AN ABSTRACT OF THE THESIS OF

RICHARD JOSEPH SHERMAN for the DOCTOR OF PHILOSOPHY
(Name) (Degree)

in Electrical and Electronics Engineering presented on April 25, 1969
(Major) (Date)

Title: ON OUTPUT STATISTICS OF NONLINEAR DEVICES: 1) THIRD AND HIGHER ORDER INFORMATION, 2) QUADRI-PHASE CARRIER RECONSTRUCTION, 3) ANALYSIS OF POINT PROCESSES

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This thesis deals with the output statistics of nonlinear devices. It develops the classical output autocorrelation function in two dimensions and extends the theory to three and four dimensions. Closed form solutions for the output correlation function in two and three dimensions are given for the full- and half-wave rectifier families while a series solution with numerical results is presented in the case of the smooth limiter in four dimensions.

A nonlinear device used to provide a coherent reference signal in a digital phase modulation system is analyzed and results presented in terms of average error rate performance degradation.

Finally, the problem of determining the average number of times per unit of time that a process consisting of a sinusoidal signal plus Gaussian noise transgresses a fixed level or has summits above or below a fixed level is investigated.

On Output Statistics of Nonlinear Devices: 1) Third and Higher Order Information, 2) Quadriphase Carrier Reconstruction, 3) Analysis of Point Processes

by

Richard Joseph Sherman

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1969

APPROVED:

Redacted for Privacy

Professor of Electrical and Electronics Engineering

in charge of major

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Date thesis is presented April 25, 1969

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ACKNOWLEDGMENTS

I would very much like to thank all those who helped me in my work on this thesis: Dr. W.M. Stone for his guidance and direction in the research, Professor L. J. Weber for his encouragement and friendly understanding, and Mr. Fred Dayton for his assistance with the digital computations. Without their help the task would have been impossible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ON OUTPUT STATISTICS OF NONLINEAR DEVICES: 1) THIRD AND HIGHER ORDER INFORMATION, 2) QUADRIPHASE CARRIER RECONSTRUCTION, 3) ANALYSIS OF POINT PROCESSES

I. INTRODUCTION

Although it is an old problem there is still interest in the output statistics of nonlinear devices. Much has been published on the several methods of determining the autocorrelation function of a random process following a nonlinear transformation, but there has been little on the higher order correlation functions. The method of Price [7, 15, 37, 41, 42] in this connection has found great acceptance but often leads to intractable integrals. Certain third and fourth order integrals have received exhaustive treatment by Kamat [25], Gupta [18], and Cheng [10]. Tukey [49] and Hasselmann, Munk, and MacDonald [21] have recently emphasized that while the spectrum (the Fourier transform of the second order correlation function) is useful for linear problems it provides insufficient information in the nonlinear case. An extension to the bispectrum (multiple Fourier transform associated with the third order correlation function) is clearly called for. The bispectrum and yet higher order spectra--called polyspectra by Tukey--represent the spectral decomposition of the third and higher order moments of a stationary, multivariate stochastic process.

Some of the better-known and successful uses of the polyspectra, the bispectrum in particular, in the study of nonlinear phenomena have occurred in the oceanographic field. A recent report by Carpenter [9] has emphasized the bispectrum in the analysis of weakly nonlinear quadratic systems.

A method for determining the n^{th} order correlation function for the output of a nonlinear device with Gaussian input will be described below and several specific examples will be worked out. The general method will also be applied to several aspects of noise analysis, originally propounded by Rice [43, 44] and refined by others. The third chapter of the thesis will deal with a practical problem in an advanced communication system-quadriphase signaling-in which nonlinear analysis is required.

II. N^{th} ORDER CORRELATION FUNCTIONS OF NONLINEAR DEVICES

2.0 The Nonlinear System Model

The nonlinear system we will consider in this chapter consists of a device whose input $X(t)$ is a stationary, continuous time, ergodic Gaussian process and whose output process $Y(t)$ results from the nonlinear transfer function, $h(X(t))$. The output correlation function is defined as

$$E[Y(t)Y(t+\tau)] = R_{Y,Y}(\tau) \tag{2.1}$$

where $E[\cdot]$ is the expected value operator and τ is an arbitrary time shift parameter. Here we are taking the time average of the quantity in brackets which, under the assumption of ergodicity, is identical with the ensemble average. This average may also be written in the form

$$R_{Y,Y}(\tau) = \int\int_{-\infty}^{\infty} h(x_1)h(x_2)p_{X_1,X_2}(x_1, x_2)dx_1 dx_2 \tag{2.2}$$

where by x_i we mean $x(t_i)$, $i = 1, 2$, and $t_2 - t_1 = \tau$. The $p_{X_1,X_2}(x_1, x_2)$ is the joint Gaussian density function associated with the input process $X(t)$.

We will use the notation of Parzen [39] where upper case subscripts refer to the appropriate random variable while the lower case arguments of the function refer to observed values of the corresponding random variables.

Assume that the Gaussian variables are standard

$$E[X(t_i)] = 0,$$
$$E[X^2(t_i)] = 1 = R_{X,X}(0) \qquad (2.3)$$

so

$$p_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(x_1^2+x_2^2-2\rho x_1 x_2)\right] \qquad (2.4)$$

and $\rho = E[X_1X_2]$ is defined as the normalized correlation function of the input variables. That is, $|\rho| \leq 1$.

2.1 An Example: The Half-Wave Rectifier

A half-wave rectifier is defined by

$$h(X) = \begin{cases} ax^m, & x \geq 0 \\ 0, & x < 0; \quad m+1 > 0 \end{cases} \qquad (2.5)$$

It is convenient here to introduce a function, say

$$G(\rho; m) = \int_0^\infty \int_0^\infty (x_1 x_2)^m p_{X_1, X_2}(x_1, x_2) dx_1 dx_2 \tag{2.6}$$

A change to polar coordinates

$$x_1 = r \cos\theta, \quad x_2 = r \sin\theta \tag{2.7}$$

is quite successful at this level as

$$\begin{aligned} G(\rho; m) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty r\,dr \int_0^{\pi/2} d\theta \left(\frac{r^2 \sin 2\theta}{2}\right)^m e^{-\frac{r^2(1-\rho\sin(2\theta))}{2(1-\rho^2)}} \\ &= \frac{1}{2\pi}\Gamma(m+1)(1-\rho^2)^{m+\frac{1}{2}} \int_0^{\pi/2} d\theta \left[\frac{\sin 2\theta}{1-\rho\sin(2\theta)}\right]^{m+1} \end{aligned} \tag{2.8}$$

The change of variable, $2\theta = \frac{\pi}{2} - \varphi$, leads to

$$G(\rho; m) = \frac{\Gamma(m+1)}{2\pi} (1-\rho^2)^{m+\frac{1}{2}} \int_0^{\pi/2} \frac{\cos^m \varphi}{(1-\rho\cos\varphi)^{m+1}} d\varphi \tag{2.9}$$

From the identity

$$\int_0^{\pi/2} \frac{d\varphi}{1-\rho\cos\varphi} = \frac{\cos^{-1}(-\rho)}{\sqrt{1-\rho^2}} = \frac{\pi/2 + \sin^{-1}\rho}{\sqrt{1-\rho^2}}, \quad |\rho| < 1 \tag{2.10}$$

it is clear that if m is restricted to integer values greater than zero then

$$G(\rho; m) = \frac{m!}{2\pi}(1-\rho^2)^{m+\frac{1}{2}} \frac{1}{m!} \frac{d^m}{d\rho^m}\left[\frac{\cos^{-1}(-\rho)}{\sqrt{1-\rho^2}}\right] \tag{2.11}$$

Some special cases are:

$$G(\rho; 0) = \frac{1}{2\pi}\cos^{-1}(-\rho) = \frac{1}{2\pi}\left[\frac{\pi}{2} + \sin^{-1}\rho\right]$$

$$G(\rho; 1) = \frac{1}{2\pi}\left[\rho\cos^{-1}(-\rho) + \sqrt{1-\rho^2}\right] \tag{2.12}$$

$$G(\rho; 2) = \frac{1}{2\pi}\left[(1+2\rho^2)\cos^{-1}(-\rho) + 3\rho\sqrt{1-\rho^2}\right], \quad |\rho| < 1$$

The autocorrelation functions of the outputs of the several half-wave rectifiers are now defined in terms of m as

$$R_{Y,Y}(\rho; m) = a^2 G(\rho; m), \quad m = 0, 1, 2. \tag{2.13}$$

2.2 Bilinear Series Expansion of Density Functions

It is advantageous to reduce a joint density function of general order N to a series of terms, each term a product of N functions which depend only on x_1, on $x_2, \ldots,$ on x_N. The starting point is the moment matrix

$$\mathcal{M} = [E(x_r x_s)], \quad r, s = 1, 2, \ldots, N \tag{2.14}$$

If there are N standard Gaussian variables present then

$$\mathfrak{M} = [\rho_{rs}], \quad \rho_{rr} = 1, \quad r, s = 1, 2, \ldots, N \tag{2.15}$$

and the moment generating function--sometimes known as the characteristic function--is

$$M_{X_1, X_2, \ldots, X_N}(\alpha_1, \alpha_2, \ldots, \alpha_N) = e^{-\frac{1}{2}\alpha' \mathfrak{M} \alpha} \tag{2.16}$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}$$

and α' is the transpose of the matrix α.

The corresponding joint density function--if the variables are Gaussian--is

$$p_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N) = \frac{1}{(2\pi)^{N/2} |\mathfrak{M}|^{1/2}} e^{-\frac{1}{2}x' \mathfrak{M}^{-1} x} \tag{2.17}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

It is easy to verify that the density function of (2.4) is derived from

$$\mathfrak{M} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad \mathfrak{M}^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \tag{2.18}$$

The probability density function and characteristic function are Fourier Transform pairs:

$$p_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \left(\frac{1}{2\pi}\right)^{N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i\alpha' x} M_{X_1,\ldots,X_N}(\alpha_1,\ldots,\alpha_N) \times d\alpha_1 \ldots d\alpha_N \tag{2.19}$$

$$M_{X_1,\ldots,X_N}(\alpha_1,\ldots,\alpha_N) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{+ix'\alpha} p_{X_1,\ldots,X_N}(x_1,\ldots,x_N) d_{X_1} \ldots d_{X_N}$$

At this point consider the set of functions

$$\varphi^{(0)}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\varphi^{(-1)}(x) = \int_0^x \varphi^{(0)}(y)dy \tag{2.20}$$

$$\varphi^{(n)}(x) = \frac{d^n}{dx^n} \varphi^{(0)}(x), \quad n = 1, 2, \ldots, \quad -\infty < x < \infty$$

The fundamental Fourier Transform pairs of interest for the standard Gaussian variable case are

$$M_x(\alpha) = e^{-\alpha^2/2} \tag{2.21}$$

and

$$p_X(x) = \varphi^{(0)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x - \alpha^2/2} d\alpha \tag{2.22}$$

Then

$$\varphi^{(n)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-\alpha)^n e^{-i\alpha x - \alpha^2/2} \, d\alpha \tag{2.23}$$

For the case of two standard Gaussian variates

$$p_{X_1, X_2}(x_1, x_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha_1 x_1 + \alpha_2 x_2) - \frac{\alpha_1^2 + \alpha_2^2 + 2\rho\alpha_1\alpha_2}{2}} \, d\alpha_1 d\alpha_2$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha_1 x_1 + \alpha_2 x_2) - \frac{\alpha_1^2 + \alpha_2^2}{2}} \sum_{n=0}^{\infty} \frac{(-\rho\alpha_1\alpha_2)^n}{n!} \, d\alpha_1 d\alpha_2$$

$$= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_1)\varphi^{(n)}(x_2)\rho^n}{n!}, \qquad |\rho| < 1 \tag{2.24}$$

and each term has the property, "function of x_1 times function of x_2". The term "error function" is reserved for $\varphi^{(-1)}(x)$; it and its first twenty derivatives have been well tabulated by the Harvard Computation Laboratory [20]. In the implementation of this formula several important integrals occur. The principle idea is that

$$\varphi^{(2n+1)}(0) = 0$$

and (2.25)

$$\varphi^{(2n)}(0) = \frac{(-1)^n (2n)!}{\sqrt{2\pi}\, 2^n n!}, \qquad n = 0, 1, 2, \ldots$$

Table 2. 1 lists these integrals.

Table 2. 1. Some special integrals.

$$\int_0^\infty \varphi^{(n)}(x)dx = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{(-1)^{n-1/2}(n-1)!}{\sqrt{2\pi}\, 2^{(n-1)/2}\, (\frac{n-1}{2})!}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$\int_0^\infty x\varphi^{(n)}(x)dx = \begin{cases} \frac{1}{\sqrt{2\pi}}, & n = 0 \\ -\frac{1}{2}, & n = 1 \\ \frac{(-1)^{(n-2)/2}(n-2)!}{\sqrt{2\pi}\, 2^{(n-2)/2} (\frac{n-2}{2})!}, & n = 2, 4, 6, \ldots \\ 0, & n = 3, 5, 7, \ldots \end{cases}$$

$$\int_0^\infty x^2\varphi^{(n)}(x)dx = \begin{cases} \frac{1}{2}, & n = 0 \\ -\sqrt{\frac{2}{\pi}}, & n = 1 \\ 1, & n = 2 \\ \frac{\sqrt{\frac{2}{\pi}}\,(-1)^{(n-1)/2}(n-3)!}{2^{(n-3)/2} (\frac{n-3}{2})!}, & n = 3, 5, 7, \ldots \\ 0, & n = 4, 6, 8 \end{cases}$$

$$\int_{-\infty}^\infty x^2\varphi^{(n)}(x)dx = \begin{cases} 1, & n = 0 \\ 0, & n = 1 \\ 2, & n = 2 \\ 0, & n \geq 3 \end{cases}$$

This means that if $h(X)$ is a polynomial defined over the half line then $\int_0^{\infty} h(x)\varphi^{(n)}(x)dx$ is easily evaluated and $\int_{-\infty}^{\infty} h(x)\varphi^{(n)}(x)dx$ will depend on the first integral in some simple fashion. Note that, at least formally, for the standard Gaussian variates

$$\frac{\partial^{m+n}}{\partial \alpha_1^m \partial \alpha_2^n} M_{X_1,X_2}(\alpha_1, \alpha_2)$$

$$= (-i)^{m+n} \iint_{-\infty}^{\infty} x_1^m x_2^n e^{-i(\alpha_1 x_1 + \alpha_2 x_2)} p_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

$$= (-i)^{m+n} E[X_1^m X_2^n e^{-i(\alpha_1 X_1 + \alpha_2 X_2)}] \tag{2. 25}$$

Setting $m=n=2$ and evaluating at $\alpha_1=\alpha_2=0$ we obtain

$$\frac{\partial^4}{\partial x_1^2 \partial x_2^2} [M_{X_1,X_2}(\alpha_1, \alpha_2)] \Big|_{\alpha_1=\alpha_2=0} = E[X_1^2 X_2^2] \tag{2. 26}$$

which is the correlation function for a full-wave, second order rectifier (within a constant scale factor). That is, such a rectifier may be defined as

$$h(x_i) = a|x_i|^m, \quad m \text{ real} \tag{2. 27}$$

and it is expedient to define a second function

$$H(\rho; m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x_1 x_2|^m p_{X_1,X_2}(x_1, x_2)dx_1 dx_2 \tag{2. 28}$$

So, in this case,

$$R_{Y,Y}(\rho; m) = a^2 H(\rho; m). \tag{2. 29}$$

From (2. 16) and (2. 26) we have that

$$H(\rho; 2) = 1 + 2\rho^2 \tag{2. 30}$$

It is worthwhile to check the forms of $G(\rho; m)$ for $m = 0, 1, 2$ by implementation of the bilinear expansion for $p_{X_1,X_2}(x_1, x_2)$. Begin with the series

$$\begin{aligned} (1-\rho^2)^{-1/2} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\rho^{2n} \\ (1-\rho^2)^{1/2} &= 1 - \sum_{n=0}^{\infty} \frac{(2n)!\rho^{2n+2}}{2^{2n+1}n!(n+1)!} . \end{aligned} \tag{2. 31}$$

Also

$$\int_0^{\rho} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}\rho = \sum_{n=0}^{\infty} \frac{(2n)!\rho^{2n+1}}{2^{2n}(n!)^2(2n+1)} , \quad |\rho| < 1 \tag{2. 32}$$

and

$$\int_0^{\rho} \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{2}[\sin^{-1}\rho - \rho\sqrt{1-\rho^2}] = \sum_{n=0}^{\infty} \frac{(2n)!\rho^{2n+3}}{2^{2n}(n!)^2(2n+3)}$$

Now we write

$$G(\rho;\,0) = \int_0^\infty\int_0^\infty \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_1)\varphi^{(n)}(x_2)\rho^n}{n!}\,dx_1\,dx_2$$

$$= \frac{1}{4} + \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{(2n)!\,\rho^{2n+1}}{2^{2n}(n!)^2(2n+1)}$$

$$= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho = \frac{1}{2\pi}\cos^{-1}(-\rho), \qquad |\rho| \leq 1 \qquad (2.33)$$

Next

$$G(\rho;\,1) = \int_0^\infty\int_0^\infty x_1 x_2 \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_1)\varphi^{(n)}(x_2)\rho^n}{n!}$$

$$= \frac{1}{2\pi} + \frac{\rho}{4} + \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\left[\frac{1}{2n+1} - \frac{1}{2n+2}\right]\rho^{2n+2}$$

$$= \frac{\rho}{4} + \frac{1}{2\pi}\left[1 - \sum_{n=0}^{\infty} \frac{(2n)!\,\rho^{2n+1}}{2^{2n+1}(n!)^2(n+1)} + \sum_{n=0}^{\infty} \frac{(2n)!\,\rho^{2n+2}}{2^{2n}(n!)^2(2n+1)}\right]$$

$$= \frac{1}{2\pi}\left[\sqrt{1-\rho^2} + \rho\left(\frac{\pi}{2} + \sin^{-1}\rho\right)\right]; \qquad (2.34)$$

and finally,

$$G(\rho;\,2) = \frac{1}{4} + \frac{2\rho}{\pi} + \frac{\rho^2}{2} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\left[\frac{1}{2n+1} - \frac{1}{n+1} + \frac{1}{2n+3}\right]\rho^{2n+3}$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{2}(1+2\rho^2)\right] + \frac{2}{\pi}\left[\rho - \rho\sum_{n=0}^{\infty} \frac{(2n)!\,\rho^{2n+2}}{2^{2n+1}\,n!\,(n+1)!}\right] +$$

$$+ \frac{\rho^2}{\pi} \sum_{n=0}^{\infty} \frac{(2n)!\rho^{2n+1}}{2^{2n} n!(2n+1)} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{\rho^{2n+3}}{2n+3}$$

$$= \frac{1}{2\pi} [(1+2\rho^2)\frac{\pi}{2} + 4\rho\sqrt{1-\rho^2} + 2\rho^2 \sin^{-1}\rho + \sin^{-1}\rho - \rho\sqrt{1-\rho^2}]$$

$$= \frac{1}{2\pi} [(1+2\rho^2)(\frac{\pi}{2} + \sin^{-1}\rho) + 3\rho\sqrt{1-\rho^2}] \tag{2. 35}$$

which agrees with (2. 12).

Correspondingly,

$$H(\rho; 0) = 1 \tag{2. 36}$$

and

$$H(\rho; 1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x_1 x_2| \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_1)\varphi^{(n)}(x_2)\rho^n}{n!} dx_1 dx_2$$

$$= 2G(\rho; 1) - \int_0^{\infty}\int_{-\infty}^{0} x_1 x_2 \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_1)\varphi^{(n)}(x_2)\rho^n}{n!} dx_1 dx_2$$

$$- \int_{-\infty}^{0}\int_0^{\infty} x_1 x_2 \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(x_1)\varphi^{(n)}(x_2)\rho^n}{n!} dx_1 dx_2 \tag{2. 36}$$

but there is just one non-zero odd term in the series, namely

$$\int_0^{\infty} x\varphi^{(1)}(-x)dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} xe^{-x^2/2}dx = \frac{1}{2} \tag{2. 38}$$

Therefore

$$H(\rho;1) = 2G(\rho;1) + 2[G(\rho;1) - \rho/2]$$

$$= 4G(\rho;1) - \rho, \tag{2.38a}$$

an identity published by Rubin [45].

2.3 Third Order Information

Although the correlation function of the output of a nonlinearity provides information about the resulting random process, additional information may be obtained by calculating higher order correlation functions. This would yield the higher moments, or cumulants, so an Edgeworth series [15, 52] might be employed to arrive at a suitable approximation to the density function associated with the output.

The multiple Fourier transform of the third order correlation function is the bispectrum, a two dimensional power spectrum which is becoming increasingly important in the physical sciences.

At the third level there are three time shift parameters, while for an arbitrary N there are $\binom{N}{2}$ such parameters. The formal expression is

$$E[\prod_{i=1}^{N} h(x_i)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} h(x_i) p_{X_1, \ldots, X_N}(x_1, \ldots, x_N) dx_1 \ldots dx_N \tag{2.39}$$

and for $N = 3$ the symmetric moment matrix associated with three standard Gaussian variables is

$$\mathfrak{M}_3 = [\rho_{ij}] \quad i, j = 1, 2, 3 \tag{2.40}$$

The determinant of the matrix is

$$|\mathfrak{M}_3| = 1 - \rho_{12}^2 - \rho_{23}^2 - \rho_{31}^2 + 2\rho_{12}\rho_{23}\rho_{31}$$

and

$$\mathfrak{M}_3^{-1} = \frac{1}{|\mathfrak{M}_3|}\begin{bmatrix} (1-\rho_{23}^2) & -(\rho_{12}-\rho_{23}\rho_{31}) & -(\rho_{31}-\rho_{12}\rho_{23}) \\ -(\rho_{12}-\rho_{23}\rho_{31}) & (1-\rho_{13}^2) & -(\rho_{23}-\rho_{12}\rho_{31}) \\ -(\rho_{31}-\rho_{12}\rho_{23}) & -(\rho_{23}-\rho_{12}\rho_{31}) & (1-\rho_{12}^2) \end{bmatrix} \tag{2.41}$$

In principle a linear transformation might be found to rotate the x_1, x_2, x_3 coordinate system so the inverse matrix which defines $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ might be reduced to a diagonal matrix. The required integration over 3-space would not, however, be appealing.

Since interest is centered on the half-and full-wave rectifiers, it is logical to extend the G and H functions, defined by (2. 6) and (2. 28), in terms of the third order density function. The extension to a trilinear series representation of $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ is straight forward.

$$p_{X_1,X_2,X_3}(x_1, x_2 \cdot x_3) = \left(\frac{1}{2\pi}\right)^3 \int\int\int_{-\infty}^{\infty} e^{-i\alpha' x - \frac{1}{2} x' \mathfrak{M}_3 x} \, dx$$

$$= \left(\frac{1}{2\pi}\right)^3 \int\int\int_{-\infty}^{\infty} e^{-i\alpha' x - \frac{1}{2}[\alpha'\alpha + 2\rho_{12}\alpha_1\alpha_2 + \ldots]} \, dx$$

$$= \left(\frac{1}{2\pi}\right)^3 \int\int\int_{-\infty}^{\infty} e^{-i\alpha' x - \frac{1}{2}\alpha'\alpha} \sum_{n=0}^{\infty} (-1)^n$$

$$\times \frac{(\rho_{12}\alpha_1\alpha_2 + \rho_{23}\alpha_2\alpha_3 + \rho_{31}\alpha_3\alpha_1)^n}{n!} \, d\alpha_1 d\alpha_2 d\alpha_3 \tag{2.42}$$

and term by term evaluation is possible.

To help the typist, let the order of the φ functions be controlled so a term like $\varphi^0\varphi^2\varphi^4$ is understood to mean $\varphi^{(0)}(x_1)\varphi^{(2)}(x_2)\varphi^{(4)}(x_3)$. Now we can write the third order analogue of the relatively simple second order case of (2. 24) as

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3)$$

$$= \varphi^0\varphi^0\varphi^0 + \rho_{12}\varphi^1\varphi^1\varphi^0 + \rho_{23}\varphi^0\varphi^1\varphi^1 + \rho_{31}\varphi^1\varphi^0\varphi^1$$

$$+ \frac{1}{2!}\Big[\rho_{12}{}^2\varphi^2\varphi^2\varphi^0 + \rho_{23}{}^2\varphi^0\varphi^2\varphi^2 + \rho_{13}{}^2\varphi^2\varphi^0\varphi^2 + 2\rho_{12}\rho_{23}\varphi^1\varphi^2\varphi^1$$

$$+ 2\rho_{23}\rho_{31}\varphi^1\varphi^1\varphi^2 + 2\rho_{31}\rho_{12}\varphi^2\varphi^1\varphi^1\Big] +$$

$$
\begin{aligned}
&+\frac{1}{3!}\Big\{\rho_{13}^{3}\varphi^{3}\varphi^{3}\varphi^{0}+\rho_{23}^{3}\varphi^{0}\varphi^{3}\varphi^{3}+\rho_{31}^{3}\varphi^{3}\varphi^{0}\varphi^{3} \\
&\qquad+3\Big[\rho_{12}^{2}(\rho_{23}\varphi^{2}\varphi^{3}\varphi^{1}+\rho_{31}\varphi^{3}\varphi^{2}\varphi^{1})+\rho_{23}^{2}(\rho_{31}\varphi^{1}\varphi^{2}\varphi^{3}+\rho_{12}\varphi^{1}\varphi^{3}\varphi^{2}) \\
&\qquad+\rho_{31}^{2}(\rho_{12}\varphi^{3}\varphi^{1}\varphi^{2}+\rho_{23}\varphi^{2}\varphi^{1}\varphi^{3})\Big]+6\rho_{12}^{2}\rho_{23}^{2}\rho_{31}^{2}\varphi^{2}\varphi^{2}\varphi^{2}\Big\} \\
&+\ldots
\end{aligned}
\tag{2.43}
$$

If $\rho_{23} = \rho_{31} = 0$ then (2.43) reduces to (2.24), the second order probability density function in bilinear form. A formally correct and compact form of (2.43) may be written as

$$
p_{X_1,X_2,X_3}(x_1,x_2,x_3) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r,s,t=0}^{n}\binom{n}{rst}\rho_{12}^{r}\rho_{23}^{s}\rho_{31}^{t}\varphi^{(t+r)}(x_1)
$$

$$
\times\ \varphi^{(r+s)}(x_2)\varphi^{(s+t)}(x_3) \tag{2.44}
$$

where

$$
\binom{n}{rst} = \frac{n!}{r!s!t!}, \qquad r+s+t = n
$$

is the set of multinomial coefficients.

The next step is to obtain the third order analogues of $G(\rho; m)$ and $H(\rho; m)$, $m = 0, 1, 2,$ the functions associated with half- and full-wave rectifiers:

$$
G(\rho_{12},\rho_{23},\rho_{31};m) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}(x_1x_2x_3)^{m}p_{X_1,X_2,X_3}(x_1,x_2,x_3)dx_1dx_2dx_3 \tag{2.45}
$$

and

$$H(\rho_{12},\rho_{23},\rho_{31};m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x_1x_2x_3|^m$$

$$\times\, p_{X_1,X_2,X_3}(x_1,x_2,x_3)dx_1dx_2dx_3 \tag{2.46}$$

Clearly

$$H(\rho_{12},\rho_{23},\rho_{31};0) = 1 \tag{2.47}$$

and

$$H(\rho_{12},\rho_{23},\rho_{31};2) = 1 + 2\rho_{12}^2 + 2\rho_{23}^2 + 2\rho_{31}^2 + 8\rho_{12}\rho_{23}\rho_{31} \tag{2.48}$$

which Kamat [25] lists as [2, 2, 2]. Note that (2.48) reduces to $H(\rho; 2)$ when $\rho_{23} = \rho_{31} = 0$. Next

$$G(\rho_{12},\rho_{23},\rho_{31};0) = \frac{1}{8} + \frac{1}{4\pi}\Big[\rho_{12}+\rho_{23}+\rho_{31}+\frac{\rho_{12}^3+\rho_{23}^3+\rho_{31}^3}{6!}$$

$$+\frac{9}{5!}(\rho_{12}^5+\rho_{23}^5+\rho_{31}^5)+\ldots\Big]$$

$$= \frac{1}{8} + \frac{1}{4\pi}(\sin^{-1}\rho_{12} + \sin^{-1}\rho_{23} + \sin^{-1}\rho_{31}) \tag{2.49}$$

(see Gupta [18]).

The next case is anything but simple.

$$G(\rho_{12},\rho_{23},\rho_{31};1)$$

$$= \left(\frac{1}{2\pi}\right)^{3/2}\Big\{1+\frac{1}{2!}(\rho_{12}^2+\rho_{23}^2+\rho_{31}^2)+\rho_{12}\rho_{23}\rho_{31}$$

$$+\frac{1}{4!}[\rho_{12}^4+\rho_{23}^4+\rho_{31}^4-6(\rho_{12}^2\rho_{23}^2+\rho_{23}^2\rho_{31}^2+\rho_{31}^2\rho_{12}^2)] +$$

$$+\frac{1}{3!}\rho_{12}\rho_{23}\rho_{31}(\rho_{12}^2+\rho_{23}^2+\rho_{31}^2)$$

$$+\frac{1}{6!}\left\{9(\rho_{12}^6+\rho_{23}^6+\rho_{31}^6)-45[\rho_{12}^4(\rho_{23}^2+\rho_{31}^2)+\rho_{23}^4(\rho_{31}^2+\rho_{12}^2)\right.$$

$$\left.+\rho_{31}^4(\rho_{12}^2+\rho_{23}^2)]-90\rho_{12}^2\rho_{23}^2\rho_{31}^2\right\}+\dots\Big\}$$

$$+\frac{1}{4\sqrt{2\pi}}[\rho_{12}+\rho_{23}+\rho_{31}+\rho_{12}\rho_{23}+\rho_{23}\rho_{31}+\rho_{31}\rho_{12}]. \tag{2.50}$$

It is convenient to write

$$M_3=\sqrt{1-\rho_{12}^2-\rho_{23}^2-\rho_{31}^2+2\rho_{12}\rho_{23}\rho_{31}}$$

$$M_{12}=\sqrt{1-\rho_{12}^2}$$

$$M_{23}=\sqrt{1-\rho_{23}^2}$$

$$M_{31}=\sqrt{1-\rho_{31}^2} \tag{2.51}$$

and conjecture that the above (i.e., (2.50)) quite horrible series may have the closed form

$$G(\rho_{12},\rho_{23},\rho_{31};1)=\left(\frac{1}{2\pi}\right)^{3/2}\left\{M_3+(\rho_{12}+\rho_{23}\rho_{31})\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho_{12}-\rho_{23}\rho_{31}}{M_{23}M_{31}}\right)\right]\right.$$

$$+(\rho_{23}+\rho_{31}\rho_{12})\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho_{23}-\rho_{31}\rho_{12}}{M_{31}M_{12}}\right)\right]$$

$$\left.+(\rho_{31}+\rho_{12}\rho_{23})\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho_{31}-\rho_{12}\rho_{23}}{M_{12}M_{23}}\right)\right]\right\} \tag{2.52}$$

The corresponding result

$$H(\rho_{12},\rho_{23},\rho_{31};1) = \left(\frac{2}{\pi}\right)^{3/2}\left\{ M_3+(\rho_{12}+\rho_{23}\rho_{31})\sin^{-1}\left(\frac{\rho_{12}-\rho_{23}\rho_{31}}{M_{23}M_{31}}\right)\right.$$

$$+ (\rho_{23}+\rho_{31}\rho_{12})\sin^{-1}\left(\frac{\rho_{23}-\rho_{31}\rho_{12}}{M_{31}M_{12}}\right)$$

$$\left.+ (\rho_{31}+\rho_{12}\rho_{23})\sin^{-1}\left(\frac{\rho_{31}-\rho_{12}\rho_{23}}{M_{12}M_{23}}\right)\right\} \tag{2.53}$$

is listed as (1, 1, 1) by Kamat. Clearly the third order analogue of the Rubin formula (2. 38a) is

$$H(\rho_{12},\rho_{23},\rho_{31};1) = 8G(\rho_{12},\rho_{23},\rho_{31};1)$$

$$- \sqrt{\frac{2}{\pi}}\,[\rho_{12}+\rho_{23}+\rho_{31}+\rho_{12}\rho_{23}+\rho_{23}\rho_{31}+\rho_{31}\rho_{12}] \tag{2.54}$$

In realistic computations the ρ_{ij} tend to be simple exponential functions so the series of ascending powers might well be best for computation if the correlation level is low (i. e., $< .6$).

The above conjecture (2. 52) has been checked out by comparing the powers of the ρ_{ij} through the sixth degree.

2. 4 Limiters and Quantizers, Fourth Order Information

Another interesting group of devices to consider by the method of bilinear, trilinear, even quadrilinear expansion of the probability density function are the limiters, clippers, and quantizers whose output functions are listed below.

The clipper

$$h(x) = \begin{cases} ab, & x \geq b \\ ax, & -b < x < b \\ -ab, & x \leq -b \end{cases} \tag{2.55}$$

has been discussed by Price [41] and the smooth limiter

$$h(x) = a\varphi^{(-1)}(\frac{x}{c}), \qquad -\infty < x < \infty \tag{2.56}$$

has been discussed by Baum [4]. The quantizer is defined by

$$h(x) = \begin{cases} b, & a - \frac{a}{2N} \leq x \\ \frac{b}{N}[\frac{Nx}{a} + \frac{1}{2}], & -(a - \frac{a}{2N}) \leq x < (a - \frac{a}{2N}) \\ -b, & x < -(a - \frac{a}{2N}) \end{cases} \tag{2.57}$$

where $[\cdot]$ is the "greatest integer function".

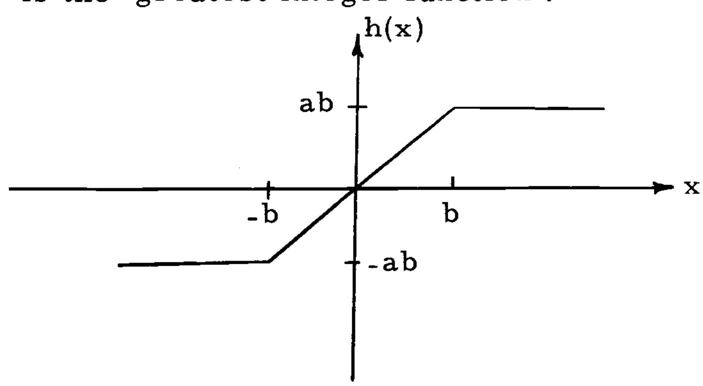

Figure 2.1. The clipper.

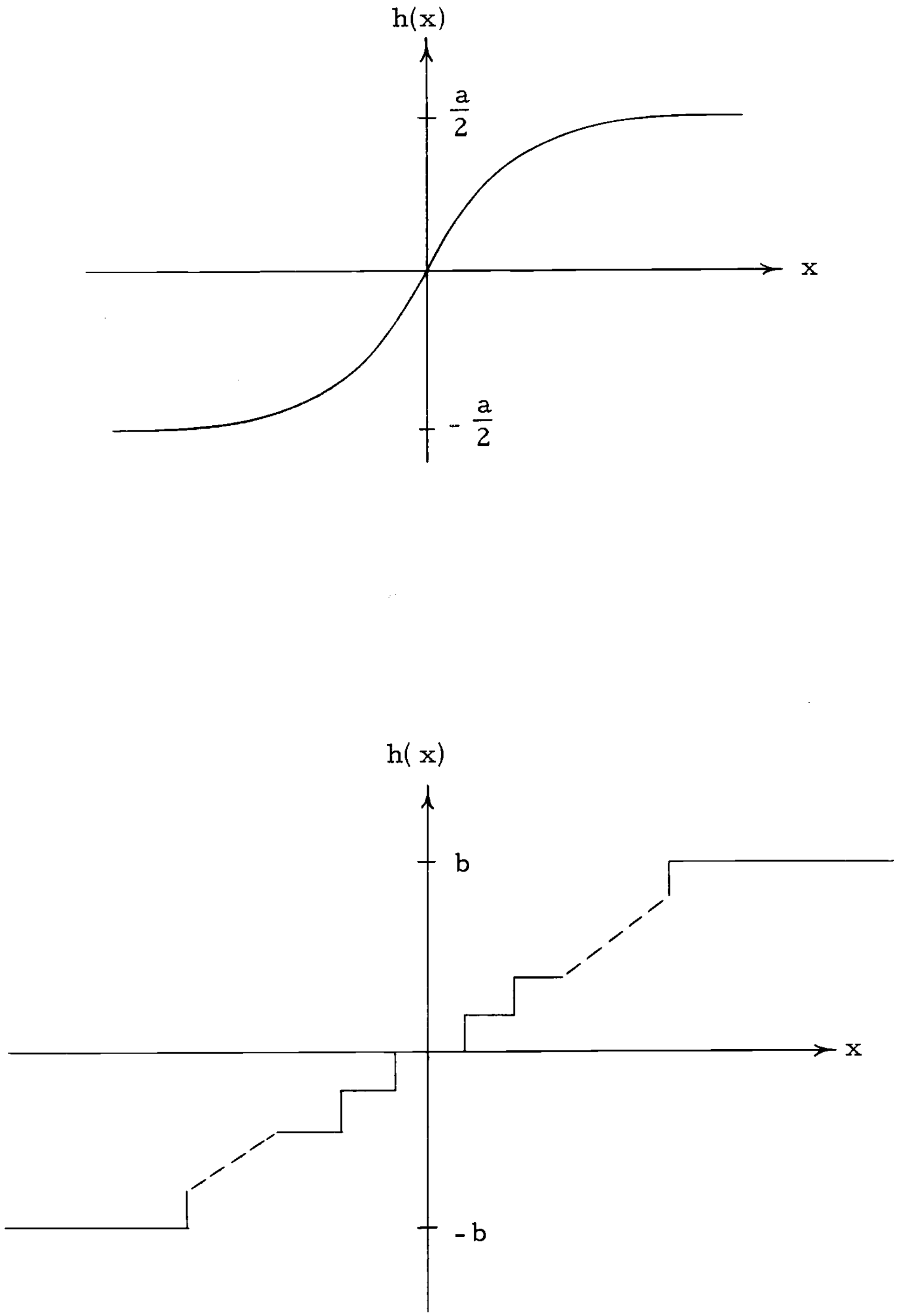

Figure 2. 3. The quantizer output function.

Due to the odd symmetry of this group of devices it can be shown that the third order information term for each is identically zero.

Moving on to the case of fourth order information we discover almost immediately that the task is to be much more formidable than that for the third order case for the rectifier family. The fourth order quadrilinear expansion for the appropriate density function must take the form

$$p_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4) = \left(\frac{1}{2\pi}\right)^4 \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} e^{-i\alpha' x - \frac{1}{2}\alpha'\alpha - \Sigma\rho_{ij}\alpha_i\alpha_j}$$

$$\times\, d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \qquad (2.58)$$

and, of course, there are $\binom{4}{2} = 6$ different ρ_{ij}'s, $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}$. Thus

$$p_{X_1, X_2, X_3, X_4}(x_1, x_2, x_3, x_4)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r,s,t,u,v,w=0}^{n} \binom{n}{rstuvw} \rho_{12}^{r}\rho_{13}^{s}\rho_{14}^{t}\rho_{23}^{u}\rho_{24}^{v}\rho_{34}^{w}$$

$$\times\, \varphi^{(r+s+t)}(x_1)\varphi^{(r+u+v)}(x_2)\varphi^{(s+u+w)}(x_3)\varphi^{(t+v+w)}(x_4) \qquad (2.59)$$

If the expansion is spelled out, an order-preserving notation could be employed such that $\varphi^{\alpha}\varphi^{\beta}\varphi^{\gamma}\varphi^{\delta}$ would mean

$\varphi^{(\alpha)}(x_1)\varphi^{(\beta)}(x_2)\varphi^{(\gamma)}(x_3)\varphi^{(\delta)}(x_4)$.

In the case of the smooth limiter (2. 54), the integral

$$\int_{-\infty}^{\infty} \varphi^{(-1)}(\frac{x}{c})\varphi^{(n)}(x)dx$$

is zero for even values of n. Let

$$I_n(c) = \int_0^{\infty} \varphi^{(-1)}(\frac{x}{c})\varphi^{(2n+1)}(x)dx$$

$$= \varphi^{(-1)}(\frac{x}{c})\varphi^{(2n)}(x)\Big|_0^{\infty} - \frac{1}{c}\int_0^{\infty} \varphi^{(0)}(\frac{x}{c})\varphi^{(2n)}(x)dx, \quad n \geq 0$$

$$= -\frac{1}{c}\int_0^{\infty} \varphi^{(0)}(\frac{x}{c})\varphi^{(2n)}(x)dx$$

$$= \frac{c^{2n+1}(2n)!\Gamma(\frac{1}{2})}{2^{n+\frac{1}{2}}n!} \tag{2. 60}$$

But one can write

$$\int_0^{\infty} e^{-x^2/2c^2} x^{2n} dx = \int_0^{\infty} (2c^2u)^{n-\frac{1}{2}} e^{-u}(c^2du)$$

$$= 2^{n-\frac{1}{2}} c^{2n+1}\Gamma(n+\frac{1}{2}). \tag{2. 61}$$

In general

$$\varphi^{(n)}(x) = (-1)^n \varphi^{(0)}(x) H_n(x)$$

$$= (-1)^n \varphi^{(0)}(x) n! \sum_{k=0}^{[n/2]} (-1)^k \frac{x^{n-2k}}{2^k k!(n-2k)!} \qquad (2.62)$$

where $H_n(x)$ are the Hermite polynomials so, finally,

$$\frac{1}{c}\int_0^\infty \varphi^{(0)}\left(\frac{x}{c}\right)\varphi^{(2n)}(x)dx$$

$$= \frac{(2n)!}{2\pi c}\int_0^\infty e^{-\frac{x^2}{2}\left(\frac{1+c^2}{c^2}\right)} \sum_{k=0}^{n} \frac{(-1)^k x^{2n-2k}}{2^k k!(2n-2k)!}\, dx$$

$$= \frac{(2n)!}{2\pi c} \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!(2n-2k)!} \frac{c}{\sqrt{1+c^2}} \quad (2n-2k+1) \quad \frac{(2n-2k)!\Gamma(\frac{1}{2})}{2^{n-k+\frac{1}{2}}(n-k)!}$$

$$= \frac{(2n)!\Gamma(\frac{1}{2})}{2\pi 2^{n+\frac{1}{2}}} \frac{c}{\sqrt{1+c^2}} \quad 2n+1 \quad \frac{1}{n!c} \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} \left(\frac{1+c^2}{c^2}\right)^k$$

$$= \frac{(2n)!}{\sqrt{2}\,\pi 2^{n+1}} \left(\frac{c^2}{1+c^2}\right)^{n+\frac{1}{2}} \frac{1}{n!c} \left(1 - \frac{1+c^2}{c^2}\right)^n \qquad (2.63)$$

and

$$I_n(c) = \frac{(-1)^{n-1}}{\sqrt{2\pi}\,(1+c^2)^{n+\frac{1}{2}}} \frac{(2n)!}{2^{n+1} n!} \qquad n = 0, 1, 2, \ldots \qquad (2.64)$$

For the fourth order information about the smooth limiter we are interested only in those terms of the quadrilinear expansion of

$p_{X_1, X_2, X_3, X_4}(x_1, x_2, x_3, x_4)$ which consist of odd order derivatives of $\varphi^{(0)}(x)$. There are many permutations of the odd indices. Table 2. 2 lists the possible combinations of odd indices through the ninth term of the expansion of the density function.

Table 2. 2. Combinations of odd order indices through the ninth term in expansion of density function.

Order of term	Combination of indices
0	none
1	none
2	(1, 1, 1, 1)
3	(3, 1, 1, 1)
4	(3, 3, 1, 1)
5	(5, 3, 1, 1), (3, 3, 3, 1)
6	(5, 5, 1, 1), (5, 3, 3, 1), (3, 3, 3, 3)
7	(7, 5, 1, 1), (5, 5, 3, 1), (7, 3, 3, 1), (5, 3, 3, 3)
8	(7, 5, 3, 1), (7, 3, 3, 3), (7, 7, 1, 1), (5, 5, 5, 1), (5, 5, 3, 3)
9	(9, 7, 1, 1), (9, 5, 3, 1), (9, 3, 3, 3), (7, 7, 3, 1), (7, 5, 5, 1), (7, 5, 3, 3), (5, 5, 5, 3)

A computer program was written to evaluate the series solution for the fourth order information of the smooth limiter for terms up to the ninth degree. In the program the constant c was chosen to be 1 and, although the program was written in general form it was also decided to obtain numerical results for the case

$$\rho_{ii} = 1, \quad \rho_{ij} = \rho; \quad i, j = 1, 2, 3, 4. \tag{2.65}$$

The Appendix lists the contributions by degree of term from two to nine as well as the running total for $\rho = .1$ to 1 in steps of 0.1. The convergence of this series is seen to be satisfactory for $\rho < .7$ however for larger values more terms would need to be added. The slow convergence for expansions of the Gaussian density function has been commented on by Cheng [10] and Gupta [18].

The quadrilinear expansion may be used to obtain fourth order information about all the devices mentioned in this chapter but one becomes painfully aware of the tediousness of the chore.

III. QUADRIPHASE CARRIER RECONSTRUCTION

3.0 Introduction

In order to coherently detect a quadriphase-modulated signal it is necessary to either reconstruct the carrier or to make use of a transmitted pilot carrier. This part of the thesis is an analysis of the carrier reconstruction section of a quadriphase demodulator.

Basically, the quadriphase (QPSK) signal may be described as

$$\sqrt{2s}\cos\left[\omega_0 t+\varphi(t)+\theta\right] \tag{3.1}$$

where

s is the average power of the signal,

ω_0 is the carrier radian frequency,

$\varphi(t)$ is the modulation (i.e., $\varphi(t)$, takes on one of the values $\pm\frac{\pi}{4}$, $\pm\frac{3\pi}{4}$ radians for a period of T seconds),

and

θ is a random phase angle uniformly distributed over $[0, 2\pi]$.

Passing this signal through a nonlinear device and filtering so as to recover the fourth harmonic yields

$$K\cos(4\omega_0 t+\pi+4\theta) \tag{3.2}$$

where we see that the modulation has been removed and in its place a constant phase term is left. To reconstruct the carrier one can divide the frequency by four and subtract out $\frac{\pi}{4}$ radians and obtain

$K_1 \cos(\omega_0 t + \theta)$ which may be used as the coherent reference.

In a physical system the implementation is generally as indicated in Figure 3. 1. The QPSK signal and additive Gaussian noise with one-sided spectral density of N_0 watts/Hz are passed through a bandpass filter sufficiently wide to pass at least the main lobe of the QPSK signal. The phase-locked loop (PLL) tracks the signal component at $4\omega_0$ from the $(\cdot)^4$ device and the reconstructed carrier is obtained from the VCO output at ω_0.

The analysis consists of deriving the signal-to-noise ratio at the input of the PLL and, knowing this, the average error rate for the detected QPSK signal.

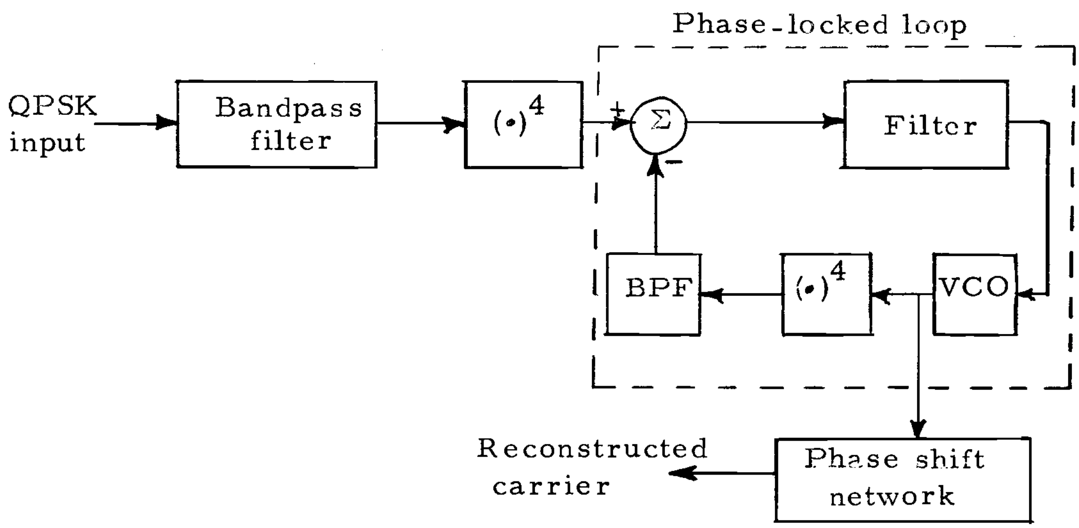

Figure 3. 1. Physical implementation of carrier reconstruction.

3.1 Signal-To-Noise Ratio at the PLL Input

Figure 3.2 illustrates the system we will be working with in this section of the analysis. $X(t)$ is the input signal plus noise $(s(t)+n(t))$. $Y(t)$ represents the filtered signal plus noise $(s'(t)+n'(t))$ and $Z(t) = Y^4(t)$. To determine the signal-to-noise ratio (SNR) at the PLL input it is necessary to take the Fourier transform of the autocorrelation function of $Z(t)$. This autocorrelation function $R_Z(\tau) = E[Z(t)Z(t+\tau)]$ (where $E[\cdot]$ is the expectation operator), may be found by expanding $Z(t)$ in terms of signal and noise components.

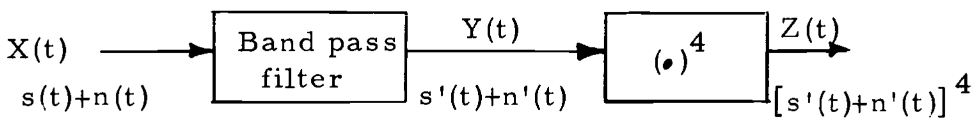

Figure 3.2. Multiplier system.

In this analysis we assume the signal and noise to be statistically independent. For notational convenience we write:

$$[\, s'(t) = s_1, \quad s'(t+\tau) = s_2, \quad n'(t) = n_1, \quad \text{and} \quad n'(t+\tau) = n_2] \tag{3.3}$$

From Figure 3.2 and the associated description we have that

$$
\begin{aligned}
R_Z(\tau) &= E[(s_1+n_1)^4(s_2+n_2)^4] \\
&= E\Big[s_1^4 s_2^4 + 4s_1^4 s_2^3 n_2 + 6s_1^4 s_2^2 n_2^2 + 4s_1^4 s_2 n_2^3 + s_1^4 n_1^4 + 4s_1^3 s_2^4 n_1 + 16 s_1^3 s_2^3 n_1 n_2 \\
&\quad + 24 s_1^3 s_2^2 n_1 n_2^2 + 16 s_1^3 s_2 n_1 n_2^3 + 4s_1^3 n_1 n_2^4 + 6s_1^2 s_2^4 n_1^2 + 24 s_1^2 s_2^3 n_1^2 n_2 \\
&\quad + 36 s_1^2 s_2^2 n_1^2 n_2^2 + 24 s_1^2 s_2 n_1^2 n_2^3 + 6 s_1^2 n_1^2 n_2^4 + 4 s_1 s_2^4 n_1^3 + 16 s_1 s_2^3 n_1^3 n_2 \\
&\quad + 24 s_1 s_2^2 n_1^3 n_2^2 + 16 s_1 s_2 n_1^3 n_2^3 + 4 s_1 n_1^3 n_2^4 + s_2^4 n_1^4 + 4 s_2^3 n_1^4 n_2 \\
&\quad + 6 s_2^2 n_1^4 n_2^2 + 4 s_2 n_1^4 n_2^3 + n_1^4 n_2^4 \Big] \qquad (3.4)
\end{aligned}
$$

The bandpass filter will be assumed ideal with transfer function $H(f)$, where

$$
H(f) = \begin{cases} 1, & f_0 - \frac{W}{2} \le |f| \le f_0 + \frac{W}{2}, \quad f_0 = \omega_0/2\pi \\ 0, & \text{otherwise.} \end{cases} \qquad (3.5)
$$

Since $n'(t)$ is a linear function of $n(t)$ (a Gaussian function) it too is Gaussian. The power density spectrum of $n'(t)$ is shown in Figure 3.3. It is easy to show that the autocorrelation function of $n'(t)$ is

$$
\begin{aligned}
R_{n'}(\tau) &= N_0 W \frac{\sin \pi\tau W}{\pi\tau W} \cos \omega_0 \tau \\
&= N_0 W \rho(\tau) \\
&= E(n_1 n_2) \qquad (3.6)
\end{aligned}
$$

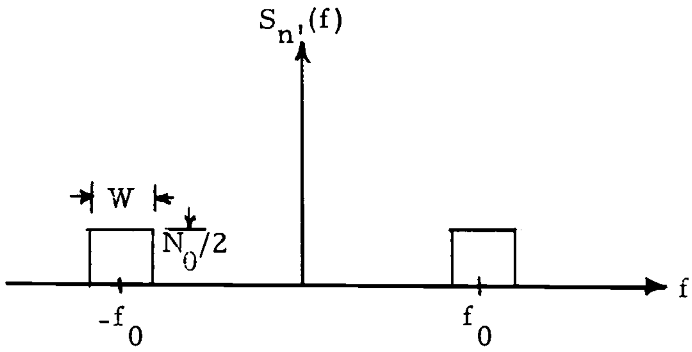

Figure 3.3. Power density spectrum of n'(t), $S_{n'}(f)$.

The various moments of n'(t) are listed below. Considerable simplification of Equation (3.4) may be made by noting that

$$E[n_1^i n_2^j] = 0 \quad \text{for} \quad i+j \text{ odd.} \tag{3.7}$$

This is a consequence of n'(t) being zero-mean Gaussian.

Table 3.1. Moments of n'(t).

$E(n_1 n_2)$	$= WN_0\rho(\tau)$
$E(n_1^2)$	$= WN_0$
$E(n_1 n_2^3)$	$= 3W^2N_0^2\rho(\tau)$
$E(n_1^2 n_2^2)$	$= W^2N_0^2(1+2\rho^2(\tau))$
$E(n_1^2 n_2^4)$	$= W^3N_0^3(1+4\rho^2(\tau))$
$E(n_1^3 n_2^3)$	$= W^3N_0^3\rho(\tau)[3+2\rho^2(\tau)]$
$E(n_1^4 n_2^4)$	$= W^4N_0^4[3+4\rho^4(\tau)]$
$E(n_1^i n_2^j)$	$= 0, \quad i+j$ odd
where	$\rho(\tau) = \dfrac{\sin(\pi\tau W)}{\pi\tau W}\cos\omega_0\tau$

3.2 Moments of the Signal Process, $s'(t)$

In order to make the mathematics tractable, we restrict the bandwidth of the filter W to be sufficiently wide to allow the assumption

$$s(t) \simeq s'(t). \tag{3.8}$$

This requires that

$$W \geq \frac{2}{T} \tag{3.9}$$

where T is the quadriphase symbol duration (seconds) and W is the filter bandwidth (Hertz).

Writing

$$R_{s'}(\tau) = E(s_1 s_2)$$

$$= E\{2s \cos(\omega_0 t+\varphi(t)+\theta) \cos[\omega_0(t+\tau)+\varphi(t+\tau)+\theta]\} \tag{3.10}$$

we first average over θ, which is uniformly distributed $(0, 2\pi)$ and obtain

$$E_\theta[s_1 s_2] = s\{\cos(\omega_0 t+\varphi(t))\cos[\omega_0(t+\tau)+\varphi(t+\tau)]$$

$$+ \sin(\omega_0 t+\varphi(t))\sin[\omega_0(t+\tau)+\varphi(t+\tau)]\}$$

$$= s \cos[\omega_0 \tau+\varphi(t+\tau)-\varphi(t)] \tag{3.11}$$

Recall now that $\varphi(u)$ is a discrete variable taking on one of the four values $(\pm \frac{\pi}{4}, \pm \frac{3\pi}{4})$ with equal probability during each successive time period of duration T seconds. $\varphi(u)$ and $\varphi(u+T)$ are assumed independent (i.e., data transmitted is random in nature).

Averaging $E_\theta[s_1 s_2]$ over time yields

$$R_{s'}(\tau) = E[s_1 s_2] = E_t[E_\theta[s_1 s_2]] = \begin{cases} s(1-\frac{|\tau|}{T}) \cos \omega_0 \tau, & |\tau| < T \\ 0, & |\tau| > T \end{cases} \tag{3.12}$$

In similar fashion one obtains the results in Table 3.2.

Table 3.2. Moments of the signal process.

$$E(s_1 s_2) = s\xi(\tau) \cos \omega_0 \tau$$

$$E(s_1^2) = s$$

$$E(s_1 s_2^3) = \frac{3}{2} s^2 \xi(\tau) \cos \omega_0 \tau$$

$$E(s_1^2 s_2^2) = s^2[1+\frac{1}{2} \xi(\tau) \cos 2\omega_0 \tau]$$

$$E(s_1^2 s_2^4) = s^3[\xi(\tau) \cos 2\omega_0 \tau + \frac{3}{2}]$$

$$E(s_1^3 s_2^3) = \frac{s^3}{4} [\xi(\tau) \cos 3\omega_0 \tau + 9\xi(\tau) \cos \omega_0 \tau]$$

$$E(s_1^4 s_2^4) = s^4[\frac{1}{8} \cos 4\omega_0 \tau + 2\xi(\tau) \cos 2\omega_0 \tau]$$

where $$\xi(\tau) = \begin{cases} 1-\frac{|\tau|}{T}, & \text{for } |\tau| < T \\ 0, & \text{otherwise} \end{cases}$$

Substituting the results of Tables 3.1 and 3.2 into Equation (3.4) we have,

$$
\begin{aligned}
R_Z(\tau) = \frac{s^4}{8}\,&[\cos\,4\omega_0\tau + 16\xi(\tau)\cos\,2\omega_0\tau] + 6WN_0 s^3[\xi(\tau)\cos 2\omega_0\tau + \tfrac{3}{2}] \\
&+ \tfrac{9}{2}W^2N_0^2s^2 + 4WN_0s^3\rho(\tau)\xi(\tau)[\cos 3\omega_0\tau + 9\cos\,\omega_0\tau] \\
&+ 72W^2N_0^2s^2\rho(\tau)\xi(\tau)\cos\omega_0\tau + 6WN_0s^3[\xi(\tau)\cos 2\omega_0\tau + \tfrac{3}{2}] \\
&+ 36W^2N_0^2s^2(1+2\rho^2(\tau))(1+\tfrac{1}{2}\xi(\tau)\cos\,2\omega_0\tau) \\
&+ 6W^3N_0^3s(1+4\rho^2(\tau)) + 72W^2N_0^2s^2\rho(\tau)\xi(\tau)\cos\,\omega_0\tau \\
&+ 16W^3N_0^3s\rho(\tau)\xi(\tau)\cos\,\omega_0\tau[3+2\rho^2(\tau)] + \tfrac{9}{2}W^2N_0^2s^2 \\
&+ 6W^3N_0^3s(1+4\rho^2(\tau)) + W^4N_0^4(3+4\rho^4(\tau))
\end{aligned} \tag{3.13}
$$

The power density spectrum $S_Z(f)$ of $Z(t)$ may be computed by taking the Fourier transform of $R_Z(\tau)$ but since we are only interested in that portion of $S_Z(f)$ near $4f_0$ we can inspect (3.13) and determine that only five terms yield components in this frequency range. They are:

$$
\begin{aligned}
R_Z^*(\tau) = \frac{s^4}{8}\cos\,4\omega_0\tau &+ 4WN_0s^3\rho(\tau)\xi(\tau)\cos\,3\omega_0\tau \\
&+ 36W^2N_0^2s^2\rho^2(\tau)\xi(\tau)\cos\,2\omega_0\tau \\
&+ 32W^3N_0^3s\rho^3(\tau)\xi(\tau)\cos\,\omega_0\tau + 4W^4N_0^4\rho^4(\tau)
\end{aligned} \tag{3.14}
$$

The first term of (3.14) yields the desired reconstructed sinusoid at $4f_0$ while the remainder produce noise. At this point it is usual to assume that the phase-locked loop has an input bandwidth $B_L \ll W$ and that the noise spectral density at $4f_0$ is essentially flat for

$$4f_0 - \frac{B_L}{2} < |f| < 4f_0 + \frac{B_L}{2}$$

so that we may say that the total noise power at the input to the PLL is $S_Z(4f_0)B_L$. $S_Z(4f_0)$ is the one-sided power spectrum evaluated at $4f_0$.

3.3 Noise Power Spectral Density at $4f_0$

Recalling that an autocorrelation function consisting of a product of n functions transforms into a spectral density function consisting of an n-fold convolution of the spectral density functions corresponding to the n functions in the autocorrelation function, i.e., if

$$R(\tau) = R_1(\tau)R_2(\tau) \ldots R_n(\tau) \tag{3.15}$$

then

$$\mathfrak{F}(R(\tau)) = S(f) = S_1(f) \otimes S_2(f) \otimes \ldots \otimes S_n(f) \tag{3.16}$$

where $\mathfrak{F}(\cdot)$ indicates Fourier transform of $(\cdot)$ and $\otimes$

signifies convolution, then we evaluate the noise spectral density at $4f_0$ as follows:

A. $4WN_0 s^3 \rho(\tau)\xi(\tau) \cos 3\omega_0 \tau$

$$\mathfrak{F}(4WN_0 s^3 \rho(\tau)) = \begin{cases} 2WN_0 s^3, & |f| - \frac{W}{2} \leq |f| \leq |f| + \frac{W}{2} \\ 0, & \text{otherwise} \end{cases} \quad (3.17)$$

and

$$\mathfrak{F}(\xi(\tau)\cos 3\omega_0 \tau) = \frac{T}{2}\left(\frac{\sin \pi T(|f|-3f_0)}{\pi T(|f|-3f_0)}\right) \qquad -\infty < f < \infty$$

Convolving these spectra, we obtain for the energy density at $4f_0$:

$$S_A(4f_0) = 4s^3 N_0 T \int_0^{W/2} \left(\frac{\sin \pi f T}{\pi f T}\right)^2 df \quad (3.18)$$

The integral in (3.18) cannot be evaluated in closed form but tables exist [17] which evaluate

$$\mathcal{S}(x) = \int_0^x \left[\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}}\right]^2 du \quad (3.19)$$

Allowing the bandwidth to be a function of the quadriphase symbol duration

$$W = \frac{a}{T} \quad (3.20)$$

we may write (3-18) as

$$S_A(4f_0) = \frac{2s^3 N_0}{\pi}\,\mathcal{S}(\pi a) \tag{3.21}$$

B. $36W^2N_0^2s^3\rho^2(\tau)\xi(\tau)\cos 2\omega_0\tau$

Convolving the transform of $\rho(\tau)$ with itself yields a triangular spectrum centered about $\pm 2f_0$ as shown in Figure 3.4.

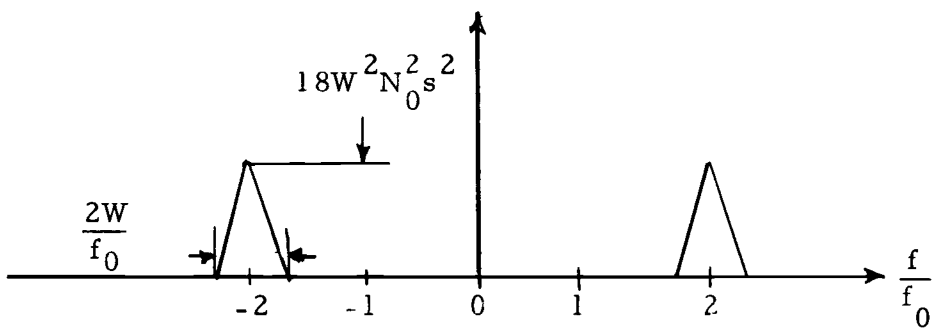

Figure 3.4. Spectrum resulting from $\mathcal{F}(\rho(\tau)) \otimes \mathcal{F}(\rho(\tau))$.

Convolving this with $\mathcal{F}[\xi(\tau)\cos 2\omega_0\tau]$ we obtain

$$S_B(4f_0) = 18s^2N_0^2WT\int_0^W \left(1-\frac{f}{W}\right)\left[\frac{\sin\left(\pi f\frac{T}{2}\right)}{\pi f\frac{T}{2}}\right]^2 df$$

$$= 18s^2N_0^2WT\left[\frac{1}{2\pi T}\,\mathcal{S}(2\pi TW) - \frac{1}{4W\pi^2T^2}\int_0^{2\pi WT} u\,\frac{\sin^2\left(\frac{u}{2}\right)}{\frac{u^2}{4}}\,du\right] \tag{3.22}$$

but

$$\int_0^x u\,\frac{\sin^2\frac{u}{2}}{\frac{u^2}{4}}\,du = x\,\mathcal{S}(x) - \mathcal{J}(x) \tag{3.23}$$

(also from the Tables) where

$$\mathcal{J}(x) = \int_0^x \mathcal{S}(u)\,du$$

Table 3.3 lists approximations to $\mathcal{J}(x)$ obtained by numerically integrating the tabular values for $\mathcal{S}(x)$.

Table 3.3 Approximate value for $\mathcal{J}(x)/\pi$.

x	$\mathcal{J}(x)/\pi$	x	$\mathcal{J}(x)/\pi$
$3\pi/2$	2.707	6π	15.71
2π	3.88	$15\pi/2$	20.31
$5\pi/2$	5.3	8π	21.81
3π	6.75	9π	24.87
4π	9.69	12π	34.11
5π	12.69		

Equation (3.22) can now be written as

$$S_B(4f_0) = \frac{9s^2N_0^2W}{\pi}\left\{\mathcal{S}(2\pi a) - \frac{1}{2\pi a}\left[2\pi a\,\mathcal{S}(2\pi a) - \mathcal{J}(2\pi a)\right]\right\}$$

$$= \frac{9s^2N_0^2W\,\mathcal{J}(2\pi a)}{2a\pi^2} \qquad (3.24)$$

C. $32W^3N_0^3 s\rho^3(\tau)\xi(\tau)\cos\omega_0\tau$

Convolving $\mathfrak{F}(\rho(\tau))$ with itself twice yields a symmetric spectrum of the form

$$S_{C'}(f) = \begin{cases} \dfrac{N_0^3 W}{8}\left(\dfrac{3W}{4} - \dfrac{2(|f|-3f_0)^2}{W}\right), & \big||f|-3f_0\big| < \dfrac{W}{2} \\[2ex] \dfrac{N_0^3 W}{8}\left[\dfrac{(|f|-3f_0)^2}{W} - 3\big||f|-3f_0\big| + \dfrac{9W}{4}\right], & \dfrac{W}{2} \le \big||f|-3f_0\big| \le \dfrac{3W}{2} \\[2ex] 0, & \text{otherwise} \end{cases} \qquad (3.25)$$

Convolving $32W^3N_0^3 s\,\mathfrak{F}(\xi(\tau)\cos\omega_0\tau)$ with $S_{C'}(f)$ yields

$$S_C(4f_0) = 8sN_0^3WT\int_0^{W/2}\left(\frac{3W}{4} - 2f^2/W\right)\frac{\sin^2(\pi f T)}{(\pi f T)^2}\,df$$

$$+ 4sN_0^3WT\int_{W/2}^{3W/2}\left(\frac{f^2}{W} - 3f + \frac{9W}{4}\right)\frac{\sin^2(\pi f T)}{(\pi f T)^2}\,df \qquad (3.26)$$

We easily show that

$$\int_0^x u^2\,\frac{\sin^2\frac{u}{2}}{\frac{u^2}{4}}\,du = 2(x - \sin x) \qquad (3.27)$$

so we have

$$S_C(4f_0) = \frac{sN_0^3W^2}{\pi}\left\{\frac{3}{2}\mathcal{S}(a\pi) + \frac{5\sin a\pi}{\pi^2a^2} - \frac{9}{2}\mathcal{S}(3a\pi)\right.$$

$$\left. - \frac{2}{a\pi} - \frac{\sin(3a\pi)}{a^2\pi^2} + \frac{3}{a\pi}\mathcal{C}(3a\pi) - \frac{3}{a\pi}\mathcal{C}(a\pi)\right\} \qquad (3.28)$$

D. $4W^4 N_0^4 \rho^4(\tau)$

Convolving $\mathfrak{F}(\rho(\tau))$ with itself twice gives a spectrum of the form (3.25). Convolving this with $4W^4 N_0^4 \mathfrak{F}(\rho(\tau))$ yields

$$S_D(4f_0) = N_0^4 W \int_0^{W/2} \left(\frac{3W}{4} - \frac{2f^2}{W}\right) df$$

$$= \frac{7N_0^4 W^3}{24} \qquad (3.29)$$

The total noise spectral density at $4f_0$ is therefore

$$S_Z(4f_0) = \frac{2s^3 N_0}{\pi} \mathcal{S}(\pi a) + \frac{9s^2 N_0^2 W \mathfrak{I}(2\pi a)}{2a\pi^2}$$
$$+ \frac{sN_0^3 W^2}{\pi}\left[\frac{3\mathcal{S}(a\pi)}{2} + \frac{5 \sin a\pi}{\pi^2 a^2} - \frac{9}{2}\mathcal{S}(3a\pi) - \frac{2}{a\pi}\right.$$
$$\left. - \frac{\sin(3a\pi)}{a^2\pi^2} + \frac{3}{a\pi}\left(\mathfrak{I}(3a\pi) - \mathfrak{I}(a\pi)\right)\right] + \frac{7N_0^4 W^3}{24} \qquad (3.30)$$

The SNR at the input to the PLL is therefore

$$\mathrm{SNR} = \frac{\frac{s^4}{8}}{S_Z(4f_0) B_L} \qquad (3.31)$$

Typical cases might be for $a = 2, 3, 4$. Table 3.4 lists the various values of $\mathcal{S}(x)$ needed in these evaluations.

Table 3.4. Several values of $\mathcal{S}(x)$.

a	$\mathcal{S}(a\pi)$	$\mathcal{S}(3a\pi)$
2	2.84	3.04
3	2.93	3.07
4	2.98	3.09

For $a = 2$ we have

$$\mathrm{SNR} = \frac{\frac{s^4}{8}}{B_L 2(.904 s^3 N_0 + 3.47 s^2 N_0^2 W + 1.28 s N_0^3 W^2 + .146 N_0^4 W^3)}$$

$$= \frac{\frac{s}{N_0}}{16 B_L \left[.904 + 3.47 \frac{N_0 W}{s} + 1.28\left(\frac{N_0 W}{s}\right)^2 + .146\left(\frac{N_0 W}{s}\right)^3\right]} \qquad (3.32)$$

Knowing that

$$sT = \text{symbol energy} = 2(\text{bit energy}) = 2E_b \qquad (3.33)$$

and

$$T = \frac{a}{W}$$

we obtain

$$\mathrm{SNR} = \frac{\left(\frac{E_b}{N_0}\right)\left(\frac{W}{B_L}\right)}{14.4 + 55.5\frac{N_0}{E_b} + 20.5\left(\frac{N_0}{E_b}\right)^2 + 2.34\left(\frac{N_0}{E_b}\right)^3} \qquad (3.34)$$

Similarly, for $a = 3$ we have

$$\mathrm{SNR}_{a=3} = \frac{(\frac{E_b}{N_0})(\frac{W}{B_L})}{22.4 + 135\,\frac{N_0}{E_b} + 73.2(\frac{N_0}{E_b})^2 + 11.8(\frac{N_0}{E_b})^3} \tag{3.34}$$

and for $a = 4$ we obtain

$$\mathrm{SNR}_{a=4} = \frac{(\frac{E_b}{N_0})(\frac{W}{B_L})}{30.2 + 250\,\frac{N_0}{E_b} + 181(\frac{N_0}{E_b})^2 + 37.4(\frac{N_0}{E_b})^3} \tag{3.35}$$

The SNR at the PLL input can be calculated for any value of $a > 2$. Values of $a < 2$ tend to become meaningless because (3.8) no longer holds.

3.4 Density Function of the PLL Output Phase Error, φ

When a sinusoid plus noise is applied to the input of a phase-locked loop the output phase of the PLL differs from the phase of the input sinusoid in a random fashion. The probability density function of this difference has been shown by Viterbi [51] to be of the form

$$p_{\Phi}(\varphi) = \frac{\exp\,(\mathrm{SNR}\,\cos\varphi)}{2\pi I_0(\mathrm{SNR})}\,, \qquad |\varphi| \leq \pi \tag{3.36}$$

where $I_0(\cdot)$ is the modified Bessel function.

We are now at a point where we can determine the probability of bit error when we have a given E_b/N_0 and W/B_L.

3. 5 Probability of Bit Error for Quadriphase Modulation Systems

Since each of the possible phases of the transmitted quadriphase signal is equally likely, we need consider only one of them in formulating the probability of making an error in the detection process. Furthermore, since the PLL is tracking a carrier at $4f_0$ we must divide its output signal by 4 yielding a reference signal at f_0. This divide-by-four operation has the advantage of also reducing the phase difference φ by 4. Figure 3. 5 shows a phasor diagram of a received signal originally transmitted as

$$\sqrt{E} \cos\left(\omega_0 t + \frac{\pi}{4} + \theta\right) . \tag{3.37}$$

where E is the symbol energy.

The reference signal used in coherently detecting this signal is of the form

$$K \cos\left(\omega_0 t + \theta + \frac{\varphi}{4}\right) \tag{3.38}$$

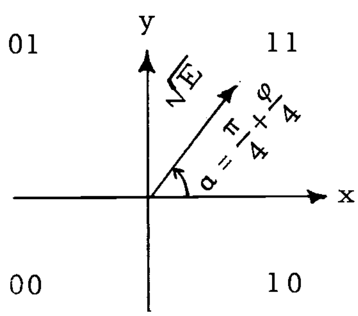

Figure 3. 5. Phasor diagram of received quadriphase signal.

The four equally likely choices for symbol in each time period T results in

$$\log_2 4 = 2 \text{ bits of information/symbol} \tag{3.39}$$

The assignment of the symbol to each pair (x, y) of binary digits transmitted may be arbitrary and we will choose to do it as indicated in Figure 3.5. Here we see that the assignments are:

x	y	$\varphi(t)$
0	0	$-3\pi/4$
0	1	$+3\pi/4$
1	0	$-\pi/4$
1	1	$\pi/4$

Since the transmitted symbol may be considered the vector addition of the x and y components we will assume that the noise is also composed of x and y components. Each component of noise (n_x, n_y) will be assumed Gaussian with zero mean and one-sided noise density N_0 watts/Hertz. Further, we assume the two components to be statistically independent.

A hard decision as to which two bits were transmitted is made by determining in which quadrant the received signal vector lies. For the case shown in Figure 3.5, one bit will be in error if the received vector lies in quadrant II or IV and two bits will be in error if it lies in quadrant III. No error will result if it lies in the first quadrant.

The conditional probability of a bit error when two bits (one symbol) are transmitted given an angle φ is

$$
\begin{aligned}
&\text{Prob (bit error per 2 bits} \mid \varphi) \\
&= \text{Prob}(n_x > -\sqrt{E}\cos a,\ n_y < -\sqrt{E}\sin a) \\
&\quad + \text{Prob}(n_x < -\sqrt{E}\cos a,\ n_y > -\sqrt{E}\sin a) \\
&\quad + 2\ \text{Prob}(n_x < -\sqrt{E}\cos a,\ n_y < -\sqrt{E}\sin \theta) \\
&= \frac{1}{\pi N_0}\int_{-\sqrt{E}\cos a}^{\infty} e^{-x^2/N_0}\,dx \int_{\sqrt{E}\sin a}^{\infty} e^{-y^2/N_0}\,dy \\
&\quad + \frac{1}{\pi N_0}\int_{\sqrt{E}\cos a}^{\infty} e^{-x^2/N_0}\,dx \int_{-\sqrt{E}\sin a}^{\infty} e^{-y^2/N_0}\,dy \\
&\quad + \frac{2}{\pi N_0}\int_{\sqrt{E}\cos a}^{\infty} e^{-x^2/N_0}\,dx \int_{\sqrt{E}\sin a}^{\infty} e^{-y^2/N_0}\,dy \\
&= \frac{1}{2}\left\{ \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\cos a\right) + \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin a\right)\right\} \qquad (3.40)
\end{aligned}
$$

where

$$\operatorname{erfc}(Z) = \frac{2}{\sqrt{\pi}}\int_{Z}^{\infty} e^{-t^2}\,dt$$

is the complementary error function.

The probability of a bit error for one bit transmitted given φ is clearly just $\frac{1}{2}$ the result of (3.40).

Using the ideas of conditional probability we can next write

$$\overline{\text{Prob(bit error/bit transmitted)}}$$

$$= \overline{\text{Prob(bit error}\,|\,\varphi)\text{Prob}(\varphi)}$$

$$= \int_{-\pi}^{\pi} \frac{\exp(\text{SNR}\cos\varphi)}{8\pi I_0(\text{SNR})} \left[\operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\cos\alpha\right) + \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\alpha\right) \right] d\varphi \qquad (3.41)$$

where, again, $\alpha = \frac{\pi + \varphi}{4}$ as in Figure 3.5 and

$$E = \text{symbol energy}$$

$$= 2(\text{bit energy}) = 2E_b$$

Equation (3.41) cannot be solved in closed form, however, a computer program was written to numerically integrate it with various values of SNR and E_b/N_0. For the sake of simplicity, SNR was taken to be of the form

$$\text{SNR} = \beta E_b/N_0 \qquad (3.42)$$

where

$$\beta = \text{constant}$$

Curves of bit error probability vs. input E_b/N_0 with several values of β are shown in Figure 3.6. It will be noted that for $\beta = 5$ there is at most a degradation in performance of 0.2 dB (i.e., the transmitted power must be increased by at most 0.2 dB to provide the error performance achieved by a noiseless reference. The

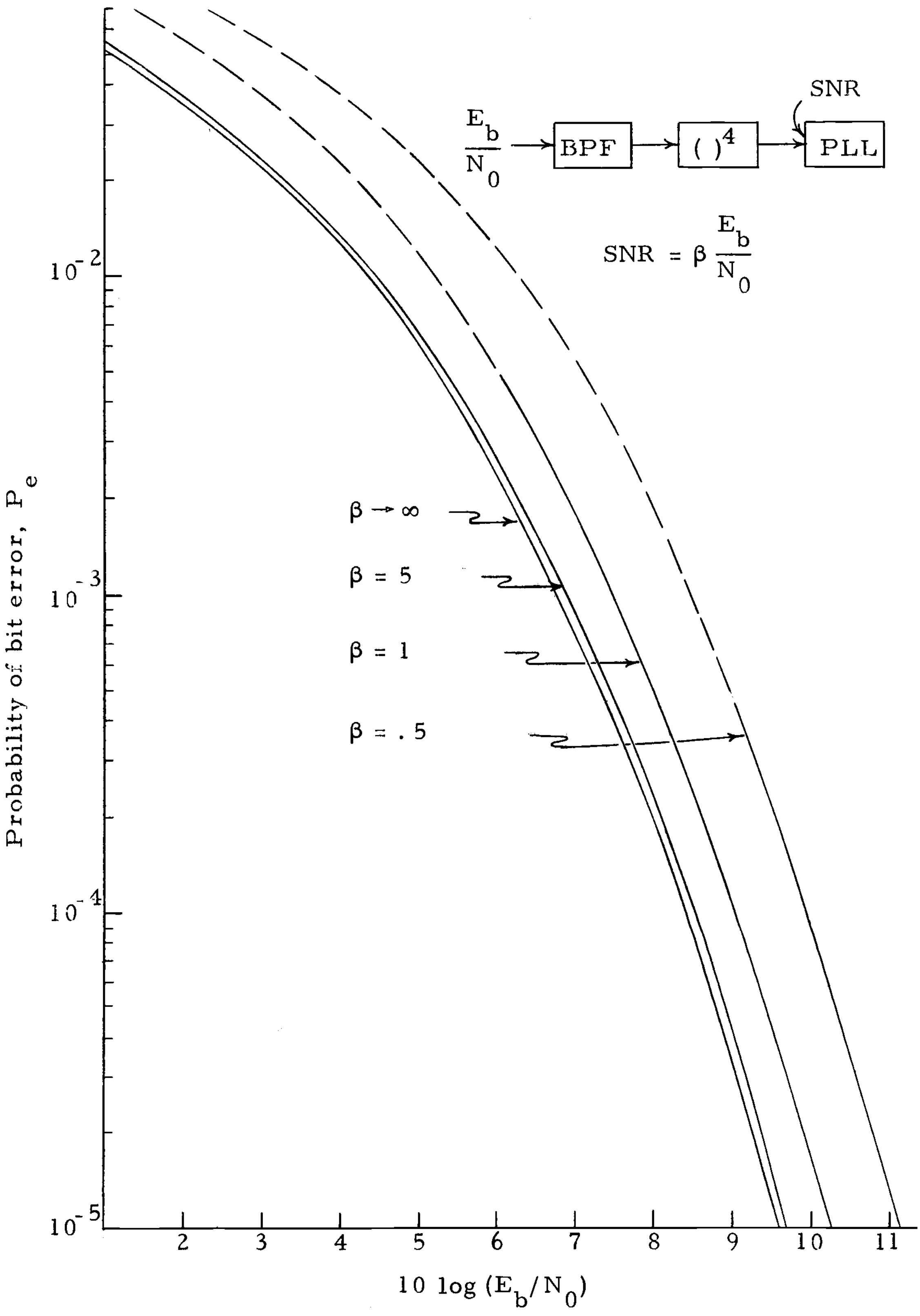

Figure 3.6. Probability of bit error vs E_b/N_0.

portions of the curves drawn broken are areas where the PLL is not in lock and should not be used for system operating points.

In physical systems one would want to keep β as large as possible to reduce performance degradation. This is usually done by lowering B_L. A compromise must be made here, however, because reducing B_L increases the acquisition time of the loop. If E_b/N_0 is small, it is to be expected that the PLL will lose lock and be required to reacquire before coherent reception is possible. One then uses the largest B_L possible consistent with low system degradation.

Table 3.5 lists the required W/B_L to maintain $\beta = 5$ for various values of E_b/N_0 and $a = 2, 3, 4$. For the lowest E_b/N_0 expected, one may select W/B_L to minimize system degradation due to a noisy reference.

Table 3.5. W/B_L required to maintain $\beta = 5$ ($a = TW$).

E_b/N_0	W/B_L		
	$a = 2$	$a = 3$	$a = 4$
1.3	347	860	1702
1.6	279	665	1274
2.1	230	525	977
2.6	196	433	785
3.7	168	357	629
4.1	146	301	515
5.1	130	258	431
6.5	117	226	367
8.1	108	201	319

IV. POINT PROCESSES

4.0 Average Number of Crossings of $X(t) = x$

The formula for the average number of times per unit of time that $X(t) = x$ has been established by Rice [43] as

$$N_Y = \int_{-\infty}^{\infty} |y| p_{X,Y}(x, y) dy \tag{4.1}$$

where it is understood that $X(t)$ represents a continuous time, differentiable, random process and $p_{X,Y}(x, y)$ is the probability density function associated with $X(t)$ and its time derivative $Y(t)$.

Let $N(t)$ represent a random noise process normally distributed with zero mean and variance N_0. It is well known that $N'(t)$ is also normally distributed with zero mean and with variance A^2N_0.

Let

$$\begin{aligned} X(t) &= N(t) + Q \sin (\omega_0 t+\theta) \\ &= N(t) + S(t) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} Y(t) &= N'(t) + \omega_0 Q \cos (\omega_0 t+\theta) \\ &= N'(t) + S'(t) \end{aligned} \tag{4.3}$$

The cross-correlation is zero:

$$E[X(t)Y(t)] = 0 \tag{4.4}$$

but this does not imply that $X(t)$ and $Y(t)$ are statistically independent.

To implement the formula of Equation (4.1) it is necessary to determine the form of the joint density function. This is accomplished as follows: consider the two dimensional random variables

$$U(t) = [N(t), N'(t)] \tag{4.5}$$

and

$$V(t) = [S(t), S'(t)] \tag{4.6}$$

with bivariate moment generating functions $M_{N,N'}(\alpha_1, \alpha_2)$ and $M_{S,S'}(\alpha_1, \alpha_2)$. The Fourier transform pairs are $p_{N,N'}(x, y)$ and $p_{S,S'}(x, y)$. The vector valued processes $U(t)$ and $V(t)$ are clearly independent of each other, thus the moment generating function for $[X(t), Y(t)]$ is defined by the product

$$M_{X,Y}(\alpha_1, \alpha_2) = M_{N,N'}(\alpha_1, \alpha_2)M_{S,S'}(\alpha_1, \alpha_2) \tag{4.7}$$

where by definition,

$$M_{N,N'}(\alpha_1, \alpha_2) = E[e^{i(\alpha_1 N+\alpha_2 N')}] \tag{4.8}$$

and

$$M_{S,S'}(\alpha_1, \alpha_2) = E[e^{i(\alpha_1 S+\alpha_2 S')}], \tag{4.9}$$

the notation of Parzen [39]. Equation (4. 7) implies that the joint density function is a convolution

$$p_{X,Y}(x, y) = p_{N,N'}(x, y) \otimes p_{S,S'}(x, y)$$

$$= \int\int_{-\infty}^{\infty} p_{N,N'}(x-u, y-v)p_{S,S'}(u, v)du\,dv \tag{4.10}$$

The covariance matrix associated with $N(t)$ and $N'(t)$ is

$$\mathfrak{M} = N_0 \begin{bmatrix} 1 & 0 \\ 0 & A^2 \end{bmatrix} \tag{4.11}$$

so

$$p_{N,N'}(x, y) = \frac{1}{AN_0} \varphi\left(\frac{x}{\sqrt{N_0}}\right)\varphi\left(\frac{y}{A\sqrt{N_0}}\right) \tag{4.12}$$

where $\varphi(\cdot)$ is the error function defined by (2. 20) with

$$\varphi^{(0)}(x) \triangleq \varphi(x) \tag{4.13}$$

Consider the fixed amplitude sinusoid $Q \sin(\omega_0 t+\theta)$. A selection of values of this continuous time process corresponding to randomly selected instants of time, $t_1, t_2, t_3, \ldots,$ is equivalent to allowing the phase angle θ to be uniformly distributed over the interval $(0, 2\pi)$. The moment generating function associated with the sinusoidal process is

$$M_S(a) = E[e^{iaQ \sin(\omega_0 t+\theta)}]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{iaQ \sin(\omega_0 t+\theta)} d\theta$$

$$= J_0(Qa) \tag{4.14}$$

where $J_0(x)$ is the Bessel function of the first kind.

The density function is the Fourier transform pair

$$p_S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iax} M_S(a)da$$

$$= \frac{1}{2\pi^2} \int_{-\infty}^{\infty} da e^{-iax} \int_0^{\pi} d\theta e^{iaQ \cos\theta}$$

$$= \frac{1}{\pi} \int_0^{\pi} \delta(x-Q \cos\theta)d\theta \tag{4.15}$$

Korevaar's [26] general formula for the Dirac delta function,

$$\int_c^d \delta[g(t)]f(t)dt = \frac{f(T)}{|g'(T)|} \tag{4.16}$$

where $g(t) \in c'$ in the open interval (c, d), $g'(T) \neq 0$, $g(T) = 0$, $c < T < d$ leads to

$$p_S(x) = \begin{cases} 0, & |x| > Q \\ \dfrac{1}{\pi\sqrt{Q^2-x^2}}, & |x| < Q \end{cases} \tag{4.17}$$

Now let $S'(t) = \omega_0 Q \cos(\omega_0 t+\theta)$. When $X(t)$ takes on a given value x, $S'(t)$ must take on one or the other of the two values $\pm\omega_0\sqrt{Q^2-x^2}$ with equal probability. The conditional probability density for $X'(t)$, given a prescribed value of $S(t)$, is then

$$p_{S'|S}(x,y) = \frac{\delta(y-\omega_0\sqrt{Q^2-x^2})+\delta(y+\omega_0\sqrt{Q^2-x^2})}{2} \tag{4.18}$$

The relationship

$$p_{S,S'}(x,y) = p_S(x)p_{S'|S}(x,y) \tag{4.19}$$

yields

$$p_{S,S'}(x,y) = \begin{cases} 0, & |x| > Q \\ \dfrac{\delta(y-\omega_0\sqrt{Q^2-x^2})+\delta(y+\omega_0\sqrt{Q^2-x^2})}{2\pi\sqrt{Q^2-x^2}}, & |x| < Q \end{cases} \tag{4.20}$$

The convolution integral of (4.10) may now be written as

$$\begin{aligned} p_{X,Y}(x,y) &= \frac{1}{2\pi AN_0}\int_{-Q}^{Q}\frac{du}{\sqrt{Q^2-u^2}}\int_{-\infty}^{\infty}dv\,\varphi\left(\frac{x-u}{\sqrt{N_0}}\right)\varphi\left(\frac{y-v}{A\sqrt{N_0}}\right) \\ &\quad\times\left[\delta(v-\omega_0\sqrt{Q^2-u^2})+\delta(v+\omega_0\sqrt{Q^2-u^2})\right] \\ &= \frac{1}{2\pi AN_0}\int_{-Q}^{Q}\varphi\left(\frac{x-u}{\sqrt{N_0}}\right)\left\{\varphi\left(\frac{y-\omega_0\sqrt{Q^2-u^2}}{A\sqrt{N_0}}\right)+\varphi\left(\frac{y-\omega_0\sqrt{Q^2-u^2}}{A\sqrt{N_0}}\right)\right\} \\ &\quad\times\frac{du}{\sqrt{Q^2-u^2}} \end{aligned} \tag{4.21}$$

Let

$$\theta = \cos^{-1}\frac{u}{Q}, \quad d\theta = -\frac{du}{\sqrt{Q^2-u^2}}, \quad 0 < \theta < \pi \tag{4.22}$$

and introduce the dimensionless parameters

$$\xi = \frac{x}{\sqrt{N_0}}, \qquad a = \frac{Q}{\sqrt{N_0}} \tag{4.23}$$

It is easily established that

$$\int_{-\infty}^{\infty} |z|\varphi(z+a)dz = 2[\varphi(z)+a\varphi^{(-1)}(a)], \tag{4.24}$$

so implementation of (4.1) leads to

$$N(\xi; a) = \frac{2A}{\pi}\int_0^{\pi}\varphi(\xi-a\cos\theta)[\varphi(\frac{\omega_0}{A}a\sin\theta)$$

$$+\frac{\omega_0}{A}a\sin\theta\,\varphi^{(-1)}(\frac{\omega_0}{A}a\sin\theta)]\,d\theta \tag{4.25}$$

a result of Rice. Let us remark that his approach was very much different in style; the present approach is interesting in itself as an application of the classical ideas in probability theory.

Some special cases are at once evident; for the noise power only case $Q = 0$ and

$$N(\xi; 0) = \frac{A}{\pi}e^{-\xi^2/2} \tag{4.26}$$

The transgression rate for the extreme $N_0 \to 0$ should be either ω_0/π or 0 and that for $Q \to \infty$ should be ω_0/π. Bendat [5] remarks about this point. The well-known properties of the error function and the delta function yield the expected results,

$$\lim_{N_0 \to 0} \varphi\left(\frac{\omega_0 Q}{A\sqrt{N_0}} \sin \theta\right) = 0$$

$$\lim_{N_0 \to 0} \varphi^{(-1)}\left(\frac{\omega_0 Q}{A\sqrt{N_0}} \sin \theta\right) = \frac{1}{2} \quad 0 < \theta < \pi \qquad (4.27)$$

so Korevaar's formula yields

$$\lim_{N_0 \to 0} N(\xi; a) = \frac{2}{\pi}\int_0^{\pi} \delta(x - Q\cos\theta)\,\frac{1}{2}\,Q\,\omega_0 \sin\theta \, d\theta$$

$$= \begin{cases} 0, & |x| > Q \\ \dfrac{\omega_0}{\pi}, & |x| < Q \end{cases} \qquad (4.28)$$

For the other extreme,

$$\lim_{Q \to \infty} N(\xi; a) = \frac{2}{\pi}\int_0^{\pi} \delta(-\cos\theta)\,\frac{1}{2}\,\omega_0 \sin\theta \, d\theta$$

$$= \frac{\omega_0}{\pi} \qquad (4.29)$$

4.1 Summit Formula

The extension of the methods of the previous section to one higher level is straightforward since the formula for the average number of summits occurring in the region $X(t) \leq x$ is known to be

$$M = \int_{-\infty}^{x} dx' \int_{-\infty}^{0} |z| p_{X,Y,Z}(x', 0, z) dz \tag{4.30}$$

where

$$Z(t) = X''(t)$$

Define the variance of the second time derivative of the noise process $N(t)$ to be $B^4 N_0$. Interest is now in the three independent processes of a continuous time parameter,

$$\begin{aligned} X(t) &= N(t) + Q \sin(\omega_0 t + \theta) \\ &= N(t) + S(t) \\ Y(t) &= N'(t) + S'(t) \\ Z(t) &= N''(t) + S''(t). \end{aligned} \tag{4.31}$$

The moment generating function for $[X(t), Y(t), Z(t)]$ is

$$M_{X,Y,Z}(a_1, a_2, a_3) = M_{N,N',N''}(a_1, a_2, a_3) M_{S,S',S''}(a_1, a_2, a_3) \tag{4.32}$$

and the corresponding convolution is

$$p_{X,Y,Z}(x, y, z) = \iiint_{-\infty}^{\infty} p_{N,N',N''}(x-u, y-v, z-w)$$

$$\times\ p_{S,S',S''}(u, v, w)\,du\,dv\,dw \tag{4.33}$$

The covariance matrix of the noise term and its first two derivatives has the form

$$\mathfrak{M}_3 = N_0 \begin{bmatrix} 1 & 0 & -A^2 \\ 0 & A^2 & 0 \\ -A^2 & 0 & B^4 \end{bmatrix} \tag{4.34}$$

and the discriminant is

$$|\mathfrak{M}_3| = N_0^3 A^2 [B^4 - A^4] \tag{4.35}$$

while the inverse matrix is

$$\mathfrak{M}_3^{-1} = \frac{1}{N_0} \begin{bmatrix} \frac{B^4}{B^4-A^4} & 0 & \frac{A^2}{B^4-A^4} \\ 0 & \frac{1}{A^2} & 0 \\ \frac{A^2}{B^4-A^4} & 0 & \frac{1}{B^4-A^4} \end{bmatrix} \tag{4.36}$$

That is

$$p_{N,N',N''}(x, y, z) = \frac{1}{(2\pi N_0)^{3/2} A\sqrt{B^4-A^4}} \exp\left[- \frac{x^2 + \frac{y^2}{A^2} + \frac{(z+A^2x)^2}{B^4-A^4}}{2N_0} \right] \tag{4.37}$$

(See Rice [44] below his Equation (66).)

An extension of the argument in the previous section leads to a signal density function

$$p_{S,S',S''}(x, y, z) = \begin{cases} 0, & |x| > Q \\ \left[\delta(y-\omega_0\sqrt{Q^2-x^2})+\delta(y+\omega_0\sqrt{Q^2-x^2})\right] \dfrac{\delta(z+\omega_0^2 x)}{2\pi\sqrt{Q^2-x^2}}, & |x| < Q \end{cases} \tag{4.38}$$

Hence

$$\begin{aligned} p_{X,Y,Z}(x,y,z) &= \frac{1}{\pi A N_0^{3/2}\sqrt{B^4-A^4}} \int_{-Q}^{Q} \frac{du}{\sqrt{Q^2-u^2}} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \varphi\left(\frac{x-u}{\sqrt{N_0}}\right)\varphi\left(\frac{y-v}{A\sqrt{N_0}}\right) \\ &\quad \times \varphi\left(\frac{z+A^2x-w}{\sqrt{N_0(B^4-A^4)}}\right)\left[\frac{\delta(v-\omega_0\sqrt{Q^2-u^2})+\delta(v+\omega_0\sqrt{Q^2-u^2})}{2}\right] \\ &\quad \times \delta(w+\omega_0^2 u) \\ &= \frac{1}{\pi A N_0^{3/2}\sqrt{B^4-A^4}} \int_{-Q}^{Q} \frac{du}{\sqrt{Q^2-u^2}}\, \varphi\left(\frac{x-u}{\sqrt{N_0}}\right) \varphi\left(\frac{z+A^2x+\omega_0^2u}{\sqrt{N_0(B^4-A^4)}}\right) \\ &\quad \times \frac{1}{2}\left[\varphi\left(\frac{y-\omega_0\sqrt{Q^2-u^2}}{A\sqrt{N_0}}\right) + \varphi\left(\frac{y+\omega_0\sqrt{Q^2-u^2}}{A\sqrt{N_0}}\right)\right] \end{aligned} \tag{4.39}$$

Since

$$\int_{-\infty}^{\infty} \varphi(x-a)dx = 1 \tag{4.40}$$

we may easily write

$$\iiint_{-\infty}^{\infty} p_{X,Y,Z}(x, y, z)dxdydz = \frac{1}{\pi}\int_{-Q}^{Q} \frac{du}{\sqrt{Q^2-u^2}} = 1 \qquad (4.41)$$

It is convenient to set

$$a = \frac{A^2}{\sqrt{B^4-A^4}}, \quad x' = \frac{x}{\sqrt{N_0}}, \quad y' = \frac{y}{A\sqrt{N_0}}, \quad z' = \frac{z}{\sqrt{N_0(B^4-A^4)}}$$

$$u = Q \cos \theta = a\sqrt{N_0} \cos \theta \qquad (4.42)$$

and change to dimensionless coordinates as defined by (4. 23). Implementation of the formula (4. 30) leads to

$$M(\xi; a) = \frac{A}{\pi a}\int_{-\infty}^{\xi} dx' \int_0^{\pi} d\theta\, \varphi(x'-a \cos \theta)\varphi(\frac{\omega_0}{A} a \sin \theta)$$

$$\times \int_0^{\infty} z'\varphi(z'-a[x'+\frac{\omega_0^2}{A^2} a \cos \theta])dz'$$

$$= \frac{A}{\pi a}\int_{-\infty}^{\xi} dx' \int_0^{\pi} d\theta\, \varphi(x'-a \cos \theta)\varphi(\frac{\omega_0}{A} a \sin \theta)$$

$$\times \left\{\varphi(a[x'+\frac{\omega_0^2}{A^2} a \cos \theta])+a[x'+\frac{\omega_0^2}{A^2} a \cos \theta]\right.$$

$$\left.\times [\frac{1}{2}+\varphi^{(-1)}(a[x'+\frac{\omega_0^2}{A^2} a \cos \theta])\right\} \qquad (4.43)$$

It is better to examine the second integral first. Integrate by parts so

$$\int_{-\infty}^{\xi} dx'\varphi(x'-a\cos\theta)[x'-a\cos\theta+(\frac{\omega_0^2}{A^2}+1)\,a\cos\theta]$$

$$\times\ [\frac{1}{2}+\varphi^{(-1)}(a[x'+\frac{\omega_0^2}{A^2}\,a\cos\theta])]$$

$$=-[\frac{1}{2}+\varphi^{(-1)}(a[\xi+\frac{\omega_0^2}{A^2}\,a\cos\theta])]\,\varphi(\xi-a\cos\theta)$$

$$+\ a\int_{-\infty}^{\xi}\varphi(x'-a\cos\theta)\varphi(a[x'+\frac{\omega_0^2}{A^2}\,a\cos\theta])$$

$$+\ \frac{\omega_0^2+A^2}{A^2}\,a\cos\theta\int_{-\infty}^{\xi}\varphi(x'-a\cos\theta)[\frac{1}{2}+\varphi^{(-1)}(a[x'+\frac{\omega_0^2}{A^2}\,a\cos\theta])]\,dx' \tag{4.44}$$

That is,

$$M(\xi;\,a)=\frac{A}{\pi a}\int_0^{\pi}d\theta\varphi(\frac{\omega_0}{A}\,a\sin\theta)\left\{-\varphi(\xi-a\cos\theta)\left[\frac{1}{2}+\varphi^{(-1)}\left(a\left[\xi+\frac{\omega_0^2}{A^2}\,a\cos\theta\right]\right)\right]\right.$$

$$+\ (1+a^2)\int_{-\infty}^{\xi}\varphi(x'-a\cos\theta)\varphi\left(a\left[x'+\frac{\omega_0^2}{A^2}\,a\cos\theta\right]\right)$$

$$\left.+\ a(\frac{\omega_0^2+A^2}{A^2})\,a\cos\theta\int_{-\infty}^{\xi}\varphi(x'-a\cos\theta)\left[\frac{1}{2}+\varphi^{(-1)}\left(a\left[x'+\frac{\omega_0^2}{B^2}\,a\cos\theta\right]\right)\right]dx'\right\} \tag{4.45}$$

It is easy to establish that the second term has the form

$$\int_{-\infty}^{\xi} \varphi(x'-a\cos\theta)\varphi\left(a\left[x'+\frac{\omega_0^2}{A^2}a\cos\theta\right]\right)dx'$$

$$= \varphi\left(\frac{a(\omega_0^2+A^2)}{A^2\sqrt{a^2+1}}a\cos\theta\right)\int_{-\infty}^{\xi}\varphi\left(x'\sqrt{a^2+1}+\frac{(a^2\omega_0^2-A^2)}{A^2\sqrt{a^2+1}}a\cos\theta\right)dx'$$

$$= \frac{1}{\sqrt{a^2+1}}\varphi\left(\frac{a(\omega_0^2+A^2)}{A^2\sqrt{a^2+1}}a\cos\theta\right)\left[\frac{1}{2}+\varphi^{(-1)}\left(\xi\sqrt{a^2+1}+\frac{a^2(\omega_0^2-A^2)}{A^2\sqrt{a^2+1}}a\cos\theta\right)\right] \tag{4.46}$$

Since

$$\frac{\sqrt{a^2+1}}{a} = \frac{B^2}{A^2} \tag{4.47}$$

the average number of summits below $x = \xi\sqrt{N_0}$ is written as

$$M(\xi;a) = \frac{1}{\pi}\int_0^{\pi} d\theta\,\varphi\left(\frac{\omega_0}{A}a\sin\theta\right)$$

$$\times\left\{\frac{B^2}{A}\varphi\left(\frac{\omega_0^2+A^2}{B^2}a\cos\theta\right)\left[\frac{1}{2}+\varphi^{(-1)}\left(a\left[\frac{B^2}{A^2}\xi+\frac{\omega_0^2-\frac{A^2}{a^2}}{B^2}a\cos\theta\right]\right)\right]\right.$$

$$-A\varphi(\xi-a\cos\theta)\left[\frac{1}{2}+\varphi^{(-1)}\left(a\left[\xi+\frac{\omega_0^2}{A^2}a\cos\theta\right]\right)\right]$$

$$\left.+\frac{\omega_0^2+A^2}{A}a\cos\theta\int_{-\infty}^{\xi}\varphi(x'-a\cos\theta)\left[\frac{1}{2}+\varphi^{(-1)}\left(a\left[x'+\frac{\omega_0^2}{A^2}a\cos\theta\right]\right)\right]dx'\right\} \tag{4.48}$$

If, in similar fashion, $K(\xi;a)$ designates the average number of summits above $x = \xi\sqrt{N_0}$ then

$$K(\xi;a) = \frac{1}{\pi}\int_0^{\pi} d\theta\,\varphi\left(\frac{\omega_0}{A}\,a\sin\theta\right)$$

$$\times\left\{\frac{B^2}{A}\,\varphi\left(\frac{\omega_0^2+A^2}{B^2}\,a\cos\theta\right)\left[\frac{1}{2}-\varphi^{(-1)}\left(\alpha\left[\frac{B^2\xi}{A}+\frac{\omega_0^2-\frac{A^2}{\alpha^2}}{B^2}\,a\cos\theta\right]\right)\right]\right.$$

$$+A\varphi(\xi-a\cos\theta)\left[\frac{1}{2}+\varphi^{(-1)}\left(\alpha\left[\xi+\frac{\omega_0^2}{A^2}\,a\cos\theta\right]\right)\right]$$

$$\left.+\frac{\omega_0^2+A^2}{A}\,a\cos\theta\int_{\xi}^{\infty}\varphi(x'-a\cos\theta)\left[\frac{1}{2}+\varphi^{(-1)}\left(\alpha\left[x'+\frac{\omega_0^2}{A^2}\,a\cos\theta\right]\right)\right]dx'\right\} \tag{4.49}$$

The total average number of summits occurring at all levels is

$$M(\xi;a)+K(\xi;a) = \frac{1}{\pi}\int_0^{\pi} d\theta\,\varphi\left(\frac{\omega_0}{A}a\sin\theta\right)\left\{\frac{B^2}{A}\,\varphi\left(\frac{\omega_0^2+A^2}{B^2}\,a\cos\theta\right)\right.$$

$$+\frac{\omega_0^2 A^2}{A^2}\,a\cos\theta\int_{-\infty}^{\infty}\varphi(x'-a\cos\theta)$$

$$\left.\times\left[\frac{1}{2}+\varphi^{(-1)}\left(\alpha\left[x'+\frac{\omega_0^2}{A^2}\,a\cos\theta\right]\right)\right]dx'\right\} \tag{4.50}$$

These formulas should be checked for extreme values of the parameters. First, let signal energy diminish to zero (i.e., $a \to 0$). Then

$$M(\xi;0) = \frac{1}{2\pi}\left\{\frac{B^2}{A}\left[\frac{1}{2}+\varphi^{(-1)}(\xi\sqrt{\alpha^2+1})\right] - Ae^{-\xi^2/2}\left[\frac{1}{2}+\varphi^{(-1)}(\xi\alpha)\right]\right\}, \tag{4.51}$$

and

$$K(\xi;0) = \frac{1}{2\pi}\left\{\frac{B^2}{A}\left[\frac{1}{2} - \varphi^{(-1)}(\xi\sqrt{a^2+1})\right] + Ae^{-\xi^2/2}\left[\frac{1}{2} + \varphi^{(-1)}(\xi a)\right]\right\} \qquad (4.52)$$

which yields

$$M(\xi;\, 0) + K(\xi;\, 0) = \frac{1}{2\pi}\,\frac{B^2}{A} \qquad (4.53)$$

all which agree with Rice.

Observe that the average number of summits occurring on the positive side, noise only, is

$$K(0;\, 0) = \frac{A^2+B^2}{4\pi A} \qquad (4.54)$$

while the average number of summits occurring on the negative side is

$$M(0;\, 0) = \frac{B^2-A^2}{4\pi A} \qquad (4.55)$$

Second, Bendat [5] points out that if ξ is relatively large in value the average number of summits occurring in the region $X(t) \geq x = \xi\sqrt{N_0}$, noise only input, should be about one-half the average number of transgressions of x. This checks out since

$$K(\xi;\, 0) \simeq \frac{A}{2\pi}\, e^{-\xi^2/2}, \quad \xi \gg 1 \qquad (4.56)$$

Third, if noise power diminishes to zero the fixed sinusoid remains. The passage to the limit is facilitated by introducing parameters

$$P_0 = A^2 N_0, \qquad R_0 = B^4 N_0 \tag{4.57}$$

which are the variances associated with $Y(t)$ and $Z(t)$. Then (4.48) may be written as

$$\begin{aligned} M(x;Q) = \frac{1}{\pi}\int_0^{\pi} d\theta\, \varphi\left(\frac{\omega_0 Q}{P_0}\sin\theta\right) & \\ \times \Bigg\{ \sqrt{\frac{R_0}{P_0}}\, \varphi\left(\frac{P_0+\omega_0^2 N_0}{N_0 R_0}\, Q\cos\theta\right) & \\ \times \left[\frac{1}{2}+\varphi^{(-1)}\left(\alpha\left[\frac{xR_0}{P_0} + \frac{\omega_0^2 - \frac{P_0}{\alpha^2 N_0}}{R_0}\, Q\cos\theta\right]\right)\right] & \\ + \frac{P_0+\omega_0^2 N_0}{N_0\sqrt{N_0}P_0}\, Q\cos\theta \int_{-\infty}^{x} dx'\varphi\left(\frac{x'-Q\cos\theta}{\sqrt{N_0}}\right) & \\ \times \left[\frac{1}{2}+\varphi^{(-1)}\left(\alpha\left[\frac{x'}{\sqrt{N_0}} + \frac{\omega_0^2\sqrt{N_0}}{P_0}\, Q\cos\theta\right]\right)\right]\Bigg\} & \end{aligned} \tag{4.58}$$

The basic formulas are

$$\lim_{\lambda\to\infty} \lambda\varphi(\lambda x) = \delta(x) \tag{4.59}$$

and

$$\lim_{\lambda\to\infty} \left[\frac{1}{2}+\varphi^{(-1)}(\lambda x)\right] = \begin{cases} 0, & x<0 \\ \frac{1}{2}, & x=0 \\ 1, & x>0 \end{cases} \tag{4.60}$$

Successive passages to the limit yield

$$\lim_{\substack{R_0 \to 0 \\ P_0 \to 0 \\ N_0 \to 0}} M(x;Q) = \frac{\omega_0^2 Q}{\pi} \int_0^{\pi/2} d\theta \int_{-\infty}^{x} dx' \delta(\omega_0 Q \sin\theta) \cos\theta\, \delta(x' - Q\cos\theta) \tag{4.61}$$

Let

$$\begin{aligned} u &= \omega_0\, Q \sin\theta \\ du &= \omega_0\, Q \cos\theta d\theta \end{aligned} \tag{4.62}$$

so

$$\lim_{N_0 \to 0} M(x;Q) = \frac{\omega_0}{\pi} \int_0^{\omega_0 Q} du \int_{-\infty}^{x} dx' \delta(x') \delta\left(x' - Q\sqrt{1 - \frac{u^2}{\omega_0^2 Q^2}}\right.$$

$$= \frac{\omega_0}{2\pi} \int_{-\infty}^{x} \delta(x' - Q) dx'$$

$$= \begin{cases} \dfrac{\omega_0}{2\pi}, & 0 < Q < x \\ 0, & 0 < x < Q \end{cases} \tag{4.63}$$

(Note that $\int_0^{\infty} \delta(t)dt = \frac{1}{2}$; see p. 71, Van der Pol and Bremmer [50].)

The formulas for $M(\xi; a)$ and $K(\xi; a)$ are somewhat formidable but for large values of certain parameters suitable approximations are available, particularly for narrow bandpass filters. From Table 4.1 it is clear that if

$$\xi > \frac{\omega_0^2 a}{A^2} + \frac{3}{a} \tag{4.64}$$

then

$$\frac{1}{2} + \varphi^{(-1)}(a[x' + \frac{\omega_0^2 a}{A^2} a \cos\theta]) \simeq 1$$

$$\frac{1}{2} + \varphi^{(-1)}(a[\xi + \frac{\omega_0^2}{A^2} a \cos\theta]) \simeq 1$$

and

$$\frac{1}{2} - \varphi^{(-1)}(a[\frac{B^2}{A^2}\xi + \frac{\omega_0^2 - \frac{A^2}{a^2}}{B^2} a \cos\theta]) \simeq 0 \tag{4.65}$$

for all value of θ. Hence the formula for the average number of summits occurring in the region $X(t) > x = \xi\sqrt{N_0}$ reduces to

$$K(\xi; a) \simeq \frac{A}{\pi}\int_0^{\pi} d\theta \varphi(\frac{\omega_0}{B} a \sin\theta)$$
$$\times \left\{ \varphi(\xi - a\cos\theta) - \frac{\omega_0^2 + A^2}{A^2} a \cos\theta\ \varphi^{(-1)}(\xi - a\cos\theta)\right\} \tag{4.66}$$

Table 4.1. Some values of $\varphi^{(-1)}(x)$.

x	$\varphi^{(-1)}(x)$
0	0
1	0.34135
2	0.47725
3	0.49867

Integration by parts of the second term is quite simple; the final form of the approximating summit formula and the transgression formula (4.25) may be exhibited side by side as

$$K(\xi; a) \simeq \frac{A}{\pi} \int_0^{\pi} \varphi(\xi - a \cos\theta) \left\{ \varphi\left(\frac{\omega_0 a \sin\theta}{A}\right) + \frac{\omega_0^2 + A^2}{A\omega_0} a \sin\theta \right.$$

$$\left. \times\ \varphi^{(-1)}\left(\frac{\omega_0}{A} a \sin\theta\right) \right\} d\theta \qquad (4.67)$$

and

$$N(\xi; a) = \frac{2A}{\pi} \int_0^{\pi} \varphi(\xi - a \cos\theta) \left\{ \varphi\left(\frac{\omega_0}{A} a \sin\theta\right) + \frac{\omega_0}{A} a \sin\theta \right.$$

$$\left. \times\ \varphi^{(-1)}\left(\frac{\omega_0}{A} a \sin\theta\right) \right\} d\theta \qquad (4.68)$$

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APPENDIX

APPENDIX

Computer results for fourth order correlation function of smooth limiter ($c = 1$, $\rho_{ii} = 1$, $\rho_{ij} = \rho$, $i, j = 1, 2, 3, 4$).

Degree of term	Contribution	Sum
$\rho = .1$		
2	.00018998	.00018998
3	$-1.2665144 \times 10^{-5}$	.00017731
4	1.1081996×10^{-6}	.00017842
5	$-0.9498854 \times 10^{-7}$	.00017833
6	0.8621529×10^{-8}	.00017833
7	$-8.7072833 \times 10^{-10}$	.00017833
8	$0.5889075 \times 10^{-10}$	.00017833
9	$-0.8852953 \times 10^{-11}$	.00017833
$\rho = .2$		
2	.00075991	.00075991
3	- .00010132	.00065859
4	1.7731196×10^{-5}	.00067632
5	$-3.0396340 \times 10^{-6}$	.00067328
6	5.5177782×10^{-7}	.00067383
7	$-1.1145321 \times 10^{-7}$	.00067372
8	1.5076034×10^{-8}	.00067373
9	$-4.5327114 \times 10^{-9}$	.00067373

Degree of term	Contribution	Sum
$\rho = .3$		
2	.00170979	.00170979
3	- .00034196	.00136784
4	0.8976419×10^{-4}	.00145760
5	$-2.3082221 \times 10^{-5}$	.00143452
6	6.2850964×10^{-6}	.00144080
7	$-1.9042830 \times 10^{-6}$	.00143890
8	3.8638242×10^{-7}	.00143928
9	$-1.7425268 \times 10^{-7}$	.00143911
$\rho = .4$		
2	.00303963	.00303963
3	- .00081057	.00222907
4	.00028370	.00251276
5	$-0.9726827 \times 10^{-4}$	.00241550
6	3.5313774×10^{-5}	.00245081
7	$-1.4266013 \times 10^{-5}$	.00243654
8	3.8594649×10^{-6}	.00244040
9	$-2.3207486 \times 10^{-6}$	.00243808
$\rho = .5$		
2	.00474943	.00474943
3	- .00158314	.00316629
4	.00069262	.00385891
5	- .00029684	.00356207
6	.00013471	.00369678
7	$-0.6802566 \times 10^{-4}$	.00362876
8	2.3004210×10^{-5}	.00365176
9	$-1.7290930 \times 10^{-5}$	.00363447

Degree of term	Contribution	Sum
$\rho = .6$		
2	.00683918	.00683918
3	- .00273567	.00410351
4	.00143623	.00553973
5	- .00073863	.00480110
6	.00040225	.00520335
7	- .00024375	.00495960
8	1.2770633×10^{-4}	.00508730
9	$-0.8921738 \times 10^{-4}$	.00499809
$\rho = .7$		
2	.00930888	.00930888
3	- .00434414	.00496474
4	.00266079	.00762552
5	- .00159647	.00602905
6	.00101431	.00704337
7	- .00071708	.00632628
8	.00043831	.00676460
9	- .00035725	.00640735
$\rho = .8$		
2	.01215854	.01215854
3	- .00648455	.00567398
4	.00453919	.01021317
5	- .00311258	.00710059
6	.00226008	.00936067
7	- .00182605	.00753462
8	.00127562	.00891024
9	- .00118822	.00762202

Degree of term	Contribution	Sum
$\rho = .9$		
2	.01538815	.01538815
3	- .00923289	.00615526
4	.00727090	.01342616
5	- .00560898	.00781718
6	.00458183	.01239901
7	- .00416467	.00823435
8	.00327297	.01150732
9	- .00342981	.00807751
$\rho = 1$		
2	.01899772	.01899772
3	- .01266514	.00633257
4	.01108200	.01741457
5	- .00949886	.00791571
6	.00862153	.01653725
7	- .00870729	.00782996
8	.00760331	.01543327
9	- .00885296	.00658031