorem 4.2C by a stepping stone V.

These are just a few illustrations of how the specialization of the general unicity theorem for the Royden topology to the concrete case produces new types of unicity theorems. Comparison with our Unicity theorem 4.2C with the Lusin-Privaloff theorem would be illustrative.

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