FOURIER ANALYSIS OF
NON-SINUSOIDAL WAVES

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1. FOURIER'S THEOREM

Fourier announced his famous theorem in his "Théorie Analytique de la Chaleur" in 1822. Essentially his theorem may be stated as follows: Any single-valued function f(x) defined over an interval $-\frac{T}{2} \leqq x \leqq \frac{T}{2}$ can be represented over this interval by a trigonometric series of the form

$$(1.1) \qquad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi kx}{T} + b_k \sin \frac{2\pi kx}{T} \right),$$

where the coefficients are computed from the function f(x) by the formulas

$$(1.2) \qquad a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(s) \cos \frac{2\pi ks}{T}\, ds, \qquad k = 0, 1, \cdots, \infty,$$

$$(1.3) \qquad b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(s) \sin \frac{2\pi ks}{T}\, ds, \qquad k = 1, 2, \cdots, \infty.$$

As stated, this theorem is not strictly true. Certain restrictions must be imposed upon the function f(x) to assure that f(x) may be represented by the series of (1.1). The discussion of Fourier series will be based upon the restatement of the theorem to include certain restrictions and a choice of the interval to be $-\pi \leqq x \leqq \pi$. The Fourier series is simplified by choosing the interval $-\pi \leqq x \leqq \pi$, with no loss of generality.

The theorem. Let $f(x)$ be a single-valued function defined in the interval $-\pi \leqq x \leqq \pi$. If $f(x)$ and $[f(x)]^2$ are Riemann integrable, and if at each point x_0 in the interval there exists two positive constants, $a = a(x_0)$ and $A = A(x_0)$, such that

$$(1.4) \qquad |f(x_0+2t)+f(x_0-2t)-2f(x_0)| \leqq At, \quad \text{for} \quad 0 \leqq t \leqq a.$$

then the series

$$(1.5) \qquad S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

with the coefficients

$$(1.6) \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s) \cos ks \, ds, \quad k = 0, 1, \cdots, \infty,$$

$$(1.7) \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s) \sin ks \, ds, \quad k = 1, 2, \cdots, \infty,$$

converges at x_0 to

$$(1.8) \qquad \tfrac{1}{2}[f(x_0 - 0) + f(x_0+0)],$$

where $f(x_0 - 0)$ is the limit of $f(x)$ as x approaches x_0 from the left and $f(x_0 + 0)$ is the limit of $f(x)$ as x approaches x_0 from the right.

Not every continuous function satisfies the condition of (1.4). For example, the function

$$(1.9) \qquad f(x) = (x - x_0)^{2/3}$$

does not satisfy this condition near x_0.

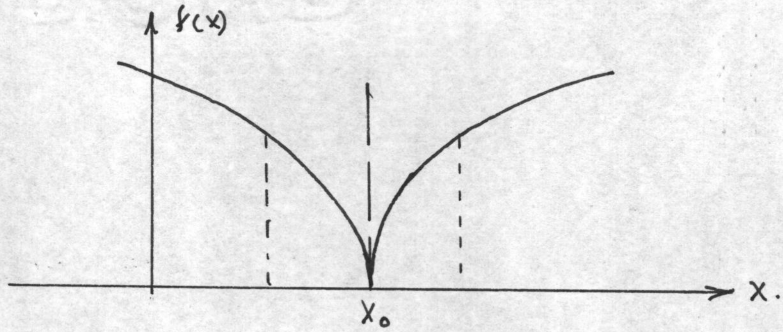


Fig. 1.1. $f(x) = (x - x_0)^{2/3}$.

Consider the left member of (1.4) divided by $4|t|$,

(1.10) $$\left| \frac{f(x_0+2t) + f(x_0-2t) - 2f(x_0)}{4t} \right|,$$

which may be rewritten as

(1.11) $$\left| \frac{f(x_0+2t) - f(x_0)}{2t} \right|,$$

since f(x) is symmetric with respect to x_0.

The limit of (1.11) as 2t goes to zero is the derivative of f(x). The derivative becomes infinite as x approaches x_0, therefore A cannot exist and the condition of (1.4) cannot be satisfied at x_0.

On the other hand, a discontinuous function may satisfy the condition of (1.4). For instance, if f(x) is continuous except for a finite jump at x_0 and has bounded right and left hand derivatives in the neighborhood of x_0, and if f(x) is defined at x_0 as the arithmetic mean of the limits approached from the right and left, then f(x) satisfies (1.4).

Under the hypothesis of the theorem, if $f(x)$ is continuous at the point $x = x_0$, then $f(x_0 - 0) = f(x_0 + 0) = f(x_0)$, so that at all points in an interval of continuity, the series converges to $f(x)$. At the points of discontinuity, it converges to the arithmetic mean of the values of the right and left hand limits. If $f(x)$ is of period 2π, that is, if $f(x + 2\pi) = f(x)$, the series converges to $\frac{1}{2}\left[f(x - 0) + f(x + 0)\right]$ for all x.

The field of mathematical applications to physics is not materially limited by these restrictions for most physical phenomena produce results which meet the restrictions for the Fourier expansion. A comparision with the Taylor series, which requires a continuous function $f(x)$ and continuous derivatives of all orders, reveals the larger class of functions to which a Fourier expansion may be applied.

The Fourier series of (1.5) will hereafter be referred to as "the series" and the Fourier coefficients of (1.6) and (1.7) as "the coefficients."

2. ORTHOGONAL FUNCTIONS

Definition. The functions $f(x)$ and $g(x)$ are orthogonal over the interval $-\pi \leqq x \leqq \pi$ if

$$(2.1) \qquad \int_{-\pi}^{\pi} f(x) \cdot g(x) \, dx = 0 .$$

The Fourier series is a series of orthogonal functions; that is, each term is orthogonal to every other term of the series. Consider cos mx, cos nx, sin mx, and sin nx, for m,n = 0, 1, 2, . . .,∞ , where

$$(2.2) \quad 2\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \int_{-\pi}^{\pi} \left[\cos (m-n)x + \cos (m+n)x\right] dx$$

$$= \left[\frac{\sin (m-n)x}{m-n} + \frac{\sin (m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \qquad m \neq n.$$

Similarly,

$$(2.3) \quad 2\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \int_{-\pi}^{\pi} \left[\cos (m-n)x - \cos (m+n)x\right] dx$$

$$= \left[\frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)}{m+n}\right]_{-\pi}^{\pi} = 0 \ , m \neq n.$$

and

$$(2.4) \quad 2\int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = \int_{-\pi}^{\pi} \left[\sin (m+n)x + \sin (m-n)x\right] dx$$

$$= \left[-\frac{\cos (m+n)x}{m+n} - \frac{\cos (m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0 \ , \text{ for all } m,n.$$

Due to these relations the terms of the series satisfy (2.1), so the Fourier series is a series of orthogonal functions in the interval $-\pi \leqq x \leqq \pi$.

3. DERIVATION OF FOURIER COEFFICIENTS

If f(x) is integrable over $(-\pi, \pi)$ and if f(x) is equal to a uniformly convergent Fourier series so that the series can be integrated term by term, the coefficients of the series are given by (1.6) and (1.7).

To obtain the coefficient a_k of the general cosine term, multiply both sides of (1.5) by ($\cos kx\, dx$) and integrate term by term from $-\pi$ to π. Since the terms of the series are orthogonal, there results

$$(3.1) \qquad \int_{-\pi}^{\pi} f(x) \cos kx \, dx = a_k \int_{-\pi}^{\pi} \cos^2 kx \, dx = a_k \pi,$$

or

$$(3.2) \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx .$$

Similarly, by multiplying (1.5) by ($\sin kx\, dx$) and performing term by term integration from $-\pi$ to π, one obtains

$$(3.3) \qquad \int_{-\pi}^{\pi} f(x) \sin kx \, dx = b_k \int_{-\pi}^{\pi} \sin^2 kx \, dx = b_k \pi,$$

or

$$(3.4) \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx .$$

Now whatever the behavior of the series of f(x), if f(x) is integrable its Fourier coefficients can be defined by (3.2) and (3.4) and the convergence of its Fourier series investigated.

4. MAGNITUDE OF COEFFICIENTS

Let f(x) be a continuous function of period 2π which has a continuous first derivative for all values of x. Consider the integrals defining the coefficients

$$(4.1) \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s) \cos ks \, ds,$$

$$(4.2) \qquad b_k = 1/\pi \int_{-\pi}^{\pi} f(s) \sin ks \, ds.$$

Applying the principle of integration by parts, one obtains

$$(4.3) \qquad a_k = \frac{1}{\pi}\left[\frac{1}{k} f(s) \sin ks\right]_{-\pi}^{\pi} - \frac{1}{\pi k}\int_{-\pi}^{\pi} f'(s) \sin ks \, ds$$

$$= -\frac{1}{\pi k}\int_{-\pi}^{\pi} f'(s) \sin ks \, ds,$$

and

$$(4.4) \qquad b_k = 1/\pi \left[-1/k \, f(s) \cos ks\right]_{-\pi}^{\pi} + \frac{1}{\pi k}\int_{-\pi}^{\pi} f'(s) \cos ks \, ds$$

$$= \frac{1}{\pi k}\int_{-\pi}^{\pi} f'(s) \cos ks \, ds,$$

since products $[f(s) \cdot \sin ks]$ and $[f(s) \cdot \cos ks]$ vanish at both ends of the interval. If M_1 is the maximum of $|f'(x)|$,

$$(4.5) \qquad |a_k| \leqq \frac{1}{\pi k}\int_{-\pi}^{\pi} |f'(s)||\sin ks| \, ds$$

$$\leqq \frac{1}{\pi k}\int_{-\pi}^{\pi} M_1 \, ds = \frac{2M_1}{k},$$

and similarly

$$(4.6) \qquad |b_k| \leqq \int_{-\pi}^{\pi} |f'(s)||\cos ks| \, ds \leqq \frac{2M_1}{k}.$$

If $f(x)$ has a continuous second derivative for all x with M_2 as the maximum of $|f''(x)|$, then integration by parts can be repeated and

$$(4.7) \qquad a_k = -\frac{1}{\pi k^2}\int_{-\pi}^{\pi} f''(s) \cos ks \, ds,$$

and

$$(4.8) \qquad |a_k| \leqq \frac{2M_2}{k^2}.$$

Likewise

$$(4.9) \qquad |b_k| \leqq \frac{2M_2}{k^2}.$$

Under this condition it can be inferred that the series is convergent. For

$$(4.10) \qquad |a_k \cos kx + b_k \sin kx| \leqq \frac{4M_2}{k^2},$$

and $\frac{4M_2}{k^2}$ is the general term of a convergent series. However, this does not prove convergence of the series to f(x).

If f(x) has a continuous nth derivative for all x, the magnitude of the coefficients

$$(4.11) \qquad |a_k| \leqq \frac{2M_n}{k^n}, \quad |b_k| \leqq \frac{2M_n}{k^n},$$

and the general term is

$$(4.12) \qquad |a_k \cos kx + b_k \sin kx| \leqq \frac{4M_n}{k^n}.$$

Thus as the number of continuous derivatives increases, the more rapidly the series converges.

5. LIMIT OF GENERAL COEFFICIENT

Let f(x) and $[f(x)]^2$ be functions which are integrable over the interval $-\pi \leqq x \leqq \pi$, and let $S_n(x)$ be the partial sum of the Fourier expansion of f(x) through terms of the nth order,

(5.1) $$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Consider the integral

(5.2) $$\int_{-\pi}^{\pi}[f(x)-S_n(x)]^2dx = \int_{-\pi}^{\pi}[f(x)]^2dx - 2\int_{-\pi}^{\pi}f(x)\,S_n(x)dx + \int_{-\pi}^{\pi}[S_n(x)]^2dx,$$

(5.3) $$\int_{-\pi}^{\pi}[f(x)-S_n(x)]^2dx = \int_{-\pi}^{\pi}[f(x)]^2dx - \pi\left[\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2)\right]$$

Transposing members of (5.3), there results

(5.4) $$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2+b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx - \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)-S_n(x)]^2dx.$$

The right member will not be decreased by dropping the last integral which is non-negative, so

(5.5) $$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2+b_k^2) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx.$$

This is known as Bessel's inequality.

Since (5.5) is true for all values of n, while the right hand side is independent of n, the series

(5.6) $$\sum_{k=1}^{n}(a_k^2+b_k^2)$$

is convergent and therefore

(5.7) $$\lim_{k\to\infty} a_k = 0, \quad \lim_{k\to\infty} b_k = 0,$$

since a necessary condition for the convergence of a series is that the general term approach zero.

That the coefficients approach zero is known as Riemann's theorem. Riemann's theorem holds under more general conditions, but the theorem under these more general conditions is not needed in this paper.

6. CONVERGENCE OF THE SERIES

To prove the convergence theorem of section 1, without loss of generality, $f(x)$ can be defined outside of the interval $(-\pi, \pi)$ to be of period 2π so that $f(x + 2\pi) = f(x)$. Consider the partial sum $S_n(x)$ defined by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \tag{6.1}$$

By substituting (1.6) and (1.7) in (6.1), one has

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds + \sum_{k=1}^{n}\Big[\frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos ks \cos kx\,ds + \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin ks \sin kx\,ds\Big], \tag{6.2}$$

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\Big[\tfrac{1}{2} + \cos s \cos x + \cdots + \cos ns \cos nx + \sin s \sin x + \cdots + \sin ns \sin nx\Big]\,ds. \tag{6.3}$$

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\Big[\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(s-x)\Big]\,ds. \tag{6.4}$$

By use of the idenity

$$\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(s-x) = \frac{\sin (n+\frac{1}{2})(s-x)}{2 \sin \left(\frac{s-x}{2}\right)}, \tag{6.5}$$

one may rewrite (6.4) as

(6.6) $$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin(n+\frac{1}{2})(s-x)}{\sin(\frac{s-x}{2})}\,ds.$$

If (6.5) is integrated over $(-\pi, \pi)$, then by the orthogonality property

(6.7) $$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})(s-x)}{\sin(\frac{s-x}{2})}\,ds.$$

Multiplying by f(x), one obtains

(6.8) $$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\frac{\sin(n+\frac{1}{2})(s-x)}{\sin(\frac{s-x}{2})}\,ds.$$

If (6.8) is subtracted from (6.6), there results

(6.9) $$S_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(s) - f(x)]\,\frac{\sin(n+\frac{1}{2})(s-x)}{\sin(\frac{s-x}{2})}\,ds.$$

A series of substitutions will change (6.9) to the form

(6.10) $$S_n(x) - f(x) = \frac{1}{\pi}\int_{0}^{\pi/2} [f(x+2t) + f(x-2t) - 2f(x)]\,\frac{\sin(2n+1)t}{\sin t}\,dt.$$

The first substitution is to let s = x + s', then ds = ds', and the limits may remain $-\pi$ to π, since f(x) is periodic. Then (6.9) written as the sum of two integrals is

(6.11) $$S_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{0} [f(x+s') - f(x)]\,\frac{\sin(2n+1)\frac{s'}{2}}{\sin\frac{s'}{2}}\,ds'$$
$$+ \frac{1}{2\pi}\int_{0}^{\pi} [f(x+s') - f(x)]\,\frac{\sin(2n+1)\frac{s'}{2}}{\sin\frac{s'}{2}}\,ds'.$$

where I_1 is the first term, and I_2 is the second term of the right side of (6.11).

In the first term, I_1, let $s' = -s''$, the $ds' = -ds''$, and I_1 becomes:

$$(6.12) \quad I_1 = \frac{1}{2\pi}\int_{\pi}^{0}[f(x-s'') - f(x)]\,\frac{\sin(2n+1)\left(\frac{-s''}{2}\right)}{\sin\left(-\frac{s''}{2}\right)}\,(-ds''),$$

$$(6.13) \quad I_1 = \frac{1}{2\pi}\int_{0}^{\pi}[f(x-s'') - f(x)]\,\frac{\sin(2n+1)\frac{s''}{2}}{\sin\left(\frac{s''}{2}\right)}\,ds''.$$

Dropping the primes and combining I_1 and I_2, one has

$$(6.14) \quad S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}[f(x+s) + f(x-s) - 2f(x)]\,\frac{\sin(2n+1)\frac{s}{2}}{\sin\frac{s}{2}}\,ds.$$

Let $s = 2t$, then $ds = 2\,dt$, and

$$(6.15) \quad S_n(x) - f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}[f(x+2t) + f(x-2t) - 2f(x)]\,\frac{\sin(2n+1)t}{\sin t}\,dt.$$

By hypothesis the function $f(x)$ satisfies the condition that at each point x_0 in the interval there exists two positive constants, $a = a(x_0)$ and $A = A(x_0)$ such that

$$(6.16) \quad |f(x_0+2t) + f(x_0-2t) - 2f(x_0)| \leqq At, \text{ for } 0 \leqq t \leqq a.$$

Therefore it follows that for any value of $\tau \leqq \frac{\pi}{2}$ in $0 < \tau < a$,

$$(6.17) \quad \left|\frac{1}{\pi}\int_0^{\tau}[f(x+2t) + f(x-2t) - 2f(x)]\,\frac{\sin(2n+1)t}{\sin t}\,dt\right|$$

$$\leq \frac{1}{\pi}\int_0^{\tau}\frac{At}{\sin t}\,dt \leq \frac{A}{\pi}\int_0^{\tau}\frac{\pi}{2}\,dt = \frac{A\tau}{2},$$

for

$$(6.18) \quad \sin(2n+1)t \leqq 1,$$

and

(6.19) $$\frac{t}{\sin t} \leqq \pi/2 \quad \text{for } 0 \leqq t \leqq \pi/2.$$

Now, given an $\varepsilon > 0$, choose $\tau < \varepsilon/A$; then this portion of the integral of (6.15) will be less in absolute value than $\varepsilon/2$. If τ is thus fixed, the rest of the integral approaches zero as n becomes infinite. This may be seen as follows. Define

(6.20) $$g(t) = f(x+2t) + f(x-2t) - 2f(x), \quad \tau \leqq t \leqq \pi/2,$$

$$= 0 \qquad \text{elsewhere in } -\pi \leqq t \leqq \pi.$$

then g(t) and $[g(t)]^2$ are integrable in the interval, so, as shown in section 5, the general coefficient b_k has the limit zero as k goes to infinity; that is

(6.21) $$\lim_{n\to\infty} b_{2n+1} = 0, \qquad k = 2n+1,$$

where

(6.22) $$b_{2n+1} = \frac{1}{\pi}\int_{\tau}^{\pi/2} \left[\frac{f(x+2t)+f(x-2t)-2f(x)}{\sin t}\right] \sin(2n+1)t \, dt.$$

If we choose n large enough the absolute value of (6.22) will be less than $\varepsilon/2$, so the absolute value of (6.15) becomes

(6.23) $$\left| S_n(x) - f(x) \right| < \varepsilon.$$

Therefore the sequence of partial sums, $S_n(x)$ of the series representing $f(x)$ converges to the value $f(x_0)$ at each point x_0 where (6.16) is satisfied. At a point of discontinuity $S_n(x)$ converges to the average of the right and left hand limits.

7. DEFINITIONS CONCERNING SERIES COMPONENTS

Consider the functions

$$\frac{a_0}{2}, \quad a_k \cos kx, \quad b_k \sin kx, \quad c_k \cos(kx - \phi_k) \tag{7.1}$$

where x is dimensionless.

The following definitions and equations are given for a clearer understanding of various symbols of (7.1) which occur in the series.

The function $f(x)$ is a periodic function with fundamental period T if $f(x + T) = f(x)$ for every x, and no number less than T has this property. The period for the trigonometric functions of (7.1) is

$$T = \frac{2\pi}{k} \quad (\text{interval}/\text{cycle}). \tag{7.2}$$

Frequency, f, is the reciprocal of T,

$$f = \frac{1}{T} = k/2\pi \quad (\text{cycles}/\text{interval}). \tag{7.3}$$

The fundamental frequency, f_1, is the frequency at which $k = 1$ in the functions of (7.1),

$$f_1 = \frac{1}{2\pi} \quad (\text{cycles}/\text{interval}). \tag{7.4}$$

A fundamental component is a component having the fundamental frequency. Examples are

$$(7.5) \qquad a_1 \cos x, \quad b_1 \sin x, \quad c_1 \cos(x - \phi_1).$$

The nth harmonic frequency, f_n, of f_1 is n times f_1,

$$(7.6) \qquad f_n = n f_1 = n/2\pi .$$

The nth harmonic components (or nth harmonics) are the components having the nth harmonic frequency,

$$(7.7) \qquad a_n \cos nx, \quad b_n \sin nx, \quad c_n \cos(nx - \phi_n).$$

A constant is said to have frequency zero. The component of zero frequency, $a_0/2$, is called the average component and is the average value of the function.

The amplitude of a function is the constant multiplier a_k, b_k, or c_k of (7.1).

The phase of the function is the angle (kx), or $(kx - \phi_k)$. The initial phase is the phase when x = 0, namely (0) in cos (kx), or $(-\phi_k)$ in cos $(kx - \phi_k)$.

If x has the dimension of time, then k will have the dimension of angular velocity, the rate of change of the phase with respect to time.

8. HARMONIC COMPONENTS

It is often convenient to combine two harmonic components of the same frequency into one component.

Consider the sum of the trigonometric functions of the kth harmonic in the series of (1.5),

(8.1) $$T_k = a_k \cos kx + b_k \sin kx.$$

One may write (8.1) as

(8.2) $$T_k = (a^2+b^2)^{1/2}\left[\frac{a_k}{(a^2+b^2)^{1/2}} \cos kx + \frac{b_k}{(a^2+b^2)^{1/2}} \sin kx\right],$$

or

(8.3) $$T_k = c_k \left[\cos \phi_k \cos kx + \sin \phi_k \sin kx\right],$$

(8.4) $$T_k = c_k \cos (kx - \phi_k],$$

where

(8.5) $$c_k = (a_k^2 + b_k^2)^{1/2},$$

(8.6) $$\phi_k = \text{arc tan } b_k/a_k.$$

Similarly (8.1) may be written

(8.7) $$T_k = c_k \sin (kx + \phi'_k),$$

where

(8.8) $$\phi'_k = \text{arc tan } a_k/b_k.$$

Thus with an introduction of phases the function f(x) of (1.5) may be expressed as a series of sines <u>or</u> cosines. The series then takes the form:

(8.9) $$f(x) = a_0/2 + \sum_{k=1}^{\infty} c_k \cos(kx - \phi_k),$$

or

(8.10) $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} c_k \sin(kx + \phi'_k),$$

where

(8.11) $$c_k = (a_k^2 + b_k^2)^{1/2}, \quad \phi_k = \tan^{-1} b_k/a_k, \quad \phi'_k = \tan^{-1} a_k/b_k.$$

All terms of the form $c_k \cos(kx - \phi_k)$ are called "harmonics" of $c_1 \cos(x - \phi_1)$, the latter being called the fundamental component.

9. ODD AND EVEN FUNCTIONS

By definition, an even function $f_1(x)$ satisfies the relation

(9.1) $$f_1(x) = f_1(-x),$$

and an odd function $f_2(x)$ satisfies

(9.2) $$f_2(x) = -f_2(-x).$$

If f(x) is even, the Fourier expansion leads to a series where all the b_k's vanish, and consists of cosine terms alone, plus a possible constant. To develop this series consider the coefficients written as the sum of two integrals:

(9.3) $a_k = \frac{1}{\pi}\int_{-\pi}^{0} f(s) \cos ks\, ds + \frac{1}{\pi}\int_{0}^{\pi} f(s) \cos ks\, ds,$

(9.4) $b_k = \frac{1}{\pi}\int_{-\pi}^{0} f(s) \sin ks\, ds + \frac{1}{\pi}\int_{0}^{\pi} f(s) \sin ks\, ds.$

In the first integral of (9.3) and (9.4) let $s = -t$. Then $ds = -dt$; $f(s) = f(-t)$, $f(-s) = f(t)$, and by (9.1) $f(s) = f(-s)$, $f(-t) = f(t)$; also $\cos ks = \cos(-ks)$, $\sin ks = \sin(-kt) = -\sin kt$, and

(9.5) $\int_{-\pi}^{0} f(s) \cos ks\, ds = \int_{0}^{\pi} f(t) \cos kt\, dt,$

(9.6) $\int_{-\pi}^{0} f(s) \sin ks\, ds = -\int_{0}^{\pi} f(t) \sin kt\, dt,$

so

(9.7) $a_k = \frac{2}{\pi}\int_{0}^{\pi} f(s) \cos ks\, ds,$

(9.8) $b_k = 0.$

The series becomes

(9.9) $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$

where a_k is defined by (9.7).

If $f(x)$ is odd, the expansion leads to a series where all the a_k's vanish and consists of sine terms alone. As before, consider the coefficients written as (9.3) and (9.4), letting $s = -t$ in the first integral of each equation. Now $ds = -dt$; $f(s) = f(-t)$, $f(-s) = f(t)$, and by (9.2) $f(s) = -f(-s)$, $f(-t) = -f(t)$; also $\cos ks =$

cos (-kt) = cos kt, sin ks = sin (-kt) = - sin kt, and

(9.10) $$\int_{-\pi}^{0} f(s)\cos ks\,ds = -\int_{0}^{\pi} f(t)\cos kt\,dt,$$

(9.11) $$\int_{-\pi}^{0} f(s)\sin ks\,ds = \int_{0}^{\pi} f(t)\sin kt\,dt,$$

so

(9.12) $$a_k = 0$$

(9.13) $$b_k = \frac{2}{\pi}\int_{0}^{\pi} f(s)\sin ks\,ds.$$

The series becomes

(9.14) $$f(x) = \sum_{k=1}^{\infty} b_k \sin kx,$$

where b_k is defined by (9.13).

Any periodic function f(x) may be analyzed into the sum of odd and even components, for f(x) may be written as

(9.15) $$f(x) = \tfrac{1}{2}\left[f(x) + f(-x)\right] + \tfrac{1}{2}\left[f(x) - f(-x)\right],$$

(9.16) $$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ is even and $f_2(x)$ is odd.

10. ABSENCE OF EVEN HARMONICS

It frequently occurs that the given periodic function satisfies the condition

(10.1) $$f(x+\pi) = -f(x).$$

If this condition is satisfied the series expansion of f(x) contains no even harmonics; the series of f(x) is developed as follows.

Consider the coefficient a_k written as the sum of two integrals:

(10.2) $$a_k = \frac{1}{\pi}\int_{-\pi}^{0} f(s)\cos ks\,ds + \frac{1}{\pi}\int_{0}^{\pi} f(s)\cos ks\,ds.$$

In the first integral of (10.2) let $s = s' - \pi$,

(10.3) $$\int_{-\pi}^{0} f(s)\cos ks\,ds = \int_{0}^{\pi} f(s'-\pi)\cos k(s'-\pi)\,ds'.$$

Dropping the primes, one may substitute the right side of (10.3) for the first integral of (10.2),

(10.4) $$a_k = \frac{1}{\pi}\int_{0}^{\pi} [f(s)\cos ks + f(s-\pi)\cos k(s-\pi)]\,ds.$$

By the identity

(10.5) $$\cos k(s-\pi) = \cos k\pi \cos ks,$$

equation (10.4) yields

(10.6) $$a_k = \frac{1}{\pi}\int_{0} [f(s)\cos ks + f(s-\pi)\cos k\pi\cos ks]\,ds.$$

By (10.1) the coefficient a_k is

(10.7) $$a_k = \frac{1}{\pi}(1-\cos k\pi)\int_{0}^{\pi} f(s)\cos ks\,ds.$$

Similarly, the coefficient b_k is

(10.8) $$b_k = \frac{1}{\pi}(1-\cos k\pi \int_{0}^{\pi} f(s)\sin ks\,ds,$$

which is determined by substituting sin ks for cos ks in each of the equations (10.2) through (10.7).

The factor $(1 - \cos k\pi)$ is zero for all even integers of k and equal to 2 for all odd integer values of k. Hence the periodic function which satisfies (10.1) has a series expansion of odd harmonics only. The average value, $a_0/2$ is also zero.

The coefficients of the odd harmonics are then given by

(10.9) $$a_k = \frac{2}{\pi}\int_0^{\pi} f(s) \cos ks \, ds, \quad k = 1, 3, 5, \cdots, \infty,$$

(10.10) $$b_k = \frac{2}{\pi}\int_0^{\pi} f(s) \sin ks \, ds, \quad k = 1, 3, 5, \cdots, \infty.$$

The question arises of what the results are when the function f(x) satisfies

(10.11) $$f(x + \pi) = f(x).$$

The factor in (10.7) and (10.8) is changed to $(1 + \cos n\pi)$ so that only even harmonics are present. But the condition given by (10.11) merely states that f(x) has the fundamental period of π instead of 2π.

11. EXPANSION OVER A FINITE INTERVAL

If the sole object is to obtain a trigonometric series which yields the correct values of the stated function over a finite interval, there is an infinite variety of ways to establish a Fourier expansion. The interval may alternatively be considered as only a part of the fundamental period, and the definition of the given function over the remainder of the period is entirely arbitrary.

If the interval over which the representation is to be maintained is only $0 \leq x \leq \pi$ the series in even functions is

(11.1) $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

and in odd functions

(11.2) $$f(x) = \sum_{k=1}^{\infty} b_k \sin kx,$$

with the coefficients

(11.3) $$a_k = 2/\pi \int_0^{\pi} f(s) \cos ks \, ds,$$

(11.4) $$b_k = 2/\pi \int_0^{\pi} f(s) \sin ks \, ds.$$

A simple example of a variety of series obtained for the function $y = \frac{a}{\pi} x$, of Fig. 11.1, defined in the interval $0 \leq x < \pi$, is illustrated in Fig. 11.2.

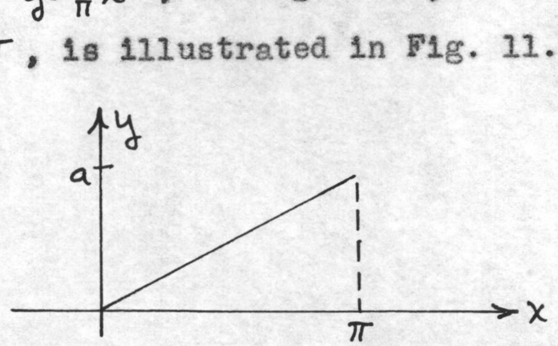
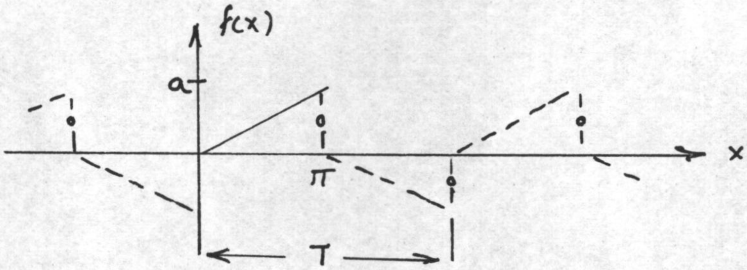


Fig. 11.1: $y = \frac{a}{\pi} x, \quad 0 \leq x < \pi.$

Fig. 11.2a: Sines and cosines, odd harmonics only.

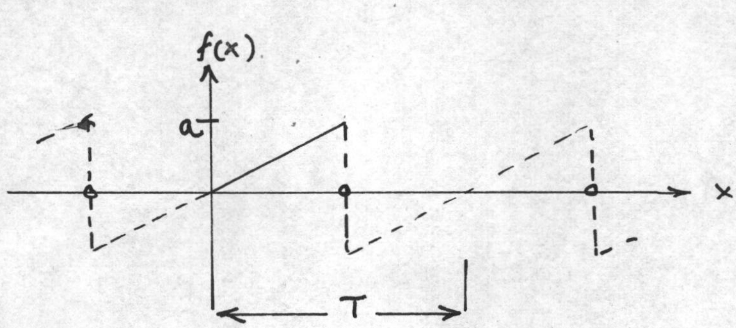


(11.5a) $$f(x) = \frac{a}{\pi}x, \quad 0 < x < \pi, \quad f(0) = -a/2,$$

$$f(x) = \frac{a}{\pi}(x-\pi), \quad -\pi < x < 0, \quad f(\pi) = a/2.$$

(11.5b) $$f(x) = \frac{a}{\pi}\left\{-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x + 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(2k-1)x\right\}.$$

Fig. 11.2b: Sines only, even and odd harmonics.

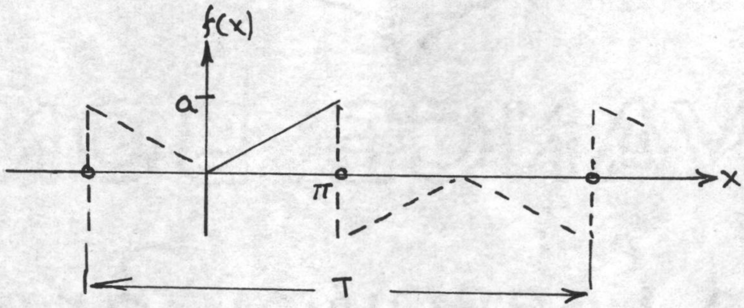


(11.6a) $$f(x) = \frac{a}{\pi}x, \quad -\pi < x < \pi,$$

$$f(\pi) = 0.$$

(11.6b) $$f(x) = \frac{2a}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx.$$

Fig. 11.2c: Cosines only, odd harmonics only.



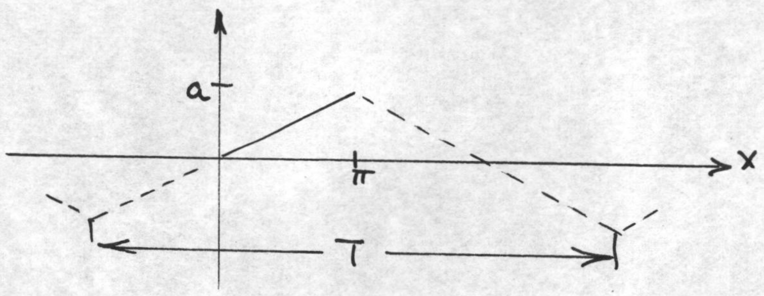
(11.7a) $f(x) = \frac{a}{\pi}|x|, \; -\pi \leqq x \leqq \pi, \qquad f(\pi) = f(3\pi) = 0,$

$f(x) = \frac{a}{\pi}(x-2\pi), \; \pi < x \leqq 2\pi, \quad f(x) = \frac{a}{\pi}(2\pi - x), \; 2\pi \leqq x \leqq 3\pi.$

(11.7b) $f(x) = \frac{16a}{\pi}\sum_{k=1}^{\infty}\left\{\frac{i^{2(k-1)}}{2k-1} - \frac{2}{\pi k^2}\right\} \cos\left(\frac{2k-1}{2}\right)x,$

where $i^2 = -1$.

Fig. 11.2d: Sines only, odd harmonics only

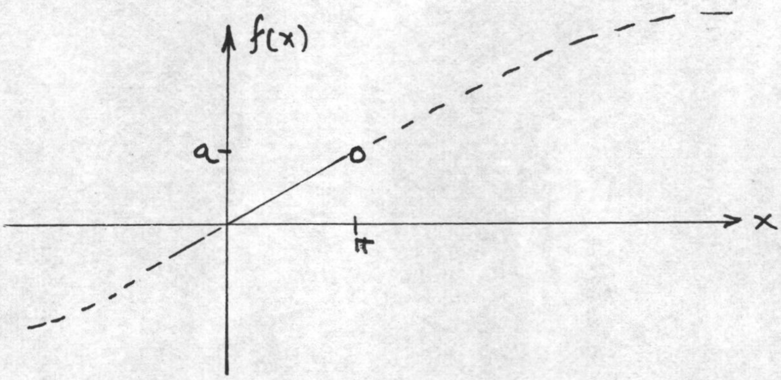


(11.8a) $f(x) = \frac{a}{\pi}x, \quad -\pi \leqq x \leqq \pi,$

$f(x) = \frac{a}{\pi}(2\pi - x), \quad \pi \leqq x \leqq 3\pi.$

(11.8b) $f(x) = \frac{32a}{\pi^2}\sum_{k=1}^{\infty}\frac{i^{2(k-1)}}{(2k-1)^2}\sin\left(\frac{2k-1}{2}\right)x.$

Fig. 11.2e: An approximation with a single sine term.



(11.9) $f(x) = a \csc \frac{\pi}{c} \sin \frac{x}{c}$, error $< 0.00035a$ for $c \geqq 180$.

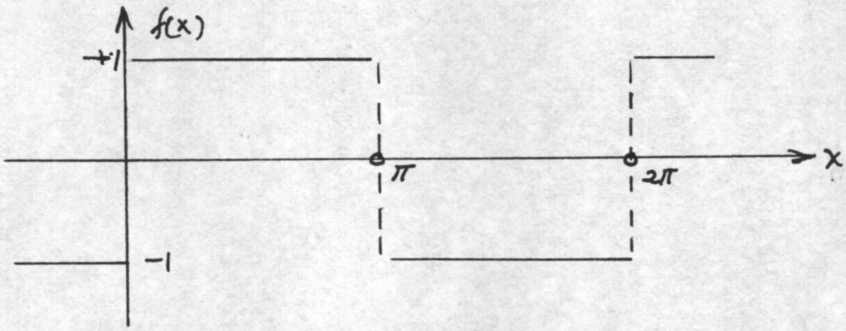
Fig. 11.2: Various possible periodic continuations of the function of Fig. 11.1.

In each case the behavior over the defining interval is the same as that of Fig. 11.1, but the series representations for the individual cases are quite different. It is also significant that the rate of convergence of the resulting series may be quite different for the various forms of periodic functions.

12. EXAMPLES OF FOURIER DEVELOPMENTS

Each of the following examples is the Fourier development of a single-valued periodic function f(x) of period 2π. The function f(x) is defined throughout a period interval and illustrated by a corresponding figure. The Fourier expansion of f(x), has been derived in accordance with the preceding formulas. Since these functions satisfy the hypothesis of the theorem, each of the series does converge and is the function f(x).

Fig. 12.1: Square wave (odd).

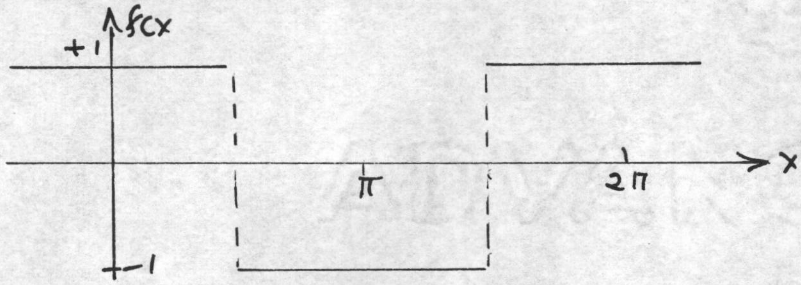


(12.1a) $f(x)=1, \quad 0<x<\pi; \quad f(0)=f(\pi)=0,$

$f(x)=-1, \quad \pi<x<2\pi.$

(12.1b) $f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin (2k-1)x.$

Fig. 12.2: Square wave (even).

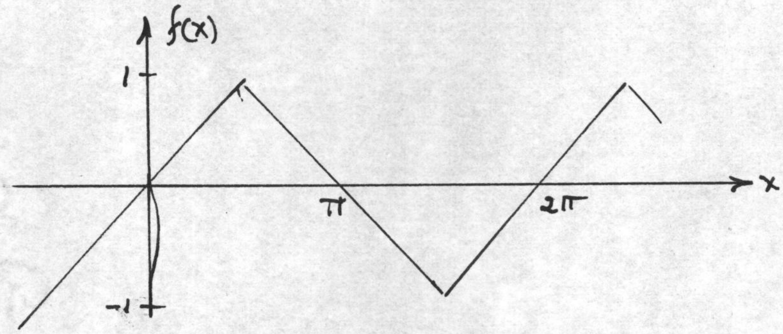


(12.2a) $f(x)=1, \quad -\frac{\pi}{2}<x<\frac{\pi}{2}; \quad f(\frac{\pi}{2})=f(\frac{3\pi}{2})=0,$

$f(x)=-1, \quad \frac{\pi}{2}<x<\frac{3\pi}{2}.$

(12.2b) $f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \cos (2k-1)x.$

Fig. 12.3: Saw-tooth wave (odd).

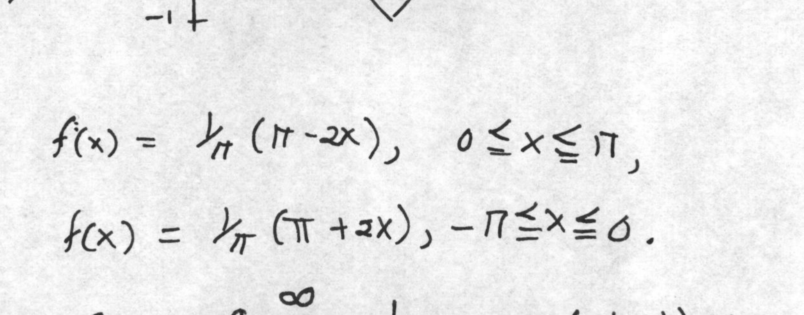


(12.3a) $$f(x) = \frac{2x}{\pi}, \quad -\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2},$$

$$f(x) = \frac{2}{\pi}(\pi - x), \quad \frac{\pi}{2} \leqq x \leqq \frac{3\pi}{2}$$

(12.3b) $$f(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x.$$

Fig. 12.4: Saw-tooth wave (even).

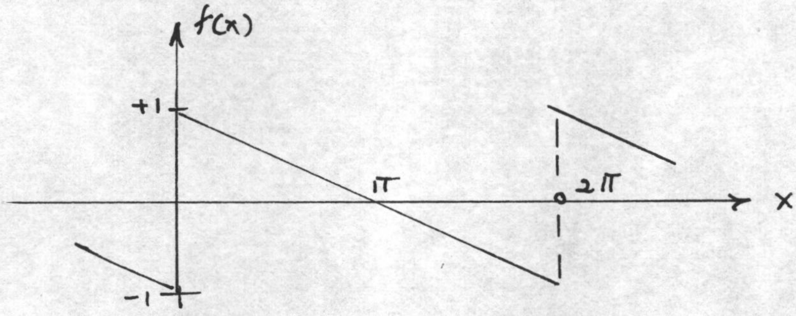


(12.4a) $$f(x) = \frac{1}{\pi}(\pi - 2x), \quad 0 \leqq x \leqq \pi,$$

$$f(x) = \frac{1}{\pi}(\pi + 2x), \quad -\pi \leqq x \leqq 0.$$

(12.4b) $$f(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x.$$

Fig. 12.5: Triangular wave #1 (odd).

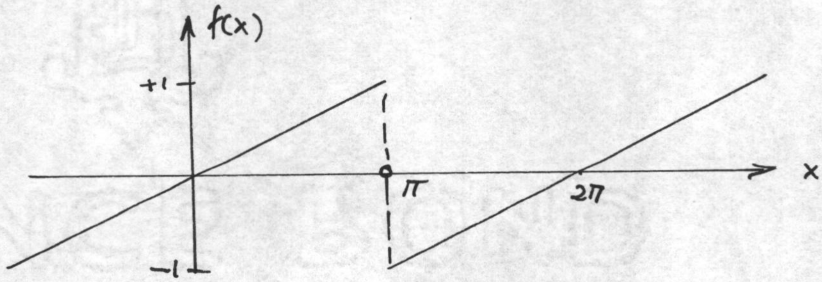


(12.5a) $$f(x) = \frac{1}{\pi}(\pi - x), \qquad 0 < x < 2\pi,$$

$$f(0) = 0$$

(12.5b) $$f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{k} \sin kx.$$

Fig. 12.6: Triangular wave #2 (odd).

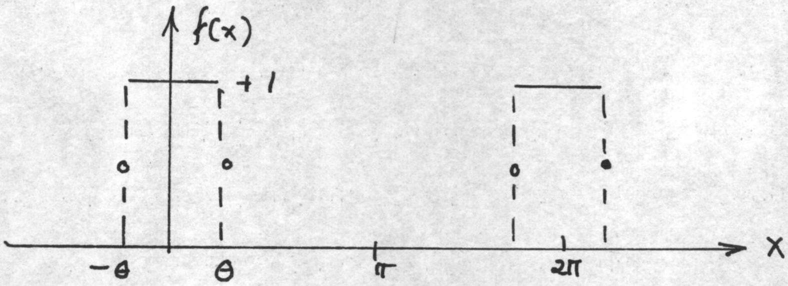


(12.6a) $$f(x) = \frac{x}{\pi}, \qquad -\pi < x < \pi$$

$$f(\pi) = 0.$$

(12.6b) $$f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx.$$

Fig. 12.7: Periodic rectangular pulse.

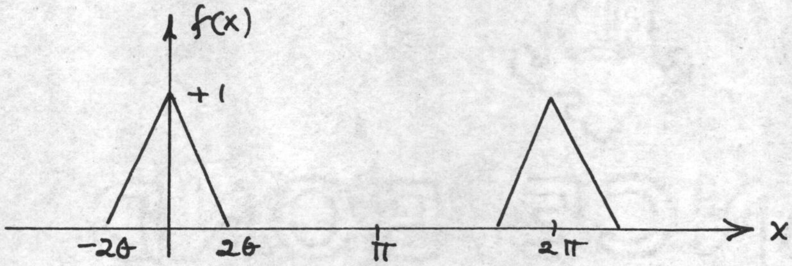


(12.7a) $f(x) = 1, \quad -\theta < x < \theta, \qquad f(-\theta) = f(\theta) = \tfrac{1}{2},$

$f(x) = 0, \quad \theta < x < 2\pi - \theta.$

(12.7b) $$f(x) = \frac{\theta}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} \cos kx .$$

Fig. 12.8: Periodic Triangular pulse.

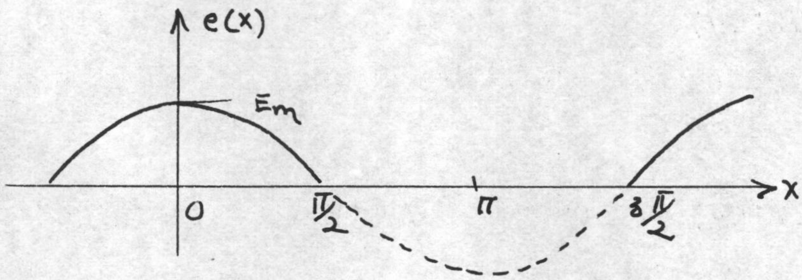


(12.8a) $f(x) = \frac{1}{2\theta}(2\theta + x), \quad -2\theta \leqq x \leqq 0,$

$f(x) = \frac{1}{2\theta}(2\theta - x), \quad 0 \leqq x \leqq 2\theta, \quad f(x) = 0, \quad 2\theta \leqq x \leqq 2\pi - 2\theta$

(12.8b) $$f(x) = \theta/\pi + \frac{2}{\pi\theta}\sum_{k=1}^{\infty} \left(\frac{\sin k\theta}{k}\right)^2 \cos kx .$$

Fig. 12.9: Output of single-phase, half-wave rectifier.

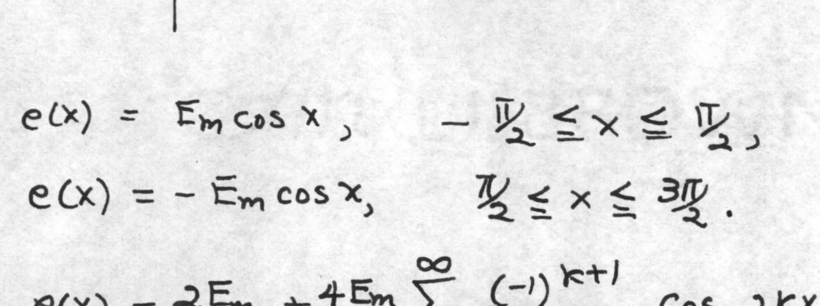


(12.9a) $e(x) = E_m \cos x \;, \qquad -\pi/2 \leq x \leq \pi/2,$

$e(x) = 0 \;, \qquad \pi/2 \leq x \leq 3\pi/2.$

(12.9b) $e(x) = \frac{E_m}{\pi} + \frac{E_m}{2}\cos x + \frac{2E_m}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2-1} \cos 2kx.$

Fig. 12.10: Output of a single-phase, full-wave rectifier.

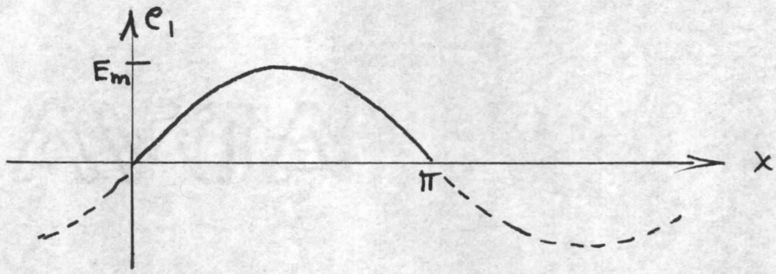


(12.10a) $e(x) = E_m \cos x \;, \qquad -\pi/2 \leq x \leq \pi/2,$

$e(x) = -E_m \cos x, \qquad \pi/2 \leq x \leq 3\pi/2.$

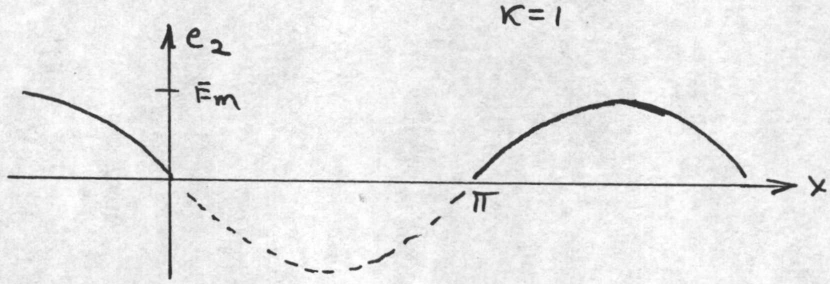
(12.10b) $e(x) = \frac{2E_m}{\pi} + \frac{4E_m}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2-1} \cos 2kx.$

Fig. 12.11: Superposition of two half-waves to form a full-wave.



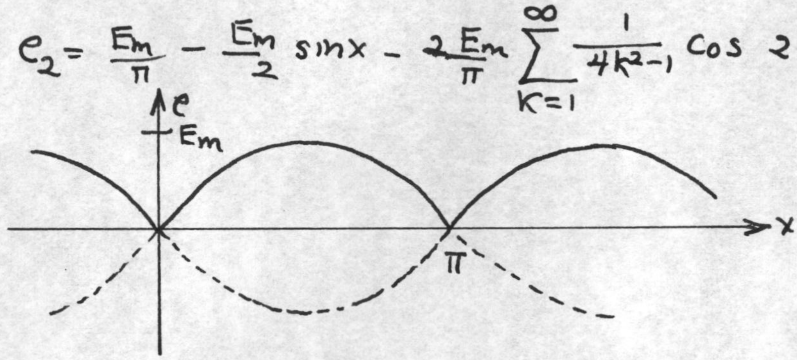
(12.11a) $e_1 = E_m \sin x \quad 0 \leqq x \leqq \pi, \quad e_1 = 0 \quad \pi \leqq x \leqq 2\pi.$

(12.11b) $e_1 = \frac{E_m}{\pi} + \frac{E_m}{2} \sin x - \frac{2E_m}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \cos 2kx.$



(12.11c) $e_2 = 0, \quad 0 \leqq x \leqq \pi, \quad e_2 = -E_m \sin x, \quad \pi \leqq x \leqq 2\pi.$

(12.11d) $e_2 = \frac{E_m}{\pi} - \frac{E_m}{2} \sin x - \frac{2E_m}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \cos 2kx.$



(12.11e) $e_1 + e_2 = e = E_m |\sin x|, \quad 0 \leqq x \leqq 2\pi,$

(12.11f) $e = \frac{2E_m}{\pi} - \frac{4E_m}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \cos 2kx.$

13. WAVES HAVING DISCONTINUITIES

In Fig. 12.1 the square-wave jumped, at $x = 0$, from the value -1 to the value $+1$. Similarly the square-wave of Fig. 12.2 is discontinuous for $x = \pi/2$. In these two examples the amplitudes of the successive components diminish very slowly; not always exactly as $1/k$, but the decrease is of the order of $1/k$. Other examples are given in Figs. 12.5, 12.6, and 12.7.

When $f(x)$ jumps discontinuously from a to b for a certain value of x, $= x_0$, the series converges to $(a + b)/2$ at $x = x_0$. For instance, in Fig. 12.1, the series converges to zero at $x = x_0$, since at this point $a = -1$ and $b = +1$.

The sum of a finite number of terms in a Fourier series for a function having a jump is only a fair approximation to the value of the function in the vicinity of the jump.

Other types of waves present no jumps, but they have sharp corners, such as the waves of Figs. 12.3, 12.4, 12.8, 12.9, 12.10, and 12.11. These waves are continuous, but their first derivatives are discontinuous, that is, they are characterized by jumps similar to the jumps which the square waves have. For such waves the successive amplitudes of the Fourier components decrease, although slowly, somewhat faster than in the case of the square waves; their decrease is of the order of $1/k^2$.

The smoother the wave, the higher the order of the derivative at which a jump first occurs. The amplitudes of the successive components then diminish more rapidly, of at least the magnitude of $1/k^n$, where n is the order of the derivative at which a jump first occurs. (See section 4: Magnitude of coefficients.)

14. OBJECT OF FOURIER ANALYSIS

If a sinusoidal voltage such as sin mt, cos mt, or cos (mt - Ø) is applied between two terminals of a linear passive network, all potential differences and currents in the network are sinusoidal and of the same frequency as the applied voltage. The amplitudes and phases of these potential differences and currents may be calculated by the ordinary alternating current theory.

However, in communication engineering periodic oscillations are used which are non-sinusoidal. The microphone current when a steady sound is sung or spoken, the output of a detector on which a sinusoidal voltage is impressed, the scanning voltage of a cathode ray tube, are important examples of periodic non-sinusoidal oscillations.

If the output voltage is desired from a certain network, to which a non-sinusoidal voltage is applied, we may use ordinary alternating current theory to obtain the sinusoidal output voltage due to any one of the sinusoidal components of the input voltage. If the network is linear, the principle of superposition holds, that is, the actual output is the sum of all the sinusoidal output components. The building up of a periodic function from its sinusoidal components of various frequencies is called Fourier synthesis.

The network problem involving periodic, non-sinusoidal voltages and currents may thus be solved in three steps: (1) Fourier analysis of input voltage or current, (2) calculation on the network according to the alternating current theory at each component frequency, and (3) Fourier synthesis of the output.

15. EFFECTIVE VALUE

A non-sinusoidal current or voltage function f(x) may be represented by the series as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} C_k \cos(kx - \phi_k), \tag{15.1}$$

with the coefficients and phase angles asdefined by (1.6), (1.7), (8.5), and (8.6).

The effective value of voltage (or current) f(x) is defined as

$$E = \left(\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx\right)^{1/2}, \tag{15.2}$$

$$E = \left[\frac{1}{2}\left(\frac{a_0^2}{2} + C_1^2 + C_2^2 + \cdots + C_n^2 + \cdots\right)\right]^{1/2}. \tag{15.3}$$

Since c_n is the maximum value of the nth harmonic of the alternating component, the effective value of the nth component is

$$E_n = \frac{c_n}{\sqrt{2}} \tag{15.4}$$

Therefore the effective value is equal to the square root of the sum of the squares of the effective components

$$E = \left[E_0^2 + E_1^2 + E_2^2 + \cdots + E_n^2 + \cdots\right]^{1/2} \tag{15.5}$$

where E_0 represents the direct current (or average) component and E_n is the effective value of the nth harmonic.

16. SERIES CIRCUIT

In the consideration of the series circuit the laws of electrical networks (Kirchhoff's Laws) are used and hence are stated here.

1. The algebraic sum of the currents flowing toward any point in a network is zero.

2. The algebraic sum of the products of the current and resistance in each of the conductors in any closed path in a network is equal to the algebraic sum of the electromotive forces in that path.

If an instantaneous current $i = i(t)$ is flowing in the series circuit of Fig. 16.1, containing the constant elements, resistance R, inductance L, and capacitance C, there is impressed a difference of potential $e = e(t)$

$$e(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int_c i\,dt, \tag{16.1}$$

where $\int_c i\,dt$ represents the total quantity of electricity on the condenser, C.

For a simple electrical example consider a periodic non-sinusoidal voltage $e(\phi)$, where $\phi = 2\pi ft$, impressed on the series circuit of Fig. 16.1, and assume that the first step in solving the network is completed; that is, $e(\phi)$ is approximated by using the nth partial sum of its Fourier series

$$e(\phi) = \frac{e_0}{2} + \sum_{k=1}^{n} e_k \cos(k\phi - \theta_k). \tag{16.2}$$

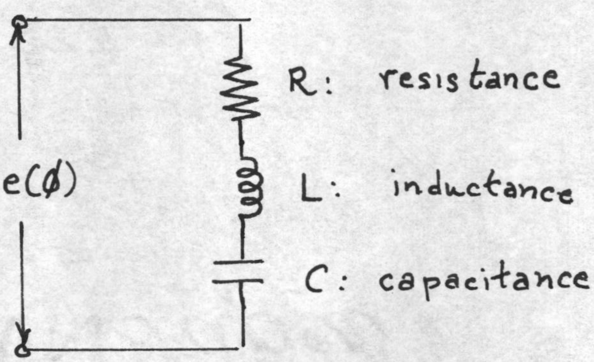


Fig. 16.1: Simple series circuit.

If $e(\phi)$ is not defined throughout a period so that one may derive its series in accordance with the preceding theory, but is an unknown voltage, then one may put the input voltage on an oscilloscope and determine experimental values of $e(\phi)$ at 2n or more points equally spaced on the ϕ axis throughout a period. Then the trigonometric sum of (16.2) may be derived in accordance with section 20. The second step in solving the network is calculating each component in the output according to alternating current theory.

Assume that the voltage $e(\phi)$ has been applied to the circuit long enough so that the steady-state condition has been reached. The voltage drops e_R, e_L, and e_C across R, L, and C respectively will be given by

(16.3) $$e_R = iR$$

(16.4) $$e_L = L\frac{di}{dt} = 2\pi f L \frac{di}{d\phi} = X_L \frac{di}{d\phi}$$

(16.5) $$e_c = \frac{1}{C}\int i\,dt = \frac{1}{2\pi f C}\int i\,d\phi = X_c \int i\,d\phi .$$

where

(16.6) $$\phi = 2\pi f t , \qquad t = \frac{\phi}{2\pi f} ,$$

(16.7) $$X_L = 2\pi f L , \qquad X_c = \frac{1}{2\pi f C} .$$

Therefore

(16.8) $$e(\phi) = X_L \frac{di}{d\phi} + iR + X_c \int i\,d\phi ,$$

or upon differentiating with respect to ϕ,

(16.9) $$\frac{de}{d\phi} = X_L \frac{d^2 i}{d\phi^2} + R\frac{di}{d\phi} + X_C\, i .$$

Since $e(\phi)$ is approximated by the first $(n + 1)$ terms of the series, we will have $n + 1$ differential equations

$$(16.10) \quad \frac{d}{d\phi}\left(\frac{e_0}{2}\right) = X_L \frac{d^2 i_0}{d\phi} + R\frac{di_0}{d\phi} + X_c i_0,$$

$$(16.11) \quad \frac{d}{d\phi}\left[e_k \cos(k\phi - \theta_k)\right] = X_L \frac{d^2 i_k}{d\phi^2} + R\frac{di_k}{d\phi} + X_c i_k, \; k = 1, \cdots, n.$$

The third step is the Fourier synthesis of the output. The current $i(\phi)$ will be the sum of the solutions of the $n + 1$ differential and have the form

$$(16.12) \quad i(\phi) = \sum_{k=1}^{n} i_k = \sum_{k=1}^{n} \sqrt{2}\, I_k \cos(k\phi - \varphi_k).$$

The direct current component, $e_0/2$, has produced a transient current which is no longer present in the steady state, but appears as a voltage across the capacitor C.

The voltage drops across the circuit components may now be calculated as

$$(16.13) \quad e_R = iR = R\sum_{k=1}^{n} \sqrt{2}\, I_k \cos(k\phi - \varphi_k),$$

$$(16.14) \quad e_L = 2\pi f L \frac{di}{d\phi} = -2\pi L \sum_{k=1}^{n} f_k k \sqrt{2}\, I_k \sin(k\phi - \varphi_k)$$

$$(16.15) \quad e_c = \frac{1}{2\pi f C}\int i\, d\phi = \frac{e_0}{2} + \frac{\sqrt{2}}{2\pi C}\sum_{k=1}^{n} \frac{I_k}{f_k k} \sin(k\phi - \varphi_k).$$

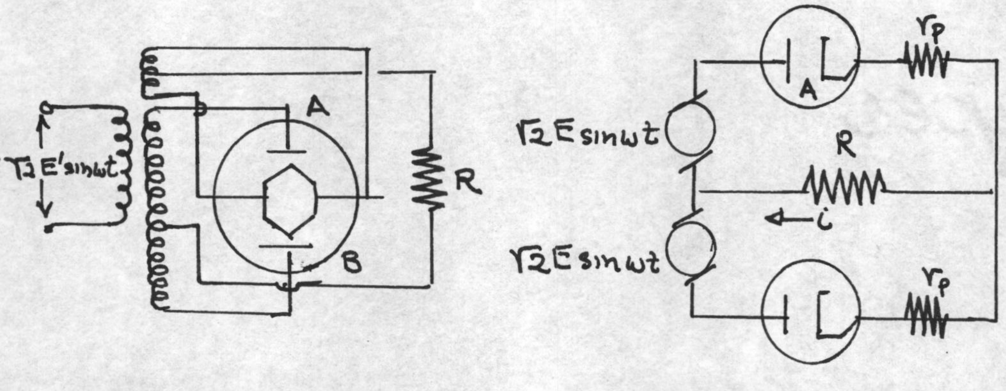
The effective value of the current is

$$(16.16) \quad I = \left[\sum_{k=1}^{n} I_k^2\right]^{1/2}$$

The electromotive force law and current law may be applied to more complicated networks to obtain as many independent equations as are necessary to determine the unknown quantities involved. If a complex voltage of the form (16.2) is applied to the network, calculation must be made on the network at each component frequency, then a Fourier synthesis of the output.

17. THE FULL WAVE RECTIFIER

For another example of an electrical application of Fourier series, consider the full wave rectifier of Fig. 17.1.



Schematic Diagram

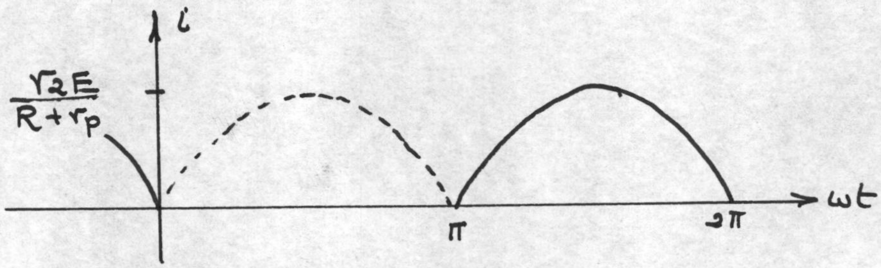
Simplified Equivalent Diagram

R: load resistance
r_p: resistance of diode and one side of the transformer

Fig. 17.1: Diagrams of full-wave rectifier.

The current through diode A consists of a half sine wave, and the current through diode B consists of a half sign wave displaced by π radians from the current through diode A. Since both currents go through the load R in the same direction, the total current through R is the sum of the currents.

If one choses the beginning of the half cycle of the current through diode A as the reference point, then the current through R is as shown in Fig. 17.1.



Broken line: current through R due to diode A
Continuous line: current through R due to diode B
Sum of both currents: total current through R

Fig. 17.1: Rectified current through R.

Therefore the current i through R may be given as

$$i = \left| \frac{\sqrt{2}\,E \sin \omega t}{R + r_p} \right| \qquad (17.1)$$

In the design of a full wave rectifier it is important to know the magnitude of the direct-current component and the magnitudes of the harmonic components of the rectified current through R.

These can be determined by expanding (17.1) in a Fourier series,

$$i = \frac{2\sqrt{2}\,E}{\pi(R + r_p)} \left[1 - 2 \sum_{k=1}^{\infty} \frac{\cos 2k\omega t}{4k^2 - 1} \right] \qquad (17.2)$$

Only a direct current and even harmonics of the supply frequency occur. The direct current component of (17.2) is

$$I_0 = \frac{2\sqrt{2}E}{\pi(R + r_p)} \tag{17.3}$$

and the effective value of the (2k)th harmonic is

$$I_{2k} = \frac{1}{\sqrt{2}} I_{(2k)max} = \frac{4E}{\pi(R + r_p)(4k^2 - 1)} \tag{17.4}$$

18. SATURATION AMPLIFIERS

Cathode limitation sometimes occurs in amplifiers at large input voltages. This condition generates harmonics of the fundamental frequency.

Cathode limitation (or saturation) occurs when all of the electrons emitted by the cathode are being absorbed by the plate. The varying plate current may be given by

$$\begin{aligned} i(mt) &= A\cos mt, & \phi \leqq mt \leqq 2\pi - \phi, \\ &= A\cos\phi, & -\phi \leqq mt \leqq \phi, \end{aligned} \tag{18.1}$$

where cathode limitation or saturation occurs in the interval $-\phi \leqq mt \leqq \phi$.

If there were no saturation, the current would be

$$i(mt) = A\cos mt. \tag{18.2}$$

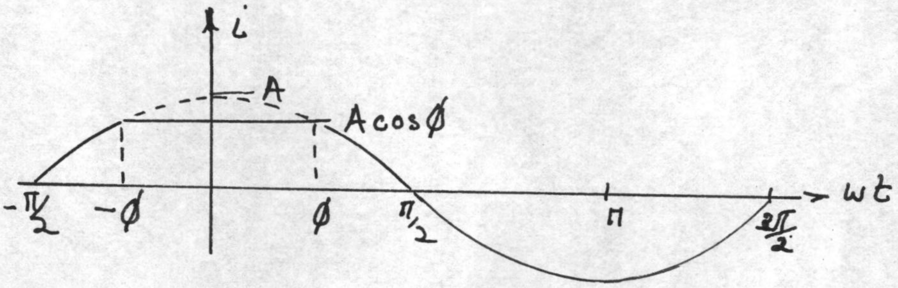


Fig. 18.1: Varying plate current component of a saturation amplifier #1.

The Fourier expansion of the current (18.1) is

$$(18.3)\qquad i(mt) = \frac{A}{\pi}(\phi\cos\phi - \sin\phi) + \frac{4}{\pi}(\pi - \phi + \sin\phi\cos\phi)\cos\omega t$$
$$+ \sum_{k=2}^{\infty} \frac{A}{\pi}\left[\frac{\sin(k+1)\phi}{k(k+1} - \frac{\sin(k-1)\phi}{k(k-1)}\right]\cos k\omega t.$$

Saturation introduces a direct current of

$$(18.4)\qquad \frac{i_0}{2} = \frac{A}{\pi}(\phi\cos\phi - \sin\phi).$$

The amplitude of the fundamental is reduced from A to

$$(18.5)\qquad A\left[1 - \frac{\phi - \sin\phi\cos\phi}{\pi}\right].$$

Harmonics are introduced as indicated by the summation term of the series of (18.2).

The current may be limited on the lower half of the cycle too, for example, i(mt) may be as Fig. 18.2, where i(mt) is defined

$$(18.6)\qquad \begin{aligned} i(mt) &= A\cos\phi, & -\phi \leqq mt \leqq \phi,\ \pi-\phi \leqq mt \leqq \pi+\phi, \\ i(mt) &= A\cos mt, & \phi \leqq mt \leqq \pi-\phi,\ \pi+\phi \leqq mt \leqq 2\pi-\phi \end{aligned}$$

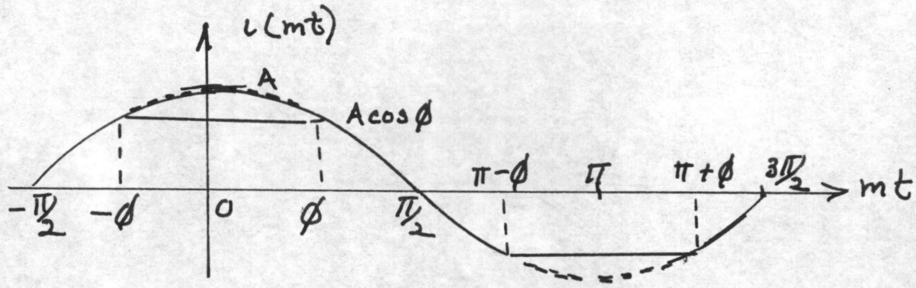


Fig. 18.2: Varying plate current component of a saturation amplifier #2.

The series expansion of (18.6) is

$$i(mt) = \frac{A}{\pi}(\pi - \phi + \sin\phi\cos\phi)\cos mt \qquad (18.7)$$

$$+\sum_{k=3,5,7}^{\infty} \frac{A}{\pi}\left[\frac{\sin(k+1)\phi}{k(k+1)} - \frac{\sin(k-1)\phi}{k(k-1)}\right]\cos kmt.$$

The current i(mt) contains no direct current component and only odd harmonics.

19. ANALYSIS OF TUNED CLASS B POWER AMPLIFIER

The object of this analysis is to derive expressions for power output and plate efficiency in terms of the rated quantities of a high vacuum thermionic triode operated as a class B tuned power amplifier.

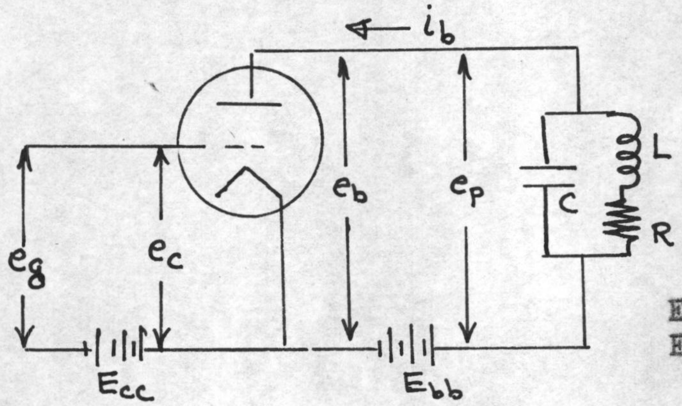
From electronic theory one may obtain a relatively simple approximate analytic expression for the tube characteristics under linear operation,

$$i_b = g_m\left(e_c + \frac{e_b}{\mu}\right), \qquad (19.1)$$

where

i_c: total instantaneous plate current,
g_m: mutual conductance of tube,
e_c: total instantaneous grid voltage,
e_b: total instantaneous plate voltage,
μ: amplification factor of the tube.

The basic diagram of an amplifier with a parallel-tuned load is shown in Fig. 19.1.



e_g: instantaneous value of varying component of grid voltage
e_p: instantaneous value of varying component of plate voltage
E_{cc}: grid bias voltage
E_{bb} plate supply voltage.

Fig. 19.1: Basic circuit diagram of a tuned amplifier.

If the plate load circuit is tuned to resonate at the frequency of the grid signal voltage e_g, and if the impedance of the parallel-tuned plate load circuit is negligible at harmonics in comparison to its impedance at the fundamental, a sinusoidal grid-signal, e_g, will produce a voltage drop across the load that is also sinusoidal.

For class B operation, the grid-bias, E_{cc}, is adjusted so that when $e_g = 0$, the plate current is zero. Equation 19.1 equated to zero becomes

$$(19.2) \quad i_c = g_m\left(e_c + \frac{e_b}{\mu}\right)\Big|_{i_c=0} = g_m\left(E_{cc} + \frac{E_{bb}}{\mu}\right) = 0 .$$

Therefore, for class B operation

(19.3) $$E_{cc} = -\frac{E_{bb}}{\mu}.$$

The voltages e_c and e_b are

(19.4) $$e_c = E_{cc} + e_g,$$

(19.5) $$e_b = E_{bb} + e_p.$$

Therefore i_b is given by

(19.6) $$i_b = g_m\left(e_c + \frac{e_b}{\mu}\right) = g_m\left(e_g + \frac{e_p}{\mu}\right), \quad i_b \geqq 0.$$

Both e_g and e_p are sinusoidal, thus the sum $(e_g + e_p)$ is sinusoidal. The wave form of the current i_b consists of a series of alternate half-sinusoid and zero-current half-cycles as shown in Fig. 19.4.

If the origin of time is chosen at the starting point of one of the current pulses, the Fourier series representation is

(19.7) $$i_b = I_{bm}\left[\frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos 2k\omega t\right].$$

The average value of the plate current I_b is

(19.8) $$I_b = \frac{I_{bm}}{\pi}.$$

The effective value ~~amplitude~~ of the fundamental component, I_{p1}, is

(19.9) $$I_{p_1} = \frac{I_{bm}}{2\sqrt{2}}.$$

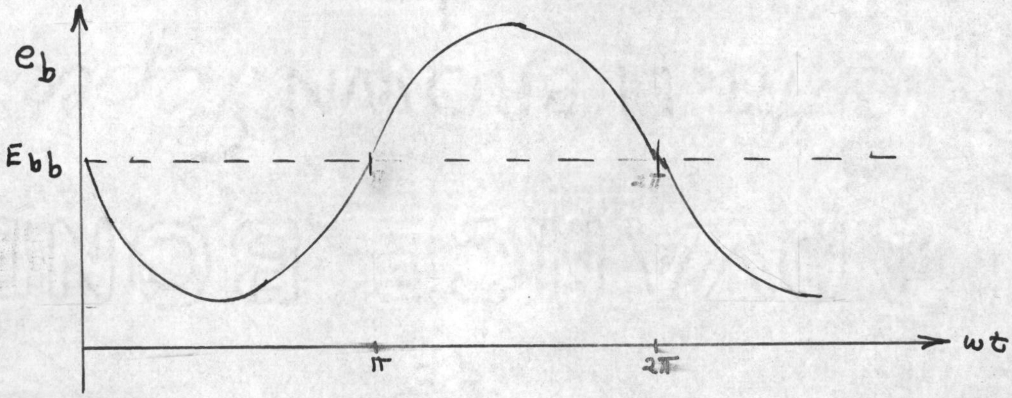


Fig. 19.2: The instantaneous plate voltage, e_b.

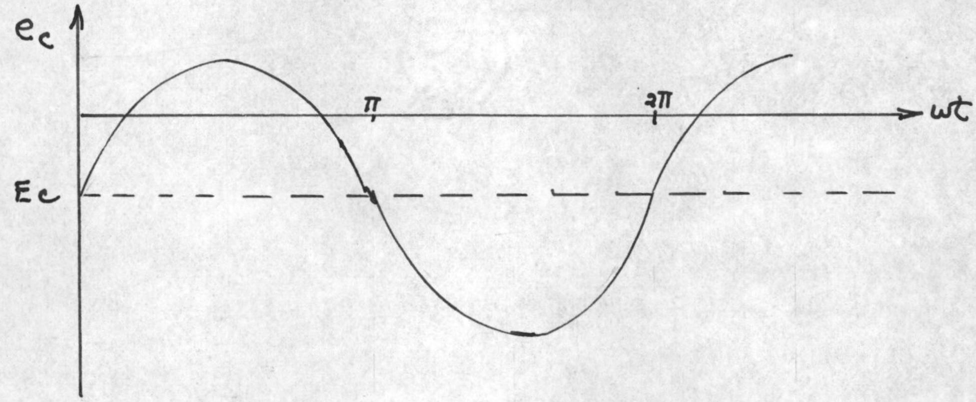


Fig. 19.3: The instantaneous grid voltage, e_c.

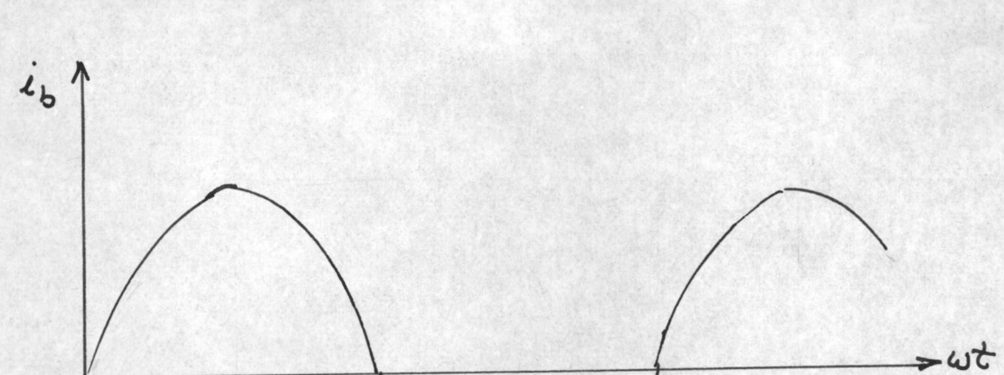


Fig. 19.4: The instantaneous plate current, i_b.

Since the circuit is tuned, the impedance at the resonant frequency is a pure resistance, R_t. Hence the effective value of load voltage, E_p, is

$$E_p = I_{p_1} R_t \tag{19.10}$$

The direct-current power input to the plate P_b, the alternating current power output to the load P_{ac}, the plate efficiency η_p are given in terms of the fundamental component of plate current I_{p1} by the following relations:

$$P_b = E_{bb} I_b = E_{bb}\left(\frac{2\sqrt{2}}{\pi} I_{p_1}\right), \tag{19.11}$$

$$P_{ac} = I_{p_1}^2 R_t, \tag{19.12}$$

$$\eta_p = \frac{P_{ac}}{P_b} \cdot 100. \tag{19.13}$$

20. REPRESENTATION OF AN EMPIRICAL CURVE

Frequently the function is not defined at every point, but is an empirical curve or set of experimental points which is to be represented by means of a trigonomentric sum analogous to the Fourier series of the form

$$u(x) = \frac{a_0}{2} + \sum_{k=1}^{r} (a_k \cos kx + b_k \sin kx). \tag{20.1}$$

Assume, for example, that the interval is and that there are n experimental values u_q taken at

$$x_q = \frac{2q\pi}{n}, \qquad q = 0, 1, 2, \cdots, n-1. \tag{20.2}$$

n being some number greater than 2r. The coordinates specifying the experimental points are

(20.3) $$\left(0, u_0\right), \left(\frac{2\pi}{n}, u_1\right), \cdots, \left(\frac{2q\pi}{n}, u_q\right), \cdots, \left(\frac{2(n-1)\pi}{n}, u_{n-1}\right).$$

The problem is to determine the (2r + 1) coefficients of (20.1). Substituting ~~the~~ each experimental point of (20.3) in (20.1), one obtains n equations (called the equations of condition) containing (r + 1) unknowns.

(20.4) $$u_q = \frac{a_0}{2} + \sum_{k=1}^{r}\left(a_k \cos\frac{k2q\pi}{n} + b_n \sin\frac{k2q\pi}{n}\right), \quad q=0,\cdots,n-1.$$

Now in order to determine anyone of the unknowns, one must form the normal equation with respect to the unknown by multiplying each of the equations of (20.4) by the coefficients of the unknown in the equation and adding together all these products.

The normal equation of a_0 is

(20.5) $$u_0 + u_1 + \cdots + u_{n-1} =$$
$$\frac{na_0}{2} + a_1\sum_{q=0}^{n-1}\cos\frac{q2\pi}{n} + a_2\sum_{q=0}^{n-1}\cos\frac{2q\cdot 2\pi}{n} + \cdots$$
$$+ a_r\sum_{q=0}^{n-1}\cos\frac{rq2\pi}{n} + \cdots + b_r\sum_{q=0}^{n-1}\sin\frac{rq2\pi}{n}.$$

Since

(20.6) $$\sum_{q=0}^{n-1}\cos\frac{kq2\pi}{n} = 0, \quad k = 1, \cdots, r.$$

and

(20.7) $$\sum_{q=0}^{n-1} \sin \frac{kq2\pi}{n} = 0, \quad k = 1, \ldots, r,$$

equation (20.5) becomes

(20.8) $$u_0 + u_1 + \cdots + u_{n-1} = \frac{n a_0}{2},$$

or

(20.9) $$a_0 = \frac{2}{n} \sum_{q=0}^{n-1} u_q.$$

The normal equation for a_1 is

(20.10) $$u_0 + u_1 \cos \frac{2\pi}{n} + u_2 \cos \frac{2\cdot 2\pi}{n} + \cdots + u_{n-1} \cos \frac{(n-1)2\pi}{n}$$
$$= a_0 \sum_{q=0}^{n-1} \cos \frac{q2\pi}{n} + a_1 \sum_{q=0}^{n-1} \left[\cos \frac{q2\pi}{n}\right]^2 + \cdots +$$
$$a_r \sum_{q=0}^{n-1} \cos \frac{q2\pi}{n} \cos \frac{rq2\pi}{n} + \cdots + b_r \sum_{q=0}^{n-1} \cos \frac{q2\pi}{n} \sin \frac{rq2\pi}{n}$$

Again, since

(20.11) $$\sum_{q=0}^{n-1} \cos \frac{iq2\pi}{n} \cos \frac{jq2\pi}{n} = \delta_{ij} \, n/2,$$

(20.12) $$\sum_{q=0}^{n-1} \cos \frac{iq2\pi}{n} \sin \frac{jq2\pi}{n} = 0,$$

(and for calculation of b_k)

(20.13) $$\sum_{q=0}^{n-1} \sin \frac{iq2\pi}{n} \sin \frac{jq2\pi}{n} = \delta_{ij} \, n/2,$$

equation (20.10) becomes

(20.14) $$u_0 + u_1 \cos \frac{2\pi}{n} + u_2 \cos \frac{2\cdot 2\pi}{n} + \cdots + u_{n-1} \cos \frac{(n-1)2\pi}{n} = \frac{n a_1}{2},$$

or

$$(20.15) \qquad a_1 = \frac{2}{n}\sum_{q=0}^{n-1} u_q \cos\frac{q2\pi}{n}.$$

The other normal equations may be obtained and reduced in the same way; finally we obtain

$$(20.16) \qquad a_k = \frac{2}{n}\sum_{q=0}^{n-1} u_q \cos\frac{kq2\pi}{n}, \quad k = 0, 1, \cdots, r.$$

$$(20.17) \qquad b_k = \frac{2}{n}\sum_{q=0}^{n-1} u_q \sin\frac{kq2\pi}{n}, \quad k = 1, 2, \cdots, r.$$

The resultant sum of (20.1) whose coefficients are defined by (20.16) and (20.17) is a trigonometric sum which represents the empirical curve.

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