In this paper we consider the behavior of certain surfaces at certain boundary points. The surfaces under consideration satisfy a topological definition and are of 2-dimension in 3-dimensional Euclidean space with the boundary a finite set of straight line segments. It is shown that the surface of minimum area with a given boundary is locally Euclidean at all non-vertex boundary points. The key to the proof is a theorem in 1 which itself concerns the behavior of a set of points under very restricted conditions. It is shown in 1 that for almost all interior points the conditions of the lemma are satisfied.

This paper first shows that the part of the given surface interior to some sphere centered at any non-vertex boundary point lies near a plane passing through the point in question. Secondly it is shown that for any point of the surface lying in the sphere the surface interior to any smaller neighborhood of that point lies near a plane. "Near" here refers to nearness with respect to the radius of the sphere or neighborhood in consideration.

From these conditions we may construct a bounded point set satisfying the hypothesis of the aforementioned theorem. The theorem of this paper follows immediately.
ON THE STRUCTURE OF
MINIMUM SURFACES AT THE BOUNDARY

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ON THE STRUCTURE OF
MINIMUM SURFACES AT THE BOUNDARY

The problem of Plateau was solved for surfaces of varying type in [1]. One of the problems considered previously was to prove that for almost all interior points the surface of minimum area is locally Euclidean. In this paper we consider surfaces satisfying the same definition as previously but with added restrictions. For a detailed description and some examples of surfaces in this class the reader is referred to the original paper. Therefore such a discussion will be omitted here and only the definition will be restated.

**Definition:** Let \( G \) be a compact Abelian group. Let \( S \) be a closed set in \( n \)-dimensional Euclidean space and \( A \) a closed subset of \( S \). Let \( n \) be a non-negative integer. Then there is defined the Čech homology group \( H_n(S; A; G) \); if \( A \) is empty this is written \( H_n(S; G) \). Let \( K \) be the kernel of the inclusion homomorphism

\[
i_n : H_{n-1}(A; G) \to H_{n-1}(S; G).
\]

Let \( L \) be any subgroup of \( H_{n-1}(A; G) \). Then we say \( S \) is a surface of class \( g^{C} \) with boundary \( \partial S \) if \( K \supset L \).

We will consider here only surfaces such that \( G \) is the group of integers modulo-2, \( n=3 \), and \( m=2 \). Further we restrict the boundary to be a finite set of straight line segments. We will show that the surface of minimum area with a given boundary is locally Euclidean at all non-vertex boundary points.

**Notation:** Throughout this paper we will adopt the following symbols:

\( S(x, r) \) is the closed solid sphere, center \( x \), radius \( r \).
s(x,r) is the surface of the sphere.

$S_0$ is a minimal surface satisfying our restrictions.

$l(x,r) = S_0 \cdot (x,r)$.

$(X,r)$ is the set of points whose distance from $X$ does not exceed $r$.

$K(x,r) = S_0 \cdot S(x,r)$.

$B$ is the set of boundary points of $S_0$.

$V$ is the set of vertex points of $S_0$.

$\Lambda^mX$ is the $m$-dimensional Hausdorff spherical measure defined as: Suppose $X$ is a set in $N$-dimensional Euclidean space.

Consider a set of spheres $S(x_1,r_1)$ such that $r_1 < \delta$ and $X \subseteq \sum_i S(x_1,r_1)$. Let $\Lambda^mX$ be the lower bound of $\sum_i W_i r_i^m$ taken over all such sets of spheres; where $W_m = 2$ if $m = 1$ and $W_m = \pi$ if $m = 2$. Let $\Lambda^mX = \lim_{\delta \to 0} \Lambda^mX$.

$S(P_0,R_0)$ or $S(P_0,r_0)$ will have no vertex points in the interior.

**Lemma 1.** Given $\epsilon_1 > 0$ and $S_0$ there exists an $r_0 > 0$.

$r_0 = r_0(P_0,\epsilon_1)$ where $P_0 \in B\cdot V$, such that

$$4r_0^2 \leq \Lambda^2K(P_0,r_0) \leq 4\pi r_0^2(1 + \epsilon_1).$$

The proof of this is similar to the relevant parts of [1]

Chapter 5.

**Lemma 2.** Let $S$ be a surface contained in $S(x,r)$ with boundary $b \in H(B)$ where $B$ consists of two rays $R_1 = xa$ and $R_2 = xb$ (where $s \in s(x,r)$ and $b \in s(x,r)$), plus a set $l(x,r) \subseteq s(x,r)$. If
\[ \Lambda(x,r) < \infty \] then either there exists \( L^2(x,r) \) such that

1) \( L(x,r) = L^2(x,r) + L_\alpha(x,r) \)

11) \( L^2(x,r) \) is a simple Jordan curve joining \( a \) to \( b \) or there exists a surface \( S^* \) with the same boundary such that

\[ S^* = \{ ax + ib \} \]

This follows from purely topological considerations.

**Corollary 1.** \( \Lambda(x,r) \geq \pi r \).

**Corollary 2.** Given any \( \epsilon > 0 \), if

\[ \pi r \leq \Lambda(x,r) \leq \pi r(1 + \epsilon) \]

then \( \Lambda^2(x,r) \geq \pi r^2 \) and \( \Lambda^2(x,r) \leq \pi r^2 r \cdot \epsilon \).

The alternative of the above lemma being excluded by lemma 1.

**Lemma 2.** If \( P_0 \in C-Y \) and there exists \( \epsilon > 0 \) such that

\[ 2 \pi r^2 \leq \int_0^r \Lambda(P_0,t)dt \leq \pi r^2 (1 + \epsilon) \]

then for some \( t \), \( 0 \leq t \leq x_0 \),

\[ \pi r \leq \Lambda(P_0,t) \leq \pi r(1 + \epsilon) \]

Write \( \Lambda(P_0,t) \leq \pi r + R(t) \), \( 0 \leq t \leq x_0 \). Then

\[ \int_0^r \Lambda(P_0,t)dt \leq \int_0^r (\pi r + R)dt = \frac{1}{2} \pi r^2 + \int_0^R Rdt \]

and by hypothesis

\[ \int_0^R Rdt \leq \frac{1}{2} \pi r^2 \epsilon. \tag{1} \]

Assume \( R > \pi r \epsilon \) for all \( t \), then

\[ \int_0^R Rd \geq \frac{1}{2} \pi r^2 \epsilon \]

contradicting (1).
Lemma 4. If $P_0 \in B-V$ and $\epsilon_4 > 0$,

$$4 \pi r_0^2 \leq \Lambda^2(P_0,E) \leq 4 \pi r_0^2(1 + \epsilon_4)$$

then for all $r < r_0$,

$$4 \pi r^2 \leq \int_0^r \Lambda 1(P_0,t) dt \leq 4 \pi r^2(1 + \epsilon_4).$$

Let $\int_0^r \Lambda 1(P_0,t) dt = F(r)$. By [1, lemma 1*]

$$F(r) \leq \frac{1}{2} r \Lambda 1(P_0,r) = \frac{1}{2} r F'(r)$$

hence

$$\frac{F'(r)}{F(r)} > \frac{2}{r}$$

and integrating from $r_2$ to $r_1$ where $r_2 < r_1 < r_0$ we obtain

$$\log \frac{F(r_1)}{F(r_2)} > \log \frac{r_1^2}{r_2^2}$$

and hence

$$\frac{F(r_1)}{r_1^2} > \frac{F(r_2)}{r_2^2}$$

and so $F(r)/r^2$ is a monotone non-decreasing function of $r$. By [1, lemma 4] and the hypothesis

$$\frac{F(r)}{r^2} = \int_0^r \frac{\Lambda 1(P_0,t) dt}{r^2} \leq \frac{\Lambda^2(P_0,r)}{r^2} \leq 4 \pi (1 + \epsilon_4)$$

for all $r < r_0$.

Lemma 5. If $P_0 \in B-V$, $0 < \epsilon_5 \leq \epsilon_4$, 

$$4 \pi r_0^2 \leq \Lambda^2(P_0,E) \leq 4 \pi r_0^2(1 + \epsilon_5^2)$$

then for all $r < r_0$. 
\[ \frac{1}{2} \pi r^2 \leq \Lambda^2 K(P_0,r) \leq \frac{1}{2} \pi r^2 (1 + 4 \varepsilon_5). \]

By lemma 4 for all \( r < r_0 \),
\[ \frac{1}{2} \pi r^2 \leq \int_0^r \Lambda 1(P_0,t) \, dt \leq \frac{1}{2} \pi r^2 (1 + \varepsilon_5^2). \] (2)

Assume that \( \frac{1}{2} \pi r^2 \leq \Lambda^2 K(P_0,r) \leq \frac{1}{2} \pi r^2 (1 + 4 \varepsilon_5) \) is false for some \( r < r_0 \). Then there exists an \( r_1 < r_0 \) such that
\[ \Lambda^2 K(P_0,r_1) \neq \frac{1}{2} \pi r_1^2 (1 + 4 \varepsilon_5) \]

hence for all \( t \) such that
\[ r_1 < t < r_1 \left( 1 + 4 \varepsilon_5 \right)^{1/3} \quad \Rightarrow \quad r_2 < r_0 \] (3)

\[ \Lambda^2 K(P_0,t) > \Lambda^2 K(P_0,r_1) > \frac{1}{2} \pi r_1^2 (1 + 4 \varepsilon_5) \]
\[ > \frac{1}{2} \pi t^2 (1 + \varepsilon_5) \]
so that for all \( t \) satisfying (3)
\[ \Lambda 1(P_0,t) > \pi t (1 + \varepsilon_5) \]
and hence
\[ \int_0^r \Lambda 1(P_0,t) \, dt = \int_0^{r_1} \Lambda 1(P_0,t) \, dt + \int_{r_1}^r \Lambda 1(P_0,t) \, dt \]
\[ > \int_0^{r_1} \Lambda 1(P_0,t) \, dt + \frac{1}{2} \pi (1 + \varepsilon_5) (r_2^2 - r_1^2) \]
and using (2)
\[ \frac{1}{2} \pi r_2^2 (1 + \varepsilon_5^2) > \frac{1}{2} \pi r_1^2 + \frac{1}{2} \pi (1 + \varepsilon_5) (r_2^2 - r_1^2) \]
which contradicts (3) if \( \varepsilon < \frac{1}{4} \) and hence our assumption is false.

**Lemma 6.** If \( P_0 \in B^V \) and \( \pi t \leq \Lambda 1(P_0,t) \leq \pi t (1 + \varepsilon_5^2) \),
then there exists some half-plane \( \Gamma^*(P_0,t) \) through \( P_0 \) such that
\[ B^*(S(P_0,t)) \subseteq \Gamma^*(P_0,t) \]
and \(1^*(P_0, t) \subset (\Gamma^*(P_0, t), \in_0 t)\).

Let \(\Delta\) be the plane through \(P_0\) orthogonal to \(B \cdot S(P_0, t)\) and let \(\Gamma(P_0, t)\) be any plane through \(B \cdot S(P_0, t)\) such that
\[
\Delta \cdot 1^*(P_0, t) \cdot \Gamma(P_0, t) \neq 0.
\]
\(\Gamma(P_0, t)\) is divided into two half-planes by \(B \cdot S(P_0, t)\) extended and let \(\Gamma^*(P_0, t)\) be that half-plane such that
\[
\Delta \cdot 1^*(P_0, t) \cdot \Gamma^*(P_0, t) \neq 0.
\]
By use of elementary spherical trigonometry it can be shown that for any point \(x \in 1^*(P_0, t)\), then
\[
x \in (\Gamma^*(P_0, t), \in_0 t).
\]

Henceforth, the symbols \(\Gamma(P_0, t)\) and \(\Gamma^*(P_0, t)\) will denote a plane and half-plane, respectively, as constructed in this lemma.

**Lemma 7.** If \(P_0 \in B \cdot V, 0 < \in_4 \leq \min (\in_0^2, \in_2^2, \in_2)\) and
\[
\frac{1}{\pi} x_0^2 \leq \Lambda^2 K(P_0, x_0) \leq \frac{1}{\pi} x_0^2 (1 + \in_4^2),
\]
Let \(T\) be the class of numbers \(t \leq x_0\) such that
\[
\Lambda 1(P_0, t) \leq \pi t (1 + \in_4^2). \tag{4}
\]
Let \(t^* \in T\) and \(t^* < (1 - \in_4) \sup T\), then for all \(t\) of \(T\) less than \(t^*
\[
1^*(P_0, t) \subset (\Gamma^*(P_0, t^*), \in_0 t^*).
\]
Let \(\gamma(P_0, t) = \Gamma^*(P_0, t) \cdot s(P_0, t)\) and let \(C\) be any finite set of components of \(\gamma(P_0, t) + 1(P_0, t)\). We can find an open set \(G^*\) such that
\[
C \subset G^* \cdot (\gamma(P_0, t) + 1(P_0, t)) \subset (C, \in_0 t^*),
\]
and
\[
\Lambda G^* \cdot (\gamma(P_0, t) + 1(P_0, t)) \leq \Lambda C + \in_0^2 t^*.
\]
By an application of the Heine Borel Theorem we can find two closed disjoint sets \(F^*\) and \(F^*_2\) such that
\[ C \subset F^* + F^* \subset E \cdot (T(P_0,t) + 1(P_0,t)) \]  
and  
\[ 1^*(P_0,t) + T(P_0,t) \subset F^* \subset (1^*(P_0,t) + T(P_0,t), E^2 t). \]  

By (4) and lemma 6 we see that \( 1^*(P_0,t) \) lies within \( E^2 t \) of \( T(P_0,t) \).

The elementary domain consisting of the points of \( s(P_0,t) \) lying within \( E^2 t \) of \( T(P_0,t) \) will contain \( F^* \) and moreover this domain is contractible to a point so that by lemma 2A [1] there will exist a surface \( J^*(P_0,t) \) with boundary containing \( F^* \). Hence

\[ \land^2 J^*(P_0,t) \leq 2 \pi t \in \E^2 t. \]  

By lemma 2 and (4)

\[ \land 1^*(P_0,t) \leq \pi t \in \E, \]  
and hence by (5) and (6)

\[ \land F^* < \pi t \in \E. \]  

By [1, lemma 8] there will exist a surface \( J^*(P_0,t) \) lying in the convex hull of \( F^* \) with boundary \( \partial \Pi^*(F^*) \) such that

\[ \land^2 J^*(P_0,t) \leq 4 \sqrt{3} (\land F^*)^2 \leq 4 \sqrt{3} \pi t^2 \in \E. \]  

Let \( J^*(P_0,t) + J^*(P_0,t) = J(P_0,t) \). Then by virtue of [1, lemmas 13A and 11A], \( J(P_0,t) + K(P_0,t) \) is a surface with boundary \( T(P_0,t) + B \cdot S(P_0,t) \).

Now \( K(P_0,t) - K(P_0,t) + J(P_0,t) + J(P_0,t) \) is a surface with boundary \( T(P_0,t) + T(P_0,t) \) plus two intervals of \( B \) lying in \( S(P_0,t) - S(P_0,t) \). Let \( \phi \) be the angle of intersection of the planes \( T(P_0,t) \) and \( T(P_0,t) \). Then

\[ \land^2 K(P_0,t) - K(P_0,t) + J(P_0,t) + J(P_0,t) \]

\[ \leq \pi t^2 - \pi t^2 \cos \phi. \]  

This is immediate by projecting this surface onto \( T(P_0,t) \). Applying lemma 5 to the hypothesis we obtain
\[ \Lambda^2 K(p_0, t) \leq \frac{1}{2} \pi t^2 (1 + 4 \epsilon_1^2) \]

for all \( t < r_0 \). Together with (7), (8) and (9) and after some manipulation we obtain

\[ \sin \phi < 2^5 \epsilon_1 \]

as long as \( t < t^* \) and \( t > \epsilon_1^2 t^* \). While if \( t = \epsilon_1^2 t^* \) the result follows from elementary geometry. Thus \( \sin \phi < 2^5 \epsilon_1^2 \) for all \( t < t^* \).

**Lemma 8.** If \( p_0 \in V - V, \epsilon_0 < \epsilon_1^4 \) and

\[ \Lambda^2 K(p_0, r_0) \leq \Lambda^2 K(p_0, r_0) \leq \frac{1}{2} \pi r_0^2 (1 + \epsilon_1^4) \]

then for \( x < r_0/4 \)

\[ K(p_0, x) \subseteq \left( \Gamma^*(p_0, r_0), 2\epsilon_1^2 \epsilon_0 x \right). \]

Let \( h = \frac{1}{2} \pi \epsilon_1^4 \) and by lemma 4 for all \( t_0 < r_0 \)

\[ \frac{1}{2} \pi t_0^2 \leq \int_0^{t_0} \Lambda_1(p_0, t) dt \leq \frac{1}{2} \pi t_0^2 + ht_0^2. \]

Let \( E \) be the set of all \( r \leq t_0 < r_0/2 \). Then there exists a set \( C \subseteq E \)

\[ \Lambda C < \frac{1}{2} t_0 \] such that for any \( r \in (E - C) \)

\[ \Lambda r \leq \Lambda r \leq \pi r + 2\epsilon_1^4 r. \]

For, if not, then

\[ \int_0^{t_0} \Lambda_1(p_0, t) dt = \int_{-E} \Lambda_1(p_0, t) dt + \int_{E} \Lambda_1(p_0, t) dt \]

\[ > \frac{1}{2} \pi t_0 (t_0 - \Lambda C) + \Lambda C (\frac{1}{2} \pi t_0 + \epsilon_1^4 t_0) \]

which contradicts (11). Hence

\[ \int_{-E} \Lambda_1(p_0, t) dt = \int_{-E} \Lambda_1(p_0, t) dt - \int_{E} \Lambda_1(p_0, t) dt. \]
Therefore
\[ \int_G \Lambda 1(P_0, t) dt < \frac{1}{2} \pi t_o^2 + h t_o^2 - \int_{E-G} \pi t dt \]
\[ < \frac{1}{2} \pi t_o^2 + h t_o^2 + h^2 t_o^2 - \int_{E} \pi t dt \]
\[ = t_o^2(h + \pi h^2). \]  \hspace{1cm} (13)

By the corollary of lemma 2 and lemmas 4 and 3
\[ \int_{E-G} \Lambda 1^a(P_0, t) dt = \]
\[ \int_E \Lambda 1(P_0, t) dt - \int_G \Lambda 1(P_0, t) dt - \int_{E-G} \Lambda 1^a(P_0, t) dt \]
\[ \geq t_o^2(\frac{1}{2} \pi - h - \pi h^2 - 2h^3). \]  \hspace{1cm} (14)

Let \( t \in (E-G) \) and assume there is a point \( x \in K(P_0, t) \) such that
\[ x \notin (\Gamma^a(P_0, t_o), 2^6 E_t o). \]

By lemma 7, for all \( r < t_o \) and \( r \in (E-G) \)
\[ S(x, 2^5 E_t o) \cdot 1^a(P_0, r) = 0. \]  \hspace{1cm} (15)

We know that
\[ \Lambda^2 K(x, 2^5 E_t o) \geq 2^{10} E_t o^2 \pi. \]

However by lemma 5 and (15)
\[ \frac{1}{2} \pi t_o^2(1 + 4 E \pi - 2^{10} E \pi) \geq \Lambda^2 K(P_0, t_o) - K(x, 2^5 E_t o) \]
\[ > \int_{E-G} \Lambda 1^a(P_0, t) dt \]
\[ \geq t_o^2(\frac{1}{2} \pi - h - \pi h^2 - 2h^3) \]

which gives a contradiction if \( \epsilon < 1. \)

This concludes that whenever lemma 1 holds for \( S(P_0, r_o) \) then
\( K(P_0, r_o/4) \) lies in a narrow slab centered about a constructed plane.
through $P_0$. It is well to note that $P_0$ has thus far remained fixed.
We will now consider behavior of the surface interior to spheres
contained in $S(P_o, r_o)$.

**Lemma 9.** Suppose in cartesian coordinates, $B$ is the line
$x = z = 0$. $P_o \in B$. $P_1 = (\alpha, 0, 0)$ with $\alpha \geq 0$, and $s(P_o, x_o) \less g S(P_1, x_1)$
then
$$\frac{1}{\alpha^2} K(P_o, x_o) - K(P_1, x_1) \leq \left[\frac{1}{\alpha^2} \left( s(P_o, x_o) - s(P_1, x_1) \right) \cdot \left\{ z = 0 \right\} \right].$$

Let $\phi$ be the mapping $y' = |y|$, $x' = \sqrt{x^2 + z^2}$, $z' = 0$. $\phi$ is
distance reducing and therefore area reducing. Since every torus
which links $B_o$ meets $S_o$
$$\phi \left[ K(P_o, x_o) - K(P_1, x_1) \right] \geq \left[ s(P_o, x_o) - s(P_1, x_1) \right] \cdot \left[ z = 0 \right].$$

**Lemma 10.** If $P_o \in B - V$, $S(P_o, x_o) \supset S(P_1, x_1)$, $x_o = k x_1$,
$P_1 \in \delta (P_o, x_o)$ and $k x_1^2 \leq \alpha^2 K(P_o, x_o) \leq \alpha^2 K(P_1, x_1)$
then
$$k \pi \pi_1^2 \leq \alpha^2 K(P_1, x_1) \leq \pi \pi_1^2 (1 + \epsilon \epsilon_1).$$
The proof is a direct application of lemma 9 with $\alpha = 0$.

**Lemma 11.** If $P_o \in S_o - B$, $x = |P_j, x|$ where $P_j$ is the nearest
point of $B$ to $P$. $K(P_j, 2x) \cdot y = 0$ and
$$2 \pi x^2 \leq \alpha^2 K(P_j, 2x) \leq 2 \pi x^2 (1 + \epsilon \epsilon_1)$$
then
$$\pi x^2 \leq \alpha^2 K(P_j, x) \leq \pi x^2 (1 + 2 \epsilon \epsilon_1).$$
The proof again follows from lemma 9 with $P_j$ as the origin and
$y = x$. Note incidentally that $P_j$ belongs to the surface since it
is a point of $B$.

**Lemma 12.** Given $\epsilon > 0$ then if $\epsilon \epsilon_1 > 0$ is sufficiently small
and for every $S(P',r) \subset S(P_0,E_0)$, $P' \in E_0$,
$$\pi r^2 \leq \lambda^2 K(P',r) \leq \pi r^2(1 + \varepsilon_{12}),$$
then for each $r < 1/16 r_0$ there exists a plane $\Gamma(P_0,r)$ such that
$$K(P_0,r) \subset (\Gamma(P_0,r), \varepsilon_{12})$$
and
$$\Gamma(P_0,r) \cdot S(P_0,E_0) \subset (K(P_0,r), \varepsilon_{12}).$$
This is proved in [1] towards the end of Chapter 3.

Lemma 13. If $P_0 \in B-V$, $\varepsilon_{12} < \min \varepsilon_{12}, \varepsilon_{10} \frac{1}{5}, \varepsilon_{10} \frac{1}{8}$ and
$$\frac{1}{4} \pi (16R_0)^2 \leq \lambda^2 K(P_0,16R_0) \leq \frac{1}{4} \pi (16R_0)^2(1 + \varepsilon_{12}),$$
then for each $R < R_0$ and each $P \in K(P_0,R)$ there exists a plane $\Gamma(P,R)$
through the segment of the boundary through $P$ such that
$$K(P,R) \subset (\Gamma(P,R), \varepsilon_{12}^* R) \cdot S(P,R).$$

The proof is by using cases of $P$ and $R$:

**Case I.** $P \in B-V$; $R < R_0$.

By lemma 10
$$\frac{1}{4} \pi (8R_0)^2 \leq \lambda^2 K(P,8R_0) \leq \frac{1}{4} \pi (8R_0)^2(1 + 4 \varepsilon_{12}).$$

Thus by lemma 5
$$\frac{1}{4} \pi R^2 \leq \lambda^2 K(P,R) \leq \frac{1}{4} \pi R^2(1 + 16 \varepsilon_{12}).$$

(16)

and hence by lemma 8
$$K(P,2R) \subset (\Gamma^*(P,2R), 2^8 \varepsilon_{12}).$$

(17)

**Case II.** $P \in (S_0 - B) \cdot S(P_0,R_0)$ and $|P_j P| < R < R_0$. By (17) for each $P_j$ corresponding to each $P \in K(P_0,R_0)$
$$K(P_j,2R) \subset (\Gamma(P_j,2R), 2^8 \varepsilon_{12}).$$

Let $\Gamma(P,R)$ be the plane through $P$ and parallel to $\Gamma(P_j,2R)$, then
$$K(P,R) \subset (\Gamma(P,R), 2^9 \varepsilon_{12}).$$

(18)

**Case III.** $P \in (S_0 - B) \cdot S(P_0,R_0)$ and $|P_j P| > R > 1/16 |P_j P|.$
From (18)
\[ K(P_i | P_j^p) \subset (r(P_i | P_j^p), 2^9 \in_{13} R) \]
then
\[ K(P, R) \subset (r(P, R), 2^{13} \in_{13} R) \]
where \( r(P, R) = r(P_i | P_j^p) \) for all \( R \) in consideration.

**Case IV.** \( P \in (S_0 - B) \cdot S(P_0, R_0) \) and \( R < 1/16 \ | P_j^p | \). By (16) for \( P_i^p \) corresponding to every \( P_i \in K(P, | P_j^p |) \)
\[ 2 \pi | P_j^p |^2 \leq \Lambda^2 K(P_i^p, 2 | P_j^p |) \leq 2 \pi | P_j^p | (1 + 2^4 \in_{13} 4) \]
then by lemma 11
\[ \pi | P_j^p |^2 \leq \Lambda^2 K(P_i^p, | P_j^p |) \leq \pi | P_j^p | (1 + 2^5 \in_{13} 4) \]
and by lemma 5
\[ \pi R^2 \leq \Lambda^2 K(P_i^p, R) \leq \pi R^2 (1 + 2^5 \in_{12} 2) \].

Apply lemma 12 with \( r = R \) and \( r_0 = | P_j^p | \) to obtain
\[ K(P, R) \subset (r(P, R), \in \ast R) \]
and \[ K(P, R) \cdot S(P, R) \subset (K(P, R), \in \ast R) \].
Taking \( \in \ast \) so small that \( \in \ast < \min (2^{-13} \in_{13} 13, \in \ast) \) the desired result is obtained.

We now quote a theorem from [1] with the notation established in this paper.

**Theorem:** If \( G \) is a bounded set of points in \( E_3 \) and \( P_o \) is a point of \( G \) such that to each \( R < R_o \) and each \( X \in G \cdot S(P_o \cdot R_o) \) there corresponds a plane \( r(X, R) \) through \( X \) such that
\[ G \cdot S(X, R) \subset (r(X, R), \in \ast R) \cdot S(X, R) \] (A)
and \[ r(X, R) \cdot S(X, R) \subset (G, \in \ast R) \cdot S(X, R) \] (B)
and \( r \) is a plane through \( P_o \) such that
\[(r, \in B) \supset \mathcal{G}. \]

Then if \( \varepsilon < 2^{-18000} \) there will exist a topological 2-disk \( \mathcal{G} \) such that

\[ G \cdot S(P_0, 1/16 R_0) \subset \mathcal{U} \subset G \cdot S(P_0, R_0). \]

We will now construct a set \( G \) and then we will show that \( G \) is a set which satisfies (A), (B) and (C) above. Define a cylindrical coordinate system \( F(r, \theta, z) \) with \( P_0 = F(0,0,0) \). Let \( \Gamma(P_0, R_0) \) be the plane \( z = 0 \) and \( B \cdot S(P_0, R_0) \) be the line \( \theta = 0 = z \).

Let \( F \) be a function whose domain is \( K(P_0, R_0) \) and such that \( F(P) = (r, \theta, z) \) for all \( P \in K(P_0, R_0) \). Let \( G \) be the set of image points of \( F \) together with \( K(P_0, R_0) \).

First we need another lemma.

**Lemma 14.** Suppose \( X \in K(P_0, R_0/2) \) and \( |XX| < 16R \). Then \( F(S(X, R)) \) will contain no points of \( K(P_0, R_0) \). In particular there will be no pair of distinct points \( X \) and \( F(X) \) both in \( K(P_0, R_0) \). That is \( F(K(P_0, R_0)) \cdot K(P_0, R_0) = \emptyset \).

By virtue of lemma 13, \( K(X, 2|XX|) \) will lie near a plane through \( B \). \( B \) will divide \( \Gamma \) into two halves \( \Gamma_1 \) which has \( X \) near it and \( \Gamma_2 \). Now either every point of \( \Gamma_1 \cdot S(X, 19/10 |XX|) \) or every point of \( \Gamma_2 \cdot S(X, 19/10 |XX|) \) is the projection of some point of \( K(P_0, R_0) \). For elementary topological reasons we could else form a closed torus linking \( B \) but not meeting \( K(P_0, R_0) \). Let \( Y \) be a point of \( K(P_0, R_0) \) in \( F(S(X, R)) \), if such exist. Then both
\[ \lambda^2 \Gamma_1 \cdot S(x_j, 19/10 \mid xx_j \mid) + \lambda^2 \Gamma(x, \frac{1}{2} \mid xx_j \mid) \]

and
\[ \lambda^2 \Gamma_2 \cdot S(x_j, 19/10 \mid xx_j \mid) + \lambda^2 \Gamma(x, \frac{1}{2} \mid xx_j \mid) \]

will substantially exceed \( \frac{1}{2} \pi \mid xx_j \mid^{2.4} \) and one of them gives a contradiction, depending on which is fully covered by the projection of \( K(P_0, r_0) \).

Now suppose that \( x \in K(P_0, r_0) \) and that \( S(x, R) \subset S(P_0, r_0) \). Consider first the case where \( \mid xx_j \mid \leq \frac{1}{16} R \). Then by cases I-III of lemma 13, \( K(x, R) \) will lie within \( 2^{13} \in R \) of \( \nabla(x_j, \mid xx_j \mid) \). Since this plane passes through \( B \) the same will be true of \( F(S \cdot R^{-1}(x, R)) \). Thus all points of \( G \cdot S(x, R) \) lie within \( 2^{14} \in R \) of the plane through \( X \) parallel to \( \nabla(x_j, \mid xx_j \mid) \).

On the other hand consider the case where \( \mid xx_j \mid > \frac{1}{16} R \). By lemma 14 and lemma 13, case IV, our condition is then satisfied. Thus by the theorem quoted, \( G \) is a disc near \( P_0 \). Now again quoting lemma 14, this disc is divided into two closed congruent sets whose common part is \( B \), namely \( K(P_0, r_0) \) and \( F(K(P_0, r_0)) \), each of which must therefore be half discs.

This completes the proof of the following theorem:

**Theorem:** For all non-vertex boundary points of \( S_0 \) as defined in the introduction of this paper, the surface is locally Euclidean.
BIBLIOGRAPHY