AN ABSTRACT OF THE THESIS OF

Theresa Kee Yu Chow for the M. A. in Mathematics
(Name) (Degree) (Major)

Date thesis is presented January 7, 1965

Title ON THE CANTOR SET

Abstract approved Redacted for Privacy
(Major professor)

The Cantor set is a compact, totally disconnected, perfect subset of the real line. In this paper it is shown that two non-empty, compact, totally disconnected, perfect metric spaces are homeomorphic. Furthermore, a subset of the real line is homeomorphic to the Cantor set if and only if it is obtained from a closed interval by removing a class of disjoint, separated from each other but sufficiently dense open intervals.

ON THE CANTOR SET

by

THERESA KEE YU CHOW

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF ARTS

June 1965

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Mathematics Department

Redacted for Privacy

Dean of Graduate School

Date thesis is presented January 7, 1965

Typed by Carol Baker

ACKNOWLEDGEMENT

The author wishes to express her deep gratitude to Dr. B. H. Arnold for his suggestion of the subject of this paper and his guidance during its writing.

TABLE OF CONTENTS

ON THE CANTOR SET

CHAPTER I

INTRODUCTION

The Cantor set plays a very important role in analysis, but it appears to us that its construction is somewhat specialized. It is our hope to generalize the construction of this set without losing any of its nice properties.

In Chapter II, there will be theorems involving properties of the Cantor set, and from some of these properties a characterization of the Cantor set is derived. In Chapter III, a generalized form of the Cantor set and some examples are given.

CHAPTER II

CHARACTERIZATION OF THE CANTOR SET

2-1. Definition of the Cantor Set

Let I be the closed unit interval on the real line. Remove from I an open interval with center at $\frac{1}{2}$ and of measure 3^{-1}. From each of the two remaining parts of I remove again a central open interval of measure 3^{-2}. There remain four mutually exclusive parts of I, and again from each remove a central open interval of measure 3^{-3}. Continuing in this way, a sequence of mutually exclusive open intervals is removed from I. The set C of the remaining points of I is the Cantor set.

2-2. Properties of the Cantor Set

Consider the real line R as a topological space with the usual topology. Then the Cantor set is a topological space with its relative topology as a subspace of R.

Theorem 1.

C is non-empty.

Proof:

Since the end-points 0 and 1 of I and all the end-points of the open intervals being removed belong to C, C is non-empty.

In fact, it is easy to give an arithmetic characterization for the points of the Cantor set. Let each point x of the closed unit interval be represented as its ternary expansion,

$$x = 0.a_1a_2a_3\cdots \qquad \text{where } a_k = 0, 1 \text{ or } 2 .$$

Each point of x of the first removed open interval $(\frac{1}{3}, \frac{2}{3})$ must have $a_1 = 1$. Each of the end-points of this interval allows two ternary expansions:

$$\frac{1}{3} = 0.1000\cdots$$

$$= 0.0222\cdots ,$$

$$\frac{2}{3} = 0.1222\cdots$$

$$= 0.2000\cdots .$$

Except these points no other points of the unit interval can have 1 immediately following the decimal point. So, at the first step of the process of construction, those and only those points are removed whose ternary expansions must have 1 immediately following the decimal point. In a similar way at the second step those and only those points x are removed for which $a_2 = 1$ necessarily, and so on. After completing the process, those and only those points remain which can be represented as ternary expansions $0. a_1 a_2 a_3 \cdots$ in which each a_k is equal to 0 or 2. In other words, the Cantor set consists of those points of the closed unit interval whose ternary expansion is possible without the use of 1.

Furthermore, since we can easily set up a one-to-one correspondence between the set $\{ 0. a_1 a_2 a_3 \cdots \mid a_i = 0 \text{ or } 2\}$ and the class of all subsets of a denumerable set which has cardinal number c (the cardinal number of continuum) (2, p. 39), we see that C also has cardinal number c.

Theorem 2.

C is a metrizable topological space.

Proof:

Define a function d for every two points x and y of R as

$$d(x, y) = | x - y | .$$

Then d satisfies

$$d(x, x) = 0 ,$$

$$d(x, y) = d(y, x) ,$$

$$d(x, z) \leq d(x, y) + d(y, z) ,$$

hence is a metric on R. This implies that R is a metric space. Since the class of open sets of the metric space R with metric d and the class of open sets of the topological space R both are the class of all unions of open intervals on R, thus R is metrizable. C is a subspace of the topological space R, therefore is also metrizable.

Theorem 3.

C is totally disconnected.

Proof:

After the pth step of our construction there remain 2^p mutually exclusive closed intervals on I, each of length less than 2^{-p}. Let x, y be any two distinct points of C. If p is sufficiently large, there is at least one removed open interval between x and y. Let this open interval be (a, b), then $([0, a] \cap C) \cup ([b, 1] \cap C)$ is a disconnection of C which separates x and y.

Theorem 4.

C is compact.

Proof:

C is bounded, and it is obtained by removing a countable disjoint class of open intervals from a closed interval hence is closed. By the Heine-Borel theorem (2, p. 114), every closed and bounded subspace of the real line is compact. Hence, C is compact.

Theorem 5.

C is perfect.

Proof:

We first consider under what condition a point of C is an isolated point. Let x be an isolated point of C, then there exists an open interval (a, b) of R containing x and containing no points of C other than x. The intervals (a, x) and (x, b) thus contain no points of C at all. There are three cases:

1. if $0 < x < 1$, then x is a common end-point of two distinct removed open intervals;
2. if $x = 0$, then x is the left end-point of a removed open interval;
3. if $x = 1$, then x is the right end-point of a removed open interval.

But in each step of our construction, we took out from the middle of each remaining closed interval an open interval of length equal to one-third of the closed interval, a point of C can not be a common end-point of two distinct removed open intervals, nor is the point 0 or 1 of C an end-point of a removed open interval. We conclude that C has no isolated points.

C is the complement of a denumerable union of open intervals relative to a closed interval, hence is closed. Being closed and having no isolated points, therefore C is perfect.

Theorem 6.

C is of measure zero.

Proof:

The part being removed from the unit interval has measure equal to

$$\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots = 1;$$

thus the remaining part, C, has measure zero.

2-3. Characterization of the Cantor Set

We have proved that the Cantor set is a totally disconnected, compact, perfect metric space. In this section, we shall prove that every two totally disconnected, compact, perfect metric spaces are homeomorphic and hence will have a topological characterization of the Cantor set.

Our steps to lead to this result are as follows. We shall first construct a new topological space from each compact totally disconnected metric space. It will follow that the compact totally disconnected metric space is in fact homeomorphic to this topological space constructed from it. Then, we can prove any two totally disconnected, perfect, compact metric spaces are homeomorphic by means of proving the two topological spaces constructed from them are homeomorphic.

Definition 1.

An open covering $\{V_\beta\}$ of a topological space X is said to be a refinement of an open covering $\{U_\alpha\}$ of X if for each open set V_β of $\{V_\beta\}$ there is an open set U_α of $\{U_\alpha\}$ such that U_α contains V_β .

Lemma 1.

Let X be a compact totally disconnected topological space. Then if P is a component of X and if U is any set containing P there is an open and closed set V lying in U and containing P .

Proof:

We first prove that P is a single point. Suppose x and y are two distinct points in P . Let $X = A \cup B$ be a disconnection of X with x in A and y in B, then $(P \cap A) \cup (P \cap B)$ is a disconnection of P which contradicts the hypothesis that P is a component. Therefore P is a single point, we denote it by x.

Since X is compact and the complement of U, (denoted by CU) is a closed subspace of X, CU is also compact. For each point z of CU, by the total disconnectedness of X, there exists a set H_z which is both open and closed and contains z but not x. Since CU is compact, there is some finite class of H_z's, which we denote by $\{H_1, H_2, \ldots, H_n\}$, with the property that its union contains CU but not x. Let V be the complement of $\bigcup_{i=1}^{n} H_i$, it is both open and closed lying in U and containing P as we wanted to prove.

Lemma 2.

If G is any open covering of a metric space M, and if n is any integer, then there is a refinement G_n of G composed of open sets of diameter $< \frac{1}{n}$. If M is compact, then G_n can be taken to be finite.

Proof:

For every point x in M, there is an open set U_x of G such that x is in U_x. Since U_x is open, there is an open sphere S_x with center x and radius less than $\frac{1}{2n}$ lying in U_x. Let $G_n = \{S_x \mid x \in M\}$, then G_n is a covering of M and it is a refinement of G composed of open sets of diameter less than $\frac{1}{n}$.

If M is compact, then G_n, being an open covering of M, has a finite subclass which also covers M.

Theorem 7:

Let M be a compact totally disconnected metric space. Then M has a sequence $G_1, G_2, \cdots$ of open coverings, each G_n being a finite collection of disjoint sets of diameter less then $\frac{1}{n}$, which are both open and closed and G_{n+1} being a refinement of G_n for each n.

Proof:

Begin with any covering G_0 of M, by lemma 2, there is a refinement Q_0 of G_0 composed of open sets of diameter less than 1. Each point x of M is a component of M and lies in an open set U_x of Q_0. By lemma 1, there is an open and closed set V_x containing x and lying in U_x. Diameter of $V_x \leq$ diameter of $U_x < 1$. By compactness, a finite number $V_1, V_2, \cdots, V_n$ of these sets covers M. Consider the sets $U_1 = V_1$, $U_2 = V_2 - V_1, \cdots, U_n = V_n - (\bigcup_{i=1}^{n-1} V_i)$, each of these is an open set minus a closed set and is open, but also each is a closed set minus an open set and is closed. No two of them intersect and diameter of $U_i \leq$ diameter of $V_i < 1$. Let $G_1 = \{U_i\}$.

Again from G_1, there is a refinement Q_1 of G_1 composed of open sets of diameter $< \frac{1}{2}$ and by the same process as above we can find G_2, and so on. Therefore the sequence exists.

Definition 2.

Let $X_0, X_1, X_2, \cdots$ be a denumerable collection of topological spaces and, for each $n > 0$, let there be given a continuous mapping $f_n : X_n \rightarrow X_{n-1}$. The sequence of spaces and mappings $\{X_n, f_n\}$ is called an inverse limit sequence.

Definition 3.

Let $\{X_n, f_n\}$ be an inverse limit sequence. The set of all points $(x_0, x_1, \cdots, x_n, \cdots)$ of the product space $P_{n=0}^{\infty} X_n$ such that each x_n is a point of X_n, and $x_n = f_{n+1}(x_{n+1})$ for all $n \geq 0$, taken with the relative topology of $P_{n=0}^{\infty} X_n$, is called the inverse limit space of the sequence $\{X_n, f_n\}$ and is denoted by X_∞.

Lemma 3.

If each topological space X_n in the inverse limit sequence

$\{X_n, f_n\}$ is a non-empty compact Hausdorff space, then X_∞ is not empty.

Proof:

For each integer $n \geq 1$, let Y_n be the set of all points $(p_0, p_1, p_2, \cdots)$ of $P_{n=0}^{\infty} X_n$ such that for $1 \leq j \leq n$, $p_{j-1} = f_j(p_j)$. We will show that every Y_n is closed in $P_{n=0}^{\infty} X_n$. Suppose that for a given n, q is not a point of Y_n. If $q = (q_0, q_1, q_2, \cdots)$, then for some $j < n$, we have $q_j \neq f_{j+1}(q_{j+1})$. Now X_j is a Hausdorff space, so there exist disjoint open sets U_j and V_j in X_j, with q_j in U_j and $f_{j+1}(q_{j+1})$ in V_j. Define $V_{j+1} = f_{j+1}^{-1}(V_j)$. Let B_q denote any rectangular basis element in $P_{n=0}^{\infty} X_n$ containing q and having U_j and V_{j+1} as factors. Then no point of Y_n lies in B_q. For if $b = (b_0, b_1, \cdots, b_n, \cdots)$ is in B_q, then b_{j+1} lies in V_{j+1}, b_j in U_j and not in V_j, hence b is not in Y_n. Thus the complement of Y_n is open; Y_n is closed. Since any finite number of these Y_n's have a non-empty intersection and $P_{n=0}^{\infty} X_n$ is compact, the intersection $\bigcap_{n=1}^{\infty} Y_n$ is not empty. But each point in $\bigcap_{n=1}^{\infty} Y_n$ satisfies the condition for being a point in X_∞. Hence X_∞ is not empty.

Lemma 4.

If each topological space X_n in the inverse limit sequence

$\{X_n, f_n\}$ is a compact Hausdorff space, then X_∞ is also a compact Hausdorff space.

Proof:

By hypothesis each X_n is a compact Hausdorff space, therefore $P_{n=0}^{\infty} X_n$ is a compact Hausdorff space. From the proof of Lemma 3, $\bigcap_{n=1}^{\infty} Y_n$ is closed in $P_{n=0}^{\infty} X_n$, hence is also a compact Hausdorff space. Since X_∞ is contained in every one of the Y_n's, it is contained in $\bigcap_{n=1}^{\infty} Y_n$. Conversely, each point of $\bigcap_{n=1}^{\infty} Y_n$ satisfies the condition for being a point in X_∞, hence $\bigcap_{n=1}^{\infty} Y_n$ is contained in X_∞. Therefore $X_\infty = \bigcap_{n=1}^{\infty} Y_n$; it is a compact Hausdorff space.

Theorem 8.

Let M be a non-empty, compact, totally disconnected metric space. Then M is homeomorphic to the inverse limit space of an inverse limit sequence of finite, discrete topological spaces.

Proof:

Let $G_1, G_2 \cdots$ be a sequence of coverings of M as given in theorem 7. For each n, let G_n^* denote the space whose points are the open sets of G_n and which has the discrete topology.

We will use the same notation for an element of G_n and the corresponding point of G_n^*. A continuous mapping $f_n: G_n^* \to G_{n-1}^*$, $n > 1$, may be defined as follows. If $U_{n,i}$ is an element of G_n, then there is a unique element $U_{n-1,j}$ of G_{n-1} containing $U_{n,i}$ because the elements of G_{n-1} are disjoint. We set $f_n(U_{n,i}) = U_{n-1,j}$. The mappings f_n are continuous since each G_n^* is discrete. With these definitions, then $\{G_n^*, f_n\}$ is an inverse limit sequence, and each G_n^* is a non-empty compact Hausdorff space since M is non-empty and G_n^* contains only a finite number of points and is discrete. By lemma 3, the inverse limit space G_∞ is non-empty.

We next define a mapping $h: G_\infty \to M$. If $p = (U_{1,n_1}, U_{2,n_2}, \cdots)$ is a point of G_∞, then the sets $U_{1,n_1}, U_{2,n_2}, \cdots$ in M form a sequence of closed sets, each containing the succeeding one. Thus the intersection $\bigcap_{j=1}^{\infty} U_{j,n_j}$ is not empty (2, p. 73-74). Since diameter of $U_{j,n_j} < \frac{1}{j}$, there can be at most one point q of M in this intersection, let $h(p) = q$. It is left to prove that h is a homeomorphism.

First, h is one-to-one, for if p is a point of G_∞, then $h(p)$ is in each of the point sets in M that are coordinates of p. Hence if two points p and p' of G_∞ differ in the nth coordinates, then $h(p) \neq h(p')$ because the elements of G_n are disjoint. Second, h is onto, for each point q of M lies in the intersection of such a sequence of sets. Third, h is continuous. Note first that the collection of all sets $U_{j,i}$ is a basis for the topology of M since it contains arbitrarily small open sets about each point. Then, if we can prove that for every $U_{j,i}$, $h^{-1}(U_{j,i})$ is open in G_∞, h is continuous. $h^{-1}(U_{j,i})$ consists of all points of G_∞ having $U_{j,i}$ for their jth coordinate, and the point $U_{j,i}$ of G_j^* is open in G_j^*, hence $h^{-1}(U_{j,i})$ is open in G_∞.

M is compact, and it is a Hausdorff space since it is totally disconnected. From lemma 4, G_∞ is a compact Hausdorff space. Then h is a one-to-one continuous mapping of a compact Hausdorff space onto a compact Hausdorff space, hence is a homeomorphism (2, p. 131).

Lemma 5.

If U is a non-empty open set in a totally disconnected perfect topological space, and n is an integer, then U is a union of n disjoint non-empty open sets.

Proof:

We prove by induction on n.

For n = 1, U itself satisfies the condition. Suppose that for n = k we have

$$U = U_1 \cup \cdots \cup U_k$$

where the U_i's are open, disjoint and non-empty.

Since the space is perfect, every point is a limiting point; so a single point is not open. And from the total disconnectedness, U_k is not connected since it contains more than one point. Thus we can find a disconnection $U_k = U_{k,1} \cup U_{k,2}$ where $U_{k,1}$ and $U_{k,2}$ are disjoint and non-empty. Each of these sets is open in U_k and hence in the space. Then $U_1, \cdots, U_{k-1}, U_{k,1}, U_{k,2}$ is a decomposition of U for $n = k + 1$. This completes the induction proof.

Lemma 6.

Let $X = P_{n=0}^{\infty} X_n$ and $Y = P_{n=0}^{\infty} Y_n$ be two product spaces and let $f_n : X_n \to Y_n$ be continuous for each n. Then the mapping $f(x) = y$, where $x = (x_1, x_2, \cdots)$ is in X and $y = (f_1(x_1), f_2(x_2), \cdots)$, is a continuous mapping of X into Y.

Proof:

Let S be the class of all open sets $B = P_{n=0}^{\infty} B_n$ of Y such that for some N, B_N is open in Y_N and $B_i = Y_i$ for $i \neq N$. Then S is an open subbase of Y. If we can show that the inverse image under f of each B in S is open in X, then f is continuous. Let $B = P_{n=0}^{\infty} B_n$ be a set in S such that B_N is open in Y_N and $B_i = Y_i$ for $i \neq N$, then by the continuity of f_N,

$A_N = f_N^{-1}(B_N)$ is open in A. Let $A = P_{n=0}^{\infty} A_n$ where $A_N = A_N$ and $A_i = X_i$ for $i \neq N$. Then $f^{-1}(B) = A$ and A is open in X, hence f is a continuous mapping of X into Y.

Theorem 9.

Any two non-empty, totally disconnected, perfect, compact metric spaces are homeomorphic.

Proof:

Let S and T be two such spaces, and let $G_1, G_2, \cdots$ and $Q_1, Q_2, \cdots$ be sequences of open coverings of S and T, respectively, where $G_k = \{U_{k,1}, \cdots, U_{k,n_k}\}$ and $Q_k = \{V_{k,1}, \cdots, V_{k,m_k}\}$ as produced in the proof of theorem 8. If G_1 and Q_1 have the same number of elements, we set $G_1' = G_1$ and $Q_1' = Q_1$. If $n_1 > m_1$, then by lemma 5, $V_{1,1}$ is the union of $n_1 - m_1 + 1$ disjoint open sets (each set is the complement of an open set, so is also closed). Take $G_1' = G_1$, and let Q_1' consist of $V_{1,2}, \cdots, V_{1,m_1}$ together with the sets into which $V_{1,1}$ has been decomposed. If $m_1 > n_1$, then the roles of G_1 and Q_1 are interchanged. Denote $G_1' = \{U_{1,1}', \cdots, U_{1,N_1}'\}$ and $Q_1' = \{V_{1,1}', \cdots, V_{1,N_1}'\}$. Let h_1 be any one-to-one correspondence between G_1' and Q_1'.

Suppose the open coverings $G_j' = \{U_{j,1}', \cdots, U_{j,N_j}'\}$ and $Q_j' = \{V_{j,1}', \cdots, V_{j,N_j}'\}$, and the one-to-one correspondence h_j

between G'_j and Q'_j have been defined. Since the elements of G'_j are disjoint closed sets, there is an integer $r_j > j$ such that no set of diameter $< \frac{1}{r_j}$ intersects any two different $U'_{j,i}$'s , and there is a similar integer s_j for Q'_j. Let m denote the larger of r_j and s_j. Then G_m refines G'_j and Q_m refines Q'_j. Consider the elements of G_m in $U'_{j,i}$ and the elements of Q_m in $h_j(U'_{j,i})$ for each i. If there are more elements of G_m in $U'_{j,i}$ than elements of Q_m in $h_j(U'_{j,i})$, then we use lemma 5 to decompose one of the elements of Q_m in $h_j(U'_{j,i})$, and vice versa. Carrying out this process for each $i \leq N_j$ yields coverings G'_{j+1} and Q'_{j+1}, which refine G'_j and Q'_j, respectively. Denote $G'_{j+1} = \{U'_{j+1,1}, \cdots, U'_{j+1,N_{j+1}}\}$ and $Q'_{j+1} = \} V'_{j+1,1}, \cdots, V'_{j+1,N_{j+1}}\}$. Let $h_{j+1} : G'_{j+1} \rightarrow Q'_{j+1}$ be defined by assigning to each $U'_{j+1,i}$ in G'_{j+1} an element of Q'_{j+1} in $h_j(p_{j+1}(U'_{j+1,i}))$, where p_{j+1} is the projection of G'_{j+1} onto G'_j. This assignment is made in such a way that h_{j+1} is a one-to-one correspondence between G'_{j+1} and Q'_{j+1}. The inductive definitions of sequences $G'_1, G'_2, \cdots$ and $Q'_1, Q'_2, \cdots$ of open coverings and the one-to-one correspondence h_n between G'_n and Q'_n, $n = 1, 2, \cdots$, are complete.

We let $G^*_1, G^*_2, \cdots$ and $Q^*_1, Q^*_2, \cdots$ be the associated sequences of discrete spaces from the sequences $G'_1, G'_2, \cdots$ and $Q'_1, Q'_2, \cdots$, respectively, as defined in the proof of theorem 8. Then each h_n can be considered as a one-to-one correspondence between G_n^* and Q_n^*. Since G_n^* and Q_n^* are discrete spaces, each h_n and its inverse are continuous.

We let G_∞, Q_∞ be the inverse limit spaces of the sequences $\{G_n^*, p_n\}$, $\{Q_n^*, q_n\}$, respectively, where p_n and q_n are the projections of G_n^* onto G_{n-1}^* and Q_n^* onto Q_{n-1}^*, respectively. We define a mapping $h: G_\infty \to Q_\infty$ as follows. For each point $U' = (U'_{1,i_1}, U'_{2,i_2}, \cdots)$ in G_∞, let $h(U') = (h_1(U'_{1,i_1}), h_2(U'_{2,i_2}), \cdots)$. For each n, since $p_n(U'_{n,i_n}) = U'_{n-1,i_{n-1}}$ and $h_n(U'_{n,i_n})$ is in $h_{n-1}(p_n(U'_{n,i_n}))$,

$$q_n(h_n(U'_{n,i_n})) = h_{n-1}(p_n(U'_{n,i_n})) = h_{n-1}(U'_{n-1,i_{n-1}}),$$

$h(U)$ is indeed a point of Q_∞. h is one-to-one since each h_n is. For every point $V' = (V'_{1,j_1}, V'_{2,j_2}, \cdots)$ in Q_∞, let $U' = (U'_{1,k_1}, U'_{2,k_2}, \cdots)$ where $h_n(U'_{n,k_n}) = V'_{n,j_n}$. We claim that U' is a point of G_∞. Since

$$h_{n-1}[p_n(U'_{n,k_n})] = q_n(V'_{n,j_n}) = V'_{n-1,j_{n-1}},$$

$$h_{n-1}(U'_{n-1,k_{n-1}}) = V'_{n-1,j_{n-1}},$$

and h_{n-1} is one-to-one,

then $p_n(U'_{n,k_n}) = U'_{n-1,k_{n-1}}$, and

U' is a point of G_∞. This implies h is onto. Furthermore, by lemma 6, h and its inverse are both continuous, therefore h

is a homeomorphism of G_∞ onto Q_∞. We have from theorem 8, G_∞ and Q_∞ are homeomorphic to S and T, respectively, it follows that S and T are homeomorphic (1, p. 99-100).

From theorems 1, 2, 3, 4, 5 of section 2-2 and theorem 9 of section 2-3, we get the result that any non-empty, compact, totally disconnected, perfect metric space is homeomorphic to the Cantor set.

CHAPTER III

GENERALIZATION OF THE CANTOR SET

3-1. Generalization of the Cantor Set

In this section, we will characterize the structure of a non-empty, compact, totally disconnected, perfect subset of the real line, R, and hence will have a generalized form (up to homeomorphism) of the Cantor set.

First, we will characterize the structure of a non-empty compact subset of R; second, the structure of a non-empty compact perfect subset of R; third, the structure of a totally disconnected subset of R; and finally, the structure of a non-empty, compact, totally disconnected, perfect subset of R.

Definition 4.

If G is an open subset of R, then an open interval which is contained in G, but whose end-points do not belong to G, is called a component interval of G.

We will use the notation $[a, b]$, (a, b), $[a, b)$, $(a, b]$, to denote a closed, open, left closed and right open, left open and right closed interval respectively, and CE to denote the complement of the set E.

Lemma 7.

Every non-empty bounded open subset G of R can be represented as the union of pairwise disjoint component intervals.

Proof:

First we shall prove that each point of G belongs to a component interval of G. Let $x \in G$ and $F = [x, +\infty) \cap CG$. Both of the sets $[x, +\infty)$ and CG are closed, hence F is closed. Since G is bounded, F is non-empty. Because none of the points of F lies to the left of x, F is bounded below. Thus F contains a left end point b, and $x \leq b$. But $x \in G$ and hence $x \notin F$, so that $x \neq b$, therefore $x < b$. We next establish that $[x, b) \subset G$. $b \notin G$, since $b \in F \subset CG$. Suppose there exists a point y such that $y \in [x, b)$ and $y \notin G$, then $y \in F$ and $y < b$, this contradicts the definition of b. We thus have $x < b$, $b \notin G$ and $[x, b) \subset G$. By a similar argument, we can prove the existence of a point a such that $a < x$, $a \notin G$ and $(a, x] \subset G$. Then (a, b) is a component interval of the set G containing the point x.

Second we shall prove that if (a, b) and (c, d) are two component intervals of G, then they are either disjoint or identical. Suppose there is a point x lying in both (a, b) and (c, d), then $a < x < b$ and $c < x < d$. Assume that $b < d$, then $c < x < b < d$, so that $b \in (c, d)$. But this is impossible, since $(c, d) \subset G$ and $b \notin G$. This implies that $b \geq d$. Since b and d are interchangeable, it follows from the same reasoning that $b \leq d$ and hence $b = d$. Similarly, we can prove $a = c$, and therefore (a, b) and (c, d) are identical.

Then G is the union of pairwise disjoint component intervals.

We can get a further result from this lemma. The set of distinct component intervals of a non-empty, bounded open set of R is finite or denumerable. Since we can choose a rational point in each of these intervals, the set of component intervals is put into one-to-one correspondence with a subset of all rational numbers, hence must be finite or denumerable.

Theorem 10.

A non-empty subset X of R is compact if and only if it is either a closed interval or is obtained from a closed interval by removing a class of pair-wise disjoint open intervals whose end points belong to X.

Proof:

If X is a non-empty compact subset of R, then X is closed and bounded. Since X is bounded, there is a smallest closed interval S containing X. Let $C_s X$ be the complement of X with respect to S, then $C_s X$ is bounded and open. By lemma 7, $C_s X$ is either empty or is the union of pairwise disjoint component intervals, the end points of which do not belong to $C_s X$. Therefore X is either a closed interval or obtained from some closed interval by removing a class of pairwise disjoint open intervals whose end points belong to X.

If X is a closed interval or is obtained from some closed interval by removing a class of pairwise disjoint open intervals whose end points belong to X, then S is closed and bounded, hence it is compact.

From the statement immediately following lemma 7, those intervals being removed from a closed interval to form a compact set are either finite or denumerable in number.

Lemma 8.

Let G be a non-empty open subset of R and $(a, b) \subset G$. Then, among the component intervals of G there exists one which contains (a, b).

Proof:

Let $x \epsilon (a, b)$, then there is a component interval (m, n) of G such that $x \epsilon (m, n)$. Assume that $b > n$, then $n \epsilon (a, b)$ which is impossible, therefore $b \leq n$. In the same way, we can prove $m \leq a$, and hence $(a, b) \subset (m, n)$.

Definition 5.

Two disjoint open intervals of R are said to be adjacent to each other if they have a common end point.

Definition 6.

Let X be a non-empty bounded closed subset of R, and let S be the smallest closed interval containing X, then a component interval of the complement $C_s X$ of X with respect to S is called a complementary interval of X.

Lemma 9.

If X contains more than one point and it is a compact subset of R and $S = [a, b]$ is the smallest closed interval containing X, then x is an isolated point of X if and only if it is either the common end point of two distinct adjacent complementary intervals of X, or x is the point a (or b) and a (or b) is an end point of a complementary interval of X.

Proof:

Let x be an isolated point of X. First suppose that $a < x < b$, then there exists an open interval (p, q) such that (p, q) contains x and contains no points of X other than x, then $(p, q) \subset [a, b]$ since a and b are in X. Let $C_s X$ be the complement of X with respect to S, then $(x, q) \subset C_s X$. By lemma 8, there is a complementary interval (m, n) of X containing the interval (x, q). If $x > m$, then x is not in X; so it is necessary that $x \leq m$. But $x < m$ would contradict the fact that $(x, q) \subset (m, n)$, therefore $x = m$, i. e., x is the left end point of the complementary interval (m, n) of X. In the same way, we can prove that x is the right end point of some complementary interval of X.

For the case $x = a$, since X contains more than one point, $a \neq b$. We can find an open interval (r, s) such that (r, s) contains no points of X other than x, then $(x, s) \subset C_s X$. By lemma 8, there exists a complementary open interval (u, v) of X such that $(x, s) \subset (u, v)$. By the same reasoning as above $x = u$. Therefore when $x = a$, it is an end point of a complementary interval of X. Similarly we can prove the case when $x = b$.

The other way around, let x be a common end point of two adjacent complementary intervals of X, or let x be the point a (or b) and be the end point of a complementary interval of X.

In either case there is an open interval which contains x and contains no points of X other than x, hence x is an isolated point.

Theorem 11.

X is a non-empty, perfect, compact subset of R if and only if it is obtained from some closed interval $[a, b]$, where $a \neq b$, by removing a class of pairwise disjoint open intervals any two of which are not adjacent to each other and none of which has a or b as end point.

Proof:

Let X be a non-empty, perfect, compact subset of R. Since X is non-empty and compact, by theorem 10, X is obtained from some closed interval $[a, b]$ by removing a class of pairwise disjoint open intervals whose end points belong to X. Since a single point is not a perfect subset of R, $a \neq b$. Suppose among the open intervals being removed, there are two of them adjacent to each other. Then their common end-point belongs to X and it is an isolated point. But this is impossible since X is perfect. By the same reasoning, none of the open intervals being removed has a or b as its end point.

If X is obtained from a closed interval $[a, b]$ by removing a class of pairwise disjoint open intervals any two of which are not adjacent to each other and none of which has a or b as end-point, then X is closed and bounded, hence it is compact. Since $a \neq b$, X contains more than one point, then by lemma 9, X has no isolated points, hence it is perfect.

Theorem 12.

A subset X of R is totally disconnected if and only if between any two different points of X there is a point which does not belong to X.

Proof:

The theorem is clearly true if X is an empty subset or contains only a single point. We may assume that X contains more than one point.

If X is a totally disconnected subset of R and $x < y$ are two different points of X, then there exist two disjoint open sets U, V of R such that $X = (X \cap U) \cup (X \cap V)$ and $x \in U$, $y \in V$. By lemma 7, U is the union of pairwise disjoint open intervals the end-points of which do not belong to U. Let (a, b) be the open interval such that $x \in (a, b) \subset U$ and $a \notin U$, $b \notin U$. Since V is also the union of pairwise disjoint open intervals the end-points of which do not belong to V, and V is disjoint from U, $b \notin V$. Since $y \notin U$ and $(a, b) \subset U$, $y \notin (a, b)$. Then $x < b < y$. Thus b is a point between x and y which does not belong to X.

If between any two points of X, there is a point which does not belong to X. Let x and y be any two points of X, with $x < y$; there is a point e such that $x < e < y$ and $e \notin X$. Then $X = [(-\infty, e), X] \cup [X, (e, +\infty)]$ is a disconnection of X separating x and y, hence X is totally disconnected.

Definition 7.

Let S be a subset of R. A class G of open intervals is said to be dense in S if for any two distinct points $x < y$ in S, there is an open interval (a, b) in G such that $(a, b) \subset (x, y)$.

Theorem 13.

A non-empty subset of R is compact, totally disconnected, perfect if andonly if it is obtained from a closed interval $[a, b]$, where $a \neq b$, by removing a disjoint class of open intervals which is dense in the remaining set, no two of the open intervals are adjacent to each other and none has common end-point with $[a, b]$.

Proof:

From theorem 12, a subset X of R is totally disconnected if and only if between any two different points of X there is a point which does not belong to X, but from theorem 11, X is non-empty perfect compact if and only if it is obtained from some closed interval [a, b], where $a \neq b$, by removing a class of pairwise disjoint open intervals any two of which are not adjacent to each other and none of which has a or b as end-point, therefore between any two points of X there must be an open interval being removed from X. Thus a non-empty subset of R is compact, totally disconnected, perfect if and only if it is obtained from a closed interval [a, b], where $a \neq b$, by removing a disjoint class of open intervals such that the class is dense in the remaining set, no two of the open intervals are adjacent to each other and none has common end-point with the closed interval [a, b].

We can also state theorem 13 as

Theorem 14.

A non-empty subset of R is compact, totally disconnected, perfect if and only if it is obtained from a closed interval [a, b], where $a \neq b$, by removing a disjoint class of open intervals such that no two are adjacent to each other and none has common end-point with [a, b], and such that every open interval contained in [a, b] contains an open interval which is in the complement of the remaining set.

Proof:

By using theorem 11, we can assume X is a compact and perfect subset of R and prove that X is totally disconnected if and only if every open interval contained in [a, b] contains an open interval which is in the complement of X. But from theorem 12 it is sufficient to prove the following two statements are equivalent for the set X :

1. between any two different points of X there is a point which does not belong to X;
2. every open interval contained in [a, b] contains an open interval which is in the complement of X.

If statement one is true, then from the structure of compact perfect set, between any two points of X there is an open interval being removed. Let (c, d) be any open interval contained in [a, b]. If (c, d) contains two different points of X, then by the above argument between these two points there is an open interval being removed, and hence this open interval is contained in the complement of X. Since X is perfect, (c, d) can not contain a single point of X. If (c, d) contains no points of X, then (c, d) itself is an open interval contained in the complement of X. Thus statement one implies statement two.

If statement two is true, then every open interval contained in [a, b] contains an open interval which is in the complement of X. Let $x < y$ be two different points of X, then (x, y) contains an open interval which is in the complement of X, hence there is a point between x and y which does not belong to X. Therefore statement two implies statement one and the two statements are equivalent. This completes the proof of the theorem.

3-2. Examples

In the following, we shall give some special constructions of compact, totally disconnected, perfect subsets of R based on theorem 13.

Example 1.

From a closed interval of R, we remove an arbitrary open interval provided that it has no common end-points with the original closed interval. From the remaining parts we remove again an arbitrary open interval provided that it has no common end-points with the original closed interval nor with the open interval already removed. Again do the same to the remaining parts. Continue in this way an infinite number of times and arrange that between any two remaining points, there is an open interval being removed. Then by theorem 13, the remaining set is a set homeomorphic to the Cantor set. Therefore from a closed interval we can construct infinitely many different forms of generalized Cantor sets.

From theorem 6 in section 1-2, the Cantor set is of measure zero. We would raise the question: Can a compact, totally disconnected, perfect subset of R have measure other than zero? The answer is yes. The following example 2 gives us a method of constructing a generalized Cantor set of arbitrary measure, and example 3 gives us a method of constructing a generalized Cantor set of any measure less than the measure of the closed interval with which we begin.

Example 2.

For any real number $r \geq 0$, take an arbitrary closed interval X of measure $r + 1$ from the real line. Remove from X a central open interval of measure 3^{-1}, and from each of the two remaining parts of X remove again a central open interval of measure 3^{-2}. And again from the four remaining parts of X remove a central open interval of measure 3^{-3}. Continuing in this way, a sequence of mutually non-adjacent open intervals which is dense in the remaining set is removed from X. By theorem 13, the remaining part is a set homeomorphic to the Cantor set. The measure of the part being removed is

$$\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots = 1.$$

Hence the measure of the remaining set is equal to r.

Therefore we can construct a subset of R of any given measure such that it is a generalized Cantor set.

Example 3.

Let S be a closed interval of measure r and let q be any real number such that $0 \leq q < 1$. Let $a = \frac{1-q}{2-q}$. Remove from S a central open interval of measure ar. From each of the remaining parts remove a central open interval such that their total measure is a^2r. And again from each of the four remaining parts remove a central open interval such that their total measure is a^3r. Continuing in this way, a sequence of mutually non-adjacent open intervals which is dense in the remaining set is removed from S. By theorem 13, the remaining set is a set homeomorphic to the Cantor set. The total measure being removed is

$$ar + a^2 r + a^3 r + \cdots = \frac{ar}{1-a} = \frac{(1-q)r}{(2-q)(1-\frac{1-q}{2-q})} = (1-q)r \quad .$$

Hence the total measure of the remaining set is

$$r - (1-q)r = qr \; .$$

Therefore we can construct from a closed interval of measure r a generalized Cantor set of measure qr for any $0 \leq q < 1$.

BIBLIOGRAPHY

1. Hocking, John G. and Gail S. Young. Topology. Reading, Massachusetts, Addison-Wesley, 1961. 374 p.

2. Simmons, George F. Introduction to topology and modern analysis. New York, McGraw-Hill, 1963. 372 p.