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An analysis was conducted to determine the effects of nonlinearity in the one dimensional Navier-Stokes equation describing a viscous fluid flow. An attempt was made to numerically determine the effect of temperature variance on the fluid, either by varying the constant viscosity factor in the energy dissipative term in the Navier-Stokes equation or by coupling the system with a thermodynamic equation. Numerical techniques used included a Fourier approximation, a finite difference scheme and the Godunov method.

Two main conclusions could be drawn from the results: the determination of the nonlinearity setting up a shock front after a finite amount of time depends on the viscosity; and the system must remain coupled for an accurate solution.
The Effects of Nonlinearity, Viscosity, and Temperature in Burgers' Equation

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THE EFFECTS OF NONLINEARITY, VISCOSITY, AND TEMPERATURE IN BURGERS’ EQUATION

INTRODUCTION

The research for this paper is part of a larger project involving the motion of an object in a flow. The behavior of the fluid is governed primarily by the Navier-Stokes equations which contain the nonlinear inertial terms, \((u \cdot \nabla)u\), where \(u\) is the velocity of the flow. Understanding the complex effects of this term interacting with viscosity in a one dimensional setting is the purpose of this paper.

Given certain assumptions the one dimensional Navier-Stokes equation is reduced to a quasi-linear hyperbolic equation involving a convective term and a viscous term,

\[
    u_t + u u_x = \nu u_{xx}. \tag{1.1}
\]

This equation is known as the viscous Burgers equation, where \(\nu\) is the viscosity of the fluid. Initially we allow the viscosity to be constant. By varying the viscosity factor a determination is made concerning when a shock is produced. If the viscosity is allowed to depend on temperature then Burgers' equation is coupled with the one dimensional equation of the thermodynamic conservation law,

\[
    \theta_t + u \theta_x = \theta_{xx} + 2\nu(u_x)^2. \tag{1.2}
\]
To arrive at the theoretical solution to Burgers' equation, we use the Hopf-Cole transformation. This reduces solving the quasi-linear hyperbolic equation, \( (1.1) \), to solving the linear heat equation

\[
    w_t = \nu w_{xx}
\]

by setting

\[
    u = -2\nu \frac{w_x}{w},
\]

where \( u \) solves \( (1.1) \) and \( w \) solves \( (1.3) \). The transformation, \( (1.4) \), is known as the Hopf-Cole transformation. We prove uniqueness for the Burgers equation making use of this transformation.

A stability analysis is obtained for Burgers' equation. Upper bounds on the kinetic energy and a disturbance between two solutions are found by making an assumption which prohibits the development of a shock.

For numerical purposes, we make use of various techniques. To solve \( (1.4) \), we use Fourier approximation. To solve \( (1.1) \) for \( \nu > 0 \), we use an implicit finite difference scheme. When \( \nu = 0 \), the numerical method for solving \( (1.1) \) employs the Godunov method. A combination of the finite difference scheme with the Godunov method solves the coupled system, \( (1.1) \) and \( (1.2) \).

The results are given graphically comparing the various numerical techniques. A discussion based on numerical observation unfolds about the effects of the nonlinear inertial term giving rise to a shock wave due to a
lack of viscosity. Observations are made concerning the need to evaluate a coupled system involving Burgers' equation and the conservation thermodynamic law. Finally, some ideas are included for areas of further research.
MATHEMATICAL FORMULATION

Various reasons suggest why we study Burgers' equation. One is that this equation, (1.1), is a simple example of an interaction between dissipative and nonlinear inertial terms in a differential equation. In addition, it may be viewed as a nonlinear heat equation. Another reason for studying (1.1) is to examine a simplified form of the Navier-Stokes equations. The derivation of Burgers' equation from the Navier-Stokes equations given here follows Burgers' physical arguments, [4], for arriving at (1.1).

A unidirectional fluid flow in a viscous, isotropic, homogeneous, infinitely compressible medium is prescribed by certain conservation laws. These equations are modified by the following assumptions: the coefficient of volumetric expansion, \( \lambda \), equals zero and no shearing motion exists. The conservation laws to be considered are as follows:

Continuity,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 ;
\]

Momentum,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{-p_x}{\rho} + \nu \frac{\partial^2 u}{\partial x^2} ;
\]

where

\( \rho = \rho(x, t) = \) density,
\[ p = p(\rho) = \) pressure,
\[ \nu = \nu(\theta) = \) kinematic coefficient of viscosity,
\[ u = u(x, t) = \) velocity,
\[ \theta = \theta(x, t) = \) heat.

If we assume further that there is no pressure on the medium then we note
that since
\[ p = p(\rho), \]
\[ \frac{\rho_x}{\rho} \to 0, \]
\[ \frac{-p_x}{\rho} = \frac{p'(\rho)}{\rho} \frac{\rho_x}{\rho} \to 0. \]
So (2.2) becomes
\[ u_t + u u_x = \nu u_{xx}, \tag{2.3} \]
the viscous Burgers equation.

The boundary and initial conditions for (2.3) are
\[ \begin{cases} 
  u(0, t) = 0 \\
  u(1, t) = 0 \\
  u(x, 0) = f(x) \\
  t \geq 0, \\
 \end{cases} \]
which requires that the motion take place between two fixed walls and
\[ u(x, 0) = f(x) \quad 0 \leq x \leq 1, \]
where \( f(x) \) is a positive, integrable curve with compatibility conditions
\[ \begin{cases} 
  f(0) = 0 \\
  f(1) = 0. \\
 \end{cases} \]
The initial condition is created to imitate a smooth surface wave held fixed at both ends, (see figure 1).
Figure 1. Initial condition for (2.3).
Note that in (2.3) the terms on the left hand side of the equation are the inertial terms and the remaining term involves the diffusivity or the kinematic viscosity, $\nu$, which is a function of $\theta$. Initially, we will assume that the temperature is constant, making $\nu$ a constant. We will use $\nu$ as a parameter and allow it to vary as $0 \leq \nu \leq 1$. Unless specified, it will be assumed that $\nu$ will be constant.

We need to nondimensionalize the measures of velocity, space and time. This is accomplished by the following technique: let $U$, $L$, and $T$ be the characteristic reference units, respectively, for the aforementioned entities. Then we arrive at the dimensionless quantities:

$$v = \frac{u}{U} : \text{velocity},$$

$$\xi = \frac{x}{L} : \text{space},$$

$$\tau = \frac{t}{T} : \text{time},$$

where $v = v(\xi, \tau)$, $u = u(x, t)$, and $u$ solves (2.3). Then we can write

$$u = U \nu,$$

$$u_t = U \nu \tau_t$$

$$= \frac{U}{T} \nu \tau,$$

$$u_x = U \nu \xi \xi_x$$

$$= \frac{U}{L} \nu \xi,$$

$$u_{xx} = \frac{U}{L^2} \nu \xi \xi_x$$

$$= \frac{U}{L^2} \nu \xi.$$

Substituting these into (2.3) we get
\[ v_T + v \xi = \frac{\nu}{\text{LU}} \nu \xi \xi. \]

Set
\[ R = \frac{\text{LU}}{\nu}, \tag{2.4} \]

where \( R \) is known as the nondimensional Reynold's number. So the nondimensional form for (2.3) becomes
\[ u_t + uu_x = \frac{1}{R} u_{xx}. \]

Note that the Reynold's number is inversely proportional to the viscosity. Thus, high Reynold's numbers correlate to low viscosities.

If we assume that the fluid is a heat conducting medium without sources or sinks and \( \nu \) is not constant, but dependent on \( \theta \), then we are lead to a coupled system involving the heat transfer equation. Hence we have the additional conservation law:

Heat transfer,
\[ c_p ( \theta_t + u \theta_x ) = \kappa \theta_{xx} + 2\nu( u_x )^2; \tag{2.5} \]

where \( \theta = \theta(x, t) = \text{heat}, \)
\( \kappa = \frac{\xi}{\rho} = \text{thermal conductivity coefficient per density}, \)
\( c_p = \text{specific heat}. \)

Setting \( \kappa = 1 \) and \( c_p = 1 \), we get
\[ \theta_t + u \theta_x = \theta_{xx} + 2\nu( u_x )^2. \tag{2.6} \]

Two sets of boundary conditions are used for \( t \geq 0 \):

\[ \begin{cases} 
\theta(0, t) = 0 \\
\theta(1, t) = 20 
\end{cases} \tag{2.7} \]

and
\[ \begin{align*}
\theta(0, t) &= 20 \\
\theta(1, t) &= 0
\end{align*} \quad t \geq 0, \tag{2.8} \]

creating a driving force either with or counter to the flow in (2.3). The initial condition is

\[ \theta(x, 0) = 1 \quad 0 \leq x \leq 1. \tag{2.9} \]

However, if one assumes that the changes in temperature are small with no pressure changes, \( \nu \) becomes constant and the system becomes uncoupled. In addition, one would have

\[ \begin{align*}
\theta_t &\approx 0, \\
\theta_x &\approx 0,
\end{align*} \]

and consequently

\[ \theta_{xx} \approx 0 \]

substituting these into (2.5) implies that the motion is rigid. Since we are describing fluid, the concept of rigid motion is untenable. From the work compiled for this study, it is apparent that one may not assume \( \nu \) is constant with respect to temperature.
1. Hopf-Cole Solution

The following solution for (2.3) comes from Cole, [7], and it is also given in Hopf, [16]. The process involves using a nonlinear transformation which enables us to reduce the nonlinear hyperbolic equation, (2.3), to a linear heat equation.

From (2.3) we have

\[
\begin{cases}
    u_t = \nu u_{xx} - u u_x \\
    u(0, t) = u(1, t) = 0 & t \geq 0, \\
    u(x, 0) = f(x) & 0 \leq x \leq 1.
\end{cases}
\] (3.1)

So we can rewrite (2.3) as

\[ u_t = (\nu u_x - \frac{u^2}{2})_x. \]

If we let

\[ u = Q_x, \]

\[ Q = \int u \, dx, \]

\[ Q_t = \int u_t \, dx, \]

and substituting from (2.3)

\[ Q_t = \int \left(\nu u_x - \frac{u^2}{2}\right)_x \, dx, \]

then

\[ Q_t = \nu Q_{xx} - \frac{Q_x^2}{2}. \] (3.2)
The transformation is derived as follows. Let

\[ Q = h(w) , \]

where \( w = w(x, t) \) then (3.2) becomes

\[
h'(w)w_t = \nu \left[ h''(w)w_x^2 + h'(w)w_{xx} \right] - \left( \frac{h'(w)w_x}{2} \right)^2
\]

\[
w_t = \left( \frac{\nu h''(w) - \left( h'(w) \right)^2}{h'(w)} \right) w_x^2 + \nu w_{xx} .
\]  

(3.3)

Choose \( h \) such that

\[
\nu h''(w) - \frac{h'(w)^2}{2} = 0 .
\]

Let

\[
h' = g
\]

then

\[
\nu g' - \frac{g^2}{2} = 0
\]

and

\[
g = -\frac{2\nu}{w + \alpha} .
\]

Since we are looking for an arbitrary transformation we set \( \alpha = 0 \) to find

\[
h = -2\nu \log(w) .
\]

Hence

\[
Q = -2\nu \log(w)
\]

and hence

\[
u = Q_x = -2\nu \frac{w_x}{w} .
\]

(3.4)

Then from (3.3) we see that \( w(x, t) \) solves

\[
w_t = \nu w_{xx} ,
\]  

(3.5)

the linear heat equation. From the boundary conditions on \( u(x, t) \) we get

\[
\begin{align*}
\{ w_x(0, t) &= 0 \\
\{ w_x(1, t) &= 0 \\
\} t \geq 0
\]  

(3.6)
and from the initial conditions on \( u(x, t) \) we get

\[
f(x) = - 2\nu \frac{w_x(x, 0)}{w(x, 0)}.
\]

Hence

\[
w(x, 0) = \exp \left[ - \frac{1}{2\nu} \int_0^x f(y) \, dy \right]
\]  \hspace{1cm} (3.7)

So, to find the solution to \( u(x, t) \) we solve

\[
\begin{align*}
  w_t &= w_{xx} \quad 0 \leq x \leq 1, \ t > 0, \\
  w_x(0, t) &= w_x(1, t) = 0 \quad t \geq 0, \\
  w(x, 0) &= \exp \left[ - \frac{1}{2\nu} \int_0^x f(y) \, dy \right] \quad 0 \leq x \leq 1.
\end{align*}
\]

Using separation of variables gives the following solution:

\[
w(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp \left[ -(n\pi)^2 \nu t \right] \cos(n\pi x),
\]

where

\[
a_n = 2 \int_0^1 \exp \left[ -\frac{1}{2\nu} \int_0^\eta f(\xi) \, d\xi \right] \cos(n\pi \eta) \, d\eta, \quad n = 1, 2, 3 \ldots
\]

\[
a_0 = 2 \int_0^1 \exp \left[ -\frac{1}{2\nu} \int_0^\eta f(\xi) \, d\xi \right] \, d\eta.
\]  \hspace{1cm} (3.8)

Hence, \( u(x, t) \) has solution
\[
\sum_{n=1}^{\infty} \int_{0}^{1} \exp\left[-\frac{1}{2\nu} \int_{0}^{\eta} f(\xi) \, d\xi \right] \cos(n\pi \eta) \, d\eta \, \exp\left[-(n\pi)^2 \nu t \right] \sin(n\pi x) \\
\frac{a_0}{2} + \sum_{n=1}^{\infty} \int_{0}^{\eta} \exp\left[-\frac{1}{2\nu} \int_{0}^{\eta} f(\xi) \, d\xi \right] \cos(n\pi \eta) \, d\eta \, \exp\left[-(n\pi)^2 \nu t \right] \cos(n\pi x)
\]

where \(a_0\) is given by (3.8).

A complete discussion of the convergence of the solution is given in [7].
A few of the highlights are included here; for large \(t\), the only significant term is for \(n = 1\) in the numerator, so for \(t \gg 1\) we have

\[
u(x, t) \approx 2\pi \nu \frac{1}{a_0} \int_{0}^{\eta} \exp\left[-\frac{1}{2\nu} \int_{0}^{\eta} f(\xi) \, d\xi \right] \cos(n\pi \eta) \, d\eta \, \exp\left[-\pi^2 \nu t \right] \sin(n\pi x)
\]
or
\[
u(x, t) \approx 2\pi \nu \frac{a_1}{a_0} \exp\left[-\pi^2 \nu t \right] \sin(n\pi x).
\]

Note that this solution is similar to the solution of the linear heat equation for large times,

\[
u_t = \nu \nu_{xx},
\]
given the same initial and boundary conditions as in (3.1). This is the linearized form of (3.1). When the amplitude of (2.3) is small, which happens after a finite amount of time, the linear and the nonlinear equations behave similarly.
The solution to the heat equation is given by

\[ a(x, t) = A_1 \exp\left(-\nu \pi^2 t\right) \sin(\pi x), \tag{3.10} \]

where

\[ A_1 = \frac{1}{\pi} \int_0^1 f(x) \sin(\pi x) \, dx, \]

for large \( t \). Note that the two solutions have the same dependence on \( x \) and \( t \) but the amplitudes are different. Cole notes that (3.9) gives a smaller amplitude for large \( t \) than (3.10) since there is increased energy dissipation during the intermediate time for the solution to Burgers' equation.

2. Uniqueness

We start with the Maximum Principle and some of its results.

**Theorem 1:** Let \( L[\phi] = a \phi_{xx} + b \phi_x + c \phi_t \), where \( a > 0 \) and \( a, b, \) and \( c \) are continuous functions in a domain, \( A \). Let \( \phi \) be a solution to

\[ L[\phi] + \alpha \phi = 0, \]

where \( \alpha \leq 0 \) in \( A \). If \( \phi \) has a positive maximum or negative minimum, \( C \), at a point of \( A \) then \( \phi = C \) in \( A \). Hence if \( \phi \) is not constant in \( A \) then \( \phi \) takes its maximum or minimum on the boundary of \( A \), denoted by \( \partial A \).

**Proof:** The proof for this theorem may be found in Nirenberg, [23], or Protter, [25]. \( \square \)

**Theorem 2:** Let \( A \) be a region in the \( x, t \) plane in which \( w \) is a solution of

\[ w_{xx} - w_t \geq 0 \]

Suppose \( P \) is a point on \( \partial A \) where the maximum of \( w \) occurs and that the normal to \( \partial A \) at \( P \) is not parallel to the \( t \) axis. Suppose that at \( P \) a circle tangent to \( \partial A \) can be constructed whose interior lies entirely in \( A \) and such
that \( w < C \), a constant, in this interior then if \( \frac{\partial}{\partial \nu} \) is any derivative in an outward direction from \( A \), then
\[
\frac{\partial w}{\partial \nu} = 0 \text{ at } P.
\]
In particular, if \( A \) is the region \( 0 < x < 1 \), \( 0 < t \leq T \) and
\[
w_x(0, t) = w_x(1, t) = 0
\]
where \( w \) satisfies
\[
w_t = w_{xx}
\]
then the maximum of \( w \) must occur at the initial condition,
\[
w(x, 0) \quad 0 \leq x \leq 1.
\]

**Proof:** The proof of this theorem may be found in [25]. \( \square \)

We can now proceed with the proof of the uniqueness of (3.1).

**Theorem 3:** Let \( u(x, t) \) be a solution to (3.1) with \( f(x) \) integrable on \( 0 \leq x \leq 1 \) and with compatibility conditions \( f(0) = f(1) = 0 \) then the solution \( u \) is unique.

**Proof:** As before we let \( u(x, t) = Q_x(x, t) \) where
\[
Q(x, t) = -2\nu \log w(x, t)
\]
and \( w(x, t) \) solves
\[
\begin{cases}
w_t = \nu w_{xx} \\
w_x(0, t) = w_x(1, t) = 0 & 0 \leq x \leq 1, t > 0 \\
w(x, 0) = F(x) & 0 \leq x \leq 1.
\end{cases}
\] (3.11)

where
\[
F(x) = \exp\left(-\frac{1}{2\nu} \int_0^x f(y) \, dy\right),
\]
and the initial and boundary conditions of \( w \) depend upon \( u \).

Assume (3.1) has two solutions, \( u \) and \( u' \), then each solution has its substitution:
\[
u = Q_x \quad \text{and} \quad u' = Q_{x'}
\]
where
\[ Q = -2\nu \log w \quad \text{and} \quad Q' = -2\nu \log w'. \]

w and w' solve (3.11). We know from Theorem 2 that w = w' hence
\[ Q = Q' \]
\[ Q_x = Q'_x \]
\[ u - u'. \]

3. **Stability Analysis**

We have a flow, u(x, t) with an initial velocity distribution. The question arises: when t = 0, if the flow is disturbed slightly, will the resulting motion differ radically or only slightly from the original flow? The analysis of this problem uses the kinetic energy formula based on the idea that if u solves (3.1), v is a perturbed flow, and w = u - v, then w → 0 as t → ∞ if the kinetic energy formula involving w also tends to zero. The hydrodynamic stability thus depends on the energy of w tending to zero as t increases.

Let \( \mathcal{E} = \frac{1}{2} \int_{0}^{1} w^2 \, dx \), the kinetic energy. Here, let w = v - u, where u solves (3.1) and v is a perturbed flow. We note that

\[ w_t = v_t - u_t, \]

and that by substituting for \( v_t \) and \( u_t \) we have
\[ w_t = \nu w_{xx} - v w_x - u_x w, \quad (3.12) \]

or
\[ w_t = \nu w_{xx} - v_x w - u w_x. \quad (3.13) \]

If we assume that u = 0 and use (3.13), integration by parts and the boundary
conditions on \( w \) we see that

\[
\frac{d\zeta}{dt} = \int w \, w_t \, dx \\
= -\int \{ \nu \, w_x^2 + w^2 \, v_x \} \, dx.
\]

Since the first term on the right hand side of the equation is always negative (\( \nu > 0 \)), we see that this term acts as an energy dissipative term. This term originates from the viscosity term of Burgers' equation, \( \nu w_{xx} \). It is for this reason that the viscosity term is often referred to as the energy dissipative term of Burgers' equation.

The second term can be positive. This term originates from the nonlinear inertial term of (3.1). With a high shear rate this term can facilitate growth of disturbances. It is the balance between these two terms which determines the stability of a flow. As the viscosity decreases the influence of the second term becomes greater, feeding more energy into the system, and thus setting up a shock.

**Lemma 1:** Let \( h \) be an arbitrary vector field in a domain, \( A \). Let \( u(x, t) \) solve (3.1). Let \( u_x \geq -M \), a constant. Let \( v(x, t) \) be the velocity of a perturbed motion and set \( w(x, t) = v(x, t) - u(x, t) \). If \( h \) is differentiable in a circle of radius of \( \frac{\pi}{2} \, C \), then

\[
\int_{0}^{1} w_x^2 \, dx \geq \int_{0}^{1} w^2 \, C^2 \, dx.
\]
Proof: We have

\[ 0 \leq (w_x + w h)^2 = w_x^2 + 2 w w_x + w^2 h^2 \]

so

\[
\int_0^1 w_x^2 \, dx \geq - \int_0^1 \left\{ 2 w w_x h + w^2 h^2 \right\} \, dx.
\]

Note that

\[
\int_0^1 2 w w_x h \, dx = 2 \int_0^1 h \, d\left(\frac{w^2}{2}\right)
\]

\[
= 2 \left\{ h \frac{w^2}{2} \right\}_0^1 - \int_0^1 \frac{h_x w^2}{2} \, dx \}
\]

The first term on the right side vanishes by the boundary conditions. Then if we set

\[
h = C \tan(Cx)
\]

\[
h_x = C^2 \sec^2(Cx)
\]

\[
h_x w^2 - h^2 w^2 = w^2 \{ C^2 \sec^2(Cx) - C^2 \tan^2(Cx) \}
\]

\[
= w^2 C^2.
\]

Thus

\[
\int_0^1 w_x^2 \, dx \geq \int_0^1 w^2 C^2 \, dx.
\]

\[ \square \]

Theorem 4: Let \( w(x, t), v(x, t), \) and \( u(x, t) \) be as in Lemma 1. Let \( u_x \geq -M, \) a constant. Then if we let

\[
\mathcal{F}_0 = \text{the initial kinetic energy},
\]

\[
V = \max_{t > 0, \; 0 \leq x \leq 1} v,
\]

then

\[
\mathcal{F} \leq \mathcal{F}_0 \exp\left(\frac{2 M + \frac{V^2}{\nu} - \nu C^2}{\nu} t\right).
\]
Moreover, if \( \nu C^2 \geq 2 M + \frac{V^2}{L} \) then \( \mathcal{F} \to 0 \) as \( t \to \infty \) and the flow is stable.

Proof: Since \( w = \nu - u \) and using (3.12), \( w \) satisfies

\[
\begin{align*}
\frac{d}{dt} \mathcal{F} &= \frac{1}{2} \frac{d}{dt} \int_0^1 w^2 \, dx \\
&= \int_0^1 w \, w_t \, dx \\
&= \int_0^1 \{ \nu w w_{xx} - w w_x v - w^2 u_x \} \, dx \\
&= I_1 + I_2 + I_3,
\end{align*}
\]

where

\[
I_1 = \int_0^1 \nu w w_{xx} \, dx,
\]

\[
I_2 = - \int_0^1 w w_x v \, dx,
\]

\[
I_3 = - \int_0^1 w^2 u_x \, dx.
\]

Using integration by parts and noting the boundary conditions for \( w \), we get

\[
I_1 = - \nu \int_0^1 w_{xx}^2 \, dx.
\]

For \( I_2 \), we note that since

\[
(\nu w_x - v w)^2 \geq 0
\]
\[
\frac{\nu w_x^2}{2} + \frac{v^2 w^2}{2\nu} \geq w_x w v,
\]

\[
l_2 \leq \int_{0}^{1} \left\{ \frac{\nu w_x^2}{2} + \frac{v^2 w^2}{2\nu} \right\} dx.
\]

Since \( u_x \geq -M \), we have

\[
l_3 \leq \int_{0}^{1} w^2 M dx.
\]

Setting \( V = \max_{t>0} v \) gives

\[
\frac{d}{dt} \mathfrak{F} \leq \mathfrak{F} \left\{ \frac{V^2}{\nu} + 2M \right\} - \int_{0}^{1} \frac{\nu w_x^2}{2} dx.
\]

From Lemma 1 we get

\[
\frac{d}{dt} \mathfrak{F} \leq \mathfrak{F} \left\{ \frac{V^2}{\nu} + 2M - \nu C^2 \right\}.
\]

Integrating with respect to \( t \) gives the desired result:

\[
\mathfrak{F} \leq \mathfrak{F}_0 \exp \left\{ \left[ 2M + \frac{V^2}{\nu} - \nu C^2 \right] t \right\}.
\]

If the viscosity is large enough, given a finite amount of time, the flow will be steady. However, for a low viscosity the two flows, \( u \) and \( v \), do not tend to a single limit flow. According to Hopf, [16], as the viscosity decreases, many solutions may appear after the effect of the initial conditions has gone.

We can get a bound on the disturbance, \( w(x, t) \).

**Theorem 5:** Let \( u, v, \) and \( w \) be as in Theorem 4. Then

\[
|w(x, t)| \leq |u(x, 0) - v(x, 0)| \exp(Mt)
\]
where \( u_x \geq -M \).

**Proof:** From (3.12)

\[
\begin{align*}
w_t + w_x v - \nu w_{xx} + w u_x &= 0.
\end{align*}
\]

Set \( w = \exp(Mt) \phi \), where \( \phi = \phi(x,t) \). Then we can rewrite this equation as

\[
\begin{align*}
\phi_t + \phi_x v - \nu \phi_{xx} + (M + u_x) \phi &= 0.
\end{align*}
\]

Let

\[
L[\phi] = \nu \phi_{xx} - \nu \phi_x - \phi_t,
\]

then we have

\[
L[\phi] - (M + u_x) \phi = 0.
\]

From Theorem 1, if \( \nu \) and \( \nu \) are bounded on a domain, \( A \), and \( (M + u_x) \geq 0 \), then \( \phi \) attains its maximum on \( \partial A \). Setting the domain, \( A \), to be \( 0 < x < 1, 0 < t \leq T \), and

\[
\begin{align*}
\phi(0, t) &= \phi(1, t) = 0 \\
\phi(x, 0) &= \begin{cases} 
0 & \text{if } u(x, 0) = v(x, 0) \\
 f(x) & \text{if } u(x, 0) - v(x, 0) = f(x).
\end{cases}
\end{align*}
\]

So, we see that

\[
|\phi(x, t)| \leq |u(x, 0) - v(x, 0)|.
\]

Hence,

\[
|w(x, t)| \exp(-Mt) \leq |u(x, 0) - v(x, 0)|
\]

\[
|w(x, t)| \leq |u(x, 0) - v(x, 0)| \exp(Mt).
\]

Here we assume \( u_x \) is bounded from below. However, as the viscosity decreases, \( u_x \) decreases rapidly in the region of the shock profile. Without this assumption the disturbance, \( w \), might not have an upper bound. This
would lead to the situation where the two solutions produce radically different results.

4. Nonviscous Burgers' Equation

As we have seen, the interplay between viscosity and the nonlinear term of the Burgers equation determine the stability of the flow. We will now examine the solution to the nonviscous Burgers equation,

$$\begin{align*}
    u_t + uu_x &= 0, \\
    u(0, t) &= u(1, t) = 0 \quad &t \geq 0, \\
    u(x, 0) &= f(x) \quad &0 \leq x \leq 1,
\end{align*} \quad (3.15)$$

to isolate the effects of nonlinearity and determine what happens as $\nu \to 0$ in the viscous Burgers equation (3.1). Note that in the analysis of the viscosity tending to zero, we are looking at

$$\lim_{\varepsilon \to 0} \nu = \varepsilon \nu_0$$

holding $\nu_0$, $x$, $t$, and $f(x)$ fixed.

**Theorem 6:** Let $u(x, t)$ be a solution to (3.1). Let $\nu = \varepsilon \nu_0$ and hold $x$, $t$, $f(x)$, and $\nu_0$ fixed. Then the solutions of the viscous Burgers equation (3.1) approach the solutions to the nonviscous Burgers equation as $\nu \to 0$.

**Proof:** The proof of this theorem may be found in Whitham, [31].

(3.15) is solved using the method of characteristics with the following system:

$$\begin{align*}
    \frac{dt}{d\tau} &= 1, \\
    \frac{dx}{d\tau} &= u, \\
    \frac{du}{d\tau} &= 0,
\end{align*}$$

and

$$x_0(s) = s, \quad t_0(s) = 0, \quad u_0(s) = f(s). \quad (3.16)$$
From the last conditions, we see that
\[ t = \tau, \]
\[ u = f(s), \]
\[ x = f(s) t + s, \]
thus
\[ x - ut = s \]
and
\[ u = f(x - ut). \]

For any fixed positive value of \( \nu \), the solutions to (3.1) are continuous and well defined. However, the solution to (3.15) is discontinuous, except locally, and can give a multivalued solution after a finite length of time. The explanation for a continuous solution converging to a discontinuous solution is given in [31]. From this result, one would expect that the behavior of \( \lim_{\nu \to 0} u(x, t; \nu) \) as \( t \to \infty \) would not be the same as reversing these two limits. The proof that reversing these limits gives different results is found in [16].

5. Numerical Techniques

Four techniques are used to evaluate the viscous Burgers equation, the nonviscous Burgers equation, and the coupled system of the fully viscous Burgers equation with the heat transfer equation. A complete analysis of the four techniques is not included in this paper. However, references are included for some of the techniques.

For the viscous Burgers equation, keeping \( \nu \) as a parameter, two different techniques are used. The first is an explicit Fourier approximation
to the solution, (3.9). The second approach comes from an implicit finite difference scheme from Sod, [28]. It is given by

$$\frac{u_i^{n+1} - u_i^n}{h} = \nu D_+ D_- (u_i^{n+1}) + u_i^n D_0 (u_i^{n+1}),$$

with initial condition

$$u_i^0 = f(ih) \quad 0 \leq i \leq 1,$$

and boundary conditions

$$u_0^n = u_N^n = 0, \quad 0 \leq nk \leq T,$$

where

$$0 \leq n \leq N,$$

$$D_+(u_i^{n+1}) = \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h},$$

$$D_-(u_i^{n+1}) = \frac{u_i^{n+1} - u_{i-1}^{n+1}}{h},$$

$$D_0(u_i^{n+1}) = \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h} \quad (3.17)$$

$$h = \Delta x,$$ \quad (3.18)

and

$$k = \Delta t. \quad (3.19)$$

The proof of convergence of the implicit finite difference approximation to the viscous Burgers equation is found in [28].

From the graphs it is apparent that for large viscosity both methods work well. For small viscosity, the number of terms necessary for convergence of the Fourier approximation is quite large while the finite difference method works well with small enough time steps. Figure 2 shows
the velocity profiles at various times using the Fourier approximation. At
time $\approx 0.1$, a shock profile is set up. By time $\approx 0.8$, the shock profile is
gone and the viscosity term dampens the velocity. Figure 3 shows the same
time frame as figure 2 except that these values are generated by the finite
difference approximation.

Comparing figures 2 and 3, we see that figure 3 shows the same profiles,
but they are much lower than those of the same time frame in figure 2. A
major difference between these two methods is that the finite difference
approximation dissipates energy much faster than the Fourier approximation.

The nonviscous Burgers equation is approximated by the Godunov
scheme. A brief outline of the technique used is included here. More
details can be found in Holt,[15], Sod, [28], and Wilson,[32].

Write (3.15) as

$$u_t + \left[ \frac{1}{2} u^2 \right]_x = 0,$$

then using the same notation as before

$$u_i^{n+1} = u_i^n - \left( \frac{k}{h} \right) \left( F\left( u_i^n \right)_{i+\frac{1}{2}} - F\left( u_i^n \right)_{i-\frac{1}{2}} \right),$$

where $F$ is the numerical flux function,

$$F(u) = \frac{u^2}{2},$$

and is determined as follows
\[
\begin{align*}
\frac{u_l^2}{2} & \quad \text{if} \quad u_l \geq 0, \ ur \geq 0 \\
\frac{u_r^2}{2} & \quad \text{if} \quad u_l \leq 0, \ ur \leq 0 \\
0 & \quad \text{if} \quad u_l < 0, \ ur > 0 \\
\frac{u_l^2}{2} & \quad \text{if} \quad u_l < 0, \ ur < 0, \ s \geq 0 \\
\frac{u_r^2}{2} & \quad \text{if} \quad u_l > 0, \ ur < 0, \ s \leq 0
\end{align*}
\] (3.20)

where \( l \) and \( r \) signify left and right respectively and \( s \) is the shock speed for (2.3) given by the Rankine-Hugoniot condition,
\[
s = u_l + u_r.
\]

After a brief period of time, the nonlinear inertial term in the nonviscous Burgers equation sets up a shock. Without the viscosity, there is no damping effect and although the height of the shock becomes smaller in time, the energy remains the same. Figure 4 shows the initially continuous solution becoming discontinuous at time \( \approx 0.4 \). The solution remains discontinuous as time increases.

Allowing \( \nu \) to depend upon \( \theta \), we see that the need arises to use the heat transfer equation, (2.6), coupled with the viscous Burgers equation, (2.3). To solve this problem numerically requires both the implicit finite difference scheme and the Godunov method. This combination is necessary since the coupling of the two equations appears to allow the nonlinear inertial term to generate enough energy to create a shock. In addition, the proof given by Sod for convergence of the approximation depends on the condition that viscosity is held constant. The finite difference scheme fails when a discontinuous solution appears; however, the Godunov scheme can give a good approximation to this situation. The numerical technique used for the solution to (2.3) is given by:
Figure 2. Fourier approximation, early times, with viscosity = 0.009

Time = 0.1, 0.2, 0.3, 0.4, 0.5, 0.8

VELOCITY $u(x,t)$

DISTANCE
Figure 3. Finite difference approximation with viscosity = 0.009
Figure 4. Godunov method for the nonviscous Burgers equation

Time = 0.1, 0.4, 0.5, 0.8, 1.0, 1.1

NONVISCOUS BURGERS SOLUTION

Godunov Method

VELOCITY $U(x,t)$

DISTANCE
\[
\frac{u_{i,n+1} - u_{i,n}}{k} + \frac{F[u_{i+\frac{1}{2}}^n] - F[u_{i-\frac{1}{2}}^n]}{h} = D_0(\nu(\theta_{i,n}) u_{i,n+1}).
\]

For (2.6) the approximation uses an implicit finite difference scheme:

\[
\frac{\theta_{i,n+1} - \theta_{i,n}}{k} + u_{i,n} D_0(\theta_{i,n+1}) = D_+D_-(\theta_{i,n+1}) + 2\nu(\theta_{i,n}) \left(D_0(u_{i,n+1})\right)^2,
\]

where \(D_0, h, k,\) and \(F\) are given by (3.17) \(\text{and} \) (3.20), respectively. Note that there is no iteration done on the viscosity factor. The difference between using one or two iterations and no iterations is negligible. The viscosity function is

\[
\nu(\theta) = \exp(-\theta),
\]

which approximates the relationship between temperature and viscosity.

Using the first set of boundary conditions, (2.7), the wave is initially damped until it reaches a region of high temperature, (see figure 5). At time \(\approx 0.5\), a shock is set up and continues as a shock for a finite amount of time, (see figure 6). However, with the second set of boundary conditions, (2.8), a shock is set up almost immediately, (see figure 7). As it travels out of the high temperature region the wave looses its discontinuity, becomes continuous, and is damped, (see figures 8 and 9).

Figures 10 and 11 show the solution to (2.6) using boundary conditions (2.7) and (2.8) respectively. After time \(\approx 0.4\), the temperature is evenly distributed for both.
Figure 5. Velocity in coupled system, early times, with boundary conditions (2.7). Time = 0.01, 0.1, 0.5, 1.0.
Figure 6. Velocity in the coupled system with boundary conditions (2.7) shows development of shock at times 0.1, 1.0, 2.0, 4.0, 6.0, 8.0.
Figure 7. Velocity in coupled system, early times, with boundary conditions (2.8). Time = 0.01, 0.1, 0.5, 1.0.
Figure 8. Velocity in the coupled system with boundary conditions (2.8)

Time = 0.1, 0.5, 1.0, 1.4, 1.8, 2.2

shows development and decay of the shock

VELOCITY $U(x,t)$
Figure 9. Velocity in the coupled system, late times, with boundary conditions (2.8) shows the damping effect of the large viscosity. Time = 1.0, 2.2, 3.0, 5.0, 7.0, 9.0.
Figure 10. Heat in coupled system with boundary conditions (2.7)

Time = 0.01, 0.1, 0.4
Figure 11. Heat in the coupled system with boundary conditions (2.8)

Time = 0.01, 0.1, 0.4
Figure 12. Fourier approximation with viscosity = 0.1

Time = 0.001, 0.01, 0.1, 0.5, 1.0, 2.0
Figure 13. Fourier approximation with viscosity = 1.0

Time = 0.0001, 0.001, 0.01, 0.1

Velocity $u(x,t)$
Figure 14. Fourier approximation, late times, with viscosity 0.009 showing beginning decay of shock profile. Time = 0.5, 0.8, 1.0, 1.5, 2.0, 4.0.
FOURIER APPROXIMATION

Viscosity = 0.009

Figure 15. Fourier approximation, late times, with viscosity = 0.009

Time = 2.0, 3.0, 4.0, 6.0, 8.0, 10.0
Fourier approximation, late times, with viscosity $0.009$, detail.

Time: 2.0, 3.0, 4.0, 6.0, 8.0, 10.0.
SUMMARY AND CONCLUSIONS

The effects of nonlinearity, viscosity, and temperature in the Burgers equation are complex. In this study, we examine the effects numerically. From the results, we find that the nonlinearity adds energy to the system, setting up a potential shock. The kinematic viscosity diffuses the energy. Temperature variation directly affects the nonlinearity and the viscosity.

For a fixed viscosity in the viscous Burgers equation, a velocity profile represented by a wave is introduced as the initial condition, (see figure 1). The wave moves to the right with a constant velocity as long as the viscosity is large enough, \( \nu \gg 0.001 \). After a finite amount of time, the wave is damped as the viscous energy dissipative term takes over, (see figure 12). For \( \nu \gg 0.1 \), the wave is damped almost immediately, (see figure 13).

As the viscosity decreases a shock profile develops after a finite period of time. The shock profile forms from the velocity of the wave propagation becoming slower as \( x \) increases. The solution remains continuous due to the positive viscosity. However, the shock profile becomes steeper as the viscosity decreases, (see figure 2). After a finite amount of time the flow becomes stable and the energy is dissipated by the viscosity, (see figures 14, 15, and 15a). Comparing figure 15 with figure 12, we see that lowering the viscosity (raising the Reynold's number) lengthens the time necessary for the energy to dissipate. Hence, raising the viscosity causes the energy in the system to dissipate more rapidly with less of the potential for a shock profile to develop. The old sailing adage of pouring oil on the water to calm the turbulent seas is a good example. Sailors probably never
thought they were raising the surface viscosity level, but they knew it worked well in deep water!

With no viscosity, a shock is set up rapidly and continues indefinitely. The height of the shock decreases, but the energy in the system remains constant, (see figure 4).

When the viscosity is allowed to vary with the heat, the behavior of the solution differs from that of a solution using a fixed viscosity. A discontinuous solution is achieved rather than a shock profile. Contrary to the nonviscous solution, the energy eventually is dissipated as the viscosity grows. So the interaction between the nonlinear inertial term and the viscous term is felt more strongly with a coupled system, i.e. each term is allowed to create its maximum effect before (or after) the opposing term takes control, (see figures 6 and 8).

Hence, in the noncoupled system, the nonlinearity adds energy to the system creating steep, but continuous, wave profiles and the energy dissipation due to viscosity is finite. In the coupled system it is possible to achieve a discontinuous solution over a finite time period as well as a finite dissipation of energy. From the stability analysis one might suspect a bifurcation in the solution since $u_x$ would not be bounded in the discontinuous solution. To ignore the effects of temperature in the Burgers equation should, therefore, be done knowing that the solution may be incomplete.

A study of why the finite difference approximation method dissipates
its energy faster than the Fourier approximation method and what significant effects this might have on approximating solutions needs to be examined. Also, a detailed analysis of the numerical technique used for the coupled system should be undertaken to determine its convergence to a solution. If we allow viscosity to depend on temperature, then we should consider that neither the thermal conductivity nor the density would remain constant as was assumed for this study. Adding density as a variable would replace the pressure gradient in (2.3). Solving this new system would add insight to the behavior of fluid.
BIBLIOGRAPHY


