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Title: Multiscale Analysis of Saturated Flow in a Porous Medium with an Adjacent Thin Channel

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Ralph E. Showalter

This thesis contains three parts addressing the asymptotic analysis of fluid flow through fully saturated porous medium in the presence of an adjacent thin channel.

In the first part the problem is modeled by Darcy's law in both the porous medium and in the channel. The permeability in the channel is scaled in order to balance the width of the channel with the high permeability in this region. The geometry of the channel is a region between a flat interface with the porous medium and a curvy top. The problem is analyzed in direct variational formulation, and the solution obeys the minimization principle. A fully-coupled model with a lower dimensional interface problem is obtained in the limit as an approximation.

The second part models the problem by means of the same Darcy's law and scaling technique. The difference consists of the geometric possibilities of the channel, which is now limited by two parallel surfaces of smoothness C^1, and the formulation of the problem as a saddle point solution of a system of first order partial differential equations in mixed formulation. A limit problem of analogous structure and lower dimensional interface is obtained as an approximation.

In the third part the channel is modeled with Stokes law. Appropriate interface conditions are given to couple the Stokes' and Darcy's flow. The tangential velocity and the pressure exhibit a discontinuity across the interface. The limit problem is a Darcy-Brinkman system which has a structure different from that of the original one.

Multiscale Analysis of Saturated Flow in a Porous Medium with an Adjacent Thin Channel

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APPROVED:

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Fernando A. Morales, Author

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MULTISCALE ANALYSIS OF SATURATED FLOW IN A POROUS MEDIUM WITH AN ADJACENT THIN CHANNEL

1 INTRODUCTION

The problem of modeling flow of a single phase fluid through a fully-saturated porous medium has been widely studied. To this end Darcy's equation is not only regarded as an acceptable model by experts of different fields but considered as an empirical-theoretical relation which effectively defines porous media. Many extensions of this law have been developed to address more complicated types of flow and types of media. Some refinements preserve the original structure and modify the equation by the introduction of physical parameters while others perturb the structure of the equation itself.

Amongst the various geometric scenarios an important case arises when the porous medium contains cracks or fractures, since the velocity of fluid in these regions is expected to be considerably higher than that of the adjacent or surrounding medium. The presence of fractures introduces the question of preferential flow, and this affects the overall permeability of the medium. The situation when there is a periodic distributed system of such fractures has been studied in [TA06]. Some other cases have been studied under the hypothesis that the medium has a microstructure such as fissured or layered media [BS89] by means of double-porosity models treating partially [RS95] or totally fissured media [SW90]. The difference is whether there is direct flow between the blocks or not, respectively; see also [Sho97b]. The previous cases are concerned with the geometry of the medium while we can quote Blavier and Mikelić for Non-Newtonian Flow [BM95], Bourgeat for Two Phase Flow [Bou86] or Ene and Sanchez-Palencia for Thermal Flow [ESP81], [ESP82] as works regarding a different type of flow. Finally we mention the work on Poroelastic Media by Auriault and Sanchez-Palencia [ASP77] as a special case since it is concerned with the material properties of the medium rather than its geometric structure.

Here we study the situation in which there is a single specified fracture which is a narrow region of high fluid velocity and low pressure gradient. Such a phenomenon takes place in the vicinity of a rigid wall of a porous medium where the particles do not pack efficiently and therefore the permeability rises sharply. As a result there exists a narrow region close to the wall where the velocity is considerably higher and the fluid flow mostly parallel to such wall; this phenomenon is known as the channeling effect [DAN99]. The same situation occurs in interior fractures, and it is applicable to several examples of the real world such as oil extraction and subsurface water flow [JRC71]. The problem in which there is a single specified fracture has been studied from a numerical point of view in [VMR05].

Such problems contain two sources of singularity, one geometric due to the thinness of the region in which the fast flow occurs, and the other, a material effect since the flow in the channel is qualitatively different from that of the porous medium. Due to the contrast in behavior it is immediate that whichever model is chosen to describe this phenomenon it will present great difficulties when trying to compute solutions. The problems may be of numerical stability when using a not so accurate choice or to spend too many computational resources in the geometric description of the phenomenon when trying to be accurate. These lead to numerical issues of stability, storage, computational time and many other aspects. Since these problems come from the inherent singularities it seems natural to use the asymptotic analysis of the problem in order to balance out these singularities and seek a model free of them which is still a good approximation.

The present dissertation is written in the "manuscript document format" as specified by the Oregon State University Thesis Guide 2006-2007. Each of the three subsequent chapters contains a manuscript on approaching the problem from different points of view, whether the formulation or the laws involved, or the discussion of interface conditions, together with its corresponding asymptotic analysis. Each approach gives deeper insight than the others in different aspects: the impact of the shape of the channel, the geometric structure of the flow under which it can be considered controlled or estimated, variational minimization principles, the interface activity, coupling PDE systems with non-uniform structure and degenerate evolution equations. The publication status of each manuscript is specified in the respective chapter heading page. In the last chapter of this dissertation the reader can find some overall conclusions and special remarks as well as open questions that did not make it into the submitted manuscripts.

I apologize in advance for the difficulties the reader will find due to the difference in notation and style between manuscripts. The chosen notation was meant to be in agreement with the preexisting literature published on the subject. Besides, each document was written at different stages of my academic career and it is aimed to different audiences. This introduction will not explain in detail the material of each document since each of them contains an appropriate introductory section. However it is pertinent to give the overall accomplishment of each work under a general perspective.

The first and second parts model the flow in both the channel and surrounding medium by means of Darcy's law with scaled permeability to balance the channel width and model the higher flow rates in it. However the formulation of the system of partial differential equations and interface conditions is different: the first document models the problem in a context of direct variational formulation, as a minimization problem, the second part takes the setting of a particular mixed formulation which to the author's best knowledge is a totally new approach in the formulation of boundary-value problems. For the similarities of the problems we mention that both cases use equivalent boundary conditions, suitable to the spaces of formulation and reasonable physical assumptions. Also a fully-coupled model with a lower dimensional interface problem is obtained in the limit as an approximation from the asymptotic analysis. In both cases the limit has the same structure and formulation of the original system but coupling the fluid flow inside an open set with a lower dimensional interface which is part of its boundary. The asymptotic analysis is presented for a stationary problem in both cases and in the first one it is extended to the evolution problem by means of analytic semigroups. The time dependent problem for the mixed formulation is still work in progress. Finally in the standard formulation the time evolution problem induces a degenerate evolution equation, since the evolution term on the interface vanishes in the limiting problem.

For the differences of the approaches we begin by mentioning that the first part presents an additional section which is the analysis of the concentrated capacity model, since its similar structure makes it a good fit in the context. In turn, the second part presents a gravity-driven flow including the appropriate forcing term since it is more convenient to handle the additional term in the mixed formulation context. Another important difference lies on the geometric description of the channel. In the first part the channel must be bounded between a flat interface and a top boundary described by a "width" function, therefore the surfaces are not parallel. The second part describes the channel as a region trapped between two parallel surfaces: the interface which may be curved or not and a top boundary. The treatment of a-priori estimates is radically different in the two cases: while in the first case a-priori estimates can be obtained by standard techniques of asymptotic analysis and they reveal standard information, the search for a-priori estimates in the second case is deeply connected to the nature of the flow and reveals valuable information about which physical entities must be preserved while the asymptotic analysis takes place. Even though in the present work both cases assume the same interface conditions of continuity of the pressure and a balance of the normal flux which are to be expected, the mixed formulation allows interface conditions of much greater generality replacing the continuity of pressure and normal flux for balance statements in both cases. These conditions are consistent with the demands and insight of homogenization theory when modeling coupled problems where two regions interact and have radically different physical properties, namely, very different values of permeability for flow in porous media, of conductivity when modeling problems of heat diffusion or different values of stiffness for elasticity problems. Interface conditions of such generality make necessary the mixed formulation and are unknown in the literature. Finally, another remarkable property of the particular mixed formulation introduced here is the fact that it leaves the spaces of functions fully decoupled, leaving the interface conditions only on the solution of the problem.

The third part uses Darcy's law in the porous medium and Stokes law for the fluid velocity in the channel, which is a more accurate model for the faster flow. The two laws have very different structure and scale of validity, making the formulation of a coupled system a more difficult task. The choice of interface conditions plays a fundamental role in the formulation of a well-posed problem as well as in the asymptotic analysis of the PDE system. The tangential velocity and the pressure of the fluid are not continuous on the interface, although conservation of fluid mass requires the normal flux to be continuous; the interface conditions address these facts and relate the normal stress balance with the exchange of fluid. The scaling by means of the physical parameters must now be introduced not only on the Stokes law modeling the narrow channel, since the interface conditions contain so much important information of the problem they must be similarly scaled. The limit solution is characterized as a fully-coupled system consisting of Darcy flow in the porous medium and Brinkman flow on the interface. Therefore, the structure of the problem itself changes. The function spaces in which the limit problem is modeled also have a different structure, and it is defined by the a-priori estimates. Several geometric possibilities for the shape of the channel and the interface are explored leading to interesting cases. The impact of these possibilities on the structure of the flux on the channel is revealed. Also, the importance of parallelism between the interface and the top boundary of the channel follows: there must exist at least a set of non-zero measure where the channel is trapped between parallel surfaces one of them being the interface and the other the top boundary.

The Relation Between Darcy, Brinkman and Stokes Equations

The present dissertation will address coupled systems containing equations of the three types: Darcy, Brinkman or Stokes. Combinations of these types will be used to model the flow in different settings as they arise from the asymptotic analysis of the fluid behavior. These three laws are related according to the scale at which the problem is analyzed. From a theoretical point of view Stokes law should suffice to model the viscous fluid flow through a porous medium as well as any other configuration including an open fracture. However, Stokes law demands a detailed description of the geometry of the medium. This requirement makes its use impractical to model flow through a porous medium, because it will hardly have an easy geometric description or even possible within reasonable investment of efforts and resources. It is therefore a necessity to use an upscaled equation which models an averaged flow which describes the main characteristics of the porous medium flow without taking in consideration a detailed geometric description of the pores. To this end, Darcy's law is considered one of the most successful models, accepted by a wide variety of experts in different fields. In its primitive form at this macro-scale, it states merely that the fluid's velocity is proportional to the negative gradient of pressure, meaning the fluid moves from regions of high pressure to regions of low pressure. The constant of proportionality is a ratio between the two main physical parameters involved in the flow, the permeability describing the overall impact of the medium's geometry on the resistance to flow and the viscosity describing the resistance of the fluid to shear. It is expressed as

$$\mathbf{v} + \frac{\kappa}{\mu}\boldsymbol{\nabla} p = 0 \tag{1.1}$$

where κ is the permeability of the medium and μ is the viscosity of the fluid. The results provided by this law are in excellent agreement with the measurements coming from experiments, and it is clear that its scale of validity is the macro-scale, meaning that Darcy's law relates quantitative averages over a typical representative cell. The size of the pores in average is much smaller than the cell and the medium is considered a homogeneous composite of these cells. It is in this sense an averaged law.

Since the same phenomenon could be modeled in theory by means of Stokes law whenever the porous medium has a relatively simple description, and since its scale of validity is considerably smaller, it is a natural question whether Darcy's law is an "averaged" version of Stokes law or how is related. Several efforts have been done in this direction from different points of view and with many different techniques. The first rigorous proof showing Darcy's law as an averaged version of Stokes law appeared in [Tar80]. Later [All89] used the two-scale convergence to show that Darcy's law is the limit problem of Stokes law under the hypothesis that the porous medium structure is periodic. However there are more general results given by Cioranescu and Murat [CM82] and [CM97] using the energy method introduced by Tartar to give a more general result. The key aspect comes from the geometry of the medium, the main parameter is a special ratio between the size of the particles ϵ and the distance between them η that can be easily identified with the period. We define according to the dimension N

$$\sigma_\epsilon = \begin{cases} \epsilon \left(\frac{\eta}{\epsilon}\right)^{N/2} & \text{for } N \geq 3 \\ \eta \left|\log\left(\frac{\epsilon}{\eta}\right)\right|^{1/2} & N = 2 \end{cases} \tag{1.2}$$

The limit behavior of the ratio σ_ϵ as $\epsilon \downarrow 0$ determines the structure of the limiting problem, since it dictates the dominant nature of the flow. Three cases are in order

$$\lim \sigma_\epsilon = +\infty : \begin{cases} \boldsymbol{\nabla} p - \mu\,\Delta\mathbf{v} = \mathbf{f} & \text{in } \Omega \\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \partial\,\Omega \end{cases} \tag{1.3a}$$

$$\lim \sigma_\epsilon = \sigma > 0 : \begin{cases} \boldsymbol{\nabla} p - \mu\,\Delta\mathbf{v} + \frac{\mu}{\sigma^2}\,\mathbf{v} = \mathbf{f} & \text{in } \Omega \\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \partial\,\Omega \end{cases} \tag{1.3b}$$

$$\lim \sigma_\epsilon = 0 : \begin{cases} \mathbf{v} = \frac{1}{\mu}\,M^{-1}\,(\mathbf{f} - \boldsymbol{\nabla} p) & \text{in } \Omega \\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \partial\,\Omega \end{cases} \tag{1.3c}$$

We see a list of momentum equations in the limit, in the first case Stokes, in the third case Darcy and the intermediate or transition case is the second one, known as Brinkman's law. In all the cases the mass conservation law $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ remains identical through the homogenizing or limiting process. An important part of the task is not only identifying the main geometric parameter, but also giving the type of convergence in which these laws can be considered as an upscaled version of Stokes law. The type of convergence also defines the type of "averaging" process executed on Stokes law to generate the other cases.

On the line of qualitative modeling rather by asymptotic analysis we recall [Lév83] who studies the problem of fluid flow through an array of fixed particles. The work does not assume periodicity of the medium, but rather the existence of a representative element of volume for the medium and with it a representative relation between the representative size of the particles ϵ and the representative space between them η. By means of asymptotic formal expansions the author discusses the order of magnitude of the different physical effects identifying which are important and which are negligible according to the geometry of the medium. For flow in three dimensions the three cases are identified

$$\epsilon \ll \eta^3 : \qquad \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} p - \mu\,\Delta\mathbf{v} = \mathbf{f} \tag{1.4a}$$

$$\epsilon = \eta^3 : \qquad \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} p - \mu\,\Delta\mathbf{v} + \mu\, c'\,\mathbf{H}\cdot\mathbf{v} = \mathbf{f} \tag{1.4b}$$

$$\eta^3 \ll \epsilon \ll \eta : \qquad \mu\, c'\,\mathbf{H}\cdot\mathbf{v} = \mathbf{f} - \boldsymbol{\nabla} p \tag{1.4c}$$

where c' is the concentration of the number of particles per unit volume c multiplied by a factor, we have $c' \equiv c\,\eta^3$ and $\mathbf{H}$ is a tensor depending only on the intrinsic geometry of the particle, dependent only on its shape. Again, the three laws are concluded according to the geometry. Other works pursuing the derivation of Darcy's law by homogenization using formal asymptotic can be found in [Kel80] and [SP80].

The mentioned works are evidence of the strong connections between these laws despite the dramatic difference in the approach when analyzing the problem. The importance of the relationship between size of the particles ϵ and space between them η as critical parameter to decide the flow regime is an agreement, though the required specifications for such relation might change from one work to another, the domain of validity for the critical case *i.e.* Brinkman's equation is an active topic of research [Aur05]. It is generally recognized that Brinkman's law does not describe a "normal" porous medium, since it would require the porosity to be at or above 0.8. It has been used with some success as a means to transition from Darcy to Stokes flow by continuously modifying the coefficients, and our results in the third part may explain in part that success. This dissertation shows the relationships of these laws are present when efficiency of the packing of the particles near a rigid wall goes down, thereby inducing transition to a different flow regime. The choice of accuracy level in the starting model will prove to be of capital importance to detect the perturbation on the flow regime.

2 THE NARROW FRACTURE APPROXIMATION BY CHANNELED FLOW

2.1 Introduction

Fluid flow through a fully-saturated porous medium is altered in the vicinity of a rigid wall by the sharp rise in permeability due to the inefficiency of the packing of the particles in the vicinity of the wall. Consequently, in a narrow region close to the wall the velocity is substantially higher and the flow is predominantly parallel to it; this phenomenon is known as the *channeling effect* [DAN99]. Related models were used previously to describe flow through a porous medium in the vicinity of a narrow fracture which is characterized similarly as a thin interior region of high permeability. Such problems arise *e.g.* from hydraulic fracturing in which narrow channels of high permeability are created in the vicinity of a well to enhance the flow rate and consequently the production. The *narrow fracture approximation* leads to a model like the one above for thin channel flow, and by taking advantage of the symmetry about the center surface defining the fracture, one can reduce such a problem to one of the type considered here with the high-permeability region located on the boundary [JRC71], [VMR05]. Analogous models of heat conduction arise from regions of high conductivity, and these may also include a *concentrated capacity*. We include these in the discussion for comparison.

For a final example, we mention saturated gravity-driven flow of subsurface water through a hillslope bounded below by sloping bedrock. A network of narrow channels of very high permeability develops in the vicinity of the impermeable bedrock, and it is observed that most of the fluid in the system flows through this region. Such systems with high flow rate over narrow regions greatly influence the transport and flow processes and are a topic of current study [WM07].

We shall describe such situations with Darcy flow for which the permeability is scaled to balance the channel width and model the higher flow rates in the channel. Due to the higher permeability, the fluid flows primarily into and then tangential to the channel. The resulting model captures the tangential boundary flow coupled to the interior flow by continuity of flux and pressure. It contains two sources of singularity: a geometric one from the the thinness of the channel and a material one due to the higher permeability of the channel. With the appropriate scaling, these two singularities are balanced, and a fully-coupled model is obtained in the limit as an approximation. See [HSP74], [SP80] for asymptotic analysis and [VMR05] for numerical analysis of these and related models.

An additional challenging issue is to account for the *shape* of the channel, especially for any *taper* near the edges or boundary of the channel. Such shapes are ubiquitous in applications, but they are not commonly included in the modeling process. They are important because the rate of the tapering at the edges determines the appropriate boundary conditions (or lack thereof) that describe the resulting model [Mey70], [Sho79].

The geometry of the model is described first. Let Ω_1 be a bounded domain in $\mathbb{R}^n$ and denote by Γ a relatively-open connected portion of its boundary $\partial\Omega_1$ along the top of the domain. For simplicity of representation, we assume this portion of the boundary is *flat*, that is, $\Gamma \subset \mathbb{R}^{n-1} \times \{0\}$ and that $x_n < 0$ for each $x = (\widetilde{x}, x_n) \in \Omega_1$, where $\widetilde{x} \in \mathbb{R}^{n-1}$. The channel is realized as a region of the form $\Omega_2^\epsilon = \{(\widetilde{x}, \omega(\widetilde{x})x_n) : \ (\widetilde{x}, x_n) \in \Gamma \times (0, \epsilon)\}$. The function $\omega(\cdot)$ shapes the width of the channel at each $\widetilde{x} \in \Gamma$, and the parameter $\epsilon > 0$ denotes its scale. We assume that this width function satisfies $0 < a \leq \omega(\widetilde{x}) \leq 1$ on each compact subset of Γ, where a depends on the set, but it may approach zero near $\partial\Gamma$ at a rate to be determined below. This assumption permits the channel to be tapered or to *pinch off* near its extremities.

For the single-phase flow of a slightly compressible fluid through $\Omega^\epsilon \equiv \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$, Darcy's law together with conservation of fluid mass lead to the interface problem

$$
\begin{aligned}
m_1 \frac{\partial u_1^\epsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\epsilon = m_1 f \text{ in } \Omega_1 \\
u_1^\epsilon = 0 \text{ on } \partial\Omega_1 - \Gamma \\
u_1^\epsilon = u_2^\epsilon,\ k_1 \partial_z u_1^\epsilon - \frac{k_2}{\epsilon}\, \partial_z u_2^\epsilon = g \text{ on } \Gamma \\
m_2 \frac{\partial u_2^\epsilon}{\partial t} - \nabla \cdot \frac{k_2}{\epsilon} \nabla u_2^\epsilon = m_2 f \text{ in } \Omega_2^\epsilon \\
\frac{k_2}{\epsilon} (\nabla\, u_2^\epsilon) \cdot \widehat{n} = 0 \text{ on } \partial\Omega_2^\epsilon - \Gamma\,,
\end{aligned} \tag{2.1.1a}
$$

at each $t > 0$ for the fluid density $u_1^\epsilon(\cdot, t)$ in Ω_1 and $u_2^\epsilon(\cdot, t)$ in Ω_2^ϵ, and these satisfy the initial conditions

$$
u_1^\epsilon(\cdot, 0) = u_1^0(\cdot) \text{ on } \Omega_1, \quad u_2^\epsilon(\cdot, 0) = u_2^0(\cdot) \text{ on } \Omega_2^\epsilon\,. \tag{2.1.1b}
$$

Thus, the region is drained along $\partial\Omega_1 - \Gamma$ and there is no flow across $\partial\Omega_2^\epsilon - \Gamma$, where the outward normal is indicated by $\widehat{n}$. This latter condition would follow if the region were symmetric about $\Gamma \times \{\epsilon\}$. The given initial density distributions $u_j^0(\cdot)$ complete the initial-boundary-value problem. Corresponding non-homogeneous problems with known pressure on $\partial\Omega_1 - \Gamma$ and flow-rate along $\partial\Omega_2^\epsilon - \Gamma$ can be reduced to this case. The permeability in Ω_2^ϵ has been scaled by $\frac{1}{\epsilon}$ to indicate the high flow rate, and this will be shown to balance the width ϵ of the channel, so the flow in Ω_2^ϵ is closely approximated by surface flow along Γ. It will be seen below that k_2 is the *effective tangential permeability* and $\frac{k_2}{\epsilon^2}$ is the *effective normal permeability* for channel flow; see [VMR05] for substantial discussion and further perspective. The coefficients $m_1,\ m_2$ are obtained from the *porosity* and from the *compressibility* of either the fluid or the medium. We include for comparison the concentrated capacity model in which also m_2 is scaled by $\frac{1}{\epsilon}$, but this has nothing to do with porous media.

2.2 Preliminaries

We use standard notation and results on function spaces. $L^2(\Omega)$ is the Hilbert space of (equivalence classes of) Lebesque square summable functions on Ω, and $H^m(\Omega)$, $m \geq 1$, with the norm $\|\cdot\|_{m,\Omega}$ is the Sobolev space of functions in $L^2(\Omega)$ for which each weak derivative up to order m belongs to $L^2(\Omega)$. The space $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of those infinitely differentiable functions which have compact support in Ω. The *trace* $\gamma(v)$ of a $v \in H^1(\Omega)$ is its boundary value in $H^{1/2}(\partial\Omega)$. The spaces with fractional exponents are defined by interpolation. Corresponding spaces of vector-valued functions are denoted by bold-face symbols, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^m(\Omega)$. The space of those functions of $\mathbf{L}^2(\Omega)$ whose divergence belongs to $L^2(\Omega)$ is denoted by $\mathbf{L}^2_{div}(\Omega)$. These have a *normal trace* on the boundary. See [Ada75], [Sho77], [Sho97a], [Tem79].

Assume the interface Γ is an open bounded connected subset of $\mathbb{R}^{n-1}$ and that it lies locally on one side of its boundary, $\partial\Gamma$, a C^1 manifold. Let $\delta(\widetilde{x})$ be the distance from $\widetilde{x} \in \Gamma$ to $\partial\Gamma$ and $0 \le \alpha < 1$. Define $W(\alpha)$ to be the space obtained by completing $H^1(\Gamma)$ in the weaker norm

$$\|v\|_{W(\alpha)} = \left\{ \int_\Gamma \left(v(\widetilde{x})^2 + \delta(\widetilde{x})^\alpha \|\widetilde{\nabla} v(\widetilde{x})\|^2 \right) d\widetilde{x} \right\}^{1/2}.$$

Here and in the following, $\widetilde{\nabla}$ denotes the $\mathbb{R}^{n-1}$-gradient in directions tangent to Γ. It is known that the embedding $W(\alpha) \to L^2(\Gamma)$ is compact and the trace operator $\gamma : W(\alpha) \to L^2(\partial\Gamma)$ is continuous [Gri63], [Mey67]. Here we assume the width function satisfies

$$\omega(\widetilde{x}) \ge c\delta^\alpha(\widetilde{x}) \text{ a.e. } \widetilde{x} \in \Gamma \tag{2.2.2}$$

for some $c > 0$, and we say Γ is *weakly tapered*. Then define $H^1_\omega(\Gamma)$ to be the completion of $H^1(\Gamma)$ with the norm

$$\|v\|_{H^1_\omega} = \left\{ \int_\Gamma \left(v(\widetilde{x})^2 + \omega(\widetilde{x}) \|\widetilde{\nabla} v(\widetilde{x})\|^2 \right) d\widetilde{x} \right\}^{1/2}.$$

As above, the embedding $H^1_\omega(\Gamma) \to L^2(\Gamma)$ is compact and the trace operator $\gamma : H^1_\omega(\Gamma) \to L^2(\partial\Gamma)$ is continuous. More generally, we have the following [Sho79].

Theorem 2.2.1. *Let the bounded domain Γ be given as above and let $0 \le \alpha < 1$. Suppose there is a function $\alpha(\cdot)$ on $\partial\Gamma$ for which $0 \le \alpha(\widetilde{x}) \le \alpha$ for each $\widetilde{x} \in \partial\Gamma$. Assume the function $\omega(\cdot)$ satisfies (2.2.2) and that at each point of $\partial\Gamma$ there is a neighborhood N in $\mathbb{R}^{n-1}$ and constants $0 < c(N) < C(N)$ such that*

1. *for each* $\widetilde{x} \in N \cap \Gamma$ *there is an* $\widetilde{x}_0 \in \partial\Gamma$ *such that* $\|\widetilde{x}_0 - \widetilde{x}\| = \delta(\widetilde{x})$, *and*

2. *for each* $\widetilde{x} \in N \cap \Gamma$, $c(N) \leq \frac{\omega(\widetilde{x})}{\delta(\widetilde{x})^{\alpha(\widetilde{x}_0)}} \leq C(N)$.

Then the trace map is continuous from $H^1_\omega(\Gamma)$ *into* $L^2(\partial\Gamma)$, *its kernel is the closure of* $C_0^\infty(\Gamma)$ *in* $H^1_\omega(\Gamma)$, *and the range is dense in* $L^2(\partial\Gamma)$.

In the contrary case we call Γ *strongly tapered* if

$$\omega(\widetilde{x}) \leq C\delta(\widetilde{x}) \text{ a.e. } \widetilde{x} \in \Gamma, \tag{2.2.3}$$

for some $C > 0$, and then $C_0^\infty(\Gamma)$ is dense in $H^1_\omega(\Gamma)$, so $H^1_\omega(\Gamma)'$ is a space of distributions on Γ and $L^2(\Gamma) \subset H^1_\omega(\Gamma)'$.

We recall some classical results for unbounded operators and the Cauchy problem; see [Kat95], [Sho77] or the first Chapter of [Sho97a] for details. Let V be a Hilbert space, and denote its dual space of continuous linear functionals by V'. A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is V*-elliptic* if there is $c_0 > 0$ for which

$$a(u,u) \geq c_0 \|u\|_V^2 \ , \qquad u \in V \ .$$

The Lax-Milgram theorem shows this is a convenient sufficient condition for the associated problem to be well-posed.

Theorem 2.2.2. *If* $a(\cdot,\cdot)$ *is bilinear, continuous and* V*-elliptic, then for each* $f \in V'$ *there is a unique*

$$u \in V : a(u,v) = f(v) \ , \qquad v \in V \ .$$

An unbounded linear operator $A : D \to H$ with domain D in the Hilbert space H is *accretive* if

$$(Ax, x)_H \geq 0 \ , \qquad x \in D \ ,$$

and it is m*-accretive* if, in addition, $A + I$ maps D onto H. Sufficient conditions for the initial-value problem to be well-posed are provided by the Hille-Yoshida theorem.

Theorem 2.2.3. *Let the operator $A : D \to H$ be m-accretive on the Hilbert space H. Then for every $u^0 \in D(A)$ and $f \in C^1([0,\infty), H)$ there is a unique solution $u \in C^1([0,\infty), H)$ of the initial-value problem*

$$\frac{du}{dt}(t) + Au(t) = f(t) \ , \qquad t > 0 \ , \quad u(0) = u^0 \ . \tag{2.2.4}$$

If additionally A is self-adjoint, then for each $u^0 \in H$ and Hölder continuous $f \in C^\beta([0,\infty), H)$, $0 < \beta < 1$, there is a unique solution $u \in C([0,\infty), H) \cap C^1((0,\infty), H)$ of (2.2.4).

Finally, the standard finite-difference approximation of (2.2.4) leads to the *stationary problem* with $\lambda > 0$,

$$u \in D(A) : \quad \lambda u + A(u) = \lambda F \text{ in } H,$$

for the resolvent of the operator A. It is precisely the m-accretive operators for which this problem is always solvable with $\|u\|_H \le \|F\|_H$.

2.3 The Stationary Problem

With the family of domains $\Omega^\epsilon = \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$ given above for each value of the parameter with $0 < \epsilon \le 1$, the stationary problem corresponding to the initial-value problem (3.1.1) takes the weak form

$$\begin{aligned} u^\epsilon \in V^\epsilon : \quad & \int_{\Omega_1} \lambda m_1 \, u^\epsilon v \, dx + \int_{\Omega_1} k_1 \nabla u^\epsilon \cdot \nabla v \, dx \\ & + \int_{\Omega_2^\epsilon} \lambda m_2 \, u^\epsilon v \, dx + \int_{\Omega_2^\epsilon} \frac{k_2}{\epsilon} \nabla u^\epsilon \cdot \nabla v \, d\widetilde{x} \, dx_N \\ & = \int_{\Omega_1} \lambda m_1 \, F v \, dx + \int_{\Omega_2^\epsilon} \lambda m_2 \, F v \, dx + \int_\Gamma g \, \gamma(v) \, d\widetilde{x} \quad \forall \, v \in V^\epsilon, \end{aligned} \tag{2.3.5}$$

in the space $V^\epsilon \equiv \{v \in H^1(\Omega^\epsilon) : v = 0 \quad \text{on} \quad \partial\Omega_1 - \Gamma\}$. This is the *exact* or ϵ-problem to be solved, and it depends on the thin domain Ω_2^ϵ and the high permeability $\dfrac{k_2}{\epsilon}$ through the scale parameter $\epsilon > 0$. We expect the last term on the left side to be approximated for small values of ϵ by averaging across the narrow channel,

$$\frac{1}{\epsilon}\int_{\Omega_2^\epsilon} k_2 \nabla u \cdot \nabla v \, dx_N \, d\widetilde{x} \approx \int_\Gamma k_2 \widetilde{\nabla} u \cdot \widetilde{\nabla} v \, \omega(\widetilde{x}) \, d\widetilde{x}, \tag{2.3.6}$$

where $\widetilde{\nabla}$ denotes the gradient in the variable $\widetilde{x}$ in Γ, and this will be established in our work below.

2.3.1 The Scaled Problem

Since our primary interest is the dependence of the solution on ϵ, we shall reformulate the problem in a space that is independent of this parameter. In order to eliminate this dependence on the domain, we scale Ω_2^ϵ in the direction normal to Γ by $x_N = \epsilon z$ to get an equivalent problem on the domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ with $\Omega_2 \equiv \Omega_2^1 = \{(\widetilde{x}, \omega(\widetilde{x})z) \in \mathbb{R}^n : (\widetilde{x}, z) \in \Gamma \times (0,1)\}$. The corresponding bilinear form is

$$a^\epsilon(u,v) \equiv \int_{\Omega_1} k_1 \nabla u \cdot \nabla v \, dx + \int_{\Omega_2} k_2 \widetilde{\nabla} u \cdot \widetilde{\nabla} v \, d\widetilde{x} \, dz + \int_{\Omega_2} \frac{k_2}{\epsilon^2} \partial_z u \partial_z v \, d\widetilde{x} \, dz. \tag{2.3.7}$$

This form is continuous on $V \equiv \{v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \partial\Omega_1 - \Gamma\}$, and the *scaled problem* is

$$\begin{aligned} u^\epsilon \in V : \quad & \int_{\Omega_1} \lambda m_1 \, u^\epsilon v \, dx + \epsilon \int_{\Omega_2} \lambda m_2 \, u^\epsilon v \, d\widetilde{x} \, dz + a^\epsilon(u^\epsilon, v) \\ & = \int_{\Omega_1} \lambda m_1 \, F \, v \, dx + \epsilon \int_{\Omega_2} \lambda m_2 \, F \, v \, dx + \int_\Gamma g \, \gamma(v) \, d\widetilde{x} \quad \forall \, v \in V. \end{aligned} \tag{2.3.8}$$

For each $\epsilon > 0$ the bilinear form (2.3.7) is clearly V-elliptic, so the problem (2.3.8) is well-posed. Moreover, the solution u^ϵ satisfies

$$\begin{aligned} \lambda m_1 \, u_1^\epsilon - \nabla \cdot k_1 \nabla u_1^\epsilon &= \lambda m_1 \, F \text{ in } \Omega_1 \,, \\ u_1^\epsilon &= 0 \text{ on } \partial\Omega_1 - \Gamma \,, \\ u_1^\epsilon = u_2^\epsilon \,, \; k_1 \partial_z u_1^\epsilon - \frac{k_2}{\epsilon^2} \partial_z u_2^\epsilon &= g \text{ on } \Gamma \,, \\ \epsilon \, \lambda m_2 \, u_2^\epsilon - \widetilde{\nabla} \cdot k_2 \widetilde{\nabla} \, u_2^\epsilon - \frac{k_2}{\epsilon^2} \partial_z \partial_z u_2^\epsilon &= \epsilon \, \lambda m_2 F \text{ in } \Omega_2 \,, \\ (\widetilde{\nabla} \, u_2^\epsilon, \frac{1}{\epsilon^2} \partial_z u_2^\epsilon) \cdot \widehat{n} &= 0 \text{ on } \partial\Omega_2 - \Gamma \,. \end{aligned} \tag{2.3.9}$$

This is the stationary form of the interface problem (3.1.1) after the rescaling. Here we see the role of the effective tangential permeability k_2 and the effective normal permeability $\frac{k_2}{\epsilon^2}$.

The Estimates

Denote by χ_j the *characteristic function* of Ω_j, $j = 1, 2$, and set $u^\epsilon \equiv u_1^\epsilon \chi_1 + u_2^\epsilon \chi_2$. Due to the boundary conditions of the space V, the gradient controls the entire $H^1(\Omega)$ norm on V. Testing (2.3.8) with $v = u^\epsilon$, we obtain

$$\begin{aligned} C_1 \left(\| u_1^\epsilon \|_{0,\Omega_1}^2 + \| u_2^\epsilon \|_{0,\Omega_2}^2 \right) \le C_2 \Big(& \| u_1^\epsilon \|_{0,\Omega_1}^2 + \| \nabla u_1^\epsilon \|_{0,\Omega_1}^2 \\ & + \epsilon \| u_2^\epsilon \|_{0,\Omega_2}^2 + \| \widetilde{\nabla} u_2^\epsilon \|_{0,\Omega_2}^2 + \left\| \frac{1}{\epsilon} \partial_z u_2^\epsilon \right\|_{0,\Omega_2}^2 \Big) \\ & \le \|F\|_{0,\Omega_1} \|u_1^\epsilon\|_{0,\Omega_1} + \|g\|_{0,\Gamma} \|u_1^\epsilon\|_{1,\Omega_1} \le \widetilde{C} \|u_1^\epsilon\|_{1,\Omega_1} \end{aligned}$$

where $C_1, C_2, \widetilde{C}$ are positive constants. It follows that

$$\| u_1^\epsilon \|_{0,\Omega_1}^2 + \| \nabla u_1^\epsilon \|_{0,\Omega_1}^2 + \epsilon \| u_2^\epsilon \|_{0,\Omega_2}^2 + \| \widetilde{\nabla} u_2^\epsilon \|_{0,\Omega_2}^2 + \left\| \frac{1}{\epsilon} \partial_z u_2^\epsilon \right\|_{0,\Omega_2}^2 \le C \quad (2.3.10)$$

for some generic positive constant C.

The Limit

The estimate (2.3.10) implies that there is a subsequence, which we denote again by $\{u^\epsilon\}$, and a $u^* = u_1 \chi_1 + u_2 \chi_2 \in V$ such that $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. For any $v \in V$, as $\epsilon \to 0$ we have

$$\begin{aligned} \int_{\Omega_1} k_1 \nabla u_1^\epsilon \cdot \nabla v \, dx &\longrightarrow \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla v \, dx, \text{ and} \\ \int_{\Omega_2} k_2 \widetilde{\nabla} u_2^\epsilon \cdot \widetilde{\nabla} v \, d\widetilde{x} \, dz &\longrightarrow \int_{\Omega_2} k_2 \widetilde{\nabla} u_2 \cdot \widetilde{\nabla} v \, d\widetilde{x} \, dz. \end{aligned}$$

Since the right side of (2.3.8) is bounded for $v \in V$ fixed, we conclude the existence of the limit

$$\ell(v) \equiv \lim_{\epsilon \downarrow 0} \int_{\Omega_2} \frac{k_2}{\epsilon^2} \partial_z u_2^\epsilon \, \partial_z v \, d\widetilde{x} dz \,,$$

and due to the a-priori estimates we conclude $\ell \in V'$. In addition, there must exist $\zeta \in L^2(\Omega_2)$ such that $\epsilon^{-1} \partial_z u_2^\epsilon \overset{w}{\rightharpoonup} \zeta$ in $L^2(\Omega_2)$. Also $\| \partial_z u_2^\epsilon \|_{0,\Omega_2} \le \epsilon C$, so $\| \partial_z u_2^\epsilon \|_{0,\Omega_2} \to 0$, and we know $\partial_z u_2^\epsilon \overset{w}{\rightharpoonup} \partial_z u_2$ in $L^2(\Omega_2)$, so $\partial_z u_2 \equiv 0$ and u_2 is independent of z in Ω_2.

Taking the limit in (2.3.8), we find that $u^* = u_1\,\chi_1 + u_2\,\chi_2$ satisfies

$$u^* \in V:\ \partial_z u_2 = 0 \text{ in } \Omega_2,\ \text{and} \int_{\Omega_1} \lambda m_1\, u_1\, v\; dx + \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla v\; dx$$
$$+ \int_{\Omega_2} k_2 \widetilde{\nabla}\, u_2 \cdot \widetilde{\nabla}\, v\; dx + \ell(v) = \int_{\Omega_1} \lambda m_1 F v\; dx + \int_{\Gamma} g\,\gamma\,(v)\, d\widetilde{x} \quad \forall v \in V. \quad (2.3.11)$$

Define now the subspace $W \equiv \{v \in V:\ \partial_z v = 0 \quad \text{on } \Omega_2\}$. We have shown that for some subsequence we obtain a weak limit, $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in V with $u^* \in W$, and since the linear functional $\ell(\cdot)$ vanishes on W, this limit satisfies

$$u^* \in W: \quad \int_{\Omega_1} \lambda m_1\, u^* v\; dx + a^0(u^*, v)$$
$$= \int_{\Omega_1} \lambda m_1\, F\, v\, dx + \int_{\Gamma} g\,\gamma(v)\, d\widetilde{x} \text{ for all } v \in W. \quad (2.3.12)$$

where the limit bilinear form on W is defined by

$$a^0(u,v) \equiv \int_{\Omega_1} k_1 \nabla u \cdot \nabla v\, dx + \int_{\Omega_2} k_2 \widetilde{\nabla} u \cdot \widetilde{\nabla} v\, d\widetilde{x}\, dz. \quad (2.3.13)$$

This continuous bilinear form is W-elliptic, so we see that u^* is the only solution and the original sequence $\{u^\epsilon\}$ converges weakly to u^*. In summary, the problem (2.3.12) characterizes the limit u^* of the stationary problems (2.3.8).

2.3.2 Strong convergence

On the space V we take the scalar product

$$\langle v, w\rangle \equiv \int_{\Omega_1} k_1 \nabla v \cdot \nabla w\, dx + \int_{\Omega_2} k_2 \nabla v \cdot \nabla w\, dx. \quad (2.3.14)$$

This scalar product $\langle \cdot, \cdot \rangle$ is equivalent to the usual $H^1(\Omega)$ scalar product, that is, the V-norm $\|\, v \,\|_V \equiv \langle\, v, v\,\rangle^{1/2}$ is equivalent to the $H^1(\Omega)$ norm, so from the weak convergence $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in $H^1(\Omega)$ we know

$$\|\, u^* \,\|_V \leq \liminf_{\epsilon \downarrow 0} \|\, u^\epsilon \,\|_V\ .$$

Now, for $0 < \epsilon \leq 1$, the solution u^ϵ of (2.3.8) satisfies

$$\|\, u^\epsilon \,\|_V^2 \leq \epsilon \int_{\Omega_2} \lambda m_2\, (u^\epsilon)^2\, d\widetilde{x}\, dz + a^\epsilon(u^\epsilon, u^\epsilon) = - \int_{\Omega_1} \lambda m_1 (u^\epsilon)^2\, dx$$
$$+ \int_{\Omega_1} \lambda m_1\, F\, u^\epsilon\; dx + \int_{\Omega_2} \epsilon \lambda m_2\, F\, u^\epsilon\; dx + \int_{\Gamma} g\,\gamma\, u^\epsilon\; d\widetilde{x}\,,$$

so from weak lower-semicontinuity of the first term we obtain

$$\limsup_{\epsilon\downarrow 0} \| u^\epsilon \|_V^2 \le -\int_{\Omega_1} \lambda m_1 (u^*)^2 \, dx + \int_{\Omega_1} \lambda m_1 \, F u^* \, dx + \int_\Gamma g\,\gamma(u^*)\, d\widetilde{x}.$$

But with (2.3.12) this gives

$$\limsup_{\epsilon\downarrow 0} \| u^\epsilon \|_V^2 \le a^0(u^*, u^*) = \|u^*\|_V^2,$$

so $\lim_{\epsilon\downarrow 0} \| u^\epsilon \|_V = \| u^* \|_V$. Together with the weak convergence of the sequence, this implies $\| u^\epsilon - u^* \|_V \to 0$, and so we have strong convergence $u^\epsilon \to u^*$ in $H^1(\Omega)$.

An Alternative System

The solution of the limiting problem can be characterized by a boundary-value problem on Ω_1 and Γ. First we rewrite (2.3.11). Since $C_0^\infty(\Omega_1) \subseteq V$, for any $\varphi \in C_0^\infty(\Omega_1)$ we obtain

$$\int_{\Omega_1} \lambda m_1 \, u_1 \, \varphi \, dx + \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla \varphi \, dx = \int_{\Omega_1} \lambda m_1 \, F \varphi \, dx,$$

i.e., $\lambda m_1 \, u_1 - \nabla \cdot k_1 \nabla u_1 = \lambda m_1 \, F$ in $L^2(\Omega_1)$, so $k_1 \nabla u_1 \in \mathbf{L}^2_{div}(\Omega_1)$ and the normal trace $k_1 \nabla u_1 \cdot \widehat{n} \in H^{-1/2}(\partial\Omega_1)$ is well-defined. Moreover, we know that for any $v \in V$ the Stokes' formula [Tem79]

$$\langle k_1 \nabla u_1 \cdot \widehat{n}, \gamma(v) \rangle_{H^{-1/2}(\partial\Omega_1),\, H^{1/2}(\partial\Omega_1)} = \int_{\Omega_1} k_1 \nabla u_1 \cdot \nabla v \, dx + \int_{\Omega_1} \nabla \cdot (k_1 \nabla u_1)\, v \, dx$$

must hold. Substituting these into (2.3.11), we conclude

$$\langle k_1 \nabla u_1 \cdot \widehat{n}, \gamma(v) \rangle_{H^{-1/2}(\Gamma),\, H^{1/2}(\Gamma)} + \int_{\Omega_2} k_2 \widetilde{\nabla}\, u_2 \cdot \widetilde{\nabla}\, v \, dx + \ell(v) = \int_\Gamma g\,\gamma(v)\, d\widetilde{x} \text{ for all } v \in V. \quad (2.3.15)$$

Since the functions in W are independent of z for $(\widetilde{x}, z) \in \Omega_2$, we have for each pair $u,\, v \in W$

$$(u, v)_{H^1(\Omega_2)} = \int_{\Omega_2} \big(u(\widetilde{x})\, v(\widetilde{x}) + \nabla\, u(\widetilde{x}) \cdot \nabla\, v(\widetilde{x}) \big)\, d\widetilde{x}\, dz = \int_\Gamma \big(u(\widetilde{x})\, v(\widetilde{x}) + \widetilde{\nabla}\, u(\widetilde{x}) \cdot \widetilde{\nabla}\, v(\widetilde{x}) \big)\, \omega(\widetilde{x})\, d\widetilde{x}\,.$$

This is equivalent to the scalar product

$$(u,v)_{H^1_\omega(\Gamma)} \equiv \int_\Gamma \big(u(\widetilde{x})\, v(\widetilde{x}) + \omega(\widetilde{x})\, \widetilde{\nabla}\, u(\widetilde{x}) \cdot \widetilde{\nabla}\, v(\widetilde{x})\big)\, d\widetilde{x}$$

of the weighted Sobolev space

$$H^1_\omega(\Gamma) \equiv \Big\{ u \in L^2(\Gamma) :\ \omega^{1/2} \widetilde{\nabla} u \in \mathbf{L}^2(\Gamma) \Big\}\,.$$

Furthermore, we see W is equivalent to the space

$$V_\Gamma \equiv \big\{ v \in H^1(\Omega_1) :\ v|_\Gamma\ \in H^1_\omega(\Gamma), \quad v|_{\partial\Omega_1 - \Gamma} = 0 \big\}$$

in the sense of boundary trace. Thus, the solution of problem (2.3.12) is characterized by

$$\begin{aligned} u^* \in V_\Gamma :\quad & \int_{\Omega_1} \lambda m_1\, u^*\, v\ dx + \int_{\Omega_1} k_1 \nabla u^* \cdot \nabla v\, dx \\ & + \int_\Gamma k_2\, \omega\, \widetilde{\nabla}\, u^* \cdot \widetilde{\nabla}\, v\ d\widetilde{x} = \int_{\Omega_1} \lambda m_1\, F\, v\, dx + \int_\Gamma g\, \gamma(v)\, d\widetilde{x} \text{ for all } v \in V_\Gamma, \end{aligned} \tag{2.3.16}$$

and this means it determines a pair $u_1 = \chi_1 u^* \in H^1(\Omega_1),\ u_2 = \gamma(u^*) \in H^1_\omega(\Gamma)$ which satisfies the system

$$\lambda m_1\, u_1 - \nabla \cdot k_1 \nabla u_1 = \lambda m_1 F \text{ in } \Omega_1 \tag{2.3.17a}$$

$$u_1 = 0 \ \text{ on } \partial\,\Omega_1 - \Gamma \tag{2.3.17b}$$

$$u_1 = u_2 \quad \text{ on } \Gamma\,, \text{ and} \tag{2.3.17c}$$

$$\begin{aligned} & \int_\Gamma k_2\, \omega\, \widetilde{\nabla}\, u_2 \cdot \widetilde{\nabla}\, \gamma v\ d\widetilde{x} + \langle k_1 \nabla u_1 \cdot \widehat{n}, \gamma v\rangle_{H^{-1/2}(\Gamma),\, H^{1/2}(\Gamma)} \\ & \qquad = \int_\Gamma g\, \gamma\,(v)\ d\widetilde{x} \text{ for all } v \in V. \end{aligned} \tag{2.3.17d}$$

In the situation of Theorem 2.2.1, the variational identity (2.3.17d) is equivalent to

$$-\widetilde{\nabla} \cdot k_2\, \omega\, \widetilde{\nabla}\, u_2 + k_1 \partial_z u_1 = g \text{ in } \Gamma \tag{2.3.17e}$$

$$u_2 = 0 \text{ on } \partial\Gamma. \tag{2.3.17f}$$

However, in the strongly tapered case of (2.2.3), the last condition (2.3.17f) is deleted, since the trace is meaningless and the variational equation is equivalent to the equation (2.3.17e) in $H^1_\omega(\Gamma)'$. See [Sho79] for such examples. Thus, the limiting form of the singular problem (2.3.8) is the elliptic boundary-value problem on Ω_1 with the (non-local and possibly degenerate) elliptic boundary constraint.

We summarize the above as follows.

Theorem 2.3.1. *Let the regions* Ω^ϵ *and the rescaled* Ω, *the constants* $k_1,\ k_2,\ m_1,\ m_2 > 0, \lambda \geq 0$, *and functions* $F \in L^2(\Omega),\ g \in L^2(\Gamma)$ *be given. Define the bilinear form* (2.3.7) *for each* $0 < \epsilon \leq 1$ *on the space* V. *Then each scaled problem* (2.3.8) *has a unique solution,* u^ϵ, *these satisfy the estimates* (2.3.10) *and converge strongly* $u^\epsilon \to u^*$ *in* V, *where* u^* *satisfies* (2.3.11). *Finally, the limit* u^* *is characterized as the solution of the well-posed limit problem* (2.3.12) *or its equivalent form* (2.3.16).

Remarks on Minimization and Penalty

Set $f^\epsilon(v) = \int_{\Omega_1} \lambda m_1\, F\, v\, dx + \epsilon \int_{\Omega_2} \lambda m_2\, F\, v\, dx + \int_\Gamma g\, \gamma(v)\, d\widetilde{x}$. The equation (2.3.8) shows that u^ϵ is characterized by the minimization of

$$\varphi^\epsilon(v) \equiv \frac{1}{2}\left(\int_{\Omega_1} \lambda m_1\, v^2\, dx + \int_{\Omega_2} \epsilon\lambda m_2\, v^2\, dx + a^\epsilon(v,v)\right) - f^\epsilon(v), \quad v \in V.$$

According to (2.3.11), the limit u^* satisfies

$$u^* \in W : \int_{\Omega_1} \lambda m_1\, u^* v\, dx + \langle u^*, v\rangle_V + \ell(v) = f^0(v) \text{ for all } v \in V$$

and is characterized by (2.3.12), that is,

$$u^* \in W : \int_{\Omega_1} \lambda m_1\, u^* v\, dx + \langle u^*, v\rangle_V = f^0(v) \text{ for all } v \in W.$$

This shows that u^* is obtained by the minimization of

$$\varphi(v) \equiv \frac{1}{2}\left(\int_{\Omega_1} \lambda m_1\, v^2\, dx + \langle v, v\rangle_V\right) - f^0(v), \quad v \in V, \tag{2.3.18}$$

over the subspace W. This is the same as minimizing $\varphi(v) + I_W(v)$ over all of V, where

$$I_W(v) \equiv \begin{cases} 0 & \text{if } v \in W\,, \\ +\infty & \text{if } v \notin W\,, \end{cases}$$

is the *indicator function* of W.

Furthermore, if $\partial I_W(\cdot)$ denotes the subgradient of the convex $I_W(\cdot)$, then $\ell \in \partial I_W(u^*)$ is the Lagrange multiplier that realizes the constraint $u^* \in W$. The last term in (2.3.7) is the *penalty* term and (2.3.8) is a *penalty method* to approximate (2.3.12).

2.3.3 The Concentrated Capacity Model

Suppose that in the interface problem (3.1.1), we assume that not only the permeability k_2 but also m_2 is scaled by $\frac{1}{\epsilon}$ in Ω_2. Such an assumption is meaningless for porous media, since the porosity is bounded by 1, but it is appropriate in analogous heat conduction problems with a concentrated capacity along the highly-conducting interface or boundary. However, the problem (2.3.8) with the factor ϵ deleted from the two terms can be used as a fracture model with highly anisotropic permeability. We include this case to show what assumptions are required to arrive at the narrow fracture model described in [JRC71].

Theorem 2.3.2. *Let the region Ω, the constants $k_1,\ k_2,\ \lambda m_1 > 0$, and functions $F \in L^2(\Omega),\ g \in L^2(\Gamma)$ be given. For each $0 < \epsilon \leq 1$, consider the problem*

$$u^\epsilon \in V : \quad \int_{\Omega_1} \lambda m_1\, u^\epsilon\, v\, dx + \int_{\Omega_2} \lambda m_2\, u^\epsilon\, v\, dx + a^\epsilon(u^\epsilon, v) = \int_{\Omega_1} \lambda m_1\, F\, v\, dx + \int_{\Omega_2} \lambda m_2\, F\, v\, dx + \int_\Gamma g\, \gamma\,(v)\, d\widetilde{x} \quad \forall\, v \in V, \tag{2.3.19}$$

This problem has a unique solution, u^ϵ, these satisfy the estimates (2.3.10) *and converge strongly $u^\epsilon \to u^*$ in V, where the limit u^* satisfies*

$$u^* \in W :$$
$$\int_{\Omega_1} \lambda m_1\, u^*\, v\ dx + \int_\Gamma \lambda m_2\, \omega\ u^*\, v\ d\widetilde{x} + \int_{\Omega_1} k_1 \nabla u^* \cdot \nabla v\, dx + \int_\Gamma k_2\, \omega\, \widetilde{\nabla}\, u^* \cdot \widetilde{\nabla}\, v\ d\widetilde{x} = \int_{\Omega_1} \lambda m_1\, F\, v\, dx + \int_\Gamma \lambda m_2\, \omega\, \widetilde{F} v\, d\widetilde{x} + \int_\Gamma g\, \gamma(v)\, d\widetilde{x} \ \text{for all } v \in W, \tag{2.3.20}$$

and the channel average *of F in Ω_2 is given by*

$$\widetilde{F}(\widetilde{x}) = \frac{1}{\omega(\widetilde{x})} \int_0^{\omega(\widetilde{x})} F(\widetilde{x}, z)\, dz, \quad \widetilde{x} \in \Gamma.$$

Note as before that the limit $u^* \in V_\Gamma$ determines a pair $u_1 \in H^1(\Omega_1),\ u_2 \in H^1_\omega(\Gamma)$ which satisfies

$$\lambda m_1\, u_1 - \nabla \cdot k_1 \nabla u_1 = \lambda m_1\, F \text{ in } \Omega_1 \tag{2.3.21a}$$

$$u_1 = 0 \ \text{ on } \partial\,\Omega_1 - \Gamma \tag{2.3.21b}$$

$$u_1 = u_2 \quad \text{ on } \Gamma, \tag{2.3.21c}$$

$$\text{and } \int_\Gamma \lambda m_2\, \omega\, u_2\, v\, d\widetilde{x} + \int_\Gamma k_2\, \omega\, \widetilde{\nabla}\, u_2 \cdot \widetilde{\nabla}\, v\, d\widetilde{x}$$
$$+ \langle k_1 \nabla u_1 \cdot \widehat{n}, \gamma v\rangle_{H^{-1/2}(\Gamma),\, H^{1/2}(\Gamma)} = \int_\Gamma \lambda m_2\, \omega\, \widetilde{F} v\, d\widetilde{x} + \int_\Gamma g\, \gamma\, (v)\; d\widetilde{x}$$
$$\text{for all } v \in V_\Gamma.$$

In the weakly tapered situation of Theorem 2.2.1, the variational identity is equivalent to

$$\lambda\, m_2\, \omega\, u_2 - \widetilde{\nabla} \cdot k_2\, \omega\, \widetilde{\nabla}\, u_2 + k_1 \partial_z u_1 = \lambda m_2\, \omega\, \widetilde{F} + g \text{ in } \Gamma, \qquad (2.3.21\text{d})$$

$$u_2 = 0 \text{ on } \partial\Gamma, \qquad (2.3.21\text{e})$$

and in the strongly tapered case of (2.2.3), the last condition (2.3.21e) is deleted.

2.4 The Evolution Problems

We apply Theorem 2.3.1 to show the dynamics of the initial-boundary-value problem (3.1.1) is governed by an analytic semigroup on the Hilbert space $H = L^2(\Omega)$, and the limiting form corresponds similarly to an analytic semigroup on the Hilbert space $H_0 = L^2(\Omega_1)$. Then we establish the convergence as $\epsilon \to 0$ of solutions of the corresponding evolution problems.

2.4.1 Well-posed problems

Let H_ϵ denote H with the norm $\|u\|_{H_\epsilon} = \|m_1^{1/2}\chi_1 u + (\epsilon\, m_2)^{1/2}\chi_2\, u\|_{L^2(\Omega)}$, so its Riesz map is the multiplication function $m_\epsilon = m_1\chi_1 + \epsilon\, m_2\chi_2$ from H_ϵ to H'_ϵ. Similarly, $m_0 = m_1$ is the Riesz map from H_0 to H'_0, where $\|u\|_{H_0} = \|m_1^{1/2} u\|_{L^2(\Omega_1)}$. Note that $V \subset H_\epsilon$ and $W \subset H_0$ are dense and continuous inclusions.

Define the operators $A^\epsilon : D^\epsilon \to H'_\epsilon$ with domains $D^\epsilon \subset V$ by $u^\epsilon \in D^\epsilon$ and $A^\epsilon(u^\epsilon) = F \in H'_\epsilon$ if $u^\epsilon \in V :\; a^\epsilon(u^\epsilon, v) = F(v)$ for all $v \in V$. Similarly the operator $A^0 : D^0 \to H'_0$ with domain $D^0 \subset W$ is determined by $u^0 \in D^0$ and $A^0(u^0) = F \in H'_0$ if $u^0 \in W :$ $a^0(u^0, w) = F(w)$ for all $w \in W$. If we set $g = 0$, then the scaled problem (2.3.8) is equivalent to $A^\epsilon(u^\epsilon) = \lambda m_\epsilon(F - u^\epsilon)$ for $F \in H$, and the limit problem (2.3.12) is equivalent to $A^0(u^*) = \lambda m_0(F - u^*)$ when $F \in H_0$.

Each of the operators $m_\epsilon^{-1}A^\epsilon$ is m-accretive on H_ϵ, that is, $\|(I+\alpha m_\epsilon^{-1}A^\epsilon)^{-1}F\|_{H_\epsilon} \leq \|F\|_{H_\epsilon}$ for each $\alpha > 0$ and $F \in H_\epsilon$. Likewise $(I + \alpha m_0^{-1}A^0)^{-1}$ is a contraction on H_0 for each $\alpha > 0$. These operators are also self-adjoint, since the corresponding bilinear forms are symmetric, so $m_\epsilon^{-1}A^\epsilon$ and $m_0^{-1}A^0$ generate analytic semigroups on H_ϵ and H_0, respectively.

The Hille-Yoshida Theorem 2.2.3 shows that the corresponding initial-value problems are well-posed. Applying it to the operator $m_\epsilon^{-1}A^\epsilon$ in H_ϵ, we obtain the scaled problem.

Theorem 2.4.1. *For every $u_0 \in L^2(\Omega)$ and $F \in C^\beta([0,\infty), L^2(\Omega))$, there is a unique $u^\epsilon \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ with $u^\epsilon(t) \in D^\epsilon$ for each $t > 0$ such that $u^\epsilon(t) = \chi_1\, u_1^\epsilon(t) + \chi_2\, u_2^\epsilon(t)$ satisfies the scaled problem*

$$\begin{aligned} m_1 \frac{\partial u_1^\epsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\epsilon &= m_1 F \ \text{ in } \Omega_1 \\ u_1^\epsilon &= 0 \ \text{ on } \partial\Omega_1 - \Gamma \\ u_1^\epsilon = u_2^\epsilon,\ k_1 \partial_z u_1^\epsilon - \frac{k_2}{\epsilon^2}\, \partial_z u_2^\epsilon &= 0 \ \text{ on } \Gamma \\ \epsilon\, m_2 \frac{\partial u_2^\epsilon}{\partial t} - \widetilde{\nabla} \cdot k_2 \widetilde{\nabla} u_2^\epsilon - \frac{k_2}{\epsilon^2}\, \partial_z \partial_z u_2^\epsilon &= \epsilon\, m_2 F \ \text{ in } \Omega_2 \\ \left(k_2 \widetilde{\nabla} u_2^\epsilon, \frac{k_2}{\epsilon^2} \partial_z u_2^\epsilon \right) \cdot \widehat{n} &= 0 \ \text{ on } \partial\Omega_2 - \Gamma\,, \end{aligned} \tag{2.4.22a}$$

at each $t > 0$, and these satisfy the initial conditions

$$u_1^\epsilon(\cdot, 0) = u_0(\cdot) \ \text{ on } \Omega_1, \quad u_2^\epsilon(\cdot, 0) = u_0(\cdot) \ \text{ on } \Omega_2\,. \tag{2.4.22b}$$

Note that this is a rather strong solution, since $\nabla \cdot k_j \nabla u_1^\epsilon(t) \in L^2(\Omega_j)$ for each $t > 0,\ j = 1, 2$.

Similarly from the operator $m_0^{-1}A^0$ in H_0 we obtain the limiting problem. When the fracture is weakly tapered, this takes the following form.

Theorem 2.4.2. *For every $u_0 \in L^2(\Omega_1)$ and $F \in C^\beta([0,\infty), L^2(\Omega_1))$, there is a unique $u^* \in C([0,\infty), L^2(\Omega_1)) \cap C^1((0,\infty), L^2(\Omega_1))$ with $u^*(t) \in D^0$ for each $t > 0$, such that the* *functions* $u_1(t) = u^*(t)|_{\Omega_1} \in H^1(\Omega_1)$, $u_2(t) = \gamma(u^*(t)) \in H^1_\omega(\Gamma)$ *satisfy*

$$m_1 \frac{\partial u_1}{\partial t} - \nabla \cdot k_1 \nabla u_1 = m_1 F \ \ in\ \Omega_1 \tag{2.4.23a}$$

$$u_1 = 0 \ \ on\ \partial\Omega_1 - \Gamma \tag{2.4.23b}$$

$$u_1 = u_2 \quad\ on\ \Gamma,\ \ and \tag{2.4.23c}$$

$$-\widetilde{\nabla} \cdot k_2\, \omega\, \widetilde{\nabla}\, u_2 + k_1\, \partial_z u_1 = 0 \ \ in\ \Gamma, \tag{2.4.23d}$$

$$u_2 = 0 \ \ on\ \partial\Gamma, \tag{2.4.23e}$$

at each $t > 0$ *and the initial condition*

$$u_1(\cdot, 0) = u_0(\cdot) \ \ on\ \Omega_1. \tag{2.4.23f}$$

In particular, each term of the equation (2.4.23a) belongs to $L^2(\Omega_1)$, so the solution is rather strong. As before, in the strongly tapered case, the last condition (2.4.23e) is deleted.

2.4.2 Strong Convergence

For the stationary problems, we have shown that $(m_\epsilon + A^\epsilon)^{-1} m_\epsilon F \to (m_0 + A^0)^{-1} m_0 F$ in the V-norm, hence, in $H^1(\Omega)$ so also in H. However, for the corresponding dynamic problems, with $\epsilon > 0$ we have an evolution in $H_\epsilon = L^2(\Omega)$ whereas the limit is an evolution in $H_0 = L^2(\Omega_1)$, and these are not immediately comparable, so we shall work directly in the corresponding evolution spaces, $\mathcal{V} \equiv L^2(0, T; V)$ and $\mathcal{W} \equiv L^2(0, T; W)$. The Cauchy problem leads to the Hilbert space

$$W^{1.2}(0, T) \equiv \left\{ u \in \mathcal{V} : \frac{du}{dt} \in \mathcal{V}' \right\}$$

with the norm $\|u\|_{W^{1.2}(0,T)} = \left(\|u\|^2_{\mathcal{V}} + \|\frac{du}{dt}\|^2_{\mathcal{V}'}\right)^{1/2}$, and this space is contained in $C([0, T], H)$ with continuous imbedding, that is,

$$\|u\|_{C([0,T],\,H)} \le C\, \|u\|_{\,W^{1.2}(0,T)}, \quad u \in W^{1.2}(0, T).$$

See any one of [Ada75, Sho97a, Tem79].

The solution of (2.4.22) satisfies

$$u^\epsilon \in \mathcal{V}: \ \forall v \in \mathcal{V} \cap W^{1.2}(0,T;H) \text{ with } v(T)=0,$$
$$-\int_0^T \left(m_\epsilon\, u^\epsilon(t), \frac{dv}{dt}(t)\right)_{L^2(\Omega)} dt + \int_0^T a^\epsilon(u^\epsilon(t), v(t))$$
$$= \int_0^T (m_\epsilon F(t), v(t))_{L^2(\Omega)}\, dt + (m_\epsilon u_0, v(0))_{L^2(\Omega)}.$$

This is the weak formulation of the Cauchy problem

$$u^\epsilon \in \mathcal{V} : m_\epsilon \frac{du^\epsilon}{dt}(\cdot) + A^\epsilon(u^\epsilon(\cdot)) = m_\epsilon\, F(\cdot) \text{ in } \mathcal{V}',\ u^\epsilon(0) = u_0,$$

and the solution u^ϵ satisfies the identity

$$\frac{1}{2}(m_\epsilon u^\epsilon(T), u^\epsilon(T))_{L^2(\Omega)} + \int_0^T a^\epsilon(u^\epsilon(t), u^\epsilon(t))\, dt$$
$$= \int_0^T (m_\epsilon F(t), u^\epsilon(t))_{L^2(\Omega)}\, dt + \frac{1}{2}(m_\epsilon u_0, u_0)_{L^2(\Omega)}. \quad (2.4.24)$$

This implies that $\|u^\epsilon\|_{\mathcal{V}},\ \|\frac{1}{\epsilon}\partial_z u^\epsilon\|_{L^2(0,T;H_0)}$ are bounded, so there is a weakly convergent subsequence, $u^\epsilon \overset{w}{\rightharpoonup} u^*$ in $\mathcal{V}$ with limit $u^* \in \mathcal{W}$. Then the evolution equation shows that $\frac{du^\epsilon}{dt} \overset{w}{\rightharpoonup} \frac{du^*}{dt}$ in $\mathcal{W}'$, so we obtain

$$u^* \in \mathcal{W}: \ \forall v \in \mathcal{W} \cap W^{1.2}(0,T;H_0) \text{ with } v(T)=0,$$
$$-\int_0^T (m_0 u^*(t), \frac{dv}{dt}(t))_{L^2(\Omega_1)}\, dt + \int_0^T a^0(u^*(t), v(t))$$
$$= \int_0^T (m_0 F(t), v(t))_{L^2(\Omega_1)}\, dt + (m_0 u_0, v(0))_{L^2(\Omega_1)}.$$

As before, this characterizes the solution of

$$u^* \in \mathcal{W} : m_0 \frac{du^*}{dt}(\cdot) + A^0(u^*(\cdot)) = m_0 F(\cdot) \text{ in } \mathcal{W}',\ u^*(0) = \chi_1\, u_0\,,$$

which has *only* one solution [Sho74], so the original sequence converges weakly to u^* and this is also the solution of (2.4.23). Moreover, we have

$$\frac{1}{2}\,(m_0\, u^*(T), u^*(T))_{L^2(\Omega_1)} + \int_0^T a^0(u^*(t), u^*(t))\, dt$$
$$= \int_0^T (m_0\, F(t), u^*(t))_{L^2(\Omega_1)}\, dt + \frac{1}{2}(m_0\, u_0, u_0)_{L^2(\Omega_1)}, \quad (2.4.25)$$

and this will be used to show strong convergence $u^\epsilon \to u^*$ in $\mathcal{V}$. From the weak convergence, we have

$$\int_0^T \langle u^*(t), u^*(t)\rangle\, dt \le \liminf_{\epsilon\downarrow 0} \int_0^T \langle u^\epsilon, u^\epsilon\rangle\, dt.$$

This follows since the V-norm from the scalar product (2.3.14) is equivlent to the $H^1(\Omega)$-norm. Also from (2.4.24) we have

$$\int_0^T \langle u^\epsilon, u^\epsilon\rangle\; dt \le \int_0^T a^\epsilon(u^\epsilon, u^\epsilon)\, dt = -\frac{1}{2}\; (m_\epsilon\, u^\epsilon(T), u^\epsilon(T))_{L^2(\Omega)} + \int_0^T (m_\epsilon F(t), u^\epsilon(t))_{L^2(\Omega)}\, dt + \frac{1}{2}\; (m_\epsilon\, u_0, u_0)_{L^2(\Omega)}\,.$$

Then using the (weak) continuity of the linear map $u \to u(T)$ from $\left\{u \in \mathcal{W} : m_0^{1/2} \frac{du}{dt} \in \mathcal{W}'\right\}$ to H_0, we take the lim sup above to get

$$\limsup_{\epsilon\downarrow 0} \int_0^T \langle u^\epsilon, u^\epsilon\rangle\, dt \le -\frac{1}{2}\; (m_0\, u^*(T), u^*(T))_{L^2(\Omega_1)} + \int_0^T (m_0\, F(t), u^*(t))_{L^2(\Omega_1)}\; dt + \frac{1}{2}\; (m_0\, u_0, u_0)_{L^2(\Omega_1)}\,.$$

Together with the limiting identity (2.4.25) this shows

$$\limsup_{\epsilon\downarrow 0} \int_0^T \langle u^\epsilon, u^\epsilon\rangle\, dt \le \int_0^T a^0(u^*(t), u^*(t))\, dt = \int_0^T \langle u^*(t), u^*(t)\rangle\, dt,$$

so we have established $\lim_{\epsilon\downarrow 0} \int_0^T \langle u^\epsilon, u^\epsilon\rangle\, dt = \int_0^T \langle u^*(t), u^*(t)\rangle\, dt$ and, hence, strong convergence in $\mathcal{V}$. Recalling that from the evolution equation we have the strong convergence $m_\epsilon \frac{du^\epsilon}{dt} \to m_0 \frac{du^*}{dt}$ in $\mathcal{W}'$, we have

Theorem 2.4.3. *In the situation of Theorem 2.4.1 and Theorem 2.4.2, the sequence converges strongly $u^\epsilon \to u^*$ in $\mathcal{V} = L^2(0, T; V)$ and in $C([0, T], H_0)$.*

2.4.3 The Concentrated Capacity Model

We obtain the analogous results for the evolution problem corresponding to Theorem 2.3.2. The approximation evolves in $H = L^2(\Omega)$ with the norm $\|u\|_H = \|(m_1^{1/2}\chi_1 + m_2^{1/2}\chi_2)\, u\|_{L^2(\Omega)}$; its Riesz map is the multiplication function $m_1\chi_1 + m_2\chi_2$ from H to H'.

Similarly, H_0 is defined to be the closure of W in H, and as above we find it is equivalent to the weighted L^2 space with the scalar product

$$(u,v)_{L^2_\omega(\Omega)} = \int_{\Omega_1} m_1\, u(x)v(x)\, dx + \int_{\Gamma} m_2\, u(\widetilde{x})\, v(\widetilde{x})\, \omega(\widetilde{x})\, d\widetilde{x}\,.$$

Note that $V \subset H$ and $W \subset H_0$ are dense and continuous inclusions.

By the same arguments given previously, we obtain the following.

Theorem 2.4.4. *For every $u_0 \in L^2(\Omega)$ and $F \in C^\beta([0,\infty), L^2(\Omega))$, there is a unique $u^\epsilon \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ with $u^\epsilon(t) \in D^\epsilon$ for each $t > 0$ such that $u^\epsilon(t) = \chi_1\, u_1^\epsilon(t) + \chi_2\, u_2^\epsilon(t)$ satisfies the scaled problem*

$$\begin{aligned} m_1 \frac{\partial u_1^\epsilon}{\partial t} - \nabla \cdot k_1 \nabla u_1^\epsilon &= m_1\, F \ \text{ in } \Omega_1 \\ u_1^\epsilon &= 0 \ \text{ on } \partial\Omega_1 - \Gamma \\ u_1^\epsilon = u_2^\epsilon,\ k_1\, \partial_z u_1^\epsilon - \frac{k_2}{\epsilon^2}\, \partial_z u_2^\epsilon &= 0 \ \text{ on } \Gamma \\ m_2 \frac{\partial u_2^\epsilon}{\partial t} - \widetilde{\nabla} \cdot k_2 \widetilde{\nabla} u_2^\epsilon - \frac{k_2}{\epsilon^2}\, \partial_z \partial_z u_2^\epsilon &= m_2 F \ \text{ in } \Omega_2 \\ \left(k_2 \widetilde{\nabla} u_2^\epsilon, \frac{k_2}{\epsilon^2} \partial_z u_2^\epsilon \right) \cdot \widehat{n} &= 0 \ \text{ on } \partial\Omega_2 - \Gamma\,, \end{aligned} \tag{2.4.26a}$$

at each $t > 0$, and these satisfy the initial conditions

$$u_1^\epsilon(\cdot, 0) = u_0(\cdot) \ \text{ on } \Omega_1, \quad u_2^\epsilon(\cdot, 0) = u_0(\cdot) \ \text{ on } \Omega_2\,. \tag{2.4.26b}$$

Also, there is a unique $u^* \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ with $u^*(t) \in D^0$ for each $t > 0$, such that the functions $u_1(t) = u^*(t)|_{\Omega_1} \in H^1(\Omega_1)$, $u_2(t) = \gamma(u^*(t)) \in H^1_\omega(\Gamma)$ satisfy

$$m_1 \frac{\partial u_1}{\partial t} - \nabla \cdot k_1 \nabla u_1 = m_1 F \ \text{ in } \Omega_1 \tag{2.4.27a}$$

$$u_1 = 0 \ \ \text{ on } \partial\,\Omega_1 - \Gamma \tag{2.4.27b}$$

$$u_1 = u_2 \quad \text{ on } \Gamma\,, \text{ and} \tag{2.4.27c}$$

$$m_2\, \omega\, \frac{\partial u_2}{\partial t} - \widetilde{\nabla} \cdot k_2\, \omega\, \widetilde{\nabla}\, u_2 + k_1\, \partial_z u_1 = m_2\, \omega\, \widetilde{F} \ \text{ in } \Gamma, \tag{2.4.27d}$$

$$u_2 = 0 \ \text{ on } \partial\Gamma, \tag{2.4.27e}$$

at each $t > 0$ *and the initial condition*

$$u_1(\cdot, 0) = u_0(\cdot) \text{ on } \Omega_1,\ u_2(\cdot, 0) = \widetilde{u}_0(\cdot) \text{ on} \tag{2.4.27f}$$

Finally, we have strong convergence $u^\epsilon \to u^*$ *in* $\mathcal{V} = L^2(0, T; V)$ *and in* $C([0, T], H_0)$.

3 LIMIT MIXED FORMULATION OF DARCY-DARCY MODEL FOR CHANNELED FLOW

3.1 Introduction

The constitutive law of Darcy is

$$a(x)\,\mathbf{u}(x,t) + \boldsymbol{\nabla} p(x,t) + \mathbf{g}(x) = \mathbf{0}\,, \tag{3.1.1a}$$

where $\mathbf{u}(x,t)$ represents the fluid flux, $p(x,t)$ the pressure, and $\mathbf{g}(x)$ is the gravity force. The flow resistance $a(x)$ is fluid viscosity times the inverse of permeability of the porous medium. The conservation law is

$$c(x)\,\frac{\partial\, p\,(x,t)}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u}(x,t) = f(x,t)\,, \tag{3.1.1b}$$

in which $c(x)$ is the (slight) compressibility and porosity of the fluid and porous medium with sources $f(x,t)$. The density factor has been dropped from each term of (3.1.1b). The system (3.1.1) is supplemented with appropriate boundary and initial conditions. The backward-difference approximation for $\frac{\partial}{\partial t}\,p$ leads to a corresponding boundary-value problem for the *stationary system*

$$\begin{aligned} a(x)\,\mathbf{u}(x) + \boldsymbol{\nabla} p(x) + \mathbf{g}(x) &= \mathbf{0}\,, \\ c(x)\,\lambda\, p(x) + \boldsymbol{\nabla}\cdot\mathbf{u}(x) &= f(x)\,, \end{aligned} \tag{3.1.2}$$

where $\lambda = h^{-1}$ is the reciprocal of the time increment $h > 0$. We study an interface problem for which the resistance coefficient $a(x)$ is of order $\epsilon > 0$ on a thin fracture with width of order ϵ and show that this is approximated by a coupled problem with tangential flow on the interface.

3.1.1 The Interface Problem

Consider a domain $\Omega^{\epsilon} = \Omega_1 \cup \Gamma \cup \Omega_2^{\epsilon}$ in $\mathrm{I\!R}^N$ representing a porous medium as the union of disjoint subdomains Ω_1, Ω_2^{ϵ} separated by a smooth $\mathrm{I\!R}^{N-1}$ manifold $\Gamma = \partial\Omega_1 \cap \partial\Omega_2^{\epsilon}$. For any function on Ω^{ϵ} we denote its restrictions to Ω_1 and to Ω_2^{ϵ} by superscripts 1 and 2, respectively. Vectors are denoted by boldface letters, as are vector-valued functions and corresponding function spaces. We use $\tilde{\mathbf{x}}$ to denote a vector in $\mathbb{R}^{N-1}$. If $\mathbf{x} \in \mathbb{R}^N$, then the $\mathbb{R}^{N-1} \times \{0\}$ projection is identified with $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_{N-1})$ so that $\mathbf{x} = (\tilde{\mathbf{x}}, x_N)$. The $\mathbb{R}^{N-1}$ gradient $\widetilde{\boldsymbol{\nabla}}$ and divergence $\widetilde{\boldsymbol{\nabla}}\cdot$ are denoted similarly.

The geometry of Ω^{ϵ} is prescribed by a domain $G \subset \mathbb{R}^{N-1}$ and a continuously differentiable map $\zeta : G \to \mathbb{R}$, *i.e.*, the interface Γ is the graph of ζ, $\Gamma \equiv \{ (\tilde{x}, \zeta(\tilde{x})) : \tilde{x} \in G \}$. Denote the thin fracture domain with width $\epsilon > 0$ by

$$\Omega_2^{\epsilon} \equiv \{ (\tilde{x}, x_N) : \zeta(\tilde{x}) < x_N < \zeta(\tilde{x}) + \epsilon,\ \tilde{x} \in G \}. \tag{3.1.3}$$

It is bounded below by Γ and above by its vertical ϵ-translate, $\Gamma + \epsilon$. Let $\Omega_1 \subset \mathbb{R}^N$ be a domain for which $\Omega_1 \cap \Omega_2^{\epsilon} = \varnothing$ and $\partial\Omega_1 \cap \partial\Omega_2^{\epsilon} = \Gamma$, and set $\Omega^{\epsilon} = \Omega_1 \cup \Gamma \cup \Omega_2^{\epsilon}$.

The *stationary interface problem* is

$$\begin{aligned} &a_1(x)\,\mathbf{u}^{\epsilon,1} + \boldsymbol{\nabla} p^{\epsilon,1} + \mathbf{g}^{\epsilon}(x) = 0 \text{ and} \\ &\qquad c_1(x)\,\lambda p^{\epsilon,1} + \boldsymbol{\nabla}\cdot\mathbf{u}^{\epsilon,1} = f^{\epsilon} \text{ in } \Omega_1, \\ &\qquad\qquad p^{\epsilon,1} = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \end{aligned} \tag{3.1.4a}$$

$$\begin{aligned} p^{\epsilon,1} - p^{\epsilon,2} &= \alpha \mathbf{u}^{\epsilon,1}\cdot\mathbf{n} \text{ and} \\ -\mathbf{u}^{\epsilon,1}\cdot\mathbf{n} + \mathbf{u}^{\epsilon,2}\cdot\mathbf{n} &= f_{\Gamma}^{\epsilon} \quad \text{on } \Gamma, \end{aligned} \tag{3.1.4b}$$

$$\begin{aligned} &\epsilon\, a_2(x)\mathbf{u}^{\epsilon,2} + \boldsymbol{\nabla} p^{\epsilon,2} + \mathbf{g}^{\epsilon}(x) = 0 \text{ and} \\ &\qquad c_2(x)\,\lambda p^{\epsilon,2} + \boldsymbol{\nabla}\cdot\mathbf{u}^{\epsilon,2} = f^{\epsilon} \text{ in } \Omega_2^{\epsilon}, \\ &\qquad\qquad \mathbf{u}^{\epsilon,2}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_2^{\epsilon} - \Gamma, \end{aligned} \tag{3.1.4c}$$

for the fluid pressure $p^{\epsilon,1}$, $p^{\epsilon,2}$ and velocity $\mathbf{u}^{\epsilon,1}$, $\mathbf{u}^{\epsilon,2}$ on the respective domains Ω_1, Ω_2^{ϵ}. The coefficients are $a_1,\ c_1$ on Ω_1 and $\epsilon\, a_2,\ c_2$ on Ω_2^{ϵ}. The interface conditions on Γ are that fluid flux from Ω_1 is driven by the pressure difference with resistance $\alpha \geq 0$ and that fluid is conserved.

For our weak formulation of the stationary system we use the spaces

$$\mathbf{V}^{\epsilon} \equiv \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega^{\epsilon}) : \ \boldsymbol{\nabla} \cdot \mathbf{v}^1 \in L^2(\Omega_1), \ \alpha \ \mathbf{v}^1 \cdot \mathbf{n}\big|_{\Gamma} \in L^2(\Gamma) \right\},$$

$$Q^{\epsilon} \equiv \left\{ q \in L^2(\Omega^{\epsilon}) : \ \boldsymbol{\nabla} q^2 \in \mathbf{L}^2(\Omega_2^{\epsilon}) \right\}$$

with the norms

$$\|\mathbf{v}\|_{\mathbf{V}^{\epsilon}} = \left(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega^{\epsilon})} + \|\boldsymbol{\nabla} \cdot \mathbf{v}^1\|^2_{L^2(\Omega_1)} + \|\alpha\, \mathbf{v}^1 \cdot \mathbf{n}\|^2_{L^2(\Gamma)} \right)^{1/2},$$

$$\|q\|_{Q^{\epsilon}} = \left(\|q\|^2_{L^2(\Omega^{\epsilon})} + \|\boldsymbol{\nabla} q^2\|^2_{\mathbf{L}^2(\Omega_2^{\epsilon})} \right)^{1/2}.$$

Our weak formulation of the interface problem (3.1.4) is

$$\begin{aligned}\mathbf{u}^{\epsilon} \in \mathbf{V}^{\epsilon},\ p^{\epsilon} \in Q^{\epsilon} :\ & \int_{\Omega_1} a_1 \mathbf{u}^{\epsilon} \cdot \mathbf{v}\, dx - \int_{\Omega_1} p^{\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{v}\, dx \\ & + \epsilon \int_{\Omega_2^{\epsilon}} a_2 \mathbf{u}^{\epsilon} \cdot \mathbf{v}\, dx + \int_{\Omega_2^{\epsilon}} \boldsymbol{\nabla} p^{\epsilon} \cdot \mathbf{v}\, dx + \int_{\Gamma} p^{\epsilon,2}\, \mathbf{v}^1 \cdot \mathbf{n}\, dS \\ & + \int_{\Gamma} \alpha \left(\mathbf{u}^{\epsilon,1} \cdot \mathbf{n} \right) \left(\mathbf{v}^1 \cdot \mathbf{n} \right)\, dS = - \int_{\Omega^{\epsilon}} \mathbf{g}^{\epsilon} \cdot \mathbf{v}\, dx\,, \quad (3.1.5a)\end{aligned}$$

$$\begin{aligned}& \int_{\Omega_1} \lambda\, c_1\, p^{\epsilon}\, q\, dx + \int_{\Omega_2^{\epsilon}} \lambda\, c_2\, p^{\epsilon}\, q\, dx + \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon} q\, dx \\ & - \int_{\Omega_2^{\epsilon}} \mathbf{u}^{\epsilon} \cdot \boldsymbol{\nabla} q\, dx - \int_{\Gamma} \mathbf{u}^{\epsilon,1} \cdot \mathbf{n}\, q^2\, dS \\ & = \int_{\Omega^{\epsilon}} f^{\epsilon}\, q\, dx + \int_{\Gamma} f_{\Gamma}^{\epsilon}\, q^2\, dS \text{ for all } \mathbf{v} \in \mathbf{V}^{\epsilon},\ q \in Q^{\epsilon}. \quad (3.1.5b)\end{aligned}$$

Remark 3.1.1. *We have coupled the $H_{div} - L^2$ formulation on Ω_1 and the $L^2 - H^1$ formulation on Ω_2^{ϵ}. Each $q \in Q^{\epsilon}$ has a well-defined* trace $q^2|_{\Gamma} \in H^{1/2}(\Gamma)$ *and similarly each $\mathbf{v} \in \mathbf{V}^{\epsilon}$ determines a* normal trace $\mathbf{v}^1 \cdot \mathbf{n} \in H^{-1/2}(\Gamma)$. *If $\alpha > 0$, then for each such $\mathbf{v}$ it is additionally required that $\mathbf{v}^1 \cdot \mathbf{n} \in L^2(\Gamma)$.*

Define operators $\mathcal{A}^{\epsilon} : \mathbf{V}^{\epsilon} \to \mathbf{V}^{\epsilon\prime}$, $\mathcal{B}^{\epsilon} : \mathbf{V}^{\epsilon} \to Q^{\epsilon\prime}$, $C^{\epsilon} : Q^{\epsilon} \to Q^{\epsilon\prime}$ by

$$\mathcal{A}^{\epsilon}\mathbf{u}\,(\mathbf{v}) = \int_{\Omega_1} a_1\, \mathbf{u} \cdot \mathbf{v}\, dx\ + \epsilon \int_{\Omega_2^{\epsilon}} a_2\, \mathbf{u} \cdot \mathbf{v}\, dx + \int_{\Gamma} \alpha \left(\mathbf{u}^{\epsilon,1} \cdot \mathbf{n} \right) \left(\mathbf{v}^1 \cdot \mathbf{n} \right)\, dS\,, \qquad (3.1.6a)$$

$$\mathcal{B}^{\epsilon}\mathbf{u}\,(q) = - \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}\, q\, dx + \int_{\Gamma} \mathbf{u}^1 \cdot \mathbf{n}\, q^2\, dS + \int_{\Omega_2^{\epsilon}} \mathbf{u} \cdot \boldsymbol{\nabla}\, q\, dx\,, \qquad (3.1.6b)$$

$$\mathcal{C}^{\epsilon} p\,(q) = \int_{\Omega_1} c_1\, p\, q\, dx + \int_{\Omega_2^{\epsilon}} c_2\, p\, q\, dx. \tag{3.1.6c}$$

Then the system (3.1.5) has the *mixed formulation*

$$\mathbf{u}^{\epsilon} \in \mathbf{V}^{\epsilon},\ p^{\epsilon} \in Q^{\epsilon} :$$

$$\mathcal{A}^{\epsilon}\mathbf{u}^{\epsilon}(\mathbf{v}) + {\mathcal{B}^{\epsilon}}'p^{\epsilon}\,(\mathbf{v}) = -\mathbf{g}^{\epsilon}(\mathbf{v}),\ \mathbf{v} \in \mathbf{V}^{\epsilon},$$

$$-\mathcal{B}^{\epsilon}\mathbf{u}^{\epsilon}(q) + \lambda\, \mathcal{C}^{\epsilon}p^{\epsilon}\,(q) = f^{\epsilon}(q),\ q \in Q^{\epsilon}.$$

For the analysis of such problems, see [GR79a, BF91, Sho10].

Theorem 3.1.1. *Assume that* $\mathbf{V}$ *and* Q *are Hilbert spaces and* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *are continuous linear operators* $\mathcal{A} : \mathbf{V} \to \mathbf{V}'$, $\mathcal{B} : \mathbf{V} \to Q'$, $C : Q \to Q'$ *such that*

- $\mathcal{A}$ *is non-negative and* $\mathbf{V}$*-coercive on* $\operatorname{Ker}\mathcal{B}$,
- $\mathcal{C}$ *is non-negative, symmetric, and*
- $\mathcal{B}'$ *is bounding,* i.e., *it is* 1-1 *and*

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{|\mathcal{B}\mathbf{v}(q)|}{\|\mathbf{v}\|_{\mathbf{V}} \|q\|_Q} \geq c_0 > 0. \tag{3.1.7}$$

Then for every $f \in Q'$, $\mathbf{g} \in \mathbf{V}'$ *and* $\lambda \geq 0$ *the system*

$$\mathbf{u} \in \mathbf{V},\ p \in Q :$$

$$\mathcal{A}\mathbf{u} + \mathcal{B}'p = -\mathbf{g} \ \text{ in } \mathbf{V}', \tag{3.1.8}$$

$$-\mathcal{B}\mathbf{u} + \lambda\, \mathcal{C}p = f \ \text{ in } Q',$$

has a unique solution, and it satisfies the estimate

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p\|_Q \leq K \left(\|\mathbf{g}\|_{\mathbf{V}'} + \|f\|_{Q'} \right). \tag{3.1.9}$$

We have the following classical result. (See Proposition 5.2 of [Sho97a].)

Lemma 3.1.2. *There is a* $c_{\epsilon} > 0$ *for which*

$$\|\boldsymbol{\nabla} q\|^2_{L^2(\Omega_2^{\epsilon})} + \|q\|^2_{L^2(\Gamma)} \geq c_{\epsilon}\, \|q\|^2_{L^2(\Omega_2^{\epsilon})} \tag{3.1.10}$$

for all $q \in H^1(\Omega_2^{\epsilon})$.

Lemma 3.1.3. *For each $\epsilon > 0$, the operator $\mathcal{B}^{\epsilon}$ satisfies the* inf-sup condition (3.1.7) *on $\mathbf{V}^{\epsilon}$ and Q^{ϵ}.*

Proof. Let $q \in Q^{\epsilon}$ and denote by ξ the unique solution of the mixed problem

$$-\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \xi = q^1 \text{ in } \Omega_1,\ \boldsymbol{\nabla} \xi \cdot \mathbf{n} = q^2 \text{ on } \Gamma,\ \xi = 0 \text{ on } \partial \Omega_1 - \Gamma.$$

Set $\mathbf{v}^1 = \boldsymbol{\nabla} \xi$. Then $-\boldsymbol{\nabla} \cdot \mathbf{v}^1 = q^1$ and $\mathbf{v}^1 \cdot \mathbf{n} = q^2$ on Γ with $c_1 \|\mathbf{v}^1\|_{\mathbf{L}^2_{div}(\Omega_1)} \leq \|q^1\|_{L^2(\Omega_1)}$ by the Poincaré inequality. Set $\mathbf{v}^2 = \boldsymbol{\nabla} q^2$. For $\mathbf{v} = [\mathbf{v}^1, \mathbf{v}^2]$ on Ω^{ϵ} we have $\mathbf{v} \in \mathbf{V}^{\epsilon}$ and with (3.1.10) the estimate

$$\begin{aligned}
\mathcal{B}^{\epsilon}\mathbf{v}(q) &= -\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1\, q^1\, dx + \int_{\Gamma} \mathbf{v}^1 \cdot \mathbf{n}\, q^2\, dS + \int_{\Omega_2^{\epsilon}} \mathbf{v}^2 \cdot \boldsymbol{\nabla} q^2\, dx \\
&= \int_{\Omega_1} \left|q^1\right|^2 dx + \int_{\Gamma} \left|q^2\right|^2 dx + \int_{\Omega_2^{\epsilon}} \left|\boldsymbol{\nabla} q^2\right|^2 dx \\
&\geq \int_{\Omega_1} \left|q^1\right|^2 dx + \frac{c_\epsilon}{2} \int_{\Omega_2^{\epsilon}} \left|q^2\right|^2 dx + \frac{1}{2}\left(\int_{\Gamma} \left|q^2\right|^2 dx + \int_{\Omega_2^{\epsilon}} \left|\boldsymbol{\nabla} q^2\right|^2 dx \right) \\
&\geq c \|\mathbf{v}\|_{V^{\epsilon}} \|q\|_{Q^{\epsilon}},
\end{aligned}$$

with $c_0 = \min(c_1, \frac{1}{2}, \frac{c_\epsilon}{2})$, and this yields the inf-sup condition (3.1.7). □

Theorem 3.1.4. *Assume that $0 < \epsilon \leq 1$, $0 \leq \lambda$, $0 \leq \alpha$, $a(\cdot)$, $c(\cdot) \in L^{\infty}(\Omega^{\epsilon})$, $a(x) \geq a^* > 0$ and $c(x) \geq 0$ on Ω^{ϵ}, $f^{\epsilon} \in L^2(\Omega^{\epsilon})$, $\mathbf{g}^{\epsilon} \in \mathbf{L}^2(\Omega^{\epsilon})$, and $f_{\Gamma}^{\epsilon} \in L^2(\Gamma)$. Then the system* (3.1.5) *has a unique solution.*

3.2 The Scaled Problem

By scaling Ω_2^{ϵ} in the vertical direction with $x_N = \epsilon\, z + (1 - \epsilon)\, \zeta(\widetilde{x})$, we shall reformulate the interface problem (3.1.5) on the regions

$$\Omega_2 \equiv \{\, (\tilde{\mathbf{x}}, z) : \zeta(\tilde{\mathbf{x}}) < z < \zeta(\tilde{\mathbf{x}}) + 1,\ \tilde{\mathbf{x}} \in G \,\}, \quad \Omega \equiv \Omega_1 \cup \Gamma \cup \Omega_2.$$

These regions and the corresponding spaces

$$\mathbf{V} \equiv \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) :\ \boldsymbol{\nabla} \cdot \mathbf{v}^1 \in L^2(\Omega_1),\ \alpha\ \mathbf{v}^1 \cdot \mathbf{n}\big|_{\Gamma} \in L^2(\Gamma) \right\},$$

$$Q \equiv \left\{ q \in L^2(\Omega) :\ \boldsymbol{\nabla} q^2 \in \mathbf{L}^2(\Omega_2) \right\}$$

are independent of ϵ. The norms on the spaces $\mathbf{V}$ and Q are given by

$$\|\mathbf{v}\|_{\mathbf{V}} = \left(\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\nabla}\cdot\mathbf{v}^1\|^2_{L^2(\Omega_1)} + \|\alpha\,\mathbf{v}^1\cdot\mathbf{n}\|^2_{L^2(\Gamma)}\right)^{1/2},$$
$$\|q\|_Q = \left(\|q\|^2_{L^2(\Omega)} + \|\boldsymbol{\nabla} q^2\|^2_{\mathbf{L}^2(\Omega_2)}\right)^{1/2}.$$

The gradient is written as $\boldsymbol{\nabla} = (\widetilde{\boldsymbol{\nabla}}, \partial_{x_N})$, and it becomes $(\widetilde{\boldsymbol{\nabla}}, \frac{1}{\epsilon}\partial_z)$ on Ω_2 under the scaling above. The *scaled interface problem* is to find

$$\begin{aligned}
\mathbf{u}^\epsilon \in \mathbf{V},\, p^\epsilon \in Q: \quad & \int_{\Omega_1} a_1\,\mathbf{u}^\epsilon\cdot\mathbf{v}\,dx - \int_{\Omega_1} p^\epsilon\,\boldsymbol{\nabla}\cdot\mathbf{v}\,dx \\
& + \epsilon^2\int_{\Omega_2} a_2\,\mathbf{u}^\epsilon\cdot\mathbf{v}\,dx + \epsilon\int_{\Omega_2}\widetilde{\boldsymbol{\nabla}}p^\epsilon\cdot\tilde{\mathbf{v}}\,dx + \int_{\Omega_2}\partial_z p^\epsilon\,v_N\,dx \\
& + \int_\Gamma p^{\epsilon,2}\,\mathbf{v}^1\cdot\mathbf{n}\,dS + \int_\Gamma \alpha\left(\mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\right)\left(\mathbf{v}^1\cdot\mathbf{n}\right)\,dS \\
& = -\int_{\Omega_1}\mathbf{g}^\epsilon\cdot\mathbf{v}\,dx - \epsilon\int_{\Omega_2}\mathbf{g}^\epsilon\cdot\mathbf{v}\,dx
\end{aligned} \tag{3.2.11a}$$

$$\begin{aligned}
& \int_{\Omega_1}\lambda\,c_1\,p^\epsilon\,q\,dx + \epsilon\int_{\Omega_2}\lambda\,c_2\,p^\epsilon\,q\,dx + \int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{u}^\epsilon q\,dx \\
& \quad - \epsilon\int_{\Omega_2}\tilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\,q\,dx - \int_{\Omega_2}u_N^{\epsilon,2}\,\partial_z\,q\,dx - \int_\Gamma \mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\,q^2\,dS \\
& = \int_{\Omega_1} f^{\epsilon,1}\,q\,dx + \epsilon\int_{\Omega_2} f^{\epsilon,2}\,q\,dx + \int_\Gamma f^\epsilon_\Gamma\,q^2\,dS \text{ for all } \mathbf{v}\in\mathbf{V},\ q\in Q.
\end{aligned} \tag{3.2.11b}$$

Theorem 3.1.4 shows that the system (3.2.11) has a unique solution for each $0 < \epsilon \le 1$. This solution satisfies the equations

$$a_1\mathbf{u}^\epsilon + \boldsymbol{\nabla}\,p^\epsilon + \mathbf{g}^\epsilon = 0 \text{ and} \tag{3.2.12a}$$

$$\lambda\,c_1\,p^\epsilon + \boldsymbol{\nabla}\cdot\mathbf{u}^\epsilon = f^\epsilon \text{ in } \Omega_1, \tag{3.2.12b}$$

$$p^\epsilon = 0 \text{ on } \partial\Omega_1 - \Gamma, \tag{3.2.12c}$$

$$p^{\epsilon,1} - p^{\epsilon,2} = \alpha\,\mathbf{u}^{\epsilon,1}\cdot\mathbf{n} \text{ and} \tag{3.2.12d}$$

$$-\mathbf{u}^{\epsilon,1}\cdot\mathbf{n} + \left(\epsilon\,\tilde{\mathbf{u}}^{\epsilon,2},\,u_N^{\epsilon,2}\right)\cdot\mathbf{n} = f^\epsilon_\Gamma \text{ on } \Gamma, \tag{3.2.12e}$$

$$\epsilon\,a_2\,\tilde{\mathbf{u}}^{\epsilon,2} + \widetilde{\boldsymbol{\nabla}}p^\epsilon + \tilde{\mathbf{g}}^\epsilon = \tilde{\mathbf{0}},\ \epsilon^2\,a_2\,u_N^{\epsilon,2} + \partial_z\,p^\epsilon + \epsilon\,g_N^\epsilon = 0 \text{ and} \tag{3.2.12f}$$

$$\epsilon\,\lambda\,c_2\,p^\epsilon + \epsilon\,\widetilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{u}}^{\epsilon,2} + \partial_z\,u_N^{\epsilon,2} = \epsilon\,f^\epsilon \text{ in } \Omega_2 \tag{3.2.12g}$$

$$\left(\epsilon\,\tilde{\mathbf{u}}^{\epsilon,2}, u_N^{\epsilon,2}\right)\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_2 - \Gamma. \tag{3.2.12h}$$

3.2.1 The Estimates

We shall assume additionally that

$$\|f^{\epsilon}\|_{L^2(\Omega^{\epsilon})} \text{ is bounded and } f^{1,\epsilon} \overset{w}{\rightharpoonup} f^1 \text{ in } L^2(\Omega_1). \tag{3.2.13a}$$

$$\|\mathbf{g}^{\epsilon}\|_{\mathbf{L}^2(\Omega)} \text{ is bounded and} \tag{3.2.13b}$$

$$\|f_{\Gamma}^{\epsilon}\|_{L^2(\Gamma)} \text{ is bounded.} \tag{3.2.13c}$$

Note that $\epsilon^{1/2} f^{2,\epsilon}$ is bounded in $L^2(\Omega_2)$, so $\epsilon f^{2,\epsilon} \to 0$.

Set $\mathbf{v} = \mathbf{u}^{\epsilon},\ q = p^{\epsilon}$ in (3.2.11) and add to obtain

$$\begin{aligned} a^* \left(\|\mathbf{u}^{\epsilon,1}\|_{0,\Omega_1}^2 + \|\epsilon\,\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2}^2\right) + \alpha\,\|\mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\|_{L^2(\Gamma)}^2 \\ + \lambda\,\|c_1^{1/2}p^{\epsilon}\|_{0,\Omega_1}^2 + \lambda\,\|c_2^{1/2}\epsilon^{1/2}p^{\epsilon}\|_{0,\Omega_2}^2 = \int_{\Omega_1} f^{\epsilon}p^{\epsilon}\,dx \\ + \int_{\Omega_2}\epsilon f^{\epsilon}p^{\epsilon}\,dx + \int_{\Gamma} f_{\Gamma}^{\epsilon}\,p^{\epsilon,2}\,dS - \int_{\Omega_1}\mathbf{g}^{\epsilon}\cdot\mathbf{u}^{\epsilon}\,dx - \int_{\Omega_2}\mathbf{g}^{\epsilon}\cdot\epsilon\,\mathbf{u}^{\epsilon}\,dx \\ \le C\left(\|f^{\epsilon}\|_{0,\Omega} + \|f_{\Gamma}^{\epsilon}\|_{0,\Gamma}\right)\|p^{\epsilon}\|_Q + \|\mathbf{g}^{\epsilon}\|_{0,\Omega}\left(\|\mathbf{u}^{\epsilon,1}\|_{0,\Omega_1} + \|\epsilon\,\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2}\right) \end{aligned} \tag{3.2.14}$$

The constant C is independent of $\epsilon \le 1$. From (3.2.12f) we have

$$\|\widetilde{\boldsymbol{\nabla}}p^{\epsilon,2}\|_{0,\Omega_2} \le \epsilon\,\|a_2\|_{L^\infty(\Omega_2)}\|\tilde{\mathbf{u}}^{\epsilon,2}\|_{0,\Omega_2} + \|\tilde{\mathbf{g}}^{\epsilon}\|_{0,\Omega_2}, \tag{3.2.15a}$$

$$\|\partial_z p^{\epsilon,2}\|_{0,\Omega_2} \le \epsilon^2\|a_2\|_{L^\infty(\Omega_2)}\|u_N^{\epsilon,2}\|_{0,\Omega_2} + \epsilon\,\|g_N^{\epsilon}\|_{0,\Omega_2}\,, \tag{3.2.15b}$$

so we obtain for $0 < \epsilon \le 1$

$$\|\boldsymbol{\nabla}p^{\epsilon,2}\|_{0,\Omega_2} \le \|a_2\|_{L^\infty(\Omega_2)}\|\epsilon\,\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2} + \|\mathbf{g}^{\epsilon}\|_{0,\Omega_2}. \tag{3.2.16}$$

From (3.2.12a) we obtain

$$\|\boldsymbol{\nabla}p^{\epsilon,1}\|_{0,\Omega_1} \le \|a_1\|_{L^\infty(\Omega_1)}\|\mathbf{u}^{\epsilon,1}\|_{0,\Omega_1} + \|\mathbf{g}^{\epsilon}\|_{0,\Omega_1}.$$

With the boundary condition (3.2.12c) and the Poincaré inequality, this shows the left side of (3.2.14) bounds $\|p^{\epsilon,1}\|_{H^1(\Omega_1)}^2$. The interface condition (3.2.12d) and (3.2.16) in (3.1.10) show that the left side of (3.2.14) bounds $\|p^{\epsilon,2}\|_{H^1(\Omega_2)}^2$. We conclude from these together with (3.2.15b) and (3.2.12b) that each of the sequences

$$\| \mathbf{u}^{\epsilon,1} \|_{0,\Omega_1}, \ \| \epsilon \mathbf{u}^{\epsilon,2} \|_{0,\Omega_2}, \ \alpha^{1/2} \| \mathbf{u}^{\epsilon,1} \cdot \mathbf{n} \|_{L^2(\Gamma)}, \tag{3.2.17}$$

$$\| p^{\epsilon,1} \|_{H^1(\Omega_1)}, \ \| p^{\epsilon,2} \|_{H^1(\Omega_2)}, \ \left\| \frac{1}{\epsilon} \partial_z p^{\epsilon} \right\|_{0,\Omega_2}, \ \| \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon,1} \|_{L^2(\Omega_1)} \tag{3.2.18}$$

is bounded. In $L^2(\Omega_2)$ we know only that the combination $\widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^{\epsilon,2} + \frac{1}{\epsilon} \partial_z u_N^{\epsilon,2}$ is bounded due to (3.2.12g).

Remark 3.2.1. *The preceding can be done even without the boundary condition* (3.2.12c) *when the coefficient* $c_1(\cdot)$ *is not identically zero and* $\lambda > 0$*.*

Lemma 3.2.1. *Assume the nonnegative function* $c_1(\cdot)$ *is non-zero in* $L^\infty(\Omega_1)$*. There is a* $c > 0$ *for which*

$$\| \boldsymbol{\nabla} q \|^2_{\mathbf{L}^2(\Omega_1)} + \| c_1^{1/2} q \|^2_{L^2(\Omega_1)} \geq c \, \| q \|^2_{H^1(\Omega_1)} \tag{3.2.19}$$

for $q \in H^1(\Omega_1)$*.*

3.2.2 The Weak Limits

We have bounds on $\mathbf{u}^{\epsilon} = \left[\mathbf{u}^{\epsilon,1}, \epsilon \mathbf{u}^{\epsilon,2} \right]$ in $\mathbf{V}$ and on $p^{\epsilon} = \left[p^{\epsilon,1}, p^{\epsilon,2} \right]$ in $H^1(\Omega_1) \times H^1(\Omega_2)$, hence, in Q. Therefore, there must exist $p \in Q, \ \mathbf{u} = [\mathbf{u}^1, \mathbf{u}^2] \in \mathbf{V}, \ \eta \in L^2(\Omega_2)$ such that for some *subsequence*, hereafter denoted the same, we have weak convergence

$$p^{\epsilon} \overset{w}{\rightharpoonup} p \text{ in } Q, \ \text{ strongly in } \ L^2(\Omega), \tag{3.2.20a}$$

$$\mathbf{u}^{\epsilon,1} \overset{w}{\rightharpoonup} \mathbf{u}^1 \text{ in } \mathbf{L}^2(\Omega_1) \text{ and } \boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon,1} \overset{w}{\rightharpoonup} \boldsymbol{\nabla} \cdot \mathbf{u}^1 \text{ in } L^2(\Omega_1), \tag{3.2.20b}$$

$$\alpha^{1/2} \mathbf{u}^{\epsilon,1} \cdot \mathbf{n} \overset{w}{\rightharpoonup} \alpha^{1/2} \mathbf{u}^1 \cdot \mathbf{n} \text{ in } L^2(\Gamma), \tag{3.2.20c}$$

$$\epsilon \mathbf{u}^{\epsilon,2} \overset{w}{\rightharpoonup} \mathbf{u}^2 \text{ in } \mathbf{L}^2(\Omega_2), \tag{3.2.20d}$$

$$\frac{1}{\epsilon} \partial_z p^{\epsilon} \overset{w}{\rightharpoonup} \eta, \quad \partial_z p^{\epsilon} \to 0 \ \text{ strongly in } \ L^2(\Omega_2). \tag{3.2.20e}$$

In the equation (3.2.11b), take limits with $q = \epsilon \phi \in C_0^\infty(\Omega_2)$; then from (3.2.20d) we conclude $\langle \epsilon \partial_z u_N^{\epsilon,2}, \phi \rangle_{D'(\Omega_2), D(\Omega_2)} \to \langle \partial_z u_N^2, \phi \rangle_{D'(\Omega_2), D(\Omega_2)} = 0$, so the component

$u_N^2 = u_N^2(\tilde{\mathbf{x}})$ is independent of z in Ω_2. Again with $\epsilon\, q$ in (3.2.11b) with a general $q \in Q$, take limits and use (3.2.20d) to conclude

$$\begin{aligned} 0 = \lim_{\epsilon \downarrow 0} \int_{\Omega_2} \epsilon\, u_N^{\epsilon,2}\, \partial_z\, q\, dx &= \int_{\Omega_2} u_N^2(\tilde{\mathbf{x}})\, \partial_z q(\tilde{\mathbf{x}}, z)\, dx \\ &= \int_G u_N^2\,(\tilde{\mathbf{x}}) \left(\int_{\zeta(\widetilde{x})}^{\zeta(\widetilde{x})+1} \partial_z\, q\,(\widetilde{x}, z)\, dz \right) d\tilde{\mathbf{x}} \\ &= \int_G u_N^2\,(\widetilde{x})\, \left[q\,(\widetilde{x}, \zeta(\widetilde{x})+1) - q\,(\widetilde{x}, \zeta(\widetilde{x})) \right] d\widetilde{x}\,. \end{aligned} \qquad (3.2.21)$$

Since this holds for all $q \in Q$, in particular with $q\,(\widetilde{x}, \zeta(\widetilde{x})) = q\,|_{\Gamma} = 0$ and $q\,(\widetilde{x}, \zeta(\widetilde{x})+1) = q\,|_{\Gamma+1} = \phi(\widetilde{x})$ for $\phi \in C_0^\infty(\Gamma)$ arbitrary, we obtain $u_N^2 = 0$.

Now consider a function $\tilde{\mathbf{v}} \in (C_0^\infty(\Omega_2))^{N-1}$, set $\mathbf{v} = (\frac{1}{\epsilon}\tilde{\mathbf{v}}, 0)$ in (3.2.11a) and let $\epsilon \downarrow 0$ to obtain

$$\begin{aligned} &\epsilon \int_{\Omega_2} a_2\,(x)\, \tilde{\mathbf{u}}^{\epsilon,2} \cdot \tilde{\mathbf{v}}\, dx + \int_{\Omega_2} \left(\widetilde{\boldsymbol{\nabla}} p^{\epsilon} + \tilde{\mathbf{g}}^{\epsilon} \right) \cdot \tilde{\mathbf{v}}\, dx \to \\ &\qquad\qquad \int_{\Omega_2} a_2\, \tilde{\mathbf{u}}^2 \cdot \tilde{\mathbf{v}}\, dx + \int_{\Omega_2} \left(\widetilde{\boldsymbol{\nabla}} p + \tilde{\mathbf{g}} \right) \cdot \tilde{\mathbf{v}}\, dx = 0\,. \end{aligned}$$

This holds for all $\tilde{\mathbf{v}} \in (C_0^\infty(\Omega_2))^{N-1}$, so we conclude the lower-dimensional Darcy-type constitutive law

$$a_2\,(x)\, \tilde{\mathbf{u}}^2 + \widetilde{\boldsymbol{\nabla}} p^2 + \tilde{\mathbf{g}} = 0 \text{ in } \Omega_2\,. \qquad (3.2.22)$$

From (3.2.20e) it is clear that p^2 does not depend on the variable z, *i.e.* $p^2 = p^2(\tilde{\mathbf{x}})$. Therefore if we assume

$$a_2 = a_2(\tilde{\mathbf{x}}),\ \ \tilde{\mathbf{g}} = \tilde{\mathbf{g}}(\tilde{\mathbf{x}}) \text{ in } \Omega_2\,, \qquad (3.2.23)$$

we conclude $\tilde{\mathbf{u}}^2 = \tilde{\mathbf{u}}^2(\tilde{\mathbf{x}})$ is independent of z in Ω_2.

3.3 The Limit Problem

Define the subspaces $\mathbf{V}_0 \equiv \{\mathbf{v} \in \mathbf{V} :\ \partial_z \mathbf{v}^2 = \mathbf{0} \text{ and } v_N = 0 \text{ in } \Omega_2\}$, $Q_0 \equiv \{q \in Q : \partial_z q = 0 \text{ in } \Omega_2\}$. That is, $\mathbf{v}^2 = [\tilde{\mathbf{v}}^2(\tilde{\mathbf{x}}), 0]$ when $\mathbf{v} \in \mathbf{V}_0$ and $q^2 = q^2(\tilde{\mathbf{x}})$ when $q \in Q_0$.

If $\mathbf{v} \in \mathbf{V}_0$ then we have $[\mathbf{v}^1, \frac{1}{\epsilon}\mathbf{v}^2] \in \mathbf{V}_0$. Using the latter and a $q \in Q_0$ as test functions in (3.2.11), we obtain

$$\begin{aligned}\mathbf{u}^\epsilon \in \mathbf{V},\, p^\epsilon \in Q: \quad & \int_{\Omega_1} a_1\, \mathbf{u}^\epsilon \cdot \mathbf{v}\, dx - \int_{\Omega_1} p^\epsilon\, \boldsymbol{\nabla} \cdot \mathbf{v}\, dx \\ & + \epsilon \int_{\Omega_2} a_2 \tilde{\mathbf{u}}^{\epsilon,2} \cdot \tilde{\mathbf{v}}\, dx + \int_{\Omega_2} \widetilde{\boldsymbol{\nabla}} p^\epsilon \cdot \tilde{\mathbf{v}}\, dx \\ & + \int_\Gamma p^{\epsilon,2}\, \mathbf{v}^1 \cdot \mathbf{n}\, dS + \int_\Gamma \alpha \left(\mathbf{u}^{\epsilon,1} \cdot \mathbf{n}\right)\left(\mathbf{v}^1 \cdot \mathbf{n}\right)\, dS \\ & = - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v}\, dx - \int_{\Omega_2} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{v}}^2 dx\,,\end{aligned}$$

$$\begin{aligned}& \int_{\Omega_1} \lambda\, c_1\, p^\epsilon\, q\, dx + \epsilon \int_{\Omega_2} \lambda\, c_2\, p^\epsilon\, q\, dx + \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}^\epsilon q\, dx \\ & \quad - \epsilon \int_{\Omega_2} \tilde{\mathbf{u}}^{\epsilon,2} \cdot \widetilde{\boldsymbol{\nabla}}\, q\, dx - \int_\Gamma \mathbf{u}^{\epsilon,1} \cdot \mathbf{n}\, q^2\, dS \\ & \quad = \int_{\Omega_1} f^\epsilon q\, dx + \int_{\Omega_2} \epsilon\, f^\epsilon q\, dx + \int_\Gamma f_\Gamma^\epsilon\, q^2 dS\,.\end{aligned}$$

Letting $\epsilon \downarrow 0$ we find that the limits $\left[\mathbf{u}^{\epsilon,1}, \epsilon\, \mathbf{u}^{\epsilon,2}\right] \to \mathbf{u}$ and $p^\epsilon \to p$ of the subsequences are a solution of the *limit problem*

$$\begin{aligned}\mathbf{u} \in \mathbf{V}_0,\, p \in Q_0: \quad & \int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v}\, dx - \int_{\Omega_1} p\, \boldsymbol{\nabla} \cdot \mathbf{v}\, dx \\ & + \int_{\Omega_2} a_2\, \tilde{\mathbf{u}}^2 \cdot \tilde{\mathbf{v}}\, dx + \int_{\Omega_2} \widetilde{\boldsymbol{\nabla}} p \cdot \tilde{\mathbf{v}}\, dx + \int_\Gamma p^2\, \mathbf{v}^1 \cdot \mathbf{n}\, dS \\ & + \int_\Gamma \alpha \left(\mathbf{u}^1 \cdot \mathbf{n}\right)\left(\mathbf{v}^1 \cdot \mathbf{n}\right)\, dS = - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v}\, dx - \int_{\Omega_2} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{v}}\, dx\,, \quad \text{(3.3.24a)}\end{aligned}$$

$$\begin{aligned}& \int_{\Omega_1} \lambda\, c_1\, p\, q\, dx + \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}\, q\, dx - \int_{\Omega_2} \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}}\, q\, dx \\ & \quad - \int_\Gamma \mathbf{u}^1 \cdot \mathbf{n}\, q^2\, dS = \int_{\Omega_1} f\, q\, dx + \int_\Gamma f_\Gamma\, q^2\, dS \\ & \quad \text{for all } \mathbf{v} \in \mathbf{V}_0,\, q \in Q_0. \quad \text{(3.3.24b)}\end{aligned}$$

This problem is a mixed formulation (3.1.8) with the operators

$$\mathcal{A}^0 \mathbf{u}(\mathbf{v}) = \int_{\Omega_1} a_1\, \mathbf{u} \cdot \mathbf{v}\, dx + \int_{\Omega_2} a_2\, \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}\, dx + \int_\Gamma \alpha \left(\mathbf{u}^1 \cdot \mathbf{n}\right)\left(\mathbf{v}^1 \cdot \mathbf{n}\right)\, dS\,,$$

$$\mathcal{B}^0 \mathbf{u}(q) = - \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}\, q\, dx + \int_\Gamma \mathbf{u}^1 \cdot \mathbf{n}\, q^2\, dS + \int_{\Omega_2} \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}}\, q\, dx\,,$$

$$\mathcal{C}^0 p\,(q) = \int_{\Omega_1} c_1\, p\, q\, dx\,.$$

Note the degeneracy in $\mathcal{C}^0$: the c_2-terms on Ω_2 have vanished in the limit. Theorem 3.1.1 applies to these operators on $\mathbf{V}_0$ and Q_0. The inf-sup condition follows from the proof of Lemma 3.1.3. As a consequence of the uniqueness of the solution of the limit problem (3.3.24), not only a subsequence but the *original* sequences $\left[\mathbf{u}^{\epsilon,1}, \epsilon\,\mathbf{u}^{\epsilon,2}\right]$, p^ϵ converge as indicated to $\left[\mathbf{u}^1, \mathbf{u}^2\right]$, p.

We summarize the above as follows.

Theorem 3.3.1. *Assume the conditions of Theorem 3.1.4 and* (3.2.13) *and* (3.2.23). *Then the sequence* $\left[\mathbf{u}^{\epsilon,1}, \epsilon\,\mathbf{u}^{\epsilon,2}\right]$, p^ϵ *of solutions of the corresponding scaled problems* (3.2.11) *converges weakly in* $\mathbf{V} \times Q$ *to the solution* $\left[\mathbf{u}^1, \mathbf{u}^2\right] \in \mathbf{V}_0$, $p \in Q_0$ *of the limit problem* (3.3.24)*, and*

$$p^\epsilon \overset{w}{\rightharpoonup} p \text{ weakly in } H^1(\Omega_1) \times H^1(\Omega_2), \text{ strongly in } L^2(\Omega).$$

3.3.1 The Strong Form

Plane area on G is related to surface area on Γ by $d\tilde{\mathbf{x}} = n_N dS$, where n_N is the N-th component of the unit inward normal on $\partial\Omega_2$, $n_N = \mathbf{n} \cdot \mathbf{e}_N$. Note $\mathbf{n} = \tilde{\mathbf{n}}(\tilde{\mathbf{x}})$ and $n_N = 0$ on $\partial\Gamma \times [0,1]$. Functions of $\tilde{\mathbf{x}}$ can be regarded as functions on Ω_2, G, or Γ, and we have the identities

$$\int_{\Omega_2} F(\tilde{\mathbf{x}})\, dx = \int_G F(\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}} = \int_\Gamma n_N\, F\, dS.$$

From (3.3.24a) we obtain

$$a_1 \mathbf{u}^1 + \boldsymbol{\nabla} p^1 + \mathbf{g} = \mathbf{0} \text{ in } \Omega_1, \tag{3.3.25a}$$

$$a_2\, \tilde{\mathbf{u}}^2 + \widetilde{\boldsymbol{\nabla}} p^2 + \tilde{\mathbf{g}} = \mathbf{0} \text{ on } \Gamma, \tag{3.3.25b}$$

and

$$-\left\langle p^1, \mathbf{v}^1 \cdot \mathbf{n} \right\rangle_{\partial\Omega_1} + \int_\Gamma \left\{ p^2 \mathbf{v}^1 \cdot \mathbf{n} + \alpha \left(\mathbf{u}^1 \cdot \mathbf{n}\right)\left(\mathbf{v}^1 \cdot \mathbf{n}\right) \right\} dS = 0$$

for all $\mathbf{v} \in \mathbf{V}$. Note that (3.3.25a) shows $p^1 \in H^1(\Omega_1)$, so it's trace is in $H^{1/2}(\partial\Omega_1)$. Thus we have

$$p^1 = p^2 + \alpha\left(\mathbf{u}^1 \cdot \mathbf{n}\right) \text{ on } \Gamma, \tag{3.3.25c}$$

$$p^1 = 0 \text{ on } \partial\Omega_1 - \Gamma \tag{3.3.25d}$$

Choosing $q \in C_0^\infty(\Omega_1) \times C_0^\infty(G)$ in (3.3.24b), we first obtain

$$\lambda c_1 p^1 + \boldsymbol{\nabla} \cdot \mathbf{u}^1 = f \text{ in } \Omega_1, \tag{3.3.25e}$$

$$n_N \widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^2 - \mathbf{u}^1 \cdot \mathbf{n} = f_\Gamma \text{ on } \Gamma. \tag{3.3.25f}$$

Since $\widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}^2 \in L^2(G)$, the third term in (3.3.24b) can be rewritten

$$\begin{aligned} -\int_{\Omega_2} \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}} q\, dx = -\int_G \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}} q\, d\tilde{\mathbf{x}} = \int_G \widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}\, q\, d\tilde{\mathbf{x}} - \langle \tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}},\, q\rangle_{\partial G} \\ = \int_\Gamma \mathbf{n}_N \widetilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{u}}\, q\, dS - \langle \tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}},\, q\rangle_{\partial\Gamma} \text{ for } q \in Q_0, \end{aligned}$$

so we obtain also

$$\tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}} = 0 \text{ in } H^{-1/2}(\partial\Gamma) \tag{3.3.25g}$$

The system (3.3.25) is the *strong form* of the limit problem (3.3.24).

The limit problem (3.3.24) on the lower dimensional interface Γ is in the mixed form (3.1.8). To see this, we define the spaces

$$\mathbf{V}_{00} \equiv \left\{\mathbf{v} = \left[\mathbf{v}^1, \tilde{\mathbf{v}}\right] \in \mathbf{L}^2(\Omega_1) \times \mathbf{L}^2(\Gamma) :\ \boldsymbol{\nabla} \cdot \mathbf{v}^1 \in L^2(\Omega_1),\ \alpha\,\mathbf{v}^1 \cdot \mathbf{n} \in L^2(\Gamma)\right\},$$

$$Q_{00} \equiv \left\{q = \left[q^1, q^2\right] \in L^2(\Omega_1) \times H^1(\Gamma)\right\}$$

and the operators

$$\begin{aligned} \mathcal{A}^{00}\mathbf{u}\,(\mathbf{v}) &= \int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v}\, dx + \int_\Gamma n_N\, a_2\, \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}\, dS + \int_\Gamma \alpha\left(\mathbf{u}^1 \cdot \mathbf{n}\right)\left(\mathbf{v}^1 \cdot \mathbf{n}\right)\, dS\,, \\ \mathcal{B}^{00}\mathbf{u}\,(q) &= -\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}\, q\, dx + \int_\Gamma \mathbf{u}^1 \cdot \mathbf{n}\, q^2\, dS + \int_\Gamma n_N\, \tilde{\mathbf{u}} \cdot \widetilde{\boldsymbol{\nabla}} q\, dx\,, \\ \mathcal{C}^{00}p\,(q) &= \int_{\Omega_1} c_1\, p\, q\, dx\,. \end{aligned}$$

It suffices then to check that (3.1.8) with these operators and spaces is equivalent to (3.3.24).

3.4 Strong Convergence of the Solutions

Assume additionally the strong convergence

$$\mathbf{g}^{\epsilon} \to \mathbf{g} \text{ in } \mathbf{L}^2(\Omega) \text{ and } f_\Gamma^{\epsilon} \to f_\Gamma \text{ in } L^2(\Gamma). \tag{3.4.26}$$

Set $\mathbf{v} = \mathbf{u}^{\epsilon},\ q = p^{\epsilon}$ in (3.2.11) and add to obtain the identity

$$\begin{aligned}\| a_1^{1/2}\mathbf{u}^{\epsilon}\|^2_{0,\Omega_1} + \epsilon^2 \| a_2^{1/2}\mathbf{u}^{\epsilon}\|^2_{0,\Omega_2} + \alpha \, \|\mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\|^2_{L^2(\Gamma)} \\ + \lambda \, \| c_1^{1/2} p^{\epsilon}\|^2_{0,\Omega_1} + \epsilon\,\lambda\,\| c_2^{1/2} p^{\epsilon}\|^2_{0,\Omega_2} = \int_{\Omega_1} f^{\epsilon}\, p^{\epsilon}\, dx \\ + \epsilon \int_{\Omega_2} f^{\epsilon}\, p^{\epsilon}\, dx + \int_\Gamma f_\Gamma^{\epsilon}\, p^{\epsilon,2}\, dS - \int_{\Omega_1} \mathbf{g}^{\epsilon}\cdot\mathbf{u}^{\epsilon}\, dx - \epsilon\int_{\Omega_2}\mathbf{g}^{\epsilon}\cdot\mathbf{u}^{\epsilon}\, dx\end{aligned} \tag{3.4.27}$$

From the strong convergence of the source terms (3.4.26) and the strong convergence of the sequence $\{p^{\epsilon} : \epsilon > 0\}$ in $L^2(\Omega)$, we can estimate

$$\begin{aligned}\limsup_{\epsilon\to 0} \left\{ \| a_1^{1/2}\mathbf{u}^{\epsilon}\|^2_{0,\Omega_1} + \| a_2^{1/2}\epsilon\,\mathbf{u}^{\epsilon}\|^2_{0,\Omega_2} + \alpha \, \|\mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\,\|^2_{L^2(\Gamma)} \right\} \\ \le -\lambda\,\| c_1^{1/2} p\,\|^2_{0,\Omega_1} + \int_{\Omega_1} f^1 p\, dx + \int_\Gamma f_\Gamma\, p^2\, dS - \int_{\Omega_1}\mathbf{g}\cdot\mathbf{u}\, dx - \int_{\Omega_2}\tilde{\mathbf{g}}\cdot\tilde{\mathbf{u}}^2 dx\,.\end{aligned} \tag{3.4.28}$$

Set $\mathbf{v} = \mathbf{u},\ q = p$ in the limit problem (3.3.24) and add. Using the resulting identity to evaluate the right side of (3.4.28), and then using the weak lower semicontinuity of the norms, we obtain

$$\begin{aligned}\limsup_{\epsilon\to 0} \left\{ \| a_1^{1/2}\mathbf{u}^{\epsilon}\|^2_{0,\Omega_1} + \| a_2^{1/2}\epsilon\,\mathbf{u}^{\epsilon}\|^2_{0,\Omega_2} + \alpha\|\mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\|^2_{L^2(\Gamma)} \right\} \\ \le \| a_1^{1/2}\mathbf{u}^1\|^2_{0,\Omega_1} + \| a_2^{1/2}\tilde{\mathbf{u}}^2\|^2_{0,\Omega_2} + \alpha\,\|\mathbf{u}^1\cdot\mathbf{n}\|^2_{L^2(\Gamma)} \\ \le \liminf_{\epsilon\to 0} \left\{ \| a_1^{1/2}\mathbf{u}^{\epsilon}\|^2_{0,\Omega_1} + \| a_2^{1/2}\epsilon\,\mathbf{u}^{\epsilon}\|^2_{0,\Omega_2} + \alpha\|\mathbf{u}^{\epsilon,1}\cdot\mathbf{n}\|^2_{L^2(\Gamma)} \right\}.\end{aligned} \tag{3.4.29}$$

But since these norms converge to their value at the weak limit, it follows that the convergence is *strong* in the indicated norm.

Theorem 3.4.1. *Under the assumptions of Theorem 3.3.1 and* (3.4.26)*, we have strong convergence*

$$\mathbf{u}^{\epsilon,1} \to \mathbf{u}^1 \text{ in } \mathbf{L}^2(\Omega_1),\ \epsilon\,\mathbf{u}^{\epsilon,2} \to \mathbf{u}^2 \text{ in } \mathbf{L}^2(\Omega_2), \tag{3.4.30a}$$

$$p^{\epsilon,1} \to p^1 \text{ in } H^1(\Omega_1),\ \text{and } p^{\epsilon,2} \to p^2 \text{ in } H^1(\Omega_2). \tag{3.4.30b}$$

4 STATIONARY DARCY-STOKES SCALED MODEL OF CHANNELED FLOW GRADIENT FORMULATION

4.1 Introduction

Consider the flow of a single phase incompressible viscous fluid through a system composed of two regions, the first being a porous structure and the second being an adjacent open channel, possibly a macropore, an isolated cavity, or a connected fracture system. Both regions are saturated with the fluid, and we need to prescribe the stress and flow couplings on the interface between the Darcy flow in the porous medium and the Stokes flow in the open channel.

The disjoint regions Ω_1 and Ω_2^ϵ in $\mathbb{R}^3$ share the common *interface*, $\Gamma \equiv \partial\Omega_1 \cap \partial\Omega_2^\epsilon$ and $\Omega_2^\epsilon \equiv \Gamma \times (0, \epsilon)$ *i.e.* the fracture Ω_2^ϵ has a cylindrical geometry. The first region Ω_1 is the fully-saturated *porous matrix* structure, and the second region Ω_2^ϵ is the fluid-filled *macro-void system*. Here we denote by $\mathbf{n}$ the unit normal vector on the boundaries, directed *out* of Ω_1 and *into* Ω_2^ϵ. The derivative with respect to time will be denoted by a superscript dot.

4.1.1 The Equations

The laminar flow of an incompressible viscous fluid through the porous medium Ω_1 is described by the *Darcy system*

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1(x,t)\,, \tag{4.1.1a}$$

$$\mathcal{Q}\,\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = \mathbf{0}\,, \tag{4.1.1b}$$

a *conservation equation* for fluid mass and *Darcy's law* for the *filtration velocity* or *fluid flux* $\mathbf{v}^1$. Here p^1 is the *pressure* of the fluid in the pores. The conductivity tensor $\mathcal{Q}$ is the reciprocal of the *permeability* of the structure, times the shear viscosity of the fluid.

The slow flow of an incompressible viscous fluid in the adjacent open channel Ω_2^ϵ is described by the *incompressible Stokes system* [Tem79, SP80]

$$\boldsymbol{\nabla}\cdot\mathbf{v}^{2}=0 \tag{4.1.2a}$$

$$-\boldsymbol{\nabla}\cdot\sigma^{2}+\boldsymbol{\nabla}p^{2}=f_{2} \tag{4.1.2b}$$

$$\sigma^{2}=2\,\epsilon\,\mu\,\mathfrak{D}(\mathbf{v}^{2})\,,\quad \text{in } \Omega_{2}^{\epsilon} \tag{4.1.2c}$$

where $\mathbf{v}^2$ is the velocity of the *fluid* and p^2 is the pressure of the fluid in Ω_2^ϵ. Amongst the above equations only two of them are constitutive. Darcy's law (4.1.1b) describes the fluid on a piece of the domain that is not subject to change with respect to the thickness of the channel, therefore it is not subject to scaling. The law (4.1.2c) describes the relationship between rate of strain tensor and stress for the fluid in the thin channel and μ is the viscosity; it is a constitutive law therefore it is subject to scaling according to the geometry. Finally, we recall that whenever $\boldsymbol{\nabla}\cdot\mathbf{v}^2=0$ we have

$$\boldsymbol{\nabla}\cdot\sigma^{2}=\boldsymbol{\nabla}\cdot\left[2\,\epsilon\,\mu\,\mathfrak{D}\left(\mathbf{v}^{2}\right)\right]=\epsilon\,\mu\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{v}^{2} \tag{4.1.3}$$

This observation transforms (4.1.2) in the classical *Stokes flow*.

4.1.2 Interface Conditions

The objectives in Section 2 are to describe a physically consistent set of interface conditions which couple these systems together in a variational statement modeling a mathematically well-posed boundary-value problem. The interface coupling conditions recognize the conservation of mass and total momentum. Thus, they will include the continuity of the normal fluid flux and account for the stress. These include the dependence of the Darcy flux on the increment of normal stress at the interface and the effect of the tangential component of stress on the velocity increment at the interface. The former is the classical *Robin* boundary condition, and the latter is the slip condition of *Beavers-Joseph-Saffman*.

The description of a free fluid in contact with a rigid but porous solid matrix requires a means to couple the slow flow to the upscaled Darcy filtration. Since a Stokes system is used for the free fluid, we have two distinct scales of hydrodynamics, and these are represented by two different systems of partial differential equations. Fluid conservation is a natural requirement at the interface, and other classically assumed conditions such as continuity of pressure or vanishing tangential velocity of the viscous fluid have been investigated [ESP75, LSP75], but these issues have been controversial. See the discussion on p. 157 of [SP80]. In fact, one can even question the *location* of the interface, since the porous medium itself is already a mixture of fluid and solid. Moreover, Beavers and Joseph [BJ67] discovered that fluid in contact with a porous medium flows faster along the interface than a fluid in contact with a solid surface: there is a substantial *slip* of the fluid at the interface with a porous medium. They proposed that the normal derivative of the tangential component of fluid velocity $\mathbf{v}_T$ satisfy

$$\frac{\partial}{\partial \mathbf{n}} \mathbf{v}_T = \frac{\gamma}{\sqrt{K}} (\mathbf{v}_T^2 - \mathbf{v}_T^1)$$

where K is the permeability of the porous medium, γ is the *slip rate coefficient* and $\mathbf{v}_T^2$ is the tangential Stokes velocity on the interface while $\mathbf{v}_T^1$ is the tangential Darcy velocity also on the interface. This condition was developed further in [Saf71, Jon73], and a substantial rigorous investigation of such interface conditions was given in [JM96, JM00]. See [DAN99, McK01] for an excellent discussion, [ASD94, DKGG96, WJLY03, ABar, TAJ04] for numerical work, [PS98] for dependence on the slip parameter, and [AL06] for homogenization results on related problems. Later, Saffman realized that the Darcy velocity of the fluid could be neglected and stated that the tangential stress is proportional to the tangential velocity on the interface this is the so called Beavers-Joseph-Saffman

$$\sigma_T^2 = \gamma \sqrt{\mathcal{Q}}\, \mathbf{v}_T^2 .$$

Finally in the present work assuming that the velocity is curl-free on the interface we give an equivalent version of this condition by

$$\epsilon \frac{\partial}{\partial \mathbf{n}} \mathbf{v}_T^2 = \epsilon \frac{\partial}{\partial x_N} \mathbf{v}_T^2 = \epsilon^2 \gamma \sqrt{\mathcal{Q}}\, \mathbf{v}_T^2 . \tag{4.1.4a}$$

Where the left hand side of the expression above express the tangential stress on a curl-free surface and on a flat horizontal interface the normal derivative becomes the derivative with respect to the third component. Finally ϵ^2 is a scaled destined to balance out the geometric singularity introduced by the thinness of the channel.

We continue with the mass-conservation requirement that the normal fluid flux be continuous across the interface. The solution is required to satisfy the *admissability constraint*

$$\mathbf{v}^1 \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n} \tag{4.1.4b}$$

for the conservation of fluid mass across the interface. The Darcy flow across Γ is driven by the difference between the total normal stress of the fluid and the pressure internal to the porous medium according to

$$\sigma_n^2 - p^2 + p^1 = \alpha\, \mathbf{v}^1 \cdot \mathbf{n}\,.$$

The constant $\alpha \geq 0$ is the *fluid entry resistance.* This last conditions expressed in terms of the rate of strain tensor and recalling the scales we have

$$\epsilon\left(\frac{\partial\, \mathbf{v}^2}{\partial\, \mathbf{n}} \cdot \mathbf{n}\right) - p^2 + p^1 = \epsilon\,\frac{\partial\, \mathbf{v}_N^2}{\partial\, x_N} - p^2 + p^1 = \alpha\, \mathbf{v}^1 \cdot \mathbf{n} \tag{4.1.4c}$$

We shall show that the *interface conditions* (4.1.4) together with adequate boundary conditions suffice precisely to couple the Darcy system (4.1.1) in Ω_1 to the Stokes system (4.1.2) in Ω_2^ϵ.

4.1.3 Boundary Conditions

We choose the *boundary conditions* on $\partial\,\Omega_1 \cup \partial\,\Omega_2^\epsilon - \Gamma$ in a classical simple form, since they play no essential role here. On the exterior boundary of the porous medium, $\partial\,\Omega_1 - \Gamma$, we shall impose *null flux* conditions, $\mathbf{v}^1 \cdot \mathbf{n} = 0$.

On the exterior boundary of the free fluid, $\partial\,\Omega_2 - \Gamma$, we impose no-slip conditions on the wall of the cylinder $\Omega_2^\epsilon = \Gamma \times (0, \epsilon)$, *i.e.* $\mathbf{v} = 0$ on $\partial\,\Gamma \times (0, \epsilon)$. On the top to the cylinder we have a mixed boundary condition: a Neumann-type condition on the tangential component of the normal derivative of the velocity

$$\frac{\partial\, \mathbf{v}^2}{\partial\, \mathbf{n}} - \left(\frac{\partial\, \mathbf{v}^2}{\partial\, \mathbf{n}} \cdot \mathbf{n}\right) \mathbf{n} = \frac{\partial\, \mathbf{v}_T^2}{\partial\, x_N} = 0 \quad \text{on } \Gamma + \epsilon \tag{4.1.5}$$

The above condition is actually a statement on the tangential stress. Besides there is a non-flux, we have:

$$\mathbf{v}^2 \cdot \mathbf{n} = \mathbf{v}_N^2 = 0 \quad \text{on } \Gamma + \epsilon \tag{4.1.6}$$

Where $\Gamma + \epsilon \equiv \{(\widetilde{x}, \epsilon) : \widetilde{x} \in \Gamma\}$

4.1.4 Preliminaries

Standard function spaces will be used [Ada75, Tem79]. Let Ω be a smoothly bounded region in $\mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$. Let $H^1(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives also in $L^2(\Omega)$. The *trace* map or restriction to the boundary is the continuous linear map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ defined by $\gamma(w) = w|_\Gamma$. Corresponding spaces of vector-valued functions will be denoted by boldface symbols. For example, we denote the product space $L^2(\Omega)^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega) \equiv H^1(\Omega)^3$. We shall also use the space $\mathbf{L}^2_{\text{div}}(\Omega)$ of vector functions $\mathbf{L}^2(\Omega)$ whose divergence belongs to $L^2(\Omega)$. Recall that for the functions $\mathbf{w} \in \mathbf{L}^2_{\text{div}}(\Omega)$ there is a *normal trace* defined on the boundary, and this is denoted by $\mathbf{w} \cdot \mathbf{n}$, since it takes this value on the smooth functions $\mathbf{w}$ in $\mathbf{L}^2_{\text{div}}(\Omega)$.

We adopt the convention that repeated indices are summed. In particular, the scalar product of two vectors is $\mathbf{v} \cdot \mathbf{w} = v_i w_i$, and that of two second-order tensors is $\sigma : \tau = \sigma_{ij}\tau_{ij}$. Let $\mathbf{n} = \{n_i\}$ be the unit normal vector on a surface. For a vector $\mathbf{w}$, we denote the normal projection $w_n = \mathbf{w} \cdot \mathbf{n}$ and the tangential component $\mathbf{w}_T = \mathbf{w} - w_n\mathbf{n}$. Likewise for a tensor τ, we have its value at $\mathbf{n}$, $\tau(\mathbf{n}) \equiv \{\tau_{ij}\, n_i\} \in \mathbb{R}^3$, and its normal and tangential parts $\tau(\mathbf{n})(\mathbf{n}) = \tau_n = \tau_{ij}n_in_j,\ \tau_T = \tau(\mathbf{n}) - \tau_n\mathbf{n}$.

For an element $x = (x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N$ we denote by $\widetilde{x} = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ the first $N-1$ components. For a vector function $\mathbf{w} : \mathbb{R}^N \to \mathbb{R}^N$ we define $\mathbf{w}_T$ the first $N-1$ components and $\mathbf{w}_N$ the last component of the function. Finally $\boldsymbol{\nabla}_T$ denotes de $\mathbb{R}^{N-1}$-gradient in directions tangent to Γ, *i.e.* $\boldsymbol{\nabla}_T = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{N-1}})$.

4.1.5 The weak formulation

We want to construct an appropriate variational formulation of the the Darcy system (4.1.1) coupled by the interface conditions (4.1.4) to the Stokes system (4.1.2). Consider now the spaces:

$$\mathbf{V}_1 \equiv \left\{ \mathbf{v} \in \mathbf{L}^{\mathbf{2}}_{\mathbf{div}}(\Omega_1) : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\,\Omega_1 - \Gamma \right\}$$

$$\mathbf{V}_2^\epsilon \equiv \left\{ \mathbf{v} \in \mathbf{H}^{\mathbf{1}}(\Omega_2^\epsilon) : \mathbf{v} = 0 \text{ on } \partial\,\Gamma \times (0,\epsilon),\, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma + \epsilon \right\}$$

$$\mathbf{X}^\epsilon \equiv \left\{ [\,\mathbf{v}^1, \mathbf{v}^2\,] \in \mathbf{V}_1 \times \mathbf{V}_2^\epsilon : \mathbf{v}^1 \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n} \text{ on } \Gamma \right\}$$

$$\mathbf{Y}^\epsilon \equiv L^2_0(\Omega_1) \times L^2_0(\Omega_2^\epsilon)$$

Where $L^2_0(U) = \left\{ p \in L^2(U) : \int_U p\, dx = 0 \right\}$ for any $U \subset \mathbb{R}^N$ open.

Multiply the Darcy law by a test function $\mathbf{w}^1 \in \mathbf{V}_1^\epsilon$ and the momentum equation by $\mathbf{w}^2 \in \mathbf{V}_2^\epsilon$ integrate and obtain:

$$\int_{\Omega_1} \left(Q\,\mathbf{v}^1 \cdot \mathbf{w}^1 - p\,\delta : \mathfrak{D}\left(\mathbf{w}^1\right) \right) dx + \int_{\Omega_2^\epsilon} \left(\epsilon\,\mu \boldsymbol{\nabla} \mathbf{v}^2 - p\,\delta\right) : \boldsymbol{\nabla} \mathbf{w}^2\, dx$$
$$+ \int_\Gamma \left(p^1\, \mathbf{n} \cdot \mathbf{w}^1 + \epsilon \left(\boldsymbol{\nabla}\, \mathbf{v}^2\, \mathbf{n}\right) \cdot \mathbf{w}^2 - p^2 \left(\mathbf{w}^2 \cdot \mathbf{n}\right)\right) dS = \int_{\Omega_2^\epsilon} \mathbf{f}^2 \cdot \mathbf{w}^2\, dx$$

For test functions satisfying the admissibility constraint (4.1.4b), *i.e* $\mathbf{w}^1 \cdot \mathbf{n} = \mathbf{w}^2 \cdot \mathbf{n}$ on Γ, the interface integral reduces to

$$\int_\Gamma \left(\epsilon\, \frac{\partial\, \mathbf{v}}{\partial\, \mathbf{n}} \cdot \mathbf{w}^2 + \left(p^1 - p^2\right) \left(\mathbf{w}^2 \cdot \mathbf{n}\right) \right) dS$$

Moreover, decomposing the velocity terms into their normal and tangential components, we obtain:

$$\int_\Gamma \left\{ \epsilon \left(\frac{\partial\, \mathbf{v}}{\partial\, \mathbf{n}} \right)_T \cdot \mathbf{w_T}^2 + \left(\epsilon \left(\frac{\partial\, \mathbf{v}}{\partial\, \mathbf{n}} \cdot \mathbf{n} \right) + p^1 - p^2 \right) \left(\mathbf{w}^2 \cdot \mathbf{n}\right) \right\} dS$$

and then, the interface conditions (4.1.4a) and (4.1.4c) yield:

$$\int_\Gamma \epsilon^2\, \gamma\, \sqrt{Q}\, \mathbf{v}_T^2 \cdot \mathbf{w_T}^2\, dS + \alpha \int_\Gamma \left(\mathbf{v}^1 \cdot \mathbf{n}\right) \left(\mathbf{w}^1 \cdot \mathbf{n}\right) dS$$

Finally, multiply the fluid conservation equations by test functions $\varphi^1 \in L^2(\Omega_1)$, $\varphi^2 \in L^2(\Omega_2^\epsilon)$, integrate over the corresponding regions and add to obtain the second of two *variational statements.*

$$\text{Find } [\mathbf{v}^{\epsilon}, p^{\epsilon}] \in \mathbf{X}^{\epsilon} \times \mathbf{Y}^{\epsilon} \text{ such that :}$$

$$\int_{\Omega_1} \left(\mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1 - p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\right)dx + \int_{\Omega_2^{\epsilon}} \left(\epsilon\,\mu\boldsymbol{\nabla}\,\mathbf{v}^{2,\epsilon} - p^{2,\epsilon}\delta\,\right):\boldsymbol{\nabla}\mathbf{w}^{2}\,d\widetilde{x}dx_N$$
$$+\,\alpha\int_{\Gamma}\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS + \int_{\Gamma}\epsilon^2\,\gamma\sqrt{\mathcal{Q}}\,\mathbf{v}_T^{2,\epsilon}\cdot\mathbf{w_T}^{2}\,dS$$
$$= \int_{\Omega_2^{\epsilon}} \mathbf{f}^{2,\epsilon}\cdot\mathbf{w}^{2}\,d\widetilde{x}\,dx_N \quad (4.1.7a)$$

$$\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\,\varphi^1\,dx + \int_{\Omega_2^{\epsilon}}\boldsymbol{\nabla}\cdot\mathbf{v}^{2,\epsilon}\,\varphi^2\,d\widetilde{x}\,dx_N = \int_{\Omega_1} h^{1,\epsilon}\,\varphi^1\,dx. \quad (4.1.7b)$$
$$\text{for all } [\mathbf{w},\Phi] \in \mathbf{X}^{\epsilon}\times\mathbf{Y}^{\epsilon}$$

Where it is understood that $\mathbf{v}^{\epsilon} = \left[\mathbf{v}^{1,\epsilon}, \mathbf{v}^{2,\epsilon}\right]$, $p^{\epsilon} = \left[p^{1,\epsilon}, p^{2,\epsilon}\right]$, $\mathbf{w} = \left[\mathbf{w}^1, \mathbf{w}^2\right]$ and $\Phi = \left[\varphi^1, \varphi^2\right]$.

4.1.6 The mixed formulation

We write the resolvent system on a product of two spaces so that it is realized as a *saddle* point problem. We define the operators:

$$A^{\epsilon} = \begin{pmatrix} Q + \gamma_n'\,\alpha\,\gamma_n & 0 \\ 0 & \epsilon^2\,\gamma_T'\,\gamma\,\sqrt{Q}\,\gamma_T + \epsilon\,(\boldsymbol{\nabla})'\,\mu\,\boldsymbol{\nabla} \end{pmatrix} \quad (4.1.8)$$

$$B^{\epsilon} = \begin{pmatrix} \boldsymbol{\nabla}\cdot & 0 \\ 0 & \boldsymbol{\nabla}\cdot \end{pmatrix} = \begin{pmatrix} \text{div} & 0 \\ 0 & \text{div} \end{pmatrix} \quad (4.1.9)$$

And the resolvent system is obtained in the form

$$[\mathbf{v}, p] \in \mathbf{X}^{\epsilon}\times\mathbf{Y}^{\epsilon} : A^{\epsilon}\,\mathbf{v} - (B^{\epsilon})'\,p = \mathbf{f} \quad (4.1.10)$$

$$B^{\epsilon}\,\mathbf{v} = \mathbf{h} \quad (4.1.11)$$

for the unknowns $\mathbf{v} \equiv [\,\mathbf{v}^1, \mathbf{v}^2\,] \in \mathbf{X}^{\epsilon}$, $p \equiv [\,p^1, p^2\,] \in \mathbf{Y}^{\epsilon}$. This formulation requires a *closed range condition* on the operator B, and it provides a natural and well established approach to the *numerical approximation* of such problems. In addition, the estimates provide a means to establish the relation with the *singular limits* below.

Lemma 4.1.1. *The operator A^{ϵ} is $\mathbf{X}^{\epsilon}$-coercive over $\mathbf{X}^{\epsilon} \cap Ker(B)$.*

Proof. The form $A^{\epsilon}\,\mathbf{v}\,(\mathbf{v}) + \int_{\Omega_1} (\boldsymbol{\nabla}\cdot\mathbf{v})^2$ is $\mathbf{X}^{\epsilon}$-coercive, and $\boldsymbol{\nabla}\cdot\mathbf{v}\,|_{\Omega_1} = 0$ if $\mathbf{v} \in \mathrm{Ker}(B)$.

□

Lemma 4.1.2. *B^{ϵ} has closed range.*

Proof. Since $\mathbf{H}_0^1(\Omega^{\epsilon}) \subseteq \mathbf{X}$ and $\|\,\mathbf{v}\,\|_{\mathbf{X}} \leq C\,\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1(\Omega^{\epsilon})}$ we have:

$$\inf_{\varphi\in L_0^2(\Omega^{\epsilon})}\ \sup_{\mathbf{v}\in\mathbf{X}} \frac{B^{\epsilon}\,\mathbf{v}\,(\varphi)}{\|\,\mathbf{v}\,\|_{\mathbf{X}}\,\|\,\varphi\,\|_{L_0^2(\Omega^{\epsilon})}} \geq \frac{1}{C}\ \inf_{\varphi\in L_0^2(\Omega^{\epsilon})}\ \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega^{\epsilon})} \frac{B^{\epsilon}\,\mathbf{v}\,(\varphi)}{\|\,\mathbf{v}\,\|_{\mathbf{H}_0^1(\Omega^{\epsilon})}\,\|\,\varphi\,\|_{L^2(\Omega^{\epsilon})}}$$

and this last term corresponds to the Stokes problem and is known to be $\geq c > 0$. □

According to the theory for problems in mixed formulation [GR79b] the problem (4.1.10) is well-posed.

4.1.7 Fixing a Domain of Reference

So far the sequence of solutions $\{[\,\mathbf{v}^{\epsilon}, p^{\epsilon}\,] : \epsilon > 0\}$ to the problem (4.1.10) have different geometric domains of definition and therefore no convergence statements can be established. On the other hand the a-priori estimates given from the well-posedness of the problem (4.1.10) depend on the geometry where the problem is defined. Therefore a domain of reference must be found; since the only part that is changing is the thickness of the channel it suffices to make a change of variable on such region.

Let $x = (\widetilde{x}, x_N) \in \Omega_2^{\epsilon}$, define $x_N = \epsilon\, z$ and notice $\frac{\partial}{\partial x_N} = \frac{1}{\epsilon}\frac{\partial}{\partial z}$ and for any $\mathbf{w} \in \mathbf{V}_2$ notice the changes on the structure of the gradient and divergence respectively:

$$\boldsymbol{\nabla}\,\mathbf{w}\;(\widetilde{x}, x_N) = \begin{pmatrix} [\,\boldsymbol{\nabla}_T\,\mathbf{w}_T] & \epsilon^{-1}\,\partial_z \mathbf{w}_{\mathbf{T}} \\ (\,\boldsymbol{\nabla}_T\,\mathbf{w}_N\,)' & \epsilon^{-1}\,\partial_z \mathbf{w}_N \end{pmatrix} (\widetilde{x}, z) \tag{4.1.12}$$

$$\boldsymbol{\nabla}\cdot\mathbf{w}\,(\widetilde{x}, x_N) = \left(\boldsymbol{\nabla}_T \cdot\ \mathbf{w}_T + \frac{1}{\epsilon}\,\partial_z \mathbf{w}_N\right)(\widetilde{x}, z) \tag{4.1.13}$$

Taking in consideration (4.1.12), (4.1.13) and combining it with (4.1.7) we conclude a family of ϵ-problems in a domain of definition given by $\Omega \equiv \Omega_1 \cup \Omega_2$, where $\Omega_1, \Omega_2 \subseteq \mathbb{R}^3$

bounded open sets, with $\Omega_2 \equiv \Gamma \times (0,1)$, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2 \subseteq \mathbb{R}^2$. The test spaces are fixed and given by:

$$\mathbf{V}_1 \equiv \left\{\mathbf{v} \in \mathbf{L}^{\mathbf{2}}_{\mathbf{div}}(\Omega_1) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_1 - \Gamma\right\}$$

$$\mathbf{V}_2 \equiv \left\{\mathbf{v} \in \mathbf{H}^{\mathbf{1}}(\Omega_2) : \mathbf{v} = 0 \text{ on } \partial\Gamma \times (0,1),\;\; \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma + 1\right\}$$

$$\mathbf{X} \equiv \left\{[\,\mathbf{v}^1, \mathbf{v}^2\,] \in \mathbf{V}_1 \times \mathbf{V}_2 : \mathbf{v}^1\cdot\mathbf{n} = \mathbf{v}^2\cdot\mathbf{n} \text{ on } \Gamma\right\}$$

$$\mathbf{Y} \equiv L^2_0(\Omega_1) \times L^2_0(\Omega_2)$$

Where $\Gamma + 1 \equiv \{(\widetilde{x}, 1) : \widetilde{x} \in \Gamma\}$. The problem (4.1.7) in this common domain of reference is given by:

$$\text{Find } [\mathbf{v}^{\epsilon}, p^{\epsilon}] \in \mathbf{X} \times \mathbf{Y} \text{ such that :}$$

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx \\
&\qquad - \epsilon\int_{\Omega_2} p^{2,\epsilon}\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x}\,dz - \int_{\Omega_2} p^{2,\epsilon}\,\partial_z\mathbf{w}^2_N\,d\widetilde{x}\,dz \\
&\quad + \epsilon^2\int_{\Omega_2} \mu\boldsymbol{\nabla}_T\,\mathbf{v}^{2,\epsilon}_T : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x}\,dz + \int_{\Omega_2} \mu\,\partial_z\mathbf{v}^{2,\epsilon}_T\cdot\partial_z\mathbf{w_T}^2\,d\widetilde{x}\,dz \\
&\quad + \epsilon^2\int_{\Omega_2} \mu\,\boldsymbol{\nabla}_T\mathbf{v}^{2,\epsilon}_N\cdot\boldsymbol{\nabla}_T\mathbf{w}^2_N\,d\widetilde{x}\,dz + \int_{\Omega_2} \mu\,\partial_z\mathbf{v}^{2,\epsilon}_N\,\partial_z\mathbf{w}^2_N\,d\widetilde{x}\,dz \\
&\quad + \alpha\int_{\Gamma}\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS + \epsilon^2\int_{\Gamma}\gamma\sqrt{\mathcal{Q}}\,\mathbf{v}^{2,\epsilon}_T\cdot\mathbf{w_T}^2\,dS \\
&\qquad\qquad = \epsilon\int_{\Omega_2}\mathbf{f}^{2,\epsilon}\cdot\mathbf{w}^2\,d\widetilde{x}\,dz
\end{aligned} \tag{4.1.14a}$$

$$\begin{aligned}
&\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\varphi^1\,dx + \epsilon\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}^{2,\epsilon}_T\,\varphi^2\,d\widetilde{x}\,dz \\
&\qquad + \int_{\Omega_2}\partial_z\mathbf{v}^{2,\epsilon}_N\,\varphi^2\,d\widetilde{x}\,dz = \int_{\Omega_1} h^{1,\epsilon}\,\varphi^1\,dx.
\end{aligned} \tag{4.1.14b}$$

$$\text{for all } [\mathbf{w}, \Phi] \in \mathbf{X} \times \mathbf{Y}$$

4.1.8 The Strong Problem on the Domain of Reference

After integrating by parts and recovering boundary and interface conditions, the problem (4.1.14) is the weak solution of the following strong problem:

$$Q\,\mathbf{v}^{1,\epsilon} + \boldsymbol{\nabla}p^{1,\epsilon} = 0\,, \tag{4.1.15a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon} = h^{1,\epsilon} \quad \text{in} \quad \Omega_1 \tag{4.1.15b}$$

$$\epsilon\, \boldsymbol{\nabla}_T p^{2,\epsilon} - \epsilon^2\, \boldsymbol{\nabla}_T \cdot \mu \boldsymbol{\nabla}_T\, \mathbf{v}_T^{2,\epsilon} - \partial_z\, \mu\, \partial_z\, \mathbf{v}_T^{2,\epsilon} = \epsilon\, \mathbf{f}_T^{2,\epsilon}\,, \tag{4.1.15c}$$

$$\partial_z\, p^{2,\epsilon} - \epsilon^2\, \boldsymbol{\nabla}_T \cdot \mu \boldsymbol{\nabla}_T \mathbf{v}_N^{2,\epsilon} - \partial_z\, \mu\, \partial_z \mathbf{v}_N^{2,\epsilon} = \epsilon\, \mathbf{f}_N^{2,\epsilon}\,, \tag{4.1.15d}$$

$$\epsilon\, \boldsymbol{\nabla}_T \cdot \mathbf{v}_T^{2,\epsilon} + \partial_z \mathbf{v}_N^{2,\epsilon} = 0 \quad \text{in} \quad \Omega_2 \tag{4.1.15e}$$

$$\epsilon\, \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon} - p^{2,\epsilon} + p^{1,\epsilon} = \alpha\, \mathbf{v}^{1,\epsilon} \cdot \mathbf{n}\,, \tag{4.1.15f}$$

$$\epsilon\, \mu\, \frac{\partial\, \mathbf{v}_\tau^{2,\epsilon}}{\partial\, \mathbf{n}} = \epsilon\, \mu\, \partial_z\, \mathbf{v}_T^{2,\epsilon} = \epsilon^2 \gamma\, \sqrt{Q}\, \mathbf{v}_T^{2,\epsilon}\,, \tag{4.1.15g}$$

$$\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} = \mathbf{v}^{2,\epsilon} \cdot \mathbf{n} \quad \text{on} \quad \Gamma \tag{4.1.15h}$$

$$\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega_1 - \Gamma \tag{4.1.15i}$$

$$\mathbf{v}^{2,\epsilon} = 0 \quad \text{on} \quad \partial\Gamma \times (0,1) \tag{4.1.15j}$$

$$\mathbf{v}^{2,\epsilon} \cdot \mathbf{n} = \mathbf{v}_N^{2,\epsilon} = 0\,, \tag{4.1.15k}$$

$$\mu\, \frac{\partial\, \mathbf{v}_\tau^{2,\epsilon}}{\partial\, \mathbf{n}} = \mu\, \partial_z\, \mathbf{v}_T^{2,\epsilon} = 0 \quad \text{on} \quad \Gamma + 1 \tag{4.1.15l}$$

4.2 The a-priori Estimates

In order to compute the a-priori estimates test (4.1.14a) with $\mathbf{w} = \mathbf{v}^\epsilon$ and (4.1.14b) with $\Phi = \mathbf{p}^\epsilon$ and add them together in order to get rid of the terms that are not necessarily positive (mixed terms) to end up with:

$$\begin{aligned} &\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^{1,\epsilon} \cdot \mathbf{v}^{1,\epsilon}\, dx \\ &\quad + \int_{\Omega_2} \mu \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_T^{2,\epsilon} \right) : \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_T^{2,\epsilon} \right)\, d\widetilde{x}\, dz + \int_{\Omega_2} \mu\, \partial_z \mathbf{v}_T^{2,\epsilon} \cdot \partial_z \mathbf{v}_T^{2,\epsilon}\, d\widetilde{x}\, dz \\ &\quad + \epsilon^2 \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \mathbf{v}_N^{2,\epsilon} \cdot \boldsymbol{\nabla}_T \mathbf{v}_N^{2,\epsilon}\, d\widetilde{x}\, dz + \int_{\Omega_2} \mu\, \partial_z \mathbf{v}_N^{2,\epsilon}\, \partial_z \mathbf{v}_N^{2,\epsilon}\, d\widetilde{x}\, dz \\ &\quad + \alpha \int_\Gamma \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} \right) \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} \right) dS + \int_\Gamma \epsilon^2\, \gamma \sqrt{\mathcal{Q}}\, \mathbf{v}_T^{2,\epsilon} \cdot \mathbf{v}_T^{2,\epsilon}\, dS \\ &\quad = \epsilon \int_{\Omega_2} \mathbf{f}^{2,\epsilon} \cdot \mathbf{v}^{2,\epsilon}\, d\widetilde{x}\, dz + \int_{\Omega_1} h^{1,\epsilon}\, p^{1,\epsilon}\, dx \end{aligned} \tag{4.2.16}$$

Assume for the respective coefficients to be positive and bounded from below and above, and for the tensors involved to be uniformly elliptic, assume $\alpha > 0$. Finally, apply the Cauchy-Schwartz inequality to the right hand side and conclude the following estimate:

$$\begin{aligned}
\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1}^2 + \left\| \boldsymbol{\nabla}_T \left(\epsilon \, \mathbf{v}_T^{2,\epsilon} \right) \right\|_{0,\Omega_2}^2 + \left\| \partial_z \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Omega_2}^2 \\
+ \left\| \epsilon \, \boldsymbol{\nabla}_T \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Omega_2}^2 + \left\| \partial_z \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Omega_2}^2 + \left\| \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Gamma}^2 + \left\| \epsilon \, \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Gamma}^2 \\
\leq \frac{1}{k} \left(\| \mathbf{f}^{2,\epsilon} \|_{0,\Omega_2} \| \epsilon \, \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2} + \int_{\Omega_1} h^{1,\epsilon} p^{1,\epsilon} \, dx \right) \qquad (4.2.17)
\end{aligned}$$

The summand involving an integral needs a special treatment in order to get the a-priori estimate.

$$\begin{aligned}
\int_{\Omega_1} h^{1,\epsilon} p^{1,\epsilon} \, dx \leq \| p^{1,\epsilon} \|_{0,\Omega_1} \| h^{1,\epsilon} \|_{0,\Omega_1} \leq C \, \| \boldsymbol{\nabla} p^{1,\epsilon} \|_{0,\Omega_1} \| h^{1,\epsilon} \|_{0,\Omega_1} \\
= \| \mathcal{Q} \, \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \| h^{1,\epsilon} \|_{0,\Omega_1} \leq \widetilde{C} \, \| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \qquad (4.2.18)
\end{aligned}$$

The second inequality holds since $\int_{\Omega_1} p^{1,\epsilon} \, dx = 0$ the equality is due to (4.1.15a) and the third inequality because the tensor $\mathcal{Q}$ is bounded and so is the family of sources $\{ h^{1,\epsilon} : \epsilon > 0 \} \subset L^2(\Omega_1)$.

Poincare-type inequalities

We need to control the $L^2(\Omega_2)$-norm of $\mathbf{v}^{2,\epsilon}$. For this we use the fundamental theorem of calculus and the trace on the interface Γ, we have

$$\| \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2} \leq \| \partial_z \, \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2} + 2 \, \| \mathbf{v}^{2,\epsilon} \|_{0,\Gamma} \qquad (4.2.19)$$

Combining (4.2.18) and (4.2.19) in (4.2.17) we have:

$$\begin{aligned}
\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1}^2 + \left\| \boldsymbol{\nabla}_T \left(\epsilon \, \mathbf{v}_T^{2,\epsilon} \right)) \right\|_{0,\Omega_2}^2 + \left\| \partial_z \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Omega_2}^2 \\
+ \left\| \epsilon \, \boldsymbol{\nabla}_T \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Omega_2}^2 + \left\| \partial_z \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Omega_2}^2 + \left\| \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Gamma}^2 + \left\| \epsilon \, \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Gamma}^2 \\
\leq C \left[\| \mathbf{f}^{2,\epsilon} \|_{0,\Omega_2} \left(\| \partial_z \, (\epsilon \, \mathbf{v}^{2,\epsilon}) \|_{0,\Omega_2} + 2 \, \| (\epsilon \, \mathbf{v}^{2,\epsilon}) \|_{0,\Gamma} \right) + \widetilde{C} \, \| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \right] \\
\leq \widehat{C} \left(\left\| \partial_z \, \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Omega_2} + \| \partial_z \, \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2} + \left\| \epsilon \, \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Gamma} \right. \\
\left. + \left\| \mathbf{v}_N^{2,\epsilon} \right\|_{0,\Gamma} + \| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \right) \qquad (4.2.20)
\end{aligned}$$

Using the equivalence of norms $\|\cdot\|_1, \|\cdot\|_2$ for 5-D vectors.

$$
\begin{aligned}
&\| \mathbf{v}^{1,\epsilon} \|^2_{0,\Omega_1} + \left\| \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}^{2,\epsilon}_T \right)) \right\|^2_{0,\Omega_2} + \left\| \partial_z \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Omega_2} \\
&\qquad + \left\| \epsilon\, \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Omega_2} + \left\| \partial_z \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Omega_2} + \left\| \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Gamma} + \left\| \epsilon\, \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Gamma} \\
&\leq C' \left\{ \left\| \partial_z\, \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Omega_2} + \| \partial_z\, \mathbf{v}^{2,\epsilon} \|^2_{0,\Omega_2} + \left\| \epsilon\, \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Gamma} + \left\| \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Gamma} + \| \mathbf{v}^{1,\epsilon} \|^2_{0,\Omega_1} \right\}^{1/2} \\
&\quad \leq C \left\{ \| \mathbf{v}^{1,\epsilon} \|^2_{0,\Omega_1} + \left\| \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}^{2,\epsilon}_T \right) \right\|^2_{0,\Omega_2} + \left\| \partial_z \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Omega_2} \right. \\
&\qquad \left. + \left\| \epsilon\, \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Omega_2} + \left\| \partial_z \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Omega_2} + \left\| \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Gamma} + \left\| \epsilon\, \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Gamma} \right\}^{1/2}
\end{aligned}
\tag{4.2.21}
$$

The expression above implies, there must exist a constant K such that:

$$
\begin{aligned}
&\| \mathbf{v}^{1,\epsilon} \|^2_{0,\Omega_1} + \left\| \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}^{2,\epsilon}_T \right) \right\|^2_{0,\Omega_2} + \left\| \partial_z \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Omega_2} \\
&\qquad + \left\| \epsilon\, \boldsymbol{\nabla}_T \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Omega_2} + \left\| \partial_z \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Omega_2} + \left\| \mathbf{v}^{2,\epsilon}_N \right\|^2_{0,\Gamma} + \left\| \epsilon\, \mathbf{v}^{2,\epsilon}_T \right\|^2_{0,\Gamma} \leq K
\end{aligned}
\tag{4.2.22}
$$

Where K is an adequate positive constant.

4.3 Convergence Statements

4.3.1 Convergence of the Velocities

Due to the general a-priori estimate (4.2.22) we assure there exists a subsequence, still denoted $\{\mathbf{v}^\epsilon : \epsilon > 0\}$ and $\mathbf{v} \in \mathbf{L^2}(\Omega)$ such that:

$$\mathbf{v}^{1,\epsilon} \to \mathbf{v}^1 \quad \text{weakly} \quad \text{in} \quad \mathbf{L}^2(\Omega_1) \tag{4.3.23}$$

$$\epsilon\, \mathbf{v}^{2,\epsilon} \to \mathbf{v}^2 \quad \text{weakly} \quad \text{in} \quad \mathbf{H}^1(\Omega_2) \quad \text{strongly} \quad \text{in} \quad \mathbf{L}^2(\Omega_2) \tag{4.3.24}$$

$$\boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}^{2,\epsilon}_T \right) \to \boldsymbol{\nabla}_T\, \mathbf{v}^2_T \quad \text{weakly} \quad \text{in} \quad \mathbf{L}^2(\Omega_2) \tag{4.3.25}$$

We also identify bounded higher order terms. There must exist $\chi \in L^2(\Omega_2)$, $\eta \in L^2(\Omega_2) \times L^2(\Omega_2)$ such that:

$$\partial_z \mathbf{v}^{2,\epsilon}_N \rightharpoonup \chi\,, \; \left\| \partial_z \left(\epsilon\, \mathbf{v}^{2,\epsilon}_N \right) \right\|_{0,\Omega_2} \to 0 \tag{4.3.26}$$

$$\partial_z \mathbf{v}^{2,\epsilon}_T \rightharpoonup \eta, \; \left\| \partial_z \left(\epsilon\, \mathbf{v}^{2,\epsilon}_T \right) \right\|_{0,\Omega_2} \to 0 \tag{4.3.27}$$

Where the weak convergence is in the sense of $L^2(\Omega_2)$. Due to (4.2.19) and (4.2.22) we know there exists $\xi \in L^2(\Omega_2)$ such that:

$$\mathbf{v}_N^{2,\epsilon} \rightharpoonup \xi\,, \ \left\|\epsilon\, \mathbf{v}_N^{2,\epsilon}\right\|_{0,\Omega_2} \to 0 \tag{4.3.28}$$

4.3.2 Convergence of $\mathbf{v}^{2,\epsilon}$

Due to (4.2.22) and (4.2.19) we can consider the sequence $\left\{\epsilon\, \mathbf{v}_T^{2,\epsilon}\Big|_\Gamma : \epsilon > 0\right\} \subset L^2(\Gamma) \times L^2(\Gamma)$ as a bounded sequence, then we have that $\epsilon\, \mathbf{v}_T^{2,\epsilon}\Big|_\Gamma \rightharpoonup \mathbf{v}_T^2\big|_\Gamma$ in $L^2(\Gamma) \times L^2(\Gamma)$. On the other hand due to (4.3.27) we have $\partial_z\, \mathbf{v}^2 = 0$, *i.e*

$$\mathbf{v}_T^2 = \mathbf{v}_T^2\,(\widetilde{x}) \tag{4.3.29}$$

Since the traces are continuous applications on this space and using (4.3.29) we conclude $\mathbf{v}_T^2 = \mathbf{v}_T^2\,|_\Gamma \in H_0^1(\Gamma) \times H_0^1(\Gamma)$.

4.3.3 Convergence and agreement of $\mathbf{v}_N^{2,\epsilon}$

Finally we show the agreement $\partial_z\, \xi = \chi$ and $\xi\,|_\Gamma = \mathbf{v}^1 \cdot \mathbf{n}$. For this, consider the Hilbert space

$$H(\partial_z, \Omega_2) \equiv \{\, u \in L^2(\Omega_2) : \partial_z\, u \in\, L^2(\Omega_2)\} \tag{4.3.30}$$

$$\langle\, u, v\,\rangle_{H(\partial_z,\Omega_2)} \equiv \int_{\Omega_2} (u\, v + \partial_z\, u\, \partial_z\, v\,)\, dx \tag{4.3.31}$$

Due to the estimates the sequence $\left\{\mathbf{v}_N^{2,\epsilon} : \epsilon > 0\right\}$ can not be considered as a bounded sequence of the space $H^1(\Omega_2)$ because we do not have estimates in $L^2(\Omega_2) \times L^2(\Omega_2)$ for $\left\{\boldsymbol{\nabla}_T\, \mathbf{v}_N^{2,\epsilon} : \epsilon > 0\right\}$. However we can still consider $\left\{\mathbf{v}_N^{2,\epsilon} : \epsilon > 0\right\} \subseteq H(\partial_z, \Omega_2)$ as a bounded sequence due to the estimates; then we conclude $\partial_z\, \xi = \chi$. Matter of fact the limiting problem will be modeled on this space in the normal direction (vertical direction in this particular case). On the other hand since the trace application $v \mapsto v\,|_\Gamma$ is well-defined and continuous in this space we conclude $\xi\,|_\Gamma = \mathbf{v}^1 \cdot \mathbf{n}\,|_\Gamma$. Finally we write:

$$\mathbf{v}_N^{2,\epsilon} \overset{w}{\rightharpoonup} \xi\,, \quad \text{in } H(\partial_z, \Omega_2) \tag{4.3.32}$$

Moreover, since $\mathbf{v}_N^{2,\epsilon}(\widetilde{x},1)=0$ and the trace on the faces $\Gamma, \Gamma+1$ are continuous in the space $H(\partial_z, \Omega_2)$ we conclude:

$$\xi\,(\widetilde{x},1)=0 \tag{4.3.33}$$

4.3.4 Convergence of $p^{1,\epsilon}$

We know from (4.1.15a) that:

$$\|\,\boldsymbol{\nabla}\, p^{1,\epsilon}\,\|_{0,\Omega_1} = \left\|\,\sqrt{Q}\,\mathbf{v}^{1,\epsilon}\,\right\|_{0,\Omega_1} \leq C$$

The positive constant on the inequality above comes from (4.2.22). Since $\int_{\Omega_1} p^{1,\epsilon}\,dx=0$ we know there exists a constant $\widetilde{C}>0$ such that:

$$\|\,p^{1,\epsilon}\,\|_{1,\Omega_1} \leq \widetilde{C}\,\|\,\boldsymbol{\nabla}\,p^{1,\epsilon}\,\|_{0,\Omega_1}$$

Combining the two inequalities above we conclude there must exist $p^1\in H^1(\Omega_1)\cap L_0^2(\Omega_1)$ such that:

$$p^{1,\epsilon}\to p^1 \quad \text{weakly in} \quad H^1(\Omega_1), \quad \text{strongly in} \quad L_0^2(\Omega_1) \tag{4.3.34}$$

In particular holds:

$$p^{1,\epsilon}\to p^1 \quad \text{weakly on} \quad L^2(\Gamma) \tag{4.3.35}$$

Notice that the fact that $\int_{\Omega_1} p^1\,dx=0$ does not imply $\int_\Gamma \gamma\,(p^1)\,d\widetilde{x}=0$. Therefore $\gamma\,(p^1)|_\Gamma$ will not belong necessarily to $L_0^2(\Gamma)$.

We turn now our attention to the convergence of the pressures on Ω_2

4.3.5 Convergence of $p^{2,\epsilon}$ in $L^2(\Omega_2)$

Take any $\phi\in C_0^\infty(\Omega_2)$, and define now a new function:

$$\varsigma(\widetilde{x},z)\equiv\int_z^1 \phi(\widetilde{x},t)\,dt \tag{4.3.36}$$

By construction we know $\|\,\varsigma\,\|_{1,\Omega_2}\leq C\|\,\phi\,\|_{0,\Omega_2}$. We know there must exist a function $\mathbf{w}^1\in\mathbf{L^2_{div}}\,(\Omega_1)$ such that $\mathbf{w}^1\cdot\mathbf{n}=\mathbf{w}^2\cdot\mathbf{n}=\varsigma(\widetilde{x},0)=\int_0^1\phi\,(\widetilde{x},t)\,dt$ on Γ and $\mathbf{w}^1\cdot\mathbf{n}=0$ on

$\partial\,\Omega_1 - \Gamma$ and $\|\,\mathbf{w}^1\,\|_{\mathbf{L^2_{div}}(\Omega_1)} \leq \|\,\varsigma\,\|_{0,\Gamma} \leq C\|\,\phi\,\|_{0,\Omega_2}$. Take the function $\mathbf{w}^2 = (0_T, \varsigma(\widetilde{x}, z))$, then $[\mathbf{w}^1, \mathbf{w}^2] \in \mathbf{X}$. Test, (4.1.14a) with this test function to end up with:

$$
\begin{gathered}
\int_{\Omega_1} Q\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx \\
+\alpha\int_{\Gamma}\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS + \int_{\Omega_2} p^{2,\epsilon}\,\phi(\widetilde{x},z)\,d\widetilde{x}\,dz \\
+\epsilon^2\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\,\mathbf{v}_N^{2,\epsilon}\cdot\,\boldsymbol{\nabla}_T\,\varsigma(\widetilde{x},z)\,d\widetilde{x}\,dz - \int_{\Omega_2}\mu\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\,\phi(\widetilde{x},z)\,d\widetilde{x}\,dz \\
= \epsilon\int_{\Omega_2}\mathbf{f}_N^{2,\epsilon}\,\varsigma\,d\widetilde{x}\,dz
\end{gathered}
\tag{4.3.37}
$$

From here we conclude the following inequality an applying the Cauchy-Schwarz inequality to the integrals we have

$$
\begin{gathered}
\left|\int_{\Omega_2} p^{2,\epsilon}\,\phi(\widetilde{x},z)\,d\widetilde{x}\,dz\right| \leq C_1\,\left\|\,\mathbf{v}^{1,\epsilon}\,\right\|_{0,\Omega_1}\left\|\,\mathbf{w}^1\,\right\|_{0,\Omega_1} + \left\|\,p^{1,\epsilon}\,\right\|_{0,\Omega_1}\left\|\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,\right\|_{0,\Omega_1} \\
+ C_2\,\left\|\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\,\right\|_{0,\Gamma}\|\,\varsigma\,\|_{0,\Gamma} + \epsilon\,C_3\,\left\|\,\boldsymbol{\nabla}_T\,\left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right)\right\|_{0,\Omega_2}\|\,\boldsymbol{\nabla}_T\,\varsigma(\widetilde{x},z)\,\|_{0,\Omega_2} \\
+ C_4\,\left\|\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\,\right\|_{0,\Omega_2}\|\,\phi\,\|_{0,\Omega_2} + \left\|\,\epsilon\,\mathbf{f}_N^{2,\epsilon}\,\right\|_{0,\Omega_2}\|\,\varsigma\,\|_{0,\Omega_2}
\end{gathered}
$$

we notice all the norms depending on $\mathbf{w}^1$ and ς due to the construction are controlled by the norm $\|\,\phi\,\|_{0,\Omega_2}$. Therefore, the above expression can be reduced to

$$
\begin{gathered}
\left|\int_{\Omega_2} p^{2,\epsilon}\,\phi(\widetilde{x},z)\,d\widetilde{x}\,dz\right| \leq C\,\Big(\left\|\,\mathbf{v}^{1,\epsilon}\,\right\|_{0,\Omega_1} + \left\|\,p^{1,\epsilon}\,\right\|_{0,\Omega_1} + \left\|\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\,\right\|_{0,\Gamma} \\
+\epsilon\,\left\|\,\boldsymbol{\nabla}_T\,\left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right)\right\|_{0,\Omega_2} + \left\|\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\,\right\|_{0,\Omega_2} + \left\|\,\epsilon\,\mathbf{f}_N^{2,\epsilon}\,\right\|_{0,\Omega_2}\Big)\,\|\,\phi\,\|_{0,\Omega_2} \leq \widetilde{C}\,\|\,\phi\,\|_{0,\Omega_2}
\end{gathered}
$$

The last inequality holds since all the summands in the parenthesis are bounded due to (4.2.22) and (4.3.34) and we assume the forcing term is bounded. Taking upper limit when $\epsilon \to 0$, on the above expression we have:

$$
\limsup_{\epsilon\downarrow 0}\left|\int_{\Omega_2} p^{2,\epsilon}\,\phi(\widetilde{x},z)\,d\widetilde{x}\,dz\right| \leq \widetilde{C}\,\|\,\phi\,\|_{0,\Omega_2} \tag{4.3.38}
$$

Since the above holds for any $\phi \in C_0^\infty(\Omega_2)$ we conclude that the sequence $\left\{\,p^{2,\epsilon} : \epsilon > 0\,\right\} \subset L^2(\Omega_2)$ is bounded and therefore it has a weakly convergent subsequence, *i.e.* there exists $p^2 \in L_0^2(\Omega_2)$ such that $p^{2,\epsilon} \overset{w}{\rightharpoonup} p^2$ in $L_0^2(\Omega_2)$.

4.3.6 Behavior of the Normal Stress Interface Condition

We want to analyze the normal interface condition in the limit. We have already shown that $\left\{ p^{2,\epsilon} : \epsilon > 0 \right\} \subset L^2(\Omega_2)$ is weakly convergent, therefore in (4.3.37) take limit as $\epsilon \downarrow 0$ to get

$$\int_{\Omega_1} Q\,\mathbf{v}^1 \cdot \mathbf{w}^1\, dx - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1\, dx + \alpha \int_\Gamma \xi \left(\mathbf{w}^1 \cdot \mathbf{n} \right) dS + \int_{\Omega_2} p^2\, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz - \int_{\Omega_2} \mu\, \partial_z\, \xi\, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz = 0$$

integrating by parts the second summand and using (4.1.15a) we get

$$-\int_\Gamma p^1\, \left(\mathbf{w}^1 \cdot \mathbf{n} \right) + \alpha \int_\Gamma \xi \left(\mathbf{w}^1 \cdot \mathbf{n} \right) dS + \int_{\Omega_2} p^2\, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz - \int_{\Omega_2} \mu\, \partial_z\, \xi\, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz = 0$$

recalling by construction that $\mathbf{w}^1 \cdot \mathbf{n}\,|_\Gamma = \int_0^1 \phi\,(\widetilde{x}, z)\, dz$ the above expression becomes

$$-\int_\Gamma p^1\,|_\Gamma\, \left(\int_0^1 \phi(\widetilde{x}, t)\, dt \right) d\widetilde{x} + \alpha \int_\Gamma \xi\,|_\Gamma\, \left(\int_0^1 \phi(\widetilde{x}, t)\, dt \right) d\widetilde{x} + \int_{\Omega_2} p^2\, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz - \int_{\Omega_2} \mu\, \partial_z\, \xi\, \phi(\widetilde{x}, z)\, d\widetilde{x}\, dz = 0$$

Since the above holds for any $\phi \in C_0^\infty(\Omega_2)$ and $\xi\,|_\Gamma$, $p^1\,|_\Gamma$ can be understood as functions in Ω_2 extended to the whole domain as constant with respect to z we conclude:

$$-p^1\,|_\Gamma + \alpha\, \xi\,|_\Gamma + p^2 - \mu\, \partial_z\, \xi = 0 \quad \text{in} \quad L^2(\Omega_2) \tag{4.3.39}$$

4.3.7 Convergence of the Equation Terms

We want to show the agreement between $\mathbf{v}^1 \cdot \mathbf{n}$ and $\xi\,|_\Gamma$. Take any $u \in H^1(\Omega_1)$ consider the identity:

$$-\int_{\Omega_1} \mathbf{v}^{1,\epsilon} \cdot \boldsymbol{\nabla} u\, dx + \langle \mathbf{v}^{1,\epsilon} \cdot \mathbf{n}, u \rangle_{H^{-1/2}(\Omega_1), H^{-1/2}(\Omega_1)} = \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon}\, u\, dx \tag{4.3.40}$$

In this specific problem we know $\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} = \mathbf{v}^{2,\epsilon} \cdot \mathbf{n}$ on Γ and $\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} = 0$ on $\partial\Omega_1 - \Gamma$ then, the identity above transforms in:

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon}\, u\, dx = -\int_{\Omega_1} \mathbf{v}^{1,\epsilon} \cdot \boldsymbol{\nabla} u\, dx + \int_\Gamma \mathbf{v}_N^{2,\epsilon}\, u\, dS \tag{4.3.41}$$

From (4.3.23) and (4.3.32) we know both terms on the right hand side converge, therefore the sequence $\{\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}:\epsilon>0\}$ is weakly convergent in $L^2(\Omega_1)$, this says the sequence $\{\mathbf{v}^{1,\epsilon}:\epsilon>0\}$ is weakly convergent in $\mathbf{L^2_{div}}(\Omega_1)$, moreover, from (4.3.28) we conclude:

$$\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^1\,u\,dx=-\int_{\Omega_1}\mathbf{v}^1\cdot\boldsymbol{\nabla}u\,dx+\int_{\Gamma}(\,\xi\,|_{\Gamma}\,)\;u\,dS \tag{4.3.42}$$

Then we conclude

$$\mathbf{v}^1\cdot\mathbf{n}=\xi\,|_{\Gamma} \tag{4.3.43}$$

Now we can rewrite (4.3.39) as

$$p^2=\mu\,\partial_z\,\xi-\alpha\,\mathbf{v}^1\cdot\mathbf{n}|_{\Gamma}+p^1\,|_{\Gamma} \tag{4.3.44}$$

Consider now a test function of the structure $\Phi=[\,0,\varphi^2\,]\in\mathbf{Y}$, test (4.1.14b) and let $\epsilon\to0$, we have:

$$\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}_T^2\,\varphi^2\,d\widetilde{x}\,dz+\int_{\Omega_2}\partial_z\,\xi\,\varphi^2\,d\widetilde{x}\,dz=0$$

i.e.

$$\boldsymbol{\nabla}_T\cdot\mathbf{v}_T^2+\partial_z\,\xi\;=c$$

Where c is a constant, we know $\mathbf{v}_T^2=\mathbf{v}_T^2(\widetilde{x})$ then we conclude

$$\partial_z\,\xi=\partial_z\,\xi\,(\widetilde{x}) \tag{4.3.45}$$

combining this with (4.3.44) we conclude:

$$p^2=p^2(\widetilde{x}) \tag{4.3.46}$$

Finally, we notice that in (4.1.14b) in the quantifier $[\,\mathbf{w}^1,\mathbf{w}^2]\in\mathbf{Y}$ since the tangential and normal effects are not coupled then we can replace $\mathbf{w}^2=[\mathbf{w_T}^2,\mathbf{w}_N^2]$, by $[\epsilon^{-1}\,\mathbf{w_T}^2,\mathbf{w}_N^2]$ and rewrite (4.1.14) as:

$$\text{Find }\;[\,\mathbf{v}^{\epsilon},p^{\epsilon}\,]\in\mathbf{X}\times\mathbf{Y}\;\text{such that :}$$

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx$$
$$- \int_{\Omega_2} p^{2,\epsilon}\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x}\,dz - \int_{\Omega_2} p^{2,\epsilon}\,\partial_z\mathbf{w}_N^2\,d\widetilde{x}\,dz$$
$$+ \int_{\Omega_2} \mu\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right) : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x}\,dz + \frac{1}{\epsilon^2}\int_{\Omega_2}\mu\,\partial_z\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\cdot\partial_z\mathbf{w_T}^2\,d\widetilde{x}\,dz$$
$$+ \epsilon\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right)\cdot\boldsymbol{\nabla}_T\mathbf{w}_N^2\,d\widetilde{x}\,dz + \int_{\Omega_2}\mu\,\partial_z\mathbf{v}_N^{2,\epsilon}\,\partial_z\mathbf{w}_N^2\,d\widetilde{x}\,dz$$
$$+ \alpha\int_\Gamma\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS + \int_\Gamma\gamma\sqrt{\mathcal{Q}}\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\cdot\mathbf{w}_T^2\,dS$$
$$= \epsilon\int_{\Omega_2}\mathbf{f}^{2,\epsilon}\cdot\mathbf{w}^2\,d\widetilde{x}\,dz \quad (4.3.47a)$$

$$\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\varphi^1\,dx + \epsilon\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}_T^{2,\epsilon}\,\varphi^2\,d\widetilde{x}\,dz$$
$$+ \int_{\Omega_2}\partial_z\mathbf{v}_N^{2,\epsilon}\,\varphi^2\,d\widetilde{x}\,dz = \int_{\Omega_1}h^{1,\epsilon}\,\varphi^1\,dx. \quad (4.3.47b)$$

for all $[\mathbf{w},\Phi]\in\mathbf{X}\times\mathbf{Y}$

4.4 The Limiting Problem

Consider now the subspaces $\mathbf{W}\subseteq\mathbf{X}$, $\Lambda\subseteq\mathbf{Y}$ defined as follows:

$$\mathbf{W}\equiv\{(\mathbf{w}^1,\mathbf{w}^2)\in\mathbf{X} : \mathbf{w}_T^2 = \mathbf{w}_T^2(\widetilde{x}),\ \partial_z\,\mathbf{w}_N^2 = \partial_z\,\mathbf{w}_N^2(\widetilde{x})\}$$

$$\Lambda\equiv\{(\varphi^1,\varphi^2)\in\mathbf{Y} : \varphi^2 = \varphi^2(\widetilde{x})\}$$

Now test the problem (4.3.47) with a function of the structure $[\mathbf{w},\Phi]\in\mathbf{W}\times\Lambda$, we have:

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx$$
$$- \int_{\Omega_2} p^{2,\epsilon}\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x}\,dz - \int_{\Omega_2} p^{2,\epsilon}\,\partial_z\,\mathbf{w}_N^2\,d\widetilde{x}\,dz$$
$$+ \int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right) : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x}\,dz + \epsilon\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right)\cdot\boldsymbol{\nabla}_T\,\mathbf{v}_N^2\,d\widetilde{x}\,dz$$
$$+ \int_{\Omega_2}\mu\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\,\partial_z\mathbf{w}_N^2\,d\widetilde{x}\,dz + \alpha\int_\Gamma\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{w}^1\cdot\mathbf{n}\right)dS$$
$$+ \int_\Gamma\gamma\sqrt{\mathcal{Q}}\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\cdot\mathbf{w_T}^2\,dS = \int_{\Omega_2}\mathbf{f}_T^{2,\epsilon}\cdot\mathbf{w_T}^2\,d\widetilde{x}\,dz + \epsilon\int_{\Omega_2}\mathbf{f}_N^{2,\epsilon}\cdot\mathbf{w}_N^2\,d\widetilde{x}\,dz \quad (4.4.48a)$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot \mathbf{v}^{1,\epsilon}\,\varphi^1\,dx + \int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \Big(\epsilon\, \mathbf{v}_T^{2,\epsilon}\Big)\,\varphi^2\,d\widetilde{x}\,dz \\ + \int_{\Omega_2} \partial_z \mathbf{v}_N^{2,\epsilon}\,\varphi^2\,d\widetilde{x}\,dz = \int_{\Omega_1} h^{1,\epsilon}\,\varphi^1\,dx \quad (4.4.48b)$$

Now let $\epsilon \downarrow 0$ to end up with:

$$\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx \\ - \int_{\Omega_2} p^2\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x}\,dz - \int_{\Omega_2} p^2\,\partial_z\,\mathbf{w}_N^2\,d\widetilde{x}\,dz \\ + \int_{\Omega_2} \mu\,\boldsymbol{\nabla}_T\,\mathbf{v}_T^2 : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x}\,dz + \int_{\Omega_2} \mu\,\partial_z\,\xi\,\partial_z\mathbf{w}_N^2\,d\widetilde{x}\,dz \\ + \alpha\int_{\Gamma}\big(\mathbf{v}^1\cdot\mathbf{n}\big)\,\big(\mathbf{w}^1\cdot\mathbf{n}\big)\,dS + \int_{\Gamma}\gamma\sqrt{\mathcal{Q}}\,\mathbf{v}_T^2\cdot\mathbf{w_T}^2\,dS \\ = \int_{\Omega_2}\mathbf{f}_T^2\,\cdot\,\mathbf{w_T}^2\,d\widetilde{x}\,dz \quad (4.4.49a)$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^1\varphi^1\,dx + \int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\big(\mathbf{v}_T^2\big)\,\varphi^2\,d\widetilde{x}\,dz \\ + \int_{\Omega_2}\partial_z\xi\,\varphi^2\,d\widetilde{x}\,dz = \int_{\Omega_1} h^1\,\varphi^1\,dx \quad (4.4.49b)$$

Notice that the limit $[\,\mathbf{v},p\,]\notin\mathbf{W}\times\Lambda$ since we have not proved $\mathbf{v}_N^2\in H^1(\Omega_2)$. Then, we need to consider a setting where the limiting solution belongs to and the above variational formulation makes sense. Define the following spaces:

$$\mathbf{V}\equiv\{\,\mathbf{w}^2=[\,\mathbf{w_T}^2,\mathbf{w}_N^2\,]:\mathbf{w_T}^2\in\big(H^1(\Omega_2)\big)^2,\,\mathbf{w_T}^2=\mathbf{w_T}^2(\widetilde{x}), \\ \mathbf{w_T}^2=0\,\text{on}\,\partial\Gamma,\,\mathbf{w}_N^2\in H(\partial_z,\Omega_2)\,,\ \partial_z\mathbf{w}_N^2=\partial_z\,\mathbf{w}_N^2(\widetilde{x}\,)\,,\,\mathbf{w}_N^2(\widetilde{x},1)=0\,\} \quad (4.4.50)$$

$$\|\,\mathbf{w}^2\,\|_{\mathbf{V}}=\Big(\,\|\,\mathbf{w_T}^2\,\|_{1,\Omega_2}^2+\|\,\mathbf{w}_N^2\,\|_{H(\partial_z,\Omega_2)}^2\Big)^{1/2} \quad (4.4.51)$$

$$\mathbf{Z}\equiv\Big\{\,(\mathbf{w}^1,\mathbf{w}^2)\in\mathbf{L}^2_{div}(\Omega_1)\times\mathbf{V}:\mathbf{w}^1\cdot\mathbf{n}=\mathbf{w}_N^2=\mathbf{w}^2\cdot\mathbf{n}\,\text{on}\,\Gamma\,\Big\} \quad (4.4.52)$$

Clearly $\mathbf{W}\subseteq\mathbf{Z}$ and we have the following result:

Lemma 4.4.1. $\mathbf{W}$ *is dense in* $\mathbf{Z}$.

Proof. Let $G\subseteq\mathbb{R}^N$ be an open set, we know from the theory that for every $g\in H^{-1/2}(\partial G)$ there exists a function $\mathbf{u}\in\mathbf{L^2_{div}}(G)$ such that $\mathbf{u}\cdot\mathbf{n}=g$ and $\|\,\mathbf{u}\,\|_{\mathbf{L^2_{div}}(G)}\leq$

$K\| g \|_{H^{1/2}(\partial G)}$. In particular if $g \in L^2(\partial G)$ the function $\mathbf{u}$ meets the estimate $\| \mathbf{u} \|_{\mathbf{L^2_{div}}(G)} \leq K\| g \|_{0, \partial G}$ with K depending only on the domain G.

Consider now an element $\mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2) \in \mathbf{Z}$, then $\mathbf{w}^2 = (\mathbf{w_T}^2, \mathbf{w}_N^2) \in \mathbf{V}$ with $\mathbf{w}_N^2 \in H(\partial_z, \Omega_2)$ and it is completely defined by its trace on the interface Γ. For $\epsilon > 0$ take $\varpi \in H_0^1(\Gamma)$ such that $\| \varpi - \mathbf{w}_N^2|_\Gamma \|_{L^2(\Gamma)} \leq \epsilon$. Now extend the function to the whole domain by $\varrho(\widetilde{x}, z) \equiv \varpi(\widetilde{x})(1 - z)$, then $\| \varrho - \mathbf{w}_N^2 \|_{H(\partial_z, \Omega_2)} \leq \epsilon$. The function $(\mathbf{w_T}^2, \varrho)$ clearly belongs to $\mathbf{W}$. By construction of ϱ we know $\| \varrho|_\Gamma - \mathbf{w}_N^2|_\Gamma \|_{0,\Gamma} = \| \varpi - \mathbf{w}_N^2|_\Gamma \|_{0,\Gamma} \leq \epsilon$. Define $g = \varrho|_\Gamma - \mathbf{w}_N^2|_\Gamma \in L^2(\Gamma)$ and find $\mathbf{u} \in \mathbf{L^2_{div}}(\Omega_1)$ such that $\mathbf{u} \cdot \mathbf{n} = g$ and $\| \mathbf{u} \|_{\mathbf{L}^2_{div}(\Omega_1)} \leq C_1 \| g \|_{0,\Gamma}$. Then, the function $\mathbf{w}^1 + \mathbf{u}$ is such that $(\mathbf{w}^1 + \mathbf{u}) \cdot \mathbf{n} = \mathbf{w}^1 \cdot \mathbf{n} + \varpi - \mathbf{w}_N^2 = \varpi$ and $\| \mathbf{w}^1 + \mathbf{u} - \mathbf{w}^1 \|_{\mathbf{L^2_{div}}(\Omega_1)} = \|\mathbf{u} \|_{\mathbf{L}^2_{div}(\Omega_1)} \leq C_1 \| g \|_{0,\Gamma} \leq C_1 \epsilon$. Moreover, we notice the function $(\mathbf{w}^1 + \mathbf{u}, [\mathbf{w_T}^2, \varrho]) \in \mathbf{W}$ and due to the previous it holds:

$$\| \mathbf{w} - (\mathbf{w}^1 + \mathbf{u}, [\mathbf{w_T}^2, \varrho]) \|_{\mathbf{Z}} = \| (\mathbf{w}^1, \mathbf{w}^2) - (\mathbf{w}^1 + \mathbf{u}, [\mathbf{w_T}^2, \varrho]) \|_{\mathbf{Z}} \leq C_1 \epsilon$$

The involved constants were dependent only on the domains Ω_1, Ω_2, then we conclude $\mathbf{W}$ is dense in $\mathbf{Z}$ as desired. □

Define now the function $\mathbf{v} = (\mathbf{v}^1, [\mathbf{v}_T^2, \xi])$, then it is clear that the limit $[\mathbf{v}, p] \in \mathbf{Z} \times \Lambda$ and the variational statement (4.4.49) holds true for all $[\mathbf{w}, \Phi] \in \mathbf{W} \times \Lambda$. Since the bilinear forms are continuous with respect to the space $\mathbf{Z} \times \Lambda$ we can extend the above variational statement by density to all test functions $[\mathbf{w}, \Phi] \in \mathbf{Z} \times \Lambda$ and formulate the problem as follows

$$\text{Find } [\mathbf{v}, p] \in \mathbf{Z} \times \Lambda$$

$$\begin{aligned}
\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 \, dx - \int_{\Omega_1} p^1 \, \boldsymbol{\nabla} \cdot \mathbf{w}^1 \, dx \\
- \int_{\Omega_2} p^2 \, \boldsymbol{\nabla} \cdot \mathbf{w}^2 \, d\widetilde{x}\, dz + \int_{\Omega_2} \mu \, \boldsymbol{\nabla}_T \, \mathbf{v}_T^2 : \boldsymbol{\nabla}_T \, \mathbf{w_T}^2 \, d\widetilde{x}\, dz \\
+ \int_{\Omega_2} \mu \, (\partial_z \xi) \, (\partial_z \mathbf{w}_N^2) \, d\widetilde{x}\, dz + \alpha \int_\Gamma (\mathbf{v}^1 \cdot \mathbf{n}) \, (\mathbf{w}^1 \cdot \mathbf{n}) \, dS \\
+ \int_\Gamma \gamma \sqrt{\mathcal{Q}} \, \mathbf{v}_T^2 \cdot \mathbf{w}_T^2 \, dS = \int_{\Omega_2} \mathbf{f}_T^2 \cdot \mathbf{w}_T^2 \, d\widetilde{x}\, dz \quad \text{(4.4.53a)}
\end{aligned}$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot \mathbf{v}^1 \varphi^1\, dx + \int_{\Omega_2} \boldsymbol{\nabla}\cdot [\, \mathbf{v}_T^2,\, \xi\,]\, \varphi^2\, d\widetilde{x}\, dz = \int_{\Omega_1} h^1\, \varphi^1\, dx \tag{4.4.53b}$$

$$\text{for all } [\,\mathbf{w}, \Phi\,] \in \mathbf{Z} \times \Lambda$$

The above problem is well-posed in mixed formulation. Define the forms

$$A = \begin{pmatrix} Q + \gamma_n' \,\alpha\, \gamma_n & 0 \\ 0 & [\, \gamma_T' \,\gamma\, \sqrt{Q}\, \gamma_T + (\boldsymbol{\nabla}_T)' \,\mu\, \boldsymbol{\nabla}_T, (\partial_z)' \,\mu\, \partial_z \,] \end{pmatrix} \tag{4.4.54}$$

$$B = \begin{pmatrix} \boldsymbol{\nabla}\cdot & 0 \\ 0 & \boldsymbol{\nabla}\cdot \end{pmatrix} = \begin{pmatrix} \text{div} & 0 \\ 0 & \text{div} \end{pmatrix} \tag{4.4.55}$$

And the resolvent system is obtained in the form

$$[\,\mathbf{v}, p\,] \in \mathbf{Z} \times \Lambda : A\,\mathbf{v} - B'\,p = \mathbf{f} \tag{4.4.56a}$$

$$B\,\mathbf{v} = \mathbf{h} \tag{4.4.56b}$$

Lemma 4.4.2. *The operator A is $\mathbf{Z}$-coercive over $\mathbf{Z} \cap Ker(B)$.*

Proof. The form $A\mathbf{v}\,(\mathbf{v}) + \int_{\Omega_1} (\boldsymbol{\nabla}\cdot \mathbf{v})^2$ is $\mathbf{Z}$-coercive, and $\boldsymbol{\nabla}\cdot \mathbf{v}\,|_{\Omega_1} = 0$ if $\mathbf{v} \in \text{Ker}(B)$. □

Lemma 4.4.3. *B has closed range.*

Proof. For an open domain $G \subseteq \mathbb{R}^N$ it is a well-known fact that for any $\varphi \in L_0^2(G)$ there exists $\mathbf{u} \in (\, H_0^1(G)\,)^N$ such that:

$$\boldsymbol{\nabla}\cdot \mathbf{u} = \varphi$$

$$\|\,\mathbf{u}\,\|_{1,G} \leq c\,\|\,\varphi\,\|_{0,G}$$

where the constant $c > 0$ depends only on the domain G.

Now choose $\Phi = [\,\varphi^1,\, \varphi^2\,] \in \Lambda$, due to the previous result and since $\varphi^2 = \varphi^2(\widetilde{x}\,) \in L^2(\Gamma)$ we know there exist a couple of functions $\mathbf{u}^1 \in \mathbf{H}_0^1(\Omega_1)$ and $\mathbf{u}_T^2 \in H_0^1(\Gamma) \times H_0^1(\Gamma)$ such that $\boldsymbol{\nabla}\cdot \mathbf{u}^1 = \varphi^1$, $\boldsymbol{\nabla}_T \cdot \mathbf{u}_T^2 = \varphi^2\,|_\Gamma$, and $\|\,\mathbf{u}^1\,\|_{1,\Omega_1} \leq c_1\,\|\,\varphi^1\,\|_{0,\Omega_1}$, $\|\,\mathbf{u}_T^2\|_{1,\Gamma} \leq c_2\,\|\,\varphi^2\,|_\Gamma\,\|_{0,\Gamma} = c_2\,\|\,\varphi^2\,\|_{0,\Omega_2}$. Extend the function $\mathbf{u}_T^2$ to $H_0^1(\Omega_2) \times H_0^1(\Omega_2)$ in the trivial way and denote it in the same way. Consider the function $\mathbf{u} = \big(\mathbf{u}^1, [\,\mathbf{u}_T^2, 0\,]\big)$, clearly this function belongs to the space $\mathbf{Z}$ and

$$\|\,\mathbf{u}\,\|_{\mathbf{Z}} \leq C\left(\|\,\mathbf{u}^1\,\|_{1,\Omega_1}^2 + \|\,\mathbf{u}_T^2\,\|_{1,\Omega_2}^2\right)^{1/2} \leq \widetilde{C}\left(\|\,\varphi^1\,\|_{0,\Omega_1}^2 + \|\,\varphi^2\,\|_{0,\Omega_2}^2\right)^{1/2}$$

Where $\widetilde{C}$ depends on the domains Ω_1, Γ and the equivalence of norms for 2-D vectors, and it is independent from $\Phi \in \Lambda$.

Consider now the following inequalities

$$\begin{aligned}
\sup_{\mathbf{w}\in\mathbf{Z}} \frac{\int_\Omega \Phi\,\boldsymbol{\nabla}\cdot\mathbf{w}\,dx}{\|\,\mathbf{w}\,\|_{\mathbf{Z}}} &\geq \frac{\int_{\Omega_1}\varphi^1\,\boldsymbol{\nabla}\cdot\mathbf{u}^1\,dx + \int_{\Omega_2}\varphi^2\,\boldsymbol{\nabla}_T\cdot\mathbf{u}_T^2\,d\widetilde{x}\,dz}{\|\,\mathbf{u}\,\|_{\mathbf{Z}}} \\
&\geq \frac{1}{\widetilde{C}}\,\frac{\|\,\varphi^1\,\|_{0,\Omega_1}^2 + \|\,\varphi^2\,\|_{0,\Omega_2}^2}{(\|\,\varphi^1\,\|_{0,\Omega_1}^2 + \|\,\varphi^2\,\|_{0,\Omega_2}^2)^{1/2}} \\
&= \frac{1}{\widetilde{C}}\,\left(\|\,\varphi^1\,\|_{0,\Omega_1}^2 + \|\,\varphi^2\,\|_{0,\Omega_2}^2\right)^{1/2} = \frac{1}{\widetilde{C}}\,\|\,\Phi\,\|_{0,\Omega}\,, \quad \forall\,\Phi\in\Lambda
\end{aligned}$$

□

From the two lemmas above and from theory of problems in mixed formulation [GR79b] we know (4.4.56) is well-posed.

4.4.1 Dimensional Reduction

Notice that the space $\mathbf{V}$ has the property $\partial_z\,\mathbf{w}_N^2 = \partial_z\,\mathbf{w}_N^2\,(\widetilde{x}) = -\mathbf{w}_N^2(\widetilde{x}, 0)$, since $\mathbf{w}_N^2(\widetilde{x}, 1) = 0$. Consider now the statement (4.4.53) written as:

$$\text{Find } [\,\mathbf{v}, p\,] \in \mathbf{Z}\times\Lambda$$

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx \\
&\qquad - \int_{\Omega_2} p^2\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x}\,dz - \int_{\Omega_2} p^2\,\partial_z\,\mathbf{w}_N^2\,d\widetilde{x}\,dz \\
&\qquad + \int_{\Omega_2} \mu\,\boldsymbol{\nabla}_T\,\mathbf{v}_T^2 : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x}\,dz \\
&\qquad + \int_{\Omega_2} \mu\,(\,\partial_z\,\xi\,)\,\left(\,\partial_z\mathbf{w}_N^2\,\right)\,d\widetilde{x}\,dz + \alpha\int_\Gamma \left(\,\mathbf{v}^1\cdot\mathbf{n}\,\right)\,\left(\,\mathbf{w}^1\cdot\mathbf{n}\,\right)\,dS \\
&\qquad + \int_\Gamma \gamma\sqrt{\mathcal{Q}}\;\mathbf{v}_T^2\,\cdot\mathbf{w}_T^2\,dS = \int_{\Omega_2} \mathbf{f}_T^2\,\cdot\,\mathbf{w_T}^2\,d\widetilde{x}\,dz
\end{aligned}$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^1\varphi^1\,dx + \int_{\Omega_2} \boldsymbol{\nabla}_T\cdot\,\mathbf{v}_T^2\,\varphi^2\,d\widetilde{x}\,dz + \int_{\Omega_2} \partial_z\,\xi\,\varphi^2\,d\widetilde{x}\,dz = \int_{\Omega_1} h^1\,\varphi^1\,dx$$

$$\text{for all } [\,\mathbf{w},\Phi\,] \in \mathbf{Z}\times\Lambda$$

introducing the previous observation in the statement above and simplifying the integrals for the functions not dependent on the variable z we have the formulation:

$$\text{Find } [\,\mathbf{v},p\,] \in \mathbf{Z}\times\Lambda$$

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx \\
&\qquad - \int_\Gamma p^2\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x} + \int_\Gamma p^2\,\mathbf{w}_N^2(\widetilde{x},0)\,d\widetilde{x} \\
&\qquad + \int_\Gamma \mu\,\boldsymbol{\nabla}_T\,\mathbf{v}_T^2 : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x} \\
&\qquad + \int_\Gamma \mu\,\xi(\widetilde{x},0)\,\mathbf{w}_N^2(\widetilde{x},0)\,d\widetilde{x} + \alpha\int_\Gamma \big(\mathbf{v}^1\cdot\mathbf{n}\big)\,\big(\mathbf{w}^1\cdot\mathbf{n}\big)\,dS \\
&\qquad + \int_\Gamma \gamma\sqrt{\mathcal{Q}}\,\mathbf{v}_T^2\,\cdot\mathbf{w}_T^2\,dS = \int_\Gamma \mathbf{f}_T^2\,\cdot\,\mathbf{w_T}^2\,d\widetilde{x}\,,
\end{aligned}$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^1\,\varphi^1\,dx + \int_\Gamma \boldsymbol{\nabla}_T\cdot\,\mathbf{v}_T^2\,\varphi^2\,d\widetilde{x} - \int_\Gamma \xi(\widetilde{x},0)\,\varphi^2\,d\widetilde{x} = \int_{\Omega_1} h^1\,\varphi^1\,dx$$

$$\text{for all } [\,\mathbf{w},\Phi\,] \in \mathbf{Z}\times\Lambda$$

Taking in consideration (4.3.43) and recalling that $\mathbf{w}_N^2(\widetilde{x},0) = \mathbf{w}^2\cdot\mathbf{n} = \mathbf{w}^1\cdot\mathbf{n}$ we rewrite the problem as:

$$\text{Find } [\,\mathbf{v},p\,] \in \mathbf{Z}\times\Lambda$$

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1\,dx - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,dx + (\,\mu+\alpha\,)\int_\Gamma \big(\mathbf{v}^1\cdot\mathbf{n}\big)\,\big(\mathbf{w}^1\cdot\mathbf{n}\big)\,d\widetilde{x} \\
&\qquad + \int_\Gamma p^2\,\big(\mathbf{w}^1\cdot\mathbf{n}\big)\,d\widetilde{x} - \int_\Gamma p^2\,\boldsymbol{\nabla}_T\cdot\mathbf{w_T}^2\,d\widetilde{x} \\
&\qquad + \int_\Gamma \mu\,\boldsymbol{\nabla}_T\,\mathbf{v}_T^2 : \boldsymbol{\nabla}_T\,\mathbf{w_T}^2\,d\widetilde{x} + \int_\Gamma \gamma\sqrt{\mathcal{Q}}\,\mathbf{v}_T^2\,\cdot\mathbf{w}_T^2\,dS = \int_\Gamma \mathbf{f}_T^2\,\cdot\,\mathbf{w_T}^2\,d\widetilde{x}
\end{aligned} \tag{4.4.59a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla}\cdot\mathbf{v}^1\,\varphi^1\,dx - \int_\Gamma \big(\mathbf{v}^1\cdot\mathbf{n}\big)\,\varphi^2\,d\widetilde{x} + \int_\Gamma \boldsymbol{\nabla}_T\cdot\,\mathbf{v}_T^2\,\varphi^2\,d\widetilde{x} = \int_{\Omega_1} h^1\,\varphi^1\,dx \tag{4.4.59b}$$

$$\text{for all } [\,\mathbf{w},\Phi\,] \in \mathbf{Z}\times\Lambda$$

Consider now the space

$$\mathbf{H} \equiv \Big\{ \mathbf{w}^2 = [\,\mathbf{w_T}^2, \mathbf{w}_N^2\,] : \mathbf{w_T}^2 \in \big(H^1(\Gamma) \big)^2 ,\ \mathbf{w_T}^2 = 0 \,\text{on}\, \partial\Gamma, \mathbf{w}_N^2 \in L^2(\Gamma) \Big\} \tag{4.4.60}$$

$$\|\, \mathbf{w}^2 \,\|_{\mathbf{H}} \equiv \big(\|\, \mathbf{w_T}^2 \,\|_{1,\Gamma}^2 + \|\, \mathbf{w}_N^2 \,\|_{0,\Gamma}^2 \,\big)^{1/2} \tag{4.4.61}$$

Notice that the space **V** is isomorphic to the space **H** and the application is given by

$$\iota : \mathbf{V} \to \mathbf{H}$$

$$\mathbf{w}^2 = [\,\mathbf{w_T}^2, \mathbf{w}_N^2\,] \mapsto [\,\mathbf{w_T}^2, \mathbf{w}_N^2(\widetilde{x}, 0)\,]$$

Define now the space **E** as follows

$$\mathbf{E} \equiv \big\{\, (\mathbf{w}^1, \mathbf{w}^2) \in \mathbf{L}^{\mathbf{2}}_{\mathbf{div}}(\Omega_1) \times \mathbf{H} : \mathbf{w}^1 \cdot \mathbf{n} = \mathbf{w}_N^2 = \mathbf{w}^2 \cdot \mathbf{n} \;\text{on}\, \Gamma \,\big\} \tag{4.4.62}$$

Then it is clear that the spaces **Z** and **E** are isomorphic too. Finally, for the pressures define the space

$$\Pi \equiv L_0^2(\Omega_1) \times L_0^2(\Gamma) \tag{4.4.63}$$

Clearly the spaces Π and Λ are isomorphic. Thus we can rewrite the statement (4.4.59) as follows:

$$\text{Find } [\,\mathbf{v}, p\,] \in \mathbf{E} \times \Pi$$

$$\begin{aligned} \int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 \, dx - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 \, dx + (\,\mu + \alpha\,) \int_\Gamma \big(\mathbf{v}^1 \cdot \mathbf{n} \big) \big(\mathbf{w}^1 \cdot \mathbf{n} \big)\, d\widetilde{x} \\ + \int_\Gamma p^2 \, \big(\mathbf{w}^1 \cdot \mathbf{n} \big)\, d\widetilde{x} - \int_\Gamma p^2\, \boldsymbol{\nabla}_T \cdot \mathbf{w_T}^2 \, d\widetilde{x} \\ + \int_\Gamma \gamma \sqrt{\mathcal{Q}}\; \mathbf{v}_T^2 \cdot \mathbf{w_T}^2 \, d\widetilde{x} + \int_\Gamma \mu\, \boldsymbol{\nabla}_T\, \mathbf{v}_T^2 : \boldsymbol{\nabla}_T\, \mathbf{w_T}^2 \, d\widetilde{x} = \int_\Gamma \mathbf{f}_T^2 \cdot \mathbf{w_T}^2 \, d\widetilde{x} \end{aligned} \tag{4.4.64a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1\, \varphi^1 \, dx - \int_\Gamma \big(\mathbf{v}^1 \cdot \mathbf{n} \big)\, \varphi^2 \, d\widetilde{x} + \int_\Gamma \boldsymbol{\nabla}_T \cdot \mathbf{v}_T^2\, \varphi^2 \, d\widetilde{x} = \int_{\Omega_1} h^1\, \varphi^1 \, dx \tag{4.4.64b}$$

$$\text{for all } [\,\mathbf{w}, \Phi\,] \in \mathbf{E} \times \Pi$$

We show now this problem is well-posed in mixed formulation.

Well-Posedeness of Reduced Limiting Problem

Consider the forms

$$A = \begin{pmatrix} Q + \gamma_n' \,(\,\mu + \alpha\,)\,\gamma_n & 0 \\ 0 & \gamma\,\sqrt{Q} + \,(\boldsymbol{\nabla}_T)'\,\mu\,\boldsymbol{\nabla}_T \end{pmatrix} \tag{4.4.65}$$

$$B = \begin{pmatrix} \boldsymbol{\nabla}\cdot & 0 \\ -\gamma_n & \boldsymbol{\nabla}_T\cdot \end{pmatrix} = \begin{pmatrix} \mathbf{div} & 0 \\ -\gamma_n & \mathbf{div_T} \end{pmatrix} \tag{4.4.66}$$

And the resolvent system is obtained in the form

$$[\,\mathbf{v}, p\,] \in \mathbf{E} \times \Pi : A\,\mathbf{v} - B'\,p = \mathbf{f} \tag{4.4.67a}$$

$$B\,\mathbf{v} = \mathbf{h} \tag{4.4.67b}$$

Lemma 4.4.4. *The operator A is $\mathbf{E}$-coercive over $\mathbf{E} \cap Ker(B)$.*

Proof. The form $A\mathbf{v}\,(\mathbf{v}) + \int_{\Omega_1} (\boldsymbol{\nabla}\cdot\mathbf{v})^2$ is $\mathbf{E}$-coercive, and $\boldsymbol{\nabla}\cdot\mathbf{v}\,|_{\Omega_1} = 0$ if $\mathbf{v} \in \mathrm{Ker}(B)$. □

Lemma 4.4.5. *B has closed range.*

Proof. For an open domain $G \subseteq \mathbb{R}^N$ it is a well-known fact that for any $\varphi \in L^2_0(G)$ there exists $\mathbf{u} \in (\,H^1_0(G)\,)^N$ such that:

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \varphi$$

$$\|\,\mathbf{u}\,\|_{1,G} \leq \; c\,\|\,\varphi\,\|_{0,G}$$

where the constant $c > 0$ depends only on the domain G.

Now choose $\Phi = [\,\varphi^1,\,\varphi^2\,] \in \Pi$, due to the previous result and since $\varphi^2 = \varphi^2(\widetilde{x}\,) \in L^2(\Gamma)$ we know there exist a couple of functions $\mathbf{u}^1 \in \mathbf{H}^1_0\,(\Omega_1)$ and $\mathbf{u}^2_T \in H^1_0\,(\Gamma) \times H^1_0\,(\Gamma)$ such that $\boldsymbol{\nabla}\cdot\mathbf{u}^1 = \varphi^1$, $\boldsymbol{\nabla}_T\cdot\mathbf{u}^2_T = \varphi^2$, and $\|\,\mathbf{u}^1\,\|_{1,\Omega_1} \leq\, c_1\,\|\,\varphi^1\,\|_{0,\Omega_1}$, $\|\,\mathbf{u}^2_T\,\|_{1,\Gamma} \leq c_2\,\|\,\varphi^2\,\|_{0,\Gamma}$. Consider the function $\mathbf{u} = \big(\mathbf{u}^1, [\,\mathbf{u}^2_T, 0\,]\big)$, clearly this function belongs to the space $\mathbf{E}$ and $\|\,\mathbf{u}\,\|_{\mathbf{E}} \leq C\left(\|\,\mathbf{u}^1\,\|^2_{1,\Omega_1} + \|\,\mathbf{u}^2_T\,\|^2_{1,\Gamma}\right)^{1/2} \leq \widetilde{C}\left(\|\,\varphi^1\,\|^2_{0,\Omega_1} + \|\,\varphi^2\,\|^2_{0,\Gamma}\right)^{1/2}$. Where $\widetilde{C}$ depends on the domains Ω_1, Γ and the equivalence of norms for 2-D vectors, and it is independent from $\Phi \in \Pi$.

Consider now the following inequalities

$$\sup_{\mathbf{w}\in\mathbf{E}} \frac{\int_\Omega \Phi\, \boldsymbol{\nabla}\cdot\mathbf{w}\, dx}{\|\,\mathbf{w}\,\|_{\mathbf{E}}} \geq \frac{\int_{\Omega_1} \varphi^1\, \boldsymbol{\nabla}\cdot\mathbf{u}^1\, dx + \int_\Gamma \varphi^2\, \boldsymbol{\nabla}_T\cdot\mathbf{u}_T^2\, d\widetilde{x}\, dz}{\|\,\mathbf{u}\,\|_{\mathbf{E}}}$$

$$\geq \frac{1}{\widetilde{C}}\, \frac{\|\,\varphi^1\,\|^2_{0,\Omega_1} + \|\,\varphi^2\,\|^2_{0,\Gamma}}{(\|\,\varphi^1\,\|^2_{0,\Omega_1} + \|\,\varphi^2\,\|^2_{0,\Gamma})^{1/2}}$$

$$= \frac{1}{\widetilde{C}}\left(\|\,\varphi^1\,\|^2_{0,\Omega_1} + \|\,\varphi^2\,\|^2_{0,\Gamma}\right)^{1/2} = \frac{1}{\widetilde{C}}\,\|\,\Phi\,\|_{0,\Omega}\,, \quad \forall\,\Phi\in\Pi$$

□

From the two lemmas above and from theory of problems in mixed formulation [GR79b] we know (4.4.56) is well-posed.

4.4.2 The Strong Reduced Limiting Problem

After integrating by parts and recovering boundary and interface conditions the problem (4.4.64) is the weak solution of:

$$Q\,\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0\,, \tag{4.4.68a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v}^1 = h^1 \quad \text{in} \quad \Omega_1 \tag{4.4.68b}$$

$$\boldsymbol{\nabla}_T p^2 + \gamma\sqrt{Q}\,\mathbf{v}_T^2 - (\boldsymbol{\nabla}_T)'\,\mu\,\boldsymbol{\nabla}_T\left(\mathbf{v}_T^2\right) = \mathbf{f}_T^2\,, \tag{4.4.68c}$$

$$\boldsymbol{\nabla}_T\cdot\mathbf{v}_T^2 - \mathbf{v}^1\cdot\mathbf{n} = 0\,, \tag{4.4.68d}$$

$$p^1 - p^2 = (\,\mu+\alpha\,)\,\mathbf{v}^1\cdot\mathbf{n} \quad \text{in} \quad \Gamma \tag{4.4.68e}$$

$$p^1 = 0 \quad \text{on} \quad \partial\Omega_1 \tag{4.4.68f}$$

$$\mathbf{v}_T^2 = 0 \quad \text{on} \quad \partial\Gamma \tag{4.4.68g}$$

4.5 Strong Convergence of the Solutions

Assume the extra hypothesis $\left\|\,\mathbf{f}^{\,2,\epsilon} - \mathbf{f}^{\,2}\,\right\|_{0,\Omega} \to 0$, $\left\|\,h^{\,1,\epsilon} - h^1\,\right\|_{0,\Omega} \to 0$. Test (4.1.14a) with $\mathbf{w} = \mathbf{v}^{\,\epsilon}$ and (4.1.14b) with $\Phi = p^{\,\epsilon}$ and add them together in order to get rid of the terms that are not necessarily positive (mixed terms) to end up with:

$$
\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{v}^{1,\epsilon}\,dx + \int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right):\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\,d\widetilde{x}\,dz\\
&\quad+\int_{\Omega_2}\mu\,\partial_z\mathbf{v}_T^{2,\epsilon}\cdot\partial_z\mathbf{v}_T^{2,\epsilon}\,d\widetilde{x}\,dz+\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right)\cdot\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_N^{2,\epsilon}\right)\,d\widetilde{x}\,dz\\
&\qquad+\int_{\Omega_2}\mu\,\partial_z\mathbf{v}_N^{2,\epsilon}\,\partial_z\mathbf{v}_N^{2,\epsilon}\,d\widetilde{x}\,dz\\
&\quad+\alpha\int_\Gamma\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)\left(\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\right)dS+\int_\Gamma\gamma\sqrt{\mathcal{Q}}\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\cdot\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)dS\\
&\qquad\qquad=\int_{\Omega_2}\mathbf{f}^{2,\epsilon}\cdot\left(\epsilon\,\mathbf{v}^{2,\epsilon}\right)d\widetilde{x}\,dz+\int_{\Omega_1}h^{1,\epsilon}p^{1,\epsilon}\,dx
\end{aligned}
$$

We realize the right hand side of the expression above converges and so does the left hand side. However we only control some terms, then taking lim sup we have:

$$
\begin{aligned}
&\limsup_{\epsilon\downarrow 0}\{\,\|\sqrt{Q}\,\mathbf{v}^{1,\epsilon}\|_{0,\Omega_1}^2+\|\sqrt{\mu}\,\boldsymbol{\nabla}_T\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\|_{0,\Omega_2}^2\\
&\quad+\|\sqrt{\mu}\,\partial_z\,\mathbf{v}_N^{2,\epsilon}\|_{0,\Omega_2}^2+\|\sqrt{\alpha}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{n}\|_{0,\Gamma}^2+\|\sqrt{\gamma}\,\sqrt[4]{Q}\left(\epsilon\,\mathbf{v}_T^{2,\epsilon}\right)\|_{0,\Gamma}^2\\
&\qquad+\|\sqrt{\mu}\left(\epsilon\,\boldsymbol{\nabla}_T\,\mathbf{v}_N^{2,\epsilon}\right)\|_{0,\Omega_2}^2+\|\sqrt{\mu}\left(\partial_z\,\mathbf{v}_T^{2,\epsilon}\right)\|_{0,\Omega_2}^2\}\\
&\qquad\qquad\le\int_{\Omega_2}\mathbf{f}^2\cdot\mathbf{v}^2\,d\widetilde{x}\,dz+\int_{\Omega_1}h^1\,p^1\,dx
\end{aligned}
\tag{4.5.69}
$$

On the other hand test (4.4.53) on the solution, add both equations together to end up with:

$$
\begin{aligned}
&\|\sqrt{Q}\,\mathbf{v}^1\|_{0,\Omega_1}^2+\|\sqrt{\mu}\,\boldsymbol{\nabla}_T\,(\mathbf{v}_T^2)\|_{0,\Omega_2}^2+\|\sqrt{\mu}\,\partial_z\,\xi\|_{0,\Omega_2}^2\\
&\qquad+\|\sqrt{\alpha}\,\mathbf{v}^1\cdot\mathbf{n}\|_{0,\Gamma}^2+\|\sqrt{\gamma}\,\sqrt[4]{Q}\,\mathbf{v}_T^2\|_{0,\Gamma}^2\\
&\qquad\qquad=\int_{\Omega_2}\mathbf{f}^2\cdot\mathbf{v}^2\,d\widetilde{x}\,dz+\int_{\Omega_1}h^1\,p^1\,dx
\end{aligned}
\tag{4.5.70}
$$

Comparing the left hand side of (4.5.69) and (4.5.70) we conclude one inequality. On the other hand, we know the application

$$
\begin{aligned}
&\mathbf{w}\mapsto\{\,\|\sqrt{\mu}\,\boldsymbol{\nabla}_T\,(\mathbf{w_T}^2)\|_{0,\Omega_2}^2+\|\sqrt{\mu}\,\partial_z\,\mathbf{w_T}^2\|_{0,\Omega_2}^2+\|\sqrt{\mu}\,\partial_z\,\mathbf{w}_N^2\|_{0,\Omega_2}^2\\
&\qquad+\|\sqrt{\alpha}\,\mathbf{w}_N^2\|_{0,\Gamma}^2+\|\sqrt{\gamma}\,\sqrt[4]{Q}\,\mathbf{w_T}^2\|_{0,\Gamma}^2\}^{1/2}\equiv\|\mathbf{w}\|_{\mathbf{V}}
\end{aligned}
\tag{4.5.71}
$$

is a norm in the space $\mathbf{V}$ as defined in (4.4.50). The norm in (4.5.71) is equivalent to the norm

$$\mathbf{w} \mapsto \left\{ \| \boldsymbol{\nabla}_T \mathbf{w_T}{}^2 \|_{0,\Omega_2}^2 + \| \partial_z \mathbf{w_T}{}^2 \|_{0,\Omega_2}^2 + \| \mathbf{w}_N^2 \|_{H(\partial_z,\Omega_2)}^2 \right\}^{1/2} \tag{4.5.72}$$

Due to the convergence statements we have seen that the sequence $\{ [\epsilon \mathbf{v}_T^{2,\epsilon}, \mathbf{v}_N^{2,\epsilon}] \} \subseteq \mathbf{V}$ is bounded and weakly convergent to $[\mathbf{v}_T^2, \xi] \in \mathbf{V} \subseteq \mathbf{M}$ it must hold:

$$\begin{aligned} \| [\mathbf{v}_T^2, \xi] \|_{\mathbf{V}}^2 &\leq \liminf_{\epsilon \downarrow 0} \left\| \left[\epsilon \mathbf{v}_T^{2,\epsilon}, \mathbf{v}_N^{2,\epsilon} \right] \right\|_{\mathbf{V}}^2 \\ &= \liminf_{\epsilon \downarrow 0} \{ \| \sqrt{\mu} \boldsymbol{\nabla}_T (\epsilon \mathbf{v}_T^{2,\epsilon}) \|_{0,\Omega_2}^2 + \| \sqrt{\mu} \partial_z \mathbf{v}_T^{2,\epsilon} \|_{0,\Omega_2}^2 \\ &\quad + \| \sqrt{\mu} \partial_z \mathbf{v}_N^{2,\epsilon} \|_{0,\Omega_2}^2 + \| \sqrt{\alpha} \mathbf{v}_N^{2,\epsilon} \|_{0,\Gamma}^2 + \| \sqrt{\gamma} \sqrt[4]{Q} \mathbf{v}_T^{2,\epsilon} \|_{0,\Gamma}^2 \} \end{aligned} \tag{4.5.73}$$

Since we have seen $\mathbf{v}^{1,\epsilon} \xrightarrow{w} \mathbf{v}^1$ in $\mathbf{L}^{\mathbf{2}}_{\mathbf{div}}(\Omega_1)$ in particular it holds

$$\left\| \mathbf{v}^1 \right\|_{0,\Omega_1}^2 \leq \liminf_{\epsilon \downarrow 0} \left\| \mathbf{v}^{1,\epsilon} \right\|_{0,\Omega_1}^2 \tag{4.5.74}$$

Putting together (4.5.73), (4.5.74), (4.5.69) and (4.5.70) we conclude the norms are convergent *i.e.*:

$$\left\| (\mathbf{v}^1, [\mathbf{v}_T^2, \xi]) \right\|_{\mathbf{L}^2(\Omega_1) \times \mathbf{V}}^2 = \lim_{\epsilon \downarrow 0} \left\| (\mathbf{v}^{1,\epsilon}, [\epsilon \mathbf{v}_T^{2,\epsilon}, \mathbf{v}_N^{2,\epsilon}]) \right\|_{\mathbf{L}^2(\Omega_1) \times \mathbf{V}}^2 \tag{4.5.75}$$

Also since $\left\{ \left(\mathbf{v}^{1,\epsilon}, \left[\epsilon \mathbf{v}_T^{2,\epsilon}, \mathbf{v}_N^{2,\epsilon} \right] \right) : \epsilon > 0 \right\} \subseteq \mathbf{L}^2(\Omega_1) \times \mathbf{V}$ is weakly convergent we conclude the strong convergence of the overall sequence; again using the equivalence of norms we have:

$$\begin{gathered} \left\| \mathbf{v}_T^{2,\epsilon} - \mathbf{v}_T^2 \right\|_{0,\Omega_2} \to 0 \\ \left\| \boldsymbol{\nabla}_T \mathbf{v}_T^{2,\epsilon} - \boldsymbol{\nabla}_T \mathbf{v}_T^2 \right\|_{0,\Omega_2} \to 0 \end{gathered} \tag{4.5.76}$$

$$\left\| \mathbf{v}_N^{2,\epsilon} - \mathbf{v}_N^2 \right\|_{H(\partial_z,\Omega_2)} \to 0 \tag{4.5.77}$$

Finally since $\boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon} = h^{1,\epsilon}$ and the forcing term converges strongly we conclude too:

$$\left\| \mathbf{v}^{1,\epsilon} - \mathbf{v}^1 \right\|_{\mathbf{L}^{\mathbf{2}}_{\mathbf{div}}(\Omega_1)} \to 0 \tag{4.5.78}$$

Strong Convergence of $p^{1,\epsilon}$

Notice that (4.5.78) together with (4.1.15a) implies $\| \boldsymbol{\nabla} p^{1,\epsilon} - \boldsymbol{\nabla} p^{1} \|_{0,\Omega_1} \to 0$. Again, since $\int_{\Omega_1} p^{1,\epsilon}\, dx = 0$, $\int_{\Omega_1} p^{1}\, dx = 0$ we know the gradient controls the $H^1(\Omega_1)$-norm, this implies:

$$\| p^{1,\epsilon} - p^{1} \|_{1,\Omega_1} \to 0 \tag{4.5.79}$$

Strong Convergence of $\boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon} \right)$, $\partial_z\, \mathbf{v}_T^{2,\epsilon}$

Notice too that (4.5.75) together with (4.5.69) imply

$$\begin{aligned} c \lim_{\epsilon \downarrow 0} & \left\{ \left\| \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon} \right) \right\|_{0,\Omega_2}^2 + \left\| \partial_z\, \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Omega_2}^2 \right\} \\ & \leq \lim_{\epsilon \downarrow 0} \left\{ \left\| \mu\, \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}_N^{2,\epsilon} \right) \right\|_{0,\Omega_2}^2 + \left\| \mu\, \partial_z\, \mathbf{v}_T^{2,\epsilon} \right\|_{0,\Omega_2}^2 \right\} = 0 \end{aligned} \tag{4.5.80}$$

with $c > 0$ an ellipticity constant coming from μ.

4.5.1 Strong Convergence of $p^{2,\epsilon}$

In order to show this convergence a previous step of localization of the function $p^{2,\epsilon}$ must be taken, consider then a function $\phi_\epsilon \in C_0^\infty(\Omega_2)$ such that:

$$\| p^{2,\epsilon} - \phi_\epsilon \|_{0,\Omega_2} < \epsilon \tag{4.5.81}$$

Observe the following:

$$\begin{aligned} & \left| \int_{\Omega_2} p^{2,\epsilon}\, p^{2,\epsilon}\, dz\, d\widetilde{x} - \int_{\Omega_2} p^{2,\epsilon}\, \phi_\epsilon\, dz\, d\widetilde{x} \right| \\ & \qquad = \left| \int_{\Omega_2} p^{2,\epsilon} \left(p^{2,\epsilon} - \phi_\epsilon \right)\, dz\, d\widetilde{x} \right| \\ & \qquad \leq \| p^{2,\epsilon} \|_{0,\Omega_2}\, \| p^{2,\epsilon} - \phi_\epsilon \|_{0,\Omega_2} < C\, \epsilon \end{aligned} \tag{4.5.82}$$

The last inequality holds due to (4.3.38). Now, for any $w \in L^2(\Omega_2)$:

$$\begin{aligned} \int_{\Omega_2} \phi_\epsilon\, w\, dz\, d\widetilde{x} & = \int_{\Omega_2} \left(\phi_\epsilon - p^{2,\epsilon} \right)\, w\, dz\, d\widetilde{x} + \int_{\Omega_2} p^{2,\epsilon}\, w\, dz\, d\widetilde{x} \\ & \to 0 + \int_{\Omega_2} p^{2}\, w\, dz\, d\widetilde{x} \end{aligned} \tag{4.5.83}$$

The expression above implies $\phi_\epsilon \overset{w}{\rightharpoonup} p^2$ in $L^2(\Omega_2)$. In particular if we take $w = w(\widetilde{x})$ in the expression above we conclude $\int_{[0,1]} \phi_\epsilon \, dz \overset{w}{\rightharpoonup} p^2$ in $L^2(\Gamma)$.

Now, for ϕ_ϵ define the function ς_ϵ by the same rule as (4.3.36). The function $\varsigma_\epsilon(\widetilde{x}, 0) = \int_{[0,1]} \phi_\epsilon \, dz \in L^2(\Gamma)$. Then, there must exist $\mathbf{w}_\epsilon^1 \in \mathbf{L^2_{div}}(\Omega_1)$ such that $\mathbf{w}_\epsilon^1 \cdot \mathbf{n} = \int_{[0,1]} \phi_\epsilon \, dz$ on Γ, $\mathbf{w}_\epsilon^1 \cdot \mathbf{n} = 0$ on $\partial \Omega_1 - \Gamma$ and $\left\| \mathbf{w}_\epsilon^1 \right\|_{\mathbf{L^2_{div}}(\Omega_1)} \leq \widetilde{C} \, \| \varsigma_\epsilon(\widetilde{x}, 0) \|_\Gamma < C$. Where $\widetilde{C}$ depends only on the domain and by construction $\varsigma_\epsilon|_\Gamma$ is bounded in $L^2(\Gamma)$.

We know the function $[\mathbf{w}_\epsilon^1, \mathbf{w}_\epsilon^2] \in \mathbf{X}$, with $\mathbf{w}_\epsilon^2 = (0_T, \varsigma_\epsilon(\widetilde{x}, z))$. Test, (4.1.14a) with this test function to end up with:

$$\begin{aligned}
\int_{\Omega_1} Q \, \mathbf{v}^{1,\epsilon} \cdot \mathbf{w}_\epsilon^1 \, dx - \int_{\Omega_1} p^{1,\epsilon} \, \boldsymbol{\nabla} \cdot \mathbf{w}_\epsilon^1 \, dx + \alpha \int_\Gamma \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} \right) \left(\mathbf{w}_\epsilon^1 \cdot \mathbf{n} \right) \, d\widetilde{x} \\
+ \int_{\Omega_2} p^{2,\epsilon} \, \phi_\epsilon(\widetilde{x}, z) \, d\widetilde{x} \, dz + \epsilon \int_{\Omega_2} \mu \, \boldsymbol{\nabla}_T \left(\epsilon \, \mathbf{v}_N^{2,\epsilon} \right) \cdot \, \boldsymbol{\nabla}_T \, \varsigma_\epsilon(\widetilde{x}, z) \, d\widetilde{x} \, dz \\
- \int_{\Omega_2} \mu \, \partial_z \, \mathbf{v}_N^{2,\epsilon} \, \phi_\epsilon(\widetilde{x}, z) \, d\widetilde{x} \, dz = \epsilon \int_{\Omega_2} \mathbf{f}_N^{2,\epsilon} \, \varsigma_\epsilon \, d\widetilde{x} \, dz
\end{aligned} \tag{4.5.84}$$

In the expression above all the summands except the fourth are known to be convergent due to the previous strong convergence statements, therefore, this last summand must be convergent too. The first two summands behave as

$$\begin{aligned}
\int_{\Omega_1} Q \, \mathbf{v}^{1,\epsilon} \cdot \mathbf{w}_\epsilon^1 \, dx - \int_{\Omega_1} p^{1,\epsilon} \boldsymbol{\nabla} \cdot \mathbf{w}_\epsilon^1 \, dx &= - \int_\Gamma p^{1,\epsilon} \, \left(\mathbf{w}_\epsilon^1 \cdot \mathbf{n} \right) \, d\widetilde{x} \\
&= - \int_\Gamma p^{1,\epsilon} \left(\int_{[0,1]} \phi_\epsilon \, dz \right) \, d\widetilde{x} \rightarrow - \int_\Gamma p^1 \, p^2 \, d\widetilde{x}
\end{aligned}$$

The limit above holds due to the strong convergence of the pressure in $H^1(\Omega_1)$ and the weak convergence of $\int_{[0,1]} \phi_\epsilon \, dz$. The third summand behaves as

$$\begin{aligned}
&\alpha \int_\Gamma \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} \right) \, \left(\mathbf{w}_\epsilon^1 \cdot \mathbf{n} \right) \, d\widetilde{x} \\
&\qquad = \int_\Gamma \alpha \, \left(\mathbf{v}^{1,\epsilon} \cdot \mathbf{n} \right) \left(\int_{[0,1]} \phi_\epsilon \, dz \right) \, d\widetilde{x} \rightarrow \int_\Gamma \alpha \, \left(\mathbf{v}^1 \cdot \mathbf{n} \right) \, p^2 \, d\widetilde{x}
\end{aligned}$$

And the strong convergence of the velocities and its traces is already proven. We know the fifth summand vanishes due to the estimates. The sixth summand behaves as

$$-\int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon}\, \phi_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz \to -\int_{\Omega_2} \mu\, \partial_z\, \xi\, \Big(wk - \lim_{\epsilon \downarrow 0} \phi_\epsilon(\widetilde{x}, z) \Big)\, d\widetilde{x}\, dz$$
$$= -\int_{\Gamma} \mu\, \partial_z\, \xi\, \Big(\int_0^1 wk - \lim_{\epsilon \downarrow 0} \phi_\epsilon(\widetilde{x}, z)\, dz \Big)\, d\widetilde{x} = -\int_{\Gamma} \mu\, \partial_z\, \xi\, p^2\, d\widetilde{x}$$

The first equality above holds true since $\partial_z\, \xi = \partial_z\, \xi(\widetilde{x})$ and because $\int_0^1 wk - \lim_{\epsilon \downarrow 0} \phi_\epsilon(\widetilde{x}, z)\, dz = wk - \lim_{\epsilon \downarrow 0} \int_0^1 \phi_\epsilon(\widetilde{x}, z)\, dz = p^2$. Finally the right hand side on (4.5.84) vanishes. Putting together all these observations we conclude:

$$\int_{\Omega_2} p^{2,\epsilon}\, \phi_\epsilon(\widetilde{x}, z)\, d\widetilde{x}\, dz \to \int_{\Omega_2} \big(\mu\, \partial_z\, \xi - \alpha\, \mathbf{v}^1 \cdot \mathbf{n}\,|_\Gamma + p^1\,|_\Gamma \big)\; p^2\, d\widetilde{x}$$

This fact together with (4.5.82) and (4.3.44) implies:

$$\|\, p^{2,\epsilon}\, \|_{0,\Omega_2}^2 \to \int_{\Omega_2} \big(\mu\, \partial_z\, \xi - \alpha\, \mathbf{v}^1 \cdot \mathbf{n}\,|_\Gamma + p^1\,|_\Gamma \big)\; p^2\, d\widetilde{x} = \int_{\Gamma} p^2\, p^2\, d\widetilde{x} = \|\, p^2\, \|_{0,\Omega_2}^2$$

again, the convergence of norms together with the weak convergence imply:

$$\big\|\, p^{2,\epsilon} - p^2 \big\|_{0,\Omega_2} \to 0 \tag{4.5.85}$$

4.5.2 Ratio of Velocities

The relationship of the velocity in the tangential direction with respect to the velocity in the normal direction is very high and tends to infinity as expected. We know $\{\, \|\, \mathbf{v}_N^{2,\epsilon}\, \|_{0,\Omega_2} : \epsilon > 0 \,\}$ is bounded, therefore $\|\, \epsilon\, \mathbf{v}_N^{2,\epsilon}\, \|_{0,\Omega_2} = \epsilon\, \|\, \mathbf{v}_N^{2,\epsilon}\, \|_{0,\Omega_2} \to 0$. Suppose first that $\mathbf{v}_T^2 \neq 0$ and consider the quotients:

$$\frac{\|\, \mathbf{v}_T^{2,\epsilon}\, \|_{0,\Omega_2}}{\|\, \mathbf{v}_N^{2,\epsilon}\, \|_{0,\Omega_2}} = \frac{\|\, \epsilon\, \mathbf{v}_T^{2,\epsilon}\, \|_{0,\Omega_2}}{\|\, \epsilon\, \mathbf{v}_N^{2,\epsilon}\, \|_{0,\Omega_2}} > \frac{\|\, \mathbf{v}_T^2\, \|_{0,\Omega_2} - \delta}{\|\, \epsilon\, \mathbf{v}_N^{2,\epsilon}\, \|_{0,\Omega_2}} > 0$$

The lower bound holds true for $\epsilon > 0$ small enough and adequate $\delta > 0$ then we conclude the quotient of tangent component over normal component L^2-norms blows-up to infinity, *i.e.* the tangential velocity is much faster than the normal one in the thin channel.

If $\mathbf{v}_T^2 = 0$ we can not use the same reasoning, so a further analysis has to be made. Suppose then that the solution $[\,(\mathbf{v}^1, \mathbf{v}^2), (p^1, p^2)\,]$ of (4.4.64) is such that $\mathbf{v}_T^2 = 0$ then (4.4.64b) imply $\mathbf{v}^1 \cdot \mathbf{n} = 0$ on Γ *i.e.* the problem on the region Ω_1 is well-posed independently from the activity on the interface Γ. The pressure on Γ becomes subordinate and must meet the following conditions:

$$p^2 = p^1 - (\mu + \alpha)\, \mathbf{v}^1 \cdot \mathbf{n} = p^1 - (\mu + \alpha)\, \mathcal{Q}^{-1}\, \boldsymbol{\nabla}\, p^1 \cdot \mathbf{n}$$

$$\boldsymbol{\nabla}_T\, p^2 = \mathbf{f}_T^2$$

We know the values of p^1 are defined by h^1 then, if we impose the condition on the forcing term $\mathbf{f}_T^2$

$$\mathbf{f}_T^2 \neq \boldsymbol{\nabla}_T\, \left(p^1 - (\mu + \alpha)\, \mathbf{v}^1 \cdot \mathbf{n} \right)$$

we conclude a contradiction and therefore it is impossible to have $\mathbf{v}_T^2 = 0$, *i.e.* restrictions on the forcing terms $\mathbf{f}^2$, h^1 can be given so that $\mathbf{v}_T^2 \neq 0$ and $\| \mathbf{v}_T^{2,\epsilon} \|_{0,\Omega_2} \gg \| \mathbf{v}_N^{2,\epsilon} \|_{0,\Omega_2}$ for $\epsilon > 0$ small enough as discussed above.

5 CONCLUSIONS AND FUTURE WORK

In the present work the problem of flow in porous media with the presence of adjacent cracks has been extensively studied with different modeling assumptions and with different mathematical techniques. The aim was to better understand the physical phenomenon of preferential flow inside porous media.

It is common in the three problems presented that the singularity introduced by the presence of the channel can be well approximated by a problem with lower dimensional interface since the geometric singularity and the physical one balance out by means of appropriate scaling. Another accomplishment of this dissertation is the treatment of the shape of the channel. The first two problems treat a large variety of potential shapes of cracks under reasonable assumptions and combinations of both geometric scenarios can be done.

Another important accomplishment of this thesis is the introduction of the mixed-mixed formulation in the treatment of coupled problems with interface. This formulation which appears to be totally new is necessary in order to study interface conditions of greater generality than the preexistent ones in the literature, and at the same time to provide a setting as reasonable as possible from the numerical approximation point of view. Important characteristics of this formulation are that it mixes the spaces of functions of both velocity and pressure and handles them in a fully decoupled fashion so that the interface conditions apply only to the solution and never on the test functions. The formulation uses boundary and interface conditions from both equations: conservative and constitutive. This is in contrast to its mixed predecessors which pick boundary and interface condition only from either the conservative or the constitutive law, but not from both.

The first work is taken further and illustrates the technique in dealing with the time dependent problem at the same time that induces in the limit a problem which is degenerate in time on the interface. It also shows how this technique can be easily applied to the analysis of the concentrated capacity model.

Finally, the third problem analyzes a much more complicated model of the same phenomenon by coupling a Darcy-Stokes system. The analysis of estimates and convergence becomes much more delicate even though the limiting problem can be identified as a problem with lower dimensional interface; this is not immediate since the identification of adequate spaces and isomorphisms to that end is of substantial difficulty. The structure of the limiting problem is different from the original because it is characterized by a Brinkman flow model in the tangential coordinates of the surface representing the collapsed fracture coupled with a Darcy flow in the interior. The Brinkman law has been successfully used in numerical simulations when coupling Darcy and Stokes flow models from the experimental point of view. The identification of a Darcy-Brinkman coupled system as a limiting problem of a Darcy-Stokes coupled system supports such numerical success and provides understanding of this phenomenon to a deeper extent.

Future Work

Amongst the future research projects we mention the analysis of the mixed-mixed formulation in evolution by means of semigroups and further explorations on the possibilities this new formulation allows at the time of modeling coupled systems with interface. These include the application of such techniques to more general settings such as deformable porous media where elasticity becomes important.

From the Numerical Analysis point of view the author has results only at the level of numerical experimentation for the first two problems. It is one of the goals for future research to achieve rigorous results on the convergence rate of the solutions and explore the theoretical aspects of the discreete mixed-mixed formulation of these problems.

For the Darcy-Stokes system our future projects include the analysis of different types of geometry for the shape of the crack. It is planned to analyze cracks limited by continuous parallel surfaces such as in the case of the mixed-mixed formulation problem as well as those limited by non parallel surfaces and finally by piece-wise linear surfaces. This third case is important from the numerical implementation point of view. Also, the behavior of the system in evolution must be studied. Another interesting question is the behavior of the system under perturbations of the interface which is clearly non-linear; since the interface separates two types of flow at a very different scale, a small perturbation of the interface can become significant. Preliminary results show that the level of technical difficulty in order to get statements of continuity can be extremely hard. However, getting stability statements is a reasonable task and the existence of such statements allows numerical simulation as a valuable tool in understanding this dependence. Finally, the scaling of the Beavers-Joseph condition in coupling the Darcy-Stokes system plays a fundamental role in the conclusion of the limiting problem. Further possibilities of such scaling must be explored.

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