A numerical technique is presented for solving the gas dynamics of a pulsating combustor. The computer solution which is developed follows directly from the fundamental conservation laws applied to discrete volumes rather than attempting either to finite difference the resulting partial differential equations, or to apply the Method of Characteristics. The integration procedure closely relates to the physical model, and various types of boundary conditions and diffusion effects can be included. The addition of diffusion terms also helps to dampen oscillations caused by near discontinuities in the flow field, spreading these discontinuities over several cell widths. Combustion processes are included in the computer solution by following the concentrations of the injected fuel and the oxygen aspirated from the atmosphere, and computing the energy released when the fuel ignites. The results
obtained using this technique show reasonable agreement with those obtained using the Method of Characteristics, and with actual test data.
A Method of Calculating Non-Steady Compressible Flow in a Propulsive Duct

by

Lawrence Decker Winiarski

A THESIS
submitted to
Oregon State University

in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

June 1972
APPROVED:

Redacted for privacy

Associate Professor of Mechanical Engineering in charge of major

Redacted for privacy

Head of Department of Mechanical and Nuclear Engineering

Redacted for privacy

Dean of Graduate School

Date thesis is presented January 4, 1972

Typed by Clover Redfern for Lawrence Decker Winiarski
ACKNOWLEDGMENT

I would like to express my sincere appreciation to my Faculty Advisor, Dr. Hans Dahlke, for his interest and personal help on this research; to Dr. Joel Davis and Dr. Larry C. Hunter of the Oregon State University Computer Center for their help in obtaining most of the necessary computer time; to Mr. Ray Lockwood, Ecology Systems, for his reports, stimulating conversations, and encouragement; to Dr. John D. Anderson, Jr., US Naval Ordnance Laboratory, for his reports and corroboration of some of my ideas; to Dr. George Rudinger, Cornell Aeronautical Laboratory, for his audience and interest; to Mr. Abbott Putnam, Professor J. M. Beér, Dr. Dennis Brown, Mr. Ron Pearson, and other conferees at the First International Symposium on Pulsating Combustion for their reviews, references, interesting papers, and information on their own extensive research; to Dr. Lorin Davis, Oregon State University, for reviewing this work; and to my wife, Flora, for her help in programming and key punching many different variations of the numerical methods.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. APPLICATIONS</td>
<td>3</td>
</tr>
<tr>
<td>III. METHODS OF ANALYSIS</td>
<td>5</td>
</tr>
<tr>
<td>Thermodynamic Analysis</td>
<td>5</td>
</tr>
<tr>
<td>Acoustic Analysis</td>
<td>10</td>
</tr>
<tr>
<td>Method of Characteristics</td>
<td>12</td>
</tr>
<tr>
<td>Theory of Characteristics</td>
<td>12</td>
</tr>
<tr>
<td>Pulse-Jet Analysis</td>
<td>18</td>
</tr>
<tr>
<td>Advantages and Disadvantages</td>
<td>19</td>
</tr>
<tr>
<td>Approximate Characteristic Methods</td>
<td>22</td>
</tr>
<tr>
<td>Non-Characteristic Computer Methods</td>
<td>25</td>
</tr>
<tr>
<td>Wilkin's Code</td>
<td>25</td>
</tr>
<tr>
<td>Anderson Three-Term Method</td>
<td>26</td>
</tr>
<tr>
<td>Anderson Two-Term Method</td>
<td>28</td>
</tr>
<tr>
<td>IV. CONTROL VOLUME METHOD</td>
<td>31</td>
</tr>
<tr>
<td>Theory</td>
<td>31</td>
</tr>
<tr>
<td>Mathematical Analysis</td>
<td>34</td>
</tr>
<tr>
<td>Examples</td>
<td>39</td>
</tr>
<tr>
<td>No Diffusion</td>
<td>39</td>
</tr>
<tr>
<td>Diffusion Added</td>
<td>40</td>
</tr>
<tr>
<td>Application to Pulse-Jet</td>
<td>58</td>
</tr>
<tr>
<td>Stability and Error</td>
<td>62</td>
</tr>
<tr>
<td>V. CONCLUSION</td>
<td>66</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>68</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>74</td>
</tr>
<tr>
<td>Appendix A: Brief History of Pulse-Jet Development</td>
<td>74</td>
</tr>
<tr>
<td>Appendix B: A Two-Dimensional Unsteady Method of Characteristics</td>
<td>85</td>
</tr>
<tr>
<td>Appendix C: Comparison of Results from Control Volume and Characteristic Methods</td>
<td>104</td>
</tr>
<tr>
<td>Appendix D: Control Volume Computer Program</td>
<td>122</td>
</tr>
</tbody>
</table>
Nomenclature

A - area
a - acoustic velocity
\( C_f \) - coefficient of wall friction
\( C_P \) - specific heat at constant pressure
\( C_V \) - specific heat at constant volume
E - total energy
e - energy per unit mass
f - force per unit mass, or frequency
h - heat transfer coefficient
i - cell index
\( K_t \) - effective eddy thermal conductivity
\( K_m \) - effective eddy mass transfer coefficient
L - length
\( f \) - co-ordinate along bicharacteristic
M - mass
MV - momentum
n - polytropic exponent, or co-ordinate tangential to mach conoid
P - pressure, or Riemann variable \( P = \frac{2a}{\gamma - 1} + u \)
Q - heat addition, or Riemann variable \( Q = \frac{2a}{\gamma - 1} - u \)
R - gas constant
S - entropy
\( T \) - temperature

\( t \) - time

\( u \) - velocity in x-direction

\( V \) - velocity

\( V_e \) - exit velocity

\( V_f \) - flame velocity

\( v \) - velocity in y-direction

\( x \) - space co-ordinate

\( y \) - space co-ordinate

\( a \) - thermal diffusion parameter

\( \beta \) - mass diffusion parameter

\( \gamma \) - specific heat ratio

\( \zeta \) - angle

\( \eta \) - cycle thermodynamic efficiency

\( \lambda \) - volume

\( \mu_t \) - effective eddy viscosity

\( \nu \) - momentum diffusion parameter

\( \rho \) - density

\( \psi \) - mass addition per unit length
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Thermal efficiency $\eta$ for air standard Sargent cycle.</td>
<td>7</td>
</tr>
<tr>
<td>2.</td>
<td>Temperature ratio for constant volume heat addition after isentropic compression from $T_1$.</td>
<td>8</td>
</tr>
<tr>
<td>3.</td>
<td>Run 3-2. Pressure distributions from control volume method using upstream properties at interface with pressure at wall equal to node pressure.</td>
<td>41</td>
</tr>
<tr>
<td>4.</td>
<td>Run 3-2. Pressure at end wall from control volume method using upstream properties at interface with pressure at wall equal to node pressure.</td>
<td>43</td>
</tr>
<tr>
<td>5.</td>
<td>Eddy motion in mainstream fluid.</td>
<td>43</td>
</tr>
<tr>
<td>6.</td>
<td>Shock tube.</td>
<td>48</td>
</tr>
<tr>
<td>7.</td>
<td>Run 5-2. Pressure at end wall from method using upstream properties at interface with pressure at wall equal to node pressure, diffusion added.</td>
<td>49</td>
</tr>
<tr>
<td>8.</td>
<td>Run 5-2. Pressure distributions from method using upstream properties at interface with pressure at wall equal to node pressure, diffusion added.</td>
<td>50</td>
</tr>
<tr>
<td>11.</td>
<td>Oscilloscope trace for downstream pressure in shock tube.</td>
<td>57</td>
</tr>
<tr>
<td>12.</td>
<td>Calculated downstream pressure in shock tube compared to measured value.</td>
<td>57</td>
</tr>
<tr>
<td>13.</td>
<td>Pressure/time traces for control volume method compared to measured values at six engine locations.</td>
<td>63</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>14.</td>
<td>Node division used for analysis of French valveless pulse-jet Model 5158.</td>
<td>64</td>
</tr>
<tr>
<td>A-1.</td>
<td>Lorin engine.</td>
<td>75</td>
</tr>
<tr>
<td>A-2.</td>
<td>Marconnet jet.</td>
<td>75</td>
</tr>
<tr>
<td>A-3.</td>
<td>Melot free piston gas generator with ejector augmentor.</td>
<td>75</td>
</tr>
<tr>
<td>A-4.</td>
<td>Reynst pulse pot.</td>
<td>80</td>
</tr>
<tr>
<td>A-5.</td>
<td>V-1 pulse-jet.</td>
<td>80</td>
</tr>
<tr>
<td>A-6.</td>
<td>Schubert valveless pulse-jet.</td>
<td>80</td>
</tr>
<tr>
<td>A-7.</td>
<td>Logan valveless pulse-jet.</td>
<td>80</td>
</tr>
<tr>
<td>A-8.</td>
<td>Escopette jet.</td>
<td>82</td>
</tr>
<tr>
<td>A-9.</td>
<td>Ecrevisse jet.</td>
<td>82</td>
</tr>
<tr>
<td>A-10.</td>
<td>Gluhareff jet.</td>
<td>82</td>
</tr>
<tr>
<td>A-11.</td>
<td>Lockwood ducted Ecrevisse jet.</td>
<td>84</td>
</tr>
<tr>
<td>A-12.</td>
<td>Melenric jet.</td>
<td>84</td>
</tr>
<tr>
<td>B-1.</td>
<td>Ripple photos.</td>
<td>87</td>
</tr>
<tr>
<td>B-2.</td>
<td>Mach cone in a two-dimensional unsteady flow formed by stacking &quot;ripple&quot; photos.</td>
<td>87</td>
</tr>
<tr>
<td>B-3.</td>
<td>Author's scheme for advancing all conoids simultaneously without interpolation.</td>
<td>89</td>
</tr>
<tr>
<td>B-4.</td>
<td>Characteristic definitions.</td>
<td>89</td>
</tr>
<tr>
<td>B-5.</td>
<td>Relationship of characteristic cones at different time intervals.</td>
<td>91</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>B-6.</td>
<td>Backward running conoid.</td>
<td>91</td>
</tr>
<tr>
<td>B-7.</td>
<td>Interpolation scheme for conoid not passing through known points.</td>
<td>93</td>
</tr>
<tr>
<td>B-8.</td>
<td>Interpolation scheme for conoid passing through known points.</td>
<td>93</td>
</tr>
<tr>
<td>B-9.</td>
<td>First and part of second time step calculation showing how characteristic surfaces are followed in time.</td>
<td>99</td>
</tr>
<tr>
<td>B-10.</td>
<td>Reflection of characteristic surfaces.</td>
<td>100</td>
</tr>
<tr>
<td>B-11.</td>
<td>Disturbance in three-dimensional unsteady flow.</td>
<td>103</td>
</tr>
<tr>
<td>B-12.</td>
<td>Three-dimensional unsteady mach conoid.</td>
<td>103</td>
</tr>
<tr>
<td>C-1.</td>
<td>Run 3-15. Pressure at end wall from method using upstream or averaged properties at interface with pressure at wall equal to node pressure.</td>
<td>107</td>
</tr>
<tr>
<td>C-2.</td>
<td>Run 3-27. Pressure at end wall for method using average properties at interface with characteristic end, more nodes.</td>
<td>107</td>
</tr>
<tr>
<td>C-3.</td>
<td>Run 4-10A. Pressure at end wall from method using upstream properties at interface with pressure at wall equal to node pressure--two trials.</td>
<td>108</td>
</tr>
<tr>
<td>C-4.</td>
<td>Run 4-10B. Pressure at end wall from method using upstream properties at interface with characteristic end--two trials.</td>
<td>108</td>
</tr>
<tr>
<td>C-5.</td>
<td>Run 4-14. Pressure at end wall from method using average properties at interface with characteristic end--two trials.</td>
<td>109</td>
</tr>
<tr>
<td>C-6.</td>
<td>Run 3-13. Pressure at end wall from method using average properties at interface with pressure at wall equal to node pressure.</td>
<td>109</td>
</tr>
<tr>
<td>C-7.</td>
<td>Run 3-15. Pressure distributions for method using upstream or averaged properties at interface with pressure at wall equal to node pressure.</td>
<td>110</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>C-8.</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>Run 3-27. Pressure distributions for method using average properties at interface with characteristic end, more nodes.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-9.</td>
<td>114</td>
<td></td>
</tr>
<tr>
<td>Run 4-10A. Pressure distributions for method using upstream properties at interface with pressure at wall equal to node pressure—two trials.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-10.</td>
<td>116</td>
<td></td>
</tr>
<tr>
<td>Run 4-10B. Pressure distributions for method using upstream properties at interface with characteristic end—two trials.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-11.</td>
<td>118</td>
<td></td>
</tr>
<tr>
<td>Run 4-14. Pressure distributions for method using average properties at interface with characteristic end—two trials.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-12.</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>Run 3-13. Pressure distributions for method using average properties at interface with pressure at wall equal to node pressure.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A METHOD OF CALCULATING NON-STEADY COMPRESSIBLE FLOW IN A PROPULSIVE DUCT

I. INTRODUCTION

Pulsating combustion devices, such as pulse-jets,\(^1\) have been known since the turn of the century. They are extremely fascinating devices in that a very simple shape--basically a contoured duct--with few or no moving parts, can produce a static thrust. Contrary to a ram jet, which must move at high velocity to operate, a pulse-jet is a self-aspirating engine once it has been started. It has been shown by Lockwood (29, 30, 31) that a ducted valveless pulse-jet can approach the efficiency of a modern gas turbine. Unfortunately, however, the configuration of this highly efficient pulse-jet engine was impractical for large forward velocities.

Although the first practical application of a pulsating combustion device was as the propulsive unit of the German V-1 bomb during World War II, modern use tends to be in the area of heaters, dryers, boiler furnaces, incinerators, etc. Huber (25) first applied a pulsating combustor as a heater at about the same time that the pulse-jet was being developed as a military engine. Recently pulsating combustors have been used in drying and incinerating, and even hole

\(^1\) Also known as propulsive ducts, resojets, sonic jets, thermo jets, pulse pots, Swingfog, Schwingfeuer, Schmidt-tubes, V-1s, buzz bombs, or pulsating combustors.
drilling and tunneling.

In the past pulse-jet engines have been developed largely by trial and error and intuition. This is because the complexity of the combustion process, with its uncertainties of mixing, kinetics of reaction, and non-equilibrium effects, makes a detailed theoretical analysis practically impossible. Even when the rate of reaction is idealized to a constant volume heat addition at one location, the prediction of the resulting gas dynamic waves is a formidable problem.

This investigation presents a numerical technique for predicting the gas dynamics of a pulsating combustor, including a provision for approximating a distributed combustion process.
II. APPLICATIONS

Although interest in pulsating combustion has not always been strong, a number of diverse applications is now being found.

In the past, interest in these devices has generally been directed to their use in jet propulsion. Valveless pulse-jets have been used for helicopter rotor tip propulsion, vertical lift devices, auxiliary power for sailplanes, and missile and drone propulsion.

Pulsating combustors may be used as replacements for the combustion chamber in gas turbines. In this application they would, at best, approximate reversible, constant volume combustion in the combustion chambers, or, at the very least, replace the pressure loss through the combustion chambers by a slight pressure gain.

The idea of using pulsating combustion chambers as gas generators for a turbine seems to have actually preceded the development of the modern gas turbine. Recent papers by Pearson and Harley (37), Servanty (52), Kentfield (26), and Swithenbank and Brown (56) indicate renewed interest in this application.

Reising (42) indicates there may also be a possibility of using pulsating combustion in power engineering. Here, however, careful attention must be paid to practical problems, such as fatigue, noise, and maintenance, incurred during operation.

Desty and Boden (12) show that a variable frequency,
spark-controlled pulsating combustor can be used to drive holes in friable ground. This is accomplished by the action of the combustion wave travelling from the burner into the ground.

Ringer, Lane, and Slawinski (46) show how pulsating combustor installations can be used to keep railway switches free of snow and ice.

Huber (25) demonstrates that pulsating combustors used as air and water heaters are practical and marketable. In some of his units the noise suppression is considerable.

Belter (5) has run large-scale coal-fired pulsating combustors as coal dryers, although noise is a problem with these units.

Lockwood (28) has developed a number of industrial drying applications for pulsating combustors. The efficiency of drying, using these engines, appears to be extremely good for small particles. It is postulated that the intense noise and air vibration continually agitate the air boundary layer around the particles, and that this increased turbulence promotes high heat and mass transfer.

Other applications for pulsating combustors are in the field of incineration and waste disposal.
III. METHODS OF ANALYSIS

The development of self-aspirating pulsating combustion engines has been held back, in part, by the lack of a practical method for analyzing the gas dynamics in a duct. Several attempts at theoretical analysis have been made in the past, but these do not model the engines with sufficient realism to show the sensitivity of these engines to minor design changes, and the importance of the proper mixing of air, fuel, and hot gas products. The analyses that have been made might roughly be classified as thermodynamic, acoustic, or Method of Characteristic.

**Thermodynamic Analysis**

In the context of this investigation a thermodynamic analysis will be denoted as one which attempts to evaluate overall engine performance without being involved with the details of the wave motion. This type of analysis is useful because it defines potential performance limits of an engine cycle and can show the influence of various parameters. For example, pulsating combustion might be particularly attractive if the mechanism of heat release could be made to approach that of a reversible, constant volume heat addition. If the pulsating combustion process could take advantage of precompression, the ideal thermodynamic cycle would then follow the steps listed below:
1-2 adiabatic compression

2-3 constant volume combustion

3-4 adiabatic expansion

4-1 constant pressure heat rejection to the atmosphere

The ideal thermodynamic efficiency for this cycle is

$$\eta = \frac{\frac{T_3}{T_2} \cdot \frac{1}{\gamma} - 1}{\frac{P_1}{P_2} \cdot \frac{\gamma - 1}{\gamma}}$$

where

\(\gamma\) - isentropic exponent

The thermal efficiency \(\eta\) can be quite large. This can be readily seen in Figure 1. The temperature before and after combustion for typical values of heat addition is shown in Figure 2.

$$\frac{T_3}{T_2} = 1 + \frac{Q_A}{C_V T_1 \left(\frac{P_2}{P_1}\right)^{\gamma - 1}}$$

where

\(Q_A\) - heat added in BTU/lb

\(C_V\) - specific heat at constant volume

This cycle, then, can have an ideal thermal efficiency considerably in excess of the usual Brayton or Otto cycles with comparable
Figure 1. Thermal efficiency $\eta$ for air standard Sargent cycle.
Figure 2. Temperature ratio for constant volume heat addition after isentropic compression from $T_1$.

$Q_A$ = heat added (BTU/lb)
$\gamma = 1.4$
$T_1 = 520 ^\circ R$
$C_V = .1714$
compression ratios. Unfortunately, it has not been proven conclusively that the engines can operate at high pressure ratios without excessive losses.

Ducarme (13) has developed equivalent expressions for the thermal efficiency of an ideal cycle and attempted to establish the actual efficiency in order to determine the extent of possible improvements. The actual efficiency is expressed in terms of time averages of instantaneous measurements over one cycle. These instantaneous measurements, however, are very difficult to obtain.

Godsey and Young (20) used a thermodynamic analysis for estimating the thrust specific fuel consumption.

Assuming constant volume combustion they derived the following approximate expression for the average or effective exit velocity.

\[
V_e \approx \frac{a_3}{\gamma} \sqrt{\frac{2}{\gamma-1}} \int_{P_0}^{P_3} \frac{1}{\gamma+1} \left[ 1 - \left( \frac{P_0}{P} \right)^{\gamma-1} \right] dP
\]

where

\[
P_0 - \text{atmospheric pressure}
\]

For \( \gamma = 4/3 \), this becomes

\[
V_e = \frac{2\sqrt{6}}{35} a_3 \sqrt{\frac{1}{7/4}} \sqrt{\frac{1}{\gamma} - P_0^{1/4}} \left( \frac{1}{P_3^{1/4}} - P_0^{1/4} \right)^3 \left( 8P_0^{1/2} + 12P_0^{1/4} P_3^{1/4} + 15P_3^{1/2} \right)
\]
The specific fuel consumption is then

\[\text{S.F.C.} = \frac{3600}{\frac{V_e}{V_1} - 1} \frac{1}{1 + \text{A/F}} \left(\frac{\text{lb fuel}}{\text{hr-lb thrust}}\right)\]

where

- \(\text{A/F}\) - air/fuel ratio
- \(V_1\) - forward speed

Foa(17) used a thermodynamic analysis to derive expressions for the thrust and specific impulse in terms of a parameter \(n\) which can be varied over a wide range. In this analysis the heating process is assumed to be represented by the polytropic expression

\[\frac{P}{\rho^n} = \text{constant}\]

For nonsteady processes the range of \(n\) is taken to be

\[-\infty < n < 0\]

**Acoustic Analysis**

The term acoustic analysis refers to one which begins by assuming apriori that the fluctuations of parameters such as pressure and velocity fit a certain wave form, and that the amplitudes of the oscillations which occur are so small that a perturbation analysis is applicable, and the equations can be linearized. Acoustic type analyses have been used in attempting to describe burner oscillations and
rocket motor oscillations. As noted by Putnam (40), however, the results of such analyses must be used with reservation because the actual processes involved are clearly non-linear.

In the case of the ramjet or valveless pulse-jet type of combustor the amplitudes of the oscillations are much greater than those for which linearized treatments are acceptable, as further noted by Putnam (39). In fact, a self-aspirating, valveless pulse-jet must necessarily have a strong oscillation so that it can suck in its own air. Any analysis which would assume small amplitude oscillations, or introduce assumed amplitude wave forms or phase relationships (time lags and space lags), must force mathematical results which are independent of the physical relationships that determine whether the valveless pulse-jet will even run.

Analyses which purport to describe the pressure oscillations of valveless pulse-jets in terms of organ tube type standing waves are particularly subject to doubt. Some simplified analyses even describe the operation in terms of a velocity node at the junction of the intake pipe and combustion chamber. Lockwood (29) points out that this approach is highly questionable since the velocity of the exhaust gas out of the "intake tube" can produce almost as much thrust as that out of the tailpipe.

Tharratt (57) presents a lengthy analysis based on acoustic theory and empirical relationships in which he postulates that the
equations of motion for large amplitude oscillations within the duct can be represented by an expression of the form used in acoustic analysis. However, as he points out, one could only reasonably look for trends with this type of theory. In general, acoustical theory has the implicit assumption that the particle velocity is much less than the speed of sound, and this is not always true in a pulse-jet.

**Method of Characteristics**

**Theory of Characteristics**

The analytical technique generally suggested for use in pulsating combustion problems is the Method of Characteristics. The Method of Characteristics can be used if the partial differential equations can be simplified to a system of quasi-linear hyperbolic equations. Descriptions of this graphical-numerical method for essentially one-dimensional unsteady flow may be found in the works of Shapiro (53), Oswatich (35), Owzcharek (36), Foa (17),\(^2\) Schultz-Grunow (51), Rudinger (47), and Abbot (1). The books by Rudinger and Foa are particularly practically oriented. Following their terminology one can briefly illustrate the steps of the calculation procedure.

\(^2\)Foa believes so strongly in the advantages of pulsating combustion propulsive engines that he has devoted a large portion of his book to their description.
Consider the equations of the conservation of mass and momentum as applied to one-dimensional or quasi-one-dimensional flow.

Mass

\[ \frac{\partial (\rho A)}{\partial t} + \frac{\partial (\rho u A)}{\partial x} + \psi = 0 \]

Momentum

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + f \]

where

\( \psi \) - is the mass flow leaving the duct walls per unit length.

\( f \) - represents all body and dissipative forces lumped into one resultant force per unit mass.

Assuming an ideal gas

\[ P = \rho RT \]

The equations of the conservation of mass and momentum can be expressed in terms of the velocity of sound

\[ a^2 = \gamma RT = \frac{\gamma P}{\rho} \]

where

\[ \gamma = \frac{C_P}{C_V} \]

\[ C_P - C_V = R \]

If entropy is desired as a variable it can be introduced as
follows. Assuming constant specific heats

\[ S_2 - S_1 = C_P \ln \left( \frac{T_2}{T_1} \right) - R \ln \left( \frac{P_2}{P_1} \right) \]

\[ \frac{P_2}{P_1} = \left( \frac{a}{a_1} \right)^{\frac{2\gamma}{\gamma-1}} e^{-\gamma(S_2 - S_1)} \]

\[ \frac{\rho_2}{\rho_1} = \left( \frac{a}{a_1} \right)^{\frac{2}{\gamma-1}} e^{-\gamma(S_2 - S_1)} \]

Using these relationships it can be shown (Rudinger (47) or Foa (17)) that the equations of continuity and momentum can be replaced by the pair of equations

\[ \frac{\partial}{\partial t} \left( \frac{2}{\gamma-1} a \pm u \right) + (u \pm a) \frac{\partial}{\partial x} \left( \frac{2}{\gamma-1} a \pm u \right) \]

\[ = -au \frac{\partial \ln A}{\partial x} - a \frac{\partial \ln A}{\partial t} + \frac{a}{R} \frac{D S}{\gamma \partial t} \pm a \frac{\partial S}{\gamma \partial t} \pm f - a^3 \frac{\Psi}{\gamma PA} \]

The left side of the equation represents the derivative of the parameters \( \frac{2}{\gamma-1} a \pm u \) in a direction in the \( x, t \) plane such that

\[ \frac{\partial x}{\partial t} = u \pm a \]

where \( \frac{D}{Dt} \) is the substantial derivative \( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \).

This direction, or characteristic line, is the path of the propagation of disturbances or signals in the \( x, t \) plane. (In the case of
two-dimensional unsteady flow these signals would form the so-called Mach conoid as shown in Figure B-2 of Appendix B. \( \frac{dx}{dt} = u + a \) and \( \frac{dx}{dt} = u - a \) may be thought of, respectively, as the extreme right and left running genetrix of a mach conoid whose axis points in the direction determined by the sign of \( u \). This axis, whose slope is given by \( \frac{dx}{dt} = u \), is the particle path. Derivatives in the characteristic direction are denoted by

\[
\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + (u + a) \frac{\partial}{\partial x}
\]

\[
\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + (u - a) \frac{\partial}{\partial x}
\]

Using these definitions the equations become

\[
\frac{\delta}{\delta t} \left( \frac{2a}{\gamma-1} \pm u \right) = -au \frac{\partial \ln A}{\partial x} - a \frac{\partial \ln A}{\partial t} + \frac{a}{\gamma R} \frac{\delta S}{\delta t} + (\gamma - 1) a \frac{DS}{\gamma R} \frac{D S}{D t} \pm f - \frac{a^3 \psi}{\gamma PA}
\]

The quantities \( P = \frac{2}{\gamma-1} a + u \) and \( Q = \frac{2}{\gamma-1} a - u \) are called Riemann variables.

Solving the equations in this form requires that the substantial derivative \( \frac{DS}{Dt} \) be defined either explicitly as a function of \( x \) and \( t \), or implicitly as a function of the flow conditions. There are generally three types of assumptions that can be made regarding \( \frac{DS}{Dt} \).
1. Entropy is the same for all particles and constant in time.

   This type of flow is called homentropic.

   \[ \frac{\partial S}{\partial x} = \frac{\partial S}{\partial t} = 0 \]

2. Entropy is not the same for all particles, but the entropy of each particle is constant, i.e., each particle undergoes an isentropic transformation. This type of flow is sometimes called multi-isentropic or, simply, isentropic.

   \[ \frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0 \]

3. Entropy of each particle varies with time in some assumed manner depending, for example, on the mode of heat addition.

   Generally computation of the entropy terms has to be based on suitable assumptions for simplification as well as because of the lack of precise information about the actual process. The use of extreme simplification is often justified since the actual process is too erratic to lend itself to exact description anyway. This is particularly true in the case of combustion, as was pointed out by Foa (17).

   Characteristic equations must be solved by numerical/graphical procedures. For instance, assume that \( a, u, S \) are known at two points (1) and (2) in the \( x, t \) plane. These have been determined from previous calculations or initial conditions. The Riemann variables \( P \) and \( Q \) are then known at points (1) and (2) also. To determine
the properties at some new point (3) in the \( x, t \) plane it is first necessary to locate point \((3')\) by the intersection of \( P \) and \( Q \) characteristics from points (1) and (2). As a first approximation these characteristics can be represented by straight line segments from points (1) and (2), located by slopes for \( P \) of \( \frac{\partial x}{\partial t} = u_1 + a_1 \) and \( Q \) of \( \frac{\partial x}{\partial t} = u_2 - a_2 \). The particle path through \((3')\) can be estimated by interpolation between \( u_1 \) and \( u_2 \). The value of \( S \) at the intersection \((4')\) of this line and a line connecting (1) and (2) can be determined by interpolation between \( S_1 \) and \( S_2 \).

\[
S_3 = S_4 + \Delta S = S_4 + \frac{DS}{Dt} \Delta t
\]

\[
P_3 = P_1 + \Delta P = P_1 + \frac{\delta P}{\delta t} \Delta t
\]

\[
Q_3 = Q_2 + \Delta Q = Q_2 + \frac{\delta Q}{\delta t} \Delta t
\]

then

\[
a_3 = \frac{\gamma - 1}{4} (P_3 + Q_3)
\]

\[
u_3 = \frac{1}{2} (P_3 - Q_3)
\]

This procedure can be repeated using the mean values of the properties between (1) and \((3')\), and (2) and \((3')\) to obtain a better estimate for (3).

Generally this procedure is applied in flow areas which are
bounded by the ends of the duct, sudden changes in cross-section, or moving discontinuities such as contact surfaces and shock waves. The basic partial differential equations do not apply across these discontinuities, therefore, the wave diagram must be, in effect, solved separately on each side of the discontinuity and patched together at the discontinuity by appropriate matching conditions. A number of possible boundary and matching conditions can occur. Information on handling some specific cases is detailed in the books by Rudinger (47) and Foa (17).

**Pulse-Jet Analysis**

One of the first attempts to provide a somewhat realistic analysis of the finite waves in a pulse-jet was made by Schultz-Grunow (51). This analysis however was not directed toward a valveless pulse-jet, but rather the valved Argus-Schmidt tube (i.e. the V-1 used in World War II). Because of the complexity of the calculations a rather coarse characteristic network was employed. No effort was made to predict the combustion process; the report states that the combustion process so completely eludes theoretical treatment that experimentation alone is decisive. In Part I of Schultz-Grunow (51) the combustion process was assumed to result in 1/7 of the pulse-jet being filled with a gas with a sonic velocity 1.144 times that in the remainder of the tube, and a pressure 2.5 times as large. The
resulting wave action was treated like that occurring upon the removal of the diaphragm in a shock tube. In Part II an experimentally determined value was used for the increase of pressure due to combustion, and a simplified assumption was made that the change of state takes place adiabatically and simultaneously along the whole charge.

Yen (62) made a characteristic analysis of an elementary type of valveless pulse-jet in which tubes of different diameter were joined together such that, in effect, a Borda type entrance was formed at the junction. In one case, wave diagrams for this tube were prepared assuming a constant volume combustion in 20 percent of the tube. In another an experimental pressure/time relationship, taken when the tube was forced to resonate with ram air, was used. Such a tube would not operate in a true static condition (i.e. self aspirating its own air), but the wave diagram did indicate that if constant volume combustion did occur the tube should aspirate its own air.

**Advantages and Disadvantages**

The main advantages in using the Method of Characteristics for analysis may be stated as follows:

1. It has potentially the largest step size.

2. Short duration problems can be done with hand computation and graphical techniques.

3. The wave action is distinctly defined.
The method however has several drawbacks which prevent its being used extensively for analysis.

1. It is extremely time consuming when done with numerical-graphical techniques and requires a high degree of computational skill.

2. The method is restricted to an idealized set of equations and when dissipative effects are likely to occur they must be treated as discontinuities, and special techniques are required to handle the interaction with the characteristic network.

3. The analyst must recognize when the equations are not applicable in the space-time domain, and assume the existence of discontinuities such as shocks, contact surfaces, and slip lines. Often the type of wave resulting from the interaction is unknown and must be postulated and checked by trial and error procedures. In unsteady flow these "discontinuities" change their location with time, and the analyst must predict their appearance or disappearance. Furthermore, the interaction with the boundary is often uncertain, as noted by Rudinger (47).

4. The method is not very amenable to machine computation as further noted by Rudinger (47). For example, Sauerwein (49) points out that approximately four man-years were required
to write the program for the calculation of two-dimensional unsteady ordinary fluid dynamics of a perfect gas. Even then, the program was reported to use linear interpolation throughout in order to obtain results so that the stability and feasibility of the procedure could be evaluated in the shortest period of time. Strom (55) also indicates how these procedures might be applied to multi-dimensional reacting gas flows, but both Sauerwein and Strom point out that limited storage and cost of computational time make it impractical to attempt to analyze them using existing computers.

5. Method of Characteristics codes tend to be restricted to a particular geometry or set of conditions. Some of the codes that have been developed are regarded as proprietary and little detailed information is available in the open literature. The works of Sauerwein (48) and Strom (55) are notable exceptions. However, these programs were written with a different purpose in mind, namely the solution of external flow around blunt bodies, and Strom advised against attempting to modify them for the internal duct flow. Manning (33) describes a code which he developed from a driven sonic oscillator but, again, he points out the difficulty in matching the boundary conditions. Practically all the characteristic codes in the past have been unable to account for the
formation of an arbitrary number of shocks. Hoskins and Lambourn (24) mention a new code which addresses itself to this problem. Unfortunately, this code is not generally available, as noted by Hoskins (23). Lockwood (28) reports that a rather extensive search was made in an effort to find existing computer programs, using the Method of Characteristics, which might be applied to the pulse-jet analysis. This search, however, did not result in locating a usable program.

**Approximate Characteristic Methods**

Characteristics, unfortunately, do not yield closed form solutions. Even approximate characteristic solutions, as in Shapiro (53), must be done with graphical or numerical techniques, and only yield rough indications of the flow process.

The method of approximating characteristics described by Shapiro (53) assumes that the flow pattern in a duct of constant area may be thought of as steady flow having the properties $u$, $p$, $\bar{a}$, and $\bar{\rho}$. One exception is the newly developed computer code using the Method of Characteristics for pulse-jets, which was presented by Pearson and Harley (37) at the First International Symposium on Pulsating Combustion, concurrently with the first presentation of the results of this thesis investigation by Winiarski (61). This code approximates the combustion process by assuming either a constant volume combustion, or empirical heat transfer across the wall. It also has the ability for accounting for an arbitrary number of shocks, however, fairly large amounts of computer time may be involved.
\[ \bar{P} \] which are independent of \( x \) and \( t \), and on which is superimposed a perturbation motion with the incremental properties \( \Delta u, \Delta \rho, \Delta a, \) and \( \Delta P \). That is

\[
\begin{align*}
    u &= \bar{u} + \Delta u \\
    \rho &= \bar{\rho} + \Delta \rho \\
    a &= \bar{a} + \Delta a \\
    P &= \bar{P} + \Delta P
\end{align*}
\]

For waves running rightward relative to the flow

\[
\frac{dx}{dt} = \bar{u} + a \quad \text{along the wave}
\]

\[
\Delta a = \frac{\gamma-1}{2} \Delta u \quad \text{across the wave}
\]

For waves running leftward relative to the flow

\[
\frac{dx}{dt} = \bar{u} - a \quad \text{along the wave}
\]

\[
\Delta a = \frac{\gamma-1}{2} \Delta u \quad \text{across the wave}
\]

The values of \( \Delta P, \Delta \rho, \) and \( \Delta T \) may be computed from the corresponding value of \( \Delta a \) by means of the isentropic relation, the equation of state, and the expression for the velocity of sound, which are, respectively,
\[ \frac{P}{\rho^\gamma} = \text{constant} \]
\[ \frac{P}{\rho T} = \text{constant} \]
\[ a = \sqrt{\gamma RT} \]

These may be expressed in differential form

\[ \frac{\Delta P}{\rho} = \frac{1}{\gamma} \frac{\Delta P}{P} = \frac{1}{\gamma - 1} \frac{\Delta T}{T} = \frac{2}{\gamma - 1} \frac{\Delta a}{a} \]

Implied in the assumption of small pressure waves are the following conditions:

1. Waves are reflected from stationary end walls in the same sense. That is, if the incident wave is a compression wave, the reflected wave is also a compression wave.

2. Waves are reflected from open ends in the opposite sense. That is, if the incident wave is a compression wave, the reflected wave is a rarefaction.

3. The reflected waves have the same strength as the incident waves.

4. Intersecting waves do not influence each other.

Lockwood (29) mentions another approximate technique developed by Oppenheim, Stern, Laderman, and Urtiew which considers only the results of wave interaction phenomena. It holds true only after all local interaction effects have died out, and small amplitude waves
have either collapsed into finite amplitude shocks or degenerated into rarefaction fans. This method differs from the usual Method of Characteristics in that it does not predict the details of the wave interaction process. This vector-polar graphical method is based on a modification of diagrams which are usually plots of \( \frac{a}{a_0} \) versus \( \frac{u}{a_0} \) or \( \frac{P}{P_0} \) versus \( \frac{u}{a_0} \), such as is shown in Shapiro (53). This modification consists of the use of the logarithm of the pressure ratio, or of the local velocity of sound ratio, as the ordinate, instead of the conventional linear scale. This assists the graphical part of the solution. Even with this labor-saving simplification, however, the calculations still tend to be rather time consuming.

During the school year, 1966-1967, as a result of an effort to better understand the physical meaning of multi-dimensional characteristics, the author conceived a possible scheme for solving two-dimensional unsteady flow by the method of intersecting surfaces. Conceptually, a similar method could be extended to three-dimensional unsteady flow. An outline of this technique is described in Appendix B.

Non-Characteristic Computer Methods

Wilkin's Code

Although a search for a characteristic code did not prove successful, it did indicate that a non-characteristic code had been
used with some limited success in the discharge portion of a pulse-jet cycle. The use of this code, developed by Wilkins, French, and Giroux was described by Lockwood (29). The code did not take advantage of characteristics, however it did handle the development of shocks automatically by introducing an artificial viscosity as a dissipative mechanism. It was reported to handle the transmissions and reflections from the interface between a hot and cold gas. Unfortunately, this computer code is not generally available. Wilkins did use it as a consultant to Lockwood, but apparently the cost prevented it from being used as a developmental tool. Lockwood (28) reported that it took over 20 minutes of computer time to evaluate a single pressure pulse.

**Anderson Three-Term Method**

Other methods for solving compressible flow problems have recently appeared in the open literature, but these have not been applied to the pulse-jet problem. Generally the computation times for these methods appear prohibitive, and it is doubtful that they possess sufficient generality to include all the processes necessary for analyzing pulse-jets. One promising technique is the method of analysis described by Anderson (2), and later referred to by him as the Three-term method. Basically the method is as follows.

At a given instant of time \((t)\) the variables \(\rho\), \(u\), and \(e\)
are known at fixed grid points of a flow field either from initial conditions or calculations from the previous time step. The value of these variables at time \((t + \Delta t)\) can be obtained from the first three terms of a Taylor's Series expansion in time.

\[
g(t + \Delta t) = g(t) + \left( \frac{\partial g}{\partial t} \right)_t \Delta t + \left( \frac{\partial^2 g}{\partial t^2} \right)_t \frac{\Delta t^2}{2} \tag{3.1}
\]

where \(g\) represents the chosen variables which, for the purpose of illustration, are taken as \(\rho, u,\) and \(e\). (The reference actually uses \(\ln \rho, \ln u,\) and \(\ln e,\) and makes these variables non-dimensional, but this does not appear to be an absolute requirement.)

Each term in Equation 3.1 is evaluated at the local value of \(x\).

The time derivatives \(\frac{\partial g}{\partial t}\) and \(\frac{\partial^2 g}{\partial t^2}\) which appear in Equation 3.1 are calculated from the quasi-one-dimensional differential form of the basic conservation equations.

**Mass**

\[
A \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u A)}{\partial x} = 0 \tag{3.2}
\]

**Momentum**

\[
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} \tag{3.3}
\]

**Energy**

\[
\rho \frac{\partial e}{\partial t} + \rho u \frac{\partial e}{\partial x} = -P \frac{\partial u}{\partial x} - Pu \frac{\partial (\ln A)}{\partial x} \tag{3.4}
\]
where \( P \) is related to \( T \) by

Equation of State

\[
P = \rho RT
\]  

(3.5)

In order to evaluate these time derivatives (i.e. \( \frac{\partial \rho}{\partial t}, \frac{\partial u}{\partial t} \) and \( \frac{\partial e}{\partial t} \)) from Equations 3.2, 3.3 and 3.4, however, it is necessary to know the \( x \)-derivatives. These \( x \)-derivatives can be obtained from central finite differences of the known flow field at time \( t \).

\[
\frac{\partial g}{\partial x} = \frac{g(x+\Delta x) - g(x-\Delta x)}{2\Delta x}
\]  

(3.6)

The second order terms in the Taylor Series expansion (i.e. \( \frac{\partial^2 g}{\partial t^2} \) in Equation 3.1) are found by differentiating Equations 3.2, 3.3 and 3.4 with respect to time. This leads to cross derivatives \( \frac{\partial^2 g}{\partial x \partial t} \) which can be obtained by differentiating the basic equations with respect to \( x \). This necessitates knowing terms \( \frac{\partial^2 g}{\partial x^2} \), but these too can be calculated from central finite differences of the known flow field at time \( t \).

\[
\frac{\partial^2 g}{\partial x^2} = \frac{g(x+\Delta x) - 2g(x) + g(x-\Delta x)}{\Delta x^2}
\]  

(3.7)

**Anderson Two-Term Method**

In an unpublished paper Anderson (3) also describes a Two-term method and compares it to the Three-term method described in his paper (2).
This Two-term method can be summarized as follows:

Let \( g \) represent the variables \( \rho, u, \) and \( e. \) Then

\[
g(x, t + \Delta t) = g(x, t) + \left( \frac{\partial g}{\partial t} \right)_{\text{avg.}} \Delta t \tag{3.8}
\]

where \( \frac{\partial g}{\partial t} \text{ avg.} \) is an approximate average of \( \frac{\partial g}{\partial t} \) over the time interval \( \Delta t, \) and is determined in the following manner. A trial or predicted value for \( g \) is found.

\[
g(x, t + \Delta t) = g(x, t) + \left( \frac{\partial g}{\partial t} \right)_{t} \Delta t \tag{3.9}
\]

where \( \frac{\partial g}{\partial t} \) can be obtained directly from the partial differential equations of mass, momentum, and energy, (Equations 3.2, 3.3 and 3.4) with the necessary spatial derivatives required to evaluate these equations computed as forward differences based on the flow field values at time \( t. \)

\[
\left( \frac{\partial g}{\partial x} \right)_{t} = \frac{g(x + \Delta x, t) - g(x, t)}{\Delta x} \tag{3.10}
\]

The trial values \( g(x, t + \Delta t) \) are used again in the partial differential equations 3.2, 3.3, and 3.4 to obtain a trial value for \( \left( \frac{\partial g}{\partial t} \right)_{t + \Delta t}, \) but in this case the \( x \)-derivatives are now obtained from backward differences at \( t + \Delta t. \)

\[
\left( \frac{\partial g}{\partial x} \right)_{t + \Delta t} = \frac{g(x, t + \Delta t) - g(x - \Delta x, t + \Delta t)}{\Delta x} \tag{3.11}
\]
Now the average value of the time derivative can be obtained.

\[
\frac{\partial g}{\partial t} \text{ avg } = \frac{1}{2} \left[ \frac{\partial g}{\partial t} \bigg|_t + \frac{\partial g}{\partial t} \bigg|_{t+\Delta t} \right] \tag{3.12}
\]

Anderson (3) points out that this is a modification of a finite difference scheme used by MacCormack for solving the Navier-Stokes equations for hypervelocity impact cratering. He further states that it was necessary to use the trial values \( g(x, t+\Delta t) \) only to obtain the \( x \)-derivatives; he used the old value of \( g(t) \) in the second application of Equations 3.2, 3.3, and 3.4. Anderson (3) reports that this Two-term method is at least 30% faster than the Three-term method.
IV. CONTROL VOLUME METHOD

Theory

In order to obtain a computer solution which would be both inexpensive and logically simple, a computational technique which would eliminate the need for solving partial differential equations was sought.

The previous methods that have been suggested for analyzing unsteady compressible flow problems generally start with the partial differential equations and attempt to use various combinations of mathematical manipulations, finite differencing techniques, and numerical integration in order to obtain a computer code. The computer codes that have been developed, however, often suffer from one or more of the following ailments:

1. Excessive computer time.
2. Inability to handle multiple shocks and contact surfaces.
3. Proprietary nature.
4. Abstruse formulation.
5. Difficulty in handling the boundary conditions.
6. Inability to predict several cycles.

It seemed that part of the difficulties that arise from numerical computations of fluid dynamics stem from the following three problems:
1. Simplifying assumptions which are used to obtain tractable partial differential equations may introduce physical inconsistencies which, when carried over many calculations in time and/or space, build up to significant violations of the basic conservation equations. Rather than specifying, apriori, some constraints on the solution, such as stipulating that the solution fit a particular temperature or pressure profile, it would appear better to use a realistic energy equation, and an equation of state. Conceptually then, gases, fluids, and solids could be handled with similar logic since all forms of matter are more or less compressible, and must obey the fundamental conservation laws and some type of equation of state.

2. A second problem may stem from attempting to apply stock mathematical integration schemes to partial differential equations without close regard to the physical concepts upon which these equations are based. Indiscriminate use of numerical integration techniques does not guarantee that the basic conservation laws hold for numerical integration of the partial differential equations.

3. A third problem consists of finding the so-called "boundary conditions" for the particular integration scheme, which are physically realistic.
These problems can be overcome if a simple correspondence between the numerical scheme and the physical problem can be established. This can best be accomplished by using the basic physical conservation laws applied directly to discrete volumes, with mass, energy, and momentum as primary variables, rather than using the differential equations derived from these basic principles. Density, temperature, pressure, and velocity are regarded as average manifestations of these primary variables.

Although this approach may not seem as exact as manipulating a partial differential equation, it may be more suitable for use by a digital computer since this computer only works with discrete numbers. A closer physical correspondence, as well as faster machine computation, could also be maintained with an explicit scheme that marched along in time. Each calculation step would be like one frame of a motion picture record of the physical process. The computer could, in effect, be "taking a limit" if $\Delta x$ and $\Delta t$ were relatively small. (To save on computational costs, however, the cell size and time step should be as large as possible.)

If a steady state solution were desired it could be obtained by starting a transient code with an arbitrary set of initial conditions and running it in time until the results did not change. Although this steady state solution would be dependent only on the fixed boundary conditions, the closer the initial conditions are to the solution, the
faster the steady state solution would be obtained.

**Mathematical Analysis**

The fundamental conservation principle may be stated very simply. For any fixed space control volume over any given period of time,

\[
\text{Increase within the control volume} = \text{Amount of inflow} - \text{Amount of outflow}
\]

If Newtonian mechanics are assumed the conservation principle can be applied independently to mass, momentum, and energy. Using vector notation the results may be expressed as follows:

**Conservation of Mass**

\[
\iiint \frac{\partial \rho}{\partial t} \, d\lambda = - \iiint \rho (\vec{V} \cdot \hat{n}) \, dA \tag{4.1}
\]

**Conservation of Momentum**

\[
\frac{\partial}{\partial t} \iiint \rho \vec{V} \, d\lambda = \Sigma \vec{F} - \iiint \rho (\vec{V} \cdot \hat{n}) \vec{V} \, dA \tag{4.2}
\]

**Conservation of Energy**

\[
\iiint \frac{\partial (e \rho)}{\partial t} \, d\lambda = \Sigma Q - \iiint (C_p T + \frac{V^2}{2}) \rho (\vec{V} \cdot \hat{n}) \, dA \tag{4.3}
\]

The logic may now be followed by examining one-dimensional or quasi-one-dimensional flow. Let \( i \) refer to the properties within a cell and \( L \) and \( R \) refer to the properties crossing the left and
right surfaces of the cell. The basic conservation equations for inviscid flow become

**Conservation of Mass**

\[ \frac{dM_i}{dt} = (\rho u)_L - (\rho u)_R \]  

(4.4)

**Conservation of Momentum**

\[ \frac{d(MV)_i}{dt} = (PA)_L - (PA)_R + (\rho u^2)_L - (\rho u^2)_R \]  

(4.5)

**Conservation of Energy**

\[ \frac{dE_i}{dt} = (\rho u)_L (C_P \frac{u^2}{2}) - (\rho u)_R (C_P \frac{u^2}{2}) \]  

(4.6)

A fast and simple method for integrating these equations is developed as follows:

\[ M_{i}^{t+\Delta t} = M_{i}^{t} + \frac{dM_{i}}{dt} \Delta t \]  

(4.7)

\[ (MV)_{i}^{t+\Delta t} = (MV)_{i}^{t} + \frac{d(MV)_{i}}{dt} \Delta t \]  

(4.8)

\[ E_{i}^{t+\Delta t} = E_{i}^{t} + \frac{dE_{i}}{dt} \Delta t \]  

(4.9)

After \( M_i, (MV)_i, \) and \( E_i \) have been computed for all \( i \) at the new time step, then density, velocity, and temperature are updated from the equations.
\[ \rho_i^{t+\Delta t} = \frac{M_i^{t+\Delta t}}{Vol_i} \]  \hspace{1cm} (4.10)

\[ u_i^{t+\Delta t} = \frac{(MV)_i^{t+\Delta t}}{M_i^{t+\Delta t}} \]  \hspace{1cm} (4.11)

\[ T_i^{t+\Delta t} = \frac{1}{C_v} \left[ \frac{E_i^{t+\Delta t}}{M_i^{t+\Delta t}} - \frac{(u_i^{t+\Delta t})^2}{2} \right] \]  \hspace{1cm} (4.12)

The pressure is computed from the approximate equation of state. Assuming an ideal gas

\[ P_i^{t+\Delta t} = \rho_i^{t+\Delta t} RT_i^{t+\Delta t} \]  \hspace{1cm} (4.13)

This procedure is repeated twice at each time step. "Trial" values of \( \overline{P}, \overline{\rho}, \overline{u}, \overline{T} \) are obtained using the "real" values \( P, \rho, u, T \) calculated at the previous time step in Equations 4.3, 4.4, and 4.5. Time averaged values can then be computed.

\[ \frac{t+\Delta t}{2} P_i = \frac{P_i^{t+\Delta t} + P_i^t}{2} \]  \hspace{1cm} (4.14)

\[ \frac{t+\Delta t}{2} \rho_i = \frac{\rho_i^{t+\Delta t} + \rho_i^t}{2} \]  \hspace{1cm} (4.15)

\[ \frac{t+\Delta t}{2} u_i = \frac{u_i^{t+\Delta t} + u_i^t}{2} \]  \hspace{1cm} (4.16)
The procedure is repeated using these time averaged values to determine the interface properties in Equations 4.3, 4.4, and 4.5. The new values of $P$, $\rho$, $u$, $T$ computed for $t + \Delta t$ are regarded as the "real" values, and the calculation procedure advances to the next step. Integrating in this manner is similar to retaining the second order term of a Taylor Series expansion. A numerical experiment indicated that there was no significant benefit in repeating the procedure three times for each time step.

Within the flow field there is no creation or destruction of the fundamental quantities of mass, momentum, and energy.\textsuperscript{4} A decrease in any of these fundamental quantities in a given control volume is accomplished by a net transfer of an amount of the quantity equal to the decrease to the neighboring control volumes, and vice versa for the increase. If an unrealistic amount of a fundamental quantity is transferred in a time interval, forces are set up which tend to correct the imbalance during subsequent time intervals. For instance, if too much mass flows out of a control volume during a given time step, the pressure within this volume will decrease, but the pressure of the

\begin{equation}
\frac{t + \Delta t}{2} = \frac{T_i^{t+\Delta t} + T_i^t}{2}
\end{equation}

\textsuperscript{4} An exception is that it is expeditious to consider the heat of combustion as a "creation" of energy within a gas with constant specific heats.
neighboring volume receiving this mass will increase. This will tend to reverse the flow between the two control volumes in the next time step. Note that although conservation does not preclude slightly incorrect spatial distributions or oscillations within a flow field during a time step, such oscillations tend to be self-correcting except when the oscillation occurs at a boundary.

It is necessary now to determine the values of pressure, density, velocity, and temperature at the left and right interfaces of the control volumes. These values determine the amount of mass, momentum, and energy transported between the control volumes, but do not affect the overall conservation principle, only the spatial distribution. On the basis of numerical experiments it was determined that the pressure at the interface could be the simple average pressure of the adjacent cells. Although the values of density, velocity, and temperature at the interface could also be determined by averaging the adjacent cell values, it was found that smoother results could be obtained if the values for density, velocity, and temperature at the interface were the values of the cell from whence the gas flowed.

Even though a cell length, $\Delta x$, and the acoustic velocity, $a$, are not used explicitly in this technique it appears that the time step is limited by the Courant-Friedrichs-Lewy stability criterion

$$\Delta t \leq \frac{\Delta x}{u + a}$$
Physically this means that the pressure wave relative to a cell grid should not travel more than one cell division in a time step.

Examples

No Diffusion

In order to test the computational method developed, the calculated results are compared with published results obtained using the Method of Characteristics. Both methods are applied to a one-dimensional tube, one square foot in cross-sectional area, and four feet long. The left end of the tube is closed and the right end is assumed to be instantaneously open to the atmosphere. Initially the pressure in the tube is higher than atmospheric.

The characteristic flow network for this problem has been published by Foa (17) and Rudinger (47) for slightly different initial conditions. Rudinger carries the solution far enough in time to show one pressure cycle. These examples are particularly useful test cases since they encompass the fundamental wave dynamics that occur in a pulse-jet, namely the following:

1. Open end outflow
2. Open end inflow
3. Closed end boundary
4. Weak shock wave formation
5. Weak shock wave reflection from closed boundary

6. Weak shock wave reflection from open boundary

It can be seen from Figures 3 and 4 that, on the average, the method follows the characteristic solution quite closely, except that discontinuities like the shock wave are spread out over several cell widths and cause some oscillations.

In making this analysis it was necessary to make certain approximations at the boundaries. The closed end wall pressure was assumed to be that of the adjacent cell, and the open end cell was assumed to adjust to the atmospheric conditions in a quasi-steady state relationship. These necessary approximations at the boundaries introduce some minor perturbations which propagate throughout the tube similar to a true physical disturbance.

Diffusion Added

The oscillations that occur as a result of trying to predict these near-discontinuities with a finite cell size can be considerably smoothed or "smeared" by introducing terms that are somewhat analogous to the "artificial viscosity" or additional friction which is sometimes used in the numerical solution of partial differential equations. For instance, one can add mass, momentum, and thermal diffusion terms to the right side of the respective equations for mass, momentum, and energy (Equations 4.4, 4.5, 4.6). These terms are
Figure 3. Run 3-2. Pressure distributions from control volume method using upstream properties at interface with pressure at wall equal to node pressure.
Figure 3. Continued.
Figure 4. Run 3-2. Pressure at end wall from control volume method using upstream properties at interface with pressure at wall equal to node pressure.

Figure 5. Eddy motion in mainstream fluid.
as follows:

\[
\beta (\rho_{i-1} - \rho_i) + \beta (\rho_{i+1} - \rho_i) \quad \text{(mass diffusion)}
\]
\[
\nu (u_{i-1} - u_i) + \nu (u_{i+1} - u_i) \quad \text{(momentum diffusion)}
\]
\[
a (T_{i-1} - T_i) + a (T_{i+1} - T_i) \quad \text{(thermal diffusion)}
\]

If at some time the numerical scheme gives a temperature, density, or velocity significantly different from that of the bordering cells, these terms force a redistribution of mass, energy and momentum which tends to iron out the oscillation. Once the oscillation is smoothed out (i.e., the density, velocity, or temperature difference is small) these terms have little effect, since in a fluid dynamics problem the bulk fluid transport terms would predominate. However, if the asymptote of the transient problem is used to approximate the steady state solution, the formulation of the diffusion terms should cause them to be reduced to the appropriate diffusion terms for the type of motion involved as the fluid stabilized.

The difficulty here lies in trying to pose the mathematical model correctly. The distinction between conduction, diffusion, and transportation by the bulk velocity is really a mathematical artifice related to the scale of the phenomena involved, relative to the area under investigation. In a sense, then, the diffusion is an attempt to account for transport on a scale of motion which is below the mean motion that is being predicted. This idea is illustrated schematically in Figure 5.
The diffusion terms here are analogous to the so-called eddy transport terms in turbulent flow. For example, the thermal diffusion terms correspond to

\[ \dot{Q} = -K_t \frac{dT}{dx} A \]

hence

\[ \alpha \sim K_t \frac{A}{\Delta x} \]

where

\( K_t \) is analogous to the eddy thermal conductivity.

\( A \) is the cross-sectional area.

\( \Delta x \) is the cell length.

The momentum diffusion is similar to an eddy viscosity term with \( \frac{du}{dy} \) replaced by \( \frac{du}{dx} \).

\[ \nu \sim \mu_t \frac{A}{\Delta x} \]

where \( \mu_t \) is like an eddy viscosity.

The mass diffusion term is

\[ \beta \sim K_m \frac{A}{\Delta x} \]

where \( K_m \) is like an eddy mass transfer coefficient.

Assuming that the turbulent Prandtl number is of the order of one,
\frac{K_t}{\rho C_p} \text{ is of the order of } \frac{\mu_t}{\rho}.

Assuming that the eddy diffusion of mass is of the same order as the eddy diffusion of heat

\frac{K_m}{\rho C_p} \text{ is of the order of } \frac{K_t}{\rho C_p}

One then might expect

\frac{K_t}{C_p} \approx \mu_t \approx \rho K_m

Unfortunately none of these terms are known for pulsating compressible flow. The author suspects, however, that the minimum values required to smooth the numerical oscillations represent reasonable first approximations to true physical values.

The effect of wall friction can be taken into account by

\[ F \approx C_f A_s \frac{pu^2}{2} \]

where

- \( C_f \) - coefficient of wall friction.
- \( A_s \) - cell surface area.

The effect of heat transfer to the wall is

\[ Q_w = hA_s (T - T_w) \]

where

- \( h \) - heat transfer coefficient
- \( T_w \) - wall temperature

The program has the capability of using constant values for
Although the program could be made to compute these values, there is no justification for the added expense in view of the uncertainty of the methods that would have to be used and the fact that even rather drastic assumptions here have little effect on the overall solution. This point is also made by Rudinger (47).

An example of a run made including diffusion terms with data from Rudinger's (47) problem is shown in Figures 7 and 8. The pressure distributions are considerably smoothed and the pressure at the end wall shows close agreement with that obtained from the Method of Characteristics.

The effect here is not so noticeable, however, as it is when the method is used on another shock tube problem in which the bursting diaphragm is inside the tube close to the closed end. Figure 9 shows the results of using the method without the diffusion terms on this problem, and Figure 10 shows how the solution is smoothed out when the diffusion terms are added.

Actual test data for the shock tube was obtained by using a piezoelectric transducer to measure pressure variations downstream. An oscilloscope trace showing the magnitude of the pressure increase and decrease below atmospheric pressure is shown in Figures 11 and 12. Although the duration of the pressure shock is smeared out in the calculated values shown in Figure 12 (as would be expected due to the artificial diffusion) we can see that the amplitude agrees with the
measured value. The model is not really applicable after the tube
begins to fill because the end of the shock tube was not flared, and the
model did not allow for the vena contracta that would form when the
tube begins to fill from the atmosphere.

The pressure trace of Figure 11 was made on a Tektronix oscil-
loscope using a Kistler pressure transducer mounted on the shock tube
as shown in Figure 6. The pertinent information for the experiment
was as follows.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial pressure behind diaphragm</td>
<td>5220 psf</td>
</tr>
<tr>
<td>Atmospheric pressure downstream</td>
<td>2120 psf</td>
</tr>
<tr>
<td>Initial temperature in both sections</td>
<td>530 °R</td>
</tr>
<tr>
<td>Charge amplifier setting</td>
<td>3475</td>
</tr>
<tr>
<td>Pressure sensitivity</td>
<td>200 psi/volt</td>
</tr>
<tr>
<td>Oscilloscope vertical scale</td>
<td>0.010 volts/div.</td>
</tr>
<tr>
<td>Oscilloscope sweep</td>
<td>0.020 sec/div</td>
</tr>
</tbody>
</table>

![Diaphragm and Pressure Transducer Diagram](image)

Figure 6. Shock tube.
Figure 7. Run 5-2. Pressure at end wall from method using upstream properties at interface with pressure at wall equal to node pressure, diffusion added.
Figure 8. Run 5-2. Pressure distributions from method using upstream properties at interface with pressure at wall equal to node pressure, diffusion added.
Figure 8. Continued.
Figure 8. Continued.
Figure 9. Run 5-27. Pressure distributions in shock tube, no diffusion.
Figure 9. Continued.
Figure 10. Run 5-30. Pressure distributions in shock tube, diffusion added.
Figure 10. Continued.
Figure 11. Oscilloscope trace for downstream pressure in shock tube.

Figure 12. Calculated downstream pressure in shock tube compared to measured value.
Application to Pulse-Jets

In order to fully analyze the pulse-jet, a combustion addition was incorporated into the basic computational method. The program was written with enough generality to enable it to keep track of the amount of unburned oxygen and fuel present in each cell. Any one cell in the program may be flagged denoting that fuel from a fuel nozzle may be added to it at a specified rate. Every cell in the engine model has the potential for combustion provided the oxygen/fuel ratio in the cell is within the prescribed flammability limits and has the temperature and pressure necessary for ignition. The fuel and oxygen in each cell at a given time is determined from (1) the product of the initial concentrations (lb/lb of mass) and the mass in the cell, (2) any transfer of fuel and oxygen across the interfaces in the cell, and (3) any fuel added to a cell by injection from the fuel nozzle. When the flow is inward, oxygen is brought into the end cells with the mass of air sucked in from the atmosphere. Similarly, when the flow is outward, any unburned fuel or oxygen in the end cells is lost to the atmosphere.

Provided the temperature in a cell is sufficiently high for combustion, a fuel burning rate is determined from a burning rate coefficient, and the temperature, density, and fuel concentration in the cell. Multiplying this by $\Delta t$ gives the maximum amount of fuel that may be burned in the cell during the calculation time step. If this is
larger than the actual amount of fuel present in the cell then the total amount present is assumed to be the amount that can be burned. The oxygen required for combustion of this fuel is then determined from the oxygen/fuel ratio. If this amount of oxygen is not present, then the actual fuel burned is reduced to the amount that can be burned using the total oxygen in the cell. The energy of combustion is then computed from the product of the heat of combustion and the actual amount of fuel burned. The fuel, oxygen, fuel concentration, and oxygen concentration are then reduced according to the amount of fuel and oxygen consumed in the energy release.

Estimating the fuel burning rate of turbulent, compressible, nonsteady flow is extremely difficult and subject to a large uncertainty. Correlation with the SNECMA data of Bertin and Salmon (7), however, was achieved by calculating the burning rate on the basis of the following model. At each calculation step all cells are assumed to be completely mixed. If appropriate conditions exist, a plane flame front is assumed to move in positive and negative directions. The velocity of the flame front is estimated from the following empirical formula expressed by Gooding (21) in Mark's Mechanical Engineers' Handbook.

\[ 2V_f \approx (\text{BRTK})T^2 \]

where \( \text{BRTK} \) is on the order of \( 6 \times 10^{-6} \text{ ft/sec} \cdot \text{R}^2 \).
The maximum amount of fuel that could be burned is that which is contained in the volume swept by this flame in the given time step. It is determined as the product of fuel concentration, cross-sectional area, density, flame velocity, and the time step.

Computing combustion in this manner permits keeping track of a distributed combustion which is dependent on the gas dynamic processes and the air/fuel ratio in the cell. The mathematical model must maintain the oxygen/fuel ratio within the appropriate flammability limits, or the total combustion within the engine will be too weak, and the engine may quit. The major controlling parameters are the longitudinal fuel nozzle location and the degree of mixing. Since fuel and oxygen are transported along with the mass diffusion term, as well as the bulk velocity term, the quasi-one-dimensional model tends to account for turbulent mixing.

The complexity of the combustion process is such that it is very difficult to generate a self-sustaining cycle. A valveless pulse-jet must operate with combustion pressure amplitudes sufficiently large to impart a velocity to exiting gases such that they will have enough momentum to overcome atmospheric pressure and continue outward by virtue of their own inertia. This outward flow must continue until the pressure in the pulse-jet has dropped far enough below atmospheric pressure so that enough fresh air will be sucked in for combustion. This combustion must be strong enough then to increase the pressure back to the original value. Fresh air (oxygen) and fuel can then be
mixed together in the proper proportions (flammability limits) for the fuel to burn. The typical pulse-jet will not be completely evacuated, however, since for self ignition it must retain enough hot burned gases from the previous cycle to bring the bulk of the air/fuel mixture up to ignition temperature. Also, the fresh air should not ignite immediately upon entering the combustion chamber as this would cause the pressure to increase prematurely, retarding aspiration of air from the atmosphere. This can cause the engine to lose the advantage of supercharging or precompression that it gains from the inertia of the inrushing air. The more serious problem, however, is that it can prevent the engine from getting sufficient oxygen to support a strong combustion. Since there is no "mechanical flywheel" effect, once a static engine has missed a cycle it ceases to operate, and must be purged of excess fuel and restarted. Ideally the combustion process should begin just when the pulse-jet has filled to maximum density. It then should proceed quite rapidly.

Although complete data on pressure, temperature, density, and oxygen and fuel distributions was not known for the SNECMA model, enough information was given by Bertin and Salmon (7) for a comparison model to be defined. By starting the calculations at the time when the engine was taking in oxygen from the ends, and the temperature within the combustion chamber was just approaching that required for burning, it was possible to describe the engine model sufficiently to
determine, by comparison with the experimental results, the validity of the method.

Figure 13 shows the results of the Control Volume method with the combustion addition, as applied to the French Valveless Pulse-jet, SNECMA Model 5158, compared to experimental results for this model reported by Bertin and Salmon (7). As can be seen from the pressure/time traces for one complete engine cycle, shown at several locations in the engine, the computational results obtained agree closely with the experimentally obtained data.

Figure 14 shows the node division used in the computational model for this comparison.

**Stability and Error**

There is no exact test for stability of non-linear partial differential equations as pointed out by Sauerwein (48). Such tests ultimately rest on a heuristic approach. The criteria generally accepted is the Courant, Friedrichs, Lewy condition

\[(u+a)\Delta t < \Delta x\]

This condition is really a physical limitation on any wave process, and holds whether the equations are expressed in finite difference or characteristic form. It is for this reason that it is so important in terms of saving computational time to use as simple an
Figure 13. Pressure/time traces for control volume method compared to measured values at six engine locations.
Figure 14. Node division used for analysis of French valveless pulse-jet Model 5158.
integration scheme as possible. This is particularly true in pulsating combustion flows where \( a \) and perhaps even \( u \) may be several thousand feet per second. One should not expect any savings in computational time if, instead of a gas, the material were a liquid or even a solid, for although the particle velocity, \( u \), would likely be low, the wave propagation speed, \( a \), would be high.

An approach that is often used in estimating the errors of a partial differential equation relies upon expressing the differentials in terms of a finite Taylor's Series expansion, and noting the order of magnitude of the terms omitted in combining the differentials. It is important to remember that there are two other sources of error which are often overlooked. These are: (1) the partial differential equations did not actually describe the process correctly in the first place, and (2) the boundary conditions may not be posed realistically when the very nature of the problem demands a complex interaction with the boundary.

A comparison of computer runs made using the Control Volume technique, for different combinations of boundary conditions and methods of selecting interface values, with solutions obtained using the Method of Characteristics, is shown in Appendix C. The results obtained show that, on the average, the Control Volume method agrees with the Method of Characteristics, except that error induced at the boundary propagates throughout the solution. No diffusion or dampening was added in these runs.
V. CONCLUSION

With the Control Volume technique developed in this dissertation it is possible to simulate the wave action and a distributed combustion process in a valveless pulse-jet. It is not necessary to use a large number of small cells; a fairly coarse grid can be used if the cells can be judiciously selected to have about the same volume. The turbulent mixing that occurs in the jet can be approximately simulated by the diffusion terms.

The integration scheme developed is fast and simple, and although the wave process is subject to a rigid stability criteria the method is not unreasonably slow. For example—the computations for the two complete pressure cycles shown in Figure 7 took about one minute of computer time on a CDC 3300 computer. The computations were made for 14 cells, with a calculation time step, \( \Delta t \), of \( 5 \times 10^{-5} \) seconds, and complete data printout every 10 \( \Delta t \). The computer time included about 1000 full lines of data printout. The time required might be reduced still further by fully optimizing the computer programming.

The physical methodology developed here is logically simple, yet it could be extended to solve two and three-dimensional unsteady problems. It should also be applicable to problems involving other states of matter, i.e., fluids and solids as well as gases, if the
physical relationships can be defined correctly. "Correctly" means that one must not make apriori stipulations that are physically inconsistent with the fundamental laws of nature. It would not be correct, for instance, to specify that a substance was incompressible or maintained a constant temperature. Instead, the appropriate equation of state and the constitutive relationships must be included in the physical model. If the substance is approximately incompressible or isothermal this will show up in the solution, provided the physical relationships were defined consistently.

For those who understand the physical processes involved, and who can correctly define a physical model for their specific problem, this method of direct integration should prove extremely useful.
BIBLIOGRAPHY


APPENDICES
Several early attempts were made to construct an intermittent jet reaction engine. In 1908, the French engineer Lorin proposed using a type of conventional engine merely to produce exhaust gas which would be expanded through a nozzle to produce a propulsive jet. No power was to have been extracted from the crank shaft. See Figure A-1. Foa (17) reports that about 1909, Marconnet proposed a jet engine similar to the Karavodine gas generator which had been developed a year earlier. The Karavodine machine as mentioned by Edelman (14) has been described as

... a long straight tube, one end of which is fitted with a check valve, the other end open and directed at a turbine. Inertia of the air column created low pressures and subsequent induction of fresh air after each explosion.

The Marconnet engine, shown in Figure A-2, might be called the forerunner of the modern pulse-jet.

Since one of the principal disadvantages of the Lorin scheme was the small mass of air handled by the engine, several attempts were made to increase the mass of air in the propulsion jet by using the reciprocating engine exhaust as a primary fluid in a jet ejector. One interesting device, invented by Melot, uses a novel type of free piston engine. See Figure A-3. Smith (54) gives the following description
Figure A-1. Lorin engine.

Figure A-2. Marconnet jet.

F. Free-flying piston
G. Venturi
H. Piston lower sleeve inlet ports
J. Piston lower sleeve exhaust ports
K. Exhaust conduit
L. Piston upper sleeve inlet ports

Figure A-3. Melot free piston gas generator with ejector augmentor.
of this device.

The air-cooled, two-stroke engine had two opposed cylinders in an axial alignment, and a common, free flying piston F having ported sleeve extensions. In the position shown, a fresh charge of air and fuel has been admitted through the venturi G, the common cylinder inlet duct and the piston ports H, and compressed in the upper cylinder which is at the point of ignition. The lower cylinder has discharged its effluents through piston exhaust ports J and the lower branch of the exhaust conduit K which delivers to the multiple nozzle device. A new charge is simultaneously being aspirated by lower cylinder by way of inlet ports L. Starting is effected by compressed air and electric ignition, but compression ignition is relied upon for continuous running.

In 1927 Melot type ejector tubes were tested at the Langley Memorial Aeronautical Laboratory using compressed air at normal temperatures for the primary fluid. The nozzle was venturi-shaped and had a throat diameter of 3/8 inch. The main diffuser tube was about 45 inches in overall length. Three annular nozzle rings were used. The diffuser tube was designed to expand air from 185 psig to atmospheric pressure at the mouth, but the best results were obtained at approximately 100 psig. At this pressure the thrust was 137% of the theoretical thrust of a free jet, used as a basis of comparison, while the actual thrust of a free jet was only 88%. At 185 psig the thrust of the Melot augmenter fell to 122% while the thrust of the free jet improved slightly to 90%. This indicates that the diffuser employed was unsuitable for use in conjunction with the Melot nozzles and did not provide for adequate expansion of the jet.
from the pressure chamber. It was found that varying the distance from the mouth of the pressure nozzle to the entrance of the diffuser between two and nine inches produced no substantial effect on the thrust. A spacing of six inches, approximately that used in the Melot system, gave the best results. Smith (54) also reports that the Melot design was better than other types of augmentors, and represented a "very creditable advance."

Figure A-4 shows one version of the Pulsating Burner or Pulse Pot developed by Reynst. This is essentially a single opening pulsating combustor. Both the inlet for the air for combustion and the outlet for the exhaust gases are the same opening. This device is discussed in detail by Reynst (43).

Reynst (43) reports that Paul Schmidt, a German engineer, was dubious about the ability of any ejector to produce a net thrust. He proposed a device in which the combustion of gases in a straight tube would cause a piston-like transfer of energy to the air in the tube. In principle, the machine he developed is almost identical to that conceived by Marconnet. The pulse-jet invented by Schmidt, and manufactured by the Argus Firm in Berlin is shown in Figure A-5. This is the famous V-1 Engine of World War II.

A series of reed valves in the front end permit air to enter, where it is mixed with fuel and ignited. The combustion pressure closes the valves and forces the air and combustion products out the
rear with instantaneous velocities on the order of 1500 ft per sec. The inertia of this "slug" of gas tends to partially evacuate the tube to a pressure about five psi below atmospheric. The reed valves then open, admitting a new charge of air. At low forward speeds some air also enters from the rear of the tube. The sudden deceleration of these fresh charges may result in a slight precompression. The charge is ignited by hot spots and residual gas remaining from the previous explosion. This again closes the front valves and the cycle repeats. For proper operation, the natural frequency of the reed valves should coincide with the natural frequency of the gas tube. If one can ascertain the average temperature of the gas in the tube, the frequency can be approximated by simple quarter-wave organ pipe theory. \[ f = \frac{a}{4L} \] where \( a \) is the acoustic velocity corresponding to the mean temperature of the gas in the tube, and \( L \) is the length of the tube. For a preliminary design one might estimate the frequency of the reed valves, considering them to be simple cantilever beams.

The first successful valved pulse-jet in this country was designed and built by William Schubert (50) and his associates. This reference gives a fairly clear description of an early constructed pulse-jet.

One of the major drawbacks of the conventional pulse-jet is the short valve life. Since the valves are excited at their natural frequency, which may be from 50-300 cycles per second, and are exposed
to very hot combustion gases, their life expectancy is understandably limited.

William Schubert (50) and associates also built one of the first successful pulse-jets in this country which did not use a mechanical valve. An intake tube was tuned to the same frequency as the main combustion chamber tailpipe in an effort to retard the reversed flow of gas. See Figure A-6.

Another valveless pulse-jet was built and tested by Logan (32). Its operation was similar to that of the Schubert jet, but the intake tube was at right angles to the combustion chamber. See Figure A-7.

A third valveless pulse-jet using a slightly re-entrant intake tube to form a type of borda inlet into the main combustion chamber is analyzed by Yen (62). He determines the wave action for a few jet tubes by the Method of Characteristics, and uses a water table analogy to illustrate the action of the borda tube in restricting the flow reversal in the intake tube.

Another valveless pulse-jet, the Escopette shown in Figure A-8, was built by the SNECMA Organization in France. In this jet a recuperator tube spaced a short distance from the mouth of the detector captures the reverse flow and turns it 180° to discharge backwards to provide additional thrust. In a multiple installation of four units, these jets were used to power the Emouchet sailplane. Fuel was supplied at 4.4 to 20.6 psi. With different fuel sprayers the thrust varied
Figure A-4. Reynst pulse pot.

Figure A-5. V-1 pulse-jet.

Figure A-6. Schubert valveless pulse-jet.

Figure A-7. Logan valveless pulse-jet.
from 6.6 to 22.0 lb with a fuel consumption of 0.24 to 0.66 lb per minute. The installed weight was 11.24 lb per unit.

A later version, the Ecrevisse shown in Figure A-9, did not have a recuperator; instead both the intake and the propulsion nozzle were directed toward the rear. This jet was 97.5 inches long, and weighed only 22 lb. Static thrust developed was 66.2 lb with a specific fuel consumption of 1.2 lb per hour per lb of thrust.

An American inventor, Eugene Gluhareff (19), has developed a jet which is easy to start, produces static thrust, and has no moving parts. In this jet, shown in Figure A-10, preheated propane expands through a three-stage ejector, drawing in the air for combustion. Mr. Gluhareff refers to his ejector as "sonic synchronized." The engines have been used by the inventor on numerous devices of his own design, including a strap-on-the-back helicopter, flying platform, jet go-cart, and orchard heater. The main drawback of the engine is its rather high fuel consumption, about one-half that of a ram jet at equivalent thrust, according to its inventor.

Lockwood (29, 30, 31) made further developments and modified the SNECMA Ecrevisse engine until he achieved the amazingly low specific fuel consumption of 0.85 lb of fuel per hour per lb of thrust. This engine is shown in Figure A-11.

Melenric (34) also sells an engine which uses the ejector action of preheated fuel to assist the air intake and facilitate starting.
Figure A-8. Escopette jet.

Figure A-9. Ecrevisse jet.

Figure A-10. Gluhareff jet.
Melenric uses multiple air intakes, similar to some versions of the SNECMA engines, and faces these intakes rearward. The air intakes are also slightly shrouded. Presumably this is so the engine will operate more favorably in forward flight. The Melenric jet is shown in Figure A-12.

The possibility of pulse-jet application for tip jet propulsion on helicopters is obvious. If a design could be found that would advantageously utilize the ram pressure and wave precompression, the pulsating jet engine might someday challenge the turbojet in numerous applications.

The thrust specific fuel consumption of these engines is given in Table A-1.

### Table A-1. Thrust specific fuel consumptions for several pulse-jets.

<table>
<thead>
<tr>
<th>Engine</th>
<th>S.F.C.</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original V-1 Pulse-jet</td>
<td>4.0</td>
<td>Reynst (43)</td>
</tr>
<tr>
<td>Improved V-1 type Pulse-jet</td>
<td>2.6</td>
<td>Reynst (43)</td>
</tr>
<tr>
<td>Estimated Possible Performance</td>
<td>1.5-1.6</td>
<td>Reynst (43)</td>
</tr>
<tr>
<td>R.P.I. Wave Engine with flapper valves</td>
<td>2.8</td>
<td>Foa (17)</td>
</tr>
<tr>
<td>Schubert's Valved Pulse-jet</td>
<td>2.86</td>
<td>Schubert (50)</td>
</tr>
<tr>
<td>Gluhareff Jet</td>
<td>6.0</td>
<td>Gluhareff (19)</td>
</tr>
<tr>
<td>Logan Valveless Pulse-jet</td>
<td>1.5</td>
<td>Logan (32)</td>
</tr>
<tr>
<td>SNECMA Escopette Valveless Pulse-jet</td>
<td>1.8</td>
<td>Bertin and Salmon (7)</td>
</tr>
<tr>
<td>SNECMA Ecrevisse Valveless Pulse-jet</td>
<td>1.2</td>
<td>Bertin and Salmon (7)</td>
</tr>
<tr>
<td>Lockwood Ducted Valveless Pulse-jet</td>
<td>0.85</td>
<td>Lockwood (29)</td>
</tr>
<tr>
<td>Melenric Jet</td>
<td>3.0</td>
<td>Melenric (34)</td>
</tr>
</tbody>
</table>
Figure A-11. Lockwood ducted Ecrevisse jet.

Figure A-12. Melenric jet.
APPENDIX B

A Two Dimensional Unsteady Method of Characteristics

The mathematical concept of the Method of Characteristics can also be extended to unsteady flows in more than one direction. The mathematical procedure for transposing the basic equations into characteristic equations has been known for some time, and is detailed in the book by Von Mises (60). Several numerical schemes have been proposed for solving these characteristic equations, but only a few computer solutions have been made. The procedure is so complex that limitations of computer time and storage seriously restrict the kinds of problems that can be solved.

In the methods that have been proposed for solving multi-dimensional, unsteady, inviscid flow it is often difficult to visualize the physical relationships between the Method of Characteristics for one-dimensional flow and the multi-dimensional model. An attempt will be made here to show that multi-dimensional flow may be solved by the intersection of characteristic surfaces just as one-dimensional flow may be solved by the intersection of characteristic lines. A physical model will first be developed to show how such characteristic surfaces might be related to the concept of the Mach conoid used in other multi-dimensional methods. Then an original calculation scheme will be proposed based on the physical model. Although this
calculation scheme would also be too cumbersome for present-day computational methods, it is believed that the physical explanation of this method will lead to a better understanding of the physical meaning of characteristics in multi-dimensional flows.

The method to be developed here requires a minimum amount of interpolation and always follows the true physical characteristics with the approximation that small sections of these surfaces may be considered planar. The characteristic spacing and time step would probably be only as large as that required to give similar accuracy for one-dimensional unsteady flow.

A simple illustration can help explain the physical model. Imagine a pebble dropped on the surface of a flowing stream; the ripple expands in an increasing radius as time progresses, but it is carried downstream by the current. Imagine taking a sequence of photographs of this ripple with a fixed camera (Figure B-1) and then stacking these photographs one on top of another with a spacing between photographs to correspond to a time scale (Figure B-2).

It can be easily seen that if the stream is flowing uniformly in a given direction with constant $u$, $v$ components, and with a constant speed of propagation of ripple disturbances, $a$, the circles in Figure B-2 can be arranged so as to envelop a cone. Of course, if the stream direction changes, the cone enveloped in Figure B-2 may
Figure B-1. Ripple photos.

Figure B-2. Mach cone in a two-dimensional unsteady flow formed by stacking "ripple" photos.
become warped or curved as time progresses. These warped cones are termed characteristic conoids. This is the physical interpretation of the characteristic conoids described by both Thornhill (58) and Ferri (16). Imagine every point in the \( x-y \) plane at any given time, \( t \), to generate a conoid. Although for any finite period of time the conoid will generally be curved, an interval of time small enough such that the start of the conoid will be a cone can still be imagined. The sequence of conoids from any arbitrary curve in space (in our case, the \( x-y \) plane) may be considered to generate two characteristic surfaces which are the tangent surfaces forming the envelope of these cones, as can be seen in Figures B-3 and B-4. On any cone the genetrices which are tangent to these surfaces form a pair of bicharacteristic lines.

Starting from an arbitrary line on an initial \( t_0 \), \( x-y \) plane, each conoid extends forward in time generating two characteristic surfaces. These characteristic surfaces extend forward in time, past their intersection on the \( t_0 + dt \) plane; this intersection may be thought of as still being generated by the same conoids emanating from vertices on the \( t_0 \), \( x-y \) plane. They can also be generated for a short distance by characteristic curves with vertices on the line of intersection of the \( t_0 + dt \), \( x-y \) plane as shown in Figure B-5.

Although physically it is fairly easy to see how the characteristic cone extends forward in time from a given point, it may be a
Figure B-3. Author's scheme for advancing all conoids simultaneously without interpolation.

Figure B-4. Characteristic definitions.
little more difficult to physically interpret the nappe of the Mach conoid extending backward in time. Consider some arbitrary point $P$ on line $A'B'$ on the $t_0+dt$, $x$-$y$ plane, and the Mach cone which has its vertex at this point and which serves to extend the characteristic surfaces through $A'B'$ for a short distance in time. This surface extends backward to the arbitrary curve $AB$ in the $t_0$, $x$-$y$ plane. Since $A'B'$ is an arbitrary direction in the $t_0+dt$, $x$-$y$ plane, there are, of course, an infinite number of directions that could be taken in the $t_0+dt$, $x$-$y$ plane and an infinite number of characteristic surfaces passing through a point. These surfaces extend backward in time to some initial $x$-$y$ plane. If the surfaces which envelop the forward time cones are considered as being extended backward a short distance, a backward running nappe of the cone would be enveloped. See Figure B-6.

Thornhill (58) proposes a numerical scheme using these backward cones. He states that, in theory, it should be possible to integrate numerically in a progressive manner along a hexahedral grid of one of the above types, starting from the known conditions of a problem on an open boundary surface. Ferri (16) gives the following equations to use with such a method.

\[
\left( \frac{\partial u}{\partial t} - \frac{\partial v}{\partial n} \right) + \frac{2}{\gamma-1} \left( \cos \xi \frac{\partial a}{\partial t} + \sin \xi \frac{\partial a}{\partial n} \right) = 0
\]
Figure B-5. Relationship of characteristic cones at different time intervals.

Figure B-6. Backward running conoid.
\[
\left( \frac{\partial u}{\partial n} + \frac{\partial v}{\partial \ell} \right) + \frac{2}{\gamma - 1} \left( \sin \frac{\partial a}{\partial \ell} - \cos \frac{\partial a}{\partial n} \right) = 0
\]

These correspond to similar equations of Thornhill (58) where

\[
\frac{\partial}{\partial \ell} = \text{variation along bicharacteristic}
\]

\[
\frac{\partial}{\partial n} = \text{variation along line in (t = constant) plane tangent to cone at the bicharacteristic}
\]

The details of the proposed scheme are not given, but by referring to Figures B-7 and B-8 (Figures 2 and 3 of Thornhill (58)) where \( P_1P_2P_3 \) are points for which the values of \( u, v, \) and \( a \) are known, the values of \( Q_1Q_2Q_3 \) might be found by some kind of interpolation. If \( P_1P_2P_3 \) lie in some \( t_0, x-y \) plane it is possible to finite difference six of the above equations along the three bicharacteristics and three of the normals. For example in Figure B-7

\[
\frac{\partial u}{\partial \ell} \left( P_4Q_3 \right) \sim \frac{u_{P_4} - u_{Q_3}}{\text{length of } P_4Q_3} = \frac{u_{P_4} - u_{Q_3}}{\sqrt{(x_{P_4} - x_{Q_3})^2 + (y_{P_4} - y_{Q_3})^2 + \Delta t^2}}
\]

\[
\frac{\partial v}{\partial n} \left( Q_3P_2 \right) \sim \frac{v_{Q_3} - v_{P_2}}{\text{length of } Q_3P_2} = \frac{v_{Q_3} - v_{P_2}}{\sqrt{(x_{Q_3} - x_{P_2})^2 + (y_{Q_3} - y_{P_2})^2 + \Delta t^2}}
\]

This would give six equations for the six unknowns of the point \( P_4 \) (i.e., \( u, v, a, x, y, \Delta t \)). Some numerical difficulties might be
Figure B-7. Interpolation scheme for conoid not passing through known points.

Figure B-8. Interpolation scheme for conoid passing through known points.
anticipated however, since the next three points taken for a second cone would not necessarily determine a new point at the same $\Delta t$. This could present a problem in the calculations for another set of points using the values of the points just determined.

In lieu of this network, Ferri (16) suggests a different network, attributed to C. Ferrari, consisting of the intersections of characteristic surfaces with planes parallel to the reference planes, as often being more convenient for calculation. A similar network was proposed by Colburn and Dolph (10) and further developed by Holt (22). These methods require additional interpolation.

Butler (9) describes a different method, employing difference relationships along four bicharacteristics and one time-like ordinary curve through each point. From these difference relationships the derivatives of the dependent variables at the unknown point are eliminated. A unique feature of this method is that each point is computed without referring to data outside its domain of dependence.

Richardson (44) reviews the general theory with some emphasis on the numerical method, and discusses how certain boundary conditions can be introduced.

Burnat, Kielbasinski, and Wakulicz (8) describe an application of the Method of Characteristics to the set of equations of a compressible, inviscid, non-isentropic, supersonic, stationary, three-dimensional gas flow.
Podladchikov (29) briefly discusses work in which he considered three-dimensional, steady, supersonic motion of a gas with a detached shock wave. It appears that he uses a network of tetrahedrons similar to that described by Thornhill (58).

In order to solve the nonsteady, inviscid, two-dimensional case described by Ferri (16) and Thornhill (58) a new method which requires no interpolation was conceived. To understand how the new method would determine values $u_f$, $v_f$, and $a_f$ at some new time $t_0 + dt$, imagine three mach cones emanating from points $A$, $B$, and $C$ in the $x$-$y$ plane, where $u_0$, $v_0$, and $a_0$ are known for each point. The geometry of each cone is determined by the simple circle equation:

$$(x-x_0 - u_0 dt)^2 + (y-y_0 - v_0 dt)^2 = (a_0 dt)^2$$

It is now possible to determine point $f$ by the intersection of characteristic surfaces $AA_iB_iB$ and $CC_iB_iB$ and the $x$-$y$ plane at a distance $dt$ from the known $x$-$y$ plane. If $A$, $B$, and $C$ are close enough, this point can be approximated quite closely by the intersection of two straight lines which lie tangent to paired circles cut by the cones in the $x$-$y$ plane. It is somewhat arbitrary which intersection of tangent lines is used. However, as it is necessary to keep track of only two family pairs of characteristic surfaces identified after the first time step, it is probably better to use the
intersection of the tangent lines representing surfaces which are on
the outsides of the circles, (e.g., point $f$ located by the intersection
of the dot-dash tangent lines in Figure B-3). The subscript $i$ is
used to indicate the points of tangency; therefore $AA_i$ is a bichar-
acteristic. Then the finite difference forms of the equations for cone
A are

$$\frac{u_{Ai} - u_{A0}}{\Delta l_A} - \frac{v_{f} - v_{Ai}}{\Delta n_A} + \frac{2}{\gamma - 1} \left[ \cos \zeta_A \left( - \frac{a_{Ai} - a_{A0}}{\Delta l_A} \right) + \sin \zeta_A \left( \frac{a_{f} - a_{Ai}}{\Delta n_A} \right) \right] = 0$$

$$\frac{u_{f} - u_{Ai}}{\Delta n_A} + \frac{v_{Ai} - v_{A0}}{\Delta l_A} + \frac{2}{\gamma - 1} \left[ \sin \zeta_A \left( - \frac{a_{Ai} - a_{A0}}{\Delta l_A} \right) - \cos \zeta_A \left( \frac{a_{f} - a_{Ai}}{\Delta n_A} \right) \right] = 0$$

There are similar pairs of equations for cones B and C. All
together these yield six equations and six of the twelve unknowns:

$${}^uAi, {}^uBi, {}^uCi, {}^u_f, {}^vAi, {}^vBi, {}^vCi, {}^v_f, {}^aAi, {}^aBi, {}^aCi, {}^a_f$$

The values of $\Delta l = \text{length } AA_i$, and $\Delta n = \text{length } A_f$ are
considered known, since they are determined beforehand from the
geometry, depending only on the conditions at the known x-y plane.

The other six required equations can be obtained from the
original equations of the form
\[ a^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -(u \frac{\partial a}{\partial x} + v \frac{\partial a}{\partial y} + \frac{\partial a}{\partial t}) \frac{2a}{\gamma-1} \]

This equation will be used six times, the \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) variation being determined by the direction of \( \partial t \) and \( \partial n \).

\[ a^2 \left( \frac{u_{i} - u_{0}}{x_{i} - x_{0}} + \frac{v_{i} - v_{0}}{y_{i} - y_{0}} \right) \]

\[ = - \left[ u_{0} \left( \frac{x_{i} - x_{0}}{x_{i} - x_{0}} \right) + v_{0} \left( \frac{y_{i} - y_{0}}{y_{i} - y_{0}} \right) + \left( \frac{a_{i} - a_{0}}{\Delta t} \right) \right] \frac{2a_{0}}{\gamma-1} \]

Since the \( x \) and \( y \) values are already determined by the geometry, no new unknowns are introduced. Thus it is possible to determine 12 unknowns with 12 equations after the geometry has been determined.

This system can be reduced to three equations and three unknowns, \( u_f', v_f', \) and \( a_f' \), but these equations are non-linear. To get a solution to these equations, an iterative approach such as Newton's method might be used. A significant improvement, however, would be to obtain explicit solutions for the \( u_f', v_f', \) and \( a_f' \), and incorporate these into the program.

In the first time step four final points on the \( t_0 + dt \) plane would be calculated for each initial point. The grid might be thought of as defining \( x_-, x_+, y_-, y_+ \) characteristic surfaces. For the next time step the flow field is swept through four times, considering for the first sweep cones originating at only those points defined by the
intersections of $y_-$ and $x_+$ characteristics. The tangents to these cones in the $t_0 + 2\Delta t$ time plane will be those that also define $y_-$ and $x_+$ characteristics, thereby continuing the line of intersection of the $x_+y_-$ surfaces up in time. The grid would again be swept through considering $y_+x_+$ intersections, again for $y_-x_-$ intersections, and finally for $y_+x_-$ intersections. Calculations would then advance to a new time plane. This is shown in Figure B-9.

The solid boundaries could probably be handled by imagining the characteristic cones, and hence the corresponding surfaces, to be reflected as though the reflection came from a virtual image appearing to come through the wall as shown in Figure B-10.

The calculation procedure could be summarized briefly as follows:

Input:

Input number of sections, the $\Delta t$ time step, and total time.

For each section, input the following information: $x$ and $y$ values for the beginning and end of section, boundary equation for solid surfaces, boundary equation for open surfaces, and initial $u$, $v$, $a$, $x$, and $y$ values for each point $(I, J, K)$ in the section.

Computation:

For each section, select three close points, testing the coordinates of each point to make certain it lies in a given section.

Compute the geometry. For example, the $x$ and $y$ values of each point where the tangent lines touch each circle, and the $x$ and
Figure B-9. First and part of second time step calculation showing how characteristic surfaces are followed in time.
Figure B-10. Reflection of characteristic surfaces.
y values of the intersection point of the pair of tangent lines.

Test the location of each point computed to make certain it does not lie beyond a solid surface. If the point does lie beyond a solid surface, consider the characteristic reflected, and recompute the location of each point inside the section. If the point lies beyond an open surface, proceed as usual.

Compute $\delta$, $\delta_n$, and $\delta_f$ for each cone.

Solve the equations for each new point at the new $\Delta t$ and label it with appropriate indices.

Print $x_f'$, $y_f'$, $u_f'$, $v_f'$, $a_f'$, and $(I, J, K)$.

Repeat the calculations for the entire section.

After all new points have been computed for a section go to the next section and repeat the calculations for this section.

So far shocks have not been considered; however, as a number of characteristics of the same family intersect, pressure gradients would steepen into shocks analogous to one-dimensional unsteady flow. The basic equations hold on either side of the shock. It should be possible to match the flow from one side to the other. This is done by assuming the flow is quasi-static through the shock, and by using the appropriate steady state shock relations in this narrow region.

Intuitively it seems that the same physical reasoning would lead
to a similar numerical scheme for the case of three space variable, unsteady flow. The disturbance might be thought of as originating under the surface of the flowing fluid, and superimposed models of spheres of ever increasing diameters might be imagined as generating the Mach conoid. See Figures B-11 and B-12. At any time interval $dt$, every point in the $x$-$y$-$z$ space might define a conoid, and any known curve of $x_0, y_0, z_0$ points would define characteristic surfaces similar to the two dimensional case. The intersection of these surfaces would then define each new point at time $t+dt$.

Although this procedure was not used for the pulse-jet analysis it might merit investigation by those who have the facilities to pursue the method further.
Figure B-11. Disturbance in three-dimensional unsteady flow.

Figure B-12. Three-dimensional unsteady mach conoid.
APPENDIX C

Comparison of Results from Control Volume and Characteristic Methods

The results of the Control Volume method without diffusion, shown in Figures 3 and 4 of Chapter IV, are only one example out of many possible variations of the method. The different ways in which the boundary conditions and the interface properties are determined each constitute separate numerical schemes.

Figures C-1 through C-12 show the results of computations made using these different variations of the Control Volume method with data from Rudinger's (47) example problem. Figures C-1 through C-6 compare non-dimensional pressure versus time at the closed end with the result Rudinger obtained using the Method of Characteristics.

In all these runs the boundary condition at the open end was approximated by setting the open end interface values of $\rho$, $V$, $T$ equal to the last cell values. $P$ at this interface was set equal to the atmospheric pressure whenever the flow in the last node was outward. When the flow in the last node was inward, and the pressure there was not greater than atmospheric, the velocity at the atmospheric interface was set equal to this node velocity and $P$, $\rho$, $T$ were compared assuming quasi-steady isentropic inflow. \footnote{The isentropic assumption would probably require a flared or rounded tube exit to prevent separation or a borda nozzle type}
in the last node is greater than atmospheric when the velocity there is inward, the velocity of the atmospheric interface is assumed to be zero.

Since the upstream method worked well as long as the spatial distributions were smooth, and averaging seemed to smooth out the wall induced undulations, a mixture of the two methods was tried. The results of this attempt are shown in curves 3-15 in Figures C-1 and C-7. Whenever the upstream velocity could be determined, upstream properties were used at the interfaces; otherwise the program simply used average properties. It will be seen in Figure C-1 that this method induced pressure oscillations at the wall, and did not remove the wall induced undulations.

Run 3-27 in Figures C-2 and C-8 was made in an attempt to determine whether characteristic logic and finer grid resolution (i.e., more cells) would remove the pressure oscillations at the wall which began to show up in Run 3-13 in Figure C-6. This did not appear to cause any significant difference in the results.

Figures C-3 through C-5 and C-9 through C-11 show the results of attempts to increase the order of the integration method which has been similar to a second order integration scheme. Although the contraction from forming during inflow. In an effort to minimize these inflow losses, flared ends are sometimes placed on the tailpipes of pulse-jets.
Equations 4.7, 4.8, and 4.9 of Chapter IV appear only to retain the first derivative of a Taylor Series Expansion, the method of calculating this derivative involves information that is normally carried in the second order term of a Taylor Series. By first calculating a trial value and then using these trial values to determine flow properties, and from them average derivatives over the time interval, information may be included on how the first derivative varies in time. This is essentially the physical meaning of the second derivative in the Taylor Series.

Since the time averages of the initial and trial values, which are used to compute the average rate of change and hence the final values, are not the same as the time averages of the initial and final values, this final value may be considered as merely a second trial value, and the new average values may be used to compute new rates of change and new final values. This two trial integration method was tried with the donor cell method using characteristic boundary conditions as shown in Figures C-4 and C-10. It was used with the donor cell method with the pressure of the adjacent cell used for the wall pressure, as shown in Figures C-3 and C-9. It was also tried with the averaging method using the characteristic wall boundary conditions as shown in Figures C-5 and C-11. There appears to be no advantage in going to the two trial integration scheme. Refinement of the boundary conditions is more important.
Figure C-1. Run 3-15. Pressure at end wall from method using upstream or averaged properties at interface with pressure at wall equal to node pressure.

Figure C-2. Run 3-27. Pressure at end wall from method using average properties at interface with characteristic end, more nodes.
Figure C-3. Run 4-10A. Pressure at end wall from method using upstream properties at interface with pressure at wall equal to node pressure—two trials.

Figure C-4. Run 4-10B. Pressure at end wall from method using upstream properties at interface with characteristic end—two trials.
Figure C-5. Run 4-14. Pressure at end wall from method using average properties at interface with characteristics end—two trials.

Figure C-6. Run 3-13. Pressure at end wall from method using average properties at interface with pressure at wall equal to node pressure.
Figure C-7. Run 3-15. Pressure distributions for method using upstream or averaged properties at interface with pressure at wall equal to node pressure.
Figure C-7. Continued.
Figure C-8. Run 3-27. Pressure distributions for method using average properties at interface with characteristic end, more nodes.
Figure C-8. Continued.
Figure C-9. Run 4-10A. Pressure distributions for method using upstream properties at interface with pressure at wall equal to node pressure--two trials.
Figure C-9. Continued.
Figure C-10. Run 4-10B. Pressure distributions for method using upstream properties at interface with characteristic end--two trials.
Figure C-10. Continued.
Figure C-11. Run 4-14. Pressure distributions for method using average properties at interface with characteristic end--two trials.
Figure C-11. Continued.
Figure C-12. Run 3-13. Pressure distributions from method using average properties at interface with pressure at wall equal to node pressure.
Figure C-12. Continued.
APPENDIX D

Control Volume Computer Program

A listing of the computer program for the Control Volume method is included here. The necessary input for running this program to analyze a valveless pulse-jet is listed below.

For each cell (I) input the following initial quantities:

- \( P(I) \) - pressure (lb/ft\(^2\))
- \( T(I) \) - temperature (°R)
- \( V(I) \) - velocity (ft/sec)
- \( A(I) \) - cross-sectional area (ft\(^2\))
- \( VOL(I) \) - volume (ft\(^3\))
- \( S(I) \) - surface area (ft\(^2\))
- \( FCON(I) \) - fuel concentration (lb fuel/lb mass)
- \( OXCON(I) \) - oxygen concentration (lb oxygen/lb mass)

It is also necessary to input the following:

- \( TIMINT \) - initial time (sec)
- \( TIMMAX \) - maximum time (sec)
- \( NSTEP \) - number of calculation steps between printout
- \( N \) - number of cells
- \( PATM \) - atmospheric pressure (lb/ft\(^2\))
- \( TATM \) - atmospheric temperature (°R)
- \( DATM \) - atmospheric density (lb/ft\(^3\))
- \( FLAG \) - 1.0 value indicates first cell is open to atmosphere
- \( DELT \) - calculation time step (sec)
- \( CP \) - specific heat at constant pressure in tube (BTU/lb °R)
- \( CV \) - specific heat at constant volume in tube (BTU/lb °R)
GK - specific heat ratio of air sucked in from atmosphere
R - gas constant for air \( \approx 53.3 \text{ ft lb/lb } °R \)
K - number of cell into which fuel is injected
FRT - fuel injection rate (lb/sec)
TREQ - ignition temperature (°R)
PREQ - ignition pressure (lb/ft²)
OXFR - stoichiometric oxygen to fuel ratio (lb/lb)
CHB - heat of combustion (BTU/lb)
BRTK - burning rate constant (i.e., twice the flame speed \( \approx BRTK \times T^2 \)) (ft/sec °R²)
DIFF - mass diffusion term \( (-K_m A/\Delta x) (\text{ft }^3/\text{lb}) \)
VISC - momentum diffusion term \( (-\mu_t A/\Delta x) (\text{lb/sec}) \)
COND - thermal diffusion term \( (-K_t A/\Delta x) (\text{BTU/sec } °R) \)
TW - average wall temperature (°R)
HC - average heat transfer coefficient (BTU/sec ft² °R)
CF - average skin friction coefficient
FLM1 - lower oxygen/fuel flammability limit (lb/lb)
FLM2 - upper oxygen/fuel flammability limit (lb/lb)
IBEG - number of beginning cell of combustion chamber
IEND - number of end cell of combustion chamber
DIFCOM - mass diffusion term in combustion chamber \( (-K_m A/\Delta x) (\text{ft }^3/\text{lb}) \)
VISCOM - momentum diffusion term in combustion chamber \( (-\mu_t A/\Delta x) (\text{lb/sec}) \)
CONCOM - thermal diffusion term in combustion chamber \( (-K_t A/\Delta x) (\text{BTU/sec } °R) \)
PROGRAM JET6

C DIMENSION A(35), VOL(35), V(35), P(35), T(35), VA(35),
* TA(35), PA(35), IA(35), M(35), MV(35), E(35), MT(35), MVT(35),
* EP(35), DV(35), VI(35), TI(35), OI(35),
* PI(35), VT(35), OT(35), PT(35), TT(35), S(35), FCI(35), OXI(35)
* , FUEL(35), OXYG(35), FCON(35), OXCON(35), ODTM(35), ECOM(35)

REAL M, MV, MT, MVT, MP, MVP, MTOT
COMMON CV, CP, R, Y, DIFF, COND, VISC, TM, HC, CF
READ (60, 10) R, CP, CV, GK, FLAG

10 FORMAT (5F10.5)
C FLAG IS ZERO FOR CLOSED END BOUNDARY
READ (60, 20) TIMINT, DELT, TIMMAX, NSTEP, N

20 FORMAT (3F10.5, 2I3)
READ (60, 25) PATM, TATH, OATM, DIFF, VISC, COND

25 FORMAT (6F10.5)
LAST = N + 1
READ (60, 30) (P(I), I=1, LAST)

30 FORMAT (6F10.5)
READ (60, 40) (T(I), I=1, LAST)

40 FORMAT (6F10.5)
READ (60, 50) (V(I), I=1, LAST)

50 FORMAT (6F10.5)
READ (60, 53) (A(I), I=1, LAST)

53 FORMAT (6F10.5)
READ (60, 54) (VOL(I), I=1, LAST)

54 FORMAT (6F10.5)
READ (60, 55) (S(I), I=1, LAST)

55 FORMAT (6F10.5)
READ (60, 58) (FCON(I), I=1, LAST)

58 FORMAT (6F10.5)
READ (60, 59) (OXCON(I), I=1, LAST)

59 FORMAT (6F10.5)
READ (60, 65) IBEG, IEND, DIFCOM, VISCOM, CONCOM

85 FORMAT (2I3, 3F10.5)
READ (60, 60) TW, HC, CF, FLM1, FLM2, K

60 FORMAT (5F10.5, 1I3)
C K IS NODE OF FUEL INJECTION
READ (60, 61) FRT, TREQ, PREQ, OXFR, CHB, BRTK

61 FORMAT (6E10.0)
WRITE (61, 62)

62 FORMAT (7X, 1HI, 11X, 4HAREA, 9X, 3HVOL, 10X, 6HSURF A, 7X,
* 4HFCON, 8X, 5HOXCON)
DO 63 I=1, LAST
63 WRITE (61, 64) I, A(I), VOL(I), S(I), FCON(I), OXCON(I)

64 FORMAT (1I7, 6X, 5E13.5)
WRITE (61, 65) DELT, CP, GV, GK, K

65 FORMAT (1H, 6HDELT=, 1E13.5, 6X, 6HCP=, 1E13.5, 6X, 6HCV=,
* 1E13.5, 6X, 6HGK=, 1E13.5, 6X, 6HK=, 1I3)
WRITE (61, 70) FRT, TREQ, OXFR, CHB, BRTK

70 FORMAT (1H, 6HFRT=, 1E13.5, 6X, 6HTREQ=, 1E13.5, 6X, 6HOXFR=
* 1E13.5, 6X, 6HCMB=, 1E13.5, 6X, 6HBRTK=, 1E13.5)
WRITE (61, 75) DIFF, VISCOM, COND, TW, HC

75 FORMAT (1H, 6HDIFF=, 1E13.5, 6X, 6HVIS=, 1E13.5, 6X, 6HCOND=
* 1E13.5, 6X, 6HTW=, 1E13.5, 6X, 6HHC=, 1E13.5)
WRITE (61, 80) CF, FLM1, FLM2, IBEG, IEND

80 FORMAT (1H, 6HCF=, 1E13.5, 6X, 6HFLM1=, 1E13.5, 6X,
* 5HFLM2=, 1E13.5, 6X, 6HIBEG=, 1I13, 6X, 6HIEND=, 1I13)
WRITE (61, 90)

90 FORMAT (1H)
WRITE(6,100)
100 FORMAT(7X,1H1,12X,1HM,12X,1HE,11X,4HPRES,9X,4HTEMP,9X, *3HODEN,10X,3HVEL,10X,4HFUEL,9X,4HOXYG,9X,4HECOM)

C INITIAL VALUES FROM INPUT
Y=19.96E6  
TIME=TIMINT 
NUDLT=0
DO 120 I=1,LAST 
(1)=P(I)/(R*T(I)) 
M(I)=Q(I)*VOL(I) 
W(I)=M(I)*V(I) 
I=CV*T(I)*M(I)*V(I) 
E(I)=CV*T(I)*M(I)+r!V4WV(I)*M(I) 
DOTM(I)=P(I)*A(I)*V(I) 
FUEL(I)=FCON(I)*M(I) 
OXYG(I)=OXCON(I)*M(I) 
ECOM(I)=0.0
C PRINT OUT
COUNT=(NUDLT*1.0)/(NSTEP*1.0)-(NUDLT/NSTEP)*1.0
IF(COUNT)150,128,150
DO 130 I=2,N 
FTOT=FTOT+FUEL(I) 
MTOT=MTOT+M(I) 
ETOT=ETOT+E(I) 
WRITE(6,132)TIME 
WRITE(6,137)I,M(I),E(I),P(I),T(I),D(I),V(I),FUEL(I), *OXYG(I),ECOM(I) 
FORMAT(1H '1I7,6)(99E13.5) 
WRITE(6,145) MTOT,ETOT,FTOT 
WRITE(6,146) 
FORMAT(1H ) 
STOT=0.0 
MTOT=0.0 
FTOT=0.0 
IF(TIME=TIMMAX)160,999,999
NUDLT=NUDLT+I 
TIME=TIME+NUDLT*DELT 
ITEM=0
DO 162 I=2,N 
VI(I)=V(I) 
DT(I)=D(I) 
PT(I)=P(I) 
IT(I)=T(I) 
C CALCULATE AVERAGE VALUES 
ITEM=ITEM+1 
DO 170 I=2,N 
VA(I)=(V(I)+VT(I))/2.0 
DA(I)=(D(I)+DT(I))/2.0 
TA(I)=(T(I)+TT(I))/2.0 
PA(I)=(P(I)+PT(I))/2.0 
C CALCULATE TRIAL VALUES 
DO 215 I=3,N 
IF(VA(I-1))185,185,200 
IF(VA(I))190,195,210
OV(I) = DA(I)*VA(I)*(A(I)+A(I+1))/(2.0*A(I))
VI(I) = OV(I)/DA(I)
FI(I) = TA(I)
FCII(I) = FCON(I)
OXCI(I) = OXCON(I)
GO TO 215

195 IF (VA(I-1)) 210, 190, 190
200 IF (VA(I)) 215, 205, 205
205 OV(I) = DA(I-1)*VA(I-1)*A(I+1)/2.0*A(I)
VI(I) = OV(I)/DAI-1)
FI(I) = TA(I-1)
FCII(I) = FCON(I-1)
OXCI(I) = OXCON(I-1)
GO TO 215

210 OV(I) = (DA(I-1)*VA(I-1)+DA(I)*VA(I))/2.0
VI(I) = OV(I)/DI(I)
FI(I) = (PA(I-1)+PA(I))/2.0*RI(I)
FCII(I) = (FCON(I)+FCON(I-1))/2.0
OXCI(I) = (OXCON(I)+OXCON(I-1))/2.0
CONTINUE
GO 217 I = 2, N

217 PI(I) = PA(I)
IF (VA(N)) 220, 225, 225
220 IF (PA(N) = PATM) 235, 230, 230
225 VI(LAST) = VA(N)
TI(LAST) = TA(N)
PI(LAST) = PATM
OV(LAST) = DA(N)*VA(N)
TA(LAST) = TI(LAST)
VA(LAST) = VI(LAST)
DA(LAST) = PATM
FCII(LAST) = FCON(N)
OXCI(LAST) = OXCON(N)
GO TO 240

230 VI(LAST) = 0.0
TI(LAST) = TATM
PI(LAST) = PATM
OV(LAST) = 0.0
TA(LAST) = TI(LAST)
VA(LAST) = VI(LAST)
DA(LAST) = PATM
FCII(LAST) = 0.0
OXCI(LAST) = 0.232
GO TO 240

235 VI(LAST) = VA(N)
TI(LAST) = TATM-(Y*VA(N)+VA(N))/0.24
PI(LAST) = 2.0*PATM*(TI(LAST)/TATM**((GK/(GK-1.0))-PI(N)
OV(LAST) = PI(LAST)*VI(LAST)/(R*TI(LAST))
TA(LAST) = TI(LAST)
VA(LAST) = VI(LAST)
DA(LAST) = PATM
FCII(LAST) = 0.0
OXCI(LAST) = 0.232
240 IF (FLAG) 245, 245, 250
245 VI(2) = 0.0
OV(2) = 0.0
TI(2) = T(1)
TA(1) = TA(2)
VA(1) = 0.0
TA(1)=0A(2)
FCI(2)=FCON(2)
OXCI(2)=0XCON(2)

247
PI(1)=PA(2)
GO TO 275

253 IF(VA(2))260,260,255
255 IF(PA(2)-PATM)265,270,270
260 VI(2)=VA(2)
TI(2)=TA(2)
PI(1)=PATM
DV(2)=DA(2)*VA(2)
TA(1)=TI(2)
VA(1)=VI(2)
FCI(2)=FCON(2)
OXCI(2)=0XCON(2)

265 VI(2)=VA(2)
TI(2)=TATM-(Y*VA(2)*VA(2))/0.24
PI(1)=2.0*PATM*(TI(2)/TATM)**(GK/(GK-1.0))-PI(2)

1V(2)=PI(1)*VI(2)/(R*TI(2))
TA(1)=TI(2)
VA(1)=VI(2)
DA(1)=DATM
GO TO 275

271 VI(2)=0.0
TI(2)=TATM
PI(1)=PATM
DV(2)=0.0
TA(1)=TI(2)
VA(1)=VI(2)
DA(1)=DATM
FCI(2)=FCON(1)
OXCI(2)=0XCON(1)
GO TO 275

275 DO 277 I=1,LAST
277 DOTM(I)=DV(I)*A(I)
C UPDATE VALUES
C PRESSURE TERM SAME AS 32.2*(((P1+P2/2)*A2)-((P2+P3/2)*A3)
C -(A2-A3)*P2)
DO 305 I=2,N
IF(I-IREF)430,420,420
420 IF(I-IEND)425,425,430
425 OIFA=OIFCOM
VISA=VISCOM
CONA=CONCOM
GO TO 450
430 OIFA=0IFF
VISA=VISC
CONA=COND
450 MP(I)=DOTM(I)-DOTM(I+1)
*+OIFA*(DA(I-1)+DA(I+1)-2.0*DA(I))
*VP(I)=16.1*(PI(I-1)*A(I)+PI(I)*(A(I+1)-A(I))-PI(I+1)*
*+PI(I+1)+DOTM(I)*VI(I)-DOTM(I+1)*VI(I+1)
*+VISA*(VA(I-1)+VA(I+1)-2.0*VA(I))-CF*S(I)*DA(I)*VA(I)*
*ARF(VA(I))
EP(I)=(CP*TI(I)+Y*VI(I)*VI(I))*DOTM(I)
*-(CP*TI(I+1)+Y*VI(I+1)*VI(I+1))*DOTM(I+1)
*CONA* (TA(I-1) * TA(I+1) - 2.0 * TA(I)) * HC* S(I) * (TA(I) - TW)

IF (ITEM) = 2, 291, 278, 278

FUEL = FCI(I) * DOTM(I) - FCI(I+1) * DOTM(I+1)

*CIF * (FCON(I-1) + DA(I-1)) + FCON(I+1) * DA(I+1) - 2.0 * FCON(I) * DA(I)

IF (I) = 0, 282, 280, 282

*FA = FRT * DELT

GO TO 283

FA = 0.0

FUEL(I) = FUEL(I) + FUEL(I) * DELT * FA

IF (FUEL(I) < 1.0E-30) 500, 510, 495

FUEL(I) = 0.0

OXYG(I) = OXYG(I) + OXYG(I) * DELT

IF (OXYG(I) < 1.0E-30) 520, 530, 500

OXYG(I) = 0.0

IF (OXYG(I) > FL(I) * 1.0E-30) 290, 540, 540

IF (OXYG(I) < FL(I) * 0.0E-30) 520, 530, 540

F9RT = 0.0

FUEL(I) = FUEL(I) / M(I)

IF (FUEL(I) < 1.0E-30) 292, 292, 292

IF (A9RT > 1.0E75) 306, 300, 305

MVT(I) = MVT(I) + MVT(I) * DELT

ET(I) = E(I) + EP(I) * DELT + ECOM(I)

IF (ET(I) < 1.0E-30) 292, 292, 292

WRITE(61, 293) TIME

FORMAT(1H ,13X, 33HLAST VALUES BEFORE NEG MASS TIME=,1E13.5)

TIME = TIMMAX

GO TO 133

CALL COMPT(VT(I), DT(I), TT(I), PT(I), MT(I), MVT(I), ET(I), VOL(I))

CALL VALUES

GO TO 750 I = 2, N

V(I) = VT(I)

T(I) = DT(I)

P(I) = PT(I)

M(I) = MT(I)

V(I) = MVT(I)

FCON(I) = FUEL(I) / M(I)

OXYCON(I) = OXYG(I) / M(I)

E(I) = ET(I)

IF (FLAG) 360, 360, 360

V(I) = 3.0

O(I) = 0(2)
T(1) = T(2)
P(1) = P(2)
GO TO 380

IF(V(2))370, 370, 375
V(1) = V(2)
T(1) = T(2)
P(1) = PATM
GO TO 380

IF(V(N))385, 390, 390
V(LAST) = V(N)
T(LAST) = TATM - (Y*V(N)*V(N))/0.24
P(LAST) = (T(LAST)/TATM)**((GK/(GK-1.0)))*PATM
D(LAST) = P(LAST)/(R*T(LAST))
GO TO 400

V(LAST) = V(N)
D(LAST) = D(N)
T(LAST) = T(N)
P(LAST) = PATM
GO TO 125

STOP
END

SUBROUTINE COMP(V, O, T, P, M, E, VOL)
COMMON CV, CP, R, Y, DIFF, COND, VISC, TH, HC, CF
REAL N, MV
V = MV/M
O = M/VOL
A = E/M
T = (A - Y*V*V)/CV
P = O*T*R
RETURN
END