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A striking feature in the study of Riemannian manifolds of positive sectional curvature is the narrowness of the collection of known examples. In this thesis, we examine the structure of the cohomology rings of three families of compact simply connected seven dimensional Riemannian manifolds that may contain new examples of positive curvature. An explicit computation of these rings reveals that there are infinitely many homotopy types represented in each family. In addition, it becomes possible to identify those manifolds to which there are associated well-known topological invariants distinguishing homeomorphism and diffeomorphism types.

All of these manifolds support an action by $S^3 \times S^3$ with orbit space a closed interval. Such manifolds are known to be diffeomorphic to the union of the total spaces of two disk bundles. This structure is exploited in two long exact cohomology sequences, which relate the cohomology of the manifold to that of the orbits of the $S^3 \times S^3$-action. These sequences, and lemmas derived from them, comprise the primary tools employed in computing the cohomology rings.
The Cohomology Rings of Seven Dimensional Primitive Cohomogeneity One Manifolds

by

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Shari K. Ultman, Author
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1 INTRODUCTION.

Finding new examples of compact simply connected Riemannian manifolds of positive sectional curvature has been an area of active interest since at least the 1960s, when the positively curved homogeneous spaces were classified. The discovery of a new example is a rare event. Recent advances in the search for such examples have focused attention on several infinite families of compact simply connected Riemannian manifolds in dimension seven. These contain several interesting subfamilies, some known to admit either positive or non-negative curvature; as well as the most recently identified example of positive curvature.

All manifolds belonging to these families carry an action by $S^3 \times S^3$ with a one dimensional orbit space. These manifolds have been classified up to $S^3 \times S^3$-equivalence; that is, an equivalence with classes determined by diffeomorphisms which respect the orbits of the action. However, there is still much work to be done in order to achieve full homeomorphism and diffeomorphism classifications. In particular, little is known about the topological invariants of these spaces.

This thesis provides an analysis of the structure of the cohomology rings belonging to members of these families. The main result is a full description of the ring structures for the most interesting manifolds. This result implies the existence of certain cohomological invariants of homeomorphism and diffeomorphism type, so determining the cohomology ring is the first step towards a classification of these manifolds.
Although the problem addressed in this thesis is topological, the motivation behind it is geometric. For this reason, we will briefly recount all currently known examples of compact simply connected manifolds with a Riemannian metric of positive sectional curvature. We will then describe a class of manifolds called cohomogeneity one manifolds. The families of interest belong to this class.

1.1 Compact simply connected manifolds with positive sectional curvature.

The diffeomorphism classification of non-compact complete Riemannian manifolds admitting positive sectional curvature is surprisingly simple: such a manifold is diffeomorphic to Euclidean space ([GM]). The compact case is significantly more complicated. To illustrate, consider that Hopf conjectured in the 1930s that $S^2 \times S^2$ does not admit a metric of positive sectional curvature. This is still an open problem.

As a consequence of the Bonnet-Myers theorem, the fundamental group of a positively curved manifold is finite (see, for example, [doC]). In addition, it has been shown that a compact manifold of non-negative curvature has a finite cover by the product $\tilde{M} \times F$ where $\tilde{M}$ is a compact simply connected manifold of non-negative curvature, and $F$ is a compact flat manifold ([CG1]). In light of this, we will focus on compact simply connected manifolds.

There are very few constructions known to give rise to manifolds of positive sectional curvature. With a single exception, all positively curved simply connected compact manifolds have been categorized as either homogeneous spaces or biquotients of compact Lie groups. The remaining example belongs to the class of cohomogeneity one manifolds.

These three types of manifolds — that is, the homogeneous spaces, the biquotients, and the cohomogeneity one manifolds — can all be defined in terms of an action by a
compact Lie group on a compact manifold. When we speak of a homogeneous space (or a biquotient, or a cohomogeneity one manifold), this action is implied. A Riemannian metric on $M$ is said to be $G$-invariant if the map $M \rightarrow M$ given by $p \mapsto g \cdot p$ is an isometry for all $g \in G$. Another way of saying that a metric is $G$-invariant, is to say that $G$ acts by isometries on the Riemannian manifold $M$. All metrics discussed below are assumed to be invariant with respect to the implicit group action. Similarly, a group acting on a Riemannian manifold is assumed to act by isometries.

1.1.1 Homogeneous spaces.

Spheres offer the obvious example of positive curvature in the compact simply connected setting. Other basic examples are the complex and quaternionic projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$, as well as the Cayley plane $\mathbb{C}aP^2$ over the octonians. These manifolds all admit metrics of positive curvature ([Be3]). As is shown in the following list of known examples of compact simply connected manifolds of positive curvature, all known manifolds of this type in dimensions higher than twenty-four are diffeomorphic to either a sphere or one of $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{C}aP^2$.

The spheres and projective spaces are all homogeneous spaces; that is, each supports a transitive action by its isometry group. A homogeneous space $M$ with isometry group $G$ is diffeomorphic to $G/H$, where $H$ is some closed subgroup of $G$. A complete classification of compact simply connected homogeneous spaces $M = G/H$ with $G$-invariant metrics of positive curvature was carried out between 1961 and 1976. Besides the spheres and simply connected projective spaces, only a few other examples were found. Two of these, one each in dimensions seven and thirteen, were identified in the course of the classification of normal homogeneous spaces of positive curvature ([Be1]). Three further examples, occurring in dimensions six, twelve and twenty-four, were found in the classification of positively curved homogeneous spaces in even dimensions ([Wa]). Finally, there is an infinite family of positively curved homogeneous seven dimensional manifolds called Aloff-Wallach spaces ([AW]). This accounts for all possible homogeneous examples ([B-B]).
1.1.2 Biquotients.

Biquotients can be thought of as a generalization of homogeneous spaces. Given a compact Lie group $G$ and a closed subgroup $H \subseteq G \times G$, define an action of $H$ on $G$ by $h \cdot g := h_1 gh_2^{-1}$ for all $h = (h_1, h_2) \in H$ and $g \in G$. The orbit space of such an action is called a biquotient. In 1982, this method was used to produce an infinite family of seven dimensional compact simply connected manifolds with positive curvature ([Es1]).

Because the Eschenburg spaces form an important and well-studied class of manifolds having a direct relation to the families studied in this thesis, we will explicitly describe their construction as biquotients. Let $k := (k_1, k_2, k_3)$ and $l := (l_1, l_2, l_3)$ be triples of integers such that $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$. There is a continuous homomorphism $\varphi$ embedding the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ in $SU(3) \times SU(3)$ given by $z \mapsto (\varphi_k(z), \varphi_l(z))$, where

$$\varphi_k(z) := \begin{bmatrix} z^{k_1} & 0 & 0 \\ 0 & z^{k_2} & 0 \\ 0 & 0 & z^{k_3} \end{bmatrix} \quad \text{and} \quad \varphi_l(z) := \begin{bmatrix} z^{l_1} & 0 & 0 \\ 0 & z^{l_2} & 0 \\ 0 & 0 & z^{l_3} \end{bmatrix}.$$  

Define an action of $S^1$ on $SU(3)$ by $\varphi(z) \cdot A := \varphi_k(z)A\varphi_l(z)^{-1}$ for all $A \in SU(3)$ and all $\varphi(z) \in S^1$. Denote by $E_{k,l}$ the biquotient of this action. When $\varphi_k(S^1)$ and $\varphi_l(S^1)$ are not conjugate in $SU(3)$, this action is free and $E_{k,l}$ is a manifold. Such a manifold is called an Eschenburg space. An Eschenburg space $E_{k,l}$ has positive sectional curvature if and only if for each $k_i \in \{k_1, k_2, k_3\}$, $k_i$ is not in the interval $[\min(l_1, l_2, l_3), \max(l_1, l_2, l_3)]$ — or if the analogous statement holds with the roles of $k$ and $l$ reversed ([Es1], [CEZ]).

The Aloff-Wallach spaces are the subfamily of the Eschenburg spaces corresponding to $k = (0, 0, 0)$ and $l = (l_1, l_2, -(l_1 + l_2))$ with $l_1l_2(l_1 + l_2) \neq 0$. However, infinitely many Eschenburg spaces do not have the homotopy type of any compact homogeneous space. In other words, the Eschenburg spaces represent infinitely many previously unknown examples of compact simply connected manifolds admitting a metric of positive curvature.
Two years after the Eschenburg spaces were revealed, a single new example, also a biquotient, was found in dimension six ([Es2]). More than a decade passed without any new discoveries. Then, in 1996, another infinite family of compact simply connected positively curved biquotients was produced, this time in dimension thirteen ([Ba]). Again, infinitely many of these manifolds are not homogeneous.

1.1.3 Cohomogeneity one manifolds.

In 1991, Grove proposed implementing a symmetry program; that is, searching for examples among spaces with “large” isometry groups. One can measure the size of an isometry group in various ways. One such measure is the cohomogeneity of an action; that is, the dimension of the orbit space $M/G$ of a smooth action by isometries of a Lie group $G$ on a connected Riemannian manifold $M$. The cohomogeneity zero spaces — that is, the homogeneous spaces — being classified, a natural next step was the investigation of compact simply connected manifolds of cohomogeneity one. As with biquotients, cohomogeneity one manifolds can be thought of as a generalization of homogeneous manifolds.

In 2004, Verdiani provided a full diffeomorphism classification of compact simply connected cohomogeneity one manifolds in even dimensions. He showed any such manifold is diffeomorphic to either a sphere or a projective space ([PV], [Ve1], [Ve2]). Although the classification in the odd dimensional case is not as advanced, at least there is an indication of where to look for new examples. If a compact simply connected cohomogeneity one manifold admits a metric of positive sectional curvature and is not diffeomorphic to one of the positively curved manifolds discussed in the preceding two sections, then it must belong to one of two infinite families $P_k$ or $Q_k$ ($k \geq 1$ an integer), or be the isolated example $R$ ([GWZ]). These three classes of candidate manifolds are all seven dimensional, and are known to support non-negative curvature. In 2007, it was shown independently in [De] and [GVZ] that the candidate manifold $P_2$ does indeed support a metric of positive sectional curvature. Although this example is known to represent a new example of positively curved Riemannian manifold, its diffeomorphism and homeomorphism types have yet to
be determined.

1.1.4 A note about non-negative curvature.

As was the case with positive curvature, the structure of open manifolds is better understood than that of compact manifolds. The soul theorem of Cheeger and Gromoll ([CG2]) states that any complete non-compact manifold with non-negative sectional curvature is diffeomorphic to the normal bundle over a compact, totally geodesic, totally convex submanifold of non-negative curvature (called the “soul”). Furthermore, by Perel-man’s positive answer to Cheeger and Gromoll’s soul conjecture ([Pe]), the existence of a single point of positive sectional curvature in such a manifold implies the manifold is diffeomorphic to Euclidean space.

The fundamental group of a non-negatively curved manifold can be infinite. In fact, finiteness of the fundamental group is one of the only known obstructions to strictly positive curvature. However, the fundamental group of an $n$ dimensional manifold of non-negative curvature is finitely generated, having at most $n2^n$ generators; and the sum of its Betti numbers (with coefficients in a field) is bounded above by $10^{10n^4}$ ([Gr1], [Gr2]).

The search for examples of non-negative curvature has been facilitated by a result of Grove and Ziller, who showed that a significant class of cohomogeneity one manifolds admit metrics of non-negative sectional curvature ([GZ]). To prove this, they generalized a metric construction used by Cheeger to produce metrics of non-negative curvature on the connected sum of two copies of a complex projective space ([Ch]). A number of infinite families of cohomogeneity one manifolds supporting non-negative curvature have been identified. As was the case with positive curvature, a seemingly disproportionate number of these occurs in dimension seven ([Ho]).
1.2 Statement of the problem and results.

We shall focus our attention on four infinite families of compact simply connected manifolds, denoted by $L(p_-,q_-),(p_+,q_+)$, $M(p_-,q_-),(p_+,q_+)$, $N(p_-,q_-),(p_+,q_+)$ and $O(p,q;:m)$ (see Section 2.1.2 for details). All support a cohomogeneity one action by $S^3 \times S^3$. The pairs of parameters $(p_\pm,q_\pm)$ (or, in the case of the family $O(p,q;m)$, the pair $(p,q)$) are relatively prime integers representing the slopes of embedded circle subgroups in specified maximal tori in $S^3 \times S^3$. These families are of interest for the following reasons:

- The positive curvature candidate family $P_k$ is the subfamily $M_{(1,1),(2k-1,2k+1)}$, and the solitary candidate $R = M_{(3,1),(1,2)}$ ([GWZ]). The new positively curved example is $P_2 = M_{(1,1),(3,5)}$.

- The candidate family $Q_k$ corresponds to the subfamily $N_{(1,1),(k,k+1)}$ ([GWZ]).

- The families $L(p_-,q_-),(p_+,q_+)$, $M(p_-,q_-),(p_+,q_+)$ and $N(p_-,q_-),(p_+,q_+)$ admit metrics of non-negative sectional curvature ([GZ]).

- The Eschenburg spaces $E_{k,l}$ where $k = (1,1,p)$ and $l = (0,0,p+2)$ for $p \geq 1$ have positive curvature, and correspond to the subfamily $O_{(p,p+1,2)}$ ([CEZ], [Es2],[GWZ]). Note that it is not known whether members of the family $O_{(p,q;m)}$ admit non-negative curvature in general.

- These four families comprise the full class of primitive cohomogeneity one manifolds (defined in Section 2.1.2) ([Ho]).

Classifications of these manifolds would be a significant achievement. As luck would have it, cohomological invariants of homeomorphism and diffeomorphism types do exist for certain seven dimensional manifolds. Let $M$ be a compact simply connected smooth seven dimensional manifold. If $M$ has non-trivial cohomology groups $H^0(M) = H^2(M) = H^5(M) = H^7(M) \cong \mathbb{Z}$ and $H^4(M) \cong \mathbb{Z}_r$ a finite group, and if
the square of a generator of $H^2(M)$ generates $H^4(M)$, then there are invariants (called Kreck-Stolz invariants) which determine the homeomorphism and diffeomorphism types of $M$ ([KS]). The Eschenburg spaces have been almost completely classified via the Kreck-Stolz invariants ([AMP],[Kr2],[CEZ]). They have cohomology groups and generators as above, with the additional characteristics that the fourth cohomology group has odd order, and the cohomology ring is fully generated by the classes in the second and fifth cohomology groups ([Es1],[Kr1]).

We now introduce the results of this thesis.

**Theorem 1.2.1** A compact simply connected seven dimensional primitive cohomogeneity one manifold $M$ is a member of:

a) the subfamily of $L_{(p_-,q_-),(p_+,q_+)}$ with the parameter $p_+$ odd and $p_+^2q_-^2 - p_-^2q_+^2 \neq 0$, or:

b) the family $N_{(p_-,q_-),(p_+,q_+)}$, or:

c) the subfamily of $O_{(p,q;m)}$ with $|p|$ and $|q|$ not both equal to one

if and only if the cohomology groups of $M$ are given by:

$$H^k(M) \cong \begin{cases} Z & k = 0, 2, 5, 7 \\ Z_r, r \neq 0, 1 & k = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the cohomology ring of any of these manifolds is completely generated (under the cup product) by cohomology group generators $x \in H^2(M)$ and $y \in H^5(M)$.

**Theorem 1.2.2** A compact simply connected seven dimensional primitive cohomogeneity one manifold $M$ is a member of the subfamily of $L_{(p_-,q_-),(p_+,q_+)}$ with the parameter $p_+$
even, if and only if the cohomology groups of $M$ are given by:

$$H^k(L) \cong \begin{cases} 
\mathbb{Z} & k = 0, 2, 7 \\
\mathbb{Z}_2 & k = 3 \\
\mathbb{Z}_r, r \neq 0, 1 & k = 4 \\
\mathbb{Z} \oplus \mathbb{Z}_2 & k = 5 \\
0 & \text{otherwise}.
\end{cases}$$

Furthermore, if the class $x$ generates $H^2(M)$ and $y$ generates the free part of $H^5(M)$, then $x^2$ generates $H^4(M)$ and $xy$ generates $H^7(M)$.

Theorem 1.2.1 leads immediately to several corollaries. The first is a list of all compact simply connected seven dimensional primitive cohomogeneity one manifolds for which the Kreck-Stolz invariants exist:

**Corollary 1.2.1** A compact simply connected seven dimensional primitive cohomogeneity one manifold admits a Kreck-Stolz invariant if and only if:

a) it is a member of the subfamily of $L_{(p_-,q_-),(p_+,q_+)}$ with the parameter $p_+$ odd and $p_+^2q_-^2 - p_-^2q_+^2 \neq 0$, or:

b) it is a member of the family $N_{(p_-,q_-),(p_+,q_+)}$, or:

c) it is a member of the subfamily $O_{(p,q;m)}$ with $|p|$ and $|q|$ not both equal to one.

By determining when the order $r$ of the fourth cohomology group is odd, we identify those compact simply connected seven dimensional primitive cohomogeneity one manifolds whose cohomology rings are indistinguishable from those of Eschenburg spaces:

**Corollary 1.2.2** A compact simply connected seven dimensional primitive cohomogeneity one manifold has the cohomology ring of an Eschenburg space if and only if:

a) it is any member of the family $N_{(p_-,q_-),(p_+,q_+)}$ or:
b) it is a member of the family $O_{(p,q;m)}$ and one of the parameters $p$ or $q$ is even.

Also through examination of the order of the fourth cohomology group, it is apparent that:

**Corollary 1.2.3** Every family of compact simply connected seven dimensional primitive cohomogeneity one manifolds contains representatives of infinitely many distinct homotopy types.

1.3 Organization of thesis.

The context in which the problem addressed in this thesis arises was provided earlier in the current chapter (Chapter 1). Section 1.1 gave an overview of the known compact simply connected Riemannian manifolds of positive sectional curvature, illustrating the scarcity of such examples. A short note on manifolds of non-negative curvature was also included in that section. In Section 1.2, the families of manifolds under consideration were introduced briefly, and the results stated.

Chapter 2 is dedicated to a description of the topological structure of cohomogeneity one manifolds, and an exploration of how this structure can be exploited in cohomology computations. Section 2.1 begins with a review of the structure imparted to a compact simply connected manifold by a cohomogeneity one action, followed by a description of the families of manifolds under consideration. Section 2.2 recounts modifications that can be made to the Mayer-Vietoris sequence and the long exact sequence of pairs associated to cohomogeneity one manifolds. These sequences are used in Section 2.3 to derive two lemmas. A commutative ladder of long exact sequences is also introduced. The sequences, lemmas and commutative ladder will be key in determining the cohomology ring structures of manifolds in the families $L_{(p_-,q_-),(p_+,q_+)}$, $N_{(p_-,q_-),(p_+,q_+)}$ and $O_{(p,q;m)}$. 
The proofs of Theorems 1.2.1 and 1.2.2 are divided between Chapters 3 and 4. In Chapter 3, the cohomology groups of members of the families $L_{(p-q), (p+q)}$ and $O_{(p,q,m)}$ are computed, and those of the family $N_{(p-q), (p+q)}$ (originally found in [GWZ]) are recalled. In particular, the order of the fourth cohomology groups are expressed in terms of the parameters of the families. Simple number-theoretic arguments are used to ascertain what restrictions on these parameters are necessary to ensure that the fourth cohomology groups are finite and non-trivial.

Cohomology ring generators are identified in Chapter 4. This leads to a full description of the cohomology rings for members of the families $N_{(p-q), (p+q)}$ and $O_{(p,q,m)}$, and the subfamily of $L_{(p-q), (p+q)}$ with $p_+$ odd, whenever the fourth cohomology group is a non-trivial finite cyclic group. An almost complete list of generators is also found for the manifolds in the family $L_{(p-q), (p+q)}$ for which $p_+$ is even. Chapter 4 concludes with a short discussion of Corollaries 1.2.1, 1.2.2 and 1.2.3, illustrating how they follow from Theorem 1.2.1.

1.4 Notation and conventions.

All manifolds are assumed to be Hausdorff, second countable, locally Euclidean spaces with a smooth structure. Unless stated otherwise, all manifolds are complete and without boundary. The only type of curvature considered is sectional curvature. When discussing cohomology, integer coefficients are assumed unless otherwise indicated.

The notation $\mathbb{Z}_r$ indicates the cyclic group of order $r$. We take $\mathbb{Z}_r$ to be infinite cyclic if $r = 0$ and trivial if $r = 1$. If $g_1, \ldots, g_n$ are elements of a group $G$, then $\langle g_1, \ldots, g_n \rangle$ denotes the subgroup of $G$ generated by $g_1, \ldots, g_n$.

The notation $\text{im} f$ refers to the image of a function; $\ker f$ and $\text{coker} f$ denote the kernel and cokernel of a homomorphism; $\det(A)$ is the determinant of the matrix $A$; and $\dim(M)$
is the dimension of a manifold. In general, the word “map” will refer to a continuous function. The symbols $\approx$, $\cong$ and $=$ will be used denote (respectively) homeomorphism, isomorphism and diffeomorphism.

Abusing notation slightly, we often use the same symbol for the total space of a fibration, as for the fibration itself. For example, $TM$ can represent both the manifold of tangent spaces over a manifold $M$, and the tangent bundle $TM \to M$. Similarly, we occasionally abuse terminology by referring to the total space of a fibration as a fibration.

Standard terminology (for example: cohomology ring, metric, bundle, Thom isomorphism, etc.) will be used without definition. The following list of texts provides a basic set of references: [Br], [doC], [DK], [Ha], [Le], [MS] and [Wal]. Several surveys were useful in researching the background material on sectional curvature, in particular: [Be2], [Be3], [Gv], [Wi1], [Wi2], [Zi1] and [Zi2].
This chapter opens with a description of the structure of a closed simply connected Riemannian manifold $M$ supporting a cohomogeneity one action by a compact Lie group. The isotropy groups of this action determine submanifolds $D(B_{\pm})$ which are the total spaces of disk bundles over two particular orbits $B_{\pm}$ of the action. Furthermore, $M = D(B_-) \cup D(B_+)$. Various modifications to the Mayer-Vietoris cohomology sequence of $M$ with respect to this cover, and to the long exact cohomology sequences of the pairs $(M, B_{\pm})$, result in long exact cohomology sequences that relate the cohomology ring of $M$ to the cohomology groups of the orbits of the action. The two lemmas stated at the end of this chapter are based on these modified long exact cohomology sequences, and are used extensively in the proofs of Theorems 1.2.1 and 1.2.2.

2.1 Structure of cohomogeneity one manifolds.

Recall from Section 1.1.3 that the smooth action of a Lie group $G$ by isometries on a connected Riemannian manifold $M$ is of cohomogeneity one if the orbit space $M/G$ is one dimensional. Because we are only interested in the case that $M$ is closed (that is, compact and without boundary), and since the isometry group of a compact manifold is a compact Lie group ([MyS]), it seems reasonable to restrict our attention to isometric actions by compact groups.

If $M$ and $G$ are both compact, the orbit space is diffeomorphic to either $S^1$ or a closed interval. If $M/G = S^1$, the projection $M \to M/G$ is a fibration with a compact fiber and connected total space ([Br, Theorem IV.8.2]). It follows from the long exact homotopy sequence of this fibration that the fundamental group of $M$ is not finite. Since
we are interested exclusively in simply connected manifolds, we need only consider actions resulting in orbit spaces diffeomorphic to a closed interval. In this section, we briefly review the structure of manifolds supporting such an action.

2.1.1 Cohomogeneity one manifolds and double disk bundles.

Let $G$ be a compact Lie group acting by isometries on a closed, connected Riemannian manifold $M$. For $x \in M$, the orbit of $x$ is $G \cdot x := \{g \cdot x | g \in G\}$. The orbit space $M/G$ is the quotient space under the projection $M \to M/G$ given by $x \mapsto G \cdot x$.

Let $K$ be the isotropy group of $x$ under the $G$-action; that is, $K$ is the subgroup of $G$ fixing $x$. Because $G$ is compact, the orbit $G \cdot x$ is a closed submanifold of $M$, and is diffeomorphic to the homogeneous space $G/K$ under the map $g \cdot x \mapsto gK$ ([Br, Corollary VI.1.3]). Identifying the homogeneous space with the submanifold, we often denote the orbit of a point $x$ by $G/K$, where $K$ is understood to be the isotropy group of $x$.

Fix a point $x \in M$ with isotropy group $K$. Let $TM|_{G/K} \to G/K$ be the restriction of the tangent bundle $TM$ to the orbit through $x$. Since $M$ is Riemannian, the normal bundle $\nu(G/K)$ over the orbit can be defined as a sub-bundle of $TM|_{G/K}$. Regarding the tangent bundle $T(G/K) \to G/K$ as a sub-bundle of $TM|_{G/K}$, the normal bundle is the sub-bundle of $TM|_{G/K}$ having total space the orthogonal complement of $T(G/K)$ in $TM|_{G/K}$. The dimension of a fiber of the normal bundle is $t = \dim(M) - \dim(G/K)$, and is called the codimension of the orbit $G/K$ in $M$.

The metric on $TM$ gives a continuous choice of inner product on the normal bundle. This permits the definition of a sub-bundle $D(G/K)$ of the normal bundle, with fibers the unit disks in the fibers of the normal bundle. We call this the normal disk bundle over $G/K$.

Let $D^t$ be the fiber in the normal disk bundle over the point $x$. The isotropy group $K$ acts smoothly on $D^t$, so we can form a twisted product $G \times_K D^t := (G \times D^t)/K$, the
orbit space of the action given by \((g, z) \cdot k = (g \cdot k, k^{-1} \cdot z)\). The total space \(D(G/K)\) is diffeomorphic to \(G \times_K D^t\). Furthermore, \(D(G/K) = G \times_K D^t\) is diffeomorphic to a neighborhood of \(G/K\) in \(M\). We call \(D(G/K) = G \times_K D^t\) a tubular neighborhood of \(G/K\) in \(M\) (see [Br, Theorems 8.1,8.2], [Wal, Propositions 10.1,10.2]).

Two orbits \(G/K\) and \(G/H\) in \(M\) are said to be of the same type if and only if \(K\) and \(H\) are conjugate in \(G\). One can check that the isotropy groups of points in the same orbit make up a full conjugacy class. When the orbit space \(M/G\) is a closed interval, there are three types of orbits. The orbits over the interior of the interval are all of the same type, and are called the principal orbits; the isotropy group of a point in a principal orbit is called a principal isotropy group. The two orbits over the endpoints account for the remaining types, and are called the non-principal orbits; an analogous definition holds for non-principal isotropy groups, as for principal isotropy groups ([Br, Theorem IV.8.2]).

**Example 2.1.1** The circle group \(S^1 = \{e^{i\theta}\} \subseteq \mathbb{C}\) acts by cohomogeneity one on the two-sphere \(S^2\). Embed \(S^2\) as the unit sphere in \(\mathbb{R}^3\). Define \(N := (0,0,1)\) and \(S := (0,0,-1)\). Embed \(S^1\) in \(SO(3)\) as the rotation subgroup fixing the axis through \(N\) and \(S\). The principal orbits of this action are circles formed by the intersection of \(S^2\) with planes in \(\mathbb{R}^3\) normal to the fixed axis. The principal isotropy groups are trivial. The non-principal orbits are the points \(N\) and \(S\), and the non-principal isotropy groups are both \(S^1\).

Suppose \(G\) is a compact Lie group acting on a closed, connected Riemannian manifold \(M\) by cohomogeneity one, with orbit space \(M/G\) diffeomorphic to a closed interval. In [GZ], it was shown that \(M\) is diffeomorphic to the quotient space formed by gluing together two tubular neighborhoods of the non-principal orbits. These tubular neighborhood share a common boundary, and the gluing map is the identity map on these boundaries. Briefly, they demonstrated that the choice of a point \(x\) in a principal orbit determines a particular geodesic \([-1,1] \rightarrow M\), such that the endpoints \(x_\pm := c(\pm 1)\) reside in different non-principal orbits. If \(H\) is the isotropy group of \(x\) and \(K_\pm\) are the isotropy groups of \(x_\pm\), then the boundaries \(\partial D(G/K_\pm)\) of the tubular neighborhoods of the non-principal orbits
$G/K_{\pm}$ are canonically diffeomorphic to the principal orbit $G/H$ through $x$. Also, the principal isotropy group $H$ is a subgroup of both non-principal isotropy groups $K_{\pm}$, and the quotients $K_{\pm}/H$ are diffeomorphic to the boundary sphere $S^{t_{\pm}-1}$ of the unit normal disks $D^{t_{\pm}}$ over the points $x_{\pm}$ (see [GZ, Section 1] for details).

This description of $M$ is an example of a more general topological construction. Suppose $D(B_{\pm})$ are the total spaces of disk bundles over paracompact spaces $B_{\pm}$, and $\varphi$ is a homeomorphism of the boundary sphere bundles $\partial D(B_{\pm})$. We call the quotient space $X = D(B_-) \cup_{\varphi} D(B_+)$ a double disk bundle. In the case of a cohomogeneity one manifold, the spaces $B_{\pm}$ are the non-principal orbits $G/K_{\pm}$, the boundaries of the disk bundles are the sphere bundle $\partial D(G/K_-) = \partial D(G/K_+) = G \times_{K_{\pm}} K_{\pm}/H = G/H$, and the attaching map $\varphi$ is the identity map. So for $M$ cohomogeneity one:

$$M = D(G/K_-) \cup_{id} D(G/K_+) = (G \times_{K_-} D^{t_-}) \cup_{id} (G \times_{K_+} D^{t_+}).$$

Conversely, suppose $G$ is a compact Lie group with closed subgroups $H \subseteq K_-, K_+$. It is shown in [GZ] that whenever $K_{\pm}/H$ are diffeomorphic to spheres $S^{t_{\pm}-1}$, there exist disk bundles $G \times_{K_{\pm}} D^{t_{\pm}} \to G/K_{\pm}$ such that $\partial(G \times_{K_{\pm}} D^{t_{\pm}}) = G/H$. Identifying these boundaries results in a smooth double disk bundle, $M = (G \times_{K_-} D^{t_-}) \cup_{id} (G \times_{K_+} D^{t_+})$. The obvious left $G$-action on $M$ is of cohomogeneity one.

Hence, every cohomogeneity one manifold can be represented by compact Lie groups $H \subseteq K_-, K_+ \subseteq G$ such that $K_{\pm}/H = S^{t_{\pm}-1}$, while every such collection of compact Lie groups determines a cohomogeneity one manifold. In this context, we call the group inclusions $H \subseteq K_-, K_+ \subseteq G$ a group diagram. The subgroup $H$ is called the principal isotropy group, and the subgroups $K_-$ and $K_+$ are called the non-principal isotropy groups.

### 2.1.2 Seven dimensional primitive cohomogeneity one manifolds.

By construction, the group diagram generated by a given cohomogeneity one $G$-action on a Riemannian manifold depends on both the choice of the point $x$ in the principal
orbit, and on the choice of metric on $M$. The first condition corresponds to conjugating all three isotropy groups by the same element of $G$. The second corresponds to conjugating the non-principal isotropy groups $K_\pm$ by (possibly different) elements of the identity component of the principal isotropy group $H$ ([GWZ]).

These conditions on the isotropy groups can be used to classify cohomogeneity one manifolds, up to a notion of equivalence called $G$-equivariant diffeomorphism. Two manifolds $M$ and $N$ are $G$-equivariantly diffeomorphic if both admit smooth actions by a Lie group $G$, and there is a diffeomorphism $M \xrightarrow{\varphi} N$ such that $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $x \in M$ and $g \in G$. In other words, there is a diffeomorphism between $M$ and $N$ that preserves the orbits of the $G$-action.

Such a classification was carried out in [Ho] for compact simply connected cohomogeneity one manifolds through dimension seven. In dimension seven, thirteen types of group diagrams were identified. Of these, all but two occur as infinite families. These manifolds are further divided into two classes: the primitive, and the non-primitive. A cohomogeneity one manifold $M$ is primitive if, for all group diagrams $H \subset K_-, K_+ \subset G$ representing $M$ (up to $G$-equivariant diffeomorphism), there is no closed, connected, proper subgroup of $G$ containing the isotropy groups $H$ and $K_\pm$. There is a relationship between the primitive and non-primitive cohomogeneity one manifolds; namely, every non-primitive cohomogeneity one manifold is diffeomorphic to the total space of a fiber bundle over a homogeneous space, with fiber a primitive cohomogeneity one manifold ([Ho]).

There are four infinite families comprising the primitive cohomogeneity one manifolds in dimension seven. Modifying the original notation of [Ho] in order to reflect that of [GWZ], we call these families $L_{(p_-,q_-),(p_+,q_+)}$, $M_{(p_-,q_-),(p_+,q_+)}$, $N_{(p_-,q_-),(p_+,q_+)}$ and $O_{(p,q;m)}$. Recall from the discussion in Section 1.2 that these are the families of interest.

All manifolds in these four families carry a cohomogeneity one action by $G = S^3 \times S^3$, where $S^3$ is regarded as the group of unit quaternions. Let $\{1, i, j, k\}$ be the canonical basis of the quaternions, with relations $i^2 = j^2 = ijk = -1$. Given a circle group
TABLE 2.1: Isotropy groups description of the compact simply connected seven dimensional primitive cohomogeneity one manifolds.

<table>
<thead>
<tr>
<th>Family</th>
<th>Isotropy groups $H \subseteq K_-, K_+$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>restrictions on parameters</td>
</tr>
<tr>
<td>$L(p_-,q_-),(p_+,q_+)$</td>
<td>$\langle (i, i) \rangle \subseteq {(e^{ip_-\theta}, e^{iq_-\theta}), (e^{ip_+\theta}, e^{iq_+\theta})} \cdot H$</td>
</tr>
<tr>
<td></td>
<td>$p_-, q_- \equiv 1 \mod 4$</td>
</tr>
<tr>
<td>$M(p_-,q_-),(p_+,q_+)$</td>
<td>$\Delta Q \subseteq {(e^{ip_-\theta}, e^{iq_-\theta})} \cdot H, {(e^{ip_+\theta}, e^{iq_+\theta})} \cdot H$</td>
</tr>
<tr>
<td></td>
<td>$\Delta Q$ the diagonal embedding of $\langle 1, i, j, k \rangle$;</td>
</tr>
<tr>
<td></td>
<td>$p_\pm, q_\pm \equiv 1 \mod 4$</td>
</tr>
<tr>
<td>$N(p_-,q_-),(p_+,q_+)$</td>
<td>$\langle (h_1, h_2), (1, -1) \rangle \subseteq {(e^{ip_-\theta}, e^{iq_-\theta})} \cdot H, {(e^{ip_+\theta}, e^{iq_+\theta})} \cdot H$</td>
</tr>
<tr>
<td></td>
<td>$h_1, h_2 \in {i, -i}$ with signs chosen so that</td>
</tr>
<tr>
<td></td>
<td>$(h_1, h_2)$ lies in ${(e^{ip_-\theta}, e^{iq_-\theta})}$;</td>
</tr>
<tr>
<td></td>
<td>$p_-$ and $q_\pm$ odd, $p_+$ even</td>
</tr>
<tr>
<td>$O(p,q;m)$</td>
<td>$\mathbb{Z}_m \subseteq {(e^{ip\theta}, e^{iq\theta})}, \Delta S^3 \cdot H$</td>
</tr>
<tr>
<td></td>
<td>either $m = 1$ (with no restrictions on $p$ or $q$)</td>
</tr>
<tr>
<td></td>
<td>or $m = 2$ and $p$ is even</td>
</tr>
</tbody>
</table>

$\{(e^{ip\theta}, e^{iq\theta})\} \subset S^3 \times S^3$, it is always assumed that the parameters $p$ and $q$ are relatively prime integers, since the circle group must embed in $S^3 \times S^3$. Table 2.1 lists the remaining data necessary to complete the group diagrams; namely, the isotropy groups together with restrictions on the parameters that guarantee an embedding of the principal isotropy group $H$ in the non-principal isotropy groups $K_\pm$ (cf [Ho, Table I]). For example, in the description of the family $L(p_-,q_-),(p_+,q_+)$, the principal isotropy group $H = \langle (i, i) \rangle$ is the
cyclic group of order four generated by the diagonal embedding of the unit quaternion \(i\), the non-principal isotropy group \(K_+ = \{(e^{i\theta p}, e^{i\theta q})\} \cdot H\) is the group whose elements are products of an element of the circle group with an element of \(H\), and the congruence of the parameters \(p_-, q_- \equiv 1 \mod 4\) of the non-principal isotropy group \(K_- = \{(e^{i\theta p}, e^{i\theta q})\}\) ensures that \(H\) is a subgroup.

2.2 Two long exact cohomology sequences.

When computing the cohomology of a double disk bundle \(X = D(B_-) \cup \varphi D(B_+)\), one has recourse to both the Mayer-Vietoris sequence and the long exact sequences of the pairs \((X, B_+)\). In this section, we will see how the disk bundle structure of the subspaces \(D(B_+)\) allows modification of these sequences. These modified sequences relate the cohomology groups of the space \(X\) to those of the subspaces \(B_\pm\). In the case of cohomogeneity one manifolds with at least one orientable non-principal orbit, these modifications make it possible to draw conclusions about the cohomology ring generators of \(X\) based on the cohomology groups of the orbits.

2.2.1 Modified Mayer-Vietoris sequence.

Let \(D(B_\pm) \xrightarrow{p_\pm} B_\pm\) be disk bundles such that \(D(B_\pm)\) share a common boundary \(\partial D(B)\). Then \(\partial D(B) \xrightarrow{\pi_\pm} B_\pm\) are sphere bundles with projections \(\pi_\pm\) the restrictions of the projections \(p_\pm\) of the disk bundles. Define the homomorphism \(\pi^* := \pi_-^* - \pi_+^*\) to be the difference of the induced homomorphisms of cohomology groups. Let \(X = D(B_-) \cup_{id} D(B_+)\) be the double disk bundle with gluing map the identity map (observe
that this is the case with a cohomogeneity one manifold). Consider the diagram:

\[
\cdots \xrightarrow{\psi} H^k(D(B_-)) \oplus H^k(D(B_+)) \xrightarrow{\phi} H^k(\partial D(B)) \xrightarrow{\delta} H^{k+1}(X) \xrightarrow{\psi} \cdots
\]

where the top row is the Mayer-Vietoris sequence of \( X \). The projections \( p_{\pm} \) are homotopy equivalences, and so induce isomorphisms of the cohomology groups. Making the appropriate substitutions in the Mayer-Vietoris sequence gives the long exact sequence:

\[
\cdots \rightarrow H^k(X) \xrightarrow{\psi} H^k(B_-) \oplus H^k(B_+) \xrightarrow{\pi^*} H^k(\partial D(B)) \xrightarrow{\delta} H^{k+1}(X) \rightarrow \cdots \tag{2.1}
\]

(cf \([GZ, \text{Sequence 3.4}]\)).

### 2.2.2 Modified long exact sequence of pairs.

Again, let \( D(B_{\pm}) \xrightarrow{p_\pm} B_{\pm} \) be disk bundles with \( \partial D(B) \) the common boundary of \( D(B_{\pm}) \), and \( X = D(B_-) \cup_{id} D(B_+) \). Embed \( B_+ \) in \( X \) as the zero section of \( D(B_+) \). The long exact cohomology sequence of the pair \( (X, B_+) \) can be modified, assuming the disk bundle \( D^t \hookrightarrow D(B_-) \rightarrow B_- \) is orientable; that is, if the structure group \( O_t(\mathbb{R}) \) can be reduced to \( SO_t(\mathbb{R}) \).

Consider the commutative diagram:

\[
\cdots \rightarrow H^{k-1}(B_+) \xrightarrow{\delta} H^k(X, B_+) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(B_+) \rightarrow \cdots
\]

\[
\begin{array}{c}
\text{incl}^* \cong \\
\text{incl}^* \cong \\
id^* \cong \\
l\text{incl}^* \cong \\
\end{array}
\]

\[
\cdots \rightarrow H^{k-1}(D(B_+)) \rightarrow H^k(X, D(B_+)) \rightarrow H^k(X) \rightarrow H^k(D(B_+)) \rightarrow \cdots
\]

\[
\begin{array}{c}
\text{incl}^* \cong \\
\text{Thom isomorphism} \cong \\
H^{k-t}(B_-)
\end{array}
\]

where the top row is the long exact sequence of the pair \( (X, B_+) \) and the bottom row is the long exact sequence of the pair \( (X, D(B_+)) \). Since the inclusion of \( B_+ \) in \( D(B_+) \) is a homotopy equivalence, the inclusion of pairs \( (X, B_+) \hookrightarrow (X, D(B_+)) \) induces isomorphisms
on the relative cohomology groups $H^k(X, B_+)$ and $H^k(X, D(B_+))$ by the five lemma. By excision, the relative cohomology groups $H^k(X, D(B_+))$ and $H^k(D(B_+), \partial D(B))$ are isomorphic. Orientability of the bundle $D(B_-) \to B_-$ guarantees the existence of a Thom isomorphism from $H^{k-t}(B_-)$ to $H^k(D(B_-), \partial D(B))$ (see, for example, [MS]).

The bundle projection $D(B_-) \to B_-$ followed by the inclusion $B_- \hookrightarrow X$ is homotopic to the inclusion $D(B_-) \hookrightarrow X$; hence, the composition of the Thom isomorphism with the inverse of the excision isomorphism is an $H^*(X)$-module homomorphism from $H^{k-t}(B_-)$ to $H^k(X, D(B_+))$ ([Do, Corollary 11.20]). Since homomorphisms induced on cohomology by topological maps respect cup products, the group isomorphism from $H^k(X, D(B_+))$ to $H^k(X, B_+)$ is also an $H^*(X)$-module isomorphism. Thus, there is an $H^*(X)$-module isomorphism from $H^{k-t}(B_-)$ to $H^k(X, B_+)$. Define $J$ to be the composition of this isomorphism with the homomorphism $j^*_+$ from $H^k(X, B_+)$ to $H^k(X)$. Checking the definition of $j^*_+$, one sees that it is a homomorphism of $H^*(X)$-modules. Thus, $J$ is an $H^*(X)$-module homomorphism. This will be key in identifying cohomology ring generators.

Making the appropriate substitutions in the sequence of the pair $(X, B_\pm)$, we have a long exact sequence:

$$
\cdots \to H^{k-t}(B_-) \overset{J}{\to} H^k(X) \overset{i^*_+}{\to} H^k(B_+) \overset{\delta}{\to} H^{k-t+1}(B_-) \to \cdots
$$

(cf [He, Sequences 4.1.a, 4.1.b]). An analogous sequence with the roles of $B_+$ and $B_-$ reversed exists if the bundle $D(B_+) \to B_+$ is orientable. If integer cohomology coefficients are replaced by $\mathbb{Z}_2$ coefficients, such a sequence exists for any pair $(X, B_\pm)$ regardless of bundle orientability. This follows from the existence of the Thom isomorphism for any disk bundle when $\mathbb{Z}_2$ coefficients are used for cohomology (again, see [MS]).

If $B$ is a closed orientable submanifold of an orientable manifold $M$, then it is known that the normal disk sub-bundle over $B$ in the tangent bundle of $M$ is an orientable bundle (see, for example, [BT, p.66]). As previously mentioned, orbits under the action of a compact group are closed submanifolds of the ambient manifold; and the manifolds
under consideration are orientable by virtue of being simply connected. We shall see
that members of the families \( L_{(p-, q-), (p+, q+)} \) and \( N_{(p-, q-), (p+, q+)} \) have one orientable non-
principal orbit, while both non-principal orbits for manifolds in the family \( O_{(p,q;m)} \) are
orientable. So there exists at least one long exact sequence of this type associated to
every member of the above families. In contrast, both non-principal orbits of members of
\( M_{(p-, q-), (p+, q+)} \) are non-orientable (see [GWZ]).

2.3 Two lemmas and a commutative ladder.

In this section, we derive two lemmas from Sequences 2.1 and 2.2, and introduce
a commutative ladder of long exact sequences. These, together with the sequences, are
the main tools on which we rely in the proofs of Theorems 1.2.1 and 1.2.2. Both lemmas
apply to double disk bundles, and require orientability of at least one of the bundles.

2.3.1 The lemmas.

Lemma 2.3.1 is used in the proofs of Theorems 1.2.1 and 1.2.2 to conclude that
the fourth cohomology groups of manifolds in the families \( L_{(p-, q-), (p+, q+)} \) and \( O_{(p,q;m)} \) are
cyclic:

**Lemma 2.3.1** Let \( X = D(B_-) \cup id D(B_+) \) be a double disk bundle where the bundle \( D^t \hookrightarrow D(B_-) \to B_- \) is orientable and \( D(B_+) \) share a common boundary \( \partial D(B) \). For a fixed integer \( \kappa \), suppose \( H^{\kappa-1}(B_-) \) is cyclic and both groups \( H^{\kappa}(B_+) \) are trivial. Furthermore,
suppose \( H^{\kappa-1}(\partial D(B)) \) is finitely generated and free, and has the same rank as the free
part of \( H^{\kappa-1}(B_-) \oplus H^{\kappa-1}(B_+) \). Let \( r \geq 0 \) be the absolute value of the determinant of
the restriction of the homomorphism \( \pi^* \) to the free part of \( H^{\kappa-1}(B_-) \oplus H^{\kappa-1}(B_+) \). Then
\( H^{\kappa}(X) \) is the cyclic group of order \( r \).
The cyclic group $H$ of $\mathbb{Z}$.

By exactness, $H \to H(\partial D)$ is surjective, so $H(\partial D)$ must be cyclic. Let $r$ be the order of $H(\partial D)$.

Set $k = \kappa - 1$ and $\kappa$ in Sequence 2.1:

$$\cdots \to H^{\kappa - 1}(B_-) \to H^{\kappa - 1}(\partial D) \to H^\kappa(B_-) \to H^\kappa(B_+) = 0.$$

By exactness, $H^{\kappa - 1}(B_-) \oplus H^{\kappa - 1}(B_+) \to H^{\kappa - 1}(\partial D) \to H^{\kappa}(B_-) \oplus H^{\kappa}(B_+) = 0$.

It is possible to choose bases for the free part of $H^{\kappa - 1}(B_-) \oplus H^{\kappa - 1}(B_+$) and for $H^{\kappa - 1}(\partial D)$ with respect to which $\varpi$ can be represented as a diagonal matrix $A$ over the non-negative integers ([Ro, Theorem 9.58 (Smith Normal Form)]). Furthermore, if the diagonal entries of this matrix $A$ are $a_1, \ldots, a_n$, then $a_i$ divides $a_{i+1}$ for $1 \leq i < n$.

To see that $|\det(\varpi)|$ is equal to the order of the cyclic group $H^\kappa(X) = \mathbb{Z}_r$, consider:

$$H^\kappa(X) \cong H^{\kappa - 1}(\partial D)/\text{im } \varpi \cong \mathbb{Z}^n/\text{im } A \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z}.$$

We proceed by considering three cases. First, suppose $r = 0$ and $H^\kappa(X)$ is infinite cyclic. Then $H^\kappa(X) \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z} \cong \mathbb{Z}$ if and only if $a_1 = \cdots = a_{n-1} = 1$ and $a_n = 0,$
so \( \det(A) = 0 \). Similarly, \( H^\kappa(X) \) is trivial (that is, \( r = 1 \)) if and only if \( a_1 = \cdots = a_n = 1 \), so \( \det(A) = 1 \). Finally, suppose the order of \( H^\kappa(X) \) is finite and non-trivial. In this case, we use the fact that the order of the direct sum of finite cyclic groups is the product of the orders of the summand groups. \( \square \)

Lemma 2.3.2 is used in the proofs of Theorems 1.2.1 and 1.2.2 to identify generators of the cohomology rings:

**Lemma 2.3.2** Let \( X = D(B_-) \cup \varphi D(B_+) \) be a double disk bundle over a connected base, where the disk bundle \( D^t \hookrightarrow D(B_-) \to B_- \) is orientable. Suppose \( H^t(X) \) is infinite cyclic, and \( H^t(B_+) \) is finite cyclic of order \( n \geq 1 \). Let \( i^*_\pm \) be the homomorphisms induced on cohomology by the inclusions of \( B_\pm \) in \( X \), and suppose \( i^*_+ : H^t(X) \to H^t(B_+) \) is a surjection. Finally, suppose \( \kappa \) is a fixed integer, \( \kappa > t \), such that the following hold:

1. \( H^\kappa(X) \) is a non-trivial cyclic group and \( H^\kappa(X) \xrightarrow{\iota^*_\pm} H^\kappa(B_\pm) \) is the zero homomorphism.

2. \( H^{\kappa-t}(B_-) \cong \mathbb{Z} \cdot \gamma \oplus T \) where \( T \) is torsion and the free part is generated by \( \gamma \). If \( H^\kappa(X) \) is finite, the orders of elements of \( T \) are relatively prime to the order of \( H^\kappa(X) \).

3. There exists a class \( \alpha \) in \( H^{\kappa-t}(X) \) with image \( i^*_-(\alpha) = s\gamma + \beta \) (for \( \beta \in T \)) such that: if \( H^\kappa(X) \) is free, then \( |s| = n \); otherwise, \( s \) is relatively prime to the order of \( H^\kappa(X) \).

Then the cohomology class \( x \sim \alpha \) generates \( H^\kappa(X) \), where \( x \) is a generator of \( H^t(X) \).

**Proof.** Let \( 1_- \) be the unit of the cohomology ring \( H^*(B_-) \). Setting \( k = t \) in Sequence 2.2, and assuming the hypotheses regarding \( H^t(X) \) and \( H^t(B_+) \) hold, one has a short exact sequence:

\[
0 \to H^0(B_-) \cong \mathbb{Z} \xrightarrow{J} H^t(X) \cong \mathbb{Z} \xrightarrow{i^*_+} H^t(B_+) \cong \mathbb{Z}_n \to 0.
\]
From this we conclude that the homomorphism $J$ from $H^0(B_-)$ to $H^t(X)$ is multiplication by $n$, and $J(1_-) = \pm nx$.

Now, let $k = \kappa$ in Sequence 2.2. By Condition 1, the homomorphism from $H^\kappa(X)$ to $H^\kappa(B_+)$ is the zero homomorphism, hence by exactness the homomorphism $J$ from $H^{\kappa-t}(B_-) \cong \mathbb{Z} \cdot \gamma \oplus T$ to $H^\kappa(X)$ is a surjection. Torsion elements of $H^{\kappa-t}(B_-)$ are in the kernel of $J$ (by Condition 2), so $J(\gamma)$ generates $H^\kappa(X)$.

We now consider separately the case in which $H^\kappa(X)$ is infinite cyclic, and that in which it is finite cyclic. First, suppose $H^\kappa(X)$ is infinite cyclic. Let $i_\kappa^*(\alpha) = \pm n\gamma + \beta \in H^{\kappa-t}(B_-)$ where $\beta$ is torsion, as required by Condition 3. Then:

$$\pm nJ(\gamma) = \pm (J(n\gamma) + J(\beta)) = \pm J(n\gamma + \beta) = \pm J(i_\kappa^*(\alpha)) = \pm J(1_- \smile i_\kappa^*(\alpha)).$$

Recall that $J$ is an $H^*(X)$-module homomorphism, so:

$$J(1_- \smile i_\kappa^*(\alpha)) = J(1_-) \smile \alpha = \pm n(x \smile \alpha).$$

Since the generator $J(\gamma)$ is non-trivial in $H^\kappa(X) \cong \mathbb{Z}$ and $n \neq 0$, cancellation implies that $x \smile \alpha = \pm J(\gamma)$; so $x \smile \alpha$ generates $H^\kappa(X)$.

On the other hand, suppose $H^\kappa(X)$ is finite cyclic. Let $i_\kappa^*(\alpha) = s\gamma + \beta$ where $s$ is relatively prime to the order of $H^\kappa(X)$, thus satisfying Condition 3. A calculation similar to the one carried out in the previous case shows that $sJ(\gamma) = \pm n(x \smile \alpha)$. The class $J(\gamma)$ generates $H^\kappa(X)$, and the order of $H^\kappa(X)$ is relatively prime to $s$, therefore $sJ(\gamma) = \pm n(x \smile \alpha)$ also generates $H^\kappa(X)$. But if a multiple of $x \smile \alpha$ generates a finite cyclic group, then $x \smile \alpha$ itself must be a generator. \qed
2.3.2 A commutative ladder.

Let \( D(B_{\pm}) \xrightarrow{p_{\pm}} B_{\pm} \) be disk bundles with \( \partial D(B) \) the common boundary of \( D(B_{\pm}) \), and \( X = D(B_{-}) \cup_{id} D(B_{+}) \). Consider the commutative diagram:

\[
\begin{array}{ccc}
\cdots & \xrightarrow{j^*} & H^k(X) & \xrightarrow{i^*} & H^k(B_{-}) & \xrightarrow{\delta} & H^{k+1}(X, B_{-}) & \xrightarrow{j^*} & \cdots \\
\text{id}^* & \cong & & & \text{incl}^* & \cong & & & \text{incl}^* & \cong \\
\cdots & \xrightarrow{\text{incl}^*} & H^k(D(B_{-})) & \xrightarrow{\text{incl}^*} & H^{k+1}(X, D(B_{-})) & \xrightarrow{\text{incl}^*} & \cdots \\
\text{incl}^* & \cong & & & \text{incl}^* & \cong & & & \text{incl}^* & \cong \\
\cdots & \xrightarrow{\text{incl}^*} & H^k(D(B_{+})) & \xrightarrow{\text{incl}^*} & H^k(\partial D(B)) & \xrightarrow{\text{incl}^*} & H^{k+1}(D(B_{+}), \partial D(B)) & \xrightarrow{\text{incl}^*} & \cdots \\
\text{incl}^* & \cong & & & & & & & & \text{incl}^* & \cong \\
\cdots & \xrightarrow{H^k(B_{+})} & & & & & & & & \text{incl}^* & \cong \\
\end{array}
\]

where the three rows are the long exact cohomology sequences of the pairs \((X, B_{-})\), \((X, D(B_{-}))\) and \((D(B_{+}), \partial D(B))\). The inclusion of pairs \((X, B_{-}) \hookrightarrow (X, D(B_{-}))\) induces a commutative ladder of long exact sequences, and since the inclusion of \( B_{-} \) in \( D(B_{-}) \) is a homotopy equivalence, the five lemma implies all vertical homomorphisms in this ladder are isomorphisms.

The inclusion of pairs \((D(B_{+}), \partial D(B)) \hookrightarrow (X, D(B_{-}))\) induces a commutative ladder of long exact sequences where the homomorphism of the relative cohomology groups is the excision isomorphism. This yields a commutative ladder between the long exact sequence of the pair \((X, B_{-})\) and the long exact sequence of the pair \((D(B_{+}), \partial D(B))\), where the vertical homomorphisms between the relative cohomology groups are isomorphisms. Finally, because the inclusion of \( B_{+} \) in \( D(B_{+}) \) is a homotopy equivalence, the cohomology groups of \( D(B_{+}) \) can be replaced with those of \( B_{+} \). So there is a commutative ladder:

\[
\begin{array}{ccc}
\cdots & \xrightarrow{j^*} & H^k(X) & \xrightarrow{i^*} & H^k(B_{-}) & \xrightarrow{\delta} & H^{k+1}(X, B_{-}) & \xrightarrow{j^*} & \cdots \\
\text{i}^* & \cong & & & & & & & & \text{incl}^* & \cong \\
\cdots & \xrightarrow{i^*} & H^k(D(B_{-})) & \xrightarrow{\delta} & H^{k+1}(D(B_{+}), \partial D(B)) & \xrightarrow{j^*} & \cdots \\
\text{i}^* & \cong & & & & & & & & \text{incl}^* & \cong \\
\cdots & \xrightarrow{i^*} & H^k(B_{+}) & \xrightarrow{\delta} & H^{k+1}(D(B_{+}), \partial D(B)) & \xrightarrow{j^*} & \cdots \\
\text{i}^* & \cong & & & & & & & & \text{incl}^* & \cong \\
\cdots & \xrightarrow{H^k(B_{+})} & & & & & & & & \text{incl}^* & \cong \\
\end{array}
\]
This ladder is a useful tool for determining whether Condition 3 of Lemma 2.3.2 holds. In particular, given a class \( x \in H^k(X) \), it can be used to determine the image of \( x \) in \( H^k(B_-) \) under \( i_* \).
3 PROOFS OF THEOREMS 1.2.1 AND 1.2.2: THE COHOMOLOGY GROUPS.

We prove Theorems 1.2.1 and 1.2.2 simultaneously. Recall Theorem 1.2.1: a compact simply connected seven dimensional primitive cohomogeneity one manifold $M$ is a member of:

a) the subfamily of $L_{(p_-,q_-),(p_+,q_+)}$ with the parameter $p_+$ odd and $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$, or:

b) the family $N_{(p_-,q_-),(p_+,q_+)}$, or:

c) the subfamily of $O_{(p,q;m)}$ with $|p|$ and $|q|$ not both equal to one

if and only if the cohomology groups of $M$ are given by:

$$H^k(M) \cong \begin{cases} 
\mathbb{Z} & k = 0, 2, 5, 7 \\
\mathbb{Z}_r, r \neq 0, 1 & k = 4 \\
0 & \text{otherwise.}
\end{cases}$$

Furthermore, the cohomology ring of any of these manifolds is completely generated (under the cup product) by cohomology group generators $x \in H^2(M)$ and $y \in H^5(M)$.

Also recall Theorem 1.2.2: a compact simply connected seven dimensional primitive cohomogeneity one manifold $M$ is a member of the subfamily of $L_{(p_-,q_-),(p_+,q_+)}$ with the parameter $p_+$ even, if and only if the cohomology groups of $M$ are given by:

$$H^k(L) \cong \begin{cases} 
\mathbb{Z} & k = 0, 2, 7 \\
\mathbb{Z}_2 & k = 3 \\
\mathbb{Z}_r, r \neq 0, 1 & k = 4 \\
\mathbb{Z} \oplus \mathbb{Z}_2 & k = 5 \\
0 & \text{otherwise.}
\end{cases}$$
Furthermore, if the class $x$ generates $H^2(M)$ and $y$ generates the free part of $H^5(M)$, then $x^2$ generates $H^4(M)$ and $xy$ generates $H^7(M)$.

**Proof.** The topology of the family $M_{(p_-,q_-),(p_+,q_+)}$ is described in [GWZ, Theorem 13.1]. A member $M$ of this family is shown to have non-trivial cohomology groups $H^0(M) = H^7(M) \cong \mathbb{Z}$, and $H^4(M) \cong \mathbb{Z}_r$, a finite group of order $r = \frac{1}{8}|p_+^2q_-^2 - p_-^2q_+^2|$, whenever $p_+^2q_-^2 - p_-^2q_+^2 \neq 0$. Otherwise, the non-trivial cohomology groups are $H^0(M) = H^3(M) = H^4(M) = H^7(M) \cong \mathbb{Z}$.

From here, the proof will proceed in two steps. In this chapter, we compute the cohomology groups of members of the families $L_{(p_-,q_-),(p_+,q_+)}$ and $O_{(p,q;m)}$. We also recalculate the cohomology groups of members of the family $N_{(p_-,q_-),(p_+,q_+)}$, originally found in [GWZ]. We show how the order of the fourth cohomology groups can be expressed in terms of the parameters, and determine what restrictions on the parameters are necessary to guarantee that these groups are non-trivial and finite.

In Chapter 4, generators of the cohomology rings are identified for those manifolds having a non-trivial finite cyclic fourth cohomology group. We achieve a full description of the cohomology ring structure in all cases, except for those members of the family $L_{(p_-,q_-),(p_+,q_+)}$ whose parameter $p_+$ is even. It will be seen that the cohomology ring generators agree with the statements of the theorems, and the results follow.

Observe that all manifolds in question are closed, seven dimensional and simply connected (see [Ho]). In particular, they are orientable. Easy calculations using Poincaré duality together with the universal coefficient theorem show that they have infinite cyclic cohomology in dimensions zero and seven and trivial cohomology in dimensions one and six. It remains to find the second through fifth cohomology groups.
3.1 Cohomology groups of the family \( L_{(p,q_\pm),(p_\pm,q_\pm)} \).

Recall that this family is described by the groups:

\[
H = \langle (i, i) \rangle \subseteq K_- = \{ (e^{ip_\theta}, e^{iq_\theta}) \}, K_+ = \{ (e^{ip_\theta}, e^{iq_\theta}) \} \cdot H \subseteq G = S^3 \times S^3
\]

where \( p_- \), \( q_- \), \( p_+ \), \( q_+ \) are pairs of relatively prime integers, and \( p_- \) and \( q_- \) are both congruent to 1 modulo 4. This family naturally splits into two subfamilies, depending on whether \( p_+ \) is even or odd. The cohomology of the principal orbit \( G/H \) and the non-principal orbit \( G/K_- \) is the same in both cases. The principal orbit \( G/H = S^3 \times S^3 / \langle (i, i) \rangle \) is homeomorphic to the product \( S^3 \times (S^3 / \langle i \rangle) \) of the 3-sphere with the lens space \( S^3 / \langle i \rangle \approx L_4(1,1) \). An explicit homeomorphism is given by \([q_1,q_2] \mapsto (q_1q_2^{-1},[q_2])\). The non-principal orbit \( G/K_- = S^3 \times S^3 / \{(e^{ip_\theta}, e^{iq_\theta})\} \) is always homeomorphic to \( S^3 \times S^2 \) by [WZ, Proposition 2.3]. The orbit \( G/K_+ \), however, varies depending on the parity of \( p_+ \).

**Case 1: \( p_+ \) is odd.**

Suppose \( p_+ \) is odd. Then the cohomology groups of the non-principal orbits \( G/K_+ \) were calculated in [GWZ, Lemma 13.3a], where they were shown to be:

\[
H^k(G/K_+) \cong \begin{cases} 
\mathbb{Z} & k = 0, 3 \\
\mathbb{Z}_2 & k = 2, 5 \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( L \) be a member of the subfamily \( L_{(p_-q_-),(p_+,q_+)} \) with \( p_+ \) odd. We know that \( H^0(L) \cong H^7(L) \cong \mathbb{Z} \). The orbit \( G/K_- \approx S^3 \times S^2 \) is a closed orientable submanifold of codimension 2, so the normal disk bundle over \( G/K_- \) is an orientable bundle with fiber \( D^2 \). Setting \( t = 2, \kappa = 4, B_\pm = G/K_\pm \) and \( \partial D(B) = G/H \), it follows from Lemma 2.3.1 that \( H^4(L) \cong \text{coker}\pi^* \cong \mathbb{Z}_r \). Recall that \( r \) is (up to sign) the determinant of the homomorphism \( \pi^* \) from the rank two free abelian group \( H^3(G/K_-) \oplus H^3(G/K_+) \cong \mathbb{Z} \oplus \mathbb{Z} \) to the rank two free abelian group \( H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z} \). Apply Sequences 2.1 and 2.2 (taking
to find the remaining cohomology groups:

\[
H^k(L) \cong \begin{cases} 
\mathbb{Z} & k = 0, 2, 5, 7 \\
\ker \pi^* & k = 3 \\
\mathbb{Z}_r & k = 4 \\
0 & \text{otherwise.}
\end{cases}
\]

Observe that \(H^3(L) \cong \ker \pi^*\) will be trivial if and only if \(\det(\pi^*) \neq 0\).

To find \(r = |\det(\pi^*)|\), we follow the example of [GZ, Proposition 3.3]. Consider the diagram:

\[
\begin{array}{c}
H^3(G) \cong \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\tau^* = \tau_-^* - \tau_+^*} H^3(G/K_-^\circ) \oplus H^3(G/K_+^\circ) \cong \mathbb{Z} \oplus \mathbb{Z} \\
\eta^* \\
H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi^* = \pi_-^* - \pi_+^*} H^3(G/K_-) \oplus H^3(G/K_+) \cong \mathbb{Z} \oplus \mathbb{Z}
\end{array}
\]

(3.1)

where the homomorphisms \(\tau^*_\pm\) and \(\eta^*\) are induced by orbit maps, and \(\mu^*_\pm\) are the homomorphisms induced by the maps \(gK_\pm \mapsto gK_\pm\) (which are themselves induced by the inclusions of the identity components \(K_\circ\) in \(K_\pm\)). In this case, the orbit \(G/K_- = S^3 \times S^2\) is connected, so \(\mu_-^*\) is the identity. And since \(\mu_+^*\) is an isomorphism by [GWZ, Lemma 13.3a], we have \(|\det(\mu^*)| = 1\).

We next wish to find \(|\det(\eta^*)|\). Recall that \(G/H\) is homeomorphic to \(S^3 \times (S^3/\langle i \rangle)\). Uniqueness of the universal cover implies the composition \(S^3 \times S^3 \xrightarrow{\eta} G/H \cong S^3 \times (S^3/\langle i \rangle)\) induces a commutative square on cohomology:

\[
\begin{array}{c}
H^3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} H^3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} \\
\eta^* \\
H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\cong} H^3(S^3 \times S^3/\langle i \rangle) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(id_{S^3} \times f)^*}
\end{array}
\]

(3.2)

where \(f\) is the projection of universal cover of \(S^3/\langle i \rangle\) by \(S^3\). An argument involving the Künneth isomorphism shows that there are bases for \(H^3(S^3 \times S^3/\langle i \rangle)\) and \(H^3(S^3 \times S^3)\) such that \((id_{S^3} \times f)^* = id_{S^3}^* \times f^*\). The covering degree \(\deg(f) = \pm 4\) implies that \(|\det(\eta^*)| = |\det(id_{S^3}^* \times f^*)| = 4\).
The determinant of $\tau^*$ follows from [GZ, Proposition 3.3]. They find a basis of $H^3(S^3 \times S^3)$ with respect to which $\operatorname{im}\tau^* = \langle (\pm q_2^2, p_2^2) \rangle$. Hence, the absolute value of the determinant of $\tau^* = \tau_+^* - \tau_-^*$ is $|p_2^2 q_2^2 - p_2^2 q_2^2|$. We conclude that the absolute value of the determinant of $\pi^* = |\det(\pi^*)| = |\det(\eta^*)||\det(\tau^*)||\det(\mu^*)|$ is $1/4 |p_2^2 q_2^2 - p_2^2 q_2^2|$.

In this family, the parameters $p_+$ and $q_+$ are odd and $p_-, q_- \equiv 1 \mod 4$. If we set $p_+ = 2k + 1$, $q_+ = 2l + 1$, $p_- = 4m + 1$ and $q_- = 4n + 1$ for some integers $k, l, m$ and $n$, we see that the parity of $r = 1/4 |p_+^2 q_-^2 - p_-^2 q_+^2|$ agrees with the parity of $k(k + 1) - l(l + 1)$, which must be even. It follows that $H^4(L) \cong \mathbb{Z}_r$ is a non-trivial cyclic group of even order. Thus, $H^3(L)$ is the trivial group and $H^4(L)$ is a non-trivial finite cyclic group if and only if $|p_+^2 q_-^2 - p_-^2 q_+^2| \neq 0$.

**Case 2: $p_+$ is even.**

On the other hand, suppose $p_+$ is even. Let $K' = \{(e^{jp_+ \theta}, e^{jq_+ \theta}) \cdot (1, -1), (i, i)\}$ be a subgroup of $S^3 \times S^3$. The inclusion of $K_+ = \{(e^{jp_+ \theta}, e^{jq_+ \theta}) \cdot (i, i)\}$ in $K'$ as a subgroup induces a continuous bijection (since $p_+$ is even) from the compact space $G/K_+$ to the Hausdorff space $G/K'$. It follows that $G/K_+$ is homeomorphic to $G/K'$. Thus, the cohomology of $G/K_+$ is the same as that of $G/K'$, which was shown in [GWZ, Lemma 13.6b] to be:

$$H^k(G/K_+) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_4 & k = 2 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_2 & k = 5 \\ 0 & \text{otherwise.} \end{cases}$$

Let $L$ be a member of the subfamily $L_{(p_-, q_-), (p_+, q_+)}$ with $p_+$ even. Using Sequence 2.1, Poincaré duality and the universal coefficient theorem, we find that $H^0(L) = H^2(L) = H^7(L) \cong \mathbb{Z}$ and $H^5(L) = \mathbb{Z} \oplus \mathbb{Z}_2$. Similarly, $H^3(L)$ and $H^4(L)$ are, respectively, the kernel and cokernel of the homomorphism $\pi^* = \pi_+^* - \pi_-^*$ from $H^3(G/K_-) \oplus H^3(G/K_+)$ to
$H^3(G/H)$ in Sequence 2.1. By Lemma 2.3.1 (with $t = 2$ and $\kappa = 4$), $H^4(L)$ is cyclic of order $r = |\det(\pi^*|_{Z_2})|$. In this case, there is a diagram:

$$
\begin{array}{c}
H^3(G) \cong \mathbb{Z} \oplus \mathbb{Z} \\
\eta^* \downarrow \quad \pi^* = \pi^*_+ - \pi^*_- \\
H^3(G/H) \cong \mathbb{Z} \oplus \mathbb{Z} \\
\mu^* = \mu^*_+ \times \mu^*_-
\end{array}
\quad
\begin{array}{c}
\pi^* = \pi^*_+ - \pi^*_- \\
H^3(G/K_\circ) \oplus H^3(G/K_\circ) \cong \mathbb{Z} \oplus \mathbb{Z} \\
\mu^* = \mu^*_+ \times \mu^*_-
\end{array}
$$

Comparing this to Diagram 3.1), we see that the homomorphisms $\eta^*$, $\pi^*$ and $\mu^*_\pm$ are the same. By [GWZ, Lemma 13.6], the homomorphism $\mu^*_\pm$ is multiplication by $\pm 4$ on the free part of $H^3(G/K_\circ)$, while the $\mathbb{Z}_2$ summand is clearly in the kernel. We conclude that $r = |p_+^2 q_-^2 - p_-^2 q_+^2|$. Since $p_+$ is even while $q_\pm$ and $p_-$ are odd, $r$ is always odd, so $H^4(L)$ is finite. Also, $r = |(p_+ q_- + p_- q_+)(p_+ q_- - p_- q_+)| \neq 1$ since the parameters $p_\pm$ and $q_\pm$ are non-zero. This can be confirmed by checking the four possible cases for the equation $r = |(a + b)(a - b)| = 1$ where $a$ and $b$ are integers. Hence, $H^4(L)$ is a non-trivial finite cyclic group of odd order; and by Poincaré duality and the universal coefficient theorem, $H^3(L) \cong \mathbb{Z}_2$. Thus, the cohomology groups of $L$ are:

$$H^k(L) \cong \begin{cases}
\mathbb{Z} & k = 0, 2, 7 \\
\mathbb{Z}_2 & k = 3 \\
\mathbb{Z}_r & k = 4 \\
\mathbb{Z} \oplus \mathbb{Z}_2 & k = 5 \\
0 & \text{otherwise.}
\end{cases}$$
3.2 Cohomology groups of the family $N_{(p-,q-),(p+,q+)}$.

The cohomology groups of a member $N$ of this family, as computed in [GWZ, Theorem 13.5], are:

$$H^k(N) \cong \begin{cases} 
    \mathbb{Z} & k = 0, 2, 5, 7 \\
    \mathbb{Z}_r & k = 4 \\
    0 & \text{otherwise}
\end{cases}$$

where the order of the cyclic group $H^4(N)$ is $r = |p_-^2 q_+^2 - p_+^2 q_-^2|$. Since $p_+$ is required to be even while $p_-, q_-$ and $q_+$ are odd, $r$ must be odd and (as was the case of the family $L_{(p-,q-),(p+,q+)}$ for $p_+$ even) cannot equal one. Thus, $H^4(N)$ is a non-trivial finite cyclic group of odd order.

3.3 Cohomology groups of the family $O_{(p,q;m)}$.

Recall that this family is described by the groups:

$$H = \mathbb{Z}_m \subseteq K_- = \{(e^{ip\theta}, e^{iq\theta})\}, \ K_+ = \Delta S^3 \cdot H \subseteq G = S^3 \times S^3$$

where $\Delta S^3$ is the diagonal embedding. The integers $p$ and $q$ are relatively prime, and either $m = 1$ (in which case $H$ is the trivial group, and there are no restrictions on the parameters), or $m = 2$ (in which case $H = \langle (1, -1) \rangle$ is isomorphic to $\mathbb{Z}_2$ and $p$ is required to be even). This family naturally splits into two subfamilies, depending on the value of $m$.

In both cases, the non-principal orbit $G/K_-$ is homeomorphic to $S^3 \times S^2$; the difference lies in the other non-principal orbit $G/K_+$, and the principal orbit $G/H$.

**Case 1: $m = 1$.**

First, suppose $m = 1$. Then $G/K_+ = S^3 \times S^3 / \Delta S^3$ is homeomorphic to $S^3$ under the map sending the coset $[(q_1, q_2)]$ to $q_1 q_2^{-1}$. The principal orbit is $G/H = S^3 \times S^3$. For
a member $O_1$ of this subfamily, recall that $H^0(O_1) \cong H^7(O_1) \cong \mathbb{Z}$. Using Sequence 2.1 and Lemma 2.3.1 (with $t = 2$ and $\kappa = 4$), one easily sees that the cohomology groups are:

$$H^k(O_1) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 5, 7 \\ \ker \pi^* & k = 3 \\ \mathbb{Z}_r & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

where again $r = |\det(\pi^*)|$ for $\pi^*$ the homomorphism from the rank two free abelian group $H^3(G/K_{-}) \oplus H^3(G/K_{+})$ to the rank two free abelian group $H^3(G/H)$. In order for $H^3(O_1)$ to be trivial, the determinant of $\pi^*$ must be non-zero.

As a preliminary step to finding this determinant, let $v$ be a generator of $H^3(S^3)$. Fix a basis $u_1, u_2$ of $H^3(G/H) = H^3(S^3 \times S^3)$ which corresponds to the images of $v \otimes 1$ and $1 \otimes v$ under the Künneth isomorphism; that is, $u_i = p_i^*(v)$ where $p_i$ is the projection of the $i^{th}$ factor of $S^3 \times S^3$ onto $S^3$ ($i = 1, 2$). Up to sign, this is the basis used in [GZ, Proposition 3.3] to show that $\text{im} \pi^+_+ = \langle (-q^2, p^2) \rangle$. We now find $\text{im} \pi^+_+ \leq H^3(S^3 \times S^3)$ with respect to the basis $u_1, u_2$.

Let $S^3 \xrightarrow{\Delta} S^3 \times S^3 \xrightarrow{\pi^+} G/K_{+} \approx (S^3 \times S^3)/\Delta S^3 \approx S^3$ be the principal $S^3$-bundle with fiber inclusion $\Delta$ the diagonal embedding of $S^3$ in $S^3 \times S^3$. The composition $\pi^+_+ \circ \Delta$ is constant, and so is a degree zero map; it follows that the induced homomorphism $\Delta^* \circ \pi^+_+$ is the trivial homomorphism from $H^3(S^3)$ to itself. Therefore, the image of $\pi^+_+$ is contained in the kernel of $\Delta^*$. If $\sigma \in C_3(S^3)$ is a singular 3-chain, then for $i = 1, 2$:

$$\Delta^*(u_i)(\sigma) = u_i(\Delta(\sigma)) = p_i^*(v)((\sigma, \sigma)) = v(\sigma).$$

So the kernel of $\Delta^*$ is the subgroup of $H^3(S^3 \times S^3)$ generated by $u_1 - u_2$, and there is an integer $n$ such that $\text{im} \pi^+_+ = \langle n(u_1 - u_2) \rangle$.

Next, consider the Serre spectral sequence $(E, d)$ of the Borel fibration $S^3 \times S^3 \xrightarrow{\pi^+} G/K_{+} \xrightarrow{\rho} \mathbb{H}P^\infty$ (here, $\rho$ is the classifying map of the previous $S^3$-bundle). The differential $E^0,3 \cong H^3(S^3 \times S^3) \xrightarrow{d_3} E^1,0 \cong H^4(\mathbb{H}P^\infty)$ can be identified with the transgression ([Mc,
Theorem 6.83). By examining the definition of the transgression (as in [Mc, p.186]), we see in this particular instance that \( \ker d_4 = \text{im} \pi^*_+ = \langle n(u_1 - u_2) \rangle \). Based on the convergence of the spectral sequence to \( H^*(G/K_+) \cong H^*(S^3) \), we observe that \( H^3(S^3 \times S^3) / \ker d_4 \) must be isomorphic to \( H^4(\mathbb{H}P^\infty) \cong \mathbb{Z} \). Using the basis \( u_1, u_1 - u_2 \) for \( H^3(S^3 \times S^3) \), we conclude that the image of \( \pi^*_+ \) in \( H^3(S^3 \times S^3) \) with respect to the basis \( u_1, u_2 \) is the subgroup \( \langle (1, -1) \rangle \).

From the above, the absolute value of the determinant of \( \pi^* = \pi^*_+ - \pi^*_- \) is \( |p^2 - q^2| \). Since neither \( p \) nor \( q \) may be zero, \( |p^2 - q^2| = |(p + q)(p - q)| \neq 1 \), so \( H^4(O_1) \) is non-trivial. The only way that \( \det(\pi^*) \) can equal zero is if \( |p| = |q| = 1 \). So as long as \( |p| \) and \( |q| \) are not both equal to one, \( H^3(O_1) \) is trivial and \( H^4(O_1) \) is a non-trivial finite cyclic group.

**Case 2: \( m = 2 \).**

Let \( m = 2 \). In this case, \( G/K_+ = (S^3 \times S^3) / (\Delta S^3 \cdot ((1, -1))) \) is homeomorphic to \( \mathbb{R}P^3 \) under the map sending the coset \([q_1, q_2]\) to the coset \([q_1 q_2^{-1}]\). The principal orbit \( G/H = S^3 \times S^3 / \langle (1, -1) \rangle \) is homeomorphic to \( S^3 \times \mathbb{R}P^3 \) under the map \([q_1, q_2] \mapsto (q_1, [q_2])\). Once again, Sequence 2.1 and Lemma 2.3.1 (with \( t = 2 \) and \( \kappa = 4 \)) are sufficient tools for determining the cohomology groups of a member \( O_2 \) of this subfamily. As in the previous case, they are:

\[
H^k(O_2) \cong \begin{cases} 
\mathbb{Z} & k = 0, 5, 7 \\
\ker \pi^* & k = 3 \\
\mathbb{Z}_r & k = 4 \\
0 & \text{otherwise}
\end{cases}
\]

for \( r \) the absolute value of the determinant of the homomorphism \( \pi^* \) from the rank two free abelian group \( H^3(G/K_-) \oplus H^3(G/K_+) \) to the rank two free abelian group \( H^3(G/H) \), and \( H^3(O_2) \cong \ker \pi^* \) is trivial when the determinant of \( \pi^* \) is not zero.

To calculate \( |\det(\pi^*)| \), we refer again to Diagram 3.1. As before, \( \mu_- \) is the identity map. Now, however, \( \mu_+ \) is the projection of the universal cover of \( \mathbb{R}P^3 \) by \( S^3 \), which has
covering degree two; so $|\det(\mu^*)| = 2$. The composition $S^3 \times S^3 \xrightarrow{\eta} G/H \xrightarrow{\cong} S^3 \times \mathbb{R}P^3$ is the universal cover, and an argument analogous to that involving Diagram 3.2 shows that $|\det(\eta^*)| = 2$.

The absolute value of the determinant of $\tau^*$ has already been computed; the homomorphism $\pi^*$ that determined the order of the fourth cohomology group in the previous subfamily $O_1$ is the same as the current homomorphism $\tau^*$. Thus, the absolute value of the determinant of the current homomorphism $\pi^*$ is $|\det(\pi^*)| = |\det(\eta^*)|^{-1}|\det(\tau^*)||\det(\mu^*)| = |p^2 - q^2|$. Recall that in this case $p$ is even, so $H^4(O_2)$ is finite cyclic of odd order $r = |p^2 - q^2|$ and $H^3(O_2) = 0$. 

4 PROOFS OF THEOREMS 1.2.1 AND 1.2.2, CONTINUED: THE COHOMOLOGY RING GENERATORS.

In Section 4.1, we identify the cohomology ring generators for members of the families \( L_{(p_-,q_-),(p_+,q_+)} \), \( N_{(p_-,q_-),(p_+,q_+)} \) and \( O_{(p,q,m)} \) having non-trivial finite cyclic fourth cohomology groups. This completes the proofs of Theorems 1.2.1 and 1.2.2. In Section 4.2, we briefly discuss how Theorem 1.2.1 implies Corollaries 1.2.1, 1.2.2 and 1.2.3.

4.1 The cohomology rings.

In Chapter 3, it was shown that a compact simply connected seven dimensional primitive cohomogeneity one manifold has cohomology groups:

\[
H^k(N) \cong \begin{cases} 
\mathbb{Z} & k = 0, 2, 5, 7 \\
\mathbb{Z}_r, r \neq 0, 1 & k = 4 \\
0 & \text{otherwise}
\end{cases}
\]

if and only if it is:

- a member of the family \( L_{(p_-,q_-),(p_+,q_+)} \) with \( p_+ \) odd and \( p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0 \), or:
- any member of the family \( N_{(p_-,q_-),(p_+,q_+)} \), or:
- a member of the family \( O_{(p,q,m)} \) for \( |p| \) and \( |q| \) not both equal to one.

Now, we show that the cohomology rings of all of these manifolds are generated by classes \( x \in H^2(M) \) and \( y \in H^5(M) \).

We also give an almost complete description of the cohomology ring structure for members of \( L_{(p_-,q_-),(p_+,q_+)} \) with \( p_+ \) even. We show that, for any such manifold \( L \) with \( x \)
and y generators of $H^2(L)$ and the free part of $H^5(L)$ respectively, the class $x^2$ generates $H^4(L)$ and $xy$ generates $H^7(L)$.

For all of the above manifolds, the non-principal orbits $G/K_-$ are closed orientable submanifolds of codimension two. The manifolds themselves are simply connected, hence orientable. We conclude that the normal disk bundles over $G/K_-$ are orientable bundles with fiber $D^2$. Thus, we have at our disposal Sequence 2.2 and (provided the remaining conditions are satisfied) Lemma 2.3.2, setting $t = 2$ in both. In all that follows, we will assume that the class $x$ generates $H^2(M)$, the class $y$ generates the free part of $H^5(M)$, and $1_{\pm}$ is the multiplicative unit of $H^*(G/K_\pm)$.

4.1.1 Cohomology ring of the family $L(p_-,q_-),(p_+,q_+)$.

Case 1: $p_+$ is odd.

Let $L$ be a member of the subfamily of $L(p_-,q_-),(p_+,q_+)$ for which $p_+$ is odd and $p_+^2q_-^2 - p_-^2q_+^2 \neq 0$. In this case, Lemma 2.3.2 cannot be called on to show that the square of the generator $x$ of $H^2(L)$ generates $H^4(L)$, as Condition 3 does not hold. To see why Condition 3 fails, we analyze the section of Diagram 2.3 corresponding to $k = 1$ and 2.

Since $H^1(G/K_-)$ and $H^1(G/H)$ are both trivial, the homomorphisms $j^*_-$ from $H^2(L,G/K_-)$ to $H^2(L) \cong \mathbb{Z}$ and $j^*$ from $H^2(D(G/K_+),G/H)$ to $H^2(G/K_+) \cong \mathbb{Z}_2$ are injective. The groups $H^2(L,G/K_-)$ and $H^2(D(G/K_+),G/H)$ are isomorphic, so injectivity of the homomorphisms $j^*_-$ and $j^*$ implies that $H^2(L,G/K_-)$ and $H^2(D(G/K_+),G/H)$ are trivial. Since $H^3(L) = 0$, the isomorphism of $H^3(L,G/K_-)$ and $H^3(D(G/K_+),G/H)$ (together with commutativity of the diagram) implies that the homomorphism $j^*$ from $H^3(D(G/K_+),G/H)$ to $H^3(G/K_+)$ is the zero homomorphism. This gives two short exact sequences:

$$0 \to H^2(L) \cong \mathbb{Z} \cdot x \xrightarrow{i_+^*} H^2(G/K_-) \cong \mathbb{Z} \cdot \gamma \xrightarrow{\delta_-} H^3(L,G/K_-) \to 0$$

$$0 \to H^2(G/K_+) \cong \mathbb{Z}_2 \xrightarrow{i_-^*} H^2(G/H) \cong \mathbb{Z}_4 \xrightarrow{\delta} H^3(D(G/K_+),G/H) \to 0$$
where the groups $H^3(L, G/K_\pm)$ and $H^3(D(G/K_\pm), G/H)$ are isomorphic. From the second sequence, it follows that $H^3(D(G/K_\pm), G/H)$ is isomorphic to $\mathbb{Z}_2$. Then, by the first sequence, the homomorphism $i^*_-$ from $H^2(L)$ to $H^2(G/K_-)$ is multiplication by 2 and $i^*_-(x) = \pm 2\gamma$ for $\gamma$ a generator of $H^2(G/K_-)$. So if Condition 3 of Lemma 2.3.2 holds, then the order of $H^4(L)$ must be relatively prime to 2. We have already shown, however, that the order of $H^4(L)$ is even.

Fortunately, there is an alternate method of showing $x^2$ generates $H^4(L)$. Setting $t = 2$, $k = 2$ and $B_\pm = G/K_\pm$ in Sequence 2.2 for the pair $(L, G/K_\pm)$ gives a short exact sequence:

$$0 \to H^0(G/K_-) \cong \mathbb{Z} \overset{J}{\to} H^2(L) \cong \mathbb{Z} \cdot x \overset{i^*_+}{\to} H^2(G/K_+) \cong \mathbb{Z}_2 \to 0$$

and we see that $J(1_-) = \pm 2x$. Setting $k = 4$ in Sequence 2.2, exactness together with the triviality of $H^4(G/K_+)$ implies $J(\gamma)$ generates $H^4(L)$.

Recalling that $J$ is an $H^*(L)$-module homomorphism:

$$2J(\gamma) = J(2\gamma) = J(i^*_+(x)) = J(1_-) \sim x = \pm 2x^2.$$  

Since $J(\gamma)$ generates $H^4(L)$, the subgroup generated by $2x^2 = \pm 2J(\gamma)$ is an index two subgroup. We show that $x^2$ is not an element of $(2x^2)$, from which it follows $x^2$ generates $H^4(L)$. Because this argument will require both integral and $\mathbb{Z}_2$ cohomology, we temporarily resort to explicitly indicating coefficients.

The short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$$

where $h$ is multiplication by two and $g$ is the natural projection, gives rise to the long exact cohomology sequence:

$$\cdots \to H^k(L; \mathbb{Z}) \xrightarrow{h^\#} H^k(L; \mathbb{Z}) \xrightarrow{g^\#} H^k(L; \mathbb{Z}_2) \xrightarrow{\beta} H^{k+1}(L; \mathbb{Z}) \to \cdots$$

where $\beta$ is the Bockstein operator, and $h^\#$ and $g^\#$ are coefficient homomorphisms. Because $J(\gamma)$ generates $H^4(L)$, and by definition $h^\#(J(\gamma)) = 2J(\gamma)$, exactness of the sequence
implies $\langle 2x^2 \rangle = \langle 2J(\gamma) \rangle$ is the kernel of $g_{\#}$. Hence, if $g_\#(x^2)$ can be shown to be non-trivial in $H^4(L; \mathbb{Z}_2)$, it will follow that $x^2$ is not in $\langle 2x^2 \rangle$.

Since $H^3(L; \mathbb{Z})$ is trivial, exactness implies that the homomorphism $g_\#$ from $H^2(L; \mathbb{Z})$ to $H^2(L; \mathbb{Z}_2)$ is surjective. Thus $g_\#$ sends the generator $x$ of $H^2(L; \mathbb{Z})$ to the generator $w$ of $H^2(L; \mathbb{Z}_2)$. Checking the definitions of the induced homomorphism $g_\#$ and the cup product reveals that $g_\#(x^2) = w^2$.

The $\mathbb{Z}_2$-cohomology of $L$ and the non-principal orbits $G/K_{\pm}$ are:

$$H^k(L; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, 2, 3, 4, 5, 7 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(G/K_-; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, 2, 3, 5 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(G/K_+; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Applying Sequence 2.2 with $\mathbb{Z}_2$-coefficients to the pair $(L, G/K_-)$ (recall that $G/K_+$ need not be orientable if we take coefficients in $\mathbb{Z}_2$) reveals that $J$ is an isomorphism from $H^0(G/K_+; \mathbb{Z}_2)$ to $H^2(L; \mathbb{Z}_2)$. Under this isomorphism, if $\mathbb{1}$ is the unit of the cohomology ring $H^*(G/K_+; \mathbb{Z}_2)$, then $J(\mathbb{1}) = w$. The corresponding long exact sequence for the pair $(L, G/K_+)$ shows the homomorphism $i^*_+(w)$ from $H^2(L; \mathbb{Z}_2)$ to $H^2(G/K_+; \mathbb{Z}_2)$ is an isomorphism, so $i^*_+(w)$ generates $H^2(G/K_+; \mathbb{Z}_2)$. Returning to the sequence of the pair $(L, G/K_-)$, we see that $H^2(G/K_+; \mathbb{Z}_2)$ is isomorphic to $H^4(L; \mathbb{Z}_2)$ under $J$; hence, $J(i^*_+(w))$ generates $H^4(L; \mathbb{Z}_2)$. Since $J$ is an $H^*(L; \mathbb{Z}_2)$-module homomorphism:

$$J(i^*_+(w)) = J(\mathbb{1} \cdot i^*_+(w)) = J(\mathbb{1}) \cdot w = w^2.$$

Thus, $w^2 = g_\#(x^2)$ generates $H^4(L; \mathbb{Z}_2)$. In particular, $g_\#(x^2)$ is non-trivial, which is what we needed to show in order to conclude that $x^2$ generates $H^4(L; \mathbb{Z})$. As $\mathbb{Z}_2$ coefficients
will no longer be needed, we return to the convention of assuming integral cohomology and no longer specify coefficients.

We now show that all of the conditions of Lemma 2.3.2 hold when \( \kappa = 7 \), from which it follows that the class \( xy \) generates \( H^7(L) \). Recall that \( t = 2 \), and observe that all conditions on the cohomology groups are met. Since \( H^1(G/K_-) \) is trivial, \( i_+^* \) from \( H^2(L) \) to \( H^2(G/K_+) \) is a surjection.

We check the remaining three conditions. Since \( H^7(G/K_+) \) is trivial, \( i_+^* \) from \( H^7(L) \) to \( H^7(G/K_+) \) is the zero homomorphism, and Condition 1 holds. Condition 2 also holds, with \( H^5(G/K_-) \cong \mathbb{Z} \cdot \nu \). The group \( H^2(G/K_+) \) is finite cyclic of order two, and \( H^7(L) \) is infinite cyclic; so to verify Condition 3, we need to show that \( i_+^*(y) = \pm 2\nu \).

Take \( k = 5 \) in Diagram 2.3. Triviality of \( H^4(G/K_-) \) and \( H^4(G/H) \) implies that \( H^5(L, G/K_-) \) injects into the infinite cyclic group \( H^5(L) \), and \( H^5(D(G/K_+), G/H) \) injects into the finite cyclic group \( H^5(G/K_+) \). Because the groups \( H^5(L, G/K_-) \) and \( H^5(D(G/K_+), G/H) \) are isomorphic, the existence of these two injections implies that \( H^5(L, G/K_-) \) and \( H^5(D(G/K_+), G/H) \) are both trivial. This yields two short exact sequences:

\[
0 \rightarrow H^5(L) \cong \mathbb{Z} \cdot y \xrightarrow{i_+^*} H^5(G/K_-) \cong \mathbb{Z} \cdot \nu \xrightarrow{\delta} H^5(L, G/K_-) \rightarrow 0
\]

\[
0 \rightarrow H^5(G/K_+) \cong \mathbb{Z}_2 \xrightarrow{i_+^*} H^5(G/H) \cong \mathbb{Z}_4 \xrightarrow{\delta} H^5(D(G/K_+), G/H) \rightarrow 0
\]

where the groups \( H^5(L, G/K_-) \) and \( H^5(D(G/K_+), G/H) \) are isomorphic. From the second sequence, we conclude \( H^6(D(G/K_+), G/H) \cong \mathbb{Z}_2 \). It is then apparent by the first sequence that \( i_+^* \) is multiplication by two; so \( i_+^*(y) = \pm 2\nu \), Condition 3 holds, and by Lemma 2.3.2, \( xy \) generates \( H^7(L) \).

**Case 2: \( p_+ \) is even.**

Suppose \( L \) is a member of the subfamily of \( L_{(p_-,q_-),(p_+,q_+)} \) with \( p_+ \) even. Let \( x \) generate \( H^2(L) \cong \mathbb{Z} \) and \( y \) the free part of \( H^5(L) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \); we show that \( x^2 \) generates
$H^4(L)$ and $xy$ generates $H^7(L)$. Whether or not the classes $x$ and $y$, together with a class $\xi$ generating $H^3(L) \cong \mathbb{Z}_2$, form a complete set of generators for the ring $H^*(L)$ is unknown at this time.

Setting $t = 2$, we confirm that the conditions of Lemma 2.3.2 hold when $\kappa = 4, 7$. The conditions on $H^2(L)$ and $H^2(G/K_+)$, as well as Conditions 1 and 2, are easily checked. Sequence 2.2 can be used to verify that the inclusion-induced homomorphism from $H^2(L) \cong \mathbb{Z}$ to $H^2(G/K_+) \cong \mathbb{Z}_4$ is a surjection, and those from $H^\kappa(L)$ to $H^\kappa(G/K_+)$ (for $\kappa = 4, 7$) are multiplication by zero. It remains to check Condition 3.

When $\kappa = 4$, $H^4(L)$ is finite cyclic. We show $i^*_\kappa(x)$ generates $H^2(G/K_-)$ when $x$ generates $H^2(L)$. Consider Diagram 2.3. The second relative cohomology groups, which are isomorphic, inject into both a free group and a finite group and so must be trivial. Because $H^3(L) \cong \mathbb{Z}_2$ is in the kernel of $i^*_\kappa$, the third relative group $H^3(L, G/K_-)$ surjects onto $H^3(L)$. By exactness of the top row, $H^3(L, G/K_-)/\text{im}\delta$ is isomorphic to $\mathbb{Z}_2$ and $\text{im}\delta$ is finite cyclic. We conclude that $H^3(L, G/K_-)$ is a non-trivial finite group.

Triviality of $H^2(D(G/K_+), G/H)$ implies $H^2(G/K_+) \cong \mathbb{Z}_4$ injects into $H^2(G/H) \cong \mathbb{Z}_4$; so $H^2(G/K_+)$ and $H^2(G/H)$ are isomorphic. It follows from the isomorphism of the third relative cohomology groups, together with exactness in the bottom row, that $H^3(L, G/K_-)$ injects into $H^3(G/K_+) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. So $H^3(L, G/K_-)$ is isomorphic to $\mathbb{Z}_2$, and the surjection of $H^3(L, G/K_-)$ onto $H^3(L)$ is an isomorphism. Then, by exactness of the top row, the inclusion-induced homomorphism $i^*_\kappa$ from $H^2(L) \cong \mathbb{Z} \cdot x$ to $H^2(G/K_-)$ must also be an isomorphism, and $i^*_\kappa(x)$ generates $H^2(G/K_-)$. This satisfies Condition 3 in the case $\kappa = 4$.

If $\kappa = 7$, $H^7(L)$ is infinite cyclic. We show that, for $y$ a generator of the free part of $H^5(L)$, $i^*_\kappa(y)$ is four times a generator of $H^5(G/K_-) \cong \mathbb{Z}$. Again turning to Diagram 2.3, we see that the fifth relative cohomology groups, which are isomorphic, inject into both $H^5(L) \cong \mathbb{Z} \cdot y \oplus \mathbb{Z}_2$ and $H^5(G/K_+) \cong \mathbb{Z}_2$. Hence, they are either trivial or cyclic of order two. Since the $\mathbb{Z}_2$ summand of $H^5(L)$ is in the kernel of the homomorphism...
from $H^5(L)$ to $H^5(G/K_-)$, we conclude that the fifth relative cohomology groups are isomorphic to $\mathbb{Z}_2$. Then exactness of the bottom row together with the isomorphism of the relative groups gives an isomorphism between $H^6(L, G/K_-)$ and $H^5(G/H) \cong \mathbb{Z}_4$. Restricting $i^*$ to the free part of $H^5(L)$ gives rise to a short exact sequence:

$$0 \to \mathbb{Z} \cdot y \xrightarrow{i^*|_y} H^5(G/K_-) \cong \mathbb{Z} \xrightarrow{\delta} \mathbb{Z}_4 \to 0.$$ 

Hence, $i^*(y)$ is four times a generator of $H^5(G/K_-)$, satisfying Condition 3 in the case $\kappa = 7$, and by Lemma 2.3.2 it follows that $x^2$ generates $H^4(L)$ and $xy$ generates $H^7(L)$.

This completes the proof of Theorem 1.2.2.  

4.1.2 Cohomology ring of the family $N(p_-, q_-), (p_+, q_+)$. 

We continue with the proof of Theorem 1.2.1.

Let $N$ be a member of the family $N(p_-, q_-), (p_+, q_+)$. The cohomology groups of the orbits (as computed in [GWZ, Lemma 13.6]) are:

$$H^k(G/K_-) \cong \begin{cases} 
\mathbb{Z} & k = 0, 3, 5 \\
\mathbb{Z} \oplus \mathbb{Z}_2 & k = 2 \\
\mathbb{Z}_2 & k = 4 \\
0 & \text{otherwise}
\end{cases}$$

$$H^k(G/K_+) \cong \begin{cases} 
\mathbb{Z} & k = 0 \\
\mathbb{Z}_4 & k = 2 \\
\mathbb{Z} \oplus \mathbb{Z}_2 & k = 3 \\
\mathbb{Z}_2 & k = 5 \\
0 & \text{otherwise}
\end{cases}$$

$$H^k(G/H) \cong \begin{cases} 
\mathbb{Z} & k = 0, 6 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_4 & k = 2, 5 \\
\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 & k = 3 \\
\mathbb{Z}_2 & k = 4 \\
0 & \text{otherwise}
\end{cases}$$
The classes \( x \) and \( y \) respectively generate the infinite cyclic groups \( H^2(N) \) and \( H^5(N) \). To show that \( x^2 \) generates \( H^4(N) \) and \( xy \) generates \( H^7(N) \), we turn to Lemma 2.3.2 (recall that \( t = 2 \)). For \( \kappa = 4 \) and 7, all of the conditions on the cohomology groups are satisfied, including Condition 2. In particular, \( H^2(G/K_+) \) is finite cyclic of order \( n = 4 \).

Taking \( k = 2 \) in Sequence 2.2, one sees that the inclusion-induced homomorphism from \( H^2(N) \) to \( H^2(G/K_+) \) is a surjection, and also that the inclusion-induced homomorphisms from \( H^\kappa(N) \) to \( H^\kappa(G/K_+) \), \( \kappa = 4 \) and 7, are the zero homomorphisms (Condition 1). It remains only to check that the requirements of Condition 3 are satisfied.

For \( \kappa = 4 \), the group \( H^4(N) \) is finite cyclic. Suppose the image of \( x \) under the inclusion-induced homomorphism \( i^-_* \) is the element \((s, \beta)\) in \( H^2(G/K_-) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \). In order for Condition 3 to hold, \( s \) must be relatively prime to the order of \( H^4(N) \). We claim this is true; that, in fact, \( |s| = 1 \). To see this, consider Diagram 2.3. Setting \( k = 2 \), we see that the second relative cohomology groups, which are isomorphic, inject into both the infinite cyclic group \( H^2(N) \) and the finite cyclic group \( H^2(G/K_+) \); so they must be trivial. Because \( H^3(N) \) is trivial, the homomorphism \( \delta_- \) from \( H^2(G/K_-) \) to \( H^3(N, G/K_-) \) is a surjection. Commutativity of the diagram together with the isomorphism of the third relative groups forces the homomorphism \( \delta \) from \( H^2(G/H) \) to \( H^3(D(G/K_+), G/H) \) to be surjective as well. This gives two short exact sequences:

\[
0 \to H^2(N) \cong \mathbb{Z} \cdot x \xrightarrow{i^-_*} H^2(G/K_-) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\delta_-} H^3(N, G/K_-) \to 0
\]

\[
0 \to H^2(G/K_+) \cong \mathbb{Z}_4 \xrightarrow{i^+_*} H^2(G/H) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \xrightarrow{\delta} H^3(D(G/K_+), G/H) \to 0
\]

where the relative cohomology groups are isomorphic. From the second sequence we see that the order of the relative groups is the order of \( H^2(G/H) \) divided by the order of \( H^2(G/K_+) \); hence, the relative groups are isomorphic to \( \mathbb{Z}_2 \).

Consider the first sequence. Because \( i^-_* \) is injective and \( i^-_*\( (x) = (s, \beta) \), \( s \) cannot be zero. By exactness, \( H^3(N, G/K_-) \cong \mathbb{Z}_2 \) is isomorphic to \( H^2(G/K_-) / \text{im} i^-_* \). If \( \beta = [0] \), the group \( H^2(G/K_-) / \text{im} i^-_* \) is clearly isomorphic to \( \mathbb{Z}_s \oplus \mathbb{Z}_2 \). If instead \( \beta = [1] \), the surjective homomorphism from \( H^2(G/K_-) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \) to \( \mathbb{Z}_2 s \), defined by \((1, [0]) \mapsto [1] \) and \((0, [1]) \mapsto [s] \), has kernel \((s, [1])) \); hence, \( H^2(G/K_-) / \text{im} i^-_* \) is isomorphic to \( \mathbb{Z}_2 s \).
In both cases, $H^2(G/K_{-})/\text{im}i_{*}$ is a finite group with $2|s|$ elements. Because we know $H^2(G/K_{-})/\text{im}i_{*}$ is isomorphic to $\mathbb{Z}_2$, we conclude $|s| = 1$. Thus, $s$ is relatively prime to the order of $H^4(N)$, Condition 3 is satisfied and Lemma 2.3.2 holds for $\kappa = 4$. We have shown $x^2$ generates $H^4(N)$.

We now show that Condition 3 of Lemma 2.3.2 holds for $\kappa = 7$, from which it follows $xy$ generates $H^7(N)$. Because $H^7(N)$ is infinite cyclic and $H^2(G/K_{+})$ is finite cyclic of order $n = 4$, Condition 3 requires the image of the generator $y$ of $H^5(N)$ under $i_{*}$ to be (up to sign) four times a generator of the infinite cyclic group $H^5(G/K_{-})$.

To show this is true, set $k = 5$ in Diagram 2.3. The sixth relative cohomology groups are isomorphic, and by exactness of the bottom row are isomorphic to the quotient of $H^5(G/H)$ by $i^{*}(H^5(G/K_{+}))$. So the orders of the sixth relative groups are equal to the order of $H^5(G/H) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ divided by the order of $i^{*}(H^5(G/K_{+}))$. Since $H^5(G/K_{+}) \cong \mathbb{Z}_2$, this is either four or eight. Observe that these relative groups cannot contain elements of order eight, since they are isomorphic to a quotient of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$; therefore, if they are eight element groups, they cannot be cyclic. But the infinite cyclic group $H^5(G/K_{-})$ surjects onto the relative cohomology groups, so we conclude they are cyclic of order four. It then follows from exactness of the top row that the homomorphism $i_{*}$ from $H^5(N)$ to $H^5(G/K_{-})$ is multiplication by four. Hence, by Lemma 2.3.2, $xy$ generates $H^7(N)$.

### 4.1.3 Cohomology ring of the family $O_{(p,q,m)}$.  

Let $O_m$ be a member of this family with $|p|$ and $|q|$ not both equal to one. The cases $m = 1$ and $m = 2$ need to be considered separately, due to differences in the orbits $G/H$ and $G/K_{+}$. However, calculations are greatly simplified by the fact that both non-principal orbits are orientable. This means Sequence 2.2 holds for both of the pairs $(O_m, G/K_{\pm})$. Also, despite the different orbits, arguments for each of the cases $m = 1, 2$ are similar; we sketch the general method.

For $\kappa = 4, 7$, all conditions of Lemma 2.3.2 applying to the cohomology groups
(including Condition 2) are met. As before, \( t = 2 \). Sequence 2.2 applied to the pair \((O_m, G/K_+)\) can be used to show that \( H^2(O_m) \) surjects onto \( H^2(G/K_+) \). This same sequence can be used to show that the homomorphisms \( i^*_\kappa \) from \( H^\kappa(O_m) \) to \( H^\kappa(G/K_+) \) for \( \kappa = 4 \) and 7 are trivial homomorphisms; consequently Condition 1 holds. Finally, applying Sequence 2.2 to the pair \((O_m, G/K_-)\), one sees that under the homomorphism \( i^*_\kappa \), the image of a generator of \( H^{\kappa-2}(O_m) \) meets the requirements of Condition 3. Thus, by Lemma 2.3.2, \( H^\kappa(O_m) \) is generated by \( x \in H^2(O_m) \) and \( y \in H^5(O_m) \).

This completes the proof of Theorem 1.2.1. \( \square \)

4.2 The corollaries.

Finally, we indicate how Corollaries 1.2.1, 1.2.2 and 1.2.3 follow from Theorem 1.2.1.

4.2.1 Corollary 1.2.1.

A compact simply connected smooth manifold \( M \) of dimension seven admits Kreck-Stolz invariants if and only if it has non-trivial cohomology groups \( H^0(M) = H^2(M) = H^5(M) = H^7(M) \cong \mathbb{Z} \) and \( H^4(M) \cong \mathbb{Z}_r \), and if the square of a generator of \( H^2(M) \) generates \( H^4(M) \). Corollary 1.2.1 states that a compact simply connected seven dimensional primitive cohomogeneity one manifold admits a Kreck-Stolz invariant if and only if:

a) it is a member of the subfamily of \( L(p_-,q_-),(p_+,q_+) \) with the parameter \( p_+ \) odd and \( p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0 \), or:

b) it is a member of the family \( N(p_-,q_-),(p_+,q_+) \), or:

c) it is a member of the subfamily \( O(p,q;m) \) with \( |p| \) and \( |q| \) not both equal to one.
This is a direct consequence of Theorem 1.2.1.

### 4.2.2 Corollary 1.2.2.

An Eschenburg space $E$ has non-trivial cohomology groups $H^0(E) = H^2(E) = H^5(E) = H^7(E) \cong \mathbb{Z}$ and $H^4(E) \cong \mathbb{Z}_r$, a finite group of odd order, and a cohomology ring generated by classes $x \in H^2(E)$ and $y \in H^5(E)$. According to Corollary 1.2.2, a compact simply connected seven dimensional primitive cohomogeneity one manifold has the cohomology ring of an Eschenburg space if and only if:

a) it is any member of the family $N_{(p_-, q_-), (p_+, q_+)}$, or:

b) it is a member of the family $O_{(p, q; m)}$ and one of the parameters $p$ or $q$ is even.

This again follows from Theorem 1.2.1, together with an examination of the order of the fourth cohomology group. In particular, the second and fifth cohomology groups of members of the family $M_{(p_-, q_-), (p_+, q_+)}$ fail to meet the criteria, as do the third and fifth cohomology groups of the manifolds belonging to $L_{(p_-, q_-), (p_+, q_+)}$ with $p_+$ even. As was shown in 3.1, members of the family $L_{(p_-, q_-), (p_+, q_+)}$ with $p_+$ odd have fourth cohomology groups of even order, so they are also ruled out. In Section 3.2, we saw that all members of the family $N_{(p_-, q_-), (p_+, q_+)}$ have fourth cohomology groups of odd order. This leaves the family $O_{(p, q; m)}$. Recall from Section 3.3 that the order of the fourth cohomology group of a member of this family is $r = |p^2 - q^2|$, which is odd if and only if either $p$ or $q$ is even. Note that this will always be the case for the subfamily $O_{(p, q; 2)}$.

### 4.2.3 Corollary 1.2.3.

Corollary 1.2.3 asserts that each of the families $L_{(p_-, q_-), (p_+, q_+)}$, $M_{(p_-, q_-), (p_+, q_+)}$, $N_{(p_-, q_-), (p_+, q_+)}$ and $O_{(p, q; m)}$ contains representatives of infinitely many distinct homotopy types. This follows from the fact that each of the families $L_{(p_-, q_-), (p_+, q_+)}$, $M_{(p_-, q_-), (p_+, q_+)}$, $N_{(p_-, q_-), (p_+, q_+)}$ and $O_{(p, q; m)}$.
\(N(p_-, q_-, (p_+, q_+))\) and \(O(p, q, m)\) contain infinitely many manifolds having fourth cohomology groups of distinct order. For example, the order of the fourth cohomology group of a member of the family \(N(p_-, q_-, (p_+, q_+))\) is given by \(r = |p_+^2 q_+^2 - p_-^2 q_-^2|\). Fix the parameters \(p_-, q_+ = 1\) and let \(p_+\) run through all even integers. Arguments for the remaining families are similar.
BIBLIOGRAPHY


<table>
<thead>
<tr>
<th>Ref.</th>
<th>Author</th>
<th>Title</th>
<th>Source and Details</th>
</tr>
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</table>
APPENDICES
A APPENDIX Summary of cohomological data.

This appendix summarizes the cohomology groups and ring generators for members of the families $L_{(p_-,q_-),(p_+,q_+)}$, $M_{(p_-,q_-),(p_+,q_+)}$, $N_{(p_-,q_-),(p_+,q_+)}$ and $O_{(p,q;m)}$ discussed in this thesis. The table entries “Parameter restrictions” are those restrictions on the parameters $p_\pm$, $q_\pm$, $p$ and $q$ necessary to ensure that the fourth cohomology group is a non-trivial, finite cyclic group, and that the free part of the third cohomology group is trivial. These restrictions are in addition to those already listed in Table 2.1, and the standing assumption that the pairs $(p_-,q_-)$, $(p_+,q_+)$ and $(p,q)$ are relatively prime.

TABLE A: Summary of cohomological data.

<table>
<thead>
<tr>
<th>Family</th>
<th>$L_{(p_-,q_-),(p_+,q_+)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter restrictions</td>
<td>$p_+$ odd &amp; $p_+^2 q_-^2 - p_-^2 q_+^2 \neq 0$</td>
</tr>
<tr>
<td>Cohomology groups</td>
<td>$H^k(L;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} &amp; k = 0, 2, 5, 7 \ \mathbb{Z}_r, \ r \neq 1, 0 &amp; k = 4 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Order of $H^4(L;\mathbb{Z})$</td>
<td>$r = \frac{1}{4}</td>
</tr>
<tr>
<td>Ring generators</td>
<td>$x \in H^2(L;\mathbb{Z})$ and $y \in H^5(L;\mathbb{Z})$</td>
</tr>
<tr>
<td>Notes</td>
<td>$r$ is always even. Kreck-Stolz invariants exits.</td>
</tr>
</tbody>
</table>
TABLE A: Summary of cohomological data (continued).

<table>
<thead>
<tr>
<th>Family</th>
<th>$L(p_-,q_-),(p_+,q_+)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter restrictions</td>
<td>$p_+\text{ even}$</td>
</tr>
<tr>
<td>Cohomology groups</td>
<td>$H^k(L;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} &amp; k = 0, 2, 7 \ \mathbb{Z}_2 &amp; k = 3 \ \mathbb{Z}_r, \ r \neq 0, 1 &amp; k = 4 \ \mathbb{Z} \oplus \mathbb{Z}_2 &amp; k = 5 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Order of $H^4(L;\mathbb{Z})$</td>
<td>$r =</td>
</tr>
<tr>
<td>Ring generators (partial list)</td>
<td>Let $H^2(L;\mathbb{Z}) = \mathbb{Z} \cdot x$, $H^5(L;\mathbb{Z}) = \mathbb{Z} \cdot y \oplus \mathbb{Z}_2$; then $x^2$ generates $H^4(L;\mathbb{Z})$, and $xy$ generates $H^7(L;\mathbb{Z})$.</td>
</tr>
<tr>
<td>Notes</td>
<td>$r$ is always odd.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Family</th>
<th>$M(p_-,q_-),(p_+,q_+)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Restrictions</td>
<td>$p_+^2q_-^2 - p_-^2q_+^2 \neq 0$</td>
</tr>
<tr>
<td>Cohomology groups</td>
<td>$H^k(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} &amp; k = 0, 7 \ \mathbb{Z}_r, \ r \neq 0, 1 &amp; k = 4 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Order of $H^4(M;\mathbb{Z})$</td>
<td>$r = \frac{1}{8}</td>
</tr>
<tr>
<td>Ring generators</td>
<td>$y \in H^4(M;\mathbb{Z})$ and $z \in H^7(M;\mathbb{Z})$</td>
</tr>
<tr>
<td>Notes</td>
<td>Computed in [GWZ]; same cohomology ring as an $S^3$-bundle over $S^4$.</td>
</tr>
</tbody>
</table>
TABLE A: Summary of cohomological data (continued).

<table>
<thead>
<tr>
<th>Family</th>
<th>$N_{(p-,q-),(p+,q+)}$</th>
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<tbody>
<tr>
<td>Parameter restrictions</td>
<td>None.</td>
</tr>
<tr>
<td>Cohomology groups</td>
<td>$H^k(N;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} &amp; k = 0, 2, 5, 7 \ \mathbb{Z}_r, r \neq 0, 1 &amp; k = 4 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Order of $H^1(N;\mathbb{Z})$</td>
<td>$r =</td>
</tr>
<tr>
<td>Ring generators</td>
<td>$x \in H^2(N;\mathbb{Z})$ and $y \in H^5(N;\mathbb{Z})$</td>
</tr>
</tbody>
</table>
| Notes | Groups computed in [GWZ].  
$r$ is always odd.  
Kreck-Stolz invariants exist. |

<table>
<thead>
<tr>
<th>Family</th>
<th>$O_{(p,q,m)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter restrictions</td>
<td>either $</td>
</tr>
<tr>
<td>Cohomology groups</td>
<td>$H^k(O;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} &amp; k = 0, 2, 5, 7 \ \mathbb{Z}_r, r \neq 0, 1 &amp; k = 4 \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>Order of $H^1(O;\mathbb{Z})$</td>
<td>$r =</td>
</tr>
<tr>
<td>Ring generators</td>
<td>$x \in H^2(O;\mathbb{Z})$ and $y \in H^5(O;\mathbb{Z})$</td>
</tr>
</tbody>
</table>
| Notes | $r$ is odd whenever either $p$ or $q$ is even.  
Kreck-Stolz invariants exist. |
APPENDIX  Example: determining cohomology ring generators using Sequence 2.2.

In the proof of Theorem 1.2.1, Lemma 2.3.2 was used to determine cohomology ring generators. For manifolds belonging to the family $O(p,q;m)$, a more direct method is available. Both non-principal orbits of members of this family are orientable. This gives two distinct long exact cohomology sequences; versions of Sequence 2.2 corresponding to each of the non-principal orbits. Once the cohomology groups have been established, it is possible to find the cohomology ring generators relying exclusively on these two sequences. We demonstrate.

Let $O_1$ be a member of $O(p,q;1)$ with either $|p|$ or $|q|$ not equal to one. The non-principal orbits are $G/K_-=S^3 \times S^2$ and $G/K_+=S^3$. The disk bundle over the non-principal orbit $G/K_-$ has fiber $D^2$ (so $t=2$ in the associated long exact sequence), while that over $G/K_+$ has fiber $D^4$. Both of these bundles are orientable. The cohomology groups of $O_1$ are:

$$H^k(O_1) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 5, 7 \\ \mathbb{Z}_r, r \neq 0, 1 & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

Suppose the class $x$ generates $H^2(O_1)$. Let $1_-$ be the unit of the cohomology ring $H^*(G/K_-)$. Set $k=2$ in the version Sequence 2.2 corresponding to the pair $(O_1, G/K_+)$:

$$\cdots \to H^0(G/K_-) \cong \mathbb{Z} \xrightarrow{J} H^2(O_1) \cong \mathbb{Z} \cdot x \to H^2(G/K_+) = 0.$$ 

Since $H^0(G/K_-)$ surjects onto $H^2(O_1)$ and $1_-$ generates $H^0(G/K_-)$, we conclude $J(1_-)$ generates $H^2(O_1)$. Without loss of generality, we may assume $J(1_-) = x$.

Next, take $k=2$ in the version of Sequence 2.2 corresponding to the pair $(O_1, G/K_-)$:

$$H^{-2}(G/K_+) = 0 \to H^2(O_1) \cong \mathbb{Z} \cdot x \xrightarrow{i_-^*} H^2(G/K_-) \cong \mathbb{Z} \to H^{-1}(G/K_+) = 0.$$ 

We see that $H^2(O_1)$ and $H^2(G/K_-)$ are isomorphic. So $i_-^*(x)$ generates $H^2(G/K_-)$. 
Finally, set $k = 4$ in the version of Sequence 2.2 corresponding to the pair $(O_1, G/K_+)$:

$$\cdots \rightarrow H^2(G/K_-) \cong \mathbb{Z} \cdot i_*^-(x) \xrightarrow{J} H^4(O_1) \cong \mathbb{Z}_4 \rightarrow H^4(G/K_+) = 0.$$ 

Since $J$ is a surjection and $i_-(x)$ generates $H^2(G/K_-)$, then $J(i_-(x))$ generates $H^2(G/K_-)$. Because $J$ is an $H^*(O_1)$-module homomorphism:

$$J(i_-(x)) = J(1 \cdot i_-(x)) = J(1) \cdot x = x^2.$$ 

Thus, if the class $x$ generates $H^2(O_1) \cong \mathbb{Z}$, the class $x^2$ generates $H^4(O_1) \cong \mathbb{Z}_4$. 