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RAY CALVIN SHIFLETT for the M.S. in Mathematics
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Title UNIFORM CONVERGENCE ON CLASSES OF SUBSETS

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In the study of uniform convergence, one is led naturally to the question of how uniform convergence on subsets relates to uniform convergence on the whole space. This paper develops theorems on how pointwise convergence relates to uniform convergence on finite sets, how uniform convergence on finite subsets relates to uniform convergence on countable sets, and how uniform convergence on countable subsets relates to uniform convergence on uncountable sets.

Questions involving uniform convergence on Cauchy sequences are also investigated. These lead to theorems concerning the continuity of limit functions of sequences of continuous functions which converge uniformly on Cauchy sequences.

Many of the theorems are generalized ultimately to uniform space.

In Chapter III, a topology equivalent to the topology of uniform

convergence on compacta on the space of all functions mapping a complete space X to a space Y is introduced.

Finally, nets of functions replace sequences of functions, and the possibility of generalizing the previously developed theorems is explored.

UNIFORM CONVERGENCE ON CLASSES OF SUBSETS

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RAY CALVIN SHIFLETT

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UNIFORM CONVERGENCE ON CLASSES OF SUBSETS

CHAPTER I

INTRODUCTION

The investigation of the relationship of uniform convergence of a sequence of functions on subsets of a set X to the uniform convergence of this sequence on the whole of X suggests itself in several well-known theorems. Moreover, it is desirable to carry out such an investigation in as many situations as possible. As the structure on the set X changes, a number of related theorems will arise from the investigation of uniform convergence.

The investigations will originate with sequences of functions which map the reals into the reals. Chapter II deals only with functions whose range space is the reals and whose domains vary from the reals, to pseudometric spaces, to generalized topological spaces. Chapter III moves on to sequences of functions whose ranges are pseudometric and general topological spaces. Chapter IV carries this process of generalization to uniform spaces, and Chapter V generalizes the sequences of functions to nets of functions.

CHAPTER II

CONVERGENCE OF SEQUENCES OF REAL VALUED FUNCTIONS

Knowledge of the basic theory of convergence of sequences of real valued functions of a real variable will be assumed and no general discussion of this point will be given in this paper. The notation $< y_n >$ will be used to denote a sequence of objects indexed by the natural numbers. The notation $\{ . . . \}$ will be used to denote a set of objects.

As a point of motivation for the initial questions dealt with here, it is noticed that sequences of real valued functions exist whose pointwise convergence on a closed and bounded set implies that the sequence converges uniformly there. This fact might well pose a question as to the nature of the relationship of pointwise convergence to that of uniform convergence on more general sets.

The following example will serve as motivation. The sequence $< f_n >$ where $f_n(x) = x^n$, x an element of the interval $(0, 1)$ converges uniformly on every closed subset of $(0, 1)$, but the convergence on all of $(0, 1)$ is not uniform. This poses a question as to the relationship of uniform convergence on subsets of a set to that of uniform convergence on the whole space.

The following theorems answer some of these questions.

<u>Theorem 1:</u> If $<f_n>$ is a sequence of functions mapping the reals to the reals such that for each r, an element of the reals, $<f_n(r)>$ converges to $g(r)$, then $<f_n>$ is uniformly convergent on any finite subset of the reals.

<u>Proof:</u> Let $\{r_1, r_2, \cdots, r_n\}$ be a finite subset of the reals. For each r_i there is an $N(\varepsilon, r_i)$ such that $|f_n(r_i) - g(r_i)| < \varepsilon$ if $n > N(\varepsilon, r_i)$. Take N as the maximum element of the set $\{N(\varepsilon, r_i)/i=1, \cdots, n\}$, written $\max\{N(\varepsilon, r_i)/i=1, \cdots, n\} = N$. Then $|f_n(r_i) - g(r_i)| < \varepsilon$ for all r_i, an element of $\{r_1, \cdots, r_n\}$ if $n > N$. Thus pointwise convergence implies uniform convergence on finite sets. ■

In the above theorem, the restriction that the domain of the functions lie in the reals was unnecessary, since none of the properties of the reals were used. Thus, a natural generalization of Theorem 1 would be to replace the domain space by a general topological space X. Most of the remaining theorems of this chapter have functions with such a domain space.

The fact that pointwise convergence implies uniform convergence on finite sets suggests the question of whether uniform convergence of a sequence of functions on every finite subset of a countable set X implies uniform convergence on all of X. The following example shows this is not the case.

Example 1: Let $f_n(x) = \frac{x}{n}$, where x is an element of the rationals. On any finite subset $\{x_1, x_2, \cdots, x_p\}$, the sequence $<f_n>$ converges uniformly to zero as shown in Theorem 1. However, $<f_n>$ does not converge uniformly on all of the rationals. This can be seen by supposing that for a given $\varepsilon > 0$, an $N(\varepsilon)$ could be found such that $|f_n(x_o) - 0| < \varepsilon$ for all $n > N(\varepsilon)$. In particular for $n = n_o > N$, $|f_{n_o}(x_o) - 0| = |\frac{x_o}{n_o} - 0| = |\frac{x_o}{n_o}| < \varepsilon$. One choice for x_o would be $x_o = 2\varepsilon n_o$. Thus $|\frac{x_o}{n_o}| = \frac{2\varepsilon n_o}{n_o} = 2\varepsilon < \varepsilon$ which is a contradiction. So uniform convergence on every finite subset of a countable set X does not imply uniform convergence on all of X. ∎

From pointwise convergence a result of uniform convergence on finite subsets was established. Example 1 dealt with uniform convergence on finite subsets of a countable set. The next reasonable step would be to investigate how uniform convergence on every countable subset of a space x relates to uniform convergence on X.

Theorem 2: If $<f_n>$ is a sequence of functions each of which maps a topological space X to the reals and if $<f_n>$

converges uniformly to g on every countable subset of X, then $<f_n>$ converges uniformly to g everywhere on X.

Proof: Assume that $<f_n>$ does not converge to g uniformly on X. Then for some $\varepsilon > 0$ there is no $N(\varepsilon)$ such that $|f_n(x) - g(x)| < \varepsilon$ for all x in X when $n > N(\varepsilon)$. That is, for some $\varepsilon = \varepsilon_o$ and any N it is always possible to find an x such that $|f_n(x) - g(x)| \geq \varepsilon_o$ for some $n > N$. In particular for $\varepsilon = \varepsilon_o$ and $N = 1$ there is an $x = x_1$ such that $|f_n(x_1) - g(x_1)| \geq \varepsilon_o$ for some $n > 1$. Also for $N = 2$ there is an $x = x_2$, not necessarily different from x_1, such that $|f_n(x_2) - f(x_2)| \geq \varepsilon_o$ for some $n > 2$. For $N = k$ there is an $x = x_k$ such that $|f_n(x_k) - g(x_k)| \geq \varepsilon_o$ for some $n > k$. By this construction two countable sets, $I = \{1, 2, \cdots, k, \cdots\}$ and $E = \{x_1, x_2, \cdots, x_k, \cdots\}$, are obtained. Now note that $<f_n>$ does not converge uniformly on E.

For $<f_n>$ to converge uniformly on E, there would have to exist an M such that $|f_n(x_i) - g(x_i)| < \varepsilon_o$ for all x_i, an element of E, when $n > M$. However there is an N which is an element of I such that $N > M$, and $|f_n(x_N) - g(x_N)| \geq \varepsilon_o$ for some $n > N > M$. Thus E is a countable set on which $<f_n>$ does not converge uniformly. ■

In the above theorem the countable subsets of X may be thought of as sequences (with repetitions if the set is finite). Theorem 2 could be restated in terms of sequences. The validity of Theorem 2 could be questioned if, rather than all sequences, only special sequences are considered. Among the most useful special sequences are the Cauchy sequences. To discuss Cauchy sequences some concept of distance is required. Thus pseudometric spaces are used for domain spaces rather than general topological spaces. A complete definition and discussion of pseudometric spaces is given in most general topology texts (Kelley, 1955). Theorem 2 suggests the question of whether uniform convergence on every Cauchy sequence of X implies uniform convergence on all of X. The following example shows this is not the case.

<u>Example 2:</u> The sequence of functions $\langle f_n \rangle$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \le \frac{2n-1}{2} \\ 2x-2n+1 & \text{if} \quad \frac{2n-1}{2} \le x \le n \\ 1 & \text{if} \quad x \ge n \end{cases}$$

converges uniformly to zero on each Cauchy sequence E of the reals. To see this suppose E converges to r, an element of the reals. Then $f_n(r) = 0$ for all $n > r+1$. Being Cauchy, E

is bounded by some number M, so for $n > \max\{M, r+1\}$, $f_n(x_i) = 0$ for all x_i in E. It is also clear that the sequence is not uniformly convergent on the reals, since for any N there is an $M > N$ such that $f_n(x) = 1$ for $n > M$. ∎

Since uniform convergence on Cauchy sequences of X does not imply uniform convergence everywhere on X, the question arises as to whether it implies uniform convergence on something less than all of X.

<u>Theorem 3:</u> Let $\langle f_n \rangle$ be a sequence of functions mapping the reals to the reals. Then $\langle f_n \rangle$ converges uniformly to g on every Cauchy sequence of reals if and only if $\langle f_n \rangle$ converges uniformly to g on every compact set of reals.

<u>Proof:</u> Any compact subset of the reals is closed and bounded. Assume the convergence on the compact subset S is not uniform. Then there is an $\varepsilon > 0$ such that for each N there is an x in S for which $|f_n(x) - g(x)| \geq \varepsilon$ for some $n > N$. Choose for values of N each natural number and obtain a corresponding element of S. The preceding construction yields two sets, $I = \{1, 2, \cdots, n, \cdots\}$ and $E = \{x_1, x_2, \cdots, x_n, \cdots\}$. The set E is a bounded sequence and so has a Cauchy subsequence $E' = \{x_i^1, x_j^2, \cdots, x_k^n, \cdots\}$.

$<f_n>$ is not uniformly convergent on E', for if it were we could find an M, given an ε, such that $|f_n(x_j^i) - g(x_j^i)| < \varepsilon$ for all x_j^i in E' when $n > M$. However there is an N in I such that $N > M$ and such that $x_N = x_k^N$ an element of E'. Thus $|f_n(x_k^N) - g(x_k^N)| \geq \varepsilon$ for some $n > N > M$.

Now suppose $<f_n>$ is uniformly convergent on every compact subset S of the reals. Note that every Cauchy sequence of reals is contained in a bounded set which is in turn contained in a compact set. Thus uniform convergence on compact subsets implies uniform convergence on Cauchy sequences of reals. ■

A basic theorem in analysis is that if a sequence of real-valued continuous functions of a real variable converges uniformly on a given interval I to a function g, then g is continuous on I. The question then naturally arises as to an analog in the suggested context of Theorem 3.

If g is a function mapping X into Y, the restriction of g to the subset K of X is denoted by g/K.

Lemma 1: If g is a function mapping the reals into the reals such that g/K is continuous for every compact K contained in the reals, then g is continuous on all of the reals.

Proof: If g is not continuous on r, then for some $\varepsilon > 0$ there is an x such that $|g(x) - g(x_o)| \geq \varepsilon$ for some x_o in the open interval $(x - \delta, x + \delta)$. However the closed interval $[x - \delta, x + \delta]$ is a compact subset of the reals on which g is continuous. So the above statement can not occur. ■

Theorem 4: If $<f_n>$ is a sequence of continuous functions mapping the reals into the reals such that $<f_n>$ is uniformly convergent to g on every Cauchy sequence of reals, then g is continuous on all of the reals.

Proof: By Theorem 3, $<f_n>$ converges uniformly to g on every compact subset K contained in the reals. On K each f_n is continuous and so g is continuous on each compact subset of the reals. By Lemma 1, g is continuous on all of the reals. ■

CHAPTER III

A GENERALIZATION TO FUNCTIONS WHOSE IMAGES LIE IN PSEUDOMETRIC AND GENERAL TOPOLOGICAL SPACES

In this chapter as many of the theorems of Chapter II as is possible will be carried over to this more general setting. New concepts suitable to the generality will be introduced in this chapter giving rise to theorems having no analog in Chapter II.

In a pseudometric space (X, d_x) where d_x is the pseudometric, a spherical neighborhood about a given point x and of radius ε will be written

$$B(x, \varepsilon) = \{y/d_x(x, y) < \varepsilon\} .$$

Theorem 5: If $< f_n >$ is a sequence of functions mapping a general topological space X to a pseudometric space (Y, d_Y) and if $<f_n>$ converges pointwise to g at each point of X, then $<f_n>$ is uniformly convergent to g on every finite subset of X.

Proof: The hypothesis states that $f_n(x)$ is in $B(g(x), \varepsilon)$ for all $n > N(\varepsilon, x)$. Take a set $\{x_1, \cdots, x_p\}$. Obtain the corresponding set of N, $\{N(\varepsilon, x_1), \cdots, N(\varepsilon, x_p)\}$. Thus $f_n(x_i)$ is in $B(g(x_i), \varepsilon)$ for all x_i in $\{x_1, \cdots, x_p\}$

whenever $n > \max\{N(\varepsilon, x_i)/i = 1, \cdots, p\}$. ■

Theorem 6: If $<f_n>$ is a sequence of functions mapping a general topological space X to the pseudometric space (Y, d_Y) and if $<f_n>$ is uniformly convergent to g on every countable subset of X, then $<f_n>$ is uniformly convergent to g on all of X.

Proof: Suppose that $<f_n>$ does not converge uniformly on X. Then for some $\varepsilon > 0$, there is no $N(\varepsilon)$ such that $f_n(x)$ is in $B(g(x), \varepsilon)$ for all x in X when $n > N(\varepsilon)$. So there is some $\varepsilon = \varepsilon_o$ such that for any N chosen there is an x for which $f_n(x)$ is not in $B(g(x), \varepsilon_o)$ for some $n > N$. In particular for $N = 1$ there is an x_1 such that $f_n(x_1)$ is not in $B(g(x_1), \varepsilon_o)$ for some $n > 1$. After systematically exhausting the natural numbers two sets are created, $I = \{1, 2, \cdots, n, \cdots\}$ the set of values chosenfor N and $E = \{x_1, x_2, \cdots, x_n, \cdots\}$ the set of corresponding x, elements of X.

If $<f_n>$ were uniformly convergent to g on E then there would have to exist an M such that $f_n(x_i)$ is in $B(g(x_i), \varepsilon_o)$ for all $n > M$ and for all x_i in E. However there is an $N > M$ such that N is in I. By the above construction $f_n(x_N)$ is not in $B(g(x_N), \varepsilon_o)$ for some $n > N > M$. Thus E is a countable on which $<f_n>$ is not uniformly convergent. ∎

By introducing the concept of sequential compactness an analog of Theorem 3 may be obtained.

Definition 1: A subspace E of topological space X is sequentially compact if and only if every sequence of points of E contains a subsequence which converges to a point of E (Pervin, 1964).

Theorem 7: If $\langle f_n \rangle$ is a sequence of functions mapping a topological space X to a pseudometric space (Y, d_Y) which converges uniformly to g on all Cauchy sequences, then $\langle f_n \rangle$ is uniformly convergent to g on sequentially compact subsets F contained in X.

Proof: Suppose that $\langle f_n \rangle$ does not converge uniformly on sequentially compact subsets. Then for some $\varepsilon = \varepsilon_o$ and for every M there is an x in F for which $f_n(x)$ is not in $B(g(x), \varepsilon_o)$ for some $n > M$. Let M assume as its value each natural number. For each natural number n find an x_n in F and in so doing create two sets $I = \{1, 2, \cdots, n, \cdots\}$ and $E = \{x_1, x_2, \cdots, x_n, \cdots\}$ where E is a subset of F. However, F is a sequentially compact set which implies that E has a convergent subsequence $E' = \{x_i^1, x_j^2, \cdots, x_k^n, \cdots\}$. Now $\langle f_n \rangle$ is not uniformly convergent on E' since for any M there is an N in I such that $N > M$, x_N in E', and $f_n(x_N)$ is not in $B(g(x_N), \varepsilon_o)$ for some $n > N > M$. ■

By adding structure to the domain space X of Theorem 7, two corollaries may be obtained. The following are, however, needed.

<u>Definition 2:</u> A topological space X is a T_1-space if and only if for distinct x and y in X there exists two open sets one containing x but not y and the other containing y but not x (Pervin, 1964).

<u>Definition 3:</u> A topological space X is a first axiom or C_1-space if and only if for every point x in X there exists a countable family $\{B_n(x)\}$ of open sets containing x such that whenever x belongs to an open set G, $B_n(x)$ is contained in G for some n (Pervin, 1964).

<u>Definition 4:</u> A subset E of a topological space will be called countably compact if and only if every infinite subset of E has at least one limit point in E (Pervin, 1964).

The following theorem and its proof may be found in general topology texts: If x is a point and E a subset of a T_1,

C_1-space X then x is a limit point of E if and only if there exists a sequence of distinct points in E converging to x (Pervin, 1964).

Corollary 1: The sequence $<f_n>$ of Theorem 7 converges uniformly to g on countably compact subsets if the space X is a T_1, C_1-space.

Proof: Theorem 7 gives uniform convergence on sequentially compact subsets. By proving that sequential compactness is equivalent to countable compactness in a T_1, C_1-space, the corollary will be established.

By the definition of sequential compactness any sequentially compact subset E is countably compact since any infinite subset of E contains an infinite sequence which has a limit point.

Now if E is countably compact, take an infinite sequence E' of E. E' has a limit point x by the countable compactness. This limit point has a subsequence of E' which converges to x, by the theorem which was just stated. Thus sequential compactness and countable compactness are equivalent in a T_1, C_1-space and the result is established. ■

Corollary 2: If the space X in Theorem 7 is metric, then the sequence $<f_n>$ converges uniformly on compact subsets.

Proof: A metric space is T_1 and C_1 (Kelley, 1955). By Corollary 1, $<f_n>$ converges uniformly on countably compact subsets. In metric spaces, countable compactness and compactness are equivalent (Pervin, 1964). Thus the desired result is established. ■

In Chapter II, Theorem 4 dealt with the question of the continuity of the limit function of a sequence of continuous functions. The analog in the generality of this chapter follows.

Definition 5: A topological space X is locally compact if and only if each point of X is contained in a compact neighborhood (Pervin, 1964).

Lemma 2: If g is a function mapping a locally compact topological space X to a topological space Y, such that g/K is continuous for all compact subsets K contained in X, then g is continuous on X.

Proof: If g were not continuous on X, then given an open set G containing $g(x)$, an x_o, element of F, would exist for all open sets F containing x, such that $g(x_o)$ is not in G. However, for each x there exists a neighborhood H which is compact. Thus g is continuous on H. So $g(H')$ is contained in G for some open set H', x an element of H'. Therefore, the above statement can not occur and g is continuous. ■

<u>Theorem 8</u>: If $\langle f_n \rangle$ is a sequence of continuous functions mapping a locally compact metric space X to a pseudometric space Y such that $\langle f_n \rangle$ is uniformly convergent to g on every Cauchy sequence, then g is continuous on X.

<u>Proof</u>: By Corollary 2, uniform convergence on Cauchy sequences implies uniform convergence on compact subsets. On each compact subset K contained in X each f_n is continuous, thus g is continuous on each compact K. Thus by Lemma 2, g is continuous on X. ■

Theorem 4 and Theorem 8 are actually discussions of the concept of completeness of a topological space. That is, discussions as to whether the space $C(X, Y)$ of all continuous functions mapping X to Y contains all its limit points. Note that in such a space as $C(X, Y)$, the points or elements of the space are in fact functions. This suggests the forming of the space of all functions from X to Y. This space is called $\mathcal{F}(X, Y)$.

There are many ways of topologizing $\mathcal{F}(X, Y)$, and these are covered in most general topology texts. However, as Theorems 4 and 8 dealt with compact subsets, it seems natural to investigate the topology of uniform convergence on compacta.

Definition 6: If $\mathcal{C}$ is the family of all compact subsets of X, the collection of all subsets of the form $B(f, \varepsilon, E) = \{g/d(f(x), g(x)) < \varepsilon$ for all x in $E\}$ for E in $\mathcal{C}$ and $\varepsilon > 0$ is a subbase for the topology of uniform convergence on compacta (Pervin, 1964).

Another collection of special subsets of X which play an important part in Theorems 4 and 8 are the Cauchy sequences. A definition analogous to Definition 3, but with $\mathcal{C}$ as the collection of all Cauchy sequences, would give another topology which Theorem 7 and its corollaries imply might be related to the topology of uniform convergence on compacta.

Definition 7: Let $\mathcal{F}(X, Y)$ be the set of all functions from the metric space X to the pseudometric space Y. Topologize $\mathcal{F}(X, Y)$ by using as a subbase for the topology all sets $W(f, \varepsilon, E) = \{g/d_Y(f(x), g(x)) < \varepsilon$ for all x in $E\}$, where E is a Cauchy sequence in X. This topology we call the topology of uniform convergence on Cauchy sequences.

As noted, Theorem 7 and its corollaries show a relationship between uniform convergence on Cauchy sequences and uniform convergence on compact subsets. An important thing to notice is that convergence of a sequence of points in $\mathcal{F}(X, Y)$ in one of these two topologies means uniform convergence of a sequence of functions on some set or sets. Thus the following theorem is the answer to an evident question.

Theorem 9: Convergence of a sequence $<f_n>$ of elements of $\mathcal{F}(X, Y)$ in the topology of uniform convergence on Cauchy sequences implies convergence of $<f_n>$ in the topology of uniform convergence on compacta.

Proof: Suppose $<f_n>$ converges in the topology of uniform convergence on Cauchy sequences. Suppose, also, that there exists a compact set F contained in X on which $<f_n>$ did not converge uniformly. Then there is an $\varepsilon > 0$ such that for any N chosen there exists an x in F for which $f_n(x)$ is not in $B(f, \varepsilon, F)$ for some $n > N$. That is, $d_Y(f_n(x), f(x)) \geq \varepsilon$ for some $n > N$. In particular, for $N = 1$ there is an x in F such that $f_n(x)$ is not in $B(f, \varepsilon, F)$ which implies that $d_Y(f_n(x_1), f(x_1)) \geq \varepsilon$ for some $n > 1$. If the values of N are allowed to range over all natural numbers, two sets are created: $I = \{1, \cdots, k, \cdots\}$ and $E = \{x_1, \cdots, x_k, \cdots\}$. Note that X is a metric and Y is a pseudometric space. F is a compact subset of X and with the induced topology forms a compact pseudometric space. Every compact metric space is countably compact and so seqentially compact. Thus, $E = \{x_1, \cdots, x_k, \cdots\}$ contained in F has a subsequence $E' = \{x_i^1, x_j^2, \cdots, x_n^k, \cdots\}$ which converges and is therefore Cauchy. The sequence $<f_n>$ does not converge uniformly on E', for if it did, given an $\varepsilon > 0$ there would exist an M such that $f_n(x_i^j)$ is in $W(f, \varepsilon, E')$ whenever $n > M$. However there is an $N > M$ such that x_N is in E', and so $f_n(x_N)$ is not in $B(f, \varepsilon, F)$ for some $n > N > M$. Thus $d_Y(f_n(x_N), f(x_N)) \geq \varepsilon$ for some $n > N > M$. Then $f_n(x_N)$ is not in $W(f, \varepsilon, E')$ for some $n > N > M$. Thus E' is a Cauchy sequence on which the convergence is not uniform. ■

Theorem 9 shows that convergence in a particular one of these two topologies implies convergence in the other. Is the convergence, in fact, equivalent in the two topologies? The following example shows this is not the case.

<u>Example 3:</u> Take the sequence $<f_n>$ where $f_n(x) = x^n$. Let X be the interval $(0, 1)$ with the induced usual topology. First note that the sequence $<f_n>$ converges uniformly on $[a, b]$, $0 < a < b < 1$. Thus, $<f_n>$ converges uniformly on every compact subset of X. However, $<f_n>$ fails to converge uniformly on any

Cauchy sequence of X which converges to 1. ■

Since the convergence is not equivalent in general, under what added structure will they be equivalent? The final theorem of this chapter gives a sufficient condition for this equivalence.

Theorem 10: If X is a complete space, then convergence of a sequence $<f_n>$ of elements of $\mathcal{F}(X, Y)$ in the topology of uniform convergence on Cauchy sequences is equivalent to the convergence of $<f_n>$ in the topology of uniform convergence on compacta.

Proof: Theorem 9 gives the result one way.

So suppose $<f_n>$ converges to g in the topology of uniform convergence on compacta, then, given a compact subset F and an $\varepsilon > 0$, there is an $N(\varepsilon)$ such that f_n is in $\{f/d_Y(f(x), g(x)) < \varepsilon$ for all x in $F\}$ if $n > N$.

Now take a Cauchy sequence $<x_n>$ contained in X. Since X is complete, $<x_n>$ converges to, say, x. The set $\{x_1, \cdots, x_n, \cdots\} \cup \{x\}$ is a sequentially compact metric subspace of X. Thus, $\{x_1, \cdots, x_n, \cdots\} \cup \{x\}$ is compact (Pervin, 1964). Thus every Cauchy sequence in a complete metric space is contained in a compact set F. ■

CHAPTER IV

A GENERALIZATION TO UNIFORM SPACES AND CAUCHY NETS

As was the case in Chapter III, Chapter IV is mainly concerned with generalizing previously established theorems. Uniform spaces allow the highest generality in which to discuss uniform convergence; there must be some notion of uniform smallness of sets.

A different approach to generalizing these theorems is also used in this chapter. The concept of Cauchy nets replaces that of Cauchy sequences.

The definition and discussion of uniform spaces is given in most general topology texts (Pervin, 1964). The notation which is used is Pervin's, and a brief explanation follows:

(1) (X, U) is a space X with uniformity U.

(2) $\{u(x)/u \text{ in } U\}$ is the family of neighborhoods of x in the topology induced by U.

As in previous chapters, the concepts of pointwise and uniform convergence shall be important.

Definition 8: Let $<f_n>$ be a sequence of functions mapping X to a uniform space (Y, U). Then, for each x in X a sequence of points $<f_n(x)>$ in Y is obtained. If for each u in U there is an $N(u, x)$ such that $f_n(x)$ is in $u(g(x))$ whenever $n > N(u, x)$, then this sequence is said to be pointwise convergent to g in the uniformity U.

Definition 9: With the sequence $<f_n>$ of Definition 8, if when given a u in U there is a single $N(u)$ such that $f_n(x)$ is in $u(g(x))$ when $n > N(u)$ for all x of a given set E contained in X, then $<f_n>$ is uniformly convergent to g on E in the uniformity U.

The next two theorems are analogs of previous results.

Theorem 11: If $<f_n>$ is a sequence of functions mapping the topological space X to the uniform space (Y, U) and $<f_n>$ is pointwise convergent to g in the uniformity U, then $<f_n>$ is uniformly convergent to g on every finite subset of X in the uniformity U.

Proof: For each x_i in $\{x_1, \cdots, x_p\}$ there is an $N(u, x_i)$ such that $f_n(x_i)$ is in $u(g(x_i))$ if $n > N(u, x_i)$. Let $N(u) = \max \{N(u, x_i)/i = 1, \cdots, p\}$. Then $f_n(x_i)$ is in $u(g(x_i))$ if $n > N(u)$. ■

Theorem 12: If $<f_n>$ is a sequence of functions mapping the topological space X to the uniform space (Y, U) and $<f_n>$ is uniformly convergent to g on every countable subset $E = \{x_1, \cdots, x_n, \cdots\}$ contained in X, then $<f_n>$ is uniformly convergent to g on all of X in the uniformity U.

Proof: Suppose the conclusion of the theorem were in fact false, then there would have to exist a u_o in U such that for every M, an x in X could be found such that $f_n(x)$ is not in $u_o(g(x))$ for some $n > M$. For $M = 1$, an x_1 exists such that $f_n(x_1)$ is not in $u_o(g(x_1))$ for some $n > 1$. For $M = k$, an x_k exists such that $f_n(x_k)$ is not in $u_o(g(x_k))$ for some $n > k$. In this manner, a set $E = \{x_1, x_2, \cdots, x_k, \cdots\}$ contained in X is constructed corresponding to the set $I = \{1, 2, \cdots k, \cdots\}$ of values of M.

If $<f_n>$ were to converge uniformly on E in U, then there would have to exist an M' such that $f_n(x_i)$ is in $u_o(g(x_i))$ for all x_i in E when $n > M'$. However, there exists an N in I such that $N > M'$ and such that x_N is in E. Then, $f_n(x_N)$ is not in $u_o(g(x_i))$ for some $n > N > M'$. Thus E is a countable set of X on which $<f_n>$ does not converge uniformly in the uniformity U. ■

As has been seen before, theorems do not carry over in their exact form. After the structure on the spaces is changed, new concepts are often needed to form theorems. One such concept needed here is total boundedness.

Definition 10: A uniform space (X, U) is totally bounded if and only if for every u in U there is a finite set of points $\{x_1, \cdots, x_n\}$ contained in X such that $X = \bigcup_{i=1}^{n} u(x_i)$ (Pervin, 1964).

A form of generalization which is usually found in uniform spaces is that of the net. A discussion of nets may be found in many general topology texts. The notation used here has been taken from Kelley (1955).

Definition 11: A net $<S_n, n \text{ in } D>$ in the uniform space (X, U) is a Cauchy net, if and only if for each member u in U there exists an N in D such that (S_m, S_n) is in u whenever both m and n follow N in the ordering of D (Kelley, 1955).

Theroem 13: If $<f_n>$ is a sequence of functions mapping a totally bounded uniform space (X, U) to a uniform space (Y, W) and $<f_n>$ is uniformly convergent to g on every Cauchy net of (X, U), then $<f_n>$ is uniformly convergent to g on all of X.

Proof: Suppose $<f_n>$ is not uniformly convergent on X. Then there sould be a u in U such that for every N,

$f_n(x)$ is not in $u(g(x))$ for some $n > N$ and some x in X. Let u_o be this u and choose $N = 1$. There is an x_1 such that $f_n(x_1)$ is not in $u_o(g(x_1))$ for some $n > 1$. Now choose $N = 2$, there is an x_2 for which $f_n(x_2)$ is not in $u_o(g(x_2))$ for some $n > 2$. Continuing this process, two sets are obtained: $I = \{1, 2, \cdots, n, \cdots\}$, as the values of N, and $E = \{x_1, \cdots, x_n, \cdots\}$, as the corresponding values of x. Since X is totally bounded, every net, including E, has a Cauchy subnet (Kelley, 1955). Let E', contained in E, be a Cauchy subset of E. If $<f_n>$ were uniformly convergent on E', there would have to be an N such that, given a $u = u_o$, $f_n(x)$ is in $u_o(g(x))$ for all $n > N$ and all x in E'. However, there is an $M > N$ such that x_M is in E' and $f_n(x_M)$ is not in $u_o(g(x_M))$ for some $n > M > N$. ■

From this theorem, the analog of Theorems 4 and 8 is immediate.

<u>Theorem 14:</u> If $<f_n>$ is a sequence of continuous functions mapping a totally bounded space (X, U) to a uniform space (Y, W), and if $<f_n>$ converges uniformly to g on all Cauchy nets of X, then g is continuous on X.

<u>Proof:</u> By Theorem 13, the sequence $<f_n>$ converges uniformly on all of X. Thus g is continuous on all of X. ■

CHAPTER V

CONVERGENCE OF NETS OF FUNCTIONS

Since uniform spaces were the ultimate generalization available to still allow discussion of uniform convergence, further generalization must occur elsewhere than in the structure of the spaces. The last thing to be generalized is the sequences of functions. The method of nets has been introduced in Chapter IV, and now the investigation changes to nets of functions in the various spaces that have been examined.

The following three theorems are analogs to Theorems 1, 5 and 11.

Theroem 15: If $\langle F_n, \geq, D\rangle$ is a net of functions mapping a topological space X to the reals which converges pointwise to G, then $\langle F_n, \geq, D\rangle$ converges uniformly to G on every finite subset of X.

Proof: For each x_i in a finite set $E = \{x_1, \cdots, x_p\}$ there is an M_i in D such that $|F_n(x_i) - G(x_i)| < \varepsilon$ if $n \geq M_i$. There is an N in D which is farther out in the order of D than any of the M_i. Thus $|F_n(x_i) - G(x_i)| < \varepsilon$ if $n \geq N$ for all x_i in E. ∎

Theorem 16: If $<F_n, \geq, D>$ is a net of functions mapping a topological space X to a pseudometric space (Y, d_Y) which converges pointwise to G, then the net converges uniformly on every finite subset of X.

Proof: For each x_i in a finite set $E = \{x_1, \cdots, x_p\}$ there is an M_i in D such that $F_n(x_i)$ is in $B(G(x_i), \varepsilon)$ if $n \geq M_i$. There is an N in D which is farther out in the ordering of D than any of the M_i. Thus, $F_n(x_i)$ is in $B(G(x_i), \varepsilon)$ if $n \geq N$ for all x_i in E. ■

Theorem 17: If $<F_n, \geq, D>$ is a net of functions mapping a topological space X to a uniform space (Y, U), and if the net is pointwise convergent to G in the uniformity U, then $<F_n, \geq, D>$ is uniformly convergent to G on every finite subset of X in the uniformity of U.

Proof: For each x_i in $E = \{x_1, \cdots, x_p\}$, there is an $M(u, x_i)$ in D such that $F_n(x_i)$ is in $u(G(x_i))$ if $n \geq M(u, x_i)$ in D. There is an $N(u)$ in D such that $N(u) \geq M(u, x_i)$ for all $i = \{1, \cdots, n\}$. Thus, $F_n(x_i)$ is in $u(G(x_i))$ for all x_i in E if $n \geq N(u)$. ■

Theorems 2, 6 and 12 do not survive this last generalization as the following example shows.

Example 4: Let $<F_{\alpha}, \geq, D>$ be a net of functions. Let $D = X$ be the set of all ordinals less than the first uncountable ordinal. Let each F_{α} map this space into the reals as follows:

$$F_{\alpha}(x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 & \text{if } \alpha \leq x \ . \end{cases}$$

This net converges uniformly to zero on all countable subsets of X since for any countable subset E contained in X there exists an element γ in X such that $\gamma \geq \lambda$ for all λ in E and such that $F_{\beta}(x) = 0$ for all x in E if $\beta \geq \lambda$. This net, however, does not converge uniformly on all of X for no matter what α in X one chooses, there is a $\beta \geq \alpha$ such that $F_{\beta}(\beta) = 1$. ■

Since Theorems 2, 6 and 12 have no direct analog, the question arises under what added conditions some sort of an analog can be reached.

Definition 12: If D is a directed set and the set E contained in D has the property that, for each m in D, there is a p in E such that $p \geq m$, then E is a cofinal subset of D (Kelley, 1955).

Theorem 18: If $<F_n, \geq, D>$ is a net of functions mapping a topological space X to the uniform space (Y, U) where D has a countable cofinal subset E and if $<F_n, \geq, D>$ converges uniformly to G on every countable subset of X in the uniformity U, then $<F_n, \geq, D>$ converges uniformly to G on X in the uniformity U.

Proof: If $<F_n, \geq, D>$ were not convergent uniformly on X then there would exist a u in U for which every M in D has the following property: There exists an x in X for which $F_n(x)$ is not in $u(G(x))$ for some $n \geq M$. In particular choose an element x_N in X for each N in E. Thus obtain a countable set $\{x_N / N \text{ in } E\}$ on which the convergence of $<F_n, \geq, D>$ is not uniform, since for any M in D there is an N in E such that $N \geq M$ and such that $F_n(x_N)$ is not in $u(G(x_N))$ for some $n \geq N \geq M$. ∎

The last question discussed here is whether or not Theorem 13 has an analog in this chapter. That is, if a net of functions $<F_n, \geq, D>$ mapping a uniform space (X, U) to a uniform space (Y, W) converges uniformly to G on every Cauchy net of X in the uniformity U, does it converge uniformly on all of X in the uniformity U? The following example shows this is not the case.

<u>Example 5:</u> Let X be the space of all ordinals less than the first uncountable ordinal. Take as the base for the uniformity the identity relation i_X. This uniformity induces the discrete topology (Pervin, 1964). In a discrete space the only Cauchy nets are those which are eventually constant.

Take a net of functions $<F_T, \geq, X>$, each F_T maps X to the reals and is defined by

$$F_T(\alpha) = \begin{cases} 0 & \text{if } \alpha < T \\ 1 & \text{if } T \leq \alpha \ . \end{cases}$$

This net converges uniformly on every Cauchy net $E = <x_\alpha, \geq, X>$ where x_α is in X and such that $x_\alpha = y$ for all $\alpha \geq A$ for some A in X. Thus for all $\alpha \geq A$, $F_T(x_\alpha) = F_T(y)$, and $<F_T(y), \geq, X>$ converges to zero. However $<F_T, \geq, X>$ is not uniformly convergent since given any A in X, there is a T such that $F_T(\alpha) = 1$ for $\alpha \geq T \geq A$. ■

BIBLIOGRAPHY

1. Fulks, Watson. Advanced calculus. New York, Wiley, 1961. 521 p.

2. Kelley, John L. General topology. Princeton, D. Van Nostrand, 1955. 298 p.

3. Pervin, William J. Foundations of general topology. New York, Academic Press, 1964. 209 p.