# BUCKLING OF THIN-WALLED PLYWOOD CYIINDERS IN TORSION 

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# BUCKLING OF THIN-WALTED PLYWOOD CYI,TNDERS IN TORSION ${ }^{-}$ 

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## Summary

This report presents a theoretical analysis and the results of tests on the buckling strengths of thin-walled plywood cylinders in torsion. The theoretical analysis leads to the expression $T=k E_{\mathrm{J}} \frac{h}{r}$ for the buckling stress. The coefficient $\underline{k}$ is determined by the plywood construction, the direction of the face grain, and a parameter which is a function of the dimensions of the cylinder. Theoretical curves are presented that can be used to find the value of the constant $k$ for a cylinder of given dimensions and type of plywood construction. Experimentally and theoretically determined values of $k$ are considered to be in satisfactory agreement if allowance is made for the scattering of experimental values associated with imperfections of construction and material and for the lack of complete agreement between the elastic constants of major and minor test specimens.

## Introduction

The buckling stress of a thin cylindrical plywood shell in torsion depends upon a number of variables, such as the dimensions of the shell,
${ }^{1}$ This report is one of a series of progress reports prepared by the Forest Products Laboratory to further the Nation's war effort. Results here reported are preliminary and may be revised as additional data become available. Original report dated June 1945.
${ }^{2}$ Maintained at Madison, Wis., in cooperation with the University of Wisconsin.
the construction and the elastic constants of the plywood, and the orientation of the orthotropic axes of the plywood. Because of the large number of combinations of these variables, a testing program designed to establish empirically a formula for determining the buckling stress would be an extremely formidable undertaking. If, however, a method of determining the buckling stress theoretically can be developed even though this method is approximate, a smaller number of tests will suffice to confirm the main features of the theoretical treatment.

In the present report an approximate theoretical treatment is given for plywood cylinders having the grain of the face plies at any angle with their axes. The principal results of the analysis are given in a set of curves for the cases where the grain of the face plies makes angles of $0^{\circ}, 90^{\circ}$, and $45^{\circ}$ with the axis of the cylinder.

The buckling of plywood cylinders in torsion is one of a large number of problems included in a unified treatment of the stability of plywood sheets given in a recent paper by L. H. Donnell. 3 The purpose of that paper and that of the present report are quite different. In that paper is given a formula intended to be applicable to a wide range of problems in which the buckling stress is expressed as a function of a number of parameters. By calculating the buckling stress for each of a number of combinations of values of the parameters, the minimum buckling stress can be determined by a series of numerical operations. The purpose of the present report is to furnish a series of curves for the single problem under consideration. From them the buckling stress can be read for a given cylinder provided that the face grain has pne of the three orientations for which the curves were drawn. Donnell 4 has also given a theoretical treatment of the stability of isotropic cylinders in torsion and the results of a series of tests on such cylinders. In his paper, references may be found to the literature of the problem for isotropic cylinders.

## Mathematical Analysis

## Elastic Constants of Wood. Choice of AxesStructure and Orientation of Plywood

As in previous reports ${ }^{2}$ issued by the Forest Products Laboratory dealing with the elastic behavior of wood and plywood, wood is considered

[^0]to be an orthotropic material. Reference should be made to these reports for details concerning the elastic constants of wood and plywood.

For definiteness, the plywood in the cylinders considered will be taken to be made of rotary-cut veneers of the same species of wood. When this is not the case, suitable modifications may be made of the constants that appear in the analysis and as parameters in the families of curves expressing the results. These modifications are explained in several of the reports to which reference has been made and in particular in the appendix to Forest Products Laboratory Report No. 1316. It is to be expected that the theoretical curves are applicable if parameters appropriate to a given construction are used, even though the curves were constructed for plywood whose veneers are rotary cut. It is assumed throughout that the plywood is of symmetrical construction with respect to its middle surface.

The choice of axes in the median surface of the cylindrical shell is shown in figure 1 , in which the coordinate $\bar{y}$ is measured along the circumference. The axis $O_{\xi}$ is taken in the direction of the grain of the face plies. The components of the displacements of points in the median surface of the cylindrical shell in the axial, circumferential, and radial directions respectively, are $\underline{u}, \underline{v}$, and $\underline{w}$, where $\underline{w}$ is positive inward. The notation for the components of stress and strain are those of Love's 6 treatise, except that in a few instances it has seemed more convenient to use the symbols $t_{x x}, t_{y y}$, and $t_{x y}$, for the components of stress instead of $X_{x}, Y_{y}$, and $X_{y}$, respectively. The thickness of the wall of the cylinder, the radius of its median surface, and its length are denoted by $\underline{h}, \underline{r}$, and $\underline{b}$, respectively.

## Energy Method. Form of the Buckled Surface

A small deflection theory will be used in applying an energy method to determine the buckling stress. The form of the buckled surface will be taken to be

$$
\begin{equation*}
\frac{W}{r}=f[\sin \alpha(y-\gamma x)+g] \sin \beta x \tag{1}
\end{equation*}
$$

[^1]where
$$
\alpha=\frac{n}{r}, \beta=\frac{\pi}{b},
$$
$\mathrm{n}=$ number of waves in the circumferential direction.
The form of the first term in the brackets in equation (1) is suggested by the form that has been used to represent the buckled surface of an infinite plane strip 7 under uniform shear. The parameter $g$ is introduced to permit a possible average change of radius of the median surface. Later it will be found to reduce to zero. The factor sin $\beta x$ is introduced to made the deflection vanish at the ends of the cylindrical shell. Further conditions at the ends are disregarded. Except for very short cylinders, these conditions are not important.

An expression, equivalent to equation (1) with the parameter gomitted, was used by L. H. Donnell 3 to represent the buckled surfaces of cylinders and of flat and curved plates in applying an energy method. In the following discussion, the procedure differs somewhat from that of Donnell in not assuming expressions for the displacements $\underline{u}$ and $\underline{v}$ of points of the median surface of the shell. Instead, the components of the induced membrane stresses in the shell are determined from a stress function that satisfies a differential equation expressing the restriction imposed on this function by the form of the assumed buckled surface in equation (1). Further, Donnell found it necessary to introduce an arbitrary additive term in the denominator of his formula for the bucking stress in order to reduce the predicted values of this stress for the torsion problem. It is not clear that this term, chosen arbitrarily, may be expected to be suitable for use with the wide variety of constructions possible in plywood cylinders.

## Membrane Stresses and Strains

When buckling begins, both bending and membrane stresses are developed. The strains associated with the membrane stresses will be

[^2]uniform across the thickness of the cylindrical shell. These strains are expressed in terms of the displacements by the following equations:-
\[

$$
\begin{equation*}
e_{x x}=\frac{\partial u}{\partial x}, \quad e_{y y}=\frac{\partial v}{\partial y}-\frac{w}{r}, \quad e_{x y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{2}
\end{equation*}
$$

\]

By eliminating $\underline{u}$ and $v$ from equations (2), the following relation is found connecting the components of strain:

$$
\begin{equation*}
\frac{\partial^{2} e_{x x}}{\partial y}+\frac{\partial^{2} e_{y y}}{\partial x}-\frac{\partial^{2} e_{x y}}{\partial x \partial y}=-\frac{1}{r} \frac{\partial^{2} w}{\partial x^{2}} \tag{3}
\end{equation*}
$$

To obtain the relations between the stress and strain components, axes of reference $O_{\xi}$ and $O \eta$ will be chosen parallel and perpendicular, respectively, to the grain of the face plies. Let $\underline{\theta}$ denote the angle between $\mathrm{OE}_{5}$ and OX , that is, the inclination of the grain of the face plies to the axis of the cylinder. From the equations of transformation of strain components2, it follows that:

$$
\begin{align*}
& e_{x x}=e_{\xi \xi} \cos ^{2} \theta+e_{\eta \eta} \sin ^{2} \theta-e_{\xi \eta} \sin \theta \cos \theta \\
& e_{y y}=e_{\xi \xi} \sin ^{2} \theta+e_{\eta \eta} \cos ^{2} \theta+e_{\xi \eta} \sin \theta \cos \theta  \tag{4}\\
& e_{x y}=2 e_{\xi \xi} \sin \theta \cos \theta-2 e_{\eta \eta} \sin \theta \cos \theta+e_{\xi \eta}\left(\cos ^{2} \theta-\sin ^{2} \theta\right)
\end{align*}
$$

${ }^{-}$See equations (1), page 7, Forest Products Laboratory Report No. 1322-A. Since the small-deflection theory is to be used, the terms containing the derivatives of the displacement $W$ do not appear. For a discussion of the approximations involved in using these expressions for the strains and expressions to be used subsequently for the changes of curvature and unit twist of the middle surface, see pages 105-108 of the paper by L. H. Donnell, referred to in footnote 4. Also the paper, . "The Buckling of Thin Cylindrical Shells under Axial Compression," by Th. von Karman and H. S. Tsien, Jour. Aero. Sciences. 8, 303, 1941. 2see for example Forest Products Laboratory Report No. 1503.

Denote the "mean moduli in stretching"10 in directions parallel and perpendicular, respectively, to the grain of the face plies by $\mathrm{E}_{\mathrm{a}}$ and $\mathrm{E}_{\mathrm{b}}$, respectively, and the mean stress components referred to the axes $\mathrm{OF}_{5}$ and $O_{\eta}$ by $\bar{t}_{\xi \xi}, \bar{t}_{\eta \eta}, \bar{t}_{\xi \eta}$. It is readily established that 11
$e \xi \xi=\frac{E_{b}}{H} \epsilon_{\xi \xi}-\frac{E_{\mathrm{L}} \sigma \mathrm{TL}}{\mathrm{H}} \overline{\mathrm{E}} \mathrm{m}_{\eta}$,
$e \eta \eta=\frac{\mathrm{E}_{\mathrm{a}}}{\mathrm{H}} \mathrm{E}_{\eta \eta}-\frac{\mathrm{E}_{\mathrm{I}} \sigma_{\mathrm{TL}}}{\mathrm{H}} \overline{\mathrm{E}}_{\xi \xi}$,
$\mathrm{e} \xi \eta=\frac{1}{\mu} \mathrm{E}_{\mathrm{LT}}$,
where $\mathbb{E}_{L_{~}}$ is Young's modulus in the longitudinal direction,

$$
\begin{equation*}
H=\frac{\mathrm{E}_{\mathrm{a}} \mathrm{E}_{\mathrm{b}}-\mathrm{E}_{\mathrm{L}}^{2} \sigma_{\mathrm{TLL}}^{2}}{\lambda} \tag{6}
\end{equation*}
$$

$$
\begin{equation*}
\lambda=1-\sigma_{\mathrm{LT}} \sigma_{\mathrm{TL}}, \tag{7}
\end{equation*}
$$

$\sigma_{\text {LTT }}$ is a Poisson's ratio representing the ratio of the tangential contraction to the longitudinal extension associated with a longitudinal tension, and
$\mu_{\mathrm{LT}}=$ modulus of rigidity associated with a shearing strain with respect to axes parallel to the longitudinal and tangential directions, respectively. In the definitions just given the subscripts I and $\underline{T}$ refer to the longitudinal and tangential directions in the wood from which the rotary-cut veneers forming the plywood were cut.

If the veneers are made of wood of different species, the constants in equations (5) can be modified as explained in the latter part of Forest Products Laboratory Report No. 1503.

[^3]
## Stress Function

The equations of equilibrium satisfied by the mean stress components $\bar{t}_{\xi \xi}, \overline{\bar{t}}_{\eta \eta}, \overline{\bar{t}}_{\xi \eta}$ imply the existence of a stress function $F$ such that

$$
\begin{equation*}
\bar{E}_{\xi \xi}=\frac{\partial^{2} F}{\partial \eta^{2}}, \quad \bar{E}_{\eta \eta}=\frac{\partial^{2} F}{\partial \xi^{2}}, \quad \bar{E}_{\xi \eta}=-\frac{\partial^{2} F}{\partial \xi \partial \eta} \tag{8}
\end{equation*}
$$

After expressing the derivatives appearing in equation (8) in terms of the derivatives $\frac{\partial^{2} F}{\partial x^{2}}, \frac{\partial^{2} F}{\partial y^{2}}$, and $\frac{\partial^{2} F}{\partial x \partial y}$ with the aid of the transformation

$$
\begin{align*}
& x=\xi \cos \theta-\eta \sin \theta  \tag{9}\\
& y=\xi \sin \theta+\eta \cos \theta
\end{align*}
$$

the results are substituted in equation (5). The results of that substitution are substituted in equation (4), and finally the results of that substitution are inserted in equation (3). The following differential equation for the stress function $\underset{F}{ }$ is obtained:

$$
\begin{equation*}
M \frac{\partial^{4} F}{\partial x^{4}}+P \frac{\partial^{4} F}{\partial x^{3} \partial y}+S \frac{\partial^{4} F}{\partial x^{2} \partial y^{2}}+Q \frac{\partial^{4} F}{\partial x^{\partial} y^{3}}+N \frac{\partial^{4} F}{\partial y^{4}}=-\frac{1}{r} \frac{\partial^{2} W}{\partial x^{2}} \tag{10}
\end{equation*}
$$

where

$$
\begin{align*}
& M=A \sin ^{4} \theta+B \cos ^{4} \theta+C \sin ^{2} \theta \cos ^{2} \theta, \\
& P=2\left[(C-2 A) \sin ^{3} \theta \cos \theta-(C-2 B) \sin \theta \cos ^{3} \theta\right], \\
& S=(6 A+6 B-4 C) \sin ^{2} \theta \cos ^{2} \theta+C\left(\sin ^{4} \theta+\cos ^{4} \theta\right)  \tag{11}\\
& Q=2\left[(C-2 A) \sin \theta \cos ^{3} \theta-(C-2 B) \sin 3 \theta \cos \theta\right] \\
& N=A \cos ^{4} \theta+B \sin ^{4} \theta+C \sin ^{2} \theta \cos ^{2} \theta \\
& A=\frac{E_{b}}{H}, \quad B=\frac{E_{a}}{H}, \quad C=\frac{1}{\mu_{L T}}-\frac{2 E_{L} \sigma_{T L}}{H} .
\end{align*}
$$

Stress and Strain Components Referred
to the Axes OX and OX
The assumed form of the buckled surface is given by equation (1). It is convenient to rewrite this equation as

$$
\begin{equation*}
\frac{W}{r}=\frac{f}{2}[\cos (\alpha y-\delta x)-\cos (\alpha y-\epsilon x)]+g f \sin \beta x \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& \delta=\alpha \gamma+\beta  \tag{13}\\
& \epsilon=\alpha \gamma-\beta
\end{align*}
$$

The following equation is obtained by substituting equation (12) in equation (10):

$$
\begin{align*}
& M \frac{\partial^{4} F}{\partial x^{4}}+P \frac{\partial^{4} F}{\partial x^{3} \partial y}+S \frac{\partial^{4} F}{\partial x^{2} \partial y^{2}}+Q \frac{\partial^{4} F}{\partial x \partial y^{3}}+N \frac{\partial^{4} F}{\partial y^{4}}  \tag{14}\\
& \quad=\frac{f}{2}\left[\delta^{2} \cos (\alpha y-\delta x)-\epsilon^{2} \cos (\alpha y-\epsilon x)\right]+g f^{2} \beta^{2} \sin \beta x
\end{align*}
$$

A solution of equation (14) is

$$
\begin{equation*}
F=a_{1} \cos (\alpha y-\delta x)+a_{2} \cos (\alpha y-\epsilon x)+a_{3} \sin \beta x \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
& a_{1}=\frac{f \delta^{2}}{2\left(M \delta^{4}-P \alpha \delta^{3}+S \alpha^{2} \delta^{2}-Q \alpha^{3} \delta+N \alpha^{4}\right)} \\
& a_{2}=-\frac{f \epsilon^{2}}{2\left(M \epsilon^{4}-P \alpha \epsilon^{3}+S \alpha^{2} \epsilon^{2}-Q \alpha^{3} \epsilon+N \alpha^{4}\right)}  \tag{16}\\
& a_{3}=\frac{g f}{M \beta^{2}}
\end{align*}
$$

The following expressions for the mean stress components referred to the axes $O X$ and $O Y$ are obtained from equation (15):

$$
\begin{align*}
& X_{x}=\frac{\partial^{2} F}{\partial y^{2}}=-a_{1} \alpha^{2} \cos (\alpha y-\delta x)-a_{2} \alpha^{2} \cos (\alpha y-\epsilon x) \\
& Y_{y}=\frac{\partial^{2} F}{\partial x^{2}}=-a_{1} \delta^{2} \cos (\alpha y-\delta x)-a_{2} \epsilon^{2} \cos (\alpha y-\epsilon x)-a_{3} \beta^{2} \sin \beta x  \tag{17}\\
& X_{y}=-\frac{\partial^{2} F}{\partial x \partial y}=-a_{1} \alpha \delta \cos (\alpha y-\delta x)-a_{2} \alpha \epsilon \cos (\alpha y-\epsilon x)
\end{align*}
$$

Next the equations connecting the strain components $e_{x x}, \ldots$ and the stress components $X_{x}, \ldots$ will be obtained by expressing the strain components of equations (5) in terms of the mean stress components $X_{x}$, ... by means of transformation (9), and substituting the resulting expressions in equations (4). For any direction of the grain of the face plies, the connecting equations are:

$$
\begin{align*}
& e_{x x}=c_{11} x_{x}+c_{12} Y_{y}+c_{13} X_{y} \\
& e_{y y}=c_{12} X_{x}+c_{22} Y_{y}+c_{23} X_{y}  \tag{18}\\
& e_{x y}=c_{13} X_{x}+c_{23} Y_{y}+c_{33} X_{y}
\end{align*}
$$

where

$$
\begin{align*}
& c_{11}=C \sin ^{2} \theta \cos ^{2} \theta+A \cos ^{4} \theta+B \sin ^{4} \theta \\
& c_{12}=\left(A+B-\frac{1}{\mu L T}\right) \sin ^{2} \theta \cos ^{2} \theta-\frac{E_{L} \sigma_{T L}}{H}\left(\sin ^{4} \theta+\cos ^{4} \theta\right) \\
& c_{13}=(C-2 B) \sin ^{3} \theta \cos \theta+(2 A-C) \sin \theta \cos ^{3} \theta \\
& c_{22}=C \sin ^{2} \theta \cos ^{2} \theta+A \sin ^{4} \theta+B \cos ^{4} \theta  \tag{19}\\
& c_{23}=(2 A-C) \sin ^{3} \theta \cos \theta+(C-2 B) \sin \theta \cos ^{3} \theta \\
& c_{33}=4(A+B-C) \sin ^{2} \theta \cos ^{2} \theta+\frac{1}{\mu L T}
\end{align*}
$$

## Strain Energy of the Shell

The strain energy associated with the membrane stresses in the shell is given by

$$
\begin{align*}
w_{1} & =\frac{h}{2} \int_{0}^{b_{0}} \int_{0}^{2}\left\{c_{11} X_{x}^{2}+c_{22} Y_{y}^{2}+c_{33} X_{y}^{2}+2 c_{12} X_{x} Y_{y}\right.  \tag{20}\\
& \left.+2 c_{13} X_{x} X_{y}+2 c_{23} X_{y} Y_{y}\right\} d y d x
\end{align*}
$$

where the coefficients $c_{\text {Il }}$, ... are given in equations (19), and $\underline{h}$ is the thickness of the shell. On substituting equations (17) in equation (20) and performing the integration, it is found that

$$
\begin{align*}
W_{1} & =\frac{h b 2 \pi r}{4}\left\{c_{11}\left[a_{1}^{2} \alpha^{4}+a_{2} \alpha^{4}\right]+c_{22}\left[a_{1}^{2} \delta^{4}+a_{2}^{2} \epsilon^{4}+a_{3}^{2} \beta^{4}\right]\right. \\
& +c_{33}\left[a_{1}^{2} \alpha^{2} \delta^{2}+a_{2}^{2} \alpha^{2} \epsilon^{2}\right]+2 c_{12}\left[a_{1}^{2} \alpha^{2} \delta^{2}+a_{2}^{2} \alpha^{2} \epsilon^{2}\right]  \tag{2I}\\
& \left.+2 c_{13}\left[a_{1}^{2} \alpha^{3} \delta+a_{2}^{2} \alpha^{3} \epsilon\right]+2 c_{23}\left[a_{1}^{2} \alpha \delta^{3}+a_{2}^{2} \alpha \epsilon^{3}\right]\right\}
\end{align*}
$$

The strain energy of bending of the shell is given by

$$
\begin{align*}
W_{2} & =\frac{h^{3}}{24} \iint_{S}\left[E_{1}\left(\frac{\partial^{2}}{\partial \xi^{2}}\right)^{2}+E_{2}\left(\frac{\partial^{2}}{\partial \eta^{2}}\right)^{2}+2 E_{I} \sigma_{M L}\left(\frac{\partial^{2} W}{\partial \xi^{2}}\right)\left(\frac{\partial^{2}}{\partial \eta^{2}}\right)\right.  \tag{22}\\
& \left.+4 \lambda \mu_{L T}\left(\frac{\partial^{2}}{\partial \xi \partial \eta}\right)_{i}^{2}\right] d \xi d \eta,
\end{align*}
$$

where $E_{1}$ and $E_{2}$ are proportional to the flexural rigidities of the plywood in the wall of the shell in the $\xi^{-}$and $\eta$-directions, respectively. The flexural rigidities in these two directions are $\mathrm{E}_{1} \mathrm{I}$ and $\mathrm{E}_{2} \mathrm{I}$, respectively, where $I$ is the moment of inertia of the entire cross section of unit width of the plywood with respect to its central line. The integration is to be extended over the entire middle surface of the shell. Transforming the expression for the strain energy of bending by means of equation (9), it is found that

$$
\begin{align*}
W_{2} & =\frac{h^{3}}{24 \lambda} \int_{0}^{b} \int_{0}^{2 \pi}\left\{b_{1}\left(\frac{\partial^{2} w}{\partial x^{2}}\right)^{2}+b_{2}\left(\frac{\partial^{2}}{\partial y^{2}}\right)^{2}+b_{3}\left(\frac{\partial^{2} w}{\partial x}\right)^{2}\right. \\
& \left.+b_{4} \frac{\partial^{2}}{\partial x^{2}} \frac{\partial^{2}}{\partial x \partial y}+b_{5} \frac{\partial^{2} w}{\partial x^{2}} \frac{\partial^{2} w}{\partial y^{2}}+b_{6} \frac{\partial^{2} w}{\partial y^{2}} \frac{\partial^{2}}{\partial x \partial y}\right\} d y d x \tag{23}
\end{align*}
$$

where

$$
\begin{align*}
b_{1}= & E_{I} \cos ^{4} \theta+E_{2} \sin ^{4} \theta+2 E_{L} \sigma_{I L} \sin ^{2} \theta \cos ^{2} \theta+4 \lambda \mu_{I T} \sin ^{2} \theta \cos ^{2} \theta \\
b_{2}= & E_{I} \sin ^{4} \theta+E_{2} \cos ^{4} \theta+2 E_{L} \sigma_{I L} \sin ^{2} \theta \cos ^{2} \theta+4 \lambda \mu_{I T} \sin ^{2} \theta \cos ^{2} \theta  \tag{24}\\
b_{3}= & 4 E_{1} \sin ^{2} \theta \cos ^{2} \theta+4 E_{2} \sin ^{2} \theta \cos ^{2} \theta-8 E_{I} \sigma_{I I} \sin ^{2} \theta \cos ^{2} \theta \\
& +4 \lambda \mu_{I T}\left(\cos ^{2} \theta-\sin ^{2} \theta\right)^{2} \\
b_{4}= & 4 E_{1} \sin \theta \cos ^{3} \theta-4 E_{2} \sin ^{3} \theta \cos \theta-4 E_{I} \sigma_{T I}\left(\cos ^{2} \theta-\sin ^{2} \theta\right) \sin \theta \cos \theta \\
- & 8 \lambda \mu_{I T}\left(\cos ^{2} \theta-\sin ^{2} \theta\right) \sin \theta \cos \theta \\
b_{5}= & 2 E_{I} \sin ^{2} \theta \cos ^{2} \theta+2 E_{2} \sin 2 \theta \cos ^{2} \theta+2 E_{L} \sigma_{I L}\left(\sin ^{4} \theta+\cos ^{4} \theta\right) \\
- & 8 \lambda \mu_{I T} \sin ^{2} \theta \cos ^{2} \theta \\
b_{6}= & 4 E_{I} \sin 3 \theta \cos \theta-4 E_{2} \sin \theta \cos ^{3} \theta+4 E_{L} \sigma_{I L}\left(\cos ^{2} \theta-\sin ^{2} \theta\right) x \\
& \sin \theta \cos \theta+8 \lambda \mu_{I T}\left(\cos ^{2} \theta-\sin ^{2} \theta\right) \sin \theta \cos \theta
\end{align*}
$$

By substituting the derivatives obtained from equation (12) in equation (23) and integrating, it is found that

$$
\begin{align*}
W_{2} & =\frac{f^{2} r^{2} h^{3} b 2 \pi r}{192 \lambda}\left\{b_{1}\left[\delta^{4}+\epsilon^{4}+4 g^{2} \beta^{4}\right]+2 b_{2} \alpha^{4}\right. \\
& +\left(b_{3}+b_{5}\right)\left[\alpha^{2} \delta^{2}+\alpha^{2} \epsilon^{2}\right]-b_{4}\left[\alpha^{3}+\alpha \epsilon^{3}\right]  \tag{25}\\
& \left.-b_{6}\left[\alpha^{3} \delta+\alpha^{3} \epsilon\right]\right\}
\end{align*}
$$

The work done by the applied load during buckling is given by

$$
\begin{equation*}
w_{3}=-\tau h \int_{0}^{b} \int_{0}^{2 \pi r} \frac{\partial w}{\partial x} \frac{\partial}{\partial} w \frac{y}{y} d y d x \tag{26}
\end{equation*}
$$

where I is the uniform shearing stress induced in the cylindrical shell by the couples applied at its ends. On substituting from equation (12), equation (26) after performing the integration becomes:

$$
\begin{equation*}
W_{3}=\frac{\operatorname{Th} b \pi r^{3} f_{f}^{2} \alpha^{2} \gamma}{2} \tag{27}
\end{equation*}
$$

## Critical Value of I

The critical value of $I$ is found from the equation

$$
\begin{equation*}
W_{3}=W_{1}+W_{2} \tag{28}
\end{equation*}
$$

Where $W_{1}, W_{2}$, and $W_{3}$ are given by equations (21), (25), and (27), respectively. On substituting these values of $W_{1}, W_{2}$, and $W_{3}$ in equation (28) and using equations (16), the following expressions for $I$ as a function of the parameters $\underline{\alpha}, B, \mathcal{Y}$, and g is found:

$$
\begin{align*}
& T=\frac{1}{4 r^{2} \alpha^{2} \gamma} \times \\
& \left\{\frac{\delta^{4}\left[c_{11} \alpha^{4}+c_{22} \delta^{4}+c_{33} a^{2} \delta^{2}+2 c_{12} \alpha^{2} \delta^{2}+2 c_{13} \alpha^{3} \delta+2 c_{23^{2}} \alpha^{3}\right]}{\left[M \delta^{4}-P \alpha \delta^{3}+S \alpha^{2} \delta^{2}-Q \alpha \alpha^{3} \delta+N \alpha^{4}\right]^{2}}\right. \\
& +\frac{\epsilon^{4}\left[c_{11} \alpha^{4}+c_{22} \epsilon^{4}+c_{33} \alpha^{2} \epsilon^{2}+2 c_{12} \alpha^{2} \epsilon^{2}+2 c_{13} \alpha^{3} \epsilon+2 c_{23^{2}} \alpha_{\epsilon}\right]}{\left[M \epsilon^{4}-P \alpha \epsilon^{3}+S \alpha^{2} \epsilon^{2}-Q \alpha 3 \epsilon+N \alpha^{4}\right]^{2}} \tag{29}
\end{align*}
$$

$$
\begin{aligned}
& \left.+\frac{4 g^{2} c_{22}}{M^{2}}\right\}+\frac{h^{2}}{48 \lambda a^{2} \gamma}\left\{b_{1}\left[\delta^{4}+\epsilon^{4}+4 g^{2} \beta^{4}\right]+2 b_{2} \alpha^{4}\right. \\
& \left.+\left(b_{3}+b_{5}\right)\left[\alpha^{2} \delta^{2}+\alpha^{2} \epsilon^{2}\right]-b_{4}\left[\alpha \delta^{3}+\alpha \varepsilon^{3}\right]-b_{6}\left[\alpha^{3} \delta+\alpha^{3} \varepsilon\right]\right\}
\end{aligned}
$$

The critical value of $I$ is given by the minimum value of the right-hand member of equation (29). It is evident from the way in which the variable $g$ enters into equation (29) that this minimum value occurs when $g=0$, that is, $\frac{\delta T}{\delta g}=0$ only when $g=0$.

In order to facilitate finding the minimum value of equation (29), the following notation will be introduced. Let

$$
\begin{equation*}
\rho=\frac{\beta}{\alpha}, \quad k=\frac{r \tau}{E_{\mathrm{L}} h}, \quad J=\frac{b^{2}}{r h} \tag{30}
\end{equation*}
$$

Then equation (29) can be written in the form:

$$
\begin{equation*}
k=J f_{1}(\rho, \gamma)+\frac{1}{J} f_{2}(\rho, \gamma) \tag{31}
\end{equation*}
$$

where

$$
\begin{align*}
& f_{1}=\frac{\rho^{2}}{4 \pi^{2} E_{L} \gamma} \times \\
& \left\{\frac{(\gamma+\rho)^{4}\left[c_{11}+2 c_{13}(\gamma+\rho)+\left(c_{33}+2 c_{12}\right)(\gamma+\rho)^{2}+2 c_{23}(\gamma+\rho)^{3}+c_{22}(\gamma+\rho)^{4}\right]}{\left[M(\gamma+\rho)^{4}-P(\gamma+\rho)^{3}+S(\gamma+\rho)^{2}-Q(\gamma+\rho)+N\right]^{2}}\right. \\
& \left.+\frac{(\gamma-\rho)^{4}\left[c_{11}+2 c_{13}(\gamma-\rho)+\left(c_{33}+2 c_{12}\right)(\gamma-\rho)^{2}+2 c_{23}(\gamma-\rho)^{3}+c_{22}(\gamma-\rho)^{4}\right]}{\left[M(\gamma-\rho)^{4}-P(\gamma-\rho)^{3}+S(\gamma-\rho)^{2}-Q(\gamma-\rho)+N\right]^{2}}\right\} \tag{32}
\end{align*}
$$

$$
\begin{aligned}
f_{2}= & \frac{\gamma^{2}}{48 \lambda E_{\mathrm{L}} \gamma \rho^{2}}\left\{b_{1}\left[(\gamma+\rho)^{4}+(\gamma-\rho)^{4}\right]+2 b_{2}+\left(b_{3}+b_{5}\right) \times\right. \\
& {\left.\left[(\gamma+\rho)^{2}+(\gamma-\rho)^{2}\right]-b_{4}\left[(\gamma+\rho)^{3}+(\gamma-\rho)^{3}\right]-2 b_{6} \gamma\right\} . }
\end{aligned}
$$

The right side of equation (31) has been expressed in terms of the three quantities $J, f_{1}$, and $f_{2}$ by the notation introduced in equations (30) and (32). The expression represented by $\bar{J}$ depends on the dimensions of the plywood cylinder, while each of the expressions represented by $f_{l}$ and $f_{2}$ depends on the construction and elastic constants of the plywood, and the two parameters $\underline{y}$ and $\underline{\rho}$. Hence for a given cylinder, $\underline{k}$, which is proportional to $\underline{I}$, can be expressed as a function of $\underline{\gamma}$ and $\underline{\rho}$ by means of equation (31). The problem is to find the minimum value of $\underline{k}$. It is convenient to do this graphically by first assigning a value $\gamma_{I}$ to $\gamma$ and then determining a minimum value of $\underline{k}$, say $\underline{k}_{\underline{1}}$, by plotting $\underline{k}$ as a function of $\rho$. A second value $\gamma$, of $\underline{\gamma}$ is assigned and a corresponding minimal k is found by the sēme method. The process is repeated until a pair of values $\underline{\gamma}_{i}$, $\mathrm{ki}_{i}$ is found (fig. 2) such that any change in the value $\underline{\gamma_{i}}$ assigned $\underline{\gamma}$ leads to a larger value of $\underline{k}$ than $k_{\underline{1}}$. This value $\mathrm{k}_{1}$ of $k$ theoretical $\overline{\mathrm{s}}$ the minimum value of the right-hand member of equation (31), and from it the theoretical buckling stress may be obtained (equation (30)). The choice of $\gamma_{1}, \gamma_{2},-\cdots-$ is guided by the fact that the arctan $y$, the angle of inclination of the wrinkles to the axis of the cylinder, lies between $10^{\circ}$ and $30^{\circ}$ for most of the plywood cylinders considered here.

## Theoretica 1 Curves

The elastic moduli used for Douglas-fir plywood are given in table 1. The values of $\mathrm{k}_{\text {theoretical }}$ were computed by the method given in the previous section for several values of $\bar{J}$ for each of several constructions of Douglas-fir plywood (table 2). The angles of inclination of face grain to the axis of the cylinder considered are $0^{\circ}, 90^{\circ}$, and $45^{\circ}$. For the constructions considered and for a given angle of face grain and value of $\underline{J}$, the values of $k_{\text {theoretical }}$ can be represented satisfactorily by a smooth curve when plotted as a function of the ratio $\frac{E_{1}}{E_{1}+E_{2}}$. Constructions that have values of $\mathrm{E}_{\mathrm{a}}$ and $\mathrm{E}_{\mathrm{b}}$ that depart markedly from those of the constructions given will have values of $k_{\text {theoretical }}$ different from those given by the theoretical curves, but this difference will be small for all practical constructions. This difference is due to the fact that $k_{\text {theoretical }}$ is a function of $\underline{E}_{\mathrm{a}}$ and $\underline{\mathrm{E}}_{\mathrm{b}}$ as well as
$\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ for a given shear modulus and construction. The constructions used in the computations are chosen to give values of $\frac{E_{1}}{E_{1}+E_{2}}$ that are approximately evenly spaced over the interval from $l / 2$ to $l$ in which the usual constructions will lie. A series of the curves for $k_{\text {theoretical }}$ are given in figure 3 where all curves in each group are drawn for the same angle of face grain.

For plywood made of other species than Douglas-fir, the ratio of $\mu_{\text {IT }}$ to $\mathrm{E}_{\mathrm{L}}$ is likely to be different, and additional curves for $\mathrm{k}_{\text {theoretical }}$ are given in figures 4 and 5 for two different shear moduli combined with the other elastic moduli of Douglas-fir in table l. The various shear moduli used are $80,000,123,800$, and 200,000 pounds per square inch, which give ratios of $\mu_{\mathrm{LT}}$ to $\mathrm{E}_{\mathrm{L}}$ of approximately $0.036,0.056$, and 0.090 , respectively. The curves (figs. 4 and 5) show that it is necessary to consider a variation of the ratio of $\mu_{\mathrm{LT}}$ to $\underline{E}_{\mathrm{L}_{\text {}}}$, since for some cases a change of from 0.036 to 0.090 in this ratio for the same construction, angle of face grain, and value of $J$ gives a difference of 45 percent in the value of $k_{\text {theoretical. }}$.

## Use of the Theoretical Curves for Plywood Cylinders

The procedure for finding the theoretical buckling stress for a given cylinder follows. Compute $E_{1}$ and $E_{2}$, the effective moduli of elasticity of the plywood in bending, by means of the equations

$$
E=\frac{1}{I} \sum_{i=1}^{i=n}\left(E_{\xi}\right)_{i} I_{i}, \quad E=\frac{1}{I} \sum_{i=1}^{i=n}\left(E_{\eta}\right)_{i} I_{i}
$$

Where $\underline{n}$ is the number of plies, $E \xi$ and $E \eta$ are Young's moduli in each ply in the $\xi$ - and $\eta$-directions, respectively (fig. 1 ), $I$ is the moment of inertia about the centerline of the total cross section perpendicular to the direction considered, and $I_{i}$ is the moment of inertia of the $i^{\text {th }} \mathrm{ply}$ about the centerline of the same total cross section. Compute next the ratio $\frac{E_{1}}{E_{1}+E_{2}}$ and the parameter $J=\frac{b^{2}}{r h}$ where $\underline{r}, \underline{b}$, and $\underline{h}$ are the radius, length, and thickness, respectively of the plywood cylinder. In figures 3 , 4 , and 5 find the group of curves drawn for the desired angle of face grain and for the ratio $\frac{\mu_{\mathrm{LT}}}{E_{T}}$ nearest that of the given cylinder. The modulus $\mathrm{E}_{\mathrm{L}}$ is Young's modulus measured parallel to the grain of the
wood from which the plywood is cut, and the modulus $\mu_{\text {LTT }}$ is the modulus of rigidity associated with a shearing strain with respect to axes parallel to the $\xi$ - and $\eta$-directions, respectively. For many plywood cylinders, the ratio $\frac{\mu_{\mathrm{LT}}}{\mathrm{E}_{\mathrm{L}}}$ will be approximately 0.056 . Then after locating the point representing the computed value of $\frac{E_{1}}{E_{1}+E_{2}}$ on the horizontal scale, the value of $k_{\text {theoretical }}$ corresponding to the given J can be found by interpolation between the ordinates of the curves drawn for various values of $J$. Then the minimum value of $I$, Tcr, obtained from equation (33), is

$$
T_{c r}=k \frac{E_{\mathrm{L}} h}{r}
$$

For the same construction, angle of face grain, and value of J, the values of k show approximately a linear variation with the ratio $\mu_{\text {LT }}$. The truth of this statement can be shown by taking values of k $\mathrm{E}_{\mathrm{L}}$ from the curves. Hence if $\mu_{\text {LT }}$ and $\underline{E}_{\text {L }}$ are known and if the ratio $\frac{\mu_{I T}}{\mathrm{E}_{\mathrm{L}}}$ differs from the ones used in figures 3, 4, or 5, the values of $\underline{k}$ can be found by interpolation.

## Experimental Analysis

## Description of Specimens

Specimens were made of yellow birch or yellow-poplar plywood. All veneer was of aircraft grade, rotary cut at the Forest Products Laboratory. All plywood was made flat, using a phenolic resin film glue set in a hot press. The cylinders were formed by bending the flat plywood around a hot mandrel (fig. 6). Prior to forming, the plywood was scarfed, then moistened to facilitate bending, and finally bent around the mandrel upon which the scarf joint was glued with a thermosetting synthetic resin glue. Cylinders were made of two diameters, 10-1/2 inches and 3 inches. Cylinders of 3 -inch diameter were tested to investigate the range in which the buckling stresses approach the shear strengths of the plywood. Most of the specimens had a diameter of $10-1 / 2$ inches. Specimens of the larger diameter had 3/4-inch snugly fitting plywood plates glued in each end. The specimens, 3 inches in diameter, were fitted with birch end plugs, which were glued to a depth of 4 inches. This was necessary to provide sufficient glue area and, therefore, sufficient strength in the joint to force the failure into the plywood shell rather than to allow a glue joint failure.

The plywood of the cylinders was of several constructions, as listed in table 3. The four-ply construction was made of $1 / 100$-inch veneers with the grain direction of the two inner plies parallel to each other and perpendicular to that of the face plies. All other plywood was manufactured with the grain directions of adjacent plies at right angles.

The grain of the face plies of the cylinders was oriented in four different directions: Face grain parallel to the axis of the cylinder ( $\theta=0^{\circ}$ ), face grain circumferential $\left(\theta=90^{\circ}\right)$, face grain oriented at $45^{\circ}$ to the axis so that the face plies were placed in tension parallel to the grain $\left(\theta=+45^{\circ}\right)$, and face grain oriented at $45^{\circ}$ to the axis so that the face plies were placed in compression parallel to the grain ( $\theta=-45^{\circ}$ ).

Three or five specimens were made for each type of plywood, diameter, length, and face grain direction as listed in table 3. The veneers in each group of specimens were matched as closely as possible by cutting them from adjacent portions of the log. In order to obtain the mechanical properties of material similar to that of the cylinders, flat sheets of matched plywood were prepared. These sheets provided specimens for testing in bending, compression, and tension.

No attempt was made to condition the specimens to a particular moisture content other than to allow them to remain in the testing laboratory for about a week before testing. All specimens of a group were tested the same day.

## Testing Methods

The apparatus used to test the cylinders is shown in figures 7 and 8.
The cylinders that were $10-1 / 2$ inches in diameter were tested in the apparatus shown in figure 7. The end plates of the cylinders were fastened to the loading arms and the end bracket by means of pins. The load was applied by means of the movable head of a testing machine and was measured on the scale placed between the loading arms and testing machine.

The cylinders that were 3 inches in diameter were tested in the apparatus shown in figure 8. The end plugs were clamped in the jaws of the torsion testing machine and the load measured on the scale. The loads were applied slowly and at a uniform rate until failure occurred.

In some instances, a troptometer was used to measure shear deformations for computations of the modulus of rigidity $\mu_{L T}$. The troptometer was of the wire-wound, differential-dial type. The wires were wound one turn around the specimen and then weighted, causing enough friction so that the wires would not slip, but would follow the twist of the
specimen without injuring it in any manner. When the shear modulus was not determined for a particular cylinder, an average value of $\frac{\mu_{L T}}{E_{T}}$, taken from tests of other cylinders of the same species, was ${ }^{L}$
used in determining the constant $\underline{k}$.
Minor specimens were tested in bending, compression, and tension to determine the mechanical properties of the material. The testing procedures followed for these coupons are described in the appendix.

## Test Results

The manner of failure of the cylinders was either by (1) formation of an elastic buckle, (2) buckling followed inmediately by breaking, or (3) shearing of the plywood. Column 13 of table 3 describes the type of failure for each specimen.

Fallure due to the forming of an elastic buckle was usually sudden. As soon as the buckle appeared, the torque immediately dropped to a low value. In some instances progressive buckling occurred, that is, one buckle would appear, then another, and so on. In general, the initial buckle was long and narrow, with its long axis at an angle to the axis of the cylinder. Some measurements of this angle showed that theoretical and experimental values agreed fairly well. If deformation was continued after formation of the initial buckle, several smaller buckles occasionally appeared inside the large initial buckle. As was true for buckles formed in thin-walled cylinders in compression, the torsion buckles also deflected inward from the original surface except at the edge of the buckle.

Buckling, followed immediately by breaking, occurred in the thicker and shorter specimens. The buckle formed, and then the plywood broke with a loud report. The breaking was caused by the high bending stresses that were induced in the plywood due to the sharp radius of curvature of the buckle.

Shearing of the plywood occurred in the short specimens of smaller diameter. This type of failure was instantaneous and explosive. A failed specimen is shown in figure 8.

## Computation of Results

The computations of test results consisted of calculating the shearing stress at buckling from the applied torque. From this stress the value of the constant $k_{\text {test }}$ was computed. The experimentally determined
values of $k$ were compared with the theoretical values. Both sets of values are presented in table 3 and are plotted in figures 9 to 13. The computations were carried out as follows:

The shearing stress at failure (column 15, table 3) was computed by means of the formula

$$
T=\frac{T}{2 \pi r^{2} h}
$$

where

$$
\begin{aligned}
& T=\text { shearing stress } \\
& T=\text { torque at fallure } \\
& T=\text { mean radius of the plywood cylinder } \\
& h=\text { plywood thickness }
\end{aligned}
$$

The experimental buckling constant (k) (column 17, table 3) was computed from the formula

$$
k=\frac{\pi r}{E_{L} h}
$$

where $\mathrm{E}_{\mathrm{L}}$ is the modulus of elasticity of the veneers of the plywood measured parallel to the grain. Average values of $\mathrm{E}_{\mathrm{L}}$ (column 16, table 3) were determined from test data of minor coupons from the formula

$$
E_{L}=\frac{E_{1}+E_{2}}{C}=\frac{E_{a}+E_{b}}{C}
$$

where
$C=a$ constant depending on the species of veneer. For yellow birch $C=1.045$, and for yellow-poplar $C=1.037$. (See Forest Products Laboratory Report 1304 "Methods of Computing the Strength and Stiffness of Plywood Strips in Bending")
$E_{I}=$ bending modulus of elasticity of the plywood with the face grain parallel to the span (table 4).
$\mathrm{E}_{2}=$ bending modulus of elasticity of the plywood with the face grain perpendicular to the span (table 4).
$\mathrm{E}_{\mathrm{a}}=$ compressive or tensile modulus of elasticity of the plywood measured parallel to the face grain (table 4).
$\mathrm{E}_{\mathrm{b}}=$ compressive or tensile modulus of elasticity of the plywood measured perpendicular to the face grain (table 4).

In order to compare the experimental velues with the theoretical values, it is necessary to compute theoretical values of the buckling constant ( $k_{\text {theoretical }}$ ). Values of
$J=\frac{b^{2}}{r h}$ (column 14, table 3) were computed from test measurements, where
$\mathrm{b}=$ length of the cylinder (column 11, table 3)
$r=$ mean radius of curvature (computed from outside diameter in column 9 table 3)
$\mathrm{h}=$ plywood thickness (column 10, table 3)

Values of $\frac{E_{1}}{E_{1}+E_{2}}$ (column 19, table 3) were computed from the formula $\frac{E_{1}}{E_{1}+E_{2}}=\frac{K_{1}}{K_{1}+K_{2}}$, where $K_{1}$ and $K_{2}$ are given by the following formula

where
$I=$ moment of inertia of the entire cross section of unit width about its centerline $\left(I=\frac{h^{3}}{12}\right)$
$I_{1}=$ moment of inertia of the $i^{\text {th }} p 1 y$ of unit width about the neutral axis of the cross section.
$\mathrm{E}_{1}$. $=$ modulus of elasticity of the $i^{\text {th }} \mathrm{ply}$, measured parallel to the span. For plies having the face grain parallel to the span $\mathrm{E}_{\mathrm{i}}=\mathrm{E}_{\mathrm{ln}}$; for plies having face grain perpendicular $\mathrm{E}_{\mathrm{i}}=0.045 \mathrm{E}_{\mathrm{L}}$ for yellow birch, and $\mathrm{E}_{\mathrm{i}}=0.037 \mathrm{E}_{\mathrm{L}}$ for yellow-poplar.

Using appropriate values of $\frac{E_{1}}{E_{1}+E_{2}}$ and $J$, the curves of figures 3,4, and 5 were entered and values of $k_{\text {theoretical }}$ were determined for the ratio $\frac{\mu_{\text {LT }}}{\mathrm{E}_{\mathrm{L}}}$ equal to $0.056,0.036$, and 0.090 , respectively. Values of $k_{\text {theoretical }}$ for test values of $\frac{\mu_{L T}}{\mathrm{E}_{\mathrm{L}}}$ (column 20, table 3 ) were then determined by interpolation.

The shearing strengths of the yellow birch plywood, with faces of $1 / 100-$ inch veneer with the grain direction axial, and with cores of two $1 / 100$ inch veneers with the grain direction circumferential $\left(\theta=0^{\circ}\right.$ or $\left.\theta=90^{\circ}\right)$, were determined by torsion tests on 3-inch diameter cylinders of short lengths, so that no buckling would occur. The shear strength for yellow birch plywood of the same construction, but having the face grain of the cylinder at $\pm 45^{\circ}$, was calculated from the formula

$$
f_{s}=\frac{f_{c}}{\sqrt{1+\left(\frac{f_{c}}{f_{t}}\right)^{2}}}
$$

where
$f_{c}=$ compressive strength of the plywood
$f_{t}=$ tensile strength of the plywood
Values of the compressive or tensile strength were obtained from tests on minor coupons.

The theoretical buckling stress (Theoretical) was computed from the formula

$$
\tau_{\text {theoretical }}=\frac{k_{\text {theoretical }}}{k_{\text {test }}} \times \tau_{\text {test }}
$$

## Discussion of Results

A comparison of theoretical buckling constants with experimental constants is shown in figure 9 where $k_{\text {theoretical }}$ is plotted against $k_{\text {test. }}$ The line drawn on the graph represents the relation $k_{\text {theoretical }}=$ Etest and does not represent the average of the points plotted. Each point represents the test results of one cylinder of 10-1/2-inch diameter. Much of the scatter of these points is attributed to the fact that
small initial imperfections not taken into account by the theory may cause premature buckling, and to variations in the dimensions of the specimen, and in the mechanical properties of the plywood. Measurements taken on several similar specimens showed that the thickness of the plywood may vary as much as 7 percent on a single cylinder:

Figures 10 to 13 show values of $k_{\text {test }}$ plotted against values of $k_{\text {theoretical }}$ for cylinders 10-1 2 inches in diameter of each different plywood construction.

The agreement of test and theoretical values of the buckling constant, as shown in figures 9 to 13, is regarded as satisfactory, considering the many possible causes that can contribute to the variation of the experimental values.

In the theory it was assumed that the ends of the cylinders were simply supported, while in test they were glued to circular plates inserted at each end. The effect of the resultant restraint should manifest itself in an increase in the value of the constant $k$ with decreasing length for a series of cylinders of the same radius, thickness, and type of plywood. In the different series of tests shown in table 3 of the cylinders $10-1 / 2$ inches in diameter and of lengths ranging from 38.5 to 8.5 inches, such an increase was not evident. In these series, the plates glued in at the ends were $3 / 4$ inch thick.

In order to obtain information regarding the influence of the shear strength on the buckling stress of the plywood cylinders, tests were made on specimens of three-ply plywood 3 inches in diameter. The face plies were one-half as thick as cores, and the lengths varied from 5 inches to $1 / 8$ inch. The shorter specimens failed in shear without buckling. Ratios of theoretical and experimental buckling stresses to shearing strengths ( $\frac{T_{\text {theoretical }}}{f_{S}}$ and $\frac{T_{\text {test }}}{f_{S}}$ ) are plotted in figure 14. It was expected that the points would fall along the straight line until the buckling stress was about 60 percent of the shearing strength, that beyond this point the theoretical ratio would exceed the experimental, and that a design curve could be drawn as it was for cylinders in compression (Forest Products Laboratory Report No. l322). Although the data are widely scattered, an approximate design curve of this character can be drawn. Buckling constants determined from specimens 3 inches in diameter showed no definite trend due to the more rigid end restraint provided by the long loading plugs and to the change in lengths of the specimens. This expected increase was not observed and was probably obscured by the approach of the shear stresses to the shear strength for the plywood.

In connection with design, it should be noted that many of the cylinders broke immediately after buckling occurred at relatively low shearing stresses. This was due to the high bending stresses induced by the sharp curvature of the buckle.

The theory presented in this report was used to compute buckling constants for thin cylinders of brass and steel reported in NACA Report No. 479, "Stability of Thin-Walled Tubes Under Torsion" by L. H. Donnell. Values of the test buckling constant versus the theoretical buckling constant, as computed by the method presented herein, are plotted in figure 15. The test values are all lower than the theoretical. In NACA Report No. 479, it was found that all observed buckling stresses were greater than 60 percent of the theoretical buckling stresses calculated by the theory of that report. By drawing the line $\mathrm{k}_{\text {test }}=0.60$ $k_{\text {theoretical }}$ in figure 15, it can be seen that this statement is also true for the buckling stresses calculated by the approximate theory of the present report. As previously indicated, the theory assumes no initial imperfections of shape in the cylinders under consideration. since the metal used in the tests of Report, No. 479 was thin, the magnitude of the departures from true cylindrical form, as compared with the thickness of the wall of the cylinder, were probably much larger than those of the plywood cylinders and consequently lower test values of the torsion buckling constant would be expected.

## Conclusions

The buckling stresses of thin plywood cylinders can be computed by the approximate formula

$$
T=k \mathrm{E}_{\mathrm{L}} \frac{\mathrm{~h}}{\mathrm{r}}
$$

where $k$ is taken from the curves presented in this report. Approximate failing stresses for cylinders in which the buckling stresses are influenced by the shear strength of the plywood can be determined by use of figure 14 in connection with the theory. The data scatter considerably, but average values obtained from test agree well with the theory.

Plywood for which the ratio $\frac{E_{1}}{E_{1}+E_{2}}$ is less than 0.9 can be used effectively in thin-walled cylinders in torsion. For ratios larger than 0.9 , the steepness of the curves (figs. 3 to 5 ) shows that caution should be exercised in their use.

The theoretical analysis may be used for any angle of inclination of the face grain of the cylinder (equation 31). The angles of face grain for which curves are given in this report were selected as representatives.

## Tests of Minor Specimens

Flat plates of plywood that matched the material of the cylinders furnished specimens for minor tests in bending, tension, and compression. Results of tests are presented in table 4.

Bending specimens were 1 inch wide and were tested on a span length 48 or more times the thickness of the specimen for face grain direction parallel to the span and 24 or more times for face grain direction perpendicular to the span, thus eliminating the necessity of correcting for shear deformation. Specimens were cut so that the face grain direction was parallel to the span, or perpendicular to the span. Specimens rested on end supports rounded to about $1 / 16$-inch radius. The load was applied at the center with a bar of about $1 / 8$-inch radius. The stiffer specimens were tested on an apparatus that measured the load by means of a platform scale reading to $1 / 100$ pound. The rate of loading and driving force were obtained from a testing machine on which the scale was mounted (fig. 16). Specimens not stiff enough to test on the platform scale apparatus were tested on an apparatus that used water as a load. The load increments could be read to 10 grams and the rate of loading was controlled by the size of nozzle (fig. 17). Immediately after testing, a sample was cut from the specimen for moisture content determination. The specific gravity was obtained from measurements and weights taken before testing and subsequently corrected for moisture content. During the test, load-deflection curves were plotted. Properties computed from the test data were the modulus of elasticity, the moisture content, and the specific gravity.

The tension specimens were 1 inch wide and 11 inches long, with face grain directions parallel and perpendicular to the direction of the load. The ends of the specimens were gripped in Templin grips fastened to the testing machine by bolts passed through spherical bearings. The strain over a 2 -inch gage length was measured at the center of the specimen with an extensometer reading to 0.0001 of an inch (fig. 18). The rate of loading was 0.05 inch per minute. During the test, load-deformation curves were plotted. Properties computed from the test data were the modulus of elasticity, the tensile strength, the moisture content, and the specific gravity. Moisture content and specific gravity were determined in the same manner as they were for the bending specimens.

The compression specimens were 1 inch wide and 4 Inches long, with face grain directions parallel or perpendicular to the direction of the load. Because of the thinness of the specimen, an apparatus consisting of thin spring fingers was used to give lateral support (fig. 19). The strain was measured over a 2-inch gage length with a Martens' mirror apparatus. Load-deformation curves were plotted as the tests were run.

Properties computed from the test data were the modulus of elasticity, the compressive stress at the proportional-limit load, the compressive strength, the moisture content, and the specific gravity. The moisture content and specific gravity were determined in the same manner as they were for the bending specimens.

Table 1.-Elastic moduli of Douglag-fin used in the calculations

$$
\begin{aligned}
& \mathrm{E}_{\mathrm{L}}=2,223,000 \text { pounds per square inch } \\
& \mathrm{E}_{\mathrm{T}}=125,000 \text { pounds per square inch } \\
& \mu_{\mathrm{LT}}=123,800 \text { pounds per square inch } \\
& \sigma_{\mathrm{LT}}=0.434 \\
& \text { l }_{\text {Ihese are tentative values of the }} \\
& \text { constants for material at } 12 \text { percent } \\
& \text { moisture content with the exception of } \\
& \text { the modulus of risidity, H }{ }_{\text {LT }} \text { The } \\
& \text { value used for this constant was taken } \\
& \text { from the results of a limited number } \\
& \text { of tests at } 10 \text { percent molsture } \\
& \text { content. }
\end{aligned}
$$

Table 2. - Thooretical buckiling constante for thin-walled olywood cylinderg in torsion


Table 2.-(continued)


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Table 3.--(continued)

Table 3.-(oontiaued)

Table 3.- (continued)



Fible 4.-(continuad)


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Table 5.-Pagultic of toaly on minor opectmena

Table 5.-(contimaod)

Table 5. --(contimuad)

Tab10 5.-(continued)

Thale 5.-(cont 1nued)




Figure 1.--Orientation of axes in the middle surface of cylindrical shell.
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Figure 2.-A typical graphical determination of $k_{\text {theoretical }}$.

2 M 61834 F


ANGLE OF FACE GRAIN $=0^{\circ}$


ANGLE OF FACE GRA, ve90.


ANGLE OF FACE GRAIN $=+45^{\circ}$


Figure 3.--Theoretical buckling constants for thin-walled plywood cylinders in torsion, $\mu_{\mathrm{Lr}} / \mathrm{E}_{\mathrm{L}}=0.056$.


Figure 4.--Theoretical buckling constants for thin-walled plywood cylinders in torsion, $\mu_{\mathrm{Lr}} / \mathrm{E}_{\mathrm{L}}=0.036$.


Figure 5.--Theoretical buckling constants for thin-walled plywood cylinders in torsion, $\mu_{\mathrm{LT}} / \mathrm{E}_{\mathrm{L}}=0.090$.


Figure 6.--Apparatus for gluing cylinder scarf joint.
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Figure 7.--Apparatus for testing thin-walled plywood cylinders, 10-1/2 inches in diameter, in torsion.



Figure 9.--Relationship of observed to theoretical torsion buckling constants for thin-walled plywood cylinders, $10-1 / 2$ inches in diameter.


Figure 10.--Relationship of observed to theoretical torsion buckling constants for thin-walled plywood cylinders, $10-1 / 2$ inches in diameter. Three-ply plywood; face veneers and core veneer $1 / 100$ inch thick.
z M 61840 F


Figure 11. - Relationship of observed to theoretical torsion buckling constants for thin-walled plywood cylinders, $10-1 / 2$ inches in diameter. Three-ply plywood. Face veneers half as thick as the core veneer.

Z M 61841 F

Figure 12.--Relationship of observed to theoretical torsion buckling constants for thin-walled plywood cylinders, $10-1 / 2$ inches in diameter.
Three-ply plywood; face veneers $1 / 100$ inch and core veneers $1 / 40$ inch thick.

$\begin{array}{lllll}0.02 & 0.04 & 0.06 & 0.08 & 0.10\end{array}$

ANGLE OF FACE GRAIN $=90^{\circ}$

2 M61842 F


Figure 13.--Relationship of observed to theoretical torsion buckling constants for thin-walled plywood cylinders, $10-1 / 2$ inches in diameter. Five-ply plywood; face veneers $1 / 100$ inch and crossband and core veneers $1 / 64$ inch thick.

2 M 61843 F



Figure 15.--Relationship of observed to theoretical torsion buckling constants for thin-walled cylinders of isotropic materials. Test data were taken from NACA report No. 479, "Stability of thin-walled tubes under torsion" by L. H. Donnell. Theoretical values were computed by the mathematical analysis presented in this report.
ZM61845 F



Figure 19.--Compression test of thin plywood.


Figure 18.--Tension test of plywood.

## 


[^0]:    3Donnell, L. H., "The Stability of Isotropic or Orthotropic Cylinders or Flat or Curved Panels -- Under any Combination of Compression and Shear." N.A.C.A. Technical Note No. 918, December 1943.
    4 Donnell, L. H. "Stability of Thin Walled Thbes in Torsion." N.A.C.A. Twentieth Annual Report No. 479, 95-116, 1934.
    ${ }^{2}$ Forest Products Laboratory Reports Nos. 1300, 1312, 1316, 1322, A, 1503.

[^1]:    ${ }^{6}$ Love, A. E. H., "Treatise on the Mathematical Theory of Elasticity." Cambridge University Press, 1927.

[^2]:    $7_{\text {Timoshenko, S., "Theory of Elastic Stability," page 360; March, H. W., }}$ Forest Products Laboratory Report No. 1316.

[^3]:    ${ }^{10}$ Forest Products Laboratory Report No. 1503, equations (53) and (55), page 19.
    11 $_{\text {Forest Products Laboratory Report No. 1503, equations (56), page } 19 .}$

