AN ABSTRACT OF THE THESIS OF

Sukosin Thongrattanasiri of the degree of Master of Science in Physics presented on November 29, 2007.

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Abstract approved:_________________________________________

Viktor A. Poldolskiy

This thesis deals with applications of uniaxial anisotropic crystals for microcavity resonators with partially chaotic underlying ray dynamics. We develop an implementation of the scattering matrix formalism, and relate the eigenvalues and eigenvectors of the scattering matrix to the field distribution of inside the system. Using the developed technique, we analyze the evolution of spatial structure of modes as functions of dielectric permittivities and shape of the resonator boundary. Numerical errors emanating are identified and discussed. The applications of this work lie in polarization control, negative refraction, and other optical phenomena.
Mode Patterns in Quadrupole Resonator with Anisotropic Core

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Sukosin Thongrattanasiri

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________________________________________________________________________
Sukosin Thongrattanasiri, Author
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Chapter 1 - Introduction

A cavity resonator is the main part in a laser, ensuring that most of the light makes many passes through the gain medium. Typically, a dielectric cavity resonator has interior surfaces that reflect an electromagnetic wave of a specific frequency. The reflection of the electromagnetic waves at the boundary of the resonator will allow standing wave modes to exist with little loss to the outside of the cavity. These standing wave modes allow certain patterns and frequencies of radiation being sustained, with other patterns and frequencies being suppressed by destructive interference. Usually, the little loss of electromagnetic waves to the outside of the cavity is occurred from the tunneling of waves through the resonator’s boundary by breaking the total internal reflection[5, 19, 37], and consequently the power of the emitted light depends on the number of bounces and mode patterns in the resonator.

In this thesis, we study mode patterns of electromagnetic waves in different types of dielectric resonators. Especially, their cladding is made with perfect electric conductor (PEC), having uniaxial anisotropic transparent crystals as cores in the resonators. With the PEC cladding, the electromagnetic waves cannot leak through the boundary. Specifically, the boundary shape of the resonators, called quadrupole billiard, has a form given by[36]

\[ R(\theta) = R_0(1 + \epsilon \cos 2\theta), \]

where $\epsilon$ in Eq. (1.1) is a deformation parameter, making a resonator reduced to a circular shape if it is limited to zero. The variation of the deformation parameter $\epsilon$ starting from zero gradually induces a transition to chaos of ray motion[15]. Note that Eq. (1.1) is represented in Cylindrical coordinates, where $\theta$ is the polar angle and $R(\theta)$ is the radius corresponding to a polar angle. $R_0$ is the radius of a circle shape when the deformation parameter $\epsilon$ is zero.

The thesis is organized as follows. The scope of the second chapter is to introduce some basic ideas of anisotropic crystals. The main interest is focused on uniaxial crystals, which yield special features when the permittivity along the
optical axis is negative. Numerical implementation tools are introduced in Chapter 3. First, formulas of electromagnetic fields are built up, using Maxwell’s equations, the separation variables method, and boundary conditions of resonators. Next, we sum all possible modes of the electromagnetic fields and then build the scattering matrix. Consequently, all information is contained in the scattering matrix giving us all possible eigenmodes of a system. Then we learn to truncate unnecessary modes out to make it finite number of modes, and learn to approximate numerical errors from plots of the scattering matrix and its eigenvalues. Poincare surface of section and Husimi projection technique are introduced as tools to explore the mode patterns. Specifically, the Poincare surface of section captures ray motion in real space, and then maps them on phase space. The Husimi projection is used as a connection between a mode pattern in real space and a mode in the surface of section. Finally, in the last chapter, we analyze some interesting behaviors of electromagnetic waves in typical quadrupole resonators with normally and extremely anisotropic transparent material cores.
Chapter 2 - A Short Introduction to Anisotropic Crystals

It is well known that an isotropic transparent material can be characterized by a real dielectric permittivity $\varepsilon$ or a refractive index $n = \sqrt{\varepsilon}$. We set the permeability $\mu = 1$, corresponding to non-magnetic and transparent materials in a given optical range of frequencies. The solution of Maxwell’s equations is monochromatic plane wave propagating with a phase velocity without a change in amplitude or polarization, regardless of the direction of propagation and initial polarization. The phase velocity is determined by dielectric permittivity in the direction of electric field, but not by dielectric permittivity in the direction of propagation of the wave. Moreover, the vector of electric induction is always parallel to the inducing electric field $\vec{D} = \varepsilon \vec{E}$.

However, many real materials are very often anisotropic. An anisotropic transparent crystal is a material having optical properties that are not the same in all directions at a point in a body. The difference between the isotropic and anisotropic transparent materials strongly depends on the dielectric constants of the crystals, shown in the relation between the electric induction and electric field in the tensor form as

$$D_i = \varepsilon_{ij} E_j, \quad (2.1)$$

or in the matrix form of Cartesian coordinates as

$$\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}. \quad (2.2)$$

In this study, we are only interested in all real $\varepsilon_{ij}$ coefficients, so one can show[1] that $\varepsilon_{ij} = \varepsilon_{ji}$, and consequently the matrix $\varepsilon_{ij}$ is symmetric. One of the basic theorems concerning such matrices is the finite-dimensional spectral theorem[2], stating that any symmetric matrix whose entries are real can be diagonalized by an orthogonal matrix. More explicitly, to every symmetric real matrix $A$, there exists a real orthogonal matrix $C$ such that $S = C^T A C$ is a diagonal matrix, where $C^T$ is
a transposed matrix of $C$. This transformation allows us to write Eq.(2.2) as a diagonal matrix,

$$
\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_x & 0 & 0 \\
0 & \varepsilon_y & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}.
$$

(2.3)

$\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_z$ are called the principal dielectric constants. In this point, we may say that anisotropic crystals are divided into uniaxial or biaxial crystals, depending on the relationship between three principal dielectric constants. For a uniaxial crystal, there is the equality of two principal dielectric constants, and another one different. Letting the ordinary dielectric constant be $\varepsilon_o$ and the extraordinary dielectric constant be $\varepsilon_e$, the axes of the ordinary permittivity are perpendicular to the optical axis and the axis of the extraordinary permittivity is parallel to the optical axis. The optical axis (O.A.) is a symmetry axis of a uniaxial crystal (see Fig. 2.1). For a biaxial crystal, all three of the principal dielectric constants are different. In this thesis, we only point our interest to uniaxial crystals, which are more widely used as polarizing crystals than biaxial crystals.

![Figure 2.1: The schematic of uniaxial anisotropic transparent crystal. The axes of the ordinary permittivity $\varepsilon_o$ are perpendicular to the optical axis, but the axis of the extraordinary permittivity $\varepsilon_e$ is parallel to the optical axis.](image)
With the absence of charges and currents, \( \rho = 0 \) and \( \vec{J} = 0 \), in non-magnetic materials, Maxwell’s equations in CGS units are,

\[
\begin{align*}
\nabla \cdot \vec{D} &= 0, \\
\nabla \cdot \vec{H} &= 0,
\end{align*}
\]

\[
\begin{align*}
\nabla \times \vec{H} &= \frac{1}{c} \frac{\partial \vec{D}}{\partial t}, \\
\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}.
\end{align*}
\] (2.4)

We may conclude directly

\[
\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2}.
\] (2.5)

If a plane wave has an electric field vector and an electric induction vector of the forms

\[
\begin{align*}
\vec{E} &= \vec{E}_0 e^{i(k \cdot \vec{r} - \omega t)}, \\
\vec{D} &= \vec{D}_0 e^{i(k \cdot \vec{r} - \omega t)}
\end{align*}
\] (2.6) (2.7)

respectively, where \( \vec{k} \) is the wave vector, \( \vec{r} \) the position vector, \( \omega \) the angular frequency, and \( t \) time, we may replace the \( \nabla \) and the \( \partial / \partial t \) operators by

\[
\begin{align*}
\nabla &\rightarrow i\vec{k} \\
\frac{\partial}{\partial t} &\rightarrow -i\omega
\end{align*}
\]

and then we rewrite Maxwell’s equations as

\[
\begin{align*}
\vec{k} \cdot \vec{D} &= 0, \\
\vec{k} \cdot \vec{H} &= 0,
\end{align*}
\]

\[
\begin{align*}
\vec{k} \times \vec{H} &= -\frac{\omega}{c} \vec{D}, \\
\vec{k} \times \vec{E} &= \frac{\omega}{c} \vec{H}.
\end{align*}
\] (2.8)

And again we may conclude directly

\[
\vec{k} \times (\vec{k} \times \vec{E}) = -\frac{\omega^2}{c^2} \vec{D}.
\] (2.9)

After using a vector relation \( \vec{k} \times (\vec{k} \times \vec{E}) = (\vec{k} \cdot \vec{E})\vec{k} - k^2 \vec{E} \), expanding components of \( \vec{k} \), \( \vec{E} \), and \( \vec{D} \) in Cartesian coordinates \( \vec{k} = (k_x, k_y, k_z) \), \( \vec{E} = (E_x, E_y, E_z) \), and
\( \bar{D} = (D_x, D_y, D_z) \), using relations Eq.(2.3), collecting terms of electric field components, the resulting matrix form becomes

\[
\begin{bmatrix}
  k_0^2 n_x^2 - k_y^2 - k_z^2 & k_y k_x & k_z k_x \\
  k_y k_x & k_0^2 n_y^2 - k_x^2 - k_z^2 & k_y k_z \\
  k_z k_x & k_y k_z & k_0^2 n_z^2 - k_x^2 - k_y^2
\end{bmatrix}
\begin{bmatrix}
  E_x \\
  E_y \\
  E_z
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix},
\tag{2.10}
\]

where \( k_0 = \omega / c \). This is a set of linear equations in \( E_x, E_y, \) and \( E_z \) and it has a non-trivial solution if its determinant vanishes[2]. Therefore, a simple calculation gives

\[
k_0^4 - k_0^2 \left( \frac{k_x^2 + k_y^2}{\epsilon_z} + \frac{k_x^2 + k_z^2}{\epsilon_y} + \frac{k_y^2 + k_z^2}{\epsilon_x} \right) + \frac{1}{\epsilon_y \epsilon_z} \left( \frac{k_x^2}{\epsilon_x} + \frac{k_y^2}{\epsilon_x} + \frac{k_z^2}{\epsilon_x} \right) \left( k_x^2 + k_y^2 + k_z^2 \right) = 0.
\tag{2.11}
\]

In the case of a uniaxial crystal, where \( \epsilon_x = \epsilon_y = \epsilon_o \) and \( \epsilon_z = \epsilon_e \neq \epsilon_o \), Eq.(2.11) may be factorized into

\[
\left( \frac{k_x^2}{\epsilon_o} + \frac{k_y^2}{\epsilon_o} + \frac{k_z^2}{\epsilon_o} - k_0^2 \right) \left( \frac{k_x^2}{\epsilon_e} + \frac{k_y^2}{\epsilon_e} + \frac{k_z^2}{\epsilon_e} - k_0^2 \right) = 0.
\tag{2.12}
\]

Each factor in Eq.(2.12) defines a surface in the space of vectors \( \bar{k} \), called wave vector surfaces. The first factor defines the surface of a sphere. However, the second factor defines the surface of an ellipsoid or hyperboloid, depending on the positive or negative of \( \epsilon_o \)[32]. Fig. 2.2 shows a cross-section on the plane \( k_x = 0 \) of these three types of surfaces. \( k_z \) is aligned parallel to the optical axis, and \( k_y \) is the horizontal axis.
Let us now examine the consequences of these results. First, consider the propagation of a wave along the optical axis; that is, $k_x = k_y = 0$. Consequently, $k_z = \sqrt{\varepsilon_\omega \omega / c}$, so the crystal behaves as though it were an isotropic medium. Waves propagating along the optical axis are called *ordinary waves*. Second, consider the propagation of a wave in any direction, but not along the optical axis. The second factor in Eq.(2.12) still contains $\varepsilon_\omega$ and $\varepsilon_\varepsilon$. Those waves propagating in any directions but not along the optical axis are called *extraordinary waves*. For more detail of this section, read Refs.[1, 3, 26].
Chapter 3 – Numerical Implementation Tools

3.1 - Formulations of Electromagnetic Fields

In this section, we plan to derive electric fields tangential to the boundary of a resonator, filled with uniaxially anisotropic transparent crystal having PEC boundaries. Results obtained are tangential components of electric fields in TM and TE modes. For the TM or transverse magnetic waves, there are no magnetic fields in the direction of propagation; for the TE or transverse electric waves, there are no electric fields in the direction of propagation (see Fig. 3.1). We follow a procedure similar to the one described in Refs. [4] and [5].

In Cylindrical coordinates, the components of permittivity tensor of the anisotropic crystals are \((\varepsilon_{r\theta}, \varepsilon_{r}, \varepsilon_{\theta})\), with \(\varepsilon_{r\theta}\) and \(\varepsilon_{\theta}\) expressing the components of permittivity tensor, transverse and parallel to the optical axis, respectively (see Fig. 3.1). The components of the electric and magnetic fields may be represented as \(\vec{E} = (E_r, E_\theta, E_z)\), and \(\vec{H} = (H_r, H_\theta, H_z)\), respectively. After substituting the components of permittivity tensor, electric field and magnetic field into Maxwell’s equations Eq.(2.4), and collecting terms, we obtain

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial \theta} \left( r E_\theta \right) - \frac{\partial E_r}{\partial z} &= -\frac{1}{c} \frac{\partial H_r}{\partial t}, \\
\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} &= -\frac{1}{c} \frac{\partial H_\theta}{\partial t}, \\
\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= -\frac{1}{c} \frac{\partial D_\theta}{\partial t}, \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_\theta \right) - \frac{\partial E_r}{\partial \theta} &= 0,
\end{align*}
\]

(3.1)

Note that \(D_r = \varepsilon_{r\theta} E_r\), \(D_\theta = \varepsilon_{r\theta} E_\theta\), and \(D_z = \varepsilon_z E_z\). In order to describe \(\vec{E}\) and \(\vec{H}\) as functions of \(r\), \(\theta\), \(z\), and \(t\), we should guess a possible form of the fields corresponding to Cylindrical coordinates. Let us go back to consider Eq.(2.5). This equation cannot be generally solved analytically because \(\nabla \cdot \vec{E} \neq 0\). However, if we change our interest to only the TE polarized modes; that is, \(E_z = 0\), then consequently
Figure 3.1: In Cylindrical coordinates, the dielectric constant components of the anisotropic crystals are \((\varepsilon_{r\theta}, \varepsilon_{\theta\theta}, \varepsilon_z)\), where \(\varepsilon_{r\theta}\) and \(\varepsilon_z\) express the dielectric constants on axes, transverse and parallel to the optical axis. For the TM waves, there are no magnetic fields in the direction of propagation, but for the TE waves, there are no electric fields in the direction of propagation.

\[
\nabla^2 \tilde{E} = \frac{\varepsilon_{r\theta}}{c^2} \frac{\partial^2 \tilde{E}}{\partial t^2} \tag{3.2}
\]

\[
\nabla^2 \tilde{H} = \frac{\varepsilon_{r\theta}}{c^2} \frac{\partial^2 \tilde{H}}{\partial t^2} \tag{3.3}
\]

Keep in mind that these equations are only for the case of TE polarized modes where \(E_z = 0\). Both equations (3.2) and (3.3) are well-known three-dimensional wave equations in the form

\[
\nabla^2 \psi = \frac{\varepsilon_0}{c^2} \frac{\partial^2 \psi}{\partial t^2} \tag{3.4}
\]

After applying the Laplacian operator,

\[
\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{3.5}
\]

and assuming that \(\tilde{E}, \tilde{H} \propto R(r)e^{im\theta}e^{(k_zz-\omega t)}\), where \(k_z\) is the z-component wave vector and \(m\) is a mode number, the resultant Bessel’s differential equation is

\[
\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} + \left( \frac{\varepsilon_0 \omega^2}{c^2} - k_z^2 - \frac{m^2}{r^2} \right) R(r) = 0, \tag{3.6}
\]

and consequently wave functions in Cylindrical coordinates are
\[ \Psi (r, \theta, z, t) = \Psi_0 J_m (\chi r) e^{i m \theta} e^{i (k_z z - \omega t)}, \]  

(3.7)

where \( J_m (\chi r) \) are Bessel functions of the first kind, \( \chi^2 = \varepsilon_0 k_z^2 - k_r^2 \), and \( k = \omega / c \).

At this point, we may use the separation of variables technique with Eqs.(3.1) by generally assuming that \( \vec{E}, \vec{H} \propto e^{i m \theta} e^{i (k_z z - \omega t)} \). The \( e^{i (k_z z - \omega t)} \) factor represents the formation of a standing wave along the \( z \)-axis and the \( e^{i m \theta} \) factor represents the symmetry in the polar angle. Substituting \( \vec{E} \) and \( \vec{H} \) into Eqs.(3.1), and expressing them as functions of \( E_z \) and \( H_z \), so

\[
E_r = -\frac{1}{\varepsilon_{r\theta} k_z^2 - k_r^2} \left( \frac{m}{r} H_z - ik_z \frac{\partial E_z}{\partial r} \right),
\]

(3.8)

\[
E_\theta = -\frac{1}{\varepsilon_{r\theta} k_z^2 - k_r^2} \left( \frac{m}{r} E_z + ik \frac{\partial H_z}{\partial r} \right),
\]

(3.9)

\[
H_r = \frac{1}{\varepsilon_{r\theta} k_z^2 - k_r^2} \left( m \varepsilon_{r\theta} \frac{k}{r} E_z + ik_z \frac{\partial H_z}{\partial r} \right),
\]

(3.10)

\[
H_\theta = -\frac{1}{\varepsilon_{r\theta} k_z^2 - k_r^2} \left( \frac{m}{r} H_z - ik \frac{\partial E_z}{\partial r} \right),
\]

(3.11)

One can see now that the electromagnetic fields break into three polarizations: TE with \( E_z = 0 \), TM with \( H_z = 0 \), and TEM with both \( E_z = H_z = 0 \). The latter one cannot be realized in a simply connected resonator geometry[6] and are not considered here. The TE and TM polarizations form a complete basis for the description of electromagnetic phenomena in the uniaxially anisotropic crystals.

Now it reaches a suitable point to consider TE and TM waves in mathematical details. Applying \( H_z = H_0 J_m (\chi_{TE} r) e^{i m \theta} e^{i (k_z z - \omega t)} \) and letting \( E_z = 0 \) in Eq.(3.8) – Eq.(3.11), we obtain electric and magnetic fields vectors of TE modes as

\[
\vec{E} = \left( -\frac{mk}{2} \frac{H_0}{\chi_{TE}} J_m (\chi_{TE} r), -\frac{ik}{2} \frac{\partial J_m (\chi_{TE} r)}{\partial r} \right) e^{i m \theta} e^{i (k_z z - \omega t)},
\]

(3.12)

\[
\vec{H} = \left( \frac{ik_z^2}{2} \frac{H_0}{\chi_{TE}} \frac{\partial J_m (\chi_{TE} r)}{\partial r}, -\frac{mk_z}{2} \frac{H_0}{r} J_m (\chi_{TE} r), \frac{H_0}{r} J_m (\chi_{TE} r) \right) e^{i m \theta} e^{i (k_z z - \omega t)},
\]

(3.13)
where the dispersion relation is $\chi_{TE}^2 = \varepsilon_r k^2 - k_z^2$. One can see that this dispersion equation depends only on the transverse dielectric constant and is identical to the one in isotropic materials with dielectric constant $\varepsilon_r$. Consequently, TE waves are ordinary waves propagating in uniaxially anisotropic crystals. Similarly, applying $E_z = E_0 J_m(\chi_{TM} r) e^{im\theta} e^{i(k_z z - \omega t)}$ and letting $H_z = 0$ in Eq.(3.8) – Eq.(3.11), we obtain electric and magnetic fields vectors of TM modes as

$$\vec{E} = \left(\frac{ik_z \varepsilon_z}{\chi_{TM}^2 \varepsilon_r} E_0 \frac{\partial J_m(\chi_{TM} r)}{\partial r}, -\frac{mk_z \varepsilon_z E_0}{\chi_{TM}^2 \varepsilon_r} r J_m(\chi_{TM} r), E_0 J_m(\chi_{TM} r)\right) e^{im\theta} e^{i(k_z z - \omega t)}, \tag{3.14}$$

$$\vec{H} = \left(\frac{mk \varepsilon_z}{\chi_{TM}^2} J_m(\chi_{TM} r), \frac{ik}{\varepsilon_z} E_0 \frac{\partial J_m(\chi_{TM} r)}{\partial r}, 0\right) e^{im\theta} e^{i(k_z z - \omega t)}, \tag{3.15}$$

where the dispersion relation is $\chi_{TM}^2 = \frac{\varepsilon_z}{\varepsilon_y}(\varepsilon_y k^2 - k_z^2)$. It is obvious that the TM waves, in contrast to TE waves, are affected by anisotropy, because of the parallel dielectric constant $\varepsilon_z$, and consequently they are also called extraordinary waves.

At the boundary of a resonator $R(\theta)$, the tangential components of electric fields $E_z$ of both TE and TM waves, $E_{m,z}^{TE}$ and $E_{m,z}^{TM}$, are calculated by

$$E_z = \frac{E_z R'(\theta) + E_\theta R(\theta)}{\sqrt{R^2(\theta) + R'^2(\theta)}}, \tag{3.16}$$

where $R'(\theta) = dR / d\theta$. Therefore,

$$E_{m,z}^{TE} = -\frac{1}{\sqrt{R^2(\theta) + R'^2(\theta)}} \frac{k}{\chi_{TE}^2} \left\{ m \frac{R'(\theta)}{R(\theta)} J_m(\chi_{TE} R(\theta)) \right\} e^{im\theta} e^{ik_z z} + iR(\theta) \frac{\partial J_m(\chi_{TE} R(\theta))}{\partial R(\theta)} \right\} e^{im\theta} e^{ik_z z},$$

\(3.17\)
\[ E_{m,z}^{\text{TM}} = \frac{1}{\sqrt{R^2(\theta) + R'^2(\theta)}} \left\{ \frac{ik_z}{\chi_{TM}} \varepsilon_{z} \frac{\partial J_m(\chi_{TM}R(\theta))}{\partial R(\theta)} R'(\theta) \right\} e^{im\theta} e^{ik_zz}, \]  

(3.18) respectively. Note that both Eq.(3.17) and Eq.(3.18) correspond only to one specified mode number \( m \). Because of no electric field in the PEC cladding, the condition that

\[ E_{m,z} = E_{m,z}^{\text{TE}} + E_{m,z}^{\text{TM}} = 0 \]  

(3.19) at the resonator boundary can be used to determine mode patterns.

In the next section, we represent \( E_{m,z}^{\text{TE}} \) and \( E_{m,z}^{\text{TM}} \) as a linear combination of an infinite number of modes, build up the scattering matrix, and use physical implement to truncate the mode spectrums.

### 3.2 - Scattering Matrix Method

*Scattering matrix* or *S-matrix* is frequently used in scattering problems of Nuclear Physics, Quantum Electrodynamics, and Quantum Field Theory. It relates a final state with an initial state \( |i_f\rangle = S |i_i\rangle \). \( S \) is the scattering matrix, \( |i_i\rangle \) is the initial state, and \( |i_f\rangle \) is the final state. In other words, the scattering matrix reveals all possible processes from the initial state to the final state. The scattering matrix method is also used to study the modes of electromagnetic waves in quantum optics\(^7, 8\) to produce a connection between the incoming waves and the outgoing waves at the boundary. In any closed systems (no leakage of any electromagnetic waves), the S-matrix is a unitary matrix; that is, \( S^\dagger S = SS^\dagger = I \), and, consequently the maximum value of elements in the S-matrix is 1. In this section, our goal is to build the scattering matrix from \( E_{m,z}^{\text{TE}} \) (Eq.(3.17)) and \( E_{m,z}^{\text{TM}} \) (Eq.(3.18)) by a
procedure in Ref.[5]. First, we represent \( E_{m,z}^{TE} \) and \( E_{m,r}^{TM} \) as a linear combination of an infinite number of modes \( m \) to include all of the possible modes:

\[
E_{r}^{TM} = \sum_{m=-\infty}^{\infty} \tilde{\alpha}_m^{TM} E_{m,r}^{TM},
\]

\[
E_{r}^{TE} = \sum_{m=-\infty}^{\infty} \tilde{\alpha}_m^{TE} E_{m,r}^{TE},
\]

where \( \tilde{\alpha}_m^{TM} \) and \( \tilde{\alpha}_m^{TE} \) are any complex number coefficients. Eq.(3.20) and Eq.(3.21) represent possible standing waves in the resonator. Moreover, the standing waves are a combination of outgoing and incoming electromagnetic waves in the resonator, represented as Hankel functions of the first and second kinds, \( H_m^+(x) \) and \( H_m^-(x) \), respectively. Therefore, Bessel functions of the first kind \( J_m(x) \) in \( E_{m,r}^{TM} \) and \( E_{m,r}^{TE} \) are represented as

\[
J_m(x) \rightarrow \alpha_m H_m^+(x) + \beta_m H_m^-(x).
\]

\( \alpha_m \) and \( \beta_m \) are any complex number constants. After using Eq.(3.19), Eq.(3.20), and Eq.(3.21), multiplying both sides by \( i e^{-i \theta} H_i^+(\chi_{TE} R(\theta)) \), and integrating with respect to \( \theta \) from 0 to \( 2\pi \) to get a matrix equation for the coefficients \( \alpha_m^{TM}, \alpha_m^{TE}, \beta_m^{TM}, \) and \( \beta_m^{TE} \), we obtain

\[
\sum_{m=-\infty}^{\infty} \left[ \alpha_m^{TM} Q_{lm}^{+,TM} + \alpha_m^{TE} Q_{lm}^{+,TE} \right] = \sum_{m=-\infty}^{\infty} \left[ \beta_m^{TM} Q_{lm}^{-,TM} + \beta_m^{TE} Q_{lm}^{-,TE} \right],
\]

such that

\[
Q_{lm}^{+,TM} = \pm \frac{2\pi}{k} \int_0^{2\pi} d\theta \, i e^{i m \theta} H_i^+(\chi_{TE} R(\theta)) \frac{1}{\sqrt{R^2(\theta) + R'^2(\theta)}} \frac{k_z}{\chi_{TE}} \left( \frac{\partial H_m^+(\chi_{TM} R(\theta))}{\partial R(\theta)} R'(\theta) - m H_m^+(\chi_{TM} R(\theta)) \right),
\]

\[
Q_{lm}^{+,TE} = \pm \frac{2\pi}{k} \int_0^{2\pi} d\theta \, i e^{i (m-1) \theta} H_i^+(\chi_{TE} R(\theta)) \frac{1}{\sqrt{R^2(\theta) + R'^2(\theta)}} \frac{k_z}{\chi_{TE}} \left( \frac{\partial H_m^+(\chi_{TM} R(\theta))}{\partial R(\theta)} R'(\theta) - m H_m^+(\chi_{TM} R(\theta)) \right).
\]
\[ x \left( \frac{i \partial H_m^+ \left( \chi_{TE} R(\theta) \right)}{\partial R(\theta)} \right) R(\theta) + m \frac{R'(\theta)}{R(\theta)} H_m^+ \left( \chi_{TE} R(\theta) \right), \tag{3.25} \]

where \( l \) is another mode number. Note that the satisfaction of the boundary condition Eq.(3.19) corresponds to the satisfaction of all other boundary conditions. \( i H_i^+ \left( \chi_{TE} R(\theta) \right) \) can be generalized in \( w_i(\theta) \), called a *adjustable function*. It should be wisely chosen for each system. With the choice of our adjustable function, \( Q_{lm}^{\pm,TE} \) are approximately Hermitian matrices and our numerical error is decreased[33].

What we have dealt with so far is the tangential components of electric fields of TE and TM modes, calculated from the \( E_r \) and \( E_\theta \) components. However, there is still another tangential component, \( E_z \), which must be included to keep all tangential components information of electric fields. Because of the absence of the \( z \) component of electric fields of TE modes, we may start with \( E_z = E_z^{TM} = 0 \). Upon applying all the same procedures as above, we then obtain

\[ \sum_{m=-\infty}^{\infty} \alpha_m^{TM} Q_{lm}^{\pm,TM} = \sum_{m=-\infty}^{\infty} \beta_m^{TM} \bar{Q}_{lm}^{\pm,TM}, \tag{3.26} \]

such that

\[ \bar{Q}_{lm}^{\pm,TM} = \pm \int_0^{2\pi} d\theta i^{(m-l)\theta} H_i^+ \left( \chi_{TM} R(\theta) \right) H_m^\pm \left( \chi_{TM} R(\theta) \right), \tag{3.27} \]

with the adjustable function \( i H_i^+ \left( \chi_{TM} R(\theta) \right) \). At this point, one may have an idea to change the linear equations (3.23) and (3.26) to matrix forms to obtain the desired scattering matrix. Consequently, it is

\[ \begin{bmatrix} \hat{Q}_{lm}^{\pm,TE} & \hat{0} \\ \hat{Q}_{lm}^{\pm,TE} & \bar{Q}_{lm}^{\pm,TE} \end{bmatrix} \begin{bmatrix} \alpha_m^{TM} \\ \beta_m^{TM} \end{bmatrix} = \begin{bmatrix} \hat{Q}_{lm}^{\pm,TE} & \hat{0} \\ \hat{Q}_{lm}^{\pm,TE} & \bar{Q}_{lm}^{\pm,TE} \end{bmatrix} \begin{bmatrix} \alpha_m^{TE} \\ \beta_m^{TE} \end{bmatrix}. \tag{3.28} \]

Because \( \hat{Q}_{lm}^{\pm,TE} \), \( \bar{Q}_{lm}^{\pm,TE} \), and \( \hat{0} \) are two-dimensional matrices with a size \((2m_{\max} + 1) \times (2m_{\max} + 1)\) where an infinite number of modes is represented as
those two big matrices have the size \(2(2m_{\text{max}} + 1) \times 2(2m_{\text{max}} + 1)\). For the \(\hat{0}\) matrix, all elements are zero. After rearranging Eq.(3.28) as

\[
\begin{bmatrix}
\beta_{m}^{\text{TM}} \\
\beta_{m}^{\text{TE}}
\end{bmatrix}
= \begin{bmatrix}
\tilde{Q}_{lm}^{\text{TM}} & \hat{0} \\
\tilde{Q}_{lm}^{\text{TM}} & \tilde{Q}_{lm}^{\text{TE}} + \hat{0}
\end{bmatrix}^{-1}
\begin{bmatrix}
\tilde{Q}_{lm}^{\text{TM}} \tilde{Q}_{lm}^{\text{TM}} \\
\tilde{Q}_{lm}^{\text{TM}} \tilde{Q}_{lm}^{\text{TE}} \alpha_{m}^{\text{TM}} \alpha_{m}^{\text{TE}}
\end{bmatrix},
\]

(3.29)

and using “blockwise matrix inversion”[2], then

\[
|\beta\rangle = S(k)|\alpha\rangle
\]

(3.30)

where

\[
|\beta\rangle \doteq \begin{bmatrix}
\beta_{m}^{\text{TM}} \\
\beta_{m}^{\text{TE}}
\end{bmatrix}, \quad
|\alpha\rangle \doteq \begin{bmatrix}
\alpha_{m}^{\text{TM}} \\
\alpha_{m}^{\text{TE}}
\end{bmatrix},
\]

(3.31)

and

\[
S(k) \doteq \begin{bmatrix}
(\tilde{Q}_{lm}^{\text{TM}})^{-1} & \hat{0} \\
-\left(\tilde{Q}_{lm}^{\text{TE}}\right)^{-1} \tilde{Q}_{lm}^{\text{TM}} \left(\tilde{Q}_{lm}^{\text{TM}}\right)^{-1} \tilde{Q}_{lm}^{\text{TM}} + \left(\tilde{Q}_{lm}^{\text{TE}}\right)^{-1} \tilde{Q}_{lm}^{\text{TE}}
\end{bmatrix}.
\]

(3.32)

This \(S(k)\) is the scattering operator, and \(|\beta\rangle\) and \(|\alpha\rangle\) are eigenvectors of the scattering operator. The scattering matrix is separated into three regions: left-upper, right-lower, and left-lower regions, corresponding to TM, TE and mixed waves, respectively. The mixed wave region corresponds to the combination of TM and TE waves. A physical interpretation of the scattering operator and its eigenvectors may be considered as Fig. 3.2.

We build two parallel infinite plates with a reference line at the center between them. These plates, left and right, are totally reflective; that is, there is no penetration or refraction of waves. We start by letting an incoming wave to a boundary, \(|\alpha\rangle\), (or an outgoing wave from the reference line) reflect or scatter perpendicularly to the right boundary into an outgoing wave from the boundary, \(|\beta\rangle\), via the scattering operator \(S_{R}(k)\). The subscript \(R\) means scattering at the right boundary,
Figure 3.2: This schematic represents a physical interpretation of the scattering operator and its eigenvectors. The parallel L and R plates are left and right infinite plates, respectively. The dashed line is a reference line sitting at the center of the L and R plates.

\[ |\beta\rangle = S_R(k)|\alpha\rangle. \quad (3.33) \]

Another scattering scenario is to scatter an incoming wave \( |\beta\rangle \) with respect to the left boundary into an outgoing wave \( |\alpha\rangle \) via the scattering operator \( S_L(k) \), where, as above, the subscript \( L \) means scattering at the left boundary,

\[ |\alpha\rangle = S_L(k)|\beta\rangle. \quad (3.34) \]

Now, consider the whole cycle, starting with the state \( |\alpha\rangle \) at the reference line, going right and then left, and ending at the reference line, so

\[ |\alpha\rangle = S_L(k)S_R(k)|\alpha\rangle. \quad (3.35) \]

This equation comes from the two scattering, right and left. Because the resonator is closed, no leakage of any electromagnetic waves, both \( |\alpha\rangle \) from the two scatterings are the same. As a result, the scattering matrix is unitary and then we only consider eigenvalues of S-matrix, which are 1. For more explanation, go back to Eq.(3.22). Hankel functions contain Bessel functions of the second kind \( Y_m(x) \) inside, making the singularity at the origin \( r = 0 \). To eliminate this singularity, we equate \( |\beta\rangle \) and \( |\alpha\rangle \) in Eq.(3.30). This gives us a secular function, \( \xi(k) \), given by

\[ \xi(k) = \det[J - S(k)] = 0. \quad (3.36) \]
How is this secular function important? It tells us a suitable $k$ for each system $\epsilon$, $k_z$, $\epsilon_z$, and $\epsilon_{\rho\theta}$, where we may use the root-search strategy to find this $k$. This suitable $k$ is related to the resonant frequency of a standing wave formed in a resonator. Not only can we consider the secular function, but we may also find the suitable $k$ from eigenvalues. In the system, there are $2(2m_{\text{max}} + 1)$ eigenvalues. However, none of them may be equal to 1, or at least one of them equal to 1, depending on the initial input $k$. Hence, our job is to adjust $k$ to make, at least, one eigenvalue equal to 1. Mode numbers, $m$, of these eigenvalues also correspond to suitable eigenvectors.

So far, we have only used theories and procedures. However, one may notice that all matrices’ sizes are $\infty \times \infty$, or $\infty \times 1$, because modes $m$ are summed from $-\infty$ to $\infty$. This cannot be solved by hand, nor exactly by computers. Hence, we have to truncate $m$ to an integer, which is suitable to each system and for a computer. To find that integer, consider Fig. 3.3a. This is a gray-scale representation of the scattering matrix Eq.(3.32), calculated for a quadrupolar

![Figure 3.3](image-url)

Figure 3.3: Two kinds of representation of the typical scattering matrix for a quadrupolar resonator at $\epsilon = 0.05$ deformation with an isotropic core, $\epsilon_z = \epsilon_{\rho\theta} = 1$, $k_z = 0$, $k = 45$, and $m_{\text{max}} = 60$. (a) The gray-scale representation of the scattering matrix. (b) The three-dimensional plot. The vertical axis represents the absolute values at each pair of mode numbers.
resonator at $\epsilon = 0.05$ deformation with an isotropic crystal, $\varepsilon_z = \varepsilon_{\rho\theta} = 1$, $k_z = 0$, $k = 45$, and $m_{\text{max}} = 60$.

The representation may be separated into three main regions as Eq.(3.32). Noticing from Eq.(3.24), Eq.(3.25), and Eq.(3.27), the mixed wave region, not plotted in this Figure, occurs because of the effect of $k_z \neq 0$. Moreover, the left-upper and right-lower regions are individually separated into three sub-regions: evanescent, transition, and scattering regions, depending on values of critical mode $m_c \approx \left\lceil \chi R_{\text{max}} \right\rceil$, and scattering mode $m_{sc} \approx \left\lceil \chi R_{\text{min}} \right\rceil$. $R_{\text{max}}$ and $R_{\text{min}}$ are the maximum and minimum radii of a resonator, and $\left\lceil \right\rceil$ stands for the integer part. If $m$ falls into an evanescent region, or in other words, grows beyond $m_c$, then the scattering matrix becomes strongly diagonal; that is, $\left[S(k)\right]_{\text{ml}} \approx \delta_{\text{ml}}$ for $|m|$ and $|l| > m_c$. This happens because the eigenvector of this $m$ corresponds to classical motion with a circular caustic of radius larger than $R_{\text{max}}$. For a transition region, $m_{sc} < |m|, |l| < m_c$, the scattering matrix is heading towards being diagonal, corresponding to evanescent components which undergo an enhanced scattering because they overlap with only certain regions of the resonator. Furthermore, there is a spread around the diagonal in a scattering region, depending on the deformation. The more the deformation, the more the spread. As a result, one may guess, for a circle shape resonator, a representation shows exactly only a straight line, because there is no deformation. Instead of the infinity of $m_{\text{max}}$, we may therefore use approximately $m_{\text{max}} \approx m_c + \Delta_{ev}$, where $\Delta_{ev}$ corresponds to the suitable evanescent region. I used $\Delta_{ev} = 13$ in the example. Note that it is important to carefully choose $\Delta_{ev}$; that is, too large $\Delta_{ev}$ may exist the machine’s numerical errors, and, consequently, some values in the S-matrix blow up. Furthermore, Fig. 3.3b represents a gray-scale representation as a three-dimensional plot. As expected, the absolute values of the S-matrix on the diagonal
line are 1, corresponding to no scattering. These values then decrease in the transition region to a minimum in the scattering region.

Before finding eigenvalues and eigenvectors, there is one part of numerical implementation we should beware of. It looks absolutely easy to use LAPACK library or IMSL Fortran library to find a scattering matrix in Eq.(3.32). However, one may encounter numerical errors after calculating the inverse of non-invertible matrices. Hence, we would like to suggest calculating the scattering matrix from Eq.(3.29), which is

$$S(k) = \begin{bmatrix} \hat{Q}_{lm}^{+,TM} & \hat{0} \\ \hat{Q}_{lm}^{-,TM} & \hat{Q}_{lm}^{-,TE} \end{bmatrix}^{-1} \begin{bmatrix} \hat{Q}_{lm}^{+,TM} \\ \hat{Q}_{lm}^{-,TM} \end{bmatrix}. \quad (3.37)$$

Again, one may use LAPACK library or IMSL Fortran library to find eigenvalues and eigenstates of a scattering matrix. A typical run of a quadrupolar resonator at $\epsilon = 0.05$ deformation with an isotropic material, $\epsilon_z = \epsilon_{r\theta} = 1$, $k_z = 0$, $k = 45$, produces the distribution of scattering eigenvalues in the complex plane $\{\phi^{\epsilon}\}$ in Fig. 3.4. All the scattering eigenvalues (black dots) are distributed on the unit circle (gray circle) $|z| = 1$, because the S-matrix is unitary. Moreover, there is a noticeable accumulation of eigenvalues on the circle at $\phi \approx 0^\circ$—in other words, where the eigenvalues are approximately equal to 1. As mentioned above, an eigenvalue for which $\phi = 0$ within the allowable numerical precision yields a stable mode of a resonator. However, we should be careful to avoid being misled all the scattering eigenvectors whose $\phi \approx 0$ are stable modes. As pointed out in Ref.[9] in the case of a closed system, there is an accumulation of scattering eigenvalues at $\phi \approx 0^\circ$, which do not correspond to proper physical modes of the resonator. These are primarily evanescent modes, and can easily be distinguished from scattering modes, because of their lack of $k$-dependence; that is, very small changing of $k$ may change positions of scattering eigenvalues on the circle, but there is not that effect with the evanescent eigenvalues.
Figure 3.4: Distribution of scattering eigenvalues (black dots) in the complex plane for $\epsilon = 0.05$, $\varepsilon_z = \varepsilon_{\rho z} = 1$, $k_z = 0$, $k = 45$. Gray circle is the unit circle.

Figure 3.5: The gray-scale representation of eigenvectors. The upper region represents eigenvector components of TE waves and the lower region represents eigenvector components of TM waves.
Not only do we consider the distribution of eigenvalues on the complex plane, but it is also of interest to consider a gray-scale representation of eigenvectors with the same system as of the eigenvalues as in Fig. 3.5. The horizontal axis shows the eigenvectors. We may separate the gray-scale representation of eigenvectors into two regions, upper and lower regions, corresponding to TE and TM waves, respectively. For each eigenvector of a mode, darkest points, representing dominant elements of the eigenvector, correspond to mode patterns in real-space, in particular the z-component of electric field intensities. Fig. 3.6 exemplifies patterns frequently found in real-space.

Figure 3.6: The real-space representations of some typical modes. Dark color represents high intensity of electric fields and bright color represents low intensity of electric fields.
3.3 - Poincare Surface of Section

As mentioned above, for the quadrupole billiard with the boundary shape

\[ R(\theta) = R_0(1 + \epsilon \cos 2\theta) , \]

the transition to chaos of ray motion depends on the variation of the deformation parameter. When the shape is gradually deformed, it becomes impossible to capture types of ray trajectory by standard ray tracing methods in real space. Therefore, Poincare surface of section (SOS), a standard tool of non-linear dynamics, is represented as a projection of ray motion in phase space. It was named after the mathematician and dynamicist Henri Poincare who had invented the method to study nonlinear dynamics[10]. To build the SOS, we start the ray trajectory with an initial condition inside the resonator and follow its propagation. Initial conditions can be specified by giving the position on the boundary at which a ray is launched, and the angle \( \chi \) it forms with the surface normal. In Fig. 3.7, we record the pair of numbers \( (\theta_i, \sin \chi_i) \) at each reflection \( i \) in real space, and map them in phase space, which is the SOS. \( \theta_i \) is the polar angle denoting the position of the \( i \)-th reflection on the boundary, and \( \chi_i \) is the angle of incidence of the ray at that position. This procedure is then evolved in time through the iteration of the SOS map \( i \rightarrow i+1 \), until patterns in the SOS obviously appear. Note that a negative value of \( \sin \chi_i \) corresponds to the backward trajectory. Additionally, the SOS depends only on the waves’ trajectory in a deformed resonator, but not waves’ eigenstates. If one represents the SOS of mixed waves, another would obtain exactly the same SOS of TM waves with same resonator. However, later we will show how to relate those eigenstates to the SOS.

To understand the patterns, which appear in the SOS, we may start by considering the easiest case, the circle shape with \( \epsilon = 0 \) (see Fig. 3.8a). Because of conservation of angular momentum, many straight lines occur on the SOS, depending on initially conserved \( \sin \chi \). These well-known structures are called whispering-gallery (WG) modes, named after an acoustic analogue in which sound
Figure 3.7: The construction of the Poincare surface of section plot. Each reflection from the boundary is represented by a point in the SOS recording the angular position of the bounce on the boundary ($\theta$) and the angle of incidence ($\sin \chi$). The $\sin \chi < 0$ region in the SOS corresponds to backward trajectories of the $\sin \chi > 0$ region.

... propagates close to the curved walls of a circular hall without being audible in its center[11]. For small deformation (Fig. 3.8b), chaotic motion results in the areas of scattered points, called chaotic sea. Moreover, islands of stable motion (closed curves) also emerge, whose centers contain only a stable fixed point, corresponding to the back-and-forth reflection of electromagnetic waves at polar angles $\theta = \pi/2$ and $\theta = 3\pi/2$. Open curves, called KAM curves or KAM tori[34, 35], describe a deformed WG-like motion, close to the perimeter of the boundary. Importantly, these stable islands and KAM tori cannot be crossed by chaotic trajectories in the SOS[12]. The periodic orbits, appearing as fixed points at the upper of the SOS, grow prominently when the deformation is increased. However, the KAM tori break one by one, while, at the same time, the islands of stable motion form around the stable fixed points, and the areas of scattered points grow. One may notice that the triangle periodic orbits, corresponding to the six-islands chain, exists at $\epsilon = 0.05$. At $\epsilon = 0.15$ (Fig. 3.8c), much of the SOS is chaotic and a typical initial condition in the chaotic sea explores a large range of $\sin \chi$. 
3.4 - Husimi Projection Technique

So far, we have talked about modes in a resonator and its corresponding Poincare Surface of Section. However, it is better to relate those modes to the phase space structures in the SOS, and the well-known Husimi Projection technique does this job well. The Husimi function is obtained by forming an overlap integral between an eigenstate and a coherent state[13, 14] corresponding to a minimum uncertainty Gaussian wavepacket. In this section, we very closely follow a procedure similar to the one described in Ref.[15]. As same as in quantum mechanics, we cannot have full information of electromagnetic fields, or in another view, photons in real space and momentum space at the same time, due to the analog of uncertainty principle,

$$\Delta x \Delta p \geq \frac{1}{2kR_c},$$

(3.38)

where $\Delta x$ and $\Delta p$ are the widths of electromagnetic fields $\psi(\vec{x})$ and $\psi(\vec{p})$ in real and momentum space, respectively. $k$ corresponds to the resonant frequency of a standing wave in an optical cavity. $R_c$ is the constant radius of a circular resonator, which is equal to 1 in this study. We use the electromagnetic fields as general $\psi$ to not yet specify the electric and magnetic fields. Our goal is to project a calculated $\psi(\vec{x})$ on a Gaussian package or coherent states $|z\rangle = |\vec{x}_c, \vec{p}_c\rangle$,

$$H_\psi(\vec{x}_c, \vec{p}_c) = |\langle z | \psi \rangle|^2,$$

(3.39)

recognizing that our resolution in real space is limited by the uncertainty relation Eq.(3.38) at its lower bound,

$$\Delta x = \frac{\sigma}{\sqrt{2k}} = \frac{\eta}{\sqrt{2}},$$

(3.40)

$$\Delta p = \frac{1}{\sqrt{2k\sigma}} = \frac{1}{\sqrt{2\eta k}},$$

(3.41)

where we set $\eta = \sigma / \sqrt{k}$. $\sigma$ is a free parameter of the method, but it must be carefully chosen based on the domains of variation of the pair $(\vec{x}, \vec{p})$. 
Figure 3.8: Some examples of the SOS where (a) $\epsilon = 0$, (b) $\epsilon = 0.05$, (c) $\epsilon = 0.1$, and (d) $\epsilon = 0.15$. Negative of $\sin \chi_i$ corresponds to the backward trajectory.
With a choice of $\sigma = 1$, resolutions are equal in both real and momentum spaces. Both $\Delta x$ and $\Delta p$ are shaped as $k$ is limited to infinity, or the wavelength goes to zero. The real-space representation of a coherent state is given in Gaussian distribution form by

$$Z_{\bar{x}, \bar{p}}(\bar{x}) = \left(\frac{1}{\pi \eta^2}\right)^{1/4} \exp\left[i\bar{p} \cdot \bar{x}\right] \exp\left[-\frac{1}{2\eta^2}(\bar{x} - \bar{x}_c)^2\right]$$

where the prefactor is the normalization factor. The first exponential factor determines the selection of the momentum $\bar{p}_c$, normalized to unity, and the second exponential factor restricts the examination to an area of size $\Delta x = \eta / \sqrt{2}$ around $\bar{x}_c$ in the real space. After projected on this coherent state as Eq.(3.42), the Husimi distribution is constructed as

$$H_{\psi}(\bar{x}_c, \bar{p}_c) = \left|\int d^2 x Z_{\bar{x}, \bar{p}}^*(\bar{x}) \psi(\bar{x})\right|^2.$$  

(3.43)

Note that the integral extends over two dimensions of $\bar{x}$, corresponding to the area in the configuration space, and this Husimi distribution is always positive in the phase space $\left(\bar{x}_c, \bar{p}_c\right)$. Eq.(3.43) is represented in general physical systems[16, 17, 18]. Hence, we should apply this Husimi distribution to project on the SOS of billiard systems, because the Poincare surface of section is used as an important interpretative tool of nonlinear dynamics in a billiard. Before doing this, we should know that the Husimi distribution is on a four-dimensional phase space of billiard; that is, $r_c$, $\theta_c$, $p_r$, and $p_\theta$ in Cylindrical coordinates. Our next goal is to project the Husimi distribution on the SOS by[19] setting $r_c = R_c$ defined as the constant radius of a circular billiard, integrating out $p_r$, choosing the spread of the coherent state around $r_c$ to be zero, and relating $p_\theta$ to $\sin \chi$. The coherent state in Cylindrical coordinates takes the form

$$Z_{c, \theta, p_r, p_\theta}(r, \theta) = Z_{r, r_c}(r) Z_{\theta, p_\theta}(\theta),$$

(3.44)

where
\[ Z_{r, p_r} (r) = \left( \frac{1}{\pi \eta^2} \right)^{1/4} \exp \left[ i p_r r \right] \exp \left[ -\frac{1}{2\eta^2} (r - r_c)^2 \right], \quad (3.45) \]

\[ Z_{\theta, p_\theta} (\theta) = \left( \frac{1}{\pi \sigma^2} \right)^{1/4} \sum_{l=0}^{\infty} \exp \left[ i p_\theta (\theta - 2\pi l) \right] \exp \left[ -\frac{1}{2\sigma^2} (\theta - \theta_c - 2\pi l)^2 \right]. \quad (3.46) \]

The sum on \( l \) is necessary to ensure periodicity in the \( \theta \) variable\([20]\). Note that the coherent state constructed is periodic only in the \( \theta \) variable, but not in the \( p_\theta \) variable. Next, we substitute Eq.(3.44) into Eq.(3.43) and obtain the projection of the full four-dimensional Husimi function onto the SOS at the boundary of the circular billiard \( r_c = R_c \),

\[ H_\psi (\theta_c, p_\theta, p_{r_c}) = \left\| \int_{-\pi}^{\pi} d\theta \int_0^r dr Z_{r, \theta, p_r, p_\theta}^* (r, \theta) \psi (r, \theta) \right\|^2. \quad (3.47) \]

To eliminate \( p_{r_c} \), not shown in the SOS, we integrate out \( p_{r_c} \), and take the limit \( \eta \to 0 \). The Husimi distribution obtained is

\[ H_\psi (\theta_c, p_\theta) = \lim_{\eta \to 0} \int_{0}^{\infty} dp_{r_c} \int_{-\pi}^{\pi} d\theta \int_0^r dr Z_{r, \theta, p_r, p_\theta}^* (r, \theta) \psi (r, \theta) \right\|^2. \quad (3.48) \]

Note that the integration only extends over positive radial momenta, because of the definition of the SOS, which only counts the trajectories that encounter the boundary inside in the outgoing direction in the resonator. The limit \( \eta \to 0 \), representative of the very short wavelength limit, is included to make certain \( \eta \) is not greater than the wavelength \( 1/k \). Generally, \( \eta \) is greater than the wavelength by the square-root factor, and is proportional to \( \Delta x \) as Eq.(3.40). As the result, these conflict with the truth to probe the coherent state to a region less than a wavelength from the boundary. Now consider the integration over \( r_c \) and its radial momenta \( p_{r_c} \) and factors dependent on them:

\[ \lim_{\eta \to 0} \int_{0}^{\infty} dp_{r_c} \int_{0}^{\infty} dr \left( \frac{1}{\pi \eta^2} \right)^{1/4} \exp \left[ -i p_c R_c \right] \exp \left[ -\frac{1}{2\eta^2} (r - R_c)^2 \right] \psi (r, \theta) \right\|^2. \]
Let $\eta \to 0$ and $1/k \to 0$ while $k\eta > 1$. These integrations may be thought of as Dirac’s delta function $\delta(r-R_c)$, and the electromagnetic fields factor may be reduced as $\psi^+(R_c,\theta)$, which is a result from the integration over $p_r$. Note that $\psi^+(R_c,\theta)$ contains only the wavefunction component with Hankel’s functions of the first kind, representing outgoing wave,

$$
\psi^+(R_c,\theta) = \sum_{m=-\infty}^{\infty} \alpha_m^{TM} H_m^+(\chi_{TM} R_c) e^{im\theta},
$$

which is the z-component of the electric fields. The resulted Husimi distribution is

$$
H_{\psi}(\theta_c, p_{\theta}) \approx \int_{-\pi}^{\pi} d\theta Z_{\theta, p_{\theta}}^* (\theta) \psi^+(R_c,\theta) \cdot \frac{\sqrt{z}}{\sqrt{\pi}}.
$$

Using Eq.(3.49) and short wavelength correspondence $p_{\theta} \leftrightarrow m = \chi_{TM} R_c \sin \chi$ [24] in Eq.(3.50), we obtain

$$
H_{\psi}(\theta_c, p_{\theta}) = \left(\frac{1}{\pi \sigma^2}\right)^{1/4} \sum_{l=0}^{\infty} \int_{-\pi}^{\pi} d\theta \exp\left[-i\chi_{TM} R_c \sin \chi (\theta - 2\pi l)\right] \times
$$

$$
\times \exp\left[-\frac{1}{2\sigma^2} (\theta - \theta_c - 2\pi l)^2\right] \psi^+(R_c,\theta) \cdot \frac{\sqrt{z}}{\sqrt{\pi}}
$$

We emphasize again that $\chi_{TM}^2 = \frac{\varepsilon_z}{\varepsilon_{r0}} (\varepsilon_{r0} k^2 - k_z^2)$ and $\chi$ is the angle of incidence of the ray at a position on boundary of a resonator. Noting that the integrand in Eq.(3.51) is $2\pi$-periodic, the integration limits can be extended to infinity[20] and the resulting Gaussian integral over $\theta$ can be evaluated analytically, yielding the final result Husimi distribution

$$
H_{\psi}(\theta_c, \sin \chi_c) = \sum_{m=-\infty}^{\infty} \alpha_m^{TM} H_m^+(\chi_{TM} R_c) \exp\left[-i\chi_{TM} R_c (\sin \chi - \sin \chi_c)\right] \theta_c \times
$$

$$
\times \exp\left[-\frac{\sigma^2}{2} (\sin \chi - \sin \chi_c)^2\right]
$$
It is recommended to choose $\sigma \sim 1/\sqrt{\chi_{TM} R_c}$, which is the optimal resolution in both SOS coordinates. For our resonator which has variable radius $R(\theta)$, $R_c$ should be chosen slightly outside the boundary $R_{\text{max}}$, and we map every pair $(\theta_c, \sin \chi_c)$ on the circular boundary to a different pair $(\theta, \sin \chi)$ on the boundary of the resonator

$$H_\psi(\theta_c, \sin \chi_c) \rightarrow H_\psi(\theta(\theta_c, \sin \chi_c), \sin \chi(\theta_c, \sin \chi_c)).$$

This job may be done by Birkhoff coordinates concept [21, 22]. It is a good idea to see some examples of this Husimi projection in this place. For a typical quadrupole resonator at $\epsilon = 0.07$ deformation with isotropic material, $\epsilon_z = \epsilon_{\theta\theta} = 1$, $k_z = 0$, $k = 45$, Fig. 3.9 shows modes and how they correspond to the false color on the Husimi-SOS projection.

In the Husimi-SOS Fig. 3.9a, those false colors settle on two stable fixed points, and correspond to bouncing back and forth of electromagnetic waves at polar angles $\theta = \pi / 2$ and $\theta = 3\pi / 2$, as expected. A diamond mode in Fig. 3.9b is settled on $\theta = 0, \pi / 2, \pi$, and $3\pi / 2$, and angular momenta are proportional to $\sin \chi$.

Figure 3.9: Typical mode patterns in real space and their Husimi-SOS projections. (a) Fabry-Perot mode and (b) Diamond mode.
Chapter 4 – Investigation

4.1 - Circular Billiard: A Case of the Rotationally Symmetric Dielectric

The dielectric circular billiard is an example where we know the exact analytic solutions. The exact eigenvectors of the scattering matrix for the circle correspond to a motion with a conserved angular momentum—in other words, a given impact angle $\sin \chi$ on the dielectric interface. The resulting scattering matrix is diagonal in the mode number representation. This means that a mode $m$ upon encountering the boundary will be scattered to the same channel $m$, corresponding to specular reflection.

Before investigating resonators with anisotropic materials, we will use the circular billiard as an example to check our numerical implementation. As in Ref. [9], the scattering matrix for a closed circular billiard can be written as

$$
\begin{bmatrix}
S(k) \\
\end{bmatrix}_{m} = -\delta_{mn} \frac{H_{m}^{-}(\chi R_{0})}{H_{m}^{+}(\chi R_{0})},
$$

(4.1)

where $H_{m}^{\pm}(\chi R_{0}) = J_{m}(\chi R_{0}) \pm i Y_{m}(\chi R_{0})$. Keep in mind that $R_{0}$ is a constant radius of the circle. We use $\chi$ as a general symbol of $\chi_{TM}$ and $\chi_{TE}$ in TM and TE fields, respectively. The quantization criterion, assuming an eigenvalue as 1, yields

$$
J_{m}(\chi R_{0}) = 0,
$$

(4.2)

which is the exact quantization condition for wavevectors $k$, obtained from

$$
k = \sqrt{\frac{\text{zero}[J_{m}]}{\varepsilon_{z} R_{0}^{2}} + \frac{k_{z}^{2}}{\varepsilon_{r \theta}}},
$$

(4.3)

where zero$[J_{m}]$ are the zeroes of Bessel functions of the first kind. With a case $m \gg \chi R_{0}$, it can be shown that[23]

$$
\begin{bmatrix}
S(k) \\
\end{bmatrix}_{mn} \sim 1 + i(1 + 2n)e^{-2m\alpha},
$$

(4.4)

where $n$ is the index of refraction of the core material and $\alpha = \cosh^{-1}\left(\frac{m}{\chi R_{0}}\right)$. This information corresponds to scattering of evanescent eigenvalues and results in
eigenphases exponentially close to zero, \( \varphi \sim (1+2n) e^{-2\alpha n} \), in the complex plane \( \{e^{i\varphi}\} \) of the distribution of the eigenvalues. Therefore, the modes of these evanescent eigenvalues are not the physical modes of the cavity. They just cling to the surface of the resonator. Fig. 4.1 gives real-space examples of the exact quantization criterion of a mode \((m = 30)\) of a circular billiard, where \( R_0 = 1 \), \( \varepsilon_z = \varepsilon_{\gamma} = 1 \), and \( k_z = 5 \). With the term *exact quantization criterion*, we expect an eigenvalue to be exactly 1, and the \( \pm m \)-th elements of its eigenvector to be non-zero but all others zero.

The number of rings depends proportionally on the \( i \)-th of \( k \), as resulted from resonant frequencies directly relative to \( k \). Note that these modes correspond perfectly to the whispering-gallery modes. After inspecting the eigenvectors of the \( 4^4 \) \( k \), the values of \(-30^4\) and \( 30^4 \) elements are \( 0.88363 + i 0 \) and \( -0.46096 + i 0.08194 \), respectively, otherwise in the machine precision range of \( 10^{-16} - 10^{-14} \), as expected. However, for the \( 4^4 k \), if we increase \( m_{\text{max}} \) by one, then the result is shown in Fig. 4.2. This situation falls to the case \( m \gg \chi R_0 \), which the eigenmode of the eigenvalue 1 is not the physical mode of the cavity, but is an evanescent mode. It really clings to the surface of the resonator. Additionally, the evanescent modes may be noticed as the two brightest points at approximately the \( \pm m_{\text{max}} \)-th elements on the TM region in the gray-scale representation of eigenvectors, as Fig. 4.3. Thus, we should emphasize that the number of such scattering eigenstates depends strongly on our choice of \( \Lambda_{ev} \) in our numerical implementation.

### 4.2 – Mode Patterns

In this section, we point our interest to study all possible modes of a deformed resonator with an anisotropic. As mentioned in Chapter 3.3, different regions in a SOS correspond to different mode patterns; as a result, physical properties of those
Figure 4.1: The real-space representations of a circular resonator with exact quantized $k$ where (a) $k = 36.443$, (b) $k = 41.3959$, (c) $k = 45.7269$, and (d) $k = 49.758$.

Figure 4.2: Increasing $m_{\text{max}}$ by 1 from Fig. 4.1d makes the mode fall to the evanescent channel.
Figure 4.3: Circled are $\pm m_{\text{max}}$ elements of the typical evanescent mode.

modes are different, as well. It was mentioned that the solutions of the wave equation for a generic shape such as the quadrupole can be classified by their association with three different kinds of motion: (1) quasi-periodic modes can be treated semiclassically by eikonal methods, (2) modes associated with unstable periodic orbits and chaotic motion cannot be treated by any analytic methods, and finally (3) modes associated with stable periodic orbits can be treated by generalizations of Gaussian optics[25]. However, we will inspect those modes by our numerical implementation.

As in Fig. 4.4a-e, a typical run of a quadrupolar resonator at $\epsilon = 0.07$ deformation with an anisotropic material, $\varepsilon_z = 1$, $\varepsilon_{\theta} = 1.5$, $k_z = 10$, $k = 75$, and $m_{\text{max}} = 103$, produces some mode patterns in real-space $z$-component of electric field intensities and their corresponding Husimi-SOS projection. A two-bounce Fabry-Perot mode in Fig. 4.4a is introduced first as a stable mode. The mode occupies two diametrical stable fixed points in the SOS, corresponding to the least
angular momentum $\sin \chi$. The name “Fabry-Perot” comes from the similarity of wave’s reflection in the Fabry-Perot resonator. Fig. 4.4b illustrates the formation of a deformed whispering-gallery mode, which is consistent with the physical behavior of light circumnavigating the perimeter of the cavity in such a way as to ensure total internal reflection at all encounters with the boundary[27]. This mode is localized on an invariant curve in the phase space, which is no longer a straight line. Nevertheless, long-lived cavity modes can still be constructed. A mode similar to the Fabry-Perot mode is a bouncing-ball mode (Fig. 4.4c). However, bouncing-ball modes occupy a larger interface of the resonator than the Fabry-Perot modes, corresponding to a larger range of angular momentum. They arise because the ray motion corresponding to this mode is a stable oscillation between the flat sides of the cavity. Note that bouncing-ball modes do not exist in the circle cavity. A stable 4-bounce orbit (Fig. 4.4d), shaped like a diamond, forms with its sharp vertices at the highest-curvature points of the resonator. This mode creates four islands of stability in the phase space. Rays launched near the diamond orbit will retain similar reflection points and angles even if they do not close onto themselves after four bounces. Furthermore, chaotic rays, which exponentially diverge from trajectories with closely neighboring initial conditions, cannot penetrate the diamond-shaped four-bounce islands. All mentioned above are stable periodic orbits. On the other hand, there are an infinite number of unstable periodic orbits in the SOS, and Fig. 4.4e is given as an example of those unstable modes. It is associated with the unstable manifolds of the unstable two-bounce orbit along the major axis of the resonator. Note that modes localized on unstable periodic orbits are termed scars in quantum chaos literature[17].

So far, we have learned typical modes of intermediate deformation range. However, there are still some modes unrevealed but found in large deformation range[28]. Let us consider a typical quadrupolar resonator at $\epsilon = 0.1$ deformation with an anisotropic core, $\epsilon_z = 1, \epsilon_{\rho\phi} = 1.3, k_z = 5, k = 56.15$ and $m_{\max} = 75$. Fig. 4.4f illustrates the formation of a chaotic whispering-gallery mode. The mode, a stable short periodic orbit, corresponds to ray trajectories which are part of the
Figure 4.4: Husimi distributions and real-space plots of modes in intermediate deformation (a) – (e), and in large deformation (f) – (h).
chaotic sea but circulate along the cavity perimeter many times before exploring other phase-space regions. In addition, the quadrupolar shape in high deformation range does have another stable short periodic orbit (other than the two-bounce Fabry-Perot mode); that is, a four-bounce bow-tie mode (Fig. 4.4g)[29]. It is seen clearly as the four islands in the SOS at $\sin(\chi) \approx 0.38$. The minimum at the very center of the islands indicates that the mode has an oscillatory motion transverse to the bow-tie path; this is consistent with the intensity pattern of Fig. 4.4g, which exhibits four transverse oscillations. The bow-tie mode therefore is rather different from the whispering-gallery mode that has the sense of rotation. The existence and stability of the bow-tie orbit is relatively insensitive to the precise shape of the boundary, so we expect these modes to be generic to deformed Cylindrical resonators. Finally, Fig. 4.4h shows one chaos mode, found abundantly in the SOS at large deformations. The mode cannot easily be associated with any particular classical phase space structure.

4.3 – Wave Propagation in Quadrupole Billiard with Anisotropic Core

With an anisotropic crystal as the core of a deformed resonator, some behavior of electromagnetic waves are changed from those in an isotropic crystal; furthermore, due to difference of $\varepsilon_z$ and $\varepsilon_{\rho\theta}$, distribution of the scattering matrix is deformed, many new mode patterns are discovered, and there are a lot of interesting effects when $k_z$ passes its critical value. Therefore, this new kind of the quadrupole resonator is still waiting to be discovered.

Let us consider a typical quadrupolar resonator at $\epsilon = 0.07$ deformation with an anisotropic core, $\varepsilon_z = 1$, $\varepsilon_{\rho\theta} = 1.5$, $k_z = 10$, $k = 75$, and $m_{\text{max}} = 103$. Fig. 4.5 illustrates a gray-scale representation of the scattering matrix. The horizontal and vertical black lines separate the representation into three regions with the same square area. According to Eq.(3.32), the first region corresponds to $z$-component of electric fields (Ez) of TM waves, the second region TE waves, and therefore the
third region the combination of TM and TE waves. Notice that the evanescent channels $\Lambda_{ev}$ of both region 1 and region 2 are different; this is a result from the difference of $\varepsilon_z$ and $\varepsilon_{r\theta}$ in an anisotropic material. Calculated $\chi_{TM} R \approx 79.773$ and $\chi_{TE} R \approx 97.702$, the evanescent channels of region 1 and region 2 are approximately 24 and 6, respectively. We still therefore emphasize that a chosen evanescent channel $\Lambda_{ev}$ should cover both that of TE and TM waves. Now let us consider the scattering channels of Ez and TE waves. Because, as mentioned in Chapter 3.2 that the dispersion relation of TE waves depends only on $\varepsilon_{r\theta}$, but of TM waves depends on both $\varepsilon_z$ and $\varepsilon_{r\theta}$, consequently, the width of spreading of both Ez and TE waves’ scattering channels are different, depending on the difference of $\varepsilon_z$ and $\varepsilon_{r\theta}$, as well. The more $\varepsilon_z$ and $\varepsilon_{r\theta}$, the more spreading out from the center line (see Fig. 4.5). Fig. 4.6 illustrates the three-dimensional plot. The height represents the scattering matrix’s absolute values of each pair of mode numbers. The second region corresponds to the combination of TM and TE waves. Nevertheless, it is only a result from $k_z \neq 0$; if $k_z = 0$, or electromagnetic waves
propagate in the $r\theta$-plane, then TM waves get the effect of only $\varepsilon_z$, and TE waves get the effect of only $\varepsilon_{r\theta}$. Consequently, the TM and TE waves propagate separately. Notice that, as another effect of the anisotropic core, the two little peaks in the region 3 are not aligned at the center. Note that circled in Fig. 4.6 are numerical error. Fig. 4.7 illustrates another scattering matrix's gray-scale representation of a quadrupolar resonator at $\varepsilon = 0.07$ deformation, $\varepsilon_z = 1.3$, $\varepsilon_{r\theta} = 1$, $k_z = 5$, $k = 59.7$, and $m_{\text{max}} = 80$.

Figure 4.6: The three-dimensional plot of Fig. 4.5. The height represents the scattering matrix’s absolute values of each pair of mode numbers.

Figure 4.7: The gray-scale representation of the scattering matrix.
Another interesting phenomenon while \( k_z \) is passing its critical value \( k_z^c \) is to switch between Ez and TE modes. In isotropic material, this critical value results in the cut-off radius \( R_c \) of a Cylindrical resonator. Therefore, the wave cannot propagate in structures with:

\[
\lambda > \frac{2\pi \sqrt{\varepsilon R_c}}{\text{zero}[J_m]},
\]

where \( \text{zero}[J_m] \) are the zeroes of Bessel functions of the first kind. However, in the case of extreme anisotropy, \( \varepsilon_z < 0 \) and \( \varepsilon_{\theta \theta} > 0 \), the cut-off radius does not exist at all; that is, any waves with different wavelengths can propagate in these structures. Moreover, waves with \( k_z > k_z^c \), propagating in this anisotropic material, are polarized to Ez waves. Also, ones with \( k_z < k_z^c \) behave as TE waves. \( k_z^c \) is defined as \( \chi_{(TM/TE)} = 0 \), consequently:

\[
k_z^c = \sqrt{\varepsilon_{\theta \theta} k}.
\]

Let us examine a typical resonator of \( \varepsilon = 0.07 \) deformation with an anisotropic core, \( \varepsilon_z = -1 \) and \( \varepsilon_{\theta \theta} = 1 \). Try the first case where \( k = 35.1, \ k_z = 50.2, \) and \( m_{\max} = 45 \). Fig. 4.8 shows that Ez modes exist and TE modes are cut off. We may intuitively interpret its physical meaning as the \( z \) component of TM waves polarized from general waves propagating in the resonator. In addition, we expect the distribution of scattering eigenvalues of Ez modes on the unit circle in the complex plane, and of TE modes at the origin with the number of dots equal to the number of the modes. Fig. 4.9a illustrates the distribution of eigenvalues of the system. After inspecting, TE modes’ eigenvalues are all in range of \( 10^{-32} \) to \( 10^{-27} \), as expected. One may notice the cut off in the representation of eigenvectors as Fig. 4.9b, as well. Ez modes distribute the Husimi-SOS projection as same as general modes, yet there is no Husimi projection on the SOS for all TE modes. Note that the strength of deformation does not have any effect with the existence of Ez modes nor the cut off of TE modes.
Figure 4.8: The gray-scale representation of the scattering matrix, showing that, for the $k_z > k_z^c$ case in an anisotropic material $\varepsilon_z = -1$ and $\varepsilon_{\theta\theta} = 1$, the $z$ component of TM modes exist but TE modes are cut off.

Figure 4.9: (a) The distribution of scattering eigenvalues. All dots at the origin are of TE modes, and equal to the number of TE modes. The dots on the unit circle represent the number of Ez modes, as usual. (b) The gray-scale representation of eigenvectors for the $k_z > k_z^c$ case. Because the eigenvalues of TE modes are not exactly zero, those dark color points in the TE region are still represented but in range of $10^{-36} - 10^{-35}$. 
Now let us move to the case \( k_z < k_z^c \) where \( k = 35.1 \), \( k_z = 2.2 \), and \( m_{\text{max}} = 39 \). Fig. 4.10 shows that TE modes exist but TM modes are cut off. Further, as expected, only eigenvalues of TE modes are distributed on the unit circle in the complex plane and of all TM modes on the center. There is a clear separation of TM and TE modes in the representation of eigenvectors as shown in Fig. 4.11, as well. However, after the inspection of the Husimi projection on the SOS, we found that the projection of each TE mode cannot be localized to any stable orbits, and it is barely related to the real space of the mode. These happens because, when solving Eq.(3.53), we use the eigenvectors of TM modes \( \alpha_{m}^{TM} \), corresponding to the exist of the z component of electric fields.

To explain the mathematical detail, try the typical \( k_z > k_z^c \) case. The constraint of an anisotropic core, \( \varepsilon_z < 0 \) and \( \varepsilon_{r\theta} > 0 \), makes \( \chi_{TM} \) a real number and \( \chi_{TE} \) a pure imaginary number. Consequently, we introduce modified Bessel functions of the first and second kinds as[31]

\[
I_m(x) = i^{-m}J_m(ix),
\]

\[
K_m(x) = \frac{\pi}{2} i^{m+1}H_m^0(ix),
\]

respectively. Unlike ordinary Bessel functions, oscillating as function of a real argument, \( I_m \) and \( K_m \) are exponentially growing and decaying functions. Hence, both exponential functions are cancelled out in our numerical implementation to cut off TE modes.
Figure 4.10: The gray-scale representation of the scattering matrix, showing that, for the $k_z < k_z^c$ case in an anisotropic material $\varepsilon_z = -1$ and $\varepsilon_{\theta\phi} = 1$, the TE modes exist but TM modes are cut off.

Figure 4.11: The gray-scale representation of eigenvectors for the $k_z < k_z^c$ case. Because of no TM modes, the representation of Husimi-SOS projection would not work in this case. As a result, the adaptation of Eq.(3.53) will be followed.
Chapter 5 – Conclusion

We consider mode patterns of electromagnetic waves in deformed quadrupolar resonators filled with anisotropic crystals. The chaos of ray motion depends strongly on the variation of the deformation. Standard tools used to study mode patterns and ray trajectories are the Poincaré surface of section and Husimi projection technique.

After using Maxwell’s equations and guessing electromagnetic wave functions, we expand the electromagnetic waves as a series of TM and TE waves, and describe them in Cylindrical coordinates by the separation variables method. After applying the boundary condition of tangential components of electric fields, the scattering matrix is introduced as an operator connecting initial states and bounced states. We then present the scattering matrix, its eigenvalues, and its eigenvectors as tools to consider evanescent channels, combination of TM and TE modes, and, importantly, numerical error.

We use our implementation tools to analyze behaviors of electromagnetic waves in different quadrupolar resonators. In the case of normal anisotropy, $\varepsilon_z > 0$ and $\varepsilon_{r\theta} > 0$, the results of different permittivities, affecting electromagnetic waves may be noticed from the representation of the scattering matrix. Moreover, we have found that in the case of extreme anisotropy, $\varepsilon_z < 0$ and $\varepsilon_{r\theta} > 0$, TM modes exist and TE modes are cut off when $k_z > k_z^c$, and vice versa. This means that electromagnetic waves are polarized by an extremely anisotropic material in a resonator. The electromagnetic waves’ resultant behaviors depend on initial frequencies and $k_z$. 
Bibliography


