AN ABSTRACT OF THE DISSERTATION OF

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Title: Splittings of Skeletal Homotopy Modules

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This dissertation is devoted to determining structure results on a group relative to a subgroup, using information about the kernel of the boundary map of associated free resolutions. If Y is a CW-complex with homotopy type $K(G,1)$ then for $n \geq 2$ the nth skeletal homotopy module, $h_n(Y) = \pi_n(Y^{(n)})$, is a kernel of the nth boundary homomorphism of a free resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules. By passing to the universal cover and using cellular homology, analogous descriptions in dimensions zero and one are available.

Let (Y, X) be a pair of connected, aspherical CW complexes of type $K(G,1)$ and type $K(S,1)$ respectively. If the map on fundamental groups induced by the topological inclusion is injective, then S can be seen as a subgroup of G and the induced skeletal homotopy module for X, $\mathbb{Z}G \otimes_S h_n(X)$, naturally injects into the nth skeletal homotopy module of Y. We define three conditions on this injection of the induced module. When it is split injective over $\mathbb{Z}S$ we say $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$ holds, when it is split over $\mathbb{Z}G$ we say $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds, and when it is split injective with $\mathbb{Z}G$-projective cokernel we say $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds.

When $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds, a theorem of Serre [Hue79] implies that every finite subgroup of G is determined by S. When $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds, a theorem of Howie and Schneebeli [HS81] implies that the intersection of S with its conjugates is torsion free.

When $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$ holds, results of Bogley and Dyer [BD93] are generalized in this thesis to show that either S is self-normalizing in G or S has cohomological dimension less than or equal to $n+1$.

We also study the relationships amongst these conditions. Our main result along these lines is that each of $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$, $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ and $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ imply the same at dimension $n+1$ and hence all higher dimensions. Meanwhile we provide an example to show that not even $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ implies $\mathrm{Sum}_{n-1}^{\mathbb{Z}S}(G,S)$. Moreover, clearly $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ implies $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ which implies $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$, but we provide examples to show neither converse holds in general.

We apply these splitting results to cyclically presented groups on n generators. We show that if $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds for the semi-direct product of a cyclically presented group on n generators with a cyclic group of order n, then the shift automorphism has order n. Using work of [BP92] we provide a family of cyclically presented groups whose shift automorphism has order n and apply a theorem of [CRS05] to determine that these groups cannot be the fundamental group of any hyperbolic 3-orbifold of finite volume.

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Nicole Sheree Seaders, Author

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1 INTRODUCTION

1.1 History and Statement of the Problem

There have been several studies devoted to the structure of modules over the integral group ring $\mathbb{Z}G$ that arise as kernels of boundary homomorphisms of a projective $\mathbb{Z}G$-resolution of the trivial module $\mathbb{Z}$. We study the kernel of the nth boundary homomorphism, denoted $K_n(G)$, in such a resolution,

$$0 \longrightarrow K_n(G) \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the P_i are projective $\mathbb{Z}G$-modules. One motivation for such a study arises from the cohomology and homology of the group G. Any projective resolution of such a kernel $K_n(G)$ extends to a projective resolution of $\mathbb{Z}$ and so the higher dimensional homology and cohomology of the group G can be expressed in terms of $K_n(G)$, see Theorem 2.1.18. Various hypotheses on the structure of $K_n(G)$ have been considered to obtain information about the higher cohomology of G. Productive hypotheses on $K_n(G)$ have been related to subgroups of G and this dissertation explores the relationships amongst, and the consequences of, such hypotheses.

An early example of this is Lyndon's Identity Theorem [Lyn50]. Lyndon considered a group G given by a one relator presentation, $\langle \mathbf{x} : r^p \rangle$ where r is not a proper power in the free group on $\mathbf{x}$. For the subgroup $S = \langle r \rangle$, Lyndon showed that S has order p and that there exists a short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{Z}[{}^{G}\!/\!{}_{S}] \longrightarrow \bigoplus_{\mathbf{x}} \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

called the Lyndon resolution for this presentation. Since $\mathbb{Z}[{}^{G}\!/\!{}_{S}] \cong \mathbb{Z}G \otimes_S \mathbb{Z}$, Shapiro's Lemma (see Theorem 2.1.17) and dimension shifting applied to Lyndon's resolution give a relationship between the cohomology of S and the cohomology of G, namely:

$$\begin{aligned} H^k(S;M) &= Ext^k_{\mathbb{Z}S}(\mathbb{Z};M) \\ &= Ext^k_{\mathbb{Z}G}(\mathbb{Z}[{}^{\mathrm{G}}\!/\!_{\mathrm{S}}];M) \\ &= Ext^{k+2}_{\mathbb{Z}G}(\mathbb{Z};M) \\ &= H^{k+2}(G;M). \end{aligned}$$

Lyndon thus described the higher cohomology of G in terms of the cohomology of the finite cyclic group S.

Several decades later, Serre proved a powerful consequence of such an isomorphism between the cohomology of G and that of some subgroup(s).

Theorem 1.1.1. *(Serre, see [Hue79]) Let G be a group and $\{S_i\}_{i\in\mathcal{I}}$ a family of subgroups such that for every $q \geq q_0$ the canonical map*

$$H^q(G,M) \to \prod H^q(S_i,M)$$

is an isomorphism for every $\mathbb{Z}G$-module M. Then for each finite subgroup K of G, there is an $i \in \mathcal{I}$, $g \in G$ such that $K \subseteq gS_ig^{-1}$ and $K \cap hS_ih^{-1} = \{1\}$ for all $h \notin gS_i$.

Therefore whenever the higher cohomology of G is carried by a family of subgroups $\{S_i\}$, then so are the finite subgroups of G. Meanwhile, the conjugates of the $\{S_i\}$ are as independent as possible. Thus conditions involving kernels in partial $\mathbb{Z}G$-resolutions that are supported in subgroups of the ambient group G give rise to both a better understanding of the cohomology of G and a better understanding of the diversity of its finite subgroups.

In 1981, Howie and Schneebeli explored one way to generalize Lyndon's one-relator case by considering when the kernel of the nth boundary map of a projective $\mathbb{Z}G$-resolution contains a permutation module as a direct summand [HS81]. A permutation module has the form $\bigoplus \mathbb{Z}[{}^{\mathrm{G}}\!/\!_{\mathrm{S}_i}]$ where the S_i are point stabilizers for the permutation G-action. Howie and Schneebeli showed that in this situation the point stabilizers must be finite and the cohomology of G contains a direct product of the cohomology of the S_i as a summand.

Thus Howie and Schneebeli were led to develop the following partial generalization of Serre's Theorem.

Theorem 1.1.2. *[HS81] Suppose $\mathcal{S} = \{S_i\}_{i\in\mathcal{I}}$ is a family of subgroups of G and suppose q, r are positive integers such that for every $\mathbb{Z}G$-module M the group $H^q(G, M)$ has a direct summand isomorphic to $\prod H^r(S_i, M)$. Suppose also that $i, j \in \mathcal{I}$ and $g \in G$ are such that $S_i \cap gS_jg^{-1}$ is not torsion-free. Then $i = j$ and $g \in S_i$.*

An important implication of Howie and Schneebeli's theorem when the S_i are finite is malnormality. A subgroup S is said to be **malnormal** if $S \cap gSg^{-1}$ is trivial for all $g \notin S$. If G is a finite group with a proper, nontrivial, malnormal subgroup S, then G is called a **Frobenius group** with Frobenius complement S. A great deal is known about these groups, see subsection 2.1.1.

Corollary 1.1.3. *[HS81] Suppose S is a non-trivial, proper subgroup of G and suppose $\mathbb{Z}[{}^{G}\!/_{S}]$ is isomorphic to a direct summand of the kernel of the nth boundary homomorphism of some $\mathbb{Z}G$-projective resolution $\mathcal{P} \to \mathbb{Z}$ for some positive integer n. Then*

(i) S is finite,

(ii) G has trivial center,

(iii) S is malnormal in G, and

(iv) S has periodic Tate cohomology with period dividing n.

The definition and main properties of the Tate cohomology theory for finite groups are presented in [Bro82, Ch. VI]. In particular, a finite group with periodic Tate cohomology is one for which all Sylow groups are either cyclic or generalized quaternion, see [Bro82, Ch. VI.9].

The relationship between the cohomology of G and the cohomology of a subgroup S needed for the hypotheses of Serre's Theorem and Howie and Schneebeli's Theorem occurs when a kernel of a $\mathbb{Z}G$-projective resolution contains an induced module of a kernel of a $\mathbb{Z}S$-projective resolution as a summand. Let $K_n(G)$ be the kernel of the nth boundary map of a $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Similarly let $K_m(S)$ be the kernel of the mth boundary map of a $\mathbb{Z}S$-resolution of $\mathbb{Z}$. If $K_n(G)$ contains $\mathbb{Z}G \otimes_S K_m(S)$ as a $\mathbb{Z}G$-module summand with complementary summand $\mathbb{Z}G$-projective (possibly trivial), Shapiro's Lemma and dimension shifting provides an isomorphism between the higher cohomology of G and the cohomology of S. If $K_n(G)$ contains $\mathbb{Z}G \otimes_S K_m(S)$ as a $\mathbb{Z}G$-module summand, then the same calculations imply the cohomology of S is a direct summand of the cohomology of G. For specifics on both of these results see Theorem 2.1.18.

A natural source of a projective resolution over a group ring, is the cellular chain complex of the universal cover of an Eilenberg Maclane space of the associated group G. Interestingly, Lyndon used that the Lyndon resolution naturally arose as the cellular chain complex of an Eilenberg Maclane space to clarify his original algebraic approach after Reidemeister pointed out the connection [ARS84, Lyn50, p. 651]. Meanwhile, Howie and Schneebeli remark in [HS81] that it would be nice to know topological circumstances under which group actions on spaces produce the phenomena they observe arising in these resolutions. With such considerations as motivation, this dissertation is dedicated to studying kernels of resolutions that arise from the cellular chain complex of the universal cover of an Eilenberg Maclane space.

Let Y be a CW complex with fundamental group $\pi_1(Y) = G$ and let R be a commutative ring with identity. All modules will be considered left modules unless otherwise indicated. We define the **nth skeletal homotopy module of Y** to be the kernel of the nth boundary homomorphism, namely, $h_n(Y)$ such that

$$0 \longrightarrow h_n(Y) \longrightarrow C_n(\widetilde{Y}; R) \xrightarrow{d_n} \cdots \xrightarrow{d_2} C_1(\widetilde{Y}; R) \xrightarrow{d_1} C_0(\widetilde{Y}; R) \xrightarrow{\varepsilon} R \longrightarrow 0.$$

Note that $h_n(Y) \cong H_n(\widetilde{Y}^{(n)}; R)$, where $Y^{(n)}$ is the n-skeleton of Y, and depends upon the given cell structure for Y.

In low dimensions, the skeletal modules $h_n(Y)$ have been the subject of considerable study, see for example [Coh72, GR75, Gru76, Dic81, BD93]. When $n = 0$ and Y has only one 0-cell, then $h_0(Y) = IG$, the augmentation ideal of the group ring RG. When $n = 1$ and the 2-skeleton of Y is modeled on a group presentation, then $h_1(Y)$ is the relation module associated to the presentation. Whenever Y is (n-1)-connected for $n \geq 2$, then Hurewicz isomorphism theorems imply that $h_n(Y) \cong \pi_n(Y^{(n)})$. For further discussion of these examples see subsection 2.2.1

We say that the pair (Y, X) ***realizes*** the group pair (G, S) if Y and X are connected CW complexes with fundamental groups G and S respectively such that the homomorphism on fundamental groups induced by the topological inclusion,

$$inc_{\#} : \pi_1(X) \to \pi_1(Y),$$

is injective where we identify $S \leq G$ via this injection. Standard arguments show that the inclusion of X into Y induces an injection of $h_n(X)$ into $h_n(Y)$ (Proposition 2.2.5) and an injection of the induced module $RG \otimes_S h_n(X)$ into $h_n(Y)$ (Theorem 2.2.8). Thus we have a short exact sequence of RG-modules

$$0 \longrightarrow RG \otimes_S h_n(X) \xrightarrow{\Phi_n} h_n(Y) \longrightarrow \operatorname{coker}(\Phi_n) \longrightarrow 0. \tag{1.1}$$

Our main focus is to study circumstances under which this short exact sequence splits, in which case we have a direct sum decomposition,

$$h_n(Y) \cong RG \otimes_S h_n(X) \oplus \operatorname{coker}(\Phi_n).$$

We will call such a splitting a **skeletal homotopy splitting** and, as we will see, it is important to distinguish between RG-splittings and RS-splittings.

Let S be a subgroup of G. It is well-known that we can build an aspherical CW-pair (Y, X) that realizes the group pair (G, S) (see Lemma 2.2.10). Some terminology for the various splitting hypotheses is useful to facilitate the discussion. If there exists a connected, aspherical CW-pair (Y, X) that realizes (G, S) where the short exact sequence (1.1) is split exact

- as RS-modules we say $\mathbf{Sum_n^{RS}(G,S)}$ holds;
- as RG-modules we say $\mathbf{Sum_n^{RG}(G,S)}$ holds; and
- with RG-projective cokernel we say $\mathbf{PSum_n^{RG}(G,S)}$ holds.

We prove the following theorem that says we do not need to consider a splitting with RS-projective cokernel as a separate condition.

Theorem 1.1.4. *Let S be a subgroup of G. Then $\mathrm{Sum}_n^{RS}(G,S)$ holds if and only if coker(Φ_n) is RS-projective for any CW pair (Y,X) that realizes the group pair (G,S).*

Note that these are progressively stronger conditions in that $\mathrm{PSum}_n^{RG}(G,S)$ implies $\mathrm{Sum}_n^{RG}(G,S)$ which in turn implies $\mathrm{Sum}_n^{RS}(G,S)$. We show that none of the reverse implications hold in general, however, see Examples 5.2.3 and 5.2.6.

The theorems of Serre and Howie and Schneebeli apply to skeletal homotopy splittings over $\mathbb{Z}G$ since skeletal homotopy modules are kernels of free resolutions. Interestingly, there are still group theoretic consequences for skeletal homotopy splittings over $\mathbb{Z}S$, one of the main results of this thesis. The following statement enumerates our general results in the case where the subgroup S is finite.

Theorem 1.1.5. *Let S be a finite, non-trivial subgroup of G and let n be a non-negative integer.*

(i) If $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds then every finite subgroup of G is contained in a unique conjugate of S.

(ii) If $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds then S is malnormal in G.

(iii) If $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$ holds then S is self-normalizing in G.

We also study the relationships between these conditions at various dimensions. Our main result along these lines is that splittings ‘go uphill’ as follows.

Theorem 1.1.6. *Let S be a subgroup of G and let n be a non-negative integer.*

(i) If $\mathrm{Sum}_n^{RS}(G,S)$ holds then $\mathrm{Sum}_k^{RS}(G,S)$ holds for all $k \geq n$.

(ii) If $\mathrm{Sum}_n^{RG}(G,S)$ holds then $\mathrm{Sum}_k^{RG}(G,S)$ holds for all $k \geq n$.

(iii) If $\mathrm{PSum}_n^{RG}(G,S)$ holds then $\mathrm{PSum}_k^{RG}(G,S)$ holds for all $k \geq n$.

We also show by example that splittings cannot be projected down even one dimension. See Example 5.2.2.

Various papers have investigated decomposition questions for skeletal homotopy modules in low dimensions. For augmentation ideals, finding $\mathbb{Z}G \otimes_S IS$ as a direct $\mathbb{Z}G$-summand of IG has been explored extensively. Cohen related $\mathbb{Z}G$-summands when G is a finitely generated torsion free group to S being a free factor of G, [Coh72]. Cohen also showed if G is finite and $\mathbb{Z}G \otimes_S IS$ is a $\mathbb{Z}G$-summand of IG then (G,S) is a Frobenius pair [Gru76, Proposition 8.9]. Meanwhile, Gruenberg and Roggenkamp noted the converse to Cohen's result [Gru76]. This research culminated in work by Dicks, who characterized fully when $\mathbb{Z}G \otimes_S IS$ is a direct $\mathbb{Z}G$-summand of IG in terms of the Bass-Serre theory of groups acting on trees for any group G finitely generated over S [Dic81].

Theorem 1.1.7. *[Dic81] If G is finitely generated over S then the following are equivalent:*

(i) $RG \otimes_S IS$ is an RG-summand of IG

(ii) G acts on a tree X

- *(a) with finite edge stabilizers,*
- *(b) with a vertex v_0 having stabilizer precisely S,*
- *(c) such that for each $v \neq v_0$ having $star(v)$ infinite, $|S \cap Stab_G(v)|$ is both finite and a unit in R for all $g \in G$,*
- *(d) and $|S \cap gSg^{-1}|$ is both finite and a unit in R for all $g \notin S$.*

Dicks' results provide a method to write down explicit RS-decompositions of the augmentation ideal when S is malnormal, see Proposition 5.1.9, and explicit RG-decompositions when G is a Frobenius group with complement S, see Proposition 5.1.14.

For finite groups, decompositions of relation modules have also been investigated. Gruenberg and Roggenkamp related direct sum decompositions of minimal relation modules, that is relation modules arising from presentations with a minimal number of generators, to certain direct sum decompositions of augmentation ideals, [GR75]. They showed that when G is solvable, minimal relation modules can be decomposed into a direct sum of two non-projective factors if and only if G is a frobenius group with cyclic complement or G is a special 2-frobenius group, [GR76, GR82]. As stated previously, Dicks' characterization allows us to write down explicit RG-decompositions of IG. By Theorem 1.1.6 these can be lifted to explicit decompositions of the relation module (for details see the proof of Lemma 3.2.1).

In a study of second homotopy modules of pairs of complexes, Bogley and Pride introduced the concept of an aspherical relative presentation. A relative presentation for a group G is a triple $\langle S, \mathbf{x} : \mathbf{r} \rangle$ where S is a group, $\mathbf{x}$ is a basis for a free group F, and $\mathbf{r}$ is a subset of the free product $S * F$ such that G is isomorphic to the quotient of $S * F$ modulo the normal closure of $\mathbf{r}$. The asphericity concept is formulated in terms of spherical pictures [BP92].

Theorem 1.1.8. *[BP92, Theorem 1.2] If a relative presentation $\langle S, \boldsymbol{x} : \boldsymbol{r} \rangle$ for G is aspherical then $Sum_1^{RG}(G, S)$ holds.*

Bogley and Dyer then considered $\mathrm{Sum}_2^{\mathbb{Z}S}(G, S)$ and showed that either the cohomological dimension of S is less than or equal to 3, or S must be self-normalizing in G [BD93]. The main investigation of this dissertation originated in an effort to frame all of the above results in a larger context and generalize them to other dimensions.

In a further application to group theory, we show that skeletal homotopy splittings influence the dynamics of group automorphisms. If a subgroup S normalizes another non-trivial subgroup Γ of G, then S acts on Γ via conjugation in G.

Theorem 1.1.9. *Consider a group pair (G, S) where S is a torsion group that normalizes a nontrivial subgroup Γ of G. If $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ holds for some n and $S \cap \Gamma = \{1\}$, then S acts freely on $\Gamma - \{1\}$. In particular, the action of S on Γ is faithful and so determines an embedding of S in the automorphism group $\mathrm{Aut}(\Gamma)$.*

In particular, this can be applied to determine the order of the shift automorphism of a cyclically presented group. A **cyclically presented group** is one having a presentation of the form

$$\Gamma_n(w) = \langle x_0, \ldots, x_{n-1} \mid w, \theta(w), \theta^2(x), \ldots, \theta^{n-1}(w) \rangle.$$

where $w = w(x_0, \ldots, x_{n-1})$ is a word representing an element of the free group F. For given n and w, the shift determines an action of $C_n = \langle a \mid a^n \rangle$ on $\Gamma_n(w)$ and we denote the resulting split extension by $G_n(w) = \Gamma_n(w) \rtimes C_n$ which has relative presentation

$$G_n(w) = \langle C_n, x \mid w(a, x) \rangle$$

See Section 4.2 for more details.

Theorem 1.1.10. *Consider a nontrivial cyclically presented group $\Gamma_n(w)$ as above. If the relative presentation constructed as above for the group $G_n(w) = \Gamma_n(w) \rtimes_\theta C_n$, is aspherical in the sense of [BP92], then:*

(i) $Sum_k^{\mathbb{Z}G_n(w)}(G_n(w), C_n)$ holds for all $k \geq 1$;

(ii) The cyclic group C_n of order n acts freely on $\Gamma_n(w) - \{1\}$ via the shift θ;

(iii) The shift automorphism $\theta \in \mathrm{Aut}(\Gamma_n(w))$ has order n.

This in turn has consequences on whether the cyclically presented group can be the fundamental group of any hyperbolic 3-orbifold of finite volume, see [CRS05].

1.2 Organization of this Dissertation

Chapter 2 lays out the algebraic and topological background for skeletal homotopy splittings. All of the results in Chapter 2 are either standard or well-known, but the presentation is tailored to suit our needs. Section 1 introduces group actions, followed by a discussion of malnormal and self-normalizing subgroups and a brief introduction to Frobenius groups. Section 1 also lists some definitions and properties of the objects in homological algebra that will be used frequently in this dissertation and concludes with a cohomological calculation for kernels of projective resolutions that contain an induced kernel as a direct summand, the consequences of which give motivation to investigate these kernels.

Section 2 of Chapter 2 gives an overview of the topological spaces, their cellular chain complexes, and gives a more thorough introduction to skeletal homotopy modules. Understanding the cellular chain complexes allows us to always find that the induced n-th skeletal homotopy module for X injects into the n-th skeletal homotopy module for Y. Moreover, when the spaces are aspherical, the skeletal homotopy modules are kernels of a free resolution and, without loss of generality, we show that we can restrict to spaces with a single zero cell.

Chapter 3 contains new results regarding relationships amongst the various skeletal homotopy splittings. In Section 1 we prove that $\mathrm{Sum}_n^{RS}(Y, X)$ is equivalent to RS-projectivity of the complementary summand, thus weak skeletal homotopy splittings are also 'projective' skeletal homotopy splittings over RS. In Section 2 we prove Theorem 1.1.6, that these hypotheses lift to higher dimensions. We also prove that if $\mathrm{coker}(\Phi_n)$ is trivial these hypotheses step down one dimension.

Chapter 4 discusses the group theoretic implications for skeletal homotopy splittings when $R = \mathbb{Z}$. In addition to the cohomological and group theoretic consequences for $\mathrm{PSum}_n^{\mathbb{Z}G}(G, S)$ and $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ (Theorem 1.1.5 (i) and (ii)), Section 2 extends Bogley and Dyer's results to show $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$ still obtains group theoretic implications even without the direct implications on the cohomology (Theorem 1.1.5 (iii)). Section 3 applies splittings to cyclically presented groups. We prove Theorem 1.1.9 and Theorem 1.1.10, both new to this thesis, and then use the combinatorial conditions developed in [BP92] and work of [CRS05] to show that a certain family of cyclically presented groups is not the fundamental group of any hyperbolic 3-orbifold of finite volume.

Chapter 5 is devoted to low dimensions and finding examples to distinguish between the conditions. Section 1 provides alternate proofs of Dicks' result for RS-splittings of augmentation ideals and RG-splittings when G is finite using permutation modules. Since we can restrict to spaces with a single zero cell, this implies in particular that $\mathrm{Sum}_0^{RG}(G,S)$ is equivalent to $\mathrm{Sum}_0^{RS}(G,S)$ when G is finite, and a necessary and sufficient condition for both is that $|S \cap gSg^{-1}|$ is finite and a unit in R for all $g \notin S$. In Section 2 we use Dicks' results to find distinguishing examples for our conditions. Namely group pairs where $\mathrm{PSum}_1^{RG}(G,S)$ holds but not $\mathrm{Sum}_0^{RS}(G,S)$, where $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds but not $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$, and where $\mathrm{Sum}_0^{RS}(G,S)$ holds but not $\mathrm{Sum}_0^{RG}(G,S)$.

Chapter 6 summarizes the key results and discusses possibilities for further research. In particular, lifting splittings could shed more light on decompositions of interesting modules at higher dimensions, given information known at lower dimensions. Also finding ways to detect such splittings for dimensions greater than zero is relatively unexplored aside from work of [BP92].

2 MATHEMATICAL BACKGROUND

We begin with some algebraic and topological background needed to discuss skeletal homotopy modules and their splittings. Section 1 discusses the algebraic side while Section 2 introduces the basic topological structure of the associated spaces. Of note is the last subsection of Section 1 on induced modules and kernels of projective resolutions which gives the cohomological calculations that motivate many of the studies of these kernels.

2.1 Group Theory and Homological Algebra

2.1.1 Group Actions, Malnormal and Self-normalizing

First we define some properties of group actions, including their connections to malnormal and self-normalizing subgroups and end with some properties of Frobenius groups. Recall that a group G **acts** on the left on a set X if there exists a function $f : G \times X \to X$, denoted $(g, x) \mapsto g \cdot x$, such that

(i) $1_G \cdot x = x$ for all $x \in X$ where 1_G is the identity in G, and

(ii) $h \cdot (g \cdot x) = (hg) \cdot x$ for all $g, h, \in G$ and for all $x \in X$.

We then say X is a **G-set**. We can similarly define a right action, $f : X \times G \to X$ $f(x, g) = x \cdot g$. Unless otherwise specified, a group action will be assumed to be a left action.

If G acts on a set X and $x \in X$, then the **orbit of x** in G is the subset of X,

$$Orb_G(x) = \{g \cdot x : g \in G\}.$$

The orbits of a group action partition the set X. If there is only one orbit, we say the action of G is **transitive**. If an orbit consists of just one point $x \in X$, then that point is a **global fixed point**. The **stabilizer of x** in G is the subset of G,

$$Stab_G(x) = \{g \in G : g \cdot x = x\},$$

that is easily seen to be a subgroup of G. We say that the G-action is **free** (or that G acts freely on X) if the stabilizer of each point is the trivial subgroup of G.

Any subgroup $S \leq G$ acts on the group G on the right (or the left) via group multiplication. If S acts on the right, the set of left cosets of G,

$$G/_S = \{gS : g \in G\},$$

is the set of all the S-orbits. The set of left cosets of G is both a left G-set and a left S-set as well.

There is a well-known correspondence between the orbits and the stabilizers of a G-set that we will often use.

Proposition 2.1.1 (Orbit-Stabilizer Correspondence)**.** *Let X be a G-set. For each $x_0 \in X$ the function*

$$\begin{aligned} \Psi : G/_{Stab_G(x_0)} &\longrightarrow Orb_G(x_0) \qquad\qquad \textit{defined by} \\ gStab_G(x_0) &\longmapsto g \cdot x_0 \end{aligned}$$

is a bijection of G-sets that respects the G-action.

Due to the above correspondence, understanding the properties of $G/_S$ as a G-set and an S-set will be helpful in understanding the structure of other G-sets and S-sets including some induced modules and topological spaces.

Example 2.1.2. Let $S \leq G$. Then G acts transitively on the left on the set of left cosets $G/_S$ via group multiplication with $Stab_G(gS) = gSg^{-1}$ for all $g \in G$. Also, the subgroup S acts on the left on $G/_S$ via group multiplication with

$$Orb_S(gS) = \{sgS : s \in S\} = SgS, \text{ and}$$

$$Stab_S(gS) = S \cap Stab_G(gS) = S \cap gSg^{-1}$$

for all $g \in G$. Note that $\{S\}$ is an S-orbit of the S-action on ${}^{G}\!/_{S}$, thus the complement ${}^{G}\!/_{S} - \{S\}$ is an S-set as well.

Example 2.1.3. Any group G acts via conjugation on the set consisting of its subgroups. A global fixed point under this action is a normal subgroup of G. For $S \leq G$, the stabilizer of S under conjugation is the **normalizer** of S in G:

$$N_G(S) = \{g \in G \mid gSg^{-1} = S\}.$$

It is clear that $S \leq N_G(S)$. The subgroup S is **self-normalizing** in G if $N_G(S) = S$.

Recall that given S a subgroup of a group G, we say S is **malnormal** in G if and only if $S \cap gSg^{-1} = 1$ for all $g \in G - S$. Notice that if $S = 1$ or $S = G$ then S is malnormal. If G has a non-trivial malnormal subgroup then there are immediate group theoretic consequences.

Lemma 2.1.4. *If S is a proper, nontrivial, malnormal subgroup of G then*

(i) S is self-normalizing in G (i.e. $gSg^{-1} = S$ if and only if $g \in S$), and

(ii) G has trivial center.

Proof. For (i), let $g \in N_G(S)$, the normalizer in G of S. Then $gSg^{-1} = S$ and so $gSg^{-1} \cap S = S \neq 1$. Since S is marlnormal, we see that $g \in S$.

For (ii) let $z \in Z(G)$, the center of G. Then $z \in N_G(S)$ and so by part (i) $z \in S$. But then for any $g \in G - S$, we have $gzg^{-1} = z \in S \cap gSg^{-1}$. Thus by malnormality $z = 1$ and therefore $Z(G) = 1$. □

In particular, part (i) implies that if S is both normal and malnormal in G then S must be trivial. The converse of part (i), however, is false as seen in Example 2.1.9.

Both malnormality and self-normalizing are related to the fixed points of the action of S on the cosets ${}^{\mathrm{G}}\!/\!_{\mathrm{S}}$. The following elementary characterization of malnormality will be useful for our purposes.

Proposition 2.1.5. *Consider the action of a non-trivial subgroup S of a group G on the cosets ${}^{G}\!/\!_{S}$.*

(i) Then S acts freely on ${}^{G}\!/\!_{S} - \{S\}$ if and only if S is malnormal in G.

(ii) If S is not self-normalizing in G, then the S-action on ${}^{G}\!/\!_{S} - \{S\}$ has a global fixed point.

(iii) If S is finite, then S is self-normalizing in G if and only if the action of S on ${}^{G}\!/\!_{S} - \{S\}$ has no global fixed point.

Proof. The claim (i) follows from Example 2.1.2 since the action is free if and only if all stabilizers are trivial. For part (ii), if S is not self-normalizing there is a $g \in G - S$ so that $gSg^{-1} = S$. But then $S = S \cap gSg^{-1} = Stab_S(gS)$ so $gS \in {}^{\mathrm{G}}\!/\!_{\mathrm{S}} - \{S\}$ is fixed by S. To prove (iii), if S has a global fixed point $gS \in {}^{\mathrm{G}}\!/\!_{\mathrm{S}} - \{S\}$ thus with $sgS = gS$ for all $s \in S$, then $S \subseteq gSg^{-1}$. Since S is finite it follows that $S = gSg^{-1}$ so g normalizes S. □

Corollary 2.1.6. *If S has prime order then S is self-normalizing in G if and only if S is malnormal in G.*

Proof. It suffices to show that when $|S|$ is prime if S is self-normalizing in G then S is malnormal in G, since the converse is always true. By Proposition 2.1.1 every S-orbit has order dividing $|S|$ and so either has order $|S|$ or 1. Since S has no global fixed points by Proposition 2.1.5 part (iii), every orbit has order $|S|$ and thus the stabilizers are all trivial. Therefore S is malnormal in G. □

Proposition 2.1.5 also allows us to provide characterizations of malnormality and self-normalizing with semi-direct products. Let $G = K \rtimes S$, where S acts on K on the left via automorphisms of K. We denote the S-action on K by ${}^sk$ for each $s \in S$ and each $k \in K$. Then any element of $g \in G$ can be written uniquely as $g = ks$ for some $k \in K$ and some $s \in S$, with multiplication

$$g_1 \cdot g_2 = k_1 s_1 k_2 s_2 = k_1 \, {}^{s_1}k_2 s_1 s_2.$$

Corollary 2.1.7. *For $G = K \rtimes S$ we have*

(i) S acts freely on $K - \{1\}$ if and only if S is malnormal in G.

(ii) If the action of S on K has no non-trivial fixed points then S is self-normalizing in G. If S is finite and self-normalizing in G then the action of S on K has no non-trivial, global, fixed points.

Proof. Since for each $g \in G$ there exists a unique $k \in K$ and a unique $s \in S$ such that $g = ks$, for each coset $gS \in {}^{G}\!/\!_{S}$ there is exactly one $k \in K$ such that $gS = kS$. Thus there is a bijection of S-sets from K to ${}^{G}\!/\!_{S}$. Consider

$$\Psi : {}^{G}\!/\!_{S} \longrightarrow K$$

given by $\Psi(gS) = k$ where $gS = kS$. Clearly Ψ is well-defined, one-to-one and onto. Moreover Ψ respects the S-action since

$$\Psi(skS) = \Psi({}^skS) = {}^sk = sk = s\Psi(kS).$$

Thus we have a correspondence between the S-sets K and ${}^{G}\!/\!_{S}$ and we can apply Proposition 2.1.5 to obtain the result. □

Example 2.1.8. *The group $S_3 \cong \mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$ has malnormal subgroup $\mathbb{Z}_2$, since the non-identity element in $\mathbb{Z}_2$ sends a non-identity element of $\mathbb{Z}_3$ to its inverse, and thus acts freely on $\mathbb{Z}_3 - \{1\}$.*

Example 2.1.9. *The group $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4$ has self-normalizing subgroup $\mathbb{Z}_4$ since a generator $x \in \mathbb{Z}_4$ sends a non-identity element of $\mathbb{Z}_3$ to its inverse. But then x^2 sends every element of $\mathbb{Z}_3$ to itself and so $\mathbb{Z}_4$ does not act freely on $\mathbb{Z}_3 - \{1\}$. Thus $\mathbb{Z}_4$ is self-normalizing but not malnormal.*

We end this section with a discussion of Frobenius groups. Recall that if S is a proper, non-trivial, malnormal subgroup of a finite group G then G is a **Frobenius group** with **Frobenius complement** S. In particular, we then call the group pair (G, S) a **Frobenius pair**.

If (G, S) is a Frobenius pair, Frobenius proved that the set $K = G - \bigcup_{g \in G} gSg^{-1}$, called the **Frobenius kernel**, is a normal subgroup of G [Hup67] . (For infinite G this set is not necessarily a subgroup of G, hence the finiteness condition [DM96, Example 3.4.2].) Frobenius also proved that a Frobenius group G is the semi-direct product of the complement and the kernel.

Theorem 2.1.10. *[Frobenius, see KvdW96, p. 404] Suppose that G is a Frobenius group with Frobenius complement S and kernel K. Then $K \lhd G$, $G = KS$, $K \cap S = \{1\}$, and so $G = K \rtimes S$. Moreover K and S have co-prime orders.*

Frobenius groups have a very restricted structure as demonstrated by the previous theorem. Much is known about the complement and kernel of a Frobenius group as well.

Theorem 2.1.11. *Suppose that G is a Frobenius group with Frobenius complement S and Frobenius kernel K. Then*

(i) The Sylow p-subgroups of S are cyclic if p is odd; the Sylow 2-subgroups of S are cyclic groups or generalized quaternion 2-groups [Bur01].

(ii) K is nilpotent [Tho59].

2.1.2 Tensor Products, Hom, Ext, and Exact Sequences

Let Λ be an associative ring with identity. Let M be a right Λ-module and N a left Λ-module. The tensor product gives rise to two covariant functors $M \otimes_\Lambda -$ and $- \otimes_\Lambda N$ from Λ-modules to abelian groups that are additive, see [DF99, Theorem 17, Section 10.4].

The functor $- \otimes_\Lambda N$ is right exact in that if

$$A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

is exact, then

$$A_2 \otimes_\Lambda N \longrightarrow A_1 \otimes_\Lambda N \longrightarrow A_0 \otimes_\Lambda N \longrightarrow 0$$

is also exact. The functor $M \otimes_\Lambda -$ is also right exact. If M (resp. N) is a free Λ-module, then $M \otimes_\Lambda -$ (resp. $- \otimes_\Lambda N$) is an *exact functor* in that it carries exact sequences to exact sequences, see [DF99, Theorem 39, Section 10.5]. When $\Lambda = RG$ we will write the tensor product over the group ring as $M \otimes_G N$.

Correspondingly, $\mathrm{Hom}_\Lambda(M, -)$ is a covariant functor from left Λ-modules to abelian groups that is left exact. Moreover, if P is a projective Λ-module, then $\mathrm{Hom}_\Lambda(P, -)$ is an exact functor, see [DF99, Proposition 30, Section 10.5]. Meanwhile, $\mathrm{Hom}_\Lambda(-, N)$ is a contravariant functor from left Λ-modules to abelian groups and is left exact in that if

$$A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0$$

is exact, then

$$\mathrm{Hom}_\Lambda(A_2, N) \xleftarrow{\partial_2^*} \mathrm{Hom}_\Lambda(A_1, N) \xleftarrow{\partial_1^*} \mathrm{Hom}_\Lambda(A_0, N) \xleftarrow{\partial_0^*} 0$$

is also exact.

For a short exact sequence of Λ-modules we will often use the following well-known characterization.

Lemma 2.1.12 (Split Short Exact Sequences)**.** *[DF99, p. 364] Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence of Λ-modules. Then the following are equivalent.

(i) The map f is split injective, that is, there exists a Λ-homomorphism $\overline{f} : B \to A$ such that $\overline{f} \circ f = 1_A$.

(ii) *The map g is split surjective, that is, there exists a Λ-homomorphism $\overline{g} : C \to B$ such that $g \circ \overline{g} = 1_C$.*

(iii) *There is a Λ-module decomposition $B \cong A \oplus C$, where f is the natural injection on A, and g is the natural projection to C.*

Note that any short exact sequence with a surjection onto a projective Λ-module is split exact since a split surjection as in (ii) always exists.

For a Λ-module A, a **projective Λ-resolution of** A is a long exact sequence of projective Λ-modules, P_i, that map onto A,

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0.$$

Let B be a Λ-module and consider $\mathrm{Hom}_\Lambda(\mathbf{P}_A, B)$,

$$\cdots \xleftarrow{\partial^*_{n+1}} \mathrm{Hom}_\Lambda(P_n, B) \xleftarrow{\partial^*_n} \cdots \xleftarrow{\partial^*_1} \mathrm{Hom}_\Lambda(P_0, B). \longleftarrow 0.$$

We define the abelian group $\mathrm{Ext}^n_\Lambda(A, B)$ to be

$$\mathrm{Ext}^n_\Lambda(A, B) = {}^{ker(\partial^*_{n+1})}\!/_{im(\partial^*_n)} \text{ for all } n \geq 1, \text{ and}$$
$$\mathrm{Ext}^0_\Lambda(A, B) = ker(\partial^*_1) = im(\partial^*_0) = \mathrm{Hom}_\Lambda(A, B).$$

$Ext_\Lambda(A, B)$ is independent of projective resolution and has some nice properties that aid in calculations. First, Ext is multiplicative in the first variable so that,

$$Ext^n_\Lambda(\bigoplus_{k \in K} A_k, B) \cong \prod_{k \in K} Ext^n_\Lambda(A_k, B).$$

Moreover if P is a projective Λ-module, $Ext^n_\Lambda(P, B) = 0$ for all $n > 0$. (For a complete discussion of these results, see [Rot02, Section 10.5, 10.6].)

The coExtology of a group with coefficients in M is a special case of Ext. That is,

$$\begin{aligned} H^*(G, M) &= Ext^*_{\mathbb{Z}G}(\mathbb{Z}, M) \\ &= ker(\partial^*_{n+1})/im(\partial^*_n) \end{aligned}$$

where ∂ is the boundary map of a projective resolution of $\mathbb{Z}$.

We say the **cohomological dimension of** G **is less than or equal to** n, $\mathbf{cd(G)} \leq \mathbf{n}$, if $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution of length n,

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

We say $\mathbf{cd(G)} = \mathbf{n}$ if $\mathbb{Z}$ admits no projective $\mathbb{Z}G$-resolution of length less than n. If no such integer exists, we say $\mathbf{cd(G)} = \infty$.

2.1.3 Induced Modules and Kernels of Projective Resolutions

This subsection proves some preliminary results about induced modules that will be needed for the induced skeletal homotopy modules, as well as proves the cohomological and group theoretic consequences of kernels of projective resolutions containing an induced module as a summand. Let S be a subgroup of G and let M be a left RS-module. The tensor product $RG \otimes_S M$ has a natural left multiplication

$$g(x \otimes m) = (gx) \otimes m,$$

making $RG \otimes_S M$ into a left RG-module called the **induced module**, [see Bro82, III.3]. There is a natural inclusion of RS-modules,

$$i : M \to RG \otimes_S M$$
$$m \mapsto 1 \otimes m$$

where the induced module is thought of as a left RS-module by restriction of scalars to the group ring RS.

Lemma 2.1.13. *Let S be a subgroup of G and let M be a left RS-module. The natural inclusion embeds M in $RG \otimes_S M$ as an RS-summand.*

Proof. The map i is split injective by an RS-homomorphism

$$\rho : RG \otimes_S M \to M$$

determined by

$$\rho(g \otimes m) = \begin{cases} gm & : g \in S \\ 0 & : g \notin S \end{cases}$$

where one checks S-bilinearity. Thus Lemma 2.1.12 gives $M \cong 1 \otimes M$ and $RG \otimes_S M \cong 1 \otimes M \oplus (RG \otimes_S M/1 \otimes M)$. □

To further understand the structure of the induced module, we need to better understand the structure of RG as an RS-module.

Proposition 2.1.14. *The group ring RG is both a free left and a free right RS-module.*

Proof. Since G is a right S-set, it is the disjoint union of left cosets $G = \bigcup_{gS \in {}^{G}/_{S}} gS$. Moreover, for each left coset there is an associated cyclic submodule of the right RS-module RG given by $g \cdot RS = \{g \cdot \xi : \xi \in RS\}$. Note that for each fixed coset representative $g \in gS$, the RS-module $g \cdot RS$ is isomorphic to RS via an RS-homomorphism that sends $1 \in S$ to $g \in gS$.

Now each element in RG can be written as an R-linear combinations of elements in each coset. Moreover, for any two distinct cosets, $gS \cap hS = 0$, the associated RS-modules are disjoint, $g \cdot RS \cap h \cdot RS = 0$. Thus there is a direct sum decomposition of abelian groups with a summand for each coset representative

$$RG \cong \bigoplus_{gS \in {}^{G}/_{S}} g \cdot RS.$$

Since each summand is a free right RS-module, this is a right RS-decomposition and RG is a free right RS-module.

Since G is also a left S-set, a similar argument shows there is a direct sum decomposition arising from the right cosets,

$$RG \cong \bigoplus_{Sg \in {}_{S}\backslash^{G}} RSg,$$

where each summand is a free left RS-module. Thus RG is also a free left RS-module. □

Note that the decomposition of RG as a free right RS-module has a left S-action arising from the group multiplication. Moreover, there is a bijection as left S-sets between the left cosets ${}^{G}\!/_{S}$ and the free RS-module summands $g \cdot RS$. Since tensor products are additive in the first variable, this helps us better understand $RG \otimes_S M$ as a left RS-module.

Corollary 2.1.15. *Let S be a subgroup of G and let M be a left RS-module. Then*

$$RG \otimes_S M \cong \bigoplus_{gS \in {}^{G}\!/_{S}} (g \otimes M)$$

as R-modules, where the left S-action permutes the summands corresponding to left cosets $gS \in {}^{G}\!/_{S} - \{S\}$ and stabilizes the RS-summand $1 \otimes M$.

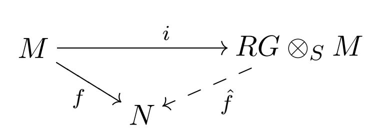
Proof. Since the tensor product is additive in the first variable we have a direct sum decomposition as R-modules

$$\begin{aligned} RG \otimes_S M &\cong \Big(\bigoplus_{gS \in {}^{G}\!/_{S}} RgS\Big) \otimes_S M \qquad \text{(by Proposition 2.1.14)} \\ &\cong \bigoplus_{gS \in {}^{G}\!/_{S}} (RgS \otimes_S M) \\ &\cong \bigoplus_{gS \in {}^{G}\!/_{S}} (g \otimes M), \end{aligned}$$

where the left S-action on the summands corresponds to the S-action on the left cosets. □

The induced module has the following important universal mapping property.

Proposition 2.1.16 (Universal mapping Property)**.** *[Bro82, section III.3 (3.2)] Let N be an RG-module, M an RS-module, and let $i : M \to RG \otimes_S M$ be the natural inclusion that sends M to $1 \otimes M$. If $f : M \to N$ is an RS-homomorphism then there exists a unique RG-homomorphism $\hat{f} : RG \otimes_S M \to N$ satisfying $\hat{f}(g \otimes m) = g \cdot f(m)$ for all $g \in G$ and $m \in M$. That is, the following diagram commutes.*



In other words, for $S \leq G$, if M is a left RS-module then for any left RG-module N the restriction

$$\mathrm{Hom}_G(RG \otimes_S M, N) \xrightarrow{res} \mathrm{Hom}_S(M, N)$$

is an isomorphism. This restriction isomorphism is natural in the strongest possible sense.

Moving to induced modules preserves both exactness and projectivity. Given a projective RS-resolution $\mathbf{P}_M \to M$ of M, we can produce a projective RG-resolution of $RG \otimes_S M$ by applying the functor $RG \otimes_S -$, which preserves exactness since RG is a free right RS-module. We then have an RG-resolution of $RG \otimes_S M$,

$$\cdots \longrightarrow RG \otimes_S P_n \xrightarrow{1 \otimes d_n} \cdots \xrightarrow{1 \otimes d_1} RG \otimes_S P_0 \xrightarrow{1 \otimes d_0} RG \otimes_S M \longrightarrow 0.$$

Moreover, if P is RS-projective then $RG \otimes_S P$ is RG-projective since the tensor product distributes over sums and $RG \otimes_S RS \cong RG$.

The induced module thus allows us to change coefficient rings within Ext via Shapiro's Lemma for group rings.

Theorem 2.1.17 (Shapiro's Lemma)**.** *[Ben91, p. 60] Let A be an RS-module and M an RG-module. Then for all $k \geq 0$*

$$\mathrm{Ext}^k_G(RG \otimes_S A, M) \cong \mathrm{Ext}^k_S(A, M).$$

Proof. Given a projective RS-resolution $\mathbf{P}_A \to A$ of A,

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0,$$

we have that $RG \otimes_S \mathbf{P}_A \to RG \otimes_S A$ is a projective RG-resolution of $RG \otimes_S A$. Thus for each RG-module M the isomorphism of Proposition 2.1.16,

$$\mathrm{Hom}_G(RG \otimes_S M, N) \xrightarrow{res} \mathrm{Hom}_S(M, N),$$

and naturality lead to an isomorphism of co-chain complexes

$$\mathrm{Hom}_G(RG \otimes_S P_A, N) \xrightarrow{res} \mathrm{Hom}_S(P_A, N).$$

Therefore

$$\begin{aligned}
\operatorname{Ext}_G^k(RG \otimes_S A, M) &= H^k(\operatorname{Hom}_G(RG \otimes_S \mathbf{P}_A, M)) \\
&= H^k(\operatorname{Hom}_S(\mathbf{P}_A, M)) \\
&= \operatorname{Ext}_S^k(A, M).
\end{aligned}$$

□

These properties of the Ext groups allow us to obtain consequences on the cohomology of G for certain hypotheses involving kernels of projective resolutions. That is, a decomposition of a kernel of a $\mathbb{Z}G$-resolution in terms of the induced module of a kernel of a $\mathbb{Z}S$-resolution provides a decomposition of the cohomology of G in terms of the cohomology of S.

Theorem 2.1.18 (Dimension Shifting)**.** *Let $K_n(G)$ be the kernel of the nth boundary map of a $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Similarly let $K_m(S)$ be the kernel of the mth boundary map of a $\mathbb{Z}S$-resolution of $\mathbb{Z}$. If N is a $\mathbb{Z}G$-module and $K_n(G) \cong \mathbb{Z}G \otimes_S K_m(S) \oplus N$ as $\mathbb{Z}G$-modules, then*

$$H^{k+(n+1)}(G, M) \cong H^{k+(m+1)}(S, M) \times \operatorname{Ext}_G^k(N, M)$$

for all $k \geq 1$ and for all $\mathbb{Z}G$-modules M. Moreover, if N is projective, then

$$H^{k+(n+1)}(G, M) \cong H^{k+(m+1)}(S, M)$$

for all $k \geq 1$ and for all $\mathbb{Z}G$-modules M.

Proof. Let $K_n(G)$ be the kernel of the nth boundary map of a projective $\mathbb{Z}G$-resolution $\mathbf{P}$ of $\mathbb{Z}$. That is,

$$\mathbf{P}: \quad \cdots \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \longrightarrow \quad \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$P_{n+1} \xrightarrow{\overline{d}_{n+1}} K_n(G) \longrightarrow P_n, \qquad K_n(G) \longrightarrow 0,$$

and thus

$$\cdots \xrightarrow{d_{n+3}} P_{n+2} \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{\overline{d}_{n+1}} K_n(G) \longrightarrow 0$$

is a projective $\mathbb{Z}G$-resolution of $K_n(G)$, say $\mathbf{Q}_{K_n(G)}$. We can then compute Ext using this resolution, giving us a dimension shift from the cohomology of the group G. For all $k \geq 1$,

$$\begin{aligned}\mathrm{Ext}_G^k(K_n(G), M) &= H^k(\mathrm{Hom}_G(\mathbf{Q}_{K_n(G)}, M)) \\ &= H^{k+(n+1)}(\mathrm{Hom}_G(\mathbf{P}, M)) = H^{k+(n+1)}(G, M).\end{aligned}$$

Similarly, for all $k \geq 1$,

$$\mathrm{Ext}_S^k(K_m(S), M) = H^{k+(m+1)}(S, M).$$

Now if $K_n(G) \cong \mathbb{Z}G \otimes_S K_m(S) \oplus N$ as $\mathbb{Z}G$-modules, then for every $\mathbb{Z}G$-module M and for all $k \geq 1$,

$$\begin{aligned}H^{k+(n+1)}(G, M) &= \mathrm{Ext}_G^k(K_n(G), M) \\ &= \mathrm{Ext}_G^k(\mathbb{Z}G \otimes_S K_m(S) \oplus N, M) \\ &= \mathrm{Ext}_G^k(\mathbb{Z}G \otimes_S K_m(S), M) \times \mathrm{Ext}_G^k(N, M) \text{ since Ext is multiplicative} \\ &= \mathrm{Ext}_S^k(K_m(S), M) \times \mathrm{Ext}_G^k(N, M) \text{ by Shapiro's Lemma (2.1.17)} \\ &= H^{k+(m+1)}(S, M) \times \mathrm{Ext}_G^k(N, M).\end{aligned}$$

Moreover, if N is projective then $\mathrm{Ext}_G^k(N, M) = 0$ since $k \geq 1$. □

But then Theorems of Serre and Howie and Schneebeli apply, obtaining implications for the structure of how the subgroup S sits inside G as well.

Corollary 2.1.19. *Let $K_n(G)$ be the kernel of the nth boundary map of a $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Similarly let $K_m(S)$ be the kernel of the mth boundary map of a $\mathbb{Z}S$-resolution of $\mathbb{Z}$. Suppose also N is a $\mathbb{Z}G$-module and $K_n(G) \cong \mathbb{Z}G \otimes_S K_m(S) \oplus N$ as $\mathbb{Z}G$-modules.*

(i) If $S \cap gSg^{-1}$ contains a non-trivial element of finite order, then $g \in S$.

(ii) If N is a projective $\mathbb{Z}G$-module and $n = m$, then for each finite subgroup K of G, there exists a $g \in G$ such that $K \subseteq gSg^{-1}$ and $K \cap hSh^{-1} = \{1\}$ for all $h \notin gS$.

Proof. If $K_n(G) \cong K_m(S) \oplus N$ as $\mathbb{Z}G$-modules then Theorem 2.1.18 says

$$H^{k+(n+1)}(G, M) = H^{k+(m+1)}(S, M) \times \mathrm{Ext}^k_G(N, M)$$

for all $k \geq 1$. But then Howie and Schneebeli's theorem (Theorem 1.1.2) applies and gives us the result for part i.). If further N is $\mathbb{Z}G$-projective and $n = m$ then

$$H^{k+(n+1)}(G, M) = H^{k+(n+1)}(S, M)$$

for all $k \geq 1$. Thus Serre's Theorem (Theorem 1.1.1) applies and gives us the result for part (ii). □

2.2 Topology

2.2.1 Cellular Chain Complexes and Skeletal Homotopy Modules

In this section we assume a basic understanding of the fundamental groups and higher homotopy groups of a topological space. Recall that the cellular chain groups of a CW complex Y are given by the relative homology groups $C_n(Y; R) = H_n(Y^n, Y^{n-1}; R)$ with boundary operators arising from the long exact sequence on homology. Excision gives an external direct sum decomposition of $C_n(Y; R)$ resulting in a free R-module basis in one-to-one correspondence with the n-cells of Y, see [Sie93, p. 76].

Let G be the fundamental group of a connected CW complex Y. Consider the universal covering projection $p : \widetilde{Y} \to Y$, where $\widetilde{Y}$ has the CW-structure induced from that of Y and so n-cells of $\widetilde{Y}$ are mapped homeomorphically by p onto n-cells of Y, see [Sie92, Chapter 15.3]. Then G is isomorphic to the group of deck transformations, Aut(p), and acts on $\widetilde{Y}$ by path lifting, depending on both a basepoint and a preferred lift in the fibre. Deck transformations are cellular and so the induced action on the cellular chain groups makes $C_*(\widetilde{Y}; R)$ into an RG-chain complex. Moreover, the deck transformations freely permute the cells that lie over a given cell of Y and so preferred lifts, $\widetilde{c}^n \in \widetilde{Y}$ of an n-cell

$c^n \in Y$, provide a preferred basis for $C_n(\widetilde{Y};R)$ as a free RG-module. With this RG-module structure derived from the covering projection, $C_*(\widetilde{Y};R)$ is called the **equivariant chain complex** of the connected CW complex Y. For a complete discussion of these results see [Sie93, Section 3.2] or [Bro82, Section I.4].

Recall that **the nth skeletal homotopy module of Y** is the kernel of the nth boundary homomorphism for the equivariant chain complex,

$$C_*(\widetilde{Y};R): \qquad \cdots \longrightarrow C_{n+1}(\widetilde{Y};R) \xrightarrow{d_{n+1}} C_n(\widetilde{Y};R) \xrightarrow{d_n} \cdots \longrightarrow C_0(\widetilde{Y};R) \xrightarrow{\epsilon} R \longrightarrow 0.$$

Note that $h_n(Y) = ker(C_n(\widetilde{Y};R) \overset{d_n}{\to} C_{n-1}(\widetilde{Y};R)) = \widetilde{H}_n(\widetilde{Y}^{(n)};R)$ can also be thought of as the nth reduced cellular homology (with coefficients in R) of the n-skeleton of the universal cover.

Example 2.2.1. *If Y has a single zero-cell, then $C_0(\widetilde{Y};R) = RG$. Moreover $RG \xrightarrow{\epsilon} R$ sends each zero-cell in $\widetilde{Y}$ to the identity $1 \in R$, thus $h_0(Y)$ is the* ***augmentation ideal*** *$IG = ker(RG \xrightarrow{\epsilon} R)$ in the group ring RG , see [Rob82, p. 321].*

Example 2.2.2. *Suppose that Y is a two-complex modeled on a group presentation $\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$ for G (see proof of Lemma 2.2.10). Then $G \cong F/\mathcal{R}$ where F is the free group with basis $\mathbf{x}$ and $\mathcal{R}$ is the normal closure in F of the relator set $\mathbf{r}$ viewed in F. Then lifting theorems imply $\pi_1(\widetilde{Y}^{(1)})$ is the kernel of the map $F \to G$ corresponding to $\pi_1(\widetilde{Y}^{(1)}) \to \pi_1(Y)$ and a Hurewicz isomorphism theorem gives that $H_1(\widetilde{Y}^{(1)})$ is isomorphic to the abelianization of $\pi_1(\widetilde{Y}^{(1)})$. Thus $h_1(Y)$ is the* ***relation module*** *$\mathcal{R}_{ab} = \mathcal{R}/[\mathcal{R},\mathcal{R}]$ with RG-action descended from conjugation in F, see [Bro82, II.5.4]*

Example 2.2.3. *If $R = \mathbb{Z}$, $n \geq 2$ and $\widetilde{Y}$ is $(n-1)$-connected, then $h_n(Y) = \pi_n(Y^{(n)})$, see [Sie93, Lemma 3.3] (hence the name skeletal homotopy modules).*

2.2.2 CW Pairs and Induced Skeletal Modules

Let (Y, X) be a pair of connected CW-complexes and suppose that the homomorphism on fundamental groups induced by the topological inclusion,

$$inc_{\#} : \pi_1(X) \to \pi_1(Y),$$

is injective. If the above holds where $G = \pi_1(Y)$ and $S = im\,(\pi_1(X) \to \pi_1(Y))$, we say the CW-pair (Y, X) **realizes** the group pair (G, S). Note that we will generally suppress the implicit basepoint for the fundamental group since our spaces are both path connected and locally path connected.

Lemma 2.2.4. *Suppose the pair of connected CW complexes (Y, X) realizes the group pair (G, S). If $p : \widetilde{Y} \to Y$ is the universal covering projection, then*

(i) Each connected component of $p^{-1}(X)$ is simply connected, and

(ii) The restriction of p to any connected component of $p^{-1}(X)$ can be identified with universal covering projection for X.

Proof. Let $\widehat{X}$ be a connected component of $p^{-1}(X)$. We have a commutative diagram of induced homomorphisms on the fundamental groups

$$\begin{array}{ccc} \pi_1(\widehat{X}) & \longrightarrow & \pi_1(\widetilde{Y}) = 1 \\ \big\downarrow p_{\#} & & \big\downarrow p_{\#} \\ \pi_1(X) & \rightarrowtail & \pi_1(Y) \end{array}$$

where the bottom row is induced by inclusion and the top row is induced by the lift of the inclusion through the covering projection p for a preferred point in the fibre in $\widehat{X}$. Since $\widetilde{Y}$ is simply connected it has trivial fundamental group. Moreover, the homomorphisms induced by covering projections are injective, and by hypothesis so is the homomorphism induced by the inclusion of X into Y. Commutativity of the diagram implies that $\pi_1(\widehat{X})$ is trivial and therefore $\widehat{X}$ is simply connected. Since the universal cover is unique and the connected component $\widehat{X}$ is a covering space for X, it must be homeomorphic to $\widetilde{X}$. Thus the restriction of p to $\widehat{X}$ can be identified with universal covering projection for X via this homeomorphism. $\square$

Since each connected component of $p^{-1}(X)$ can be seen as a copy of $\widetilde{X}$, choosing a set of preferred lifts of n-cells of X and expanding to preferred lifts of n-cells on Y, gives us a way to see the cellular chain groups of $\widetilde{X}$ inside the cellular chain groups of $\widetilde{Y}$.

Proposition 2.2.5. *Suppose the connected CW-pair (Y, X) realizes the group pair (G, S). Then for each $n \geq 0$ the topological inclusion induces*

(i) An injective map of RS-modules, $C_n(\widetilde{X}; R) \to C_n(\widetilde{Y}; R)$, that sends generators corresponding to n-cells in X to generators corresponding to the inclusion of those n-cells into Y,

(ii) An injective map on skeletal homotopy modules, $h_n(X) \to h_n(Y)$, and

(iii) An injective map of RG-modules, $RG \otimes_S C_n(\widetilde{X}; R) \to C_n(\widetilde{Y}; R)$, with cokernel a free RG-module whose generators correspond to the n-cells of $Y - X$.

Proof. Fixing a basepoint in the fibre determines a preferred copy of $\widetilde{X}$ inside $p^{-1}(X)$, as in Lemma 2.2.4. Choose preferred lifts of n-cells in X and extend to a set of preferred lifts of n-cells inside Y. Then the inclusion induced map

$$\begin{array}{ccc} C_n(\widetilde{X}; R) & \longrightarrow & C_n(\widetilde{Y}; R) \\ \| & & \| \\ \bigoplus_{\text{n-cells of X}} RS & & \bigoplus_{\text{n-cells of Y}} RG \end{array}$$

includes each copy of RS generated by an n-cell of X into the copy of RG generated by the corresponding n-cell in Y, where these are both thought of as RS-modules. Clearly this is injective and a chain map on the equivariant chain complexes, $C_*(\widetilde{X}; R) \to C_*(\widetilde{Y}; R)$. Thus the above map sends kernels to kernels and restricts to an injective map on skeletal homotopy modules, $h_n(X) \to h_n(Y)$ for all n.

Now consider the induced chain complex $RG \otimes_S C_*(\widetilde{X}; R)$. By Proposition 2.1.16 the inclusion induces a chain map $RG \otimes_S C_*(\widetilde{X}; R) \to C_*(\widetilde{Y}; R)$. Note that

$$\begin{aligned} RG \otimes_S C_n(\widetilde{X}; R) &\cong RG \otimes_S \{ \bigoplus_{\text{n-cells of X}} RS \} \\ &\cong \bigoplus_{\text{n-cells of X}} RG \otimes_S RS \\ &\cong \bigoplus_{\text{n-cells of X}} RG. \end{aligned}$$

Thus the induced homomorphism includes each copy of RG generated by an n-cell of X into the copy of RG generated by the corresponding n-cell in Y

$$\begin{array}{ccc} RG \otimes_S C_n(\widetilde{X}; R) & \longrightarrow & C_n(\widetilde{Y}; R). \\ \| & & \| \\ \bigoplus_{\text{n-cells of X}} RG & & \bigoplus_{\text{n-cells of Y}} RG \end{array}$$

Clearly this map is injective with a natural split injection given by the projection and thus by Lemma 2.1.12 we can decompose $C_n(\widetilde{Y}; R)$ as a RG-module by those n-cells in X and those in $Y - X$,

$$C_n(\widetilde{Y}; R) \cong \bigoplus_{\text{n-cells of X}} RG \oplus \bigoplus_{\text{n-cells of Y-X}} RG. \quad \square$$

A better understanding of $p^{-1}(X)$ will give insight to the induced module $RG \otimes_S h_n(X)$. Thus we want to explore the action of G on $p^{-1}(X)$ by path-lifting after choice of base-point and fibre point (see Figure 2.1).

Proposition 2.2.6. *Suppose X is a connected sub-complex of a connected CW-complex Y where the inclusion induced homomorphism on fundamental groups is injective. Fix a basepoint $x_0 \in X$. Then $G = \pi_1(Y, x_0)$ acts on the set of path components of $p^{-1}(X)$ via path lifting and a preferred point in the fibre where*

(i) The action is transitive, and

(ii) The preferred path component of $p^{-1}(X)$ (determined by the preferred point in the fibre) has stabilizer $S = im\,(\pi_1(X, x_0) \to \pi_1(Y, x_0))$.

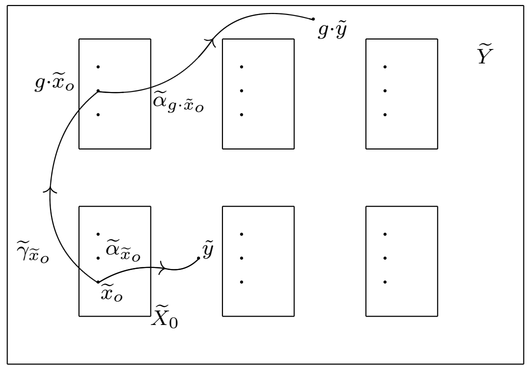


FIGURE 2.1: The action by path lifting

Proof. For (i) recall that deck transformations are homeomorphisms that respect the covering projection, thus the G-action permutes the set of connected components of $p^{-1}(X)$. Moreover since the action on the fibre is transitive, the action on the connected components of $p^{-1}(X)$ is also transitive.

For (ii), first let $\widetilde{x}_0$ be the preferred fibre point and $\widetilde{X}_0$ be the path component containing $\widetilde{x}_0$ and we will show $\pi_1(X, x_0) \subseteq Stab_G(\widetilde{X}_0)$. Consider $s = [\sigma] \in \pi_1(X, x_0)$. Then σ is a loop based at x_0 contained entirely in X and so it lifts to a path beginning at $\widetilde{x}_0$ that is contained entirely in the path connected component of $p^{-1}(X)$ that contains $\widetilde{x}_0$, namely $\widetilde{X}_0$. And so $s \cdot \widetilde{x}_0 = \widetilde{\sigma}_{\widetilde{x}_0}(1)$ is also in $\widetilde{X}_0$ thus $s \cdot \widetilde{X}_0 \cup \widetilde{X}_0 \neq \emptyset$. But then $s \cdot \widetilde{X}_0 = \widetilde{X}_0$ and so $s \in Stab_G(\widetilde{X}_0)$.

To show $Stab_G(\widetilde{X}_0) \subseteq \pi_1(X, x_0)$. Let $g = [\gamma] \in Stab_G(\widetilde{X}_0)$. In particular, $g \cdot \widetilde{x}_0 = \widetilde{\gamma}_{\widetilde{x}_0}(1) \in \widetilde{X}_0$ and so $\widetilde{\gamma}_{\widetilde{x}_0}$ is a path in $\widetilde{Y}$ with endpoints in $\widetilde{X}_0$. Since $\widetilde{X}_0$ is path-connected, there exists a path $\overline{\gamma}$ between these endpoints that lies entirely in $\widetilde{X}_0$. Moreover, since $\widetilde{Y}$ is simply connected, $\overline{\gamma}$ is path homotopic to $\widetilde{\gamma}_{\widetilde{x}_0}$. But then $p(\overline{\gamma})$ is path homotopic to γ and so $[\gamma] = [p(\overline{\gamma})]$. Since $p(\overline{\gamma})$ is a loop based at x_0 that lies entirely in X, $[p(\overline{\gamma})] \in \pi_1(X, x_0)$. Therefore $[\gamma] \in \pi_1(X, x_0)$ and thus $Stab_G(\widetilde{X}_0) = \pi_1(X, x_0)$. $\square$

Now the Orbit-Stabilizer Correspondence (Proposition 2.1.1) gives a bijection as G-sets between the orbit of our preferred connected component, $Orb_G(\widetilde{X}_0)$, and the left cosets ${}^{G}\!/_{Stab_G(\widetilde{X}_0)} = {}^{G}\!/_{S}$. Thus $Stab_G(g \cdot \widetilde{X}_0) = Stab_G(gS) = gSg^{-1}$ and

$$p^{-1}(X) = \dot{\bigcup_{gS \in {}^{G}\!/_{S}}} g \cdot \widetilde{X}_0$$

where the G-action permutes the connected components of $p^{-1}(X)$ according to the action of G on the cosets ${}^{G}\!/_{S}$. Note that by Proposition 2.1.5, this implies that S acts freely on the connected components of $p^{-1}(X)$ if and only if S is malnormal in G. Moreover, if S is not self-normalizing in G then S fixes a connected component of $p^{-1}(X)$. The following proposition in [Bro82] allows us to leverage our understanding of $p^{-1}(X)$ to recognize its homology group as an induced module, which we can then show is a submodule of $h_n(Y)$.

Proposition 2.2.7. *[Bro82, III.5.3] Suppose N is a G-module whose underlying abelian group is the direct sum $\bigoplus_{i \in \mathcal{I}} M_i$ and the G-action transitively permutes the summands and the index set $\mathcal{I}$ so that $gM_i = M_{gi}$. Fix $j \in \mathcal{I}$ and let S be the isotropy group of j. Then M_j is an S-module and $N \cong RG \otimes_{RS} M_j$.*

Theorem 2.2.8. *If the connected CW-pair (Y, X) realizes the group pair (G, S), then*

(i) The homology of $p^{-1}(X)$ is the induced module, $H_n(p^{-1}(X)^{(n)}; R) \cong RG \otimes_S h_n(X)$,

(ii) The induced map $\Phi_n : RG \otimes_S h_n(X) \to h_n(Y)$ is injective.

Proof. First we relate $RG \otimes_S h_n(X)$ to the homology module $H_n(p^{-1}(X)^{(n)}; R)$. Since $\pi_1 X \to \pi_1 Y$ is injective, we have

$$p^{-1}(X)^{(n)} = \dot{\bigcup_{gS \in {}^{G}\!/_{S}}} g \cdot \widetilde{X}_0^{(n)},$$

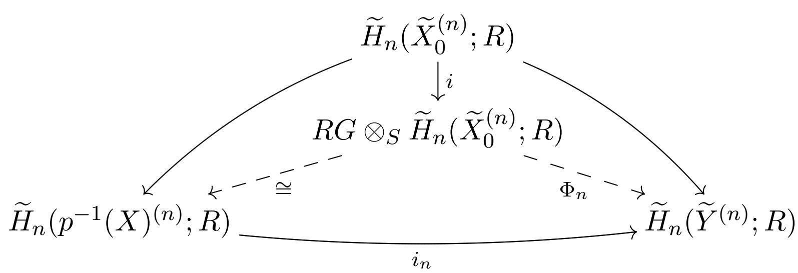
where each $g \cdot \widetilde{X}_0^{(n)}$ is a copy of $\widetilde{X}^{(n)}$. But then

$$H_n(p^{-1}(X)^{(n)}; R) = \bigoplus_{gS \in {}^{G}\!/_{S}} H_n(g \cdot \widetilde{X}_0^{(n)}; R)$$

is a direct sum of abelian groups with G-action induced by that on $p^{-1}(X)^{(n)}$, which transitively permutes the summands $H_n(g \cdot \widetilde{X}_0^{(n)}; R)$ according to the G-action on the left cosets ${}^{\mathrm{G}}\!/_{\mathrm{S}}$. Applying Proposition 2.2.7 to the above direct sum, where the index set $\mathcal{I} = {}^{\mathrm{G}}\!/_{\mathrm{S}}$ and the subgroup S is the isotropy group of the left coset $S \in {}^{\mathrm{G}}\!/_{\mathrm{S}}$, we have

$$H_n(p^{-1}(X)^{(n)}; R) \cong RG \otimes_S H_n(\widetilde{X}_0^{(n)}; R) \cong RG \otimes_S h_n(X).$$

By the Universal Mapping Property for Induced Modules (Proposition 2.1.16) applied to maps induced by the topological inclusions (see Proposition 2.2.5 (*ii*) and Lemma 2.2.4 (*ii*)) we have the following commutative diagram of RS and RG-modules:



Note that the long exact sequence of homology modules for the pair $(\widetilde{Y}^{(n)}, p^{-1}(X)^{(n)})$, has trivial homology for dimensions strictly greater than n since we are only considering the n-skeletons, thus the map induced by the inclusion $H_n(p^{-1}(X)^{(n)}; R) \xrightarrow{i_n} H_n(\widetilde{Y}^{(n)}; R)$ is injective. Therefore Φ_n is the composition of an isomorphism followed by the injective i_n, and thus $\Phi_n : RG \otimes_S h_n(X) \to h_n(Y)$ is also injective. $\square$

Therefore when a connected CW-pair (Y, X) realizes a group pair (G, S), we can identify $RG \otimes_S h_n(X)$ as a submodule of $h_n(Y)$ and we have the following short exact sequence of RG-modules

$$0 \longrightarrow RG \otimes_S h_n(X) \xrightarrow{\Phi_n} h_n(Y) \longrightarrow {}^{h_n(Y)}\!/_{RG \otimes_S h_n(X)} \longrightarrow 0.$$

We will often refer to the quotient module ${}^{h_n(Y)}\!/_{RG \otimes_S h_n(X)}$ as $\operatorname{coker}(\Phi_n)$.

2.2.3 Aspherical Spaces

If Y is an aspherical CW complex, its universal cover $\widetilde{Y}$ is contractible and so the equivariant chain complex is a free RG-resolution of R, see [Sie93, Theorem 4.5, p. 90]. Thus the module $h_n(Y)$ is a kernel of this free resolution. Moreover the skeletal homotopy modules are both kernels and images of the boundary homomorphisms of this free resolution, giving rise to an **nth short exact sequence of skeletal homotopy modules** for every positive integer n:

$$0 \longrightarrow h_n(Y) \longrightarrow C_n(\widetilde{Y}) \xrightarrow{\partial_n} h_{n-1}(Y) \longrightarrow 0.$$

If $h_n(Y)$ is projective over $\mathbb{Z}G$ there are immediate consequences for the cohomological dimension of G.

Proposition 2.2.9. *Suppose $R = \mathbb{Z}$ and Y is a connected, aspherical CW complex. If $h_n(Y)$ is a projective $\mathbb{Z}G$-module then $cd(G) \leq n + 1$.*

Proof. If R= $\mathbb{Z}$, Y is a connected, aspherical CW complex, and $h_n(Y)$ is a projective $\mathbb{Z}G$-module then

$$0 \longrightarrow h_n(Y) \longrightarrow C_n(\widetilde{Y}) \longrightarrow \cdots \longrightarrow C_0(\widetilde{Y}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of length $n + 1$. □

For applications to group theory, it is important to see that given any group pair (G, S) we can construct an aspherical CW-pair (Y, X) that realizes the group pair due to the existence of Eilenberg Maclane spaces for any group. Then $h_n(Y)$ is a kernel of a free resolution that contains the induced kernel $RG \otimes_S h_n(X)$ as a submodule.

Lemma 2.2.10. *For any group pair (G, S), there exists a pair of connected, aspherical CW complexes (Y, X) that realizes that group pair.*

Proof. Find a presentation for S and extend this to a presentation for G by adding the necessary generators and relators. Build an Eilenberg Maclane space, $X = K(S, 1)$, by first building the 2-complex associated to the presentation for S and then attaching cells of dimension $n \geq 3$ to kill off the generators of the higher homotopy groups (as in Hatcher Example 4.17 p. 354).

Next attach 1-cells to this X, one for each additional generator in G, and 2-cells by the loops specified by each additional relator in G. Again, attach cells of dimension $n \geq 3$ to kill off the generators of the higher homotopy groups and call this new space Y. Since $Y^{(2)}$ is exactly the 2-dimensional complex associated to the presentation for G, $\pi_1(Y) \cong \pi_1(Y^{(2)}) = G$, thus Y is a $K(G, 1)$. □

It is helpful to be able to understand skeletal homotopy splittings for a space with more than one zero-cell in terms of a space with only a single zero-cell. Fortunately, contracting a maximal tree in the one-skeleton doesn't change the existence of skeletal homotopy splittings.

Lemma 2.2.11. *Let (Y, X) be a pair of connected, aspherical CW complexes that realizes the group pair (G, S). Let T_X be a maximal tree in X that extends to a maximal tree, T, in Y and consider the pair of quotient spaces $({}^{Y}\!/_{T}, {}^{X}\!/_{T_X})$. Then $({}^{Y}\!/_{T}, {}^{X}\!/_{T_X})$ is a pair of connected, aspherical CW complexes that also realizes the group pair (G, S). Moreover,*

(i) $\mathrm{Sum}_n^{RS}(Y, X)$ *holds if and only if* $\mathrm{Sum}_n^{RS}({}^{Y}\!/_{T}, {}^{X}\!/_{T_X})$ *holds,*

(ii) $\mathrm{Sum}_n^{RG}(Y, X)$ *holds if and only if* $\mathrm{Sum}_n^{RG}({}^{Y}\!/_{T}, {}^{X}\!/_{T_X})$ *holds, and*

(iii) $\mathrm{PSum}_n^{RG}(Y, X)$ *holds if and only if* $\mathrm{PSum}_n^{RG}({}^{Y}\!/_{T}, {}^{X}\!/_{T_X})$ *holds.*

Proof. Since the quotient map contracts a maximal tree, $q : Y \to {}^{\mathrm{Y}}\!/_{\mathrm{T}}$ is a homotopy equivalence that restricts to a homotopy equivalence, $q| : X \to {}^{\mathrm{X}}\!/_{\mathrm{T}_X}$, on X. Thus $({}^{\mathrm{Y}}\!/_{\mathrm{T}}, {}^{\mathrm{X}}\!/_{\mathrm{T}_X})$ is still a pair of connected, aspherical CW complexes with fundamental group pair (G, S) where the inclusion induced homomorphism is injective.

Moreover, the map q takes open $(n+1)$-cells to $(n+1)$-cells for all $n \geq 1$, thus the induced chain maps on equivariant chain complexes, $C_*(\widetilde{Y}; R) \xrightarrow{q_*} C_*(\widetilde{{}^{\mathrm{Y}}\!/_{\mathrm{T}}}; R)$ and

$C_*(\widetilde{X}; R) \overset{q|_*}{\to} C_*(\widetilde{{}^{X}\!/_{T_X}}; R)$, naturally identifies the $n+1$ cellular chain groups and their preferred generators as RG and RS-modules for all $n \geq 1$. But then the induced map also naturally identifies the cokernels of the boundary homomorphisms which, since the spaces are aspherical, are the skeletal homotopy modules, $h_n(Y) \overset{q_*}{\equiv} h_n({}^{Y}\!/_{T})$ and $h_n(X) \overset{q|_*}{\equiv} h_n({}^{X}\!/_{T_X})$ for all $n \geq 1$.

Now the lift of q on the universal covers, $\widetilde{q} : \widetilde{Y} \to \widetilde{{}^{Y}\!/_{T}}$, carries $p^{-1}(X)$ to $p^{-1}({}^{X}\!/_{T_X})$. We then have the following commutative diagram on homology groups arising from the long exact sequences for the pairs $(\widetilde{Y}^{(n)}, p^{-1}(X)^{(n)})$ and $(\widetilde{{}^{Y}\!/_{T}}^{(n)}, p^{-1}({}^{X}\!/_{T_X})^{(n)})$

$$\begin{array}{ccc} H_n(p^{-1}(X)^{(n)}) & \xrightarrow{i_n} & H_n(\widetilde{Y}^{(n)}) \\ \downarrow{\scriptstyle \tilde{q}|_*} & & \downarrow{\scriptstyle \tilde{q}_*} \\ H_n\left(p^{-1}({}^{X}\!/_{T_X})^{(n)}\right) & \xrightarrow{i'_n} & H_n\left(\widetilde{{}^{Y}\!/_{T}}^{(n)}\right) \end{array}$$

By Lemma 2.2.8, this diagram corresponds to the following commutative diagram

$$\begin{array}{ccc} RG \otimes_S h_n(X) & \xrightarrow{\Phi_n} & h_n(Y) \\ \downarrow{\scriptstyle \tilde{q}|_*} & & \downarrow{\scriptstyle \tilde{q}_*} \\ RG \otimes_S h_n({}^{X}\!/_{T_X}) & \xrightarrow{\Phi'_n} & h_n({}^{Y}\!/_{T}) \end{array}$$

Since the induced maps of q are natural identifications of the skeletal homotopy modules for all $n \geq 1$, they are natural identifications of the induced skeletal homotopy modules as well. Thus we have Φ_n is split injective if and only if Φ'_n is split injective as either RS or RG-module homomorphisms for all $n \geq 1$. Moreover coker(Φ_n) is projective if and only if the coker(Φ'_n) is projective as either RS or RG-modules for all $n \geq 1$. Thus we have the result for all $n \geq 1$.

It remains to show the theorem is true when $n = 0$. Let $x_0, x_i \in X^{(0)}$ and $y_j \in Y^{(0)} - X^{(0)}$ be the zero cells of X and $Y - X$ respectively with $i \in \mathcal{I}$ and $j \in \mathcal{J}$ for some index sets $\mathcal{I}$ and $\mathcal{J}$. Then $C_0(\widetilde{Y}; R)$ is a free RG-module with generators corresponding to the 0-cells $x_0, x_i, y_i \in Y^{(0)}$ and $C_0(\widetilde{X}; R)$ is a free RS-module with generators corresponding to the 0-cells $x_0, x_i \in X^{(0)}$.

Consider $h_0(X) = ker\left(C_0(\widetilde{X};R) \xrightarrow{\varepsilon} R\right)$. Each $\alpha \in h_0(X)$ can be written as a sum, $\alpha = \xi x_0 + \sum_{i\in\mathcal{I}} \xi_i x_i$, where $\xi, \xi_i \in RG$ and $\varepsilon(\alpha) = 0$. But then $\xi + \sum_{i\in\mathcal{I}} \xi_i \in IG$ and so

$$\alpha = \left(\xi + \sum_{i\in\mathcal{I}} \xi_i\right) x_0 + \sum_{i\in\mathcal{I}} \xi_i \left(x_i - x_0\right) \in IG[x_0] + \bigoplus_{i\in\mathcal{I}} RG[x_i - x_0].$$

Moreover, this sum is direct since the 0-cells are distinct and so the summands intersect trivially. Thus as RS-modules

$$h_0(X) \cong IS[x_0] \oplus \bigoplus_{i\in\mathcal{I}} RS[x_i - x_0].$$

A similar argument shows that the 0th skeletal homotopy module for Y also decomposes as a direct sum of RG-modules

$$h_0(Y) \cong IG[x_0] \oplus \bigoplus_{i\in\mathcal{I}} RG[x_i - x_0] \oplus \bigoplus_{j\in\mathcal{J}} RG[y_j - x_0].$$

Also, since $({}^{\mathrm{Y}}\!/_{\mathrm{T}}, {}^{\mathrm{X}}\!/_{\mathrm{T}_X})$ has only one 0-cell, say v, the skeletal homotopy modules are the augmentation ideals. Thus when $n = 0$ the commutative diagram with rows from the inclusion in the long exact homology sequence for each pair becomes

$$\begin{array}{ccc} RG \otimes_S IS[x_0] \oplus \bigoplus_{i\in\mathcal{I}} RG[x_i - x_0] & \xrightarrow{\Phi_n} & IG[x_0] \oplus \bigoplus_{i\in\mathcal{I}} RG[x_i - x_0] \oplus \bigoplus_{j\in\mathcal{J}} RG[y_j - x_0] \\ \downarrow \tilde{q}|_* & & \downarrow \tilde{q}_* \\ RG \otimes_S IS[v] & \xrightarrow{\Phi'_n} & IG[v] \end{array}$$

Since the quotient map q takes all 0-cells in X and Y to the single 0-cell v in ${}^{\mathrm{Y}}\!/_{\mathrm{T}}$, the maps induced by the quotient on the columns are simply projections onto the augmentation ideal and induced augmentation ideal summands. But then these split by the inclusion and by commutativity of the diagram the map Φ_0 is split injective if and only if the map Φ'_0 is split injective either as RS or RG-module homomorphisms. Moreover,

$$\mathrm{coker}(\Phi_0) = {}^{IG[x_0]}\!/_{RG \otimes_S IS[x_0]} \oplus \bigoplus_{j\in\mathcal{J}} RG[y_j - x_0] \text{ and}$$

$$\mathrm{coker}(\Phi'_0) = {}^{IG[v]}\!/_{RG \otimes_S IS[v]}.$$

Since a direct sum is projective if and only if each summand is projective, coker(Φ_0) is projective if and only if coker(Φ'_0) is projective either as RS or RG-modules. Thus the result is proved for all $n \geq 0$. $\square$

3 NEW RESULTS AMONGST THE CONDITIONS

Let (Y, X) be an aspherical CW pair that realizes the group pair (G, S). We then have a short exact sequence of RG-modules from Theorem 2.2.8

$$0 \longrightarrow RG \otimes_S h_n(X) \xrightarrow{\Phi_n} h_n(Y) \longrightarrow \text{coker}(\Phi_n) \longrightarrow 0.$$

We say the CW pair (Y, X) has a weak skeletal homotopy splitting at dimension n if the above sequence is split exact as RS-modules. If further the above short exact sequence is split exact as RG-modules we say the CW pair has a skeletal homotopy splitting at dimension n. Last, we say the CW pair has a projective skeletal homotopy splitting at dimension n if the above sequence is split exact as RG-modules with RG-projective cokernel (possibly trivial). We denote these progressively stronger hypotheses by $\text{Sum}_n^{RS}(Y, X)$, $\text{Sum}_n^{RG}(Y, X)$, and $\text{PSum}_n^{RG}(Y, X)$ respectively. Note that if Φ_n is an isomorphism then all three of these conditions hold since $\text{coker}(\Phi_n)$ is trivial.

In this chapter I consider the three conditions $\text{Sum}_n^{RS}(Y, X)$, $\text{Sum}_n^{RG}(Y, X)$, and $\text{PSum}_n^{RG}(Y, X)$. Section 1 is dedicated to when $\text{coker}(\Phi_n)$ is projective and Section 2 discusses lifting splittings up dimensions and pushing them down one dimension.

3.1 Projective Cokernels

One of the advantages of this topological approach are the extremely useful commutative diagrams arising from the topological inclusion (see Proposition 2.2.5 and Theorem 2.2.8). To begin, we will use such diagrams to recognize that a weak skeletal homotopy splitting at dimension n is equivalent to $\text{coker}(\Phi_n)$ being RS-projective, and so we do not need to consider a 'projective' weak skeletal homotopy splitting as a separate condition. Characterizing $\text{Sum}_n^{RS}(Y, X)$ by the RS-projectivity of $\text{coker}(\Phi_n)$ is also extremely useful for determining when splittings hold for $n = 0$ (see Theorem 5.1.7).

First we need the following lemma that the quotient module $h_n(Y)/h_n(X)$ is always RS-projective when the spaces are aspherical.

Lemma 3.1.1. *Let (Y, X) be a connected, aspherical pair of CW complexes that realizes the group pair (G, S). Then ${}^{h_n(Y)}\!/\!{}_{h_n(X)}$ is RS-projective for all n.*

Proof. We will use induction on n. For the base case we prove that ${}^{h_0(Y)}\!/\!{}_{h_0(X)}$ is actually a free RS-module and hence RS-projective. By Proposition 2.2.5 (i) and (ii) the topological inclusion of X into Y induces an injection of the 0th short exact sequence of skeletal homotopy modules for X into the 0th short exact sequence of skeletal homotopy modules for Y, giving rise to the following commutative diagram of RS-modules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_0(X) & \longrightarrow & \bigoplus_{\text{0-cells of X}} RS & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & h_0(Y) & \longrightarrow & \bigoplus_{\text{0-cells of Y}} RG & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}^{h_0(Y)}\!/\!{}_{h_0(X)} & \xrightarrow{\cong} & \bigoplus_{\text{0-cells of } Y-X} RG \oplus \bigoplus_{\text{0-cells of } X} {}^{\mathrm{RG}}\!/\!{}_{\mathrm{RS}} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the bottom row is exact since the top two rows are exact. Thus

$$ {}^{h_0(Y)}\!/\!{}_{h_0(X)} \cong \bigoplus_{\text{0-cells of } Y-X} RG \oplus \bigoplus_{\text{0-cells of } X} {}^{\mathrm{RG}}\!/\!{}_{\mathrm{RS}}. $$

By Proposition 2.1.14, we have RG is a free left RS-module where $RG \cong \bigoplus_{Sg \in {}_S\backslash^G} RSg$. Moreover the short exact sequence of left RS-modules

$$ 0 \longrightarrow RS \xrightarrow{i} RG \longrightarrow {}^{\mathrm{RG}}\!/\!{}_{\mathrm{RS}} \longrightarrow 0 $$

coincides with

$$0 \longrightarrow RS \xrightarrow{i} \bigoplus_{Sg \in {}_S\backslash^G} RSg \longrightarrow \bigoplus_{Sg \in {}_S\backslash^G - \{S\}} RSg \longrightarrow 0$$

since the inclusion maps RS isomorphically onto the summand of RG associated with S in ${}_S\backslash^G$. Thus the quotient ${}^{RG}/_{RS}$ is a direct sum of copies of RS and thus is also a free left RS-module. But then ${}^{h_0(Y)}/_{h_0(X)}$ is a sum of free RS-modules and hence a free RS-module as well.

Next assume that ${}^{h_n(Y)}/_{h_n(X)}$ is RS-projective. Again, the topological inclusion induces an injection on the nth skeletal homotopy sequence for X into the nth skeletal homotopy sequence for Y by Proposition 2.2.5 *i.*) and *ii.*), giving rise to the following commutative diagram of RS-modules with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_{n+1}(X) & \longrightarrow & \bigoplus_{\text{n+1-cells of X}} RS & \longrightarrow & h_n(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_{n+1}(Y) & \longrightarrow & \bigoplus_{\text{n+1-cells of Y}} RG & \longrightarrow & h_n(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}^{h_{n+1}(Y)}/_{h_{n+1}(X)} & \longrightarrow & \bigoplus_{\text{n+1-cells of } Y-X} RG \oplus \bigoplus_{\text{n+1-cells of } X} {}^{RG}/_{RS} & \longrightarrow & {}^{h_n(Y)}/_{h_n(X)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since ${}^{h_n(Y)}/_{h_n(X)}$ is RS-projective the bottom row is split exact and by Lemma 2.1.12

$$\bigoplus_{\text{n+1-cells of } Y-X} RG \oplus \bigoplus_{\text{n+1-cells of } X} {}^{RG}/_{RS} \cong {}^{h_{n+1}(Y)}/_{h_{n+1}(X)} \oplus {}^{h_n(Y)}/_{h_n(X)}.$$

Thus ${}^{h_{n+1}(Y)}/_{h_{n+1}(X)}$ is a summand of a free RS-module and hence RS-projective. $\square$

Notice that RS-projectivity of coker(Φ_n) is a sufficient condition for $\text{Sum}_n^{RS}(Y,X)$, since surjective maps to a projective module always split. Thus it remains to show RS-projectivity is also a necessary condition for $\text{Sum}_n^{RS}(Y,X)$. The module ${}^{h_n(Y)}/_{h_n(X)}$ being

RS-projective for all n allows us to relate weak skeletal homotopy splittings to $\operatorname{coker}(\Phi_n)$ being RS-projective.

Theorem 3.1.2. *Let (Y, X) be a pair of connected, aspherical CW complexes that realizes the group pair (G, S). For each fixed, non-negative integer n, $\mathrm{Sum}_n^{RS}(Y, X)$ holds if and only if coker(Φ_n) is RS-projective.*

Proof. Suppose $\mathrm{Sum}_n^{RS}(Y, X)$ holds. By Proposition 2.2.5, the topological inclusion induces an injection of $h_n(X)$ into $h_n(Y)$ that then agrees with the injection of the induced skeletal homotopy module from Theorem 2.2.8. Also, by Lemma 2.1.13, $h_n(X)$ naturally sits inside $RG \otimes_S h_n(X)$ as an RS-module summand and so we have the following commutative diagram of exact sequences of RS-modules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_n(X) & \longrightarrow & RG \otimes_S h_n(X) & \longrightarrow & RG \otimes_S h_n(X)/h_n(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \Phi_n} & & \downarrow{\scriptstyle \Phi_n'} & & \\
0 & \longrightarrow & h_n(X) & \longrightarrow & h_n(Y) & \xrightarrow{\pi_n} & h_n(Y)/h_n(X) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \gamma}\downarrow \uparrow{\scriptstyle \overline{\gamma}} & & \downarrow{\scriptstyle \gamma'} & & \\
 & & 0 & \longrightarrow & \operatorname{coker}(\Phi_n) & \underset{h}{\xrightarrow{\cong}} & \operatorname{coker}(\Phi_n') & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

where the middle column is split exact by hypothesis.

Since the bottom row is exact, the map h is an isomorphism and we can define $\overline{\gamma'} = \pi_n \circ \overline{\gamma} \circ h^{-1}$. But then

$$\gamma' \circ \overline{\gamma'} = \gamma' \circ \pi_n \circ \overline{\gamma} \circ h^{-1} = h \circ \gamma \circ \overline{\gamma} \circ h^{-1} = 1.$$

Thus the right column is also split exact and therefore $\operatorname{coker}(\Phi_n')$ is an RS-summand of $h_n(Y)/h_n(X)$. Since $h_n(Y)/h_n(X)$ is RS-projective by Lemma 3.1.1, the RS-summand $\operatorname{coker}(\Phi_n) \cong \operatorname{coker}(\Phi_n')$ is also RS-projective. □

Thus weak skeletal homotopy splittings are equivalent to 'projective' weak skeletal homotopy splittings. Moreover, projectivity of coker(Φ_n) lifts to higher dimensions, or 'goes uphill', as seen in the following lemma.

Lemma 3.1.3. *[Projectivity goes uphill] Let (Y, X) be a connected, aspherical pair of CW complexes that realizes the group pair (G, S) and let n be a fixed non-negative integer.*

(i) If coker(Φ_n) is RS-projective then coker(Φ_k) is also RS-projective for all $k \geq n$.

(ii) If coker(Φ_n) is RG-projective then coker(Φ_k) is also RG-projective for all $k \geq n$.

Proof. Suppose coker(Φ_n) is RG-projective. Since RG is RS-free, $RG \otimes_S -$ is an exact functor and applying it to the $(n+1)$-th short exact sequence of skeletal homotopy modules for X we obtain another short exact sequence

$$0 \longrightarrow RG \otimes_S h_{n+1}(X) \longrightarrow RG \otimes_S C_n(\widetilde{X}; R) \longrightarrow RG \otimes_S h_n(X) \longrightarrow 0.$$

But then the topological inclusion induces an injection of the induced sequence into the $(n+1)$-th short exact sequence of skeletal homotopy modules for Y and by Theorem 2.2.8 and Proposition 2.2.5 we have the following commutative diagram of RG-modules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG \otimes_S h_{n+1}(X) & \longrightarrow & \bigoplus_{\text{n+1-cells of X}} RG & \longrightarrow & RG \otimes_S h_n(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_{n+1}} & & \downarrow & & \downarrow{\scriptstyle \Phi_n} & & \\
0 & \longrightarrow & h_{n+1}(Y) & \longrightarrow & \bigoplus_{\text{n+1-cells of Y}} RG & \longrightarrow & h_n(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{coker}(\Phi_{n+1}) & \longrightarrow & \bigoplus_{\text{n+1-cells of } Y-X} RG & \longrightarrow & \text{coker}(\Phi_\text{n}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Since coker(Φ_n) is RG-projective, the bottom row is a split exact sequence of RG-modules and thus both coker(Φ_n) and coker(Φ_{n+1}) are RG-summands of the free RG-module $\bigoplus_{Y-X} RG$. Therefore coker(Φ_{n+1}) is also RG-projective.

Moreover, if we replace G with S in the preceding paragraph, we obtain the same conclusion for RS-projectivity since RG is also a free RS-module. □

3.2 Going Uphill and Stepping Down

Since projectivity of the cokernel goes uphill, that is, lifts to higher dimension by Lemma 3.1.3, it follows that both projective skeletal homotopy splittings and weak skeletal homotopy splittings go uphill as well. When coker(Φ_n) is not projective over either the group ring RG or RS, we can still lift splittings. In general, we can lift splittings in a commutative diagram of short exact sequences whenever a particular module in the diagram is projective as seen in the following lemma.

Lemma 3.2.1. *Lifting Splittings*

Assume that the following commutative diagram of Λ-modules and Λ-homomorphisms has exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma_1} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_2} & & \\
0 & \longrightarrow & A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Suppose further that the right column is a split exact sequence and B_3 is Λ-projective.

(i) Then β_1 is split injective via a Λ-homomorphism that lifts through g_1, g_2. (I.e. if there exists $\overline{\gamma}_1 \in Hom_\Lambda(C_2, C_1)$ such that $\overline{\gamma}_1 \circ \gamma_1 = 1_{C_1}$ then there exists $\overline{\beta_1} \in Hom_\Lambda(B_2, B_1)$ such that $\overline{\beta}_1 \circ \beta_1 = 1_{B_1}$ with $\overline{\gamma}_1 \circ g_2 = g_1 \circ \overline{\beta}_1$.)

(ii) Moreover, α_1 is also split injective as a Λ-homomorphism.

Proof. Suppose B_3 is Λ-projective in the above commutative diagram and suppose γ_1 has a splitting $\overline{\gamma}_1 \in Hom_\Lambda(C_2, C_1)$. Since B_3 is Λ-projective, the middle column is a split exact sequence and so there is a splitting of β_1, $\sigma \in Hom_\Lambda(B_2, B_1)$, such that $\sigma \circ \beta_1 = 1_{B_2}$, see Lemma 2.1.12. In fact, there are several possible splittings, but we want to find one that induces $\overline{\gamma}_1$ to prove (i). More than likely $\overline{\gamma}_1 \circ g_2 \neq g_1 \circ \sigma$, but we can use σ to build our desired splitting that does.

Applying the contravariant functor $Hom_\Lambda(-, C_1)$ to the middle column of the above diagram gives us the following sequence:

$$Hom_\Lambda(B_1, C_1) \xleftarrow{\beta_1^*} Hom_\Lambda(B_2, C_1) \xleftarrow{\beta_2^*} Hom_\Lambda(B_3, C_1) \longleftarrow 0$$

which is exact at $Hom_\Lambda(B_2, C_1)$ since $Hom_\Lambda(-, C_1)$ is left exact.

While we might not have $\overline{\gamma}_1 \circ g_2 = g_1 \circ \sigma$, we do have

$$\beta_1^*(g_1 \circ \sigma) = (g_1 \circ \sigma) \circ \beta_1 = g_1 \circ (\sigma \circ \beta_1) = g_1 \circ 1_{B_1} = g_1, \text{ and}$$
$$\beta_1^*(\overline{\gamma}_1 \circ g_2) = \overline{\gamma}_1 \circ g_2 \circ \beta_1 = \overline{\gamma}_1 \circ \gamma_1 \circ g_1 = 1_{C_1} \circ g_1 = g_1,$$

and thus

$$\beta_1^*(\overline{\gamma}_1 \circ g_2 - g_1 \circ \sigma) = 0.$$

But then $\overline{\gamma}_1 \circ g_2 - g_1 \circ \sigma \in ker(\beta_1^*) = im(\beta_2^*)$ and so there exists $\tau \in Hom_\Lambda(B_3, C_1)$ such that $\beta_2^*(\tau) = \overline{\gamma}_1 \circ g_2 - g_1 \circ \sigma$.

Now g_1 is surjective and B_3 is projective, so there also exists $\tau' : B_3 \to B_1$ such that $\tau' \circ g_1 = \tau$,

$$\begin{array}{ccc} & & B_3 \\ & \overset{\tau'}{\swarrow} & \downarrow \tau \\ B_1 & \underset{g_1}{\twoheadrightarrow} & C_1. \end{array}$$

Define $\overline{\beta}_1 = \sigma + \tau' \circ \beta_2 \in Hom_\Lambda(B_2, B_1)$. Then

$$\overline{\beta}_1 \circ \beta_1 = \sigma \circ \beta_1 + \tau' \circ \beta_2 \circ \beta_1 = 1_{B_1} + 0 = 1_{B_1},$$

since σ is a splitting and since $\beta_2 \circ \beta_1 = 0$ by exactness of the original sequence. Thus $\overline{\beta}_1$ is a splitting of β_1 and it remains to check that $\overline{\gamma}_1 \circ g_2 = g_1 \circ \overline{\beta}_1$. For this,

$$\begin{aligned} g_1 \circ \overline{\beta}_1 &= g_1 \circ \sigma + g_1 \circ \tau' \circ \beta_2 \\ &= g_1 \circ \sigma + \tau \circ \beta_2 \\ &= g_1 \circ \sigma + \beta_2^*(\tau) \\ &= g_1 \circ \sigma + \overline{\gamma}_1 \circ g_2 - g_1 \circ \sigma \\ &= \overline{\gamma}_1 \circ g_2. \end{aligned}$$

Thus $\overline{\beta}_1$ is a split injection of β_1 that induces $\overline{\gamma}_1$, as desired.

To prove (ii) we start with the following commutative diagram from (i)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\ & & \alpha_1 \downarrow & & \beta_1 \downarrow \uparrow \overline{\beta}_1 & & \gamma_1 \downarrow \uparrow \overline{\gamma}_1 & & \\ 0 & \longrightarrow & A_2 & \xrightarrow[f_2]{} & B_2 & \xrightarrow[g_2]{} & C_2 & \longrightarrow & 0 \end{array}$$

where $g_1 \circ \overline{\beta}_1 = \overline{\gamma}_1 \circ g_2$, and both $\overline{\beta}_1 \circ \beta = 1_{B_1}$ and $\overline{\gamma}_1 \circ \gamma = 1_{C_1}$. Since the rows are exact

$$g_1 \circ \overline{\beta}_1 \circ f_2 = \overline{\gamma}_1 \circ g_2 \circ f_2 = 0,$$

and so $im(\overline{\beta}_1 \circ f_2) \subseteq ker(g_1) = im(f_1)$. But since f_1 is injective it is invertible on its image, thus there exists $\overline{\alpha}_1 : A_2 \to A_1$ such that $f_1 \circ \overline{\alpha}_1 = \overline{\beta}_1 \circ f_2$. But then,

$$\begin{aligned} f_1 \circ \overline{\alpha}_1 \circ \alpha_1 &= \overline{\beta}_1 \circ f_2 \circ \alpha_1 \\ &= \overline{\beta}_1 \circ \beta_1 \circ f_1 \\ &= f_1 \\ &= f_1 \circ 1_{A_1}. \end{aligned}$$

Since f_1 is injective, it is left cancellable and we have $\overline{\alpha}_1 \circ \alpha_1 = 1_{A_1}$. □

Applying the above lemma gives us a way to lift skeletal homotopy splittings.

Theorem 3.2.2 (Skeletal homotopy splittings go uphill)**.** *Let (Y, X) be a connected, aspherical pair of CW complexes that realizes the group pair (G, S) and let n be a fixed, non-negative integer.*

(i) If $\mathrm{Sum}_n^{RS}(Y, X)$ holds, then $\mathrm{Sum}_k^{RS}(Y, X)$ holds for all $k \geq n$.

(ii) If $\mathrm{Sum}_n^{RG}(Y, X)$ holds, then $\mathrm{Sum}_k^{RG}(Y, X)$ holds for all $k \geq n$.

(iii) If $\mathrm{PSum}_n^{RG}(Y, X)$ holds, then $\mathrm{PSum}_k^{RG}(Y, X)$ holds for all $k \geq n$.

Proof. First, weak skeletal homotopy splittings imply $\mathrm{coker}(\Phi_n)$ is RS-projective by Theorem 3.1.2. Since projectivity of the cokernel goes uphill by Lemma 3.1.3, both weak skeletal homotopy splittings and projective skeletal homotopy splittings go uphill as well. For skeletal homotopy splittings, the topological inclusion induces an inclusion of the induced (n+1)-th skeletal homotopy sequence for X to the (n+1)-th skeletal homotopy sequence for Y by Proposition 2.2.5 and Theorem 2.2.8, giving rise to the following commutative diagram of RG-modules with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG \otimes_S h_{n+1}(X) & \longrightarrow & \bigoplus_{\text{n+1-cells of X}} RG & \longrightarrow & RG \otimes_S h_n(X) & \longrightarrow & 0 \\
 & & \downarrow \Phi_{n+1} & & \downarrow & & \downarrow \Phi_n & & \\
0 & \longrightarrow & h_{n+1}(Y) & \longrightarrow & \bigoplus_{\text{n+1-cells of Y}} RG & \longrightarrow & h_n(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{coker}(\Phi_{n+1}) & \longrightarrow & \bigoplus_{\text{n+1-cells of } Y-X} RG & \longrightarrow & \text{coker}(\Phi_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since Φ_n is split injective and $\bigoplus_{\text{n+1-cells of Y-X}} RG$ is a free and hence projective RG-module, applying Lemma 3.2.1 gives that Φ_{n+1} is also split injective. But then the same commutative diagram for all higher dimensions implies $\text{Sum}_k^{RG}(Y, X)$ holds for all $k \geq n$. □

Thus the fact that the cokernel of $RG \otimes_S C_{n+1}(\widetilde{X}; R) \to C_{n+1}(\widetilde{Y}; R)$ is a free RG-module is enough to lift a skeletal homotopy splitting from dimension n to dimension $n+1$. Note that the proof of Lemma 3.2.1 provides a method to take an explicit splitting at one dimension and produce an explicit splitting for one dimension higher, which we will use in Chapter 4 (see Example 5.1.15).

On the other hand, if $\text{coker}(\Phi_n)$ is trivial, that is, Φ_n is an RG-isomorphism, we can also say something about dimension $n - 1$.

Theorem 3.2.3. *Stepping down*

Let (Y, X) be a connected, aspherical pair of CW complexes that realizes the group pair (G, S) and let n be a fixed, non-negative integer. If Φ_n is an isomorphism then $\text{PSum}_k^{RG}(Y, X)$ holds for all $k \geq n - 1$.

Proof. Suppose Φ_n is an isomorphism. We have the same commutative diagram of exact sequences of RG-modules from the proof of Theorem 3.2.2

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG \otimes_S h_n(X) & \longrightarrow & \bigoplus_{\text{n-cells of X}} RG & \longrightarrow & RG \otimes_S h_{n-1}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_n} & & \downarrow & & \downarrow{\scriptstyle \Phi_{n-1}} & & \\
0 & \longrightarrow & h_n(Y) & \longrightarrow & \bigoplus_{\text{n-cells of Y}} RG & \longrightarrow & h_{n-1}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \bigoplus_{\text{n-cells of Y-X}} RG & \xrightarrow{\cong} & \operatorname{coker}(\Phi_{n-1}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with $\operatorname{coker}(\Phi_n) = 0$. But then

$$\operatorname{coker}(\Phi_{n-1}) \cong \bigoplus_{\text{n-cells of Y-X}} RG$$

is a free (and hence projective) RG-module, or it is trivial. In either case, $\mathrm{PSum}_{n-1}^{RG}(Y, X)$ holds and by Theorem 3.2.2, $\mathrm{PSum}_k^{RG}(Y, X)$ holds for all $k \geq n - 1$. $\square$

4 APPLICATIONS TO GROUP THEORY

Recall for the group pair (G,S), we say respectively that $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$, $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$, or $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds if there exists a connected, aspherical CW-pair (Y,X) that realizes (G,S) such that $\mathrm{Sum}_n^{\mathbb{Z}G}(Y,X)$, $\mathrm{Sum}_n^{\mathbb{Z}G}(Y,X)$, or $\mathrm{PSum}_n^{\mathbb{Z}G}(Y,X)$ holds. In this Chapter we consider the group theoretic implications for these three conditions. We first apply Theorems of Serre and Howie and Schneebeli and then in Section 2 we extend results of Bogley and Dyer to other dimensions, showing there are still group theoretic implications for $\mathrm{Sum}_n^{\mathbb{Z}S}(G,S)$. Section 3 applies the results of $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ to cyclically presented groups, giving implications for the geometrization of 3-manifolds.

4.1 Strong Skeletal Homotopy Splittings

Both skeletal homotopy splittings and projective skeletal homotopy splittings have consequences for the cohomology of G since the skeletal homotopy modules are kernels of a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Theorem 4.1.1. *Cohomological Implications*

Let (G,S) be a group pair and let n be a fixed, non-negative integer.
If $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds, then

$$H^{k+(n+1)}(G,M) = H^{k+(n+1)}(S,M) \times Ext_G^k(coker(\Phi_n),M),$$

for all $k \geq 1$, for all $\mathbb{Z}G$-modules M.
Moreover, if $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds, then

$$H^{k+(n+1)}(G,M) = H^{k+(n+1)}(S,M),$$

for all $k \geq 1$ and for all $\mathbb{Z}G$-modules M.

Proof. Since $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds, there exists a connected, aspherical pair of CW complexes (Y,X) that realize (G,S) such that $h_n(Y) \cong \mathbb{Z}G \otimes_S h_n(X) \oplus \mathrm{coker}(\Phi_n)$. Moreover, the skeletal homotopy modules $h_n(Y)$ and $h_n(X)$ are kernels of the free resolutions arising from the cellular chain complexes of the associated universal covers, thus Theorem 2.1.18 applies. □

Applying the work of Serre and Howie and Schneebeli to the above theorem gives us the main group theoretic consequences for skeletal homotopy splittings and projective skeletal homotopy splittings.

Corollary 4.1.2. *Let (G,S) be a group pair and let n be a fixed, non-negative integer.*

(i) Suppose $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds. Then for each finite subgroup K of G, there exists a $g \in G$ such that $K \subseteq gSg^{-1}$ and $K \cap hSh^{-1} = \{1\}$ for all $h \notin gS$, [Hue79]. Moreover, $cd(G) \leq max\{n+2, cd(S)\}$.

(ii) Suppose $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds. If $S \cap gSg^{-1}$ contains a non-trivial element of finite order, then $g \in S$.

Proof. Most of (i) and all of (ii) is an application of Corollary 2.1.19. For the latter conclusion in (i) we have if $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds, then for all $k \geq 1$ and for all $\mathbb{Z}G$-modules M, $H^{k+(n+1)}(G,M) = H^{k+(n+1)}(S,M)$. Note that an equivalent definition of cohomological dimension is the smallest integer for which the cohomology of the group vanishes, $\mathrm{cd}(G) = \inf\{m : H^i(G,-) = 0 \text{ for } i > m\}$ [see Bro82, VIII.2]. Thus if $\mathrm{cd}(S) \geq n+2$, then $H^j(G,M) = H^j(S,M) = 0$ for all $j \geq cd(S)$ and so $cd(G) \leq cd(S)$. Otherwise, $\mathrm{cd}(S) < n+2$ and so $H^j(G,M) = H^j(S,M) = 0$ for all $j \geq n+2$, thus $cd(G) \leq n+2$. Therefore $cd(G) \leq \max\{n+2, cd(S)\}$. □

Note that in general S can be an infinite subgroup of G and our conditions $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ and $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ could still hold. When S is finite, however, the implications are stronger.

Corollary 4.1.3. *[HS81]. Consider the group pair (G,S) where S is finite and let n be a fixed, non-negative integer. If $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds then S is malnormal in G,*

Proof. If S is finite, for every $g \in G$, the subgroup $S \cap gSg^{-1}$ is also finite. Thus by Corollary 4.1.2, if $S \cap gSg^{-1} \neq \{1\}$ then $g \in S$. Therefore S is malnormal in G. □

Corollary 4.1.4. *[HS81] Consider the group pair (G,S) where S is finite and G has non-trivial center. Then for a fixed, non-negative integer n, if $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds then either $S = 1$ or $S = G$.*

Proof. Recall that if G has a proper, non-trivial, malnormal subgroup then G has trivial center by Lemma 2.1.4. □

Finally, if G is also finite, we have even stronger implications. If S is a proper, non-trivial subgroup of a finite group G then $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ implies S is malnormal in G and so (G,S) is a Frobenius pair, in which case we have considerable information about the structure of G, see Theorem 2.1.11. Moreover we have the following corollary for $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$.

Corollary 4.1.5. *Let (G,S) be a group pair and let n be a fixed, non-negative integer. If G is finite and $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds then $S = G$.*

Proof. If G is finite and $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ holds, then there exists $g \in G$ such that $G \subseteq gSg^{-1}$. But then S has the same number of elements as G and so $S = G$. □

4.2 Weak Skeletal Homotopy Splittings

Interestingly, a weak skeletal homotopy splitting still gives us group theoretic consequences. To see these consequences we must first prove a series of lemmas.

Let $\alpha : S \to G$ be a group homomorphism. This determines a functor from left RG-modules N to left RS-modules ${}_{\alpha}N$ that is additive in the sense that it distributes over direct sums of RG-modules. The RS-module ${}_{\alpha}N$ is the module with underlying abelian group N and S-action given by

$$s * n = \alpha(s)n.$$

Lemma 4.2.1. *Let $\alpha : S \to S$ be an automorphism of the group S. A left RS-module N is projective if and only if ${}_{\alpha}N$ is also RS-projective.*

Proof. Note that ${}_{\alpha^{-1}}({}_{\alpha}N) = N$ since $s*n = \alpha^{-1}(\alpha(s))n = sn$. Thus it suffices to prove the forward direction since if ${}_{\alpha}N$ is RS-projective then the same proof implies ${}_{\alpha^{-1}}({}_{\alpha}N) = N$ is RS-projective as well.

Suppose N is a projective RS-module. Then N is a summand of a free RS-module, $F = N \oplus K$, where K is an RS-module. Recall that F is a free RS-module if and only if it is the direct sum of copies of RS. Since the functor that sends N to ${}_{\alpha}N$ is additive, then ${}_{\alpha}N \oplus_{\alpha} K =_{\alpha} F$, where F is itself a direct sum of copies of ${}_{\alpha}RS$. Thus it suffices to prove that ${}_{\alpha}RS$ is a free RS-module. In fact, ${}_{\alpha}RS \cong RS$.

First note that S is an R-basis for the module RS, and thus the set $\alpha(S)$ is an R-basis for the module ${}_{\alpha}RS$. Define

$$\phi : RS \to_{\alpha} RS \qquad \text{by}$$

$$\phi(1) = 1$$

and extend RS-linearly. Then for all $s \in S$,

$$\phi(s) = s * \phi(1) = s * 1 = \alpha(s)1 = \alpha(s).$$

Thus ϕ agrees with α on an R-basis, S, of RS. Since α is bijective, ϕ carries an R-basis of RS bijectively onto an R-basis of ${}_{\alpha}RS$ and thus ϕ is bijective. Therefore ϕ is an RS-isomorphism.

Thus ${}_{\alpha}F$ is a free RS-module, and so ${}_{\alpha}N$ is a summand of a free RS-module, and therefore ${}_{\alpha}N$ is RS-projective. □

The next lemma provides conditions under which an R-summand $g \otimes M$ is also an RS-summand of the induced module.

Lemma 4.2.2. *Let (G, S) be a group pair and M be a left RS-module. If for some $g \in G - S$, we have $gSg^{-1} \subseteq S$, then*

(i) $g^{-1} \otimes M$ is an RS-summand of ${}^{RG \otimes_S M}/_{1 \otimes M}$, and

(ii) ${}_{\alpha}M \cong g^{-1} \otimes M$ as RS-modules,
where $\alpha : S \to S$ is the monomorphism arising from conjugation by g, $\alpha(s) = gsg^{-1}$.

Proof. Let $g \in G - S$ such that $gSg^{-1} \subseteq S$.

For (i), by Lemma 2.1.13 and Corollary 2.1.15 there is an R-module decomposition

$$ {}^{RG \otimes_S M}/_{1 \otimes M} = \bigoplus_{xS \in {}^{G}/_{S} - \{S\}} (x \otimes M), $$

where the left action by S permutes these R-summands according to the left action of S on ${}^{\mathrm{G}}/_{\mathrm{S}} - \{S\}$. We want to show that $g^{-1} \otimes M$ is a left RS-submodule of ${}^{RG \otimes_S M}/_{1 \otimes M}$ and that the remaining R-summands are S-stable and hence a left RS-submodule as well. For each $s \in S$, it suffices to show that $s \cdot xS = g^{-1}S$ if and only if $xS = g^{-1}S$. The following statements are equivalent:

$$
\begin{aligned}
s \cdot xS &= g^{-1}S \\
gsx &\in S \\
gsg^{-1}gx &\in S \\
gx &\in S \qquad \text{since } gsg^{-1} \in S \text{ by hypothesis} \\
xS &= g^{-1}S.
\end{aligned}
$$

Thus $g^{-1} \otimes M$ is an RS-summand of ${}^{RG \otimes_S M}/_{1 \otimes M}$.

For (ii), consider how S acts on $g^{-1} \otimes M$. Let $s \in S$ and $g^{-1} \otimes m \in g^{-1} \otimes M$. Then

$$\begin{aligned} s(g^{-1} \otimes m) &= (sg^{-1}) \otimes m \\ &= g^{-1}gsg^{-1} \otimes m \\ &= g^{-1}\alpha(s) \otimes m = g^{-1} \otimes \alpha(s)m. \end{aligned}$$

Define $\phi_\alpha : M \to g^{-1} \otimes M$ by $\phi_\alpha(m) = g^{-1} \otimes m$ and extend R-linearly. Then,

$$\begin{aligned} \phi(s * m) &= \phi\left(\alpha(s)m\right) \\ &= g^{-1} \otimes (\alpha(s)m) \\ &= s(g^{-1} \otimes m) = s\phi(m). \end{aligned}$$

Therefore ϕ respects the S-action and thus is an RS-homomorphism. Since ϕ is clearly bijective, ϕ is an RS-isomorphism. □

Putting these two lemmas together we obtain the following result for $\mathrm{Sum}_n^{RS}(Y, X)$.

Theorem 4.2.3. *Let (Y, X) be a connected, aspherical pair of CW complexes that realizes the group pair (G, S) and let n be a fixed, positive integer. If $\mathrm{Sum}_n^{RS}(Y, X)$ holds and $g \in N_G(S) - S$, then $h_n(X)$ is RS-projective.*

Proof. Suppose $\mathrm{Sum}_n^{RS}(Y, X)$ holds for some fixed n and suppose $gSg^{-1} = S$ for some $g \notin S$. Let $\alpha : S \to S$ be the automorphism of S arising from conjugation by g,

$$\alpha(s) = gsg^{-1}.$$

By Lemma 4.2.2, since $gSg^{-1} \subseteq S$ for some $g \notin S$ we have that $g^{-1} \otimes h_n(X) \cong {}_{\alpha}h_n(X)$ as RS-modules, and $g^{-1} \otimes h_n(X)$ is an RS-summand of ${}^{RG \otimes_S h_n(X)}/_{1 \otimes h_n(X)}$.

Now $\mathrm{Sum}_n^{RS}(Y, X)$ holds, thus $\mathrm{coker}(\Phi_n)$ is RS-projective by Theorem 3.1.2. We then have the following commutative diagram with exact rows and columns arising from

Lemma 2.1.13 and Theorem 2.2.8

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_n(X) & \longrightarrow & RG \otimes_S h_n(X) & \longrightarrow & {}^{RG\otimes_S h_n(X)}\!/_{1\otimes h_n(X)} & \longrightarrow & 0\\
 & & \| & & \downarrow \Phi_n & & \downarrow \Phi'_n & & \\
0 & \longrightarrow & h_n(X) & \longrightarrow & h_n(Y) & \longrightarrow & {}^{h_n(Y)}\!/_{h_n(X)} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \operatorname{coker}(\Phi_n) & \xrightarrow{\cong} & \operatorname{coker}(\Phi'_n) & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the right column is a split exact sequence of RS-modules. Thus ${}^{RG\otimes_S h_n(X)}\!/_{1\otimes h_n(X)}$ is an RS-summand of ${}^{h_n(Y)}\!/_{h_n(X)}$.

But then $g^{-1}\otimes h_n(X)$ is an RS-summand of ${}^{RG\otimes_S h_n(X)}\!/_{1\otimes h_n(X)}$ which is in turn an RS-summand of ${}^{h_n(Y)}\!/_{h_n(X)}$, a projective RS-module by Proposition 3.1.1. Thus $g^{-1}\otimes h_n(X)$ is RS-projective, and hence the isomorphic ${}_{\alpha}h_n(X)$ is RS-projective as well. But the functor α preserves projectivity by Lemma 4.2.1 and therefore $h_n(X)$ is RS-projective. □

We then have the following theorem for a group pair (G, S).

Corollary 4.2.4. *Let (G, S) be a group pair, let n be a fixed, non-negative integer and suppose $\mathrm{Sum}_n^{\mathbb{Z}S}(G, S)$ holds. Then either $N_G(S) = S$ or $cd(S) \leq n + 1$.*

Proof. Suppose that $\mathrm{Sum}_n^{\mathbb{Z}S}(G, S)$ holds for some fixed n. Then there exists a connected, aspherical pair of CW complexes (Y, X) that realizes the group pair (G, S). If S is not self-normalizing in G, then by Theorem 4.2.3, $h_n(X)$ is $\mathbb{Z}S$-projective. But then we have a $\mathbb{Z}S$-projective resolution of $\mathbb{Z}$ of length $n + 1$ and so $cd(S) \leq n + 1$. □

When S has torsion, the implications are stronger.

Corollary 4.2.5. *Consider the group pair (G, S) where S has torsion and let n be a fixed, non-negative integer. If $\mathrm{Sum}_n^{\mathbb{Z}S}(G, S)$ holds then S is self-normalizing in G.*

Proof. If $cd(S)$ is finite then S is torsion free [Bro82, p. 187]. Thus if S has non-trivial torsion, $cd(S)$ must be infinite and by Corollary 4.2.4 S is self-normalizing in G. □

In particular, if S is finite then S is self-normalizing in G.

Corollary 4.2.6. *Consider the group pair (G, S) where S is non-trivial, finite and G has non-trivial center. Then for a fixed, non-negative integer n if $\mathrm{Sum}_n^{\mathbb{Z}S}(G, S)$ holds then either $Z(G) \lneq S$ or $S = G$.*

Proof. If $N_G(S) = S$ then the center of G is contained in S, that is, $Z(G) \leq N_G(S) = S$. But if $Z(G) = S$ then for $g \in G$, $gSg^{-1} = S$ and so $g \in N_G(S) = S$. Thus $G = S$. □

4.3 Applications to Cyclically Presented Groups

When S and Γ are subgroups of a group G and S normalizes Γ, then S acts on Γ via conjugation in G. Since the action is by automorphisms, we have S acting on $\Gamma - \{1\}$ as well, as long as Γ is nontrivial. Conversely, if S acts on an abstract group Γ by automorphisms, then S normalizes Γ in the semidirect product $G = \Gamma \rtimes S$ where we have $S \cap \Gamma = \{1\}$. In this section we investigate the impact of homotopy splittings on the dynamics of finite (or periodic) groups of automorphisms. The results are applied in the case of shift automorphisms defined on cyclically presented groups.

A general result along these lines is the following.

Theorem 4.3.1. *Consider a group pair (G, S) where S is a torsion group that normalizes a nontrivial subgroup Γ of G. If $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ holds for some n and $S \cap \Gamma = \{1\}$, then S*

acts freely on $\Gamma - \{1\}$*. In particular, the action of* S *on* Γ *is faithful and so determines an embedding of* S *in the automorphism group* $\mathrm{Aut}(\Gamma)$*.*

Proof. Let $s \in S$ and suppose $s\gamma s^{-1} = \gamma$ for some $\gamma \in \Gamma - \{1\}$. Then $s = \gamma s \gamma^{-1} \in S \cap \gamma S \gamma^{-1}$ and s has finite order since S is a torsion group. But then by Corollary 2.1.19 (i) either $\gamma \in \Gamma \cap S = \{1\}$ or $s = 1$. Since we assumed $\gamma \neq 1$, we have $s = 1$ and so only the trivial element in S fixes any element of $\Gamma - \{1\}$. □

Note that these results apply whenever an induced kernel of a projective $\mathbb{Z}S$-resolution of $\mathbb{Z}$ is a $\mathbb{Z}G$-summand of a kernel of a projective $\mathbb{Z}G$-resolution of G.

4.3.1 Cyclically Presented Groups

For a positive integer n, let θ be an n-cycle operating on the set $\{x_0, \ldots, x_{n-1}\}$ where $\theta(x_i) = x_{i+1}$ and subscripts are taken modulo n. Then θ extends to an automorphism of order n for the free group F with basis $\{x_0, \ldots, x_{n-1}\}$. A **cyclically presented group** is one having a presentation of the form

$$\Gamma_n(w) = \langle x_0, \ldots, x_{n-1} \mid w, \theta(w), \theta^2(x), \ldots, \theta^{n-1}(w) \rangle. \tag{4.1}$$

where $w = w(x_0, \ldots, x_{n-1})$ is a word representing an element of the free group F. Thus the group $\Gamma = \Gamma_n(w)$ presented by (4.1) is the quotient of F modulo the smallest normal subgroup that contains w and all of its shifts $\theta^i(w)$ under $\theta \in \mathrm{Aut}(F)$.

Fundamental questions, considered by many authors, include determining conditions on n and w under which one of the following holds:

- $\Gamma_n(w)$ is trivial;
- $\Gamma_n(w)$ is finite;
- θ has order n as an automorphism of $\Gamma_n(w)$.

It is obvious that $\Gamma_n(w)$ is trivial if $w = x_0$. Non-obvious examples for which $\Gamma_n(w)$ is trivial are known, with some of the first appearing in [Hig51]. Criteria for non-triviality have been investigated; see [Edj03] and the references cited there. Many authors have investigated the structure of cyclically presented groups that happen to be finite. An early study of this sort appeared in [JWW74], where the question of the order of θ was also addressed. Viewed as an automorphism of $\Gamma_n(w)$, the shift θ has order dividing n and it can happen that this order is strictly less than n, for example if $\Gamma_n(w)$ happens to be trivial. If this order is $t = t(n, w)$, then $\Gamma_n(w) \cong \Gamma_t(w)$ via an isomorphism that reduces indices modulo t.

Let C_n be a cyclic group of order n generated by a. For given n and w, the shift determines an action of C_n on $\Gamma_n(w)$ and we denote the resulting split extension by $G_n(w) = \Gamma_n(w) \rtimes C_n$. As many have noted, this group has a two-generator two-relator presentation

$$G_n(w) = \Gamma_n(w) \rtimes_\theta C_n = \langle x, a \mid a^n, w(a, x)\rangle \tag{4.2}$$

where $w(a, x)$ is obtained from $w(x_0, \ldots, x_{n-1})$ by rewriting $x_0 = x$ and $x_i = a^i x a^{-i}$ for $i = 1, \ldots, n-1$. The generator a occurs with exponent sum zero in the word $w(a, x)$.

We can view the two-generator two-relator presentation (4.2) as a relative presentation (in the sense of [BP92]) with coefficients in the cyclic group $C_n = \langle a \mid a^n\rangle$ of order n:

$$G_n(w) = \langle C_n, x \mid w(a, x)\rangle. \tag{4.3}$$

There are many results concerning asphericity of relative presentations; see [BBP97, BP92, Edj94, HM01, Kim08a, Kim08b]. The significance of these results in the current context is the following.

Theorem 4.3.2. *Consider a nontrivial cyclically presented group $\Gamma_n(w)$ as in (4.1). If the relative presentation (4.3), constructed as above for the group $G_n(w) = \Gamma_n(w) \rtimes_\theta C_n$, is aspherical, then:*

(i) $Sum_k^{\mathbb{Z}G_n(w)}(G_n(w), C_n)$ holds for all $k \geq 1$;

(ii) The cyclic group C_n *of order* n *acts freely on* $\Gamma_n(w) - \{1\}$ *via the shift* θ*;*

(iii) The shift automorphism $\theta \in \mathrm{Aut}(\Gamma_n(w))$ *has order* n*.*

Proof. Let X be an Eilenberg-Maclane space of type $K(C_n, 1)$, which can be chosen to have two-skeleton modelled on the presentation $\langle a \mid a^n \rangle$ for C_n and with exactly one cell in each dimension. Let Y be obtained from X by attaching a one-cell c_x^1 and a two-cell c_w^2 modeling the word $w(a, x)$ from the relative presentation (4.3). Then the pair (Y, X) models the group pair $(G_n(w), C_n)$. By [BP92, Theorem 1.2], asphericity of the relative presentation (4.3) implies that $\mathrm{Sum}_1^{\mathbb{Z}G_n(w)}(Y, X)$ holds. But then $\mathrm{Sum}_k^{\mathbb{Z}G_n(w)}(Y, X)$ holds for all $k \geq 1$ by Theorem 3.2.2.

Now $\Gamma_n(w) \neq \{1\}$ sits in $G_n(w) = \Gamma_n(w) \rtimes C_n$, so $\Gamma_n(w) \cap C_n = \{1\}$ and C_n normalizes $\Gamma_n(w)$ in $G_n(w)$. Theorem 4.3.1 implies that the cyclic group C_n acts freely on $\Gamma_n(w) - \{1\}$. Since the orbit $\{\theta^k(\gamma)\} = \{a^k \gamma a^{-k}\}$ of a nontrivial element $\gamma \in \Gamma_n(w)$ has n distinct elements, it follows that $\theta \in \mathrm{Aut}(\Gamma_n(w))$ has order n. □

4.3.2 Three-Manifolds

Cyclic presentations and their groups arise in the study of three-manifold topology. Although this has not yet been fully explored, the following families of examples suggest that the connection lies very deep. The Fibonacci groups are those given by the cyclic presentations

$$F(n) = \Gamma_n(x_0 x_1 x_2^{-1}).$$

The Fibonacci group $F(n)$ is finite if and only if $n = 2, 3, 4, 5$, or 7 [Bru74, CWLF67, HRS79, Hol95, Lyn, New90]. It turns out that the two-complexes modeled on the cyclic presentations $F(2)$ (resp. $F(4), F(6)$) serve as spines for the three-sphere (resp. a lens space, a Euclidean three-manifold). A 1994 preprint of H. Helling, A. C. Kim, and J. L. Mennicke reported in addition that $F(2m)$ is the fundamental group of a hyperbolic three-manifold if $m \geq 4$. See [HKM98] for the published result. As noted in [Hol95], this geometric result implies that the corresponding groups are automatic. The situation was further elucidated by C. Maclachlan [Mac95], who studied arithmeticity for these groups, but also showed that none of the groups $F(2m+1)$ can be fundamental groups of any hyperbolic three-orbifold of finite volume.

Another one-parameter family of cyclically presented groups was introduced by A. J. Sieradski in 1986 [Sie86]. These are the groups

$$S(n) = \Gamma_n(x_0 x_2 x_1^{-1}).$$

Sieradski showed that all of the two-complexes modeled on these presentations $S(n)$, $n \geq 2$, are three-manifold spines. R. M. Thomas later showed that $S(n)$ is finite if and only if $n \leq 6$ [Tho91] and the groups $S(n)$, $n \geq 2$, have since been identified as fundamental groups of n-fold cyclic covers of the three-sphere branched over the trefoil knot [CHK98].

In [CRS05], A. Cavicchioli, D. Repovš, and F. Spaggiari introduced the following family of cyclic presentations. Given $\epsilon = (a, b, r, s) \in \mathbb{Z}^4$, $n \geq 2$, and integer parameters m, k, h taken modulo n, the group $\Gamma_n^\epsilon(m, k, h)$ is that having cyclic presentation $\Gamma_n(w)$ where

$$w = x_0^a x_k^b x_{h+m}^a (x_h^r x_m^r)^{-s}. \tag{4.4}$$

The Fibonacci and Sieradski groups occur among these for suitable choice of parameters. Take $a = b = s = 1$, $r = 2$, and $h = 0$. Setting $(m, k) = (1, 2)$ leads to $F(n)$ and setting $(m, k) = (2, 1)$ produces $S(n)$, after cyclic reduction of relators. The main result of [CRS05] builds on the argument of [Mac95] to obtain the following negative result concerning three-dimensional geometrization.

Theorem 4.3.3 ([CRS05]). *Suppose that n and b are odd. The cyclically presented group $\Gamma_n^\epsilon(m, k, h)$ cannot be the fundamental group of any hyperbolic three-orbifold of finite volume if either of the following conditions are met:*

(i) n is coprime to $2k - h - m$, or

(ii) n *does not divide* $2k-h-m$ *and the shift* θ *has order* n *as an automorphism of the cyclically presented group* $\Gamma_n^{\epsilon}(m,k,h)$.

4.3.3 Applications

Theorem 4.3.2 establishes a connection between asphericity of relative presentations and the order and dynamics of the shift on cyclically presented groups. We exploit this connection to construct new families of examples to which Theorem 4.3.3(ii) applies.

Consider the cyclic presentation $\Gamma_n^{\epsilon}(m,k,h)$ as in (4.4) where $\epsilon = (a,b,r,s) = (1,1,0,0)$, and let $\ell = h+m$. Following [CRS05], denote this cyclic presentation and its group by

$$\Gamma_n(k,\ell) = \Gamma_n(x_0x_kx_\ell) = \langle x_0,\ldots,x_{n-1} \mid x_ix_{i+k}x_{i+\ell} \text{ (subscripts mod } n)\rangle.$$

We use the shift θ to form the split extension

$$G_n(k,\ell) = \Gamma_n(k,\ell) \rtimes_\theta C_n = \langle a,x \mid a^n, xa^kxa^{\ell-k}xa^{-\ell}\rangle = \langle C_n,x \mid xa^kxa^{\ell-k}xa^{-\ell}\rangle. \quad (4.5)$$

The cyclically presented group $\Gamma_n(k,\ell)$ is nontrivial for all n,k,ℓ since $\Gamma_n(k,\ell)$ maps onto the cyclic group of order three in an obvious way. Thus Theorem 4.3.2 applies whenever the relative presentation $G_n(k,\ell) = \langle C_n,x \mid xa^kxa^{\ell-k}xa^{-\ell}\rangle$, with coefficients in $C_n = \langle a \mid a^n\rangle$, is aspherical.

Theorem 4.3.4. *If the relative presentation* $G_n(k,\ell) = \langle C_n,x \mid xa^kxa^{\ell-k}xa^{-\ell}\rangle$ *is aspherical, then the shift on the cyclically presented group* $\Gamma_n(k,\ell)$ *has order* n *and* C_n *acts freely on* $\Gamma_n(k,\ell)-\{1\}$.

Necessary and sufficient criteria for asphericity of relative presentations of the form (4.5) were given in [BP92, Theorem 3.1]. The conditions are expressed in terms of the elements

$$\alpha_1 = a^ka^{-(\ell-k)} = a^{2k-\ell},\ \alpha_2 = a^{\ell-k}a^{-(-\ell)} = a^{2\ell-k}, \text{and } \alpha_3 = a^{-\ell}a^{-k} = a^{-\ell-k}$$

in the cyclic group C_n. Let p_i be the order of $\alpha_i \in C_n$. As an application of [BP92, Theorem 3.1], if $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \le 1$, then the relative presentation (4.5) is aspherical. This holds, for example, if $p_i \ge 3$ for each i. These remarks enable us to employ case (ii) of Theorem 4.3.3, which requires a determination of the order of the shift.

Theorem 4.3.5. *For each $p \ge 2$, if we set $n = (2p+1)^2$, $k = 2p+2$, and $\ell = 2p+3$, then for each $p \ge 2$, the cyclically presented group $\Gamma_n(k,\ell) = \Gamma_{(2p+1)^2}(2p+2, 2p+3)$ with presentation*

$$\langle x_0, \ldots, x_{(2p+1)^2-1} \mid x_i x_{i+2p+2} x_{i+2p+3} \text{ (subscripts mod } (2p+1)^2) \rangle$$

satisfies the conditions of Theorem 4.3.3(ii) (but not Theorem 4.3.3(i)), and so is not the fundamental group of any hyperbolic three-orbifold of finite volume.

Proof. In these examples, we have that $n = (2p+1)^2$ and $b = 1$ are odd. The parameters k, h, m, and ℓ are such that $2k-h-m = 2k-\ell = 4p+4-(2p+3) = 2p+1$, so $n = (2p+1)^2$ is not coprime to $2k-h-m = 2p+1$. Thus Theorem 4.3.3(i) does not apply. However, $n = (2p+1)^2$ does not divide $2k-h-m = 2p+1$, so it remains to show that the shift has order n whenever $p \ge 2$. For this, we verify asphericity by showing that the elements

$$\alpha_1 = a^{2k-\ell} = a^{2p+1},\ \alpha_2 = a^{2\ell-k} = a^{2p+4}, \alpha_3 = a^{-\ell-k} = a^{-(4p+5)}$$

all have order at least three. This is routinely done, using the fact that $p \ge 2$, by showing that none of the elements α_i^j are equal to the identity in the cyclic group $C_n = C_{(2p+1)^2}$ generated by a for $i, j = 1, 2, 3$. □

5 AUGMENTATION IDEALS AND DISTINCTIONS

Chapter 4 explores more fully the conditions $\mathrm{PSum}_n^{RG}(G,S)$, $\mathrm{Sum}_n^{RG}(G,S)$ and $\mathrm{Sum}_n^{RS}(G,S)$ and provides distinguishing examples. When $n=0$, this involves splittings of the augmentation ideal IG where both necessary and sufficient conditions for such splittings have been characterized by [Dic81]. Using information from $n=0$, Section 2 provides examples to distinguish between the splitting conditions in various dimensions.

5.1 Augmentation Ideals

Recall that if the CW-pair (Y,X) has a single zero-cell, then the 0-th skeletal homotopy modules are the augmentation ideals of G and S respectively, $h_0(Y)=IG$ and $h_0(X)=IS$ (see Example 2.2.1). Thus by Lemma 2.2.11, to determine when any of $\mathrm{PSum}_0^{RG}(Y,X)$, $\mathrm{Sum}_0^{RG}(Y,X)$, $\mathrm{Sum}_0^{RS}(Y,X)$ hold, it suffices to understand splittings of the short exact sequence of RG-modules

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \longrightarrow {}^{IG}\!/_{RG \otimes_S IS} \longrightarrow 0.$$

Splittings of the augmentation ideal IG were explored extensively by Gruenberg and Roggenkamp and eventually fully characterized by Warren Dicks, see [Dic81]. In this section I provide alternate proofs of some of Dick's results to characterize splittings of augmentation ideals which can then be used to characterize $\mathrm{Sum}_0^{RS}(Y,X)$ and $\mathrm{Sum}_0^{RG}(Y,X)$.

Permutation modules play a key role in splittings of augmentation ideals. Let X be a set and let R$[X]$ denote the free R-module with basis X. If a group G acts on the set X, then R$[X]$ is also a module over the group ring RG by extending the G-action to an R-linear G-action, called a **permutation module** over RG. (Note that the permutation module R$[G]$, where G is a left G-set via group multiplication, is exactly the RG-module

RG.) By the Orbit Stabilizer Correspondence (Lemma 2.1.1), every permutation module is RG-isomorphic to one of the form

$$R[X] \cong \bigoplus_{x_i} R\left[{}^{\mathrm{G}}\!/_{Stab_G(x_i)}\right],$$

where the sum is taken over a set of orbit-representatives of the G-action on X. Thus a permutation module is free if and only if G acts freely on the set X.

If $S \leq G$, then ${}^{G}\!/_{S}$ is both a left G-set and a left S-set, thus $\mathrm{R}[{}^{G}\!/_{S}]$ is both a left RG-permutation module and a left RS-permutation module. Now consider the augmentation

$$\begin{array}{rcl} R[{}^{G}\!/_{S}] & \xrightarrow{\varepsilon} & R \\ gS & \longmapsto & 1 \end{array}$$

and let $I[{}^{G}\!/_{S}] = ker(\varepsilon)$. Note that $R[{}^{\mathrm{G}}\!/_{\mathrm{S}}] \cong RG \otimes_S R$ as left RG-modules via $gS \mapsto g \otimes 1$, where R has trivial left S-action.

Lemma 5.1.1. *Let $S \leq G$. There is an exact sequence of RG-modules*

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \xrightarrow{\pi} I[{}^{G}\!/_{S}] \longrightarrow 0$$

where $\pi(1-g) = S - gS$*. Thus* $\mathrm{coker}(\Phi_0) \cong I[{}^{G}\!/_{S}]$.

Proof. We have the following commutative diagram of RG-modules with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & I[{}^{G}\!/_{S}] & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG \otimes_S IS & \longrightarrow & RG & \xrightarrow{\pi} & R[{}^{G}\!/_{S}] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_0} & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \\
0 & \longrightarrow & IG & \longrightarrow & RG & \xrightarrow{\varepsilon} & R & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}(\Phi_0) & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Here $\pi(1) = S \in R[{}^{\mathrm{G}}\!/_{\mathrm{S}}]$. By the Snake Lemma $I[{}^{G}\!/_{S}] \cong \mathrm{coker}(\Phi_0)$. □

Lemma 5.1.2. *As an RS-module,* $I[{}^{G}\!/_{S}]$ *is a permutation module that decomposes as a direct sum,*

$$I[{}^{G}\!/_{S}] \cong \bigoplus_{SgS \neq S} R[{}^{S}\!/_{S \cap gSg^{-1}}],$$

where the sum is taken over the S-S double cosets that are distinct from S.

Proof. First we will prove that the set $X = \{S - gS : S \neq gS\}$ is a R-basis for $I[{}^{G}\!/_{S}]$. Let $\xi \in ker(\varepsilon) \subseteq R[{}^{G}\!/_{S}]$. Then we can write ξ as the finite sum

$$\xi = \sum_{i=1}^{n} r_i g_i S$$

where $g_i S \in {}^{G}\!/_{S}$ and $r_i \in R$. But then

$$\varepsilon(\xi) = \sum_{i=1}^{n} r_i \varepsilon(g_i S) = \sum_{i=1}^{n} r_i = 0.$$

Thus $r_n = -\sum_{i=1}^{n-1} r_i$, and so

$$\xi = \sum_{i=1}^{n-1} r_i g_i S - \sum_{i=1}^{n-1} r_i g_n S = \sum_{i=1}^{n-1} r_i (g_i S - g_n S).$$

Note that $g_i S - g_n S = (S - g_i S) + (S - g_n S) \in R[X]$ for $i = 1, \cdots, n-1$. Thus every element of $I[{}^{G}\!/_{S}]$ can be written as an R-linear combination of elements of X.

Now suppose $\sum_{i=1}^{m} r_i (S - g_i S) = 0$ where the $g_i S$ are distinct cosets in ${}^{G}\!/_{S} - \{S\}$. Then

$$\begin{aligned} \sum_{i=1}^{m} r_i (S - g_i S) &= \sum_{i=1}^{m} r_i S - \sum_{i=1}^{m} r_i g_i S \\ &= (\sum_{i=1}^{m} r_i) S - \sum_{i=1}^{m} r_i g_i S \\ &= rS - \sum_{i=1}^{m} r_i g_i S = 0. \end{aligned}$$

Since the $g_i S$ were distinct cosets in ${}^{G}\!/_{S} - \{S\}$, and distinct cosets in ${}^{G}\!/_{S}$ are R-linearly independent, we have $r_i = 0$ for all i. Thus X is R-linearly independent and $I[{}^{G}\!/_{S}] = R[X]$.

Next we see that X is an S-set since S acts on ${}^{G}\!/_{S}$ fixing $S = 1 \cdot S \in {}^{G}\!/_{S}$. Therefore $I[{}^{G}\!/_{S}] = R[X]$ is an RS-permutation module where the S-action corresponds exactly to the S-action on the left cosets ${}^{G}\!/_{S} - \{S\}$. But then the set of S-orbits in X is in one-to-one correspondence with the set of double cosets ${}_{S}\backslash^{G}\!/_{S} - \{S\}$. Moreover, $Stab_S(S - gS) = S \cap gSg^{-1}$ for all $g \notin S$. $\square$

Recall that $\mathrm{Sum}_0^{RS}(Y, X)$ is equivalent to $\mathrm{coker}(\Phi_0)$ being RS-projective by Theorem 3.1.2. Thus we consider when permutation modules are projective.

Theorem 5.1.3. *[Dic81] Let $S \leq G$ and consider the RG-permutation module $R[{}^{G}\!/_{S}]$. Then the following are equivalent.*

(i) $R[{}^{G}\!/_{S}]$ is RG-projective

(ii) $R[{}^{G}\!/_{S}]$ is RS-projective

(iii) S is finite and $|S|$ is a unit in R.

Proof. That (i) implies (iii) is obvious. To (ii) implies (iii), our aim is to show that if $R[{}^{G}\!/_{S}]$ is RS-projective, then the augmentation $\varepsilon : RS \to R$ is RS-split. We have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & IS & \longrightarrow & RS & \xrightarrow{\varepsilon} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \nwarrow \sigma & & \text{inc}\downarrow \uparrow \varepsilon_{G/S} & & \\
0 & \longrightarrow & RG \otimes_S IS & \longrightarrow & RG & \xrightarrow{\pi} & R[{}^{G}\!/_{S}] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG \otimes_S IS/_{IS} & \longrightarrow & {}^{RG}\!/_{RS} & \longrightarrow & R[{}^{G}\!/_{S}]/_{R} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Notice that inclusion in the right-hand column has $\mathrm{inc}(1) = S$. This is an RS-homomorphism since S fixes $1 \cdot S = S$. Consider the augmentation $\varepsilon_{G/S} : R[{}^{G}\!/_{S}] \longrightarrow R$

where $\varepsilon_{G/S}(gS) = 1$ for all $g \in G$. Then inc is RS-split by the augmentation since $\varepsilon_{G/S} \circ \text{inc}(1) = \varepsilon_{G/S}(S) = 1$.

Now ${}^{\mathrm{RG}}\!/_{\mathrm{RS}}$ is a free RS-module (see the proof of Lemma 3.1.1). Thus we can apply Lemma 3.2.1-(i) to obtain an RS-splitting $\sigma : RG \longrightarrow RS$ such that $\varepsilon \circ \sigma = \varepsilon_{G/S} \circ \pi$. Moreover, if $R[{}^{\mathrm{G}}\!/_{\mathrm{S}}]$ is RS-projective then π is RS-split and there exists an RS-homomorphism $\overline{\pi}$ where $\pi \circ \overline{\pi} = 1_{R[G/S]}$. But then $\overline{\varepsilon} = \sigma \circ \overline{\pi} \circ \text{inc}$ is an RS-splitting of the augmentation ε,

$$\varepsilon \circ \overline{\varepsilon} = \varepsilon \circ \sigma \circ \overline{\pi} \circ \text{inc} = \varepsilon_{G/S} \circ \pi \circ \overline{\pi} \circ \text{inc} = \varepsilon_{G/S} \circ \text{inc} = 1_R.$$

Now consider $1 \in R$ and its image under the RS-splitting of the augmentation, $\overline{\varepsilon}(1) \in RS$, which must be finitely supported and non-zero. Since S acts trivially on R and $\overline{\varepsilon}$ is an RS-homomorphism, S acts trivially on $\overline{\varepsilon}(1)$. Then for some $s' \in S$ in the support of $\overline{\varepsilon}(1)$, the trivial S-action implies ss' is also in the support of $\overline{\varepsilon}(1)$ for any $s \in S$. But then every element of S is in the support of $\overline{\varepsilon}(1)$, and so S must be finite.

Moreover $s\overline{\varepsilon}(1) = \overline{\varepsilon}(1)$ for all $s \in S$ implies the terms in the support of $\overline{\varepsilon}(1)$ all occur with the same R-coefficient, say $r' \in R$. Then $\overline{\varepsilon}(1) = r' \cdot N_S$ where N_S is the norm element of S, that is, $N_S = \sum_{s \in S} s$. Thus

$$1 = \varepsilon \circ \overline{\varepsilon}(1) = r'\varepsilon(N_S) = r'|S|,$$

and therefore $|S|$ is a unit in R.

To prove (iii) implies (i) note that if S is finite and $|S| = r_s$ is a unit in R then the projection $RG \xrightarrow{\pi} R[{}^{\mathrm{G}}\!/_{\mathrm{S}}]$ is split by the RG-module homomorphism

$$\begin{aligned} \overline{\pi} : R[{}^{\mathrm{G}}\!/_{\mathrm{S}}] &\longrightarrow RG \\ gS &\longmapsto r_s^{-1} g N_S. \end{aligned}$$

□

Corollary 5.1.4. *Let X be a G-set and let $R[X]$ be the associated permutation module. Then R[X] is RG-projective if and only if for each $x \in X$, the point-wise stabilizer $Stab_G(x)$ is finite and $|Stab_G(x)|$ is a unit in R.*

Corollary 5.1.5. *A permutation module over $\mathbb{Z}G$ is $\mathbb{Z}G$-projective if and only if G acts freely on a permutation module basis. In particular, a projective permutation module over $\mathbb{Z}G$ is $\mathbb{Z}G$-free.*

Combining these results with Lemma 5.1.2 leads to the following.

Corollary 5.1.6. *$I[{}^{G}\!/_{S}]$ is RS-projective if and only if $|S \cap gSg^{-1}|$ is finite and a unit in R for all $g \in G - S$.*

5.1.1 Projective coker(Φ_0)

In this subsection we consider projectivity of coker(Φ_0) (see Theorem 3.1.2).

Theorem 5.1.7. *[Dic81] The short exact sequence*

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \longrightarrow I[{}^{G}\!/_{S}] \longrightarrow 0$$

is split exact as RS-modules if and only if $|S \cap gSg^{-1}|$ is finite and a unit in R for all $g \notin S$.

Proof. Combine Lemma 5.1.1, and Corollary 5.1.6. □

Since $\mathbb{Z}$ has no non-trivial units, when $R = \mathbb{Z}$ the stabilizers must be trivial.

Corollary 5.1.8. *The short exact sequence*

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \longrightarrow I[{}^{G}\!/_{S}] \longrightarrow 0$$

is split exact as $\mathbb{Z}S$-modules if and only if S is malnormal in G.

In particular, if the stabilizers $S \cap gSg^{-1}$ are all trivial then I[${}^{\mathrm{G}}\!/_{\mathrm{S}}$] is a free RS-module on the set of nontrivial double cosets $SgS \neq S$. Thus when S is malnormal, we can easily write down RS-splittings of the surjection $IG \to I[{}^{G}\!/_{S}]$, which has kernel $RG \otimes_S IS$.

Proposition 5.1.9. *We can define RS-splittings of $IG \to I[{}^{G}/_{S}]$ by choosing double coset representatives $g_i \in G$ such that $S \neq Sg_iS$. Since $I[{}^{G}/_{S}]$ is a free RS-module with basis the $S - g_iS$, we can send each basis element to any element of its pre-image, and extend RS-linearly to obtain a splitting. In particular, for each i if we send*

$$S - g_iS \longmapsto s_{i_1} - g_i s_{i_2} + \xi_i$$

for any $s_{i_1}, s_{i_2} \in S$ and $\xi_i \in RG \otimes_S IS$ and extend RS-linearly, then we have a splitting.

Note that $I[{}^{\mathrm{G}}/_{\mathrm{S}}]$ is not, in general, an RG-permutation module and so we cannot use our understanding of projective permutation modules to determine when coker(Φ_0) is RG-projective. If $G = S * F$, then it is well-known that $IG \cong RG \otimes_S IS \oplus RG \otimes_F IF$ and there is a split exact sequence of RG-modules

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \longrightarrow RG \otimes_F IF \longrightarrow 0.$$

If further F is a free group, then IF is a free RF-module and so $RG \otimes_F IF$ is a free RG-module. It is impossible for a proper subgroup, however, for coker(Φ_0) to be trivial.

Theorem 5.1.10. *If the injective map $RG \otimes_S IS \xrightarrow{\Phi_0} IG$ is an isomorphism then $S = G$.*

Proof. If Φ_0 is an isomorphism, then $\text{coker}(\Phi_0) \cong I[{}^{\mathrm{G}}/_{\mathrm{S}}] = \ker(R[{}^{G}/_{S}] \xrightarrow{\varepsilon} R)$ is trivial. Thus $S = G$. □

Also, it is impossible when G is finite for coker(Φ_0) to be RG-projective when S is a proper subgroup.

Theorem 5.1.11. *Let G be finite and suppose coker(Φ_0) as defined above is RG-projective. Then $S = G$.*

Proof. The short exact sequence of free R-modules

$$0 \longrightarrow I[{}^{G}/_{S}] \longrightarrow R[{}^{G}/_{S}] \longrightarrow R \longrightarrow 0$$

shows that

$$1 - \frac{|G|}{|S|} + rk_R(I[{}^{G}/_{S}]) = 0.$$

If $I[{}^{G}/_{S}]$ is RG-projective, $|G|$ divides its R-rank [Swa59] and thus $|G|$ divides $\frac{|G|}{|S|} - 1$. Therefore $|S| = |G|$ and thus $S = G$. $\square$

5.1.2 Strong augmentation ideal splittings

We now see $|S \cap gSg^{-1}|$ being a unit in R is a necessary condition for RG-splittings of the augmentation ideal. We will show it is also a sufficient condition for RG-splittings when G is finite. Cohen proved that S malnormal in G is a necessary condition for splittings of the augmentation ideal when G is finite and $R = \mathbb{Z}$ [Gru76, Proposition 8.9]. Meanwhile Gruenberg used semi-local rings and local projectivity to prove that malnormality of S is a sufficient condition to obtain a $\mathbb{Z}G$-splitting when G is finite [Gru76, Proposition 8.5]. Dicks rederived both results as an example of more general splittings of the augmentation ideal over RG when G is finitely generated over S. His proof involves Bass-Serre Theory and Dunwoody's Almost Stability Theorem [DD89]. Here we re-prove this result for finite G using the characterization of $I[{}^{G}/_{S}]$ as a permutation module and then appeal to his more general result to understand splittings when G is infinite.

Consider the following commutative diagram of RG-modules,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG \otimes_S IS & \xrightarrow{\Phi_0} & IG & \xrightarrow{\sigma} & {}^{IG}/_{RG \otimes_S IS} \cong I[{}^{G}/_{S}] & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc_{G/S}} & & \\
0 & \longrightarrow & RG \otimes_S IS & \longrightarrow & RG & \xrightarrow{\pi} & R[{}^{G}/_{S}] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon_{G/S}} & & \\
 & & 0 & \longrightarrow & R & = & R & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We want to show the top row is split exact as RG-modules. Notice that commutativity implies we have a splitting $\overline{\sigma}$ of σ if and only if $\pi \circ inc \circ \overline{\sigma} = inc_{G/S}$ since σ is surjective and hence right cancellable. Thus we can focus our attention on the interaction of a right inverse with π. There is a family of RG-homomorphisms from $R[{}^{G}/_{S}] \rightarrow RG$ that restrict to a map from $I[{}^{G}/_{S}] \rightarrow IG$. Some satisfy the condition $\pi \circ inc \circ \overline{\sigma}(S - gS) = S - gS$ for all $g \in G - S$, which will give us the RG-splitting we desire.

In general, if S is finite, we can define a family of RG-homomorphisms from $R[{}^{G}/_{S}]$ to RG, using the norm element for S, that is $N_S = \sum_{s \in S} s$.

Lemma 5.1.12. *Let S be a finite subgroup of G. Then for any $\xi \in RG$ the map*

$$\varphi_\xi : R[{}^{G}/_{S}] \longrightarrow RG$$

$$gS \longmapsto gN_S\xi$$

is an RG-homomorphism with $\varphi_\xi(I[{}^{G}/_{S}]) \subseteq IG$.

Proof. First define an RG-homomorphism $\phi : RG \longrightarrow RG$ by $\phi(1) = N_S$ and RG-linearity. For each $s \in S$ we have

$$\phi(s - 1) = (s - 1)N_S = sN_S - N_S = N_S - N_S = 0.$$

Thus IS$\subseteq ker(\phi)$ and since ϕ is RG-linear, $RG \otimes_S IS \subseteq ker(\phi)$. But since ${}^{RG}/_{RG \otimes_S IS} \cong R[{}^{G}/_{S}]$, the map ϕ induces an RG-homomorphism $\varphi : R[{}^{G}/_{S}] \rightarrow RG$ where $\varphi(S) = N_S$.

Now for any $\xi \in RG$, right multiplication by ξ is a left RG-homomorphism. That is

$$\xi : RG \longrightarrow RG$$

$$g \longmapsto g\xi$$

Let $\varphi_\xi = \xi \circ \varphi$. Then

$$\varphi_\xi : R[{}^{G}/_{S}] \longrightarrow RG$$

$$gS \longmapsto gN_S\xi$$

is an RG-homomorphism as desired.

Moreover, for an R-generating element $S - gS \in I[{}^{G}\!/_{S}]$ and for the augmentation map $\varepsilon : RG \to R$ as above,

$$\varepsilon \circ \varphi_\xi(S - gS) = \varepsilon(N_S \cdot \xi - g \cdot N_S \cdot \xi) = \varepsilon(1-g)\varepsilon(N_S\xi) = 0$$

since $\varepsilon(1-g) = 0$. Thus, $\varphi_\xi(I[{}^{G}\!/_{S}]) \subseteq ker(\varepsilon) = IG$. $\square$

A particular member of this family of RG-homomorphisms has the properties we need to obtain an RG-splitting of our sequence and so we have the following theorem.

Theorem 5.1.13. *[Dic81] Consider the short exact sequence*

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \longrightarrow I[{}^{G}\!/_{S}] \longrightarrow 0.$$

Then the following are equivalent:

(i) The above sequence is split exact as RG-modules.

(ii) The above sequence is split exact as RS-modules.

(iii) $|S \cap gSg^{-1}|$ *is both finite and a unit in R for all* $g \notin S$.

Proof. We know that (i) implies (ii) and by Theorem 5.1.7 (ii) implies (iii). For (iii) implies (i) we consider splittings of the surjection $\sigma : IG \to I[{}^{G}\!/_{S}]$. Choose non-trivial orbit representatives, $g_i \notin S$, of the double cosets $\{S, Sg_1S, Sg_2S, ..., Sg_kS\}$. There are only finitely many coset representatives since G is finite. Let $r_{g_i} = |S \cap g_iSg_i^{-1}|$ and note that by hypothesis $r_{g_i}^{-1} \in R$.

Let $\xi = -\sum_{i=1}^{k} r_{g_i}^{-1} g_i \in RG$ and consider

$$\begin{aligned} \varphi_\xi : R[{}^{G}\!/_{S}] &\longrightarrow RG \\ S &\longmapsto -\sum_{i=1}^{k} N_S r_{g_i}^{-1} g_i. \end{aligned}$$

Note that to define $\varphi_\xi(S)$ in this manner is exactly where we need both S and G to be finite.

Now for $\pi : RG \to R[{}^{G}\!/_{S}]$ as above,

$$\pi \circ \varphi_\xi(S) = -\sum_{i=1}^{k} r_{g_i}^{-1} N_S g_i S.$$

Consider the term $N_S g_i S$. This term is itself a sum with terms ranging through every element of the left S-orbit of $g_i S$. Moreoever since $Stab_S(g_i S) = S \cap g_i S g_i^{-1}$, the term $N_S g_i S$ contains every member of the orbit of $g_i S$ exactly r_{g_i} times. Thus

$$N_S g_i S = \sum_{gS \in Orb_S(g_i S)} r_{g_i} gS.$$

Returning to $\pi \circ \varphi_\xi(S)$ we see that

$$\begin{aligned}\pi \circ \varphi_\xi(S) &= -\sum_{i=1}^{k} r_{g_i}^{-1} \sum_{gS \in Orb_S(g_i S)} r_{g_i} gS \\ &= -\sum_{i=1}^{k} \sum_{gS \in Orb_S(g_i S)} gS\end{aligned}$$

Since we chose the g_i to be non-trivial double coset representatives, this sum includes every non-trivial left coset, $gS \neq S$. Thus

$$\pi \circ \varphi_\xi(S) = -\sum_{gS \neq S} gS.$$

Now consider $N_{G/_S} = \sum_{gS \in {}^{G}\!/_{S}} gS$, the norm element for ${}^{G}\!/_{S}$. Note that for all $g \in G$, $gN_{G/_S} = N_{G/_S}$ since G permutes the cosets. Then we can re-write

$$\pi \circ \varphi_\xi(S) = S - N_{G/_S}.$$

But then for $g \in G - S$

$$\begin{aligned}\pi \circ \varphi_\xi(S - gS) &= (1 - g)(S - N_{G/_S}) \\ &= S - N_{G/_S} - gS + gN_{G/_S} = S - gS.\end{aligned}$$

Recall that we have a splitting $\overline{\sigma}$ of σ if and only if $\pi \circ inc \circ \overline{\sigma} = inc_{G/S}$. Define $\overline{\sigma}$ to be the restriction of φ_ξ to $I[{}^{G}/_{S}]$. Then for any $g \in G - S$

$$\pi \circ inc \circ \overline{\sigma}(S - gS) = S - gS$$

and therefore $\overline{\sigma}$ is the RG-splitting we desired. □

Thus if $|S \cap gSg^{-1}|$ is both finite and a unit in R we can find an RG-splitting. Moreover, not only does such a splitting exist, but the proof of Theorem 5.1.13 gives an explicit description of such a splitting.

Proposition 5.1.14. *We can define RG-splittings of $IG \to I[{}^{G}/_{S}]$ by choosing double coset representatives $g_i \in G$ such that $S \neq Sg_iS$. Let $r_{g_i} = |S \cap g_iSg_i^{-1}|$ and let $\xi = -\sum_{i=1}^{k} r_{g_i}^{-1} g_i$. Define the map φ_ξ by*

$$\begin{aligned} \varphi_\xi : R[{}^{G}/_{S}] &\longrightarrow RG \\ gS &\longmapsto gN_S\xi. \end{aligned}$$

Restricting φ_ξ to $I[{}^{G}/_{S}]$ gives us the splitting we desire.

Interestingly, not every RS-splitting of the augmentation ideal is also an RG-splitting, even when RG-splittings exist.

Example 5.1.15. Consider $S_3 = \mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$ with subgroup $\mathbb{Z}_2$. Then we have presentations

$$\begin{aligned} G &= \langle x, y : x^2, y^3, xyx^{-1} = y^{-1} \rangle, \text{ and} \\ S &= \langle x : x^2 \rangle \end{aligned}$$

[Wei77, Example 4.1, p. 101-102]. In this case both $\mathrm{Sum}_0^{RS}(G,S)$ and $\mathrm{Sum}_0^{RG}(G,S)$ holds by Theorem 5.1.13 but not every RS-splitting is an RG-splitting.

By Lemma 2.2.10 we can build an aspherical CW-pair (Y, X) that realizes the group pair (G, S) and by construction has a single zero cell. Then $h_0(Y) = IG$ and $h_0(X) = IS$.

Recall by Example 2.1.8, (G, S) is a frobenius pair and so Theorem 5.1.13 applies. We can then use Proposition 5.1.9 to write down explicit RS-splittings for $n = 0$, which are determined by the structure of $\operatorname{coker}(\Phi_0) = I[{}^{\mathrm{G}}\!/_{\mathrm{S}}]$. The action of S on ${}^{\mathrm{G}}\!/_{\mathrm{S}} - \{S\} = \{yS, y^2S\}$ swaps yS to y^2S via multiplication by x, thus $I[{}^{\mathrm{G}}\!/_{\mathrm{S}}]$ is a free RS-module with basis $\{S - yS\}$. Then we have an RS-splitting of $\sigma : IG \to I[{}^{\mathrm{G}}\!/_{\mathrm{S}}]$ given by

$$\overline{\sigma}(S - yS) = s_1 - ys_2 + \xi$$

for any $s_1, s_2 \in S$ and any $\xi \in RG \otimes_S IS$. In particular, $\overline{\sigma}(S - yS) = 1 - y$ is an RS-splitting. Note that this particular RS-splitting is not also a RG-splitting since

$$\begin{aligned}\overline{\sigma}(y(S - yS)) &= \overline{\sigma}(-(S - yS) + (S - y^2S)) \\ &= \overline{\sigma}(-(S - yS) + x(S - yS)) \\ &= -(1 - y) + x(1 - y) \neq y(1 - y).\end{aligned}$$

To obtain an RG-splitting, we use Proposition 5.1.14 and so choose a double coset representative for the double coset SyS, say y where $r_y = 1$ since S is malnormal in G. Then $\varphi_{(-y)}$ is an RG-splitting of σ, with

$$\varphi_{(-y)}(S - yS) = (1 - y)N_S(-y) = x - y + y^2(1 - x).$$

Recall that if G is finite and $\mathrm{Sum}_n^{\mathbb{Z}G}(Y, X)$ holds for some n then by Corollary 4.1.3, S is malnormal in G and thus (G, S) is a Frobenius pair. By Theorem 5.1.13 and Lemma 2.2.11 this is also a sufficient condition for $\mathrm{Sum}_0^{\mathbb{Z}G}(Y, X)$ when G is finite. Moreover, since splittings go uphill by Theorem 3.2.2, this is also a sufficient condition for $\mathrm{Sum}_n^{\mathbb{Z}G}(Y, X)$ for all n.

Corollary 5.1.16. *Consider the group pair (G, S) where G is finite. Then $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ holds for some n if and only if (G, S) is a Frobenius pair if and only if $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ holds for all n.*

Recall that when G is a Frobenius group with complement S, there is a Frobenius kernel K with $G = K \rtimes S$ (see Theorem 2.1.10). Work by Gruenberg and Roggenkamp used the RG-module decomposition of the augmentation ideal $IG \cong \mathbb{Z}G \otimes_S IS \oplus IK$ to find an RG-module decomposition of the relation module, but their approach is indirect [GR75]. They appeal to Schanuel's Lemma to say that

$$h_1(Y) \oplus (\mathbb{Z}G \oplus \mathbb{Z}G \otimes_S \mathbb{Z}S) \cong \mathbb{Z}G \otimes_S h_1(X) \oplus \operatorname{coker}(\Phi_1) \oplus (\mathbb{Z}G \oplus \mathbb{Z}G).$$

They proceed to consider these as induced modules over a semi-local ring that contains $\mathbb{Z}G$, this and a few more conditions on the augmentation ideal decomposition allows them to cancel the group rings from both sides and then restrict scalars back down to $\mathbb{Z}G$. By Theorem 3.2.2, however, an explicit splitting at the augmentation ideal level gives rise to an explicit splitting at the relation module level and all higher dimensions.

When G is infinite, we cannot use the norm element of G to obtain a splitting as we did in Theorem 5.1.13. Thus we appeal to Dicks' theorem which characterizes when the augmentation ideal is split over RG in terms of the action of G on a tree, see [Dic81, Theorem A]. Dicks proves the following corollary that he credits to Dunwoody after Cohen.

Corollary 5.1.17. *[Dic81, after [Dun79] and [Coh72]] Suppose G is finitely generated over S where S is torsion free. Then the short exact sequence*

$$0 \longrightarrow RG \otimes_S IS \xrightarrow{\Phi_0} IG \longrightarrow I[G/S] \longrightarrow 0$$

is split exact as RG-modules if and only if S is a free factor of G.

5.2 Distinguishing between conditions

In this section we provide examples to distinguish between our conditions. First, While each of $\mathrm{PSum}_n^{RG}(Y, X)$, $\mathrm{Sum}_n^{RG}(Y, X)$, and $\mathrm{Sum}_n^{RS}(Y, X)$ imply the same at all higher dimensions, it is not the case that they imply the same at even one dimension lower. Moreover, projective skeletal homotopy splittings, skeletal homotopy splittings and weak skeletal homotopy splittings are distinct for a given n. In fact, for a given group pair, the existence of an appropriate CW-pair (Y, X) is sometimes impossible and thus even the conditions $\text{PSum}_n^{RG}(G, S)$, $\text{Sum}_n^{RG}(G, S)$, and $\text{Sum}_n^{RS}(G, S)$ are distinct.

First we provide a situation where we have a projective skeletal homotopy splitting at dimension $n = 1$ but not at dimension $n = 0$. For this we consider the general context of one-relator groups.

Let $G = \langle \mathbf{x}, y : w \rangle$ where $\mathbf{x}$ is a set and w is an element in the free group with basis $\mathbf{x} \cup y$ that **strictly involves** y in the sense that w does not lie in the free group S with basis $\mathbf{x}$. Then S embeds naturally in G by the Magnus Freiheitssatz, [MKS76, Theorem 4.10]. Further the two-complex Y modeled on the one-relator presentation for G is aspherical by the Lyndon Identity Theorem [Lyn50] (See [Bro82, p. 37] and the references cited there.)

Proposition 5.2.1. *Let $G = \langle \mathbf{x}, y : w \rangle$ be a one-relator group as above where w strictly involves y and let S be the free subgroup of G generated by $\mathbf{x}$. Then $\text{PSum}_1^{RG}(G, S)$ holds.*

Proof. Let X be the one-point union of circles indexed by $\mathbf{x}$ sitting inside the two-complex Y that models the one-relator presentation. Then (Y, X) is a pair of Eilenberg-Maclane complexes that realizes the group pair (G, S). Because S is free with basis $\mathbf{x}$, we have $h_1(X) = 0$ and so it is clear that $\text{coker}(\Phi_1) = h_2(Y)$ and hence that $\text{Sum}_1^{RG}(Y, X)$ holds. Since Y is aspherical and has just a single two-cell, we have the short exact sequence of RG-modules

$$0 = h_2(Y) \to RG \to h_1(Y) \to 0$$

This means that $\text{coker}(\Phi_1) \cong RG$ is a free RG-module. □

Example 5.2.2. Let $w = gyhy^{-1}$ where g and h are non-trivial in the free group on the set $\mathbf{x}$ and consider $G = \langle \mathbf{x}, y : w \rangle$. Then $S = \mathbb{Z} = \langle \mathbf{x} : - \rangle$ is a subgroup of G and $\text{PSum}_1^{RG}(G, S)$ holds by Proposition 5.2.1. But $1 \neq g = yh^{-1}y^{-1} \in S \cap ySy^{-1}$ and so

$S \cap ySy^{-1}$ is not finite. Therefore by Theorem 5.1.7 and Lemma 2.2.11, $\mathrm{Sum}_0^{RS}(G,S)$ does not hold.

We know $\mathrm{PSum}_n^{RG}(G,S)$ implies $\mathrm{Sum}_n^{RG}(G,S)$, the following example proves that the converse is, in general, false.

Example 5.2.3. Let (G,S) be a Frobenius pair, so that S is a nontrivial, proper malnormal subgroup of the finite group G. Then $\mathrm{Sum}_0^{\mathbb{Z}G}(G,S)$ holds (by Theorem 5.1.13, above, due to Dicks). Thus $\mathrm{Sum}_n^{\mathbb{Z}G}(G,S)$ holds for all n by Theorem 3.2.2. However, $\mathrm{PSum}_n^{\mathbb{Z}G}(G,S)$ does not hold for any n since S is a proper subgroup of the finite group G. See Corollary 4.1.5.

Last, we know that $\mathrm{Sum}_n^{RG}(G,S)$ implies $\mathrm{Sum}_n^{RS}(G,S)$ but we will show the converse is false when $n=0$. First we prove that for a free group, a word that is not a proper power in that free group generates a subgroup that is malnormal.

Lemma 5.2.4. *Let $G = F(x_1, ..., x_k)$ be the free group on n generators and let $S = \langle w \rangle$ be a nontrivial cyclic subgroup generated by an element $w \in G$. Then S is malnormal in G if and only if w is not a proper power in G.*

Proof. To prove one direction suppose that w is a proper power, that is, $w = u^k$ for some element $u \notin S$ and integer k. Then $uwu^{-1} = w \in S \cap uSu^{-1}$ and so S is not malnormal in G.

For the other direction, suppose $1 \neq w \in G$ is not a proper power. Note that if S is malnormal, then its conjugate hSh^{-1} is also malnormal. Recall also that a word is conjugate to its cyclic reduction. Thus without loss of generality we can assume that w is cyclically reduced. We want to show that S is malnormal. Let $g \in G$ and consider $w^m = gw^ng^{-1} \in S \cap gSg^{-1}$ where $m, n \neq 0$. We will show $g \in S$.

Now both w^m and w^n are cyclically reduced and conjugate in the free group G, so w^m is a cyclic permutation of w^n (see [MKS76, Theorem 1.3]). Since these words have the same length it follows that $m = n$. Thus we have $w^m g = g w^m$, which in turn implies that $wg = gw$ (see [MKS76, Section 1.4, Problem 4]). Now we conclude that w and g are powers of a common element $u \in G$ (see [MKS76, Section 1.4, Problem 6]). But since w is not expressible as a proper power, it follows that g is a power of w, that is, $g \in \langle w \rangle = S$. □

Corollary 5.2.5. *Let $G = F(x_1, ..., x_k)$ be the free group on n generators and let $S = \langle w \rangle$ be a nontrivial cyclic subgroup generated by an element $w \in G$. Then $\mathrm{Sum}_0^{RS}(G, S)$ holds if and only if w is not a proper power in G.*

Proof. Apply Lemma 2.2.11, Lemma 5.2.4, and Theorem 5.1.7. □

Example 5.2.6. Let G be the free group on two generators $G = F(a, b)$ and let $S = \langle w \rangle$ be the subgroup generated by the commutator element $w = aba^{-1}b^{-1}$. Then $\mathrm{Sum}_0^{RS}(G, S)$ holds since w is not a proper power. Note, however, that w is not an element of a basis for G, since if it were, it would also be an element of a basis in the abelianization. In the abelianization, a free abelian group of rank 2, w is trivial. Thus S cannot be a free factor of G and therefore by Corollary 5.1.17, $\mathrm{Sum}_0^{RG}(G, S)$ does not hold.

In general, whenever G is a free group and S is a cyclic subgroup generated by an element of G that is not *primitive* (i.e. not in a set of free generators for that group) and is not a proper power in G, then $\mathrm{Sum}_0^{RS}(G, S)$ holds but $\mathrm{Sum}_0^{RG}(G, S)$ does not.

6 CONCLUSIONS AND FURTHER RESEARCH

6.1 Conclusions

Multiple studies have considered various conditions on kernels of projective $\mathbb{Z}G$-resolutions. Many of these studies, including [Lyn50, GR75, Dic81, HS81, BP92], can be framed in terms of considering when kernels of a projective $\mathbb{Z}G$-resolution contain an induced kernel of a $\mathbb{Z}S$-resolution as a $\mathbb{Z}G$-summand. Such a decomposition gives rise to interesting cohomological information and ultimately group theoretic information via Theorems of Serre and Howie and Schneebeli (see Corollary 2.1.19). Interestingly, investigations into decompositions of augmentation ideals and minimal relation modules often found a connection to Frobenius pairs which can be seen in this context as a direct consequence of Howie and Schneebeli's Theorem when G is finite. Three main questions arise in these considerations: when such decompositions exist, how they are related, and what implications they have for the group pair (G, S).

Since the equivariant chain complex of a CW complex of type $K(G, 1)$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, it is a natural place to look for topological connections. Any group pair (G, S) can be realized by a pair of connected, aspherical CW-complexes (see Lemma 2.2.10) and so no generality is lost with this approach. Moreover, the induced kernels of the associated $\mathbb{Z}S$-resolution always embed as submodules of the kernels of the associated $\mathbb{Z}G$-resolution (see Theorem 2.2.8) a distinctive of the topology that is not guaranteed by considering kernels of projective resolutions in general. Determining when such skeletal homotopy module decompositions exist is equivalent to detecting when the corresponding short exact sequence is split exact.

In particular, projective skeletal homotopy splittings and strong skeletal homotopy splittings for a pair of connected aspherical CW complexes (Y, X) that realize the group pair (G, S) give rise to decompositions on cohomology and thus Theorems of Serre and

Howie and Schneebeli apply. Moreover, interesting group theoretic consequences can also be obtained for weak skeletal homotopy splittings, when the induced kernel is merely a $\mathbb{Z}S$-module summand rather than a $\mathbb{Z}G$-module summand, a relatively unexplored question in previous research. In summary, for a finite subgroup S of G, a projective skeletal homotopy splitting implies every finite subgroup of G is contained in a unique conjugate of S, a skeletal homotopy splitting implies S is malnormal in G, and a weak skeletal homotopy splitting implies that S is self-normalizing in G.

An interesting application of the condition $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ involves determining the order of the shift automorphism of a cyclically presented group. The order of the shift automorphism for a large family of cyclically presented groups has been related by [CRS05] to the possibility of being the fundamental group of a hyperbolic three-orbifold of finite volume. Thus determining when $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ holds for a certain split extension via work of [BP92] gives a family of cyclically presented groups that cannot be the fundamental group of any hyperbolic three-orbifold of finite volume.

There are interesting relationships between the conditions themselves. In particular, $\mathrm{PSum}_n^{RG}(G, S)$, $\mathrm{Sum}_n^{RG}(G, S)$ and $\mathrm{Sum}_n^{RS}(G, S)$ each go uphill but do not always step down a dimension (Theorem 3.2.2 and Example 5.2.2). Thus the smallest integer n for which each of these conditions hold is a non-trivial invariant of the group pair (G, S). Moreover, while each stronger condition implies the weaker condition, the converses do not hold in general (Example 5.2.3 and 5.2.6), thus each is of distinct interest. However, considering 'projective' weak skeletal homotopy splittings is equivalent to considering weak skeletal homotopy splittings due to the commutative diagrams arising from the topology (see Theorem 3.1.2).

Skeletal homotopy splittings are completely determined by spaces with a single zero-cell and thus determining splittings when $n = 0$ reduces to sufficient conditions for splittings of augmentation ideals. Since these splittings lift to higher dimensions, these are also sufficient conditions for splittings at higher dimensions. In particular, for a Frobenius pair (G, S), $\mathrm{Sum}_n^{RG}(G, S)$ holds for all n.

6.2 Further Research

One important question is determining when skeletal homotopy splittings exist for a given group pair. The results in [Dic81] give sufficient conditions for RS-splittings and RG-splittings when $n = 0$. Determining sufficient conditions for skeletal homotopy splittings for $n > 0$ remains an open question, however. One result in this direction is [BP92, Theorem 1.2] which gives combinatorial conditions on a relative presentation for G to detect $\mathrm{Sum}_1^{\mathbb{Z}G}(G, S)$. Further exploration of decompositions of relation modules in the finite case could lead to sufficient conditions for skeletal homotopy splittings for $n = 1$ as well.

Interestingly, Lemma 3.2.1 gives an explicit way to lift splittings given a splitting a dimension lower. For Frobenius pairs, this could be used to determine possible generators for the relation module and thus possible relators for a presentation given a sub-presentation. Similarly, if a decomposition of the relation module were known, this could be used to determine possible generators for a second homotopy group of a 2-complex relative to a sub-complex, and the same for higher homotopy groups.

Another interesting consideration in this research is the relationship between RG and RS-splittings. In particular, RG and RS-splittings are equivalent for finite G (but not for infinite G) when $n = 0$. The natural next step is to determine whether RG and RS-splittings are distinct for finite G when $n > 0$. A necessary condition for a $\mathbb{Z}S$-splitting that is not a $\mathbb{Z}G$-splitting is a self-normalizing subgroup that is not malnormal by Corollary 4.1.3, as in Example 2.1.9. But $\mathrm{Sum}_n^{\mathbb{Z}S}(G, S)$ is also equivalent to the projectivity of $\mathrm{coker}(\Phi_n)$ and thus results of [Swa59] provide additional cardinality conditions at the various levels due to rank considerations. One place to look for $\mathbb{Z}S$-splittings that are not $\mathbb{Z}G$-splittings at the level of relation modules is special 2-frobenius groups as defined in [GR82]. Finding such examples would hopefully lead to a better understanding of the differences between RG and RS-splittings.

As mentioned, the smallest integer n for which each of these conditions hold is an invariant of the group pair (G, S). Another avenue for further inquiry is how these compare to other invariants arising from projective resolutions over a group ring. Also, $\mathrm{PSum}_n^{\mathbb{Z}G}(G, S)$ and $\mathrm{Sum}_n^{\mathbb{Z}G}(G, S)$ are detectable in the higher cohomology of G and so it would be interesting to learn whether $\mathrm{Sum}_n^{\mathbb{Z}S}(G, S)$ is also detectable in the cohomology of G or some other homological invariant.

BIBLIOGRAPHY

ARS84. Kenneth I. Appel, John G. Ratcliffe, and Paul E. Schupp (eds.), *Contributions to group theory*, Contemporary Mathematics, vol. 33, American Mathematical Society, Providence, R.I., 1984.

BBP97. Y. G. Baik, W. A. Bogley, and S. J. Pride, *On the asphericity of length four relative group presentations*, Internat. J. Algebra Comput. **7** (1997), no. 3, 277–312.

BD93. W.A. Bogley and M. N. Dyer, *Splittings of homotopy modules and normalizers in groups*, Preprint (1993), 1–25.

Ben91. David J. Benson, *Representations and cohomology*, Cambridge University Press, 1991.

BP92. W. A. Bogley and S. J. Pride, *Aspherical relative presentations*, Proc. Edinburgh Math. Soc. (2) **35** (1992), no. 1, 1–39.

BP93. William A. Bogley and Steven J. Pride, *Two-dimensional homotopy and combinatorial group theory*, London Mathematical Society Lecture Note Series, vol. 197, ch. Calculating Generators of Π_2, pp. 157–188, Cambridge University Press, 1993.

Bro82. Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982.

Bru74. A. M. Brunner, *The determination of Fibonacci groups*, Bull. Austral. Math. Soc. **11** (1974), 11–14.

Bur01. W.H. Burnside, *On some properties of groups of odd order*, Proc. London Math. Soc. **33** (1901), 162–185.

CHK98. A. Cavicchioli, F. Hegenbarth, and A. C. Kim, *A geometric study of Sieradski groups*, Algebra Colloq. **5** (1998), no. 2, 203–217.

Coh72. Daniel E. Cohen, *Groups of cohomological dimension one*, vol. 245, Springer-Verlag, 1972.

CRS05. Alberto Cavicchioli, Dušan Repovš, and Fulvia Spaggiari, *Families of group presentations related to topology*, J. Algebra **286** (2005), no. 1, 41–56.

CWLF67. J. H. Conway, J. A. Wenzel, R. C. Lyndon, and Harley Flanders, *Problems and Solutions: Solutions of Advanced Problems: 5327*, Amer. Math. Monthly **74** (1967), no. 1, 91–93.

DD89. Warren Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics, vol. 17, Cambridge University Press, Cambridge, 1989.

DF99. David S. Dummit and Richard M. Foote, *Abstract algebra*, second ed., Prentice Hall, 1999.

Dic81. Warren Dicks, *On splitting augmentation ideals*, Proc. Amer. Math. Soc. **83** (1981), no. 2, 221–227.

DM96. John D. Dixon and Brian Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996.

Dun79. M. J. Dunwoody, *Recognizing free factors*, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge, 1979, pp. 245–249.

Edj94. Martin Edjvet, *On the asphericity of one-relator relative presentations*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 4, 713–728.

Edj03. ———, *On irreducible cyclic presentations*, J. Group Theory **6** (2003), no. 2, 261–270.

GR75. K. W. Gruenberg and K. W. Roggenkamp, *Decomposition of the augmentation ideal and of the relation modules of a finite group*, Proc. London Math. Soc. (3) **31** (1975), no. 2, 149–166.

GR82. ———, *A correction: "Decomposition of the augmentation ideal and of the relation modules of a finite group"*, Proc. London Math. Soc. (3) **45** (1982), no. 1, 89–96.

GR76. ———, *Decomposition of the relation modules of a finite group*, J. London Math. Soc. (2) **12** (1975/76), no. 3, 262–266.

Gru76. Karl W. Gruenberg, *Relation modules of finite groups*, American Mathematical Society, Providence, R.I., 1976, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 25.

Hig51. Graham Higman, *A finitely generated infinite simple group*, J. London Math. Soc. **26** (1951), 61–64.

HKM98. H. Helling, A. C. Kim, and J. L. Mennicke, *A geometric study of Fibonacci groups*, J. Lie Theory **8** (1998), no. 1, 1–23.

HM01. J. Howie and V. Metaftsis, *On the asphericity of length five relative group presentations*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 173–194.

Hol95. Derek F. Holt, *An alternative proof that the Fibonacci group $F(2,9)$ is infinite*, Experiment. Math. **4** (1995), no. 2, 97–100.

HRS79. George Havas, J. S. Richardson, and Leon S. Sterling, *The last of the Fibonacci groups*, Proc. Roy. Soc. Edinburgh Sect. A **83** (1979), no. 3-4, 199–203.

HS81. James Howie and Hans Rudolf Schneebeli, *Permutation modules and projective resolutions*, Comment. Math. Helv. **56** (1981), no. 3, 447–464.

Hue79. Johannes Huebschmann, *Cohomology theory of aspherical groups and of small cancellation groups*, J. Pure Appl. Algebra **14** (1979), no. 2, 137–143.

Hup67. B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

JWW74. D. L. Johnson, J. W. Wamsley, and D. Wright, *The Fibonacci groups*, Proc. London Math. Soc. (3) **29** (1974), 577–592.

Kim08a. Seong Kun Kim, *On the asphericity of certain relative presentations over torsion-free groups*, Internat. J. Algebra Comput. **18** (2008), no. 6, 979–987.

Kim08b. ______, *On the asphericity of length-6 relative presentations with torsion-free coefficients*, Proc. Edinb. Math. Soc. (2) **51** (2008), no. 1, 201–214.

KvdW96. Eric B. Kuisch and Robert W. van der Waall, *Modular Frobenius groups*, Manuscripta Math. **90** (1996), no. 4, 403–427.

LS77. Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

Lyn. Roger C. Lyndon, *On a family of infinite groups introduced by conway*, 1974.

Lyn50. ______, *Cohomology theory of groups with a single defining relation*, Annals of Mathematics. Second Series **52** (1950), 650–665.

Mac95. C. Maclachlan, *Generalisations of Fibonacci numbers, groups and manifolds*, Combinatorial and geometric group theory (Edinburgh, 1993), London Math. Soc. Lecture Note Ser., vol. 204, Cambridge Univ. Press, Cambridge, 1995, pp. 233–238.

MKS76. Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory*, revised ed., Dover Publications Inc., New York, 1976, Presentations of groups in terms of generators and relations.

New90. M. F. Newman, *Proving a group infinite*, Arch. Math. (Basel) **54** (1990), no. 3, 209–211.

Rob82. Derek John Scott Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1982.

Rot02. Joseph J. Rotman, *Advanced modern algebra*, Prentice Hall Inc., Upper Saddle River, NJ, 2002.

Sie86. Allan J. Sieradski, *Combinatorial squashings, 3-manifolds, and the third homology of groups*, Invent. Math. **84** (1986), no. 1, 121–139.

Sie92. ______, *An introduction to topology and homotopy*, The Prindle, Weber & Schmidt Series in Advanced Mathematics, PWS-KENT Publishing Co., Boston, MA, 1992.

Sie93. Allan J. Sieradski, *Two-dimensional homotopy and combinatorial group theory*, London Mathematical Society Lecture Note Series, vol. 197, ch. Algebraic Topology for Two Dimensional Complexes, pp. 51–96, Cambridge University Press, 1993.

Swa59. Richard Swan, *Projective modules over finite groups*, Bull. Amer. Math. Soc. **65** (1959), 465–467.

Tho59. John Thompson, *Finite groups with fixed-point-free automorphisms of prime order*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 578–581.

Tho91. Richard M. Thomas, *On a question of Kim concerning certain group presentations*, Bull. Korean Math. Soc. **28** (1991), no. 2, 219–224.

Wei77. Michael Weinstein, *Examples of groups*, Polygonal Publishing House, Passaic, N.J., 1977.