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Dr. Gunnar Bodvarsson

In this thesis, the interpretation and reduction of ocean heat flow measurements are discussed on the basis of theoretical models. The instrument effect on heat flow measurements is investigated for the case of long period measurements by studying the heat conduction along the measurement probe for both steady and unsteady state bottom temperatures. This effect is found to be unimportant. Measurement errors due to recent bottom temperature transients are studied and the possible magnitude of such errors is estimated. Moreover, effects of climatic variation on the ocean floor temperature are estimated on the basis of diffusion models. It is shown that climatic variations with periods longer than one thousand years will be unattenuated and will affect the entire ocean floor.

The perturbation method is used to study the effects of an irregular topography and a variable thickness of ocean floor sediments on
the heat flow. Some special examples are given to provide a comparison between the perturbation solutions and exact solutions of similar problems. The perturbation method is also applied to a buried body with different thermal conductivity from its surroundings and the reliability of the perturbation solution is examined.

Heat flow anomalies due to heat transport by magma intruded into crustal layers is studied by solving the heat conduction equation. It is shown that magmatic intrusions can lead to very large surface heat flow anomalies.

Finally, the possibility of deriving the ocean floor thermal gradient on the basis of on-ship measurements performed on sediment cores is investigated. The results appear positive. The temperature variations in flowing wells and the temperature variation in a cylindrical sediment core influenced by the movement of water along the axis of the core are also studied.
On the Reduction and Interpretation of Ocean-Floor Temperature and Heat Flow Data

by

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ON THE REDUCTION AND INTERPRETATION OF OCEAN-FLOOR TEMPERATURE AND HEAT FLOW DATA

INTRODUCTION

Magnitude of Terrestrial Heat Flow

The average solar heat flow reaching the surface of the earth is of the order of \(10^{-2}\) cal/cm\(^2\) sec while the heat flow from the earth's interior to the surface is of the order of \(10^{-6}\) cal/cm\(^2\) sec. Although the terrestrial heat flow is small compared with solar heat flow, it is the most important form of energy which flows from the inside of the earth to the surface. The total outflow of heat through the surface of the earth is about \(10^{28}\) ergs per year which is about 1000 times larger than the total mechanical energy dissipated by earthquake activity. For this reason, the measurement of terrestrial heat flow becomes important in the study of the interior of the earth to obtain information on the internal distribution of temperature, heat sources, and dynamic processes.

Measurement of Heat Flow at the Ocean Bottom

Heat flow by conduction, \(\vec{q}\), in a solid is found experimentally to be proportional to the temperature gradient, \(\text{grad } T\):

\[
\vec{q} = - K \cdot \text{grad } T
\]  

(1)
where $K$ is called the thermal conductivity. In general the thermal conductivity is a matrix which reduces to a scalar in the case of an isotropic solid. For the measurement of terrestrial heat flow, the heat conduction can be expressed as

$$q = -K \frac{\partial T}{\partial z}$$

where $\frac{\partial T}{\partial z}$ is the vertical thermal gradient. At the surface of the earth both $K$ and $\frac{\partial T}{\partial z}$ can be measured experimentally. The thermal gradient in the sediments of the sea bottom is of the order of $10^{-4} \, ^\circ C/cm$. For the top of the sediments in the bottom of the oceans, the thermal conductivity is between 0.0016 and 0.0027 cal/cm sec $^\circ C$ (Gutenberg, 1951, p. 125) while for most rocks, the thermal conductivity is about 0.005 cal/cm sec $^\circ C$. The range of the thermal conductivity for rocks is from 0.00325 to 0.01605 cal/cm sec $^\circ C$ (Clark, 1965).

The measurement of heat flow at the ocean bottom has the advantage of uniformity and constancy of temperature and absence of seasonal and diurnal fluctuations. The temperature is near $0 \, ^\circ C$ at the surface of the deep ocean bottom. Measurement of the thermal gradient may be made in the uppermost few meters of sediment which is easily penetrated in many areas by thermal probes. For these reasons, the collection of oceanic heat flow data has increased rapidly in recent years.
On the basis of ocean bottom heat flow measurements it is found that the average heat flow on land and the average heat flow at sea are approximately equal. This is somewhat surprising because of the great difference in the crustal structure of the two regions. Another important discovery is the variability of the oceanic heat flow measurements.

Study of the distribution of heat flow values on a histogram shows that the most frequent value is 1.1 units (Lee and Uyeda, 1965). The unit of heat flow is defined as $10^{-6}$ cal/cm$^2$ sec and will be used in this paper. The mean values are shown to be $1.58 \pm 1.14$ s.d. for the whole world on the basis of 1150 measurements, $1.43 \pm 0.56$ s.d. for continents on the basis of 131 measurements, and $1.60 \pm 1.18$ s.d. for oceans on the basis of 913 measurements. In this paper a mean heat flow of $q_0 = 1.5$ units will be assumed for the calculations and will not be mentioned further.

Heat flow values over oceanic ridge crests have been observed to vary from very low values up to a maximum value of ten. The large variation and the high heat flow over the crestal zone of the ridges show some correlation of heat flow and topography. Vacquier and Von Herzen (1964) showed that most of the values of high heat flow on the Mid-Atlantic Ridge were obtained within 100 km of the ridge axis. Later, Nason and Lee (1964) and Langseth et al. (1966) confirmed this observation. Menard (1965) mentioned that on the
East Pacific Rise the band of high heat flow parallel to the crest is only about 200 to 300 km wide. Von Herzen and Uyeda (1963) have pointed out that the values of high heat flow appear to be in two continuous narrow zones on either side of the crest of the East Pacific Rise. The source of high heat flow is thought to be a region of unusually high temperature, tens of kilometers wide and located 10 km beneath the ocean floor (Lee and Uyeda, 1965). The dimensions of the heat sources indicate that they are dikes (Menard, 1965).

Relatively low values of heat flow are observed over the flanks of the ridges. On the East Pacific Rise, Von Herzen and Uyeda (1963) found two approximately similar regions near the equator, and to each side of the rise, which show generally low heat flow. In the Atlantic Ocean low heat flow values were observed to be predominant between 300 and 400 km from the crest of the Mid-Atlantic Ridge (Vacquier and Von Herzen, 1964). It is apparent that, in general, the heat flow values for a ridge are high over the crest and relatively low on both sides of the crest. This type of heat flow pattern has been considered as evidence of convection in upper mantle.

In general, ocean basin areas yield a low mean value of heat flow. Lee and Uyeda (1965) showed that the world-wide mean value for these areas is $1.28 \pm 0.53$ s.d., based on 273 measurements.

The mean value of $0.99 \pm 0.61$ s.d. (based on 21 measurements) for trenches (Lee and Uyeda, 1965) seems to show correlation
between trench areas and low values of heat flow.

The large variation of heat flow values at the ocean bottom gives rise to an interesting geophysical study of the possible causes of such variations.

The possible causes of heat flow anomalies are listed as follows:

a. Topographic relief.
b. Variable thickness of sediment layers.
c. Differences in the thermal conductivity in the crust.
d. Differences of heat production in the rocks.
e. Sedimentation and erosion.
f. Magmatic activity, such as cooling of intrusions and extrusions.
g. Mass transport by liquid, e.g. water flow through permeable formations.
h. Thermal convection in the upper mantle.
i. Temperature variation at the ocean bottom.
j. Others: such as turbidity currents near the ocean bottom and earthquake activity.

Although these possible causes of the disturbance of heat flow have been considered by various authors (Bullard et al. 1956; Von Herzen and Uyeda, 1963), little is known about their quantitative influence.
Physical Constants Within the Crust and Upper Mantle

The variation with depth of the longitudinal velocity \( v_p \) and of the shear velocity \( v_s \) is now fairly well known on the basis of seismological observations. These two wave velocities are the basic observables in the physics of the solid earth. In the case of an isotropic linearly elastic solid, \( v_p \) and \( v_s \) can be expressed as

\[
v_p = \sqrt{\frac{k+4\mu}{3\rho}} \quad (3)
\]

\[
v_s = \sqrt{\frac{k}{\rho}} \quad (4)
\]

where \( k \) is the bulk modulus and \( \mu \) is the modulus of rigidity. The modulus of rigidity can be eliminated from the above two equations to give

\[
\frac{v_p^2}{3} - \frac{4}{3} v_s^2 = \frac{k}{\rho} = v_b^2 = (\text{bulk sound velocity})^2 \quad (5)
\]

This relation shows that the ratio \( k/\rho \) can be obtained from the basic observables \( v_p \) and \( v_s \).

Since \( k/\rho \) is known on the basis of the above equation, the density distribution in the earth can be computed (Bullen, 1963, p. 229). Hence, the observables \( v_p \) and \( v_s \) yield both the density and the bulk modulus as functions of depth.

The thermal conductivity in the upper parts of the earth may be estimated on the basis of experimental data. It is known that there
are mainly three agents which contribute to the transport of heat through crystalline solids. They are lattice waves, free electrons, and electromagnetic waves. The quantized lattice waves are called phonons and the quantized electromagnetic waves are called photons. The phonons dominate heat transport in electrical insulators, in particular in rock-forming silicates at temperatures below 1000 °C and at moderate pressures. At these conditions, heat transport by free electrons and by photons is insignificant. Heat conduction in the upper part of the earth can therefore be considered as a phonon effect and the thermal conductivity will be referred as phonon conductivity. The ratio of phonon conductivities at two locations in the mantle may possibly be expressed in terms of the density and the seismic velocities $v_p$ and $v_s$. The relationship is (Bodvarsson, 1966, p. A.11)

$$\frac{K_1}{K_2} = \left(\frac{\rho_1}{\rho_2}\right)^{2/3} \left(\frac{v_{b1}}{v_{b2}}\right)^2$$

Since $v_p$ and $v_s$ are fairly well known quantities, it appears that the ratio of $K_1/K_2$ can be roughly estimated.

At greater depth, the photon conductivity should also be taken into account. In general, the total conductivity is expressed as (Bodvarsson, 1966, p. A.15)
where \( a_1 \) and \( b_1 \) are constants which depend on the pressure, frequency of radiation, etc., and \( T \) is in \( ^{0}\text{K} \). The second term on the right hand side represents the photon conductivity. The phonon conductivity predominates at temperatures below 1000 \(^{0}\text{C} \), but the photon conductivity increases rapidly with the temperature and predominates at temperatures above 2000 \(^{0}\text{C} \).

Actually, thermal conductivity is a function of temperature, pressure, and other variables such as water content. However, in this paper it will be assumed to be constant in the upper part of the earth. The thermal conductivity is about 0.005 cal/cm sec \(^{0}\text{C} \) for many rocks in the crust, but near 0.01 cal/cm sec \(^{0}\text{C} \) for the ultrabasic rocks (Gutenberg, 1959, p. 125).

The generation of heat in the upper part of the earth is the main source of terrestrial heat flow. The rate of generation of heat for the decay of radioactive material is a function of depth.

In general, heat production \( A \) and density \( \rho \) of the rocks have a negative correlation, that is, if the material has a high rate of heat production per unit volume, its density is expected to be low. This correlation between \( A \) and \( \rho \) is important in the study of the relation between heat flow and gravity anomalies.

The correlation between \( A, K \) and \( \rho \) for a simplified model of the crust and upper mantle (see Figure 3.15, Gutenberg, 1959, p. 64)
are given in the following Table.

Table 1. Average values of physical constants for rocks.

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<tr>
<th>Materials</th>
<th>Physical constants</th>
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<tr>
<td>Granite</td>
<td>$A = 145 \times 10^{-14}\text{cal/cm}^3\text{sec}$</td>
<td>Gutenberg, 1959, p.134</td>
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<tr>
<td></td>
<td>$K = 5.5 \times 10^{-3}\text{cal/cm sec}^\circ\text{C}$</td>
<td>Clark, 1965, T. 21-4</td>
</tr>
<tr>
<td></td>
<td>$\rho = 2.65\text{gm/cm}^3$</td>
<td>Clark, 1965, T. 21-4</td>
</tr>
<tr>
<td>Basalt or Gabbro</td>
<td>$A = 45 \times 10^{-14}\text{cal/cm}^3\text{sec}$</td>
<td>Gutenberg, 1959, p.134</td>
</tr>
<tr>
<td></td>
<td>$K = 4.6 \times 10^{-3}\text{cal/cm sec}^\circ\text{C}$</td>
<td>Birch, 1955, p. 104</td>
</tr>
<tr>
<td></td>
<td>$\rho = 3\text{gm/cm}^3$</td>
<td>Birch, 1955, p. 104</td>
</tr>
<tr>
<td>Dunite</td>
<td>$A = 17 \times 10^{-14}\text{cal/cm}^3\text{sec}$</td>
<td>Gutenberg, 1959, p.134</td>
</tr>
<tr>
<td></td>
<td>$K = 8.1 \times 10^{-3}\text{cal/cm sec}^\circ\text{C}$</td>
<td>Clark, 1965, T. 21-4</td>
</tr>
<tr>
<td></td>
<td>$\rho = 3.26\text{gm/cm}^3$</td>
<td>Clark, 1965, T. 21-4</td>
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<tr>
<td>Deep sea sediments</td>
<td>$A = 3 \times 10^{-14}\text{cal/cm}^3\text{sec}$</td>
<td>Von Herzen et al. 63</td>
</tr>
<tr>
<td></td>
<td>$K = 2 \times 10^{-3}\text{cal/cm sec}^\circ\text{C}$</td>
<td>Clark, 1965, T. 21-5</td>
</tr>
<tr>
<td></td>
<td>$\rho = 1.5\text{gm/cm}^3$</td>
<td>Clark, 1965, T. 21-5</td>
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**General Theory of the Interpretation for Heat Flow Anomalies**

In order to interpret heat flow anomalies, the heat transport equation with given boundary conditions is solved on the basis of idealized models. To arrive at the general heat transport equation, we treat a body $D$, with boundary $B$, which is composed of an incompressible, homogeneous, and isotropic material. The material of the body is assumed to yield to external stresses and to be moving with velocity $\mathbf{u}(P, t)$ at the point $P$ and time $t$. 
The temperature is $T(P, t)$. Moreover, the body is porous and is permeated by an incompressible fluid which moves relative to the material. The pores or cracks through which the fluid permeates are supposed to be very finely distributed and the averaged specific fluid flow relative to the body is represented by $\vec{q}_f(P, t)$ in gm/cm$^2$ sec. Furthermore, it is being assumed that the thermal contact between the fluid and the solid is perfect, that is, both phases have at the same point the same temperature $T(P, t)$. Under these assumptions, the general heat transport equation in an incompressible, homogeneous, and isotropic medium is given as (Bodvarsson, 1966, p. A-29)

$$\rho c \frac{\partial T}{\partial t} + \rho c_f \vec{f} \cdot \text{grad} \ T = \text{div} (K \ \text{grad} \ T) + \rho cS$$

(8)

where $\rho cS$ is the rate of heat production per unit volume within the solid earth and $\vec{f}$ is the transport vector defined as

$$\vec{f} = \vec{u} + \frac{c_f q_f}{\rho c}$$

(9)

where

- $c_f =$ specific heat of liquid,
- $S =$ temperature production in °C/sec.

The above equation is based on the assumption that both phases are
incompressible and that \( c \) and \( c_f \) are independent of temperature. If \( f = 0 \) and \( K \) is scalar and constant, Equation (8) simplifies to the general heat conduction equation for a homogeneous and isotropic conductor as

\[
\frac{\partial T}{\partial t} = a \nabla^2 T + \frac{A}{K}
\]  

(10)

where

\[
a = \frac{K}{\rho c}
\]

is the thermal diffusivity in cm\(^2\)/sec.

In order to interpret a heat flow anomaly on the basis of a model calculation, the ocean bottom is considered as a flat boundary of a semi-infinite, homogeneous, and isotropic solid. The undisturbed temperature is assumed to be linear in the form

\[
T = g_0 z
\]

(11)

Because of uniformity and constancy of temperature at the ocean bottom and absence of seasonal and diurnal fluctuations, the temperature along the ocean bottom is generally assumed to be constant. The temperature is near 0°C at the deep ocean bottom. Therefore the surface boundary condition is

\[
T = 0, \quad \text{at} \quad z = 0
\]

(12)

The image method is used in order to satisfy the surface boundary condition.
Various solutions of Equations (8) and (10) with given initial and boundary conditions are found and discussed. In the case of steady state, solutions of Equation (10) can be obtained by using the known solutions of Laplace's equation or Poisson's equation. For transient temperature fields the instantaneous- and continuous-source solutions can be used as a solution of Equation (10).

No general solutions are known for the heat transport Equation (8), and it is quite difficult to find solutions for specific cases. In particular it is difficult to find solutions for cases which involve variations in the physical parameters $f$ and $K$. However, some cases, where $f$ is small and variations in $K$ are small, can be treated with the perturbation method. This is considered below.

One will assume a steady state case where $S = 0$ and the thermal diffusivity, $a$, can be divided into two terms

\[ a = a_0 + a_1 \]

where $a_0$ is constant and $a_1$ is much smaller than $a_0$. Moreover, one assumes that $f$ is small so that the transport term on the left hand side of Equation (8) is relatively small. One can then, in the first approximation, write

\[ T = T_0 + T_1 \]  \hspace{1cm} (13)

where $T_0$ is a solution of the simple equation
\[ \nabla^2 T_0 = 0 \]

and \( T_1 \) is a small perturbation due to \( a_1 \) and \( \vec{f} \). \( T \) is inserted into Equation (8) and higher order terms in \( T_1 \) are cancelled. This gives an equation in the first order terms

\[ - a_0 \nabla^2 T_1 = \text{div} (a_1 \text{grad } T_0) - \vec{f} \cdot \text{grad } T_0 \] (14)

Equation (14) is an equation of the Poisson type and therefore, its solution can be expressed as

\[ T_1 = - \frac{1}{a_0} \iiint F(Q)G(P,Q)dV_Q \] (15)

where

\[ - F(P) = \text{div} (a_1 \text{grad } T_0) - \vec{f} \cdot \text{grad } T_0, \]

\[ G(P,Q) = \text{Green's function}. \]

The Green's function for a homogeneous and isotropic half-space with zero surface temperature is

\[ G(P,Q) = \frac{1}{4\pi r_{PQ}} - \frac{1}{4\pi r_{PQ'}} \] (16)

where \( r_{PQ} \) and \( r_{PQ'} \) are defined in Figure 1.
Figure 1. Coordinates of the body buried in a half-space.

The corresponding disturbed thermal gradient is

\[ \frac{\partial T}{\partial z} = -\frac{1}{a_o} \int \int \int F(Q) \frac{\partial}{\partial z} G(P, Q) dV_Q \]  

Equation (17) is the general solution obtained by using the perturbation method.

Units and Notations

All units are in cgs, calorie, and °C and will be understood in this paper unless some other unit is specifically mentioned. The common notations and their units used in this paper are listed as
follows:

\[ a = \frac{K}{\rho c} = \text{thermal diffusivity in } \text{cm}^2/\text{sec} \]

\[ A = \text{heat production in } 10^{-14} \text{ cal/cm}^3\text{sec} \]

\[ c = \text{specific heat of solids in cal/gm}^\circ\text{C} \]

\[ g_0 = \text{undisturbed thermal gradient in } ^\circ\text{C/cm} \]

\[ K = \text{thermal conductivity in cal/cm sec}^\circ\text{C} \]

\[ q_0 = \text{undisturbed heat flow in } 10^{-6} \text{ cal/cm}^2\text{sec} \]

\[ q = \text{disturbed heat flow in } 10^{-6} \text{ cal/cm}^2\text{sec} \]

\[ \Delta q = \text{heat flow anomaly in } 10^{-6} \text{ cal/cm}^2\text{sec} \]

\[ q_m = \text{maximum heat flow in } 10^{-6} \text{ cal/cm}^2\text{sec} \]

\[ \Delta q_m = \text{maximum heat flow anomaly in } 10^{-6} \text{ cal/cm}^2\text{sec} \]

\[ \rho = \text{density in gm/cm}^3 \]

\[ T = \text{temperature in } ^\circ\text{C} \]

\[ T_0 = \text{initial temperature in } ^\circ\text{C}. \]
INSTRUMENT EFFECT

Heat Conduction Along the Probe

Determination of oceanic terrestrial heat flow requires measurement of the temperature gradient in the uppermost two to twenty meters of bottom sediments and sampling of the sediments in order to obtain the thermal conductivity. The thermal gradient is generally obtained by penetrating the probe into the sediments at the ocean floor. Two types of probes in common use are the Bullard type and the Ewing type. Both of these use two or more temperature elements which are placed vertically some known distance apart in or on a probe. On the Bullard type probe the thermal elements are placed inside a long, slender, hollow probe two centimeters or more in diameter and two meters or more in length. It is driven into the sediments by the momentum of the lowering and the instrument's weight. On the Ewing type instrument the thermal elements are placed in fine needle probes which are mounted on outriggers on the penetrating vehicle. Piston or gravity coring devices are used to carry the probe into the sediments. The piston corer can achieve penetrations up to 20 meters, thus allowing wider separation of the thermal sensing elements and removal of the probes from the possible near surface disturbances. These conditions lead to more accurate gradient measurements.
The temperature field around the probe will be disturbed by frictional heat generated when the probe is penetrating into the sediments. About 80 percent to 90 percent of this effect is eliminated by leaving the probe in the sediments for 14 to 40 minutes during which the frictional heat is dissipated to the surrounding material. Heat conduction along the probe is of no importance during such short periods of time. However this effect has to be considered if the probe is left in the sediments for longer time intervals. This will be discussed below.

**Steady State--Zero Surface Temperature**

Since the conductivity of the thermal probe is different from that of the sediments, the temperature field in the sediments will be disturbed by the probe. In order to arrive at an estimate of this effect, the probe can be considered as an elongated semi-ellipsoid, and the temperature at the surface of the ocean bottom can be assumed to be zero. The problem of the probe placed vertically in the sediments can be reduced to the problem of a conducting ellipsoid placed in a linear temperature field. Thus a solution of Laplace's equation in ellipsoidal coordinates should be found.

Let the temperature in the sediments be $T_1$ and let the temperature in the probe be $T_2$. Moreover, let the axes of the ellipsoidal conductor be $L, M, N$. In the case of steady state heat
conduction with constant conductivities both $T_1$ and $T_2$ have to be solutions of Laplace's equation

$$\nabla^2 T_1 = 0$$

$$\nabla^2 T_2 = 0$$

with the boundary conditions

\[\begin{align*}
\text{a} & \quad T_1 = T_2 \quad \text{at the interface}, \\
\text{b} & \quad K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n} \quad \text{at the interface}, \\
\text{c} & \quad T_1 = g_0 z \quad \text{at large distance from the probe},
\end{align*}\]

where $n$ is the normal and $K_1$ and $K_2$ are respectively the thermal conductivities of the sediments and the effective conductivity of the probe.

In order to find the solutions of Laplace's equation in ellipsoidal coordinates, $U_1, U_2, U_3$ are defined as curvilinear, orthogonal, ellipsoidal coordinates which are the roots of the following equation

$$\frac{x^2}{U^2 - a^2} + \frac{y^2}{U^2 - b^2} + \frac{z^2}{U^2} = 1$$

When the conductor is described by the above equation, then one has
\[ a^2 = N^2 - L^2, \quad b^2 = N^2 - M^2 \]

It is being assumed that \( L < M < N \). The relations between \( x, y, z \) and \( U_1, U_2, U_3 \) are given as

\[
x = \sqrt{\frac{(U_1^2 - a^2)(U_2^2 - a^2)(U_3^2 - a^2)}{a^2(b^2 - a^2)}}
\]

\[
y = \sqrt{\frac{(U_1^2 - b^2)(U_2^2 - b^2)(U_3^2 - b^2)}{b^2(b^2 - a^2)}}
\]

\[
z = \frac{U_1 U_2 U_3}{ab}
\]

where \( \infty > U_1 > a > U_2 > b > U_3 > -b \).

If the longest semi-axis of the ellipsoid is along the direction of the undisturbed heat flow, then, by using Lame functions (Morse and Feshbach, 1953, p. 1305) satisfying boundary condition (18c), the temperature outside the probe is found to be

\[
T_1 = A_1 E_1^0(U_1)E_1^0(U_2)E_1^0(U_3) + A_2 F_1^0(U_1)E_1^0(U_2)E_1^0(U_3) \quad (19)
\]

where \( A_1, A_2 \) are constants and

\[
E_1^0(t) = t
\]

\[
F_1^0(t) = 3E_1^0(t) \int_0^\infty \frac{du}{[E_1^0(u)]^2 (u^2 - a^2)^{1/2}(u^2 - b^2)^{1/2}}
\]
Thus, Equation (19) can be expressed as

\[ T_1 = g_0 z [1 + BF(U_1)] \]  \hspace{1cm} (20)

where

\[ F(t) = \int_{\infty}^{t} \frac{du}{u^2 \sqrt{(u^2 - a^2)(u^2 - b^2)}} \]

Because of boundary condition (18a), \( T_2 \) has to be in the form

\[ T_2 = Cz \]  \hspace{1cm} (21)

By using boundary conditions (18a, b), it is easy to find the constants B and C. Hence

\[ T_1 = g_0 z + \frac{K_2}{K_1} F(U_1) g_0 z, \quad U_1 > N \]  \hspace{1cm} (22)

\[ T_2 = \frac{g_0 z}{1 - LMN(1 - \frac{K_2}{K_1}) F(N)}, \quad U_1 < N \]  \hspace{1cm} (23)

The corresponding thermal gradient within the ellipsoid is in the form

\[ \frac{\partial T_2}{\partial z} = \frac{g_0 z}{K \frac{1}{1 + (\frac{K_2}{K_1} - 1) I_1}} \]  \hspace{1cm} (24)

where
If the probe is considered as a semi-spheroid, then by evaluating Equation (25) and letting $L = M$, $I_1$ has the form (Byrd and Friedman, 1954, p. 52 and 191)

$$I_1 = \frac{MN}{2} \int_{N}^{\infty} \frac{dt}{t^2 (t^2 - N^2 + L^2)^{1/2} (t^2 - N^2 + M^2)^{1/2}}$$

(25)

where

$$F(\phi, k) = \text{elliptic integral of the first kind},$$

$$E(\phi, k) = \text{elliptic integral of the second kind}.$$ 

It should be noted that $I_1$ can also be expressed in the following approximate form (Carslaw and Jaeger, 1959, p. 428)

$$I_1 = \frac{M^2}{N^2} \left( \ln \frac{2N}{M} - 1 \right)$$

(26)

(27)

For computation, it is more convenient to use Equation (27) than Equation (26). The results obtained by these two equations are very close. For example, for the case of $N = 200$ cm and $M = 4$ cm, $I_1 = 0.00142$ by using Equation (26) while $I_1 = 0.00144
by using Equation (27).

In the above, the probe has been assumed to be homogeneous. Since it consists of a metal tube filled with oil, one has to estimate the effective conductivity \( K_2 \). Thus assumptions must be made for the computation of \( K_2 \). Assume a hollow cylinder filled with oil is used as the probe. \( R_1 \) is the outer radius of the cylinder and \( R_2 \) is the inner radius. If the heat conduction is along the vertical direction only, then the total rate of heat conduction through the cross section of the probe is

\[
q_1 = (K_s A_s + K_o A_o) \frac{\Delta T}{\Delta z}
\]  

(28)

where \( A_s = \pi (R_1^2 - R_2^2) \) and \( A_o = \pi R_2^2 \). Since the probe is assumed to be a homogeneous rod of thermal conductivity \( K_2 \), the rate of heat conduction through the cross section of the homogeneous rod is

\[
q_2 = K_2 A_2 \frac{\Delta T}{\Delta z}
\]

where \( A_2 = \pi R_1^2 \). Now, assuming \( q_2 \) equal to \( q_1 \), then

\[
K_2 = \frac{K_s (R_1^2 - R_2^2) + K_o R_2^2}{R_1^2}
\]  

(30)

where

\[
K_s = \text{thermal conductivity of steel},
\]

\[
K_o = \text{thermal conductivity of oil}.
\]
A two-meter long probe with inner radius 3.5 cm and outer radius 4 cm will be used as an example to illustrate the effect of the probe on heat flow measurement. If the values of $K_s$, $K_0$, and $K_1$ are assumed to be 0.06, 0.00025, and 0.0025 (all in unit cal/cm sec $^\circ$C), then it turns out that $K_2$ is about 0.014 cal/cm sec $^\circ$C and the steady state disturbance due to the probe is less than one percent. Therefore, it may be concluded that in the case of steady state the effect of the probe on the thermal gradient measurement is negligible.

**Unsteady State--Surface Temperature as Harmonic Function of Time**

Assume that the probe is so thin compared with its length that the temperature at all points of the cross section may be considered equal. The problem is thus one of linear flow in which the temperature is a function of time and vertical distance $z$. The probe is still considered as a homogeneous rod. Since the probe loses heat to the surrounding medium, the equation of heat conduction along the probe has the form

$$K_2A \frac{\partial^2 T}{\partial z^2} = A \rho P \frac{\partial^2 T}{\partial t^2} + K'(T_p - T_s)$$

(31)

where

- $K_2$ = effective thermal conductivity of the probe,
- $A_P$ = cross section of the probe,
- $\rho_P$ = density of the probe,
\( c_p = \) specific heat of the probe,

\( T_p(z,t) = \) temperature distribution within the probe,

\( T_s(z,t) = \) temperature of the undisturbed medium,

\( K^t = \) coefficient for the transfer of heat from the probe to the surroundings.

It is being assumed that the ocean bottom surface temperature is a harmonic function of time

\[ T_s(0,t) = A_0 \exp(i\omega t), \quad \text{at} \quad z = 0 \]

Hence (Carslaw and Jaeger, 1959, p. 65)

\[ T_s(z,t) = A_0 \exp \left( -\sqrt{\frac{\omega}{2a}}z + i\omega t - i\sqrt{\frac{\omega}{2a}}z \right) \]  \hspace{1cm} (32)

The temperature of the probe, \( T_p \), can be determined by solving Equation (31) subject to the following boundary conditions.

At the top of the probe, the instrument weight will absorb heat from the surrounding material but part of this heat will be transmitted to the probe. For the present purpose, the mass of the instrument weight can be lumped into a single mass \( m \) and the thermal resistance of the weight can be lumped into a single thermal resistor. Thus the boundary condition at the top, where \( z = 0 \), is

\[ \frac{K_m A_m (T_w - T_m)}{h} = m \frac{dT_m}{dt} - K_p A_p \left( \frac{\partial T_p}{\partial z} \right)_{z=0} \] \hspace{1cm} (33)
where

\[ K_m = \text{thermal conductivity of the instrument weight}, \]
\[ m = \text{the mass of the instrument weight}, \]
\[ c_m = \text{specific heat of the instrument weight}, \]
\[ A_m = \text{cross section of the instrument weight}, \]
\[ h = \text{effective thickness of the instrument weight}. \]

If the lower end of the probe is assumed to be a hemisphere, then on the basis of potential theory the outflow of heat can be approximately expressed as

\[ 2\pi K r \frac{(T_p - T_s)}{p} \]

where \( K \) is the conductivity of the sediments. Therefore the boundary condition at the lower end is

\[ \frac{\partial T}{\partial z} - K A_p \frac{p}{p} = 2\pi K r \frac{(T_p - T_s)}{p} \]

(34)

Assume \( T_p = v(z) \exp (i\omega t) \) and substitute it into Equation (31), then \( T_p \), satisfying boundary conditions (33) and (34), is in the form

\[ T_p = \frac{m_3 n_2 - m_2 n_3}{m_1 n_2 - m_2 n_1} \exp (b_1 z + i\omega t) + \frac{m_3 n_1 - m_1 n_3}{m_2 n_1 - m_1 n_2} \exp (-b_1 z + i\omega t) + b_3 \exp (-d_1 z + i\omega t) \]

(35)
where

\[ d_1 = (1+i)\sqrt{\frac{\omega}{2a}}, \]

\[ m_1 = 1 + b_1 c_2 + i c_1 \omega, \]

\[ m_2 = 1 - b_1 c_2 + i c_1 \omega, \]

\[ m_3 = A_o - b_3 + d_1 c_2 b_3 - i c_1 b_3 \omega, \]

\[ n_1 = (b_1 - c_3) \exp(b_1 N), \]

\[ n_2 = -(b_1 + c_3) \exp(-b_1 N), \]

\[ n_3 = (b_3 d_1 + b_3 c_3 - A_o c_3) \exp(-d_1 N), \]

with

\[ b_1 = \sqrt{(K' + iA_{o p} A_{p p} c_1 \omega)/K'} \]

\[ b_3 = K' A_o / (K' + iA_{o p} A_{p p} c_1 \omega - iK' A_{p p} \omega / a), \]

\[ c_1 = m c_{m' h}/K_{m'} m', \]

\[ c_2 = -K_{p p} h/K_{m'} m', \]

\[ c_3 = -2K_{r p}/K_{2 p}. \]

It should be noted that \( T_{ww} \) is assumed to be \( A_o \exp(i\omega t) \) and \( T_m = T_p \) for the application of boundary condition (33).

It seems possible that for such a long probe the outflow of heat at the lower end of the probe can be negligible. Thus the boundary condition (34) can be replaced by
The solution of Equation (31) satisfying boundary conditions (33) and (34') still has the same form shown in Equation (35) except one should redefine $n_1$, $n_2$, and $n_3$ as following:

\[ n_1 = \exp(b_1 N) \]
\[ n_2 = \exp(-b_1 N) \]
\[ n_3 = (A_0 - b_3) \exp(-d_1 N) \]

In order to estimate the effect of the probe on heat flow measurement, $K'$ should be estimated first. Since the unsteady state solution of prolate spheroidal rod is not available, the steady state solution of the prolate spheroidal rod will be used for the estimation of $K'$. The heat lost per unit time to the surroundings through an element $dz$ on the lateral surface of the probe is $K'(T_p - T_s)dz$. This amount of the lost heat is assumed to be equal the difference of the rate of heat flow through two cross section $A_1$ and $A_2$ of the rod. Thus

\[ K'(T_p - T_s)dz = K_p \left( \frac{dT}{dz} \right) (A_2 - A_1) \]  

where
$T_s$ = the undisturbed temperature at depth $z$ in the sediments,

$A_2$ = cross section of the probe at $z + dz$,

$A_1$ = cross section of the probe at $z$.

$A_2$ and $A_1$ can be determined by using $\frac{x^2}{M^2} + \frac{z^2}{N^2} = 1$. Hence, one obtains

$$K' = \frac{2\pi K \frac{z}{p}}{(T_p - T_s) \left( \frac{M^2}{N^2} \right) \left( \frac{dP}{dz} \right)}$$

Equation (37)

For the case of the two-meter long probe previously described, one obtains by using Equation (23) and Equation (24)

$$\frac{dT}{dz} \left( \frac{P}{dz} \right) = 0.993 g_o$$

$$T_p - T_s = 0.007 g_o z$$

Inserting the above values into Equation (37), $K'$ is found to be about 1.9K where $K$ is the conductivity of sediments.

Another method to estimate $K'$ is to consider the probe as a solid cylinder with a steady radial outflow of heat. Assume that the temperature of the rod satisfies

$$\frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0$$

Equation (38)

The boundary conditions are $T = T_p$ at $r = r_p$ and $T = T_s$ at
\[ r = x_o, \] where \( x_o \) is the distance from the center of the rod to the position where the temperature field is practically undisturbed. By solving the above equation, the rate of heat flow through the lateral surface is

\[
-K(\frac{dT}{dr})dS = \frac{K(T_p - T_s)}{r \ln(x_o/r_p)} dS \tag{39}
\]

where \( dS = 2\pi r_p dz \). Since \(-K(\frac{dT}{dr})dS = K'(T_p - T_s)dz\), it is apparent that

\[
K' = 2\pi K/\ln(x_o/r_p) \tag{40}
\]

Although the estimation of the value of \( x_o \) is uncertain, the results obtained by using (37) and (40) are comparable. The reason is that \( \ln(x_o/r_p) \) is a slowly varying function even though the range of the values of \( x_o \) is large. For computation one assumes \( r_p = 4 \text{ cm} \) and \( x_o = 200 \text{ cm} \), then one has \( K' = 1.6K \).

The computation of \( T_p \) using a slide rule is very difficult. In order to estimate the magnitude of the effect in the case of the present probe, the following values are assumed in the computations:

- \( K_2 = 0.016 \text{ cal/cm sec } ^\circ C, \quad \rho_p = 2.6 \text{ gm/cm}^3, \quad c_p = 0.21 \text{ cal/gm } ^\circ C, \)
- \( K = 0.0027 \text{ cal/cm sec } ^\circ C, \quad a = 0.005 \text{ cm}^2/\text{sec}, \quad K' = 0.0027 \text{ cal/cm sec } ^\circ C, \quad r_p = 4 \text{ cm}, \quad N = 200 \text{ cm}, \quad m = 60,000 \text{ gm}, \quad c_m = 0.031 \text{ cal/gm } ^\circ C, \quad K_m = 0.083 \text{ cal/cm sec } ^\circ C, \quad h = 7.5 \text{ cm} \) and \( r_m = 15 \text{ cm} \).

From the computations for the cases of one and 100 day periods
one finds that Equation (35) can be approximately expressed as

\[ T_p = \frac{m_3}{m_2} \exp(-b_1 z + i\omega t) + b_3 \exp(-d_1 z + i\omega t) \]  

(41)

The computed results are shown in Figure 2. This indicates that for a period of one day the effect is important only when the temperature elements are placed in the upper 40 cm of the probe. The effect of the probe alone on temperature measurement is determined by \( |T_p| - |T_s| \). The results are shown in Figure 3. For a period of one day, the effect of the probe on temperature measurement is significant when the temperature element is placed in the upper 40 cm of the probe. The maximum effect is about \( 0.2A_o \). However, if the temperature element is placed below 40 cm, the effect in this case is negligible. For a period of 100 days, the difference \( |T_p| - |T_s| \) is negligible and the measurement is largely undisturbed by the probe. Thus, from these calculations it may be concluded that if the period is longer than 100 days the effect of the probe on heat flow measurement is very small.

The effect of the instrument weight on the thermal measurements can be investigated by using Equation (41). Since \( m_3/m_2 \) is a function of the mass and the height of the instrument weight, the change of the mass and the height of the instrument weight will affect the thermal gradient. For the case of above examples, if the values of \( m \) and \( h \) are doubled, the results nearly remain unchanged. Thus the instrument weight has only a small effect on the measurement.
Figure 2. Effect of the variation of the temperature at the ocean bottom on the temperature within the probe.
Figure 3. Effect of the probe on temperature measurement for the case of a period of one day.
TEMPERATURE TRANSIENTS IN THE OCEAN--DETECTABILITY ON THE BASIS OF A LINEAR T-z DIAGRAM

Since the product of thermal conductivity and thermal gradient gives the present heat flow through the ocean bottom at each station, the question arises whether it is representative of the true terrestrial heat flow. A necessary condition is a long term stability of the temperature environment at the bottom. Temperature fluctuations would lead to heat flow transients and the measured data could deviate considerably from the true terrestrial values. The questions arising in the consideration of this problem have been discussed at length by a number of authors (see Lubimova et al., 1965). In general, it is believed that conditions at the floor of deep oceans are sufficiently stable for the observed data to be practically equal to the terrestrial heat flow values.

It is well known that the temperature data obtained by the probes provide a certain temperature check on the stability of the environment. Any deviation from the linearity of the measured temperature profile may be indicative of recent fluctuations, and non-linear profiles will in general have to be regarded as unreliable, and in most cases rejected. Fluctuations of a certain amplitude-period range can therefore be detected depending on the type of probe and its dimensions. This can be illustrated on the basis of the following considerations of the three element probe which is used quite often.
Let the temperature in the sediments be $T(z, t)$. Moreover, let the depth to the temperature elements be $z_1, z_2, z_3$ respectively. Hence, at time $t$, the upper two elements measure the temperature $T(z_1, t)$ and $T(z_2, t)$ respectively. On the $T$-$z$ plot, a line can be drawn through these two points and, on the basis of elementary geometry, it is possible to show that the deviation from this line of the temperature at the third element $T(z_3, t)$ is

$$
\Delta T_3 = T_3 - (T_1 + (T_2 - T_1) \frac{z_3 - z_1}{z_2 - z_1})
$$

(42)

In most cases $z_3 - z_2 = z_2 - z_1$ and hence the deviation is

$$
\Delta T_3 = T_3 - 2T_2 + T_1
$$

(43)

In order that a non-linearity be undetected $\Delta T_3$ has to be smaller than the accuracy of the elements. On the average, the temperature gradient in the sediments is of the order of 0.05 °C/m, and the temperature sensing elements must therefore have an accuracy of the order of $10^{-3}$ °C.

Let the non-linearity result from a sudden jump $A_o$ of the bottom temperature occurring at time $t$ before the measurement is performed. If the temperature remains constant before and after the jump, then the temperature transient is in the form (Carslaw and Jaeger, 1959, p. 60)
The resulting heat flow transient at the interface is

\[ q = KA_o / \sqrt{\pi at} \] (45)

On the basis of Equations (43) and (45) one is now able to compute the errors induced by an undetected non-linearity in the profile obtained by the three element device and resulting from the temperature jump. Substituting (44) into (43), the time \( t \) can be computed in order that \( \Delta T_3 \) is smaller than the accuracy of the elements for given \( A_o \). The relation between \( A_o \) and \( t \) is in the form

\[ A_o = \frac{\Delta m}{2 \text{erf}\left(\frac{z_2}{2\sqrt{at}}\right) - \text{erf}\left(\frac{z_3}{2\sqrt{at}}\right) - \text{erf}\left(\frac{z_1}{2\sqrt{at}}\right)} \] (46)

The transient heat flow can be computed on the basis of Equation (45). The results in the case of a given probe are given in Figure 4 in which the computations are based on \( z_1 = 0.6 \text{ m}, \ z_2 = 1.6 \text{ m}, \ \text{and} \ z_3 = 2.6 \text{ m}. \) The figure shows the amplitude \( A_o \) and the time \( t \) for which the non-linearity will remain undetected if \( A_o \) and \( t \) lie below the solid curve for a given measurement accuracy. For example, from Figure 4, it can be seen that if \( A_o = 0.03^\circ\text{C} \) and if the measurement accuracy is \( \Delta m = 0.006^\circ\text{C}, \) then \( \Delta T_3 \) is undetectable when \( t \) is greater than 130 days and \( a \) is assumed to be
Figure 4. Detectability of a non-linearity in the temperature distribution as a function of time, amplitude of the temperature jump, and of measurement accuracy. The numbers on the solid curves are values of the measurement accuracy in °C and the numbers on the dashed curves are values of the error induced in gradient in °C/meter.
0.001 cm$^2$/sec. The time corresponding to the left part of the minimum point of the solid curve shown in Figure 4 is not of practical interest, since it represents the initial phase of the transient due to the temperature jump at the surface. The error induced in the thermal gradient can also be read from Figure 4. If $A_o = 0.03^\circ$C, then the induced error in the thermal gradient is $0.03^\circ$C/m when $t = 36$ days and less than $0.016^\circ$C/m when $t$ is greater than 130 days.
In this section, the effect of climatic changes at the surface on the temperature of the water taking part in the thermo-haline circulation will be studied. The main purpose is to examine what period of climatic variation at the surface will have a noticeable influence on the temperature at the ocean bottom.

In order to do this, a model must be constructed first. The deep water of the Pacific is formed in the Antarctic Ocean and moves northward into the Pacific more or less uniformly and with relatively little mixing with overlying water masses, until it begins to rise in the central and northern regions of the North Pacific. The mean northward speed is from 0.05 to 0.1 cm/sec (Knauss, 1962). The temperature of the deep water near the southern boundary will depend upon the temperature at the surface in the Wedell Sea. The temperature distribution in the Pacific at a depth of 3500 meters is from 1.4 to 1.5°C in most regions north of latitude 50°S. Also, the water is heated from underneath by heat flow from the earth's surface at a rate of $10^{-6}$ cal/cm² sec. Thus, the model will be assumed as a layer of bottom water (Figure 5) with the following boundary conditions:

\[
\begin{align*}
\text{a} & : T = A_0 \exp(i\omega t), \quad \text{at} \quad x = 0 \\
\text{b} & : T = T_0', \quad \text{at} \quad z = h \\
\text{c} & : q = q_0', \quad \text{at} \quad z = 0
\end{align*}
\]  

(47)
where $x$ is measured horizontally from the Antarctic Ocean and $z$ is measured vertically upward from the bottom. The water is assumed to be flowing horizontally with speed $u$. The quantity $q$ represents the flow of heat up through the ocean bottom.

In the oceans, the heat transport has to be regarded as a process of eddy diffusion. The vertical transport per unit area may be represented by

$$q_v = -K_v \frac{\partial T}{\partial z}$$

(48)

where $K_v$ is defined as the eddy conductivity in vertical direction. $K_h$ may be defined similarly for turbulent heat transport in horizontal direction. In general, $K_v$ and $K_h$ differ from one another,
vary with position, and depend on the type and scale of motion and on the stability. Their numerical values cover a very wide range. In the oceans, \( K_v \) is usually in the range \( 0.1 \) to \( 10^3 \) cal/cm sec \( ^°C \), while \( K_h \) falls within the range \( 10^4 \) to \( 10^8 \) cal/cm sec \( ^°C \) (Bowden, 1965). The order of magnitude of \( K_h \) is several times larger than that of the vertical eddy conductivity \( K_v \). Although the vertical temperature gradient, \( \frac{\partial T}{\partial z} \), is considerably larger than the horizontal temperature gradient, \( \frac{\partial T}{\partial x} \), the horizontal transport, \( q_h \), may still be of the same order as the vertical transport, \( q_v \), since the product of \( K_h \) and \( \frac{\partial T}{\partial x} \) is essential. This appears to be the case in reality so that the lateral mixing is no less important than the vertical (Defant, 1961, p. 105).

On the basis of the above discussion, if the \( x \)-axis is taken as the direction of the turbulent flow, \( v = w = 0 \) and \( f = u \), and all quantities constant in the \( y \)-direction, then the temperature distribution should satisfy the heat transport Equation (8) in the following form

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = a_h \frac{\partial^2 T}{\partial x^2} + a_v \frac{\partial^2 T}{\partial z^2}
\]

provided that \( K_h \) and \( K_v \) are assumed to be constants and \( S = 0 \).

Here \( a_h = K_h / \rho_c \) and \( a_v = K_v / \rho_c \) are the horizontal and vertical diffusivities respectively.

To solve Equation (49) with boundary conditions (47a, b, c), the
following technique is used. Let

\[ T(x, z, t) = T_1(x, z) + T_2(x, z, t) \]

where \( T_1 \) satisfies

\[ u \frac{\partial T_1}{\partial x} = a_h \frac{\partial^2 T_1}{\partial x^2} + a_v \frac{\partial^2 T_1}{\partial z^2} \]  \hspace{1cm} (50) \]

with the boundary conditions

\[ a \hspace{1cm} T_1 = T_0, \hspace{1cm} z = h \]
\[ b \hspace{1cm} q = q_0, \hspace{1cm} z = 0 \]
\[ c \hspace{1cm} T_1 = 0, \hspace{1cm} x = 0 \]

and \( T_2 \) satisfies

\[ \frac{\partial T_2}{\partial t} + u \frac{\partial T_2}{\partial x} = a_h \frac{\partial^2 T_2}{\partial x^2} + a_v \frac{\partial^2 T_2}{\partial z^2} \]  \hspace{1cm} (52) \]

with the boundary conditions

\[ a \hspace{1cm} T_2 = 0, \hspace{1cm} z = h \]
\[ b \hspace{1cm} q_2 = 0, \hspace{1cm} z = 0 \]
\[ c \hspace{1cm} T_2 = A \exp(i\omega t), \hspace{1cm} x = 0 \]

Therefore the solution of Equation (49) is obtained in the form
\[
T = T_o + \frac{q_o (h-z)}{K_v} + \sum_{n=1}^{\infty} D_n \exp \left( \frac{u x}{2a_h} - n_1 x \right) \cos \frac{(2n-1)\pi z}{2h} + \sum_{n=1}^{\infty} C_n \exp \left( \frac{u x}{2a_h} - d_1 x + i\omega t \right) \cos \frac{(2n-1)\pi z}{2h}
\]

where

\[C_n = \frac{(-1)^{n+1} 4A}{(2n-1)\pi}\]

\[D_n = \frac{(-1)^{n+1} 4T_o}{(2n-1)\pi} + \frac{8hq_o}{(2n-1)^2 \pi^2 K_v}\]

\[n_1^2 = \frac{u^2}{4a_h^2} + \frac{(2n-1)^2 \pi^2 a_v}{4h^2 a_h}\]

\[d_1 = \sqrt{n_1^2 + i\omega / a_h}\]

The first two terms on the right hand side of the above equation are solutions of Equation (50) and the last term is the solution of Equation (52).

In order to determine the effect of the climatic variation on ocean bottom temperature, one can use Equation (54) so that the amplitude of the temperature variation is as follows:
\[ |T_2|^2 = \sum_{n=1}^{\infty} C_n \exp\left(\frac{ux}{2a_h} - p_1 x\right) \cos p_2 x \cos \left(\frac{(2n-1)\pi z}{2h}\right)^2 \]

\[ + \sum_{n=1}^{\infty} C_n \exp\left(\frac{ux}{2a_h} - p_1 x\right) \sin p_2 x \cos \left(\frac{(2n-1)\pi z}{2h}\right)^2 \]  \hspace{1cm} (55)

where

\[ p_1 = \left(n_1 + \frac{\omega}{2}\right)^2 \cos \frac{\theta}{2} \frac{1}{a_h} \]

\[ p_2 = \left(n_1 + \frac{\omega}{2}\right)^2 \sin \frac{\theta}{2} \frac{1}{a_h} \]

with

\[ \theta = \tan^{-1} \frac{\omega}{\frac{2}{n_1 a_h}} \]

By using Equation (55), the computed results, at \( z = 0 \), are shown in Figures 6, 7 and 8. In the computation, \( \omega = 10^{-6} \) sec\(^{-1} \) and \( \omega = 10^{-8} \) sec\(^{-1} \), \( a_v = 1 \) cm/sec, \( a_h = 10^4 \) cm\(^2\)/sec and \( 10^8 \) cm\(^2\)/sec, and \( u = 0.1 \) cm/sec. From these figures, it can be seen that for periods less than 70 days the amplitude of the variation of ocean bottom temperature is very small beyond \( x = 1000 \) km and the influence will be uniformly extended to the whole ocean bottom when periods are greater than 2000 years.

In the same problem, a second model will also be considered. Let the temperature oscillation at \( x = 0 \) start at \( t = 0 \) and let the
Figure 6. Amplitude of the variation of the ocean bottom temperature for $a_h = 10^8 \text{ cm}^2/\text{sec}$. 
Figure 7. Amplitude of the variation of the ocean bottom temperature for $a_h = 10^4 \text{ cm}^2/\text{sec}$. 
Figure 8. Amplitude of the variation of ocean bottom temperature for different depths of the ocean bottom layer. The numbers on the curves are values of the depth in meters.
The initial temperature be

\[ T = T_0 + \frac{q_o (h-z)}{K_v} \]  

(56)

The temperature distribution in this case satisfies Equation (49) with boundary conditions (47a, b, c) and initial condition (56). Thus, one can obtain the solution by using Laplace transform techniques. The solution is in the form

\[ T = T_0 + \frac{q_o (h-z)}{K_v} + \sum_{n=1}^{\infty} \frac{1}{2} C_n \left\{ \exp\left[ \frac{ux}{2a_h} - d_1 x + i\omega t \right] \text{erfc}\left( \frac{x-d_1 a_h t}{2\sqrt{a_h t}} \right) \right. \]

\[ + \exp\left[ \frac{ux}{2a_h} + d_1 x + i\omega t \right] \text{erfc}\left( \frac{x+d_1 a_h t}{2\sqrt{a_h t}} \right) \cos \left( \frac{(2n-1)\pi z}{2h} \right) \]

\[ + \frac{1}{2} D_n \left[ \exp\left[ \frac{ux}{2a_h} - n_1 x \right] \text{erfc}\left( \frac{x-n_1 a_h t}{2\sqrt{a_h t}} \right) \right. \]

\[ + \exp\left[ \frac{ux}{2a_h} + n_1 x \right] \text{erfc}\left( \frac{x+n_1 a_h t}{2\sqrt{a_h t}} \right) \cos \left( \frac{(2n-1)\pi z}{2h} \right) \]  

(57)

where \( C_n, D_n, n_1, d_1 \) have been defined in Equation (54).

Because of the complex error function shown in the above equation, the computation is very complex. However, if \( t \) is assumed to be very large, Equation (57) can be reduced to Equation (54).

The third model of this problem is similar to the first model except the top surface boundary is changed to
In order to find a solution of Equation (49) with boundary conditions (47a, c) and (58), let

\[ T(x, z, t) = C_1 + C_2 x + C_3 z + C_4 z^2 + U(x, z) + V(x, z, t) \]

where \( U(x, z) \) and \( V(x, z, t) \) both satisfy Equation (49). Thus

\[
T = \frac{q_o}{K_v} (-z + \frac{z^2}{2h}) + \frac{q_o a_h x}{h u K_v} + \sum_{n=0}^{\infty} D_n' \exp \left[ \left( \frac{u}{2a_h} - n_1 \right) x \right] \cos \frac{n \pi z}{h} \]

\[
+ A \exp \left[ \left( \frac{u}{2a_h} - \sqrt{\frac{u^2}{4a_h^2} + \frac{i \omega}{a_h}} \right) x + i \omega t \right] \]

(59)

where

\[
D_n' = \frac{2}{h} \int_0^h q_o K_v (z - \frac{z^2}{2h}) \cos \frac{n \pi z}{h} dz
\]

The above equation shows that the term corresponding to the effect due to the climatic change over the surface is independent of the depth \( h \) of the layer of water. This was to be expected since no heat flow has been assumed at \( z = h \).
THE EFFECT OF TOPOGRAPHY AND SEDIMENTS ON
OCEAN FLOOR HEAT FLOW

Introduction

The simplest model assumed in the interpretation of oceanic heat flow is an isothermal, flat, oceanic bottom of infinite horizontal extent. Since, in many regions the topography of the ocean bottom is quite irregular, the flat isothermal surface is an oversimplification. Moreover, in many oceanic regions the crystalline basement is covered by sediments of a variable thickness. The thermal gradient will be disturbed by the irregular surface and because of the difference of the thermal conductivities of the basement and the sediments. In this section, the effect of the topography and the non-uniform thickness of sediments will be discussed.

For the discussion of the topographic effect on the heat flow, the bottom is assumed to be an isothermal surface with zero temperature and the temperature below the ocean bottom is assumed to satisfy the potential equation

$$\text{div}(K \text{ grad } T) = 0$$

(60)

when no heat sources are present.

The boundary conditions are

$$T = 0, \quad \text{on the surface of ocean bottom},$$
\[ T = g_0 z, \quad \text{for large } z. \]

In order to get an exact solution for this kind of problem, the form of the ocean floor can in some cases be considered as having a regular geometric shape, such as a semi-spherical depression or a semi-ellipsoidal depression in a flat ocean bottom. This kind of problem can be easily solved by using the known solutions of potential theory.

**An Exact Solution**

The exact solution for a semi-ellipsoidal depression in a homogeneous semi-infinite solid will be considered below. Since the problem can be reduced to the problem of a perfect ellipsoidal conductor placed in a linear potential field, the solution can be obtained by using the same method used for deriving Equation (24). Thus, the temperature distribution satisfying the boundary conditions (11) and (12) is in the form

\[ T = g_0 x - \frac{F(U_1)g_0 x}{F(N)} \]  \hspace{1cm} (61)

where the \( x \)-axis, and therefore the shortest semi-axis, is taken in the vertical direction and \( F(t) \) is defined as

\[ F(t) = \int_{t}^{\infty} \frac{du}{(u - a)^2 \sqrt{(u - a)(u - b)}} \]  \hspace{1cm} (62)
The corresponding thermal gradient on the surface of the ellipsoidal depression is

\[ g = \frac{g_0}{LMNF(N)} \]  \hfill (63)

\( F(t) \) can be obtained by evaluating Equation (62). The result is

\[ F(t) = \frac{(t^2-N^2+M^2)^{\frac{1}{2}}}{t(M^2-L^2)(t^2-N^2+L^2)^{\frac{1}{2}}} - \frac{E(\phi, k)}{(N^2-L^2)^{\frac{1}{2}}(M^2-L^2)} \]  \hfill (64)

where

\[ k^2 = \frac{N^2-M^2}{N^2-L^2} \]

\[ \phi = \sin^{-1}\frac{(N^2-L^2)^{\frac{1}{2}}}{t} \]

\( L, M, N \) and \( E(\phi, k) \) have been defined in the section on instrument effect.

**Irregular Surface of the Ocean Floor**

The disturbance due to an irregular topography will also be discussed by seeking an approximate solution. The topography for this case is shown in Figure 9. The plane \( z = 0 \) is assumed to be tangent to the highest peak of the irregular surface and this plane is regarded as the surface of a homogeneous solid with the undisturbed or zeroth order temperature distribution.
\[ T_0 = g_0 z \]

In many practical cases the deviation of the real surface from the plane \( z = 0 \) is not great; that is, \( h(x, y) \) is small compared to the horizontal scale of the irregularities. Moreover, the real surface can be considered to be smooth and \( h(x, y) \) can be assumed to be a slowly varying continuous function. The maximum deviation \( H \) is therefore assumed to be small compared to the horizontal scale.

Let the temperature field in the earth be

\[ T = g_0 z + T_1(x, y, z) \] (65)
where $T_1$ is the disturbance caused by the topography. Since

$T = 0$ on the real surface, $z = h(x, y)$, then

$$T_1(x, y, h) = -gh$$

Since $h(x, y)$ is small and slowly varying one can, in the first order approximation, assume that $T_1(x, y, z)$ is also a slowly varying function. Hence one can assume

$$T_1(x, y, 0) = -g_0 h, \text{ at } z = 0 \quad (66)$$

Now the problem is equivalent to the problem of solving

$$\nabla^2 T_1 = 0$$

in a semi-infinite medium with the surface boundary condition (66).

The solution is then in the form (Duff and Naylor, 1966, p. 276)

$$T_1(x, y, z) = -\frac{g_0}{2\pi} \int \frac{z h(x', y')}{{r'}^3} dx'dy' \quad (67)$$

where

$$r^2 = (x-x')^2 + (y-y')^2 + z^2$$

The temperature distribution due to the irregular topography is then in the approximate form

$$T(x, y, z) = gz - \frac{g_0}{2\pi} \int \frac{z h(x', y')}{{r'}^3} dx'dy' \quad (68)$$
which holds for \( z \geq h(x, y) \).

The corresponding thermal gradient can, in the first approximation, be expressed as

\[
\frac{\partial T}{\partial z} = g_o - \frac{g_o}{2\pi} \int \frac{h(x', y')}{r^3} dx' dy', \quad z \geq h
\]

(69)

In general, the above integral has to be evaluated by numerical methods. As a simple example where an exact evaluation is possible we will consider the case below.

Let us assume an elongated depression of the form

\[
h(x, y) = H(1 - x^2/L')^2.
\]

In this case an evaluation of Equation (69) gives

\[
g = g_o + \frac{g_o H}{\pi L'} \left[ \left( z - \frac{x^2}{z} \right) (\tan^{-1} \frac{z}{x+L'} - \tan^{-1} \frac{z}{x-L'}) + 2L' + x \ln \left( \frac{(x-L')^2 + z^2}{(x+L')^2 + z^2} \right) \right] \\
+ \frac{g_o H}{\pi z} (\tan^{-1} \frac{z}{x+L'} - \tan^{-1} \frac{z}{x-L'}) \quad z \geq h
\]

(70)

The thermal gradient at \((0, 0, H)\) becomes

\[
g = g_o + \frac{2g_o}{\pi} \left( \frac{H}{L'} + \tan^{-1} \frac{H}{L'} \right)
\]

(71)

where the higher order terms have been omitted.

This result can be compared with the exact solution of the ellipsoidal depression where \( N >> M >> L \). By using Equation (63), one
finds the thermal gradient to be of the following form

\[ g = g_o \left[ 1 + \frac{L}{M} \left( \frac{L}{M} \right)^2 + \left( \frac{L}{M} \right)^3 \right] \quad (72) \]

For the case of \( L/M = H/L' = 0.1 \), Equation (71) gives a gradient anomaly of \( 0.128g_o \) whereas Equation (72) gives about \( 0.1g_o \).

The results are comparable as is to be expected.

![Figure 10. Ocean bottom depression described by the parabolic function \( z = h(x) = H(1-x^2/L'^2) \).]

**Basement Covered by Sediments with a Variable Thickness and a Flat Surface**

The effect on the heat flow of a basement covered by sediments will also be discussed below. This kind of problem has been discussed by Bullard et al. (1956), Von Herzen and Uyeda (1963) and Lachenbruch and Marshall (1966). The solution for the problem of a basement surface of an arbitrary shape has not yet been worked out;
however, some cases where the surface is of a regular geometric shape have been solved. For example, Von Herzen and Uyeda (1963) give the solution for the case of a semi-spheroidal depression filled with sediments. This case can be treated by the method given in the section on instrument effect.

Since the difference in the conductivities of the sediments and the basement is not great, the perturbation method discussed in the general introduction can be applied to the case where the surface of the basement is irregular. The approximate solution is given by Equation (17). By assuming $\overline{T} = 0$, one obtains for the thermal gradient:

$$g = g_0 - \frac{1}{4\pi a_0} \int \int \int \frac{\partial a_1}{\partial z'} \frac{\partial T_0}{\partial z'} \frac{(z-z') dx' dz' dy'}{r_1^3}$$

where

$$r_1^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

In order to satisfy the condition of $T = 0$ at $z = 0$, the integration has to be carried over the whole space with an image in the upper half plane.

Let $K_1$ and $K_2$ be the thermal conductivities of the basement and the sediments respectively. One assumes that $K_1$ and $K_2$ and $\rho c$ are constants. Then by using the delta function one writes
\[ \frac{\partial a_1}{\partial z} = \frac{a_0}{K_1} \frac{\partial K}{\partial z} = -\frac{a_0}{K_1} (K_2 - K_1) \delta(z-h) \]  

(74)

Inserting (74) into Equation (73) and integrating with respect to 
\( z' \) from \(-\infty\) to \(+\infty\), because of the image in the upper half plane, 
the surface thermal gradient is obtained in the form

\[ g_{z=0} = g_0 - \frac{(K_2 - K_1)g_0}{2\pi K_1} \int \frac{h(x', y')dx'dy'}{r_1^3(h)} \]

(75)

where

\[ r_1^2(h) = (x-x')^2 + (y-y')^2 + h^2(x', y') \]

and \( h(x, y) \) is defined similar to \( h(x, y) \) shown in Figure 9.

**Basement Covered by Sediments with a Variable Thickness and an Irregular Surface**

A more general situation concerning topography and sediments, 
shown in Figure 11, will also be considered. One can derive an ap- 
proximate solution for this situation by using the same procedure as 
was used in deriving the solution for the case of a free irregular 
basement surface. The plane \( z = 0 \) is again assumed to be just 
above the highest peak of the irregular surface of a homogeneous 
semi-infinite solid with a thermal conductivity \( K_1 \) and the undis- 
turbed zeroth order temperature distribution.
Figure 11. Basement covered with sediments of a variable thickness.

The disturbed temperature is

\[ T = T_0 + T_1(x, y, z) \]

where \( T_1(x, y, z) \) is a small perturbation of the temperature due to the sediments and the irregular surface. The temperature at \( P \) is

\[ T(P) = g_0 (h_1 + h_2) + T_1(x, y, h_1 + h_2) \]  

In the zeroth order approximation, one has
\[
\frac{T(P) - T(Q)}{h_2} = \frac{K_1 g_0}{K_2}
\]

and since \( T(Q) = 0 \), therefore

\[
T(P) = \frac{K_1 g_0 h_2}{K_2}
\]

(77)

Hence from Equations (76) and (77), one has

\[
T_1(x, y, h_1 + h_2) = (\frac{K_1}{K_2} - 1)g_0 h_2 - g_0 h_1
\]

(78)

Since \( h_1(x, y) \) and \( h_2(x, y) \) are assumed to be small and slowly varying functions, one can in the first approximation put

\[
T_1(x, y, 0) = (\frac{K_1}{K_2} - 1)g_0 h_2 - g_0 h_1
\]

Now the problem can again be treated as the case of a semi-infinite solid with a given temperature distribution at the boundary.

Hence

\[
T(x, y, z) = g_0 z - \frac{(K_2 - K_1)g_0}{2\pi K_2} \int \int \frac{zh_2(x', y') dx' dy'}{r^3} - \frac{g_0}{2\pi} \int \int \frac{zh_1(x', y') dx' dy'}{r^3} \quad z \geq h_1 + h_2
\]

(79)

where
\[ r^2 = (x-x')^2 + (y-y')^2 + z^2 \]

The corresponding thermal gradient is

\[
g = g_o - \frac{(K_2 - K_1)g_o}{2\pi K_2} \int \int \frac{h_2(x', y') \, dx' \, dy'}{r^3} - \frac{g_o}{2\pi} \int \int \frac{h_1(x', y') \, dx' \, dy'}{r^3} \tag{80} \]

\[ z \geq h_1 + h_2 \]

Figure 12 is used as an example to illustrate the case where

\[ h_1 = H_1(1 - \frac{x^2}{L_1^2}), \quad -L_1 < x < L_1, \quad -D < y < D \]

\[ h_2 = \left( \frac{H_1 - H_2}{L_2^2} + \frac{H_1}{L_1^2} \right) x^2 + H_2, \quad -L_1 < x < L_1, \quad L_2 > L_1 \]

By assuming \( D \gg L_1 \gg H_1 \) and integrating with respect to \( y \) in (80), one obtains

\[
g = g_o - \frac{(K_2 - K_1)g_o}{2\pi K_2} \int_{-L_1}^{L_1} \left[ \frac{H_1 - H_2}{L_2^2} + \frac{H_1}{L_1^2} \right] x'^2 + H_2 \left( \frac{\left( \frac{H_1 - H_2}{L_2^2} + \frac{H_1}{L_1^2} \right) x'^2 + H_2}{(x-x')^2 + z^2} \right) \, dx' \]

\[- \frac{g_o}{2\pi} \int_{-L_1}^{L_1} \frac{2H_1(1 - \frac{x^2}{L_1^2}) \, dx'}{(x-x')^2 + z^2} \quad z \geq h_1 + h_2 \tag{81} \]
The evaluation of these integrals gives

\[
g = g_0 - \frac{(K_2 - K_1)g_0}{\pi K_2} \left\{ \left[ \frac{H_1 - H_2}{L_2^2} + \frac{H_1}{L_1^2} \right] \left( \frac{x}{z} - z \right) + \frac{H_2}{z} \right\} \left[ \tan^{-1} \frac{z}{x - L_1} \right.
\]

\[
- \tan^{-1} \frac{z}{x + L_1} \right] + \left( \frac{H_1 - H_2}{L_2^2} + \frac{H_1}{L_1^2} \right) \left( 2L_1 + x \ln \frac{(x - L_1)^2 + z^2}{(x + L_1)^2 + z^2} \right)
\]

\[
- \frac{g_0}{\pi} \left\{ \frac{H_1}{L_1^2} \left[ 2L_1 + x \ln \frac{(x - L_1)^2 + z^2}{(x + L_1)^2 + z^2} \right] + \left[ \frac{H_1}{L_1^2} \frac{2}{z} \left( \frac{x}{z} - z \right) \right] \left[ \tan^{-1} \frac{z}{x - L_1} - \tan^{-1} \frac{z}{x + L_1} \right] \right\} \quad z \geq h_1 + h_2 \quad (82)
\]

In order to determine the magnitude of this effect, the thermal gradient at \( x = 0 \) and \( z = H_1 + H_2 \) is evaluated. Thus

\[
g = g_0 - \frac{2(K_2 - K_1)g_0}{\pi K_2} \left\{ \left( - \frac{H_2}{H_1 + H_2} \right) \tan^{-1} \frac{H_1 + H_2}{L_1} \right.
\]

\[
+ \left( \frac{H_1 - H_2}{L_2^2} + \frac{H_1}{L_1^2} \right) \left[ L_1 + (H_1 + H_2) \tan^{-1} \frac{H_1 + H_2}{L_1} \right]
\]

\[
+ \frac{2g_0}{\pi} \left\{ \frac{H_1}{L_1} + \left( \frac{H_1^2 + H_1 H_2}{L_1^2} + \frac{H_1}{H_1 + H_2} \right) \tan^{-1} \frac{H_1 + H_2}{L_1} \right\} \quad (83)
\]

For numerical illustration, let \( H_2 = 4H_1, \ L_1 = 10H_1 \).
\[ L_2 = 20H_1 \quad \text{and} \quad K_2/K_1 = 0.6. \]  The calculation in this case indicates that the thermal gradient at \((0, 0, H_1 + H_2)\) decreases by about 14.2 percent due to the sediment effect and increases by about 13.5 percent due to the topographic effect; that is, the combined topographic effect and sediment effect lowers the total thermal gradient by about one percent. The heat flow at \((0, 0, H_1 + H_2)\) also decreases by about one percent in this case. Since \(h_1\) and \(h_2\) are slowly varying functions, therefore the heat flow at \((0, 0, H_1)\) is expected to be 0.99\(g_0\) but the thermal gradient is 1.67\(g_0\).

\begin{figure}
\centering
\begin{tikzpicture}
\draw[->] (-3,0) -- (3,0) node[right] {x};
\draw[->] (0,-2) -- (0,2) node[below] {z};
\draw (-1.5,1) parabola bend (1.5,1) (3,0);
\draw (-1.5,0) -- (-1.5,1) node[midway, above] {Sediments, \(K_2\)};
\draw (1.5,0) -- (1.5,1) node[midway, above] {Rock, \(K_1\)};
\draw (-2,0) -- (-2,1) node[midway, below] {(-2,0,2H_1)};
\draw (2,0) -- (2,1) node[midway, below] {(2,0,2H_1)};
\draw (-1.5,1) -- (-1.5,0) node[midway, left] {(-L_1,0,0)};
\draw (1.5,1) -- (1.5,0) node[midway, right] {(L_1,0,0)};
\draw (-1.5,0) -- (1.5,0) node[midway, above] {(0,0,H_1+H_2)};
\draw (-2,0) -- (2,0) node[midway, above] {(L_2,0,2H_1)};
\end{tikzpicture}
\caption{Sediment and basement surface both described by parabolic functions.}
\end{figure}
In order to study an example of the solution (80) for the case $H_1 = 0$, a sediment filled parabolic depression will be considered. By using Equation (83), and letting $L_2 = L_1$, $H_1 = 0$ one arrives at the following form for the disturbed heat flow at $z = H_2$:

$$q = q_o + \frac{2(K_2 - K_1)q_o H_2^2}{\pi K_2} \left[ \frac{H_2}{L_2} + \left(1 + \frac{H_2^2}{L_2^2}\right) \tan^{-1} \frac{H_2}{L_2} \right]$$  

(84)

If $H_2/L_2 = 0.1$ and $K_2/K_1 = 0.6$, then $q = 0.915q_o$; that is, the heat flow decreases by about 8.5 percent at $(0, 0, H_2)$. Since the parabolic surface is assumed to be slowly varying, the heat flow at the surface of the sediments should decrease by about eight percent.

This result can now be compared to the case where the depression is assumed to be an elliptical cylinder depression which should give similar results. For this case, the thermal gradient in the sediments can be derived in a manner similar to that used in the deriving Equation (24) in the section on instrument effect. Thus the heat flow is

$$q = \frac{K_2}{K_1} q_o \frac{K_2}{1 + (\frac{L_2}{K_1} - 1)LMN} F(N)$$  

(85)

where $L, M, N$ are the semi-axes of the ellipsoid. Since the shortest axis is in the vertical direction and $N >> M >> L$, $F(N)$ can be
found by using (64). Hence

\[
q = \frac{K_2}{K_1} q_o \frac{1}{1+(\frac{K_2}{K_1}-1)(1-\frac{L}{M}+\frac{L^2}{M^2}-\frac{L^3}{M^3})}
\]  

(86)

Using \( L/M = 0.1 \) and \( K_2/K_1 = 0.6 \) in Equation (86), it was found that \( q = 0.94q_o \); that is heat flow at the surface of sediments decreases by about six percent. This indicates that the result obtained by using Equation (80) is comparable with the result obtained on the basis of an exact solution for the case of the ellipsoid provided that \( \frac{L}{M} \), \( \frac{H}{L} \) are small.
SCATTERING OF HEAT FLOW BY NON-UNIFORM CONDUCTIVITY

In this section we will discuss the effects of a non-uniform conductivity on the outward heat flow in a semi-infinite solid. The discussion will be restricted to the case of a steady state scattering due to a body embedded in a homogeneous and isotropic semi-infinite solid with conductivity $K_1$. The body is assumed to have the conductivity $K_2$. At great depth the outward heat flow is assumed to be uniform. The temperature at the surface of the solid is assumed to be zero. In order to satisfy the surface boundary condition, the half-space is extended and an image object in the above upper half-space is introduced. Exact solutions to this type of problem are in general very difficult to obtain.

In order to get approximate solutions, one can consider the body and its image as two isolated, non-interacting bodies placed in a uniform heat flow field. The temperature distribution of either one of these is calculated on the assumption that the other one is absent; that is, the solution for a conductor placed in a linear potential field can be used.

Approximate solutions obtained under this assumption satisfy the boundary condition (12) but do not satisfy the boundary conditions on the surface of the body. They can therefore not be applied to the cases of relatively shallow bodies.
Since the distortion of a linear potential field by a sphere, differing in conductivity from the surrounding solid, is equivalent to a field disturbed by a dipole placed at the center of the sphere, the approximate solution for the sphere in a half-space with zero surface temperature can be obtained by placing another dipole in the upper half-space at the center of the image body so that the surface boundary condition will be satisfied. The temperature in the solid is, by superimposing the two dipole solutions,

\[ T = g_0 z - \frac{g R^3 (K_1 - K_2)}{2K_1 + K_2} \left( \frac{h - z}{r^3} - \frac{h + z}{r_1^3} \right) \]  

(87)

\( r, r_1, h \) are defined in Figure 13.

Figure 13. A body and its image for a half-space.

The disturbed surface heat flow is found to be
\[
q_{z=0} = q_0 \left[ 1 - \frac{2(K_1-K_2)(2h^2-x^2)R_o^3}{(2K_1+K_2)(h^2+x^2)^{5/2}} \right]
\]  

(88)

According to Grant and West (1965, p. 425) the error involved in Equation (88) will be less than ten percent if \( h \geq 1.3R_o \).

Some results computed on the basis of Equation (88) are given in Figure 14.

A different type of approximate solution to the scattering problem can be obtained on the basis of the perturbation method. This will be considered below in relation to a spherical model and a finite vertical cylinder model. The general expression of the disturbed thermal gradient obtained by using the perturbation method has been given in Equation (17). If one assumes that \( t = 0 \), then, in the first approximation, the perturbation of the temperature is

\[
T_1 = + \frac{1}{4\pi a_o} \iiint \frac{\nabla T \cdot \nabla a_1 \, dV}{r_{PQ}} - \frac{1}{4\pi a_o} \iiint \nabla T_0 \cdot \nabla a_1 \, dV_{Q'}
\]

(89)

where \( r_{PQ} \) and \( r_{PQ'} \) are defined in Figure 15. The second term on the right hand side of the above equation is due to image solution so that the surface boundary condition (12) will be satisfied.

It has been mentioned that the method of perturbation can be applied only when the difference between \( K_1 \) and \( K_2 \) is small. In relation to this it is interesting to note that conductivity contrasts
Figure 14. Maximum disturbed surface heat flow at a point P due to a sphere with different thermal conductivity from its surrounding (see Figure 13). The numbers on the curves are values of $K_2/K_1$. 
may be very small at depths greater than 20 km (Lachenbruch and Marshall, 1966).

Figure 15. Spherical coordinates.

The perturbation solution for a sphere of thermal conductivity \( K_2 \) buried in the semi-infinite homogeneous medium of thermal conductivity \( K_1 \) is found here.

For a sphere in a homogeneous medium, one has

\[
\nabla T_0 \cdot \nabla a_1 = \frac{a_0}{K_1} g_0 \left( \frac{dK}{dr} \right) \cos \theta \tag{90}
\]

If \( K_1 \) and \( K_2 \) are assumed to be constant, then

\[
\nabla T_0 \cdot \nabla a_1 = \frac{a_0}{K_1} g_0 (K_1 - K_2) \delta (r - R_0) \cos \theta \tag{91}
\]
Therefore the first order perturbation of temperature is (see Figure 15).

\[
T_1 = \frac{-g_o(K_2-K_1)}{4\pi K_1} \iiint \frac{\delta(r'-R_o) \cos \theta' \sin \theta' r'^2 dr' d\theta' d\phi'}{(r^2 + r'^2 - 2rr' \cos \gamma)^{\frac{1}{2}}} \tag{92}
\]

Since

\[
(r^2 + r'^2 - 2rr' \cos \gamma)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos \gamma), \quad \text{for} \quad r > r' \tag{93}
\]

Equation (92) can be expressed as an infinite series. Thus

\[
T_1 = \frac{-g_o(K_2-K_1)}{4\pi K_1} \frac{R^2}{r} \sum_{n=0}^{\infty} \frac{R^n}{r^n} \iiint P_n(\cos \gamma) \cos \theta' \sin \theta' d\theta' d\phi' \tag{94}
\]

for \( r > R_o \)

where \( P_n(\cos x) = \) Legendre polynomial.

According to the addition theorem for Legendre functions (Grant and West, 1965, p. 223), one has

\[
P_n(\cos \gamma) = P_n(\cos \theta) P_n(\cos \theta') \tag{95}
\]

provided that \( T \) is independent of \( \phi \).

Inserting Equation (95) into (94) and integrating, one obtains the solution for the sphere in a half-space.
Here the summation has been terminated at \( n = 2 \) under the assumption that \( r \) and \( r_1 \) are much greater than \( R_o \).

The corresponding disturbed thermal gradient at surface is

\[
T_1 = \frac{g_o (K_2 - K_1) R_o^3}{3K_1} \left( \frac{h-z}{r^3} - \frac{h+z}{r_1^3} \right)
\]

\( (96) \)

Equation (97) and Equation (88) have the same form except that the coefficients differ. This is because the higher order terms in Equation (96) are neglected.

In order to compare solution (97) with solution (88), computations for both solutions have been made for the case of \( h = 2R_o \).

The results for the maximum heat flow anomaly at surface are summarized in Table 2.

<table>
<thead>
<tr>
<th>( \frac{K_2}{K_1} )</th>
<th>Maximum heat flow anomaly as percent of mean heat flow</th>
<th>difference in percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>perturbation method</td>
<td>method according to Equation (88)</td>
</tr>
<tr>
<td>0.5</td>
<td>-8.33</td>
<td>-10.00</td>
</tr>
<tr>
<td>0.7</td>
<td>-5.00</td>
<td>-5.55</td>
</tr>
<tr>
<td>0.9</td>
<td>-1.67</td>
<td>-1.72</td>
</tr>
<tr>
<td>1.1</td>
<td>1.67</td>
<td>1.61</td>
</tr>
<tr>
<td>1.3</td>
<td>5.00</td>
<td>4.55</td>
</tr>
<tr>
<td>1.5</td>
<td>8.33</td>
<td>7.15</td>
</tr>
</tbody>
</table>

Table 2. Comparison of the perturbation solution with solution (88).
If the approximate solution (88) is considered to be reliable at the depth $h = 2R_o$, then the calculations in Table 2 indicate that the perturbation solution (97) can be applied when

$$0.7 \frac{K_2}{K_1} < 1.3$$

provided that the solutions (88) and (97) agree to within ten percent. This was expected because the perturbation method assumes that $K_2$ and $K_1$ differ by a small amount.

The example of a finite vertical cylinder may also be used to illustrate the perturbation method. For the case of a vertical cylinder in half-space, the disturbed surface thermal gradient, obtained by using Equation (89), is

$$g_{z=0} = g_o + \frac{g_o}{2\pi K_1} \int_0^{R_o} \int_0^\infty \int_0^{2\pi} \frac{\partial K}{\partial z'} r'^2 dr' dz' d\theta' \frac{z'}{[(r-r')^2 + z'^2 + 4r'^2 \sin^2 \frac{1}{2}(\theta - \theta')]}^{3/2}$$

(98)

The cylindrical coordinates are defined in Figure 16.

The above equation is difficult to evaluate. However for the case $r = 0$ and $z = 0$, the gradient can be expressed as

$$g_{z=0} = g_o + \frac{g_o}{K_1} \int_0^{R_o} \int_0^\infty \int_0^{2\pi} \frac{\partial K}{\partial z'} z' r'^2 dr' dz'$$

(99)
In order to evaluate Equation (99), one assumes \( K_1 \) and \( K_2 \) to be constants. Hence

\[
\frac{\partial K}{\partial z} = \begin{cases} 
0 & z \neq h_1 \text{ or } z \neq h_2 \\
(K_2 - K_1)\delta(z-h_1) & z = h_1 \\
(K_1 - K_2)\delta(z-h_2) & z = h_2 
\end{cases}
\]  

(100)

Substituting (100) into Equation (99), one has

\[
g_{z=0} = g_o + \frac{(K_2 - K_1)g_o}{K_1} \left[ \frac{h_2}{\sqrt{R_o^2+h_2^2}} - \frac{h_1}{\sqrt{R_o^2+h_1^2}} \right] 
\]  

(101)

We will now compare heat flow anomalies caused by a sphere
and a finite vertical cylinder when these bodies have the same radius, the same volume, and are placed at the same depth (from the surface to the center of the body). In the case of \( h = 2R_o \), the ratio of the sphere anomaly to the cylinder anomaly is found to be about 1.2. This means that the maximum effect of the sphere is about 20% higher than the maximum effect of the vertical cylinder. The reason for this is that the depth of the top of the cylinder is greater than the depth of the top of the sphere. If the tops of the sphere and vertical cylinder are placed at same level, the effect of the cylinder will be larger than that of the sphere.
EFFECT OF MAGMATIC INTRUSIONS ON SURFACE HEAT FLOW

The total volume of magma transported to the earth's surface since Pre-Cambrian era has been estimated to be $3 \times 10^7$ km$^3$ (Bodvarsson, 1966, p. 75). This is a relatively small amount of material and the average rate of heat flow transport by volcanism is therefore not great. Volcanism is confined to certain relatively small active belts where the transport per unit area is substantial. The greatest activity is in the circum-Pacific belt and on the Mid-Atlantic Ridge. The volcanic belts are very narrow. For example, the active belt of the Mid-Atlantic Ridge appears to be only 30 km to 50 km wide. But the upward transport of magma within this narrow zone is very great, and the average rate of transport of heat by magma may be significant.

The effect of magma transport on heat flow will be discussed by using an idealized model. One assumes that the process starts at $t = 0$ that the individual intrusions, such as dikes and sills, are relatively small and are being formed at a constant rate in space and time. Moreover, it is assumed that the continuous intrusions are uniformly distributed. On the other hand, the volume change due to these intrusions is assumed to be negligible.

Let $M$ be the mass intrusion per unit time and per unit volume of country rock, $L$ the latent heat of melting, $T_m$ its
melting temperature. Also, let the density and specific heat of the solidifying lava be equal to that of the country rock. If the heat released by the solidifying lava is conducted rapidly into the surrounding rock, the temperature $T$ will be a function of the depth only. This results from the smallness and uniform distribution of the intrusions. Under these assumptions the heat transport equation takes the simple form (Bodvarsson, 1966, p. 77)

$$\rho c \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial z^2} + M[L + c(T_m - T)]$$

(102)

where $T$ is the temperature of the rock. The second term on the right hand side of the above equation is the rate of transport of heat into the country rock. Equation (102) can be written

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2} + b(T_o - T)$$

(103)

where $b = M/\rho$, $T_o = T_m + L/c$. $b$ is assumed to be constant in space and time.

The boundary condition and initial condition are taken to be

$$T = 0, \quad \text{at} \quad z = 0$$

$$T = g_o z, \quad \text{when} \quad t = 0$$

To solve the above equation one makes the substitution
Thus

\[ T = T_0 + U(z, t) \exp(-bt) \]  

(104)

Thus

\[ \frac{\partial U}{\partial t} = a \frac{\partial^2 U}{\partial z^2} \]  

(105)

with

\[
\begin{align*}
\text{a) } & \quad U = -T_0 \exp(bt), \quad \text{at } z = 0 \\
\text{b) } & \quad U = -T_0 + g_0 z, \quad \text{when } t = 0
\end{align*}
\]

(106)

In order to solve Equation (105), let

\[ U(z, t) = V(z, t) + W(z, t) \]

where \( V \) satisfies

\[ \frac{\partial V}{\partial t} = a \frac{\partial^2 V}{\partial z^2} \]

with boundary and initial conditions

\[ V = 0 \quad \text{at } z = 0 \]

\[ V = -T_0 + g_0 z, \quad \text{when } t = 0 \]

and \( W \) satisfies

\[ \frac{\partial W}{\partial t} = a \frac{\partial^2 W}{\partial z^2} \]
with boundary and initial conditions

\[ W = 0, \quad \text{when} \quad t = 0 \]
\[ W = -T_o \exp(bt), \quad \text{at} \quad z = 0 \]

\( V \) can be found by using the formula given by Carslaw and Jaeger (1959, p. 59) and \( W \) can be found by using the formulae given by Carslaw and Jaeger (1959, p. 63) and Abramowitz and Stegun (1965, p. 304). Therefore the solution of Equation (103) is of the form

\[
T = T_o - T_o \text{erf} \left( \frac{z}{2\sqrt{at}} \right) \exp(-bt) + g_o z \exp(-bt) \\
- \frac{1}{2} T_o \exp \left( \frac{b}{\sqrt{a}} z \right) \exp \left( \frac{z+2\sqrt{abt}}{2\sqrt{at}} \right) \\
- \frac{1}{2} T_o \exp \left( -\frac{b}{\sqrt{a}} z \right) \text{erfc} \left( \frac{z-2\sqrt{abt}}{2\sqrt{at}} \right)
\] (107)

The corresponding disturbed surface heat flow is

\[
q_{z=0} = \frac{K(T_m + \frac{L_t}{\rho})}{\sqrt{\frac{M}{a \rho}}} \text{erf} \left( bt \right) + q_o \exp(-bt)
\] (108)

As already stated, the magmatic activity is generally confined to rather narrow belts in the crust and upper mantle. The total volume of the material intruded into the crust may, in the active belts, amount to a measureable fraction of the volume of the country rock. Let us assume that the dikes formed at a constant rate during a time \( t \) amount to a total of ten percent by volume of the country rock.
$K = 0.006 \text{ cal/cm sec } ^\circ \text{C}, \ a = 0.01 \text{ cm}^2/\text{sec}, \ \rho = 3 \text{ gm/cm}^3,$

$T_\infty = 1500 ^\circ \text{C} \ (\text{including latent heat}),$ and $q_0 = 1.5 \text{ units},$ then one can calculate the value of the surface heat flow at the time $t.$ The values of the heat flow are as follows:

$q_{z=0} = 3.17 \quad \text{for} \quad t = 0.1 \text{ m. y.}$

$q_{z=0} = 1.99 \quad \text{for} \quad t = 1 \text{ m. y.}$

$q_{z=0} = 1.60 \quad \text{for} \quad t = 10 \text{ m. y.}$

In many cases it is possible to assume that $T \ll T_m$ near the surface and also to neglect the last term in Equation (101). As an illustration of this case one considers the intrusions to be confined to a thin vertical section $-d \leq x \leq d, \ -\infty < y < \infty, \ h_1 \leq z \leq h_2$ (see Figure 17).

![Figure 17. Thin dikes confined in a slab.](image-url)
Since the heat flow is assumed to be uniformly flowing into this slab and to be quickly absorbed by the rock, the rise of temperature of the rock can be considered due to continuous heat source in it. The heat production, starting at $t = 0$, is now assumed to be at a constant rate per unit volume $ρcbT_o$. The slab is assumed to be in a medium with zero temperature everywhere. The temperature distribution is then (Carslaw and Jaeger, 1959, Chapter 10)

\[
T = \int_0^t \int_{-d}^d \int_{-h_1}^h \frac{bT_o}{4πa(t-t')} \exp\left(-\frac{(x-x')^2+(z-z')^2}{4a(t-t')}\right) dx'dz'dt' \quad (109)
\]

The corresponding surface heat flow anomaly is then obtained by differentiating $T$ in the above equation

\[
Δq_{z=0} = \frac{KT_b}{2πa} \int_{-d}^d \left[ E_1\left(\frac{(x-x')^2+h_1^2}{4at}\right) - E_1\left(\frac{(x-x')^2+h_2^2}{4at}\right) \right] dx' \quad (110)
\]

where $E_1(x)$ is the exponential integral and is defined as

\[
E_1(x) = \int_x^∞ \frac{e^{-t}}{t} dt \quad (111)
\]

In making an estimation, Equation (110) must be evaluated by numerical methods. However, if the value of $h_1$ is much larger than $d$, Equation (110) can be simplified as
Below is an example of the effect on heat flow for this case. If one assumes $h_1 = 4 \text{ km}$, $h_2 = 40 \text{ km}$, $d = 0.5 \text{ km}$, $a = 0.01 \text{ cm}^2/\text{sec}$, and intrusions of 50% by volume, then one can calculate the maximum heat flow anomaly at the time $t$. The high volume fraction of intrusions is assumed since they are confined to a relatively small total volume. Results are as follows:

\[
\Delta q_{\text{max}} = 0.32 \quad \text{when} \quad t = 0.1 \text{ m. y.}
\]

\[
\Delta q_{\text{max}} = 0.38 \quad \text{when} \quad t = 1.0 \text{ m. y.}
\]

\[
\Delta q_{\text{max}} = 0.08 \quad \text{when} \quad t = 10.0 \text{ m. y.}
\]

For the case where the intrusions are confined to the region $0 \leq x < \infty$, $-\infty < y < \infty$, $h_1 \leq z \leq h_2$ (see Figure 18), Equation (110) gives

\[
\frac{dq}{dx} = \frac{KT_o b}{2\pi a} \left[ E_1 \left( \frac{x^2 + h_1^2}{4at} \right) - E_1 \left( \frac{x^2 + h_2^2}{4at} \right) \right]
\]

(113)

where $\frac{dq}{dx}$ is the slope of the anomaly profile.

The maximum slope of the anomaly profile is at $x = 0$, thus

\[
\left( \frac{dq}{dx} \right)_{\text{max}} = \frac{KT_o b}{2\pi a} \left[ E_1 \left( \frac{h_1^2}{4at} \right) - E_1 \left( \frac{h_2^2}{4at} \right) \right]
\]

(114)
This equation indicates that if the maximum slope of the anoma-
ly profile is known, the depth to the intrusive zone can be estimated.

Figure 18. Thin dikes uniformly distributed in a semi-infinite
sheet.
The thermal gradient in ocean bottom sediments is usually obtained by a thermal probe as described in the chapter on instrument effect. It would appear possible to obtain the same data by measuring the temperature in a core taken from the bottom sediments. The core would have to be brought quickly to the surface in order that its temperature remains unaffected by changes in the ambient temperature. In order to investigate the possibility of this technique the variation of the temperature with time within a sediment core will be considered. The model used for this purpose is a very long cylinder of radius \( R_o \) surrounded by a partial insulation of thickness \( d \).

The temperature distribution within the cylindrical core satisfies the equation

\[
\frac{\partial T}{\partial t} = a \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right), \quad 0 \leq r < R_o
\]

provided that it contains no heat source and heat conduction along the axis is neglected.

The initial temperature of the sediments is \( T_0 \). Both \( T \) and \( T_0 \) can be slowly varying functions of the coordinate along the axis of the core. Since the rate of heat flow through the insulator is given as \( K_i(T-T_1)/d \), where \( T \) and \( T_1 \) are the temperatures inside
and outside of the insulation and $K_1$, its thermal conductivity (Carslaw and Jaeger, 1959, p. 20), the boundary condition is

$$K_1 \frac{\partial T}{\partial r} + \frac{K_1 T}{d} = 0, \text{ at } r = R_o$$

(116)

when $T_1$ is assumed to be zero. The heat capacity of the insulation is neglected.

The solution of Equation (115), satisfying the initial condition and boundary condition (116) is given as (Carslaw and Jaeger, 1959, p. 202)

$$T = 2A_1 T_o \sum_{n=1}^{\infty} \frac{J_o (\beta_n r/R_o)}{(\beta_n^2 + A_1^2) J_o (\beta_n)} \exp \left( - \frac{\beta_n^2 r}{R_o} \right)$$

(117)

where

$$A_1 = \frac{R_o K_i}{K_s d}$$

and $\beta_n$ are the roots of the equation

$$\beta J_1 (\beta) - A_1 J_0 (\beta) = 0$$

(118)

The values of $\beta_n$ depend on $A_1$. $\beta_1$ varies from about 0.4 to 2.1 as $A_1$ varies from 0.1 to 10 while the other $\beta$'s are large and vary slowly with $A_1$ (Carslaw and Jaeger, 1967, p. 493).

In order to estimate the time variation of the temperature at the center of the core, the following numerical values are considered:
\[ R_0 = 5 \text{ cm, 3 cm, 7 cm, } d = 2 \text{ cm, } K_1 = 0.15 \text{ K, and } \]
\[ a = 0.0035 \text{ cm}^2/\text{sec.} \]

The computed results are shown in Figure 19. From this figure, it can be seen that the temperature at the center of the sediment core changes quickly as time increases. For example, in the case of \( R_0 = 5 \text{ cm} \), the temperature drops by about five percent after the core is cooled for 17 minutes. However, if \( A_1 \) is made smaller while \( R_0 \) is kept large, the temperature at the center will decrease more slowly as time increases. Thus it appears possible to obtain the temperature gradient in the ocean bottom sediment from the measurements of temperature within the sediment core if the core can be brought to the surface within about 20 minutes. It should be noted that it may be necessary to take the frictional heat into consideration.

The case of the non-insulated core is treated by putting \( T = 0 \) for \( r = R_0 \). For this case, the temperature distribution is given as (Carslaw and Jaeger, 1959, p. 199)

\[
T = \frac{2T_0}{R_0} \sum_{n=1}^{\infty} \frac{J_0(a_n r)}{a_n J_1(a_n R_0)} \exp(-a_n^2 at) \tag{119}
\]

To satisfy the surface boundary condition at \( r = R_0 \), \( a_n \) must be a root of
Figure 19. Temperature variation at the center of a partially insulated sediment core. The numbers on the curves are values of the radius $R_o$ in cm.
The comparison of the results of the model without insulation with the results of the model with insulation is shown in Figure 20.

Temperature Variation in a Flowing Well with Variable Bottom Temperature

In connection with the above study of heat conduction in cylindrical coordinates the following problem has been considered. A well of radius $R_0$ and depth $h$ are given. A liquid enters the well at the bottom and flows out at the surface. The mass flow is $q_w$ in gm/sec and the bottom-hole temperature varies harmonically with time; that is, at the bottom

$$T_h = A \exp (i\omega t)$$

The problem is to derive the temperature of the liquid at the top.

It will be assumed that $h \gg R_0$ and that $\omega$ is small so that heat conduction in the direction of the axis of the well can be neglected. Moreover, due to turbulent exchange of heat in the fluid, the temperature of the liquid will be assumed constant over the cross section of the well.

Let $T_1(r, z, t)$ be the temperature of the rock surrounding the well and $T_2(z, t)$ be the temperature of liquid. At the given conditions, the temperature $T_1$ satisfies
Figure 20. Temperature variation at the center of the sediment core with and without insulation. $R_o = 3$ cm.
\[
\frac{\partial T_1}{\partial t} = a \left( \frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r} \frac{\partial T_1}{\partial r} \right), \quad T_1 = T_1(r, z, t), \quad r > R_0
\]  

(121)

with boundary conditions

\[
\begin{align*}
\text{a} & \quad T_1 = T_2, \quad \text{at } r = R_0 \\
\text{b} & \quad 2\pi K_1 R_0 \frac{\partial T_1}{\partial r} = c_w q_w \frac{\partial T_2}{\partial z}, \quad \text{at } r = R_0 \\
\text{c} & \quad T_1 = A_o \exp(i\omega t), \quad \text{at } r = R_0 \text{ and } z = h
\end{align*}
\]

(122)

where \( c_w \) is the specific heat of the liquid.

Let \( T_1 = V(r)W(z) \exp(i\omega t) \), then Equation (121) becomes

\[
\frac{d^2 V}{dr^2} + \frac{1}{r} \frac{dV}{dr} - ib^2 V = 0
\]  

(123)

where

\[
b^2 = \omega/a \quad \text{and} \quad a = K_1/\rho c
\]

Since \( R_o < r < \infty \), the solution of Equation (123) is given as (Mc-Lachlan, 1939, p. 119)

\[
V(r) = B[\text{ker}(br) + i\text{kei}(br)]
\]  

(124)

where \( B \) is a constant.

Applying the boundary condition (122c), one obtains
By using boundary condition (122b), \( W(z) \) is found in the form

\[
W(z) = \exp \left\{ -\frac{2\pi K_0 b_R \left[ \text{ker}'(b_R) + i\text{kei}'(b_R) \right]}{c_w q_w \left[ \text{ker}(b_R) + i\text{kei}(b_R) \right]} z \right\}
\]  

(126)

Thus the solution of Equation (121) is

\[
T_1 = \frac{A_0 \left[ \text{ker}(br) + i\text{kei}(br) \right]}{\text{ker}(b_R) + i\text{kei}(b_R)} W(z-h) \exp (i\omega t)
\]  

(127)

The temperature of the liquid is

\[
T_2 = A_0 W(z-h) \exp (i\omega t)
\]  

(128)

Figure 21. The well model and the corresponding lumped model.
For the computation of the amplitude variation of the temperature distribution, Equation (128) can be rewritten as

\[ T_2 = A \exp (i\omega t + i\phi) \]  

(129)

where

\[ A = A_0 \exp \frac{2\pi K_1 R_0 b(h-z)Y}{c_w q_w} \]  

(130)

\[ \phi = \frac{2\pi K_1 R_0 b(h-z)Z}{c_w q_w} \]  

(131)

with

\[ Y = \frac{\ker(X)\ker'(X) + \kei(X)\kei'(X)}{\ker^2(X) + \kei^2(X)} \]  

(132)

\[ Z = \frac{\ker(X)\kei'(X) - \ker'(X)\kei(X)}{\ker^2(X) + \kei^2(X)} \]  

(133)

and

\[ X = bR_0 \]

\( X, Y, Z \) are dimensionless numbers and \( \ker(X) \) and \( \kei(X) \) represent the real and imaginary parts respectively of the modified Bessel function of the second kind.

In order to estimate the variation of the amplitude \( A \) at \( z = 0 \), the following values will be used in the calculation:

\( R_0 = 10 \text{ cm}, \ h = 1000 \text{ meters}, \ q_w = 10000 \text{ gm/sec} \) and \( c_w = 1 \text{ cal/gm}^\circ \text{C} \). The computed results are shown in Figure 22. From this figure, it can be seen that the temperature amplitude at the top suffers
Figure 22. Amplitude of the surface temperature for a flowing well. The numbers on the curves are the values of the thermal conductivity of the surrounding rock in cal/cm sec °C.
practically no attenuation for $\omega < 10^{-5}$; that is, for periods longer than a few days.

For the purpose of simplification it is in many cases convenient to introduce a lumped model of the well (see Figure 20). In this model the thermal conductance of the surrounding rock is lumped into a massless conducting sheet along the walls and the thermal capacitance of the rock is lumped into a mass sheet behind the conductor.

The differential equations for the lumped model are

$$K(T_w - T_m) = \frac{dT_w}{dz} = m \frac{dT_m}{dt}$$

and the boundary condition is $T_w = A_o \exp(i\omega t)$ at $z = h$,

where

$K$ = thermal conductance of the insulation,

$m$ = lumped thermal capacitance per unit length in cal/cm $^\circ$C,

$q_w$ = mass flow of liquid in gm/sec,

$T_w(z, t)$ = temperature of liquid,

$T_m(z, t)$ = temperature of the lumped mass.

$T_w$, satisfying the above boundary conditions, is found to be

$$T_w = A_o \exp \left[ \frac{K(z-h)}{c_w q_w (1 - \frac{iK}{m \omega})} + i\omega t \right]$$

Equation (135) can also be rewritten as
where

\[ A_1 = A_o \exp \left[ \frac{K(z-h)}{\frac{c_w q_w}{m \omega} (1+\frac{K^2}{m \omega^2})} \right] \]

and

\[ \phi_1 = \frac{K^2(z-h)}{c_w q_w m \omega (1+\frac{K^2}{m \omega^2})} \]

In order that the lumped model gives the same results as Equation (130), one determines \( m \) and \( K \) by solving the two equations

\[ A = A_1 \quad \text{and} \quad \phi = \phi_1 \]

and arrives at the following values

\[ K = -2\pi XYZ(1+\frac{Z^2}{Y^2})K_1 \]

\[ m = -\frac{2\pi XZ(1+\frac{Z^2}{Y^2})}{XZ} \frac{K_1 R_o^2}{a} \]

\( K \) and \( m \) depend upon the values of \( \omega, R_o, a \) and \( K_1 \) but they are independent of \( c_w, q_w \) and \( h \).

The computed results are shown in Figure 23. The values

\( q_w = 10000 \text{ gm/sec}, \ c_w = 1 \text{ cal/gm}^0\text{C}, \ R_o = 10 \text{ cm} \quad \text{and} \quad h = 1000 \)
Figure 23. Values of m and K for the lumped parameter well model.
meters were used in order to obtain these results.

The Temperature Variation in a Core Due to Variable Surface Temperature and Axial Movement of Water

In this case the heat transport equation takes the form

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial z} = a \left( \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r} \frac{\partial T}{\partial r} \right), \quad 0 \leq r < R_o
\]  

(141)

The boundary condition at \( z = 0 \) is assumed to be \( T = 0 \).

The surface temperature is assumed to be harmonic in time; that is,

\[
T = A \exp(i\omega t), \quad \text{at} \quad r = R_o
\]  

(142)

In order to solve Equation (141), the following technique is used.

Let

\[
T(r, z, t) = T_1(r, t) + T_2(r, z, t)
\]

where

\[
\frac{\partial T_1}{\partial t} = a \left( \frac{1}{r} \frac{\partial T_1}{\partial r} + \frac{1}{r} \frac{\partial T_1}{\partial r} \right)
\]  

(143)

\[
\frac{\partial T_2}{\partial t} + u \frac{\partial T_2}{\partial z} = a \left( \frac{1}{r} \frac{\partial T_2}{\partial r} + \frac{1}{r} \frac{\partial T_2}{\partial r} \right)
\]  

(144)

and \( T_1 \) and \( T_2 \) are required to satisfy the boundary conditions

\[
T_1 = A \exp(i\omega t), \quad \text{at} \quad r = R_o
\]  

(145)
\[ T_2 = -T_1, \quad \text{at } z = 0 \]  
\[ T_2 = 0, \quad \text{at } r = R_0 \]  

(146)

Solution of Equation (143) is given as (Carslaw and Jaeger, 1959, p. 193)

\[
T_1 = \frac{A[\text{ber}(br)+i\text{bei}(br)] \exp (i\omega t)}{\text{ber}(bR_0)+i\text{bei}(bR_0)}
\]  

(147)

where \( b^2 = \omega/a \), and \( a = \text{diffusivity of sediments} \).

Solution of (144) is found to be

\[
T_2 = \sum_{j=1}^{\infty} B_j J_0\left(\sqrt{\frac{m_j}{a}}r\right) \exp \left(i\omega t - i\omega z/u - m_j^2 z\right)
\]  

(148)

where

\[
J_0\left(\sqrt{\frac{m_j}{a}}R_0\right) = 0
\]  

(149)

Since \( T_2 = -T_1 \) at \( z = 0 \), \( B_j \) is found to be

\[
B_j = \frac{-A \int_0^{R_0} \frac{I_0(\sqrt{ikr})J_0\left(\sqrt{\frac{m_j}{a}}r\right)rdr}{I_0(\sqrt{ikR_0})}}{\int_0^{R_0} J_0^2\left(\sqrt{\frac{m_j}{a}}r\right)rdr}
\]  

(150)

where \( I_0(\sqrt{ibr}) = \text{ber}(br) + i\text{bei}(br) \) and \( \text{ber}(br) \) and \( \text{bei}(br) \)
represent the real and imaginary parts respectively of the modified Bessel function of the first kind.

After evaluating the above integral (integral formulae are taken from Sagan, 1961, p. 232; McLachlan, 1939, p. 134), one obtains

\[ B_j = \frac{-2A\sqrt{\frac{u}{a}} m_j}{\sum_{j=0}^{\infty} \frac{m_j \sqrt{\frac{\pi}{a}}}{\sqrt{\frac{\pi}{a}} \sqrt{m_j \, R}} \left( \frac{\pi}{a} m_j R \right) \left( \frac{\pi}{a} m_j R \right)} \]  

(151)

Hence the solution of Equation (141) is in the form

\[ T = \frac{A \text{ber}(br) + ib\text{ei}(br)) \exp(i\omega t)}{\text{ber}(bR_o) + ib\text{ei}(bR_o)} \]

\[ - \frac{2A\sqrt{\frac{u}{a}} \sum_{j=1}^{\infty} \frac{m_j J_1 m_j \sqrt{\frac{\pi}{a}} m_j r}}{R_o} \exp(i\omega t - i\omega z / u - m_j^2) \]

(152)

The quantity \( m_j \) can be determined by using Equation (149).
BIBLIOGRAPHY


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