AN ABSTRACT OF THE DISSERATION OF

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Title: An Investigation of Tenth Grade Students' Views of the Purposes of Geometric Proof

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Margaret L. Niess

The purpose of this investigation was to describe tenth grade students' views of the purposes of geometric proof within the context of their learning. Classroom observations, the curriculum, assessment tools, journal questions, and a preconceptions questionnaire were used to provide context for the views expressed by students from a single classroom. Eleven classroom episodes selected from the classroom observations were used to describe the instructional context as well as discourse among the students during group work. The episodes provided details about how and when the classroom teacher addressed various purposes of proofs involving geometry concepts throughout two instructional units on coordinate geometry proofs and two-column proofs. The episodes also consisted of student discourse relating to

the purposes of geometric proof as students worked on assigned proof problems. The students' views were examined through journal questions given at the beginning of selected days and through a post-instruction questionnaire and individual interviews.

There were three main findings of the study. First, several students experienced difficulty in expressing their views of the purposes of geometric proof when asked directly. One-third of the students could only list properties or theorems they encountered during the unit on geometric proof. However, when these students were asked to describe the purpose for each column, all of the students listed both explanation and verification. Second, the students expressed limited views of the purposes of proof, referring mainly to verification. Only a few students mentioned explanation, systematization, and communication. However, students generally referred to at least two purposes of proof (explanation, verification, and communication) when describing the proving process involved in coordinate geometry. Third, the students' views of various purposes of geometric proof were diverse.

Recommendations for future research include the examination of students' views of the purposes of geometric proofs for students who use paragraph form and studies to investigate the development of students' views of the purposes of proof as they gain more experience with formal proof writing and other methods of proof.

An Investigation of Tenth Grade Students' Views of the Purpose of Geometric Proof

by

Mary Katherine Gfeller

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I understand that my dissertation will become part of the permanent collection of the Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

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Mary Katherine Gfeller, Author

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CHAPTER I

THE PROBLEM

In *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics [NCTM] (2000) distinguished the understanding of, and the ability to write mathematical proof as an important topic and a desirable goal. NCTM proposed that certain topics, such as mathematical proof, are important in the curriculum when the topic has the "utility to develop other mathematical ideas, in linking different areas of mathematics, or in deepening students' appreciation of mathematics as a discipline and a human creation" (p. 15). NCTM argued that reasoning and proof, which rely heavily on making connections between established ideas, are helpful in understanding newly developed concepts as well as in the discovery of new ideas.

In an epistemological sense, proving a mathematical conjecture is one of several processes involved in developing and justifying new knowledge within the discipline of mathematics (Schwab, 1978). Unlike the natural sciences, mathematics is based on non-empirical objects. While empirical methods such as measuring and experimentation may be used in the proving process to achieve some level of conviction in mathematics, the validity of mathematical conjectures is not based solely on observation. The validity of mathematical conjectures is based on mathematical proof.

Mathematical Proof

In terms of school mathematics, mathematical proof is defined as "a formal way of expressing particular kinds of reasoning and justification" (NCTM, 2000, p. 56). The research literature on mathematical proof includes the following methods of proof:

1. logical deductive proof (including induction and proof by cases) defined as a finite sequence of formulae or propositions derived from axioms on the basis of establishing rules (Hanna & Jehnke, 1993);

2. indirect proof whereby a contradiction is found when the negation of a conjecture has been made (Williams, 1979);

3. proof by exhaustion in a finite situation (Maher & Martino, 1996).

However, mathematical proof is defined differently by philosophers of mathematics. Lakatos (1998) stated that there are really two kinds of proofs, informal (pre-formal and post-formal) and formal (logical deductive proof). According to Lakatos, working mathematicians use informal proofs:

> Now this so-called 'informal proof' is nothing other than a proof in an axiomatized mathematical theory which has already taken the shape of the hypothetico-deductive system, but which leaves the underlying logic unspecified. At the present stage of development in mathematical logic, a competent logician can grasp in a very short time what the necessary underlying logic of a theory is, and can formalize any such proof without too much brain-racking. (p. 156)

Using Lakatos' definition of proof, it is the formal proof that high school students are asked to accomplish, but informal proof that is mainly used by working mathematicians.

Students' Understanding of Mathematical Proof

One way to explore student's understanding of mathematical proof is through Skemp's (1979) conceptualization for understanding mathematics: instrumental, relational, and formal. *Instrumental understanding* refers to the kind of understanding achieved by students who are able to produce correct answers or procedures in a mathematical problem without necessarily understanding what has been achieved. Students with only an instrumental understanding of proof may resort to the memorization of proofs or the memorization of a sequence of steps in proof writing. Harel and Sowder (1998) believed that instruction emphasizing the structure and symbols of proofs might reinforce students' desires to memorize proofs, which may, in turn, lead to ritualism. Students who engage in ritualism may determine that a proof is valid based solely on its appearance rather than for its content.

Relational understanding refers to the kind of understanding achieved when students are able to conceptualize the process or procedures of an algorithm or mathematical process. Students who have achieved a relational understanding of proof understand that once a proof has been given for a generalization, no further justification of a particular case within the domain of the generalization is needed. Students who do not possess a relational understanding of proof often produce inductive arguments in the form of empirical checks even after a proof of a generalization has been given (Fishbein & Kedem, 1982; Porteous, 1991; Vinner, 1983).

Formal understanding refers to the kind of understanding achieved by students who acknowledge the symbols and methods needed to communicate their ideas within a mathematical community. Students who have a formal understanding of mathematics understand the role of mathematical proof in a public forum as a method for convincing others of the validity of conjectures and to explain and communicate to others why a conjecture is true.

In the research literature on mathematical proof, there are several aspects of proof that may be considered formal understanding of proof and have been referred to as the functions (or purposes) of proof (Bell, 1976; de Villiers, 1999; Hanna & Jehnke, 1993; Schoenfeld, 1992). Hanna (2000) has provided a comprehensive list of the various purposes of mathematical proof:

1. verification (concerned with the truth of a statement);
2. explanation (providing insight into why it is true);
3. systematization (the organization of various results into a deductive system of axioms, major concepts and theorems);
4. discovery (the discovery or invention of new results);
5. communication (the transmission of mathematical knowledge);
6. construction of an empirical theory;
7. exploration of the meaning of a definition or the consequences of an assumption;
8. incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective. (p. 9).

The Problem

In the mathematics education research literature, assessment of students' understanding of proof has focused mainly on students' abilities to write proofs or to evaluate various types of arguments. Empirical studies that have investigated high school students' instrumental understanding of mathematical proof have documented that students generally have difficulty producing algebraic and geometric proofs (Healy & Hoyles, 2000; Kahan, 1999; Senk, 1989). For example, Senk reported that only 30% of high school students were able to produce simple geometry proofs after a year-long course in high school geometry. She attributed this finding to the lack of content knowledge in geometry and recommended that geometric concepts be taught as early as the elementary grades.

In terms of relational understanding, researchers have mainly agreed that high school students often fail to apply previously proven generalizations to particular cases of generalizations. For example, when high school students were asked to evaluate a fictitious students' solutions to a homework assignment involving a particular case of a proven algebraic generalization, approximately 50% of the students who said they understood the proof preferred to use computation or a re-enactment of the proof given for the generalization rather than to deduce the case from the generalization (Vinner, 1983). The results of three other studies (Fischbein & Kedem, 1982; Porteous, 1991; Williams, 1979) also indicated high school students did not fully grasp the power of proven algebraic and geometric generalizations. In a recent study, Healy and Hoyles (2000) found that nearly 40% of high school students in England and Wales, when given a choice to check a particular case of a proven algebraic generalization or to deduce it, chose to empirically check the particular case. Kahan (1999) claimed that most students are aware of the power of generalizations, but do not accept the truth of particular cases if students have only been shown a proof or have been told that one exists.

Researchers have also generally characterized high school students as empiricists, that is, relying mainly on the demonstration of specific examples to establish the validity of a mathematical conjecture. This finding has been evidenced in several studies where students were asked to explain, justify, or prove a mathematical conjecture (Balacheff, 1988; Bell, 1976; Galbraith, 1981; Schoenfeld, 1983) and to convince others about the truth of a mathematical conjecture (Williams, 1979). However, the nature of these studies raises serious concerns when interpreting these findings. First, some of the students in these studies might have wanted to produce a mathematical proof, but turned to empirical arguments due to a lack of language skills. In a study regarding students' views of the purposes of algebraic proof, Healy and Hoyles (2000) asked students to choose the argument that their classroom teacher would give the highest grade. This line of questioning enabled the researchers to get a better notion of whether students valued deductive proofs, even if they were not able to write proofs. Second, none of the studies revealed any details about the instruction students had received on proof or even if they were given the opportunity to learn about the purposes of mathematical proof.

Only one study focused on instructional issues and students' views of the purposes of geometric proof (Chazen, 1993). This investigation examined the explanatory and verification features of geometric proof within the context of an experimental classroom using dynamic geometry software with high school students. Chazen found that, even with explicit instruction on the explanatory feature of deductive geometric proofs, some students still did not seem to value the explanatory nature of proof or the limitations of the software in establishing the truth of a conjecture. Unfortunately, this study did not provide specific details about the instruction to determine the extent of the results. For example, the discourse between the classroom teacher and the students, which would have provided insight into the explicitness of the instruction, was not examined, nor was attention given, if any, to determine the level of implicit instruction on the explanatory and verification features of geometric proof.

While Chazen's (1993) examination of high school students' views of verification and explanatory features of geometric proof is important, examining students' views of other purposes of proof is also needed to provide a more holistic description. This study will examine the views of tenth grade students regarding the purposes of geometric proof throughout four weeks of instruction on proofs involving geometry. Classroom events, curriculum materials, and assessments will contribute to the context of the students' views. Thus, the primary purpose of this study will be guided by the main research question: What are the views of tenth graders regarding the purposes of geometric proof in the context of learning geometric proof?

Significance of the Study

This study aims to describe the breadth and depth of high school students' views of the purposes of geometric proof in the context of learning geometric proof. This investigation will be a natural extension of the current knowledge base in the area of student understanding of geometric proof. While the classroom chosen for this study is traditional in the sense that two-column geometric proofs will be taught, Herbst (2002) argued that glimpses into traditional classroom settings have potential in enabling researchers and mathematics teacher educators to find ways to improve current instruction by showing limitations in traditional practices. High school teachers can also benefit from learning about this group of high school students' views of the purposes of geometric proof by learning about alternative views that might be expressed by the students.

In addition to providing descriptions of students' formal understanding, this study provides a basis for further clarification of Skemp's (1979) conceptual framework for understanding. Within this conceptual framework, a student may be able to produce a proof, but not understand that it provides generality to the conjecture or that it is a mathematical way of communicating the validity of the conjecture to others. Thus, it makes sense to investigate students' formal understanding of proof in isolation of instrumental and relational understanding. However, further development of a framework for understanding mathematical proof might include possible relationships among the three constructs. While researchers (Hiebert & Lefevre, 1986) have investigated possible relationships between instrumental and relational understanding, formal understanding has been absent from this body of literature.

This study is significant in two distinctive ways. First of all, the study provides the breadth and depth of the students' views of the purposes of proof. Rather than focusing solely on the verification and explanatory aspects of proof, this study examines the extent of the students' views in terms of other purposes of proof, such as systematization, discovery, and communication. Second, this study examines students' understanding of geometric proof from a naturalistic perspective. A naturalistic perspective acknowledges the potential progress of the students' understanding rather than their deficiencies (Moschkovich & Brenner, 2000). This investigation examined the students' views from their own perspective using terms and constructs provided mainly by the students and their learning environment prior to comparing it to the established views held by mathematics education researchers. Context is provided from the natural classroom events, assessment tools, and curriculum materials.

CHAPTER II

REVIEW OF THE LITERATURE

Introduction

This chapter reviews the literature related to high school students' understanding of mathematical proof. High school students were the focus of many of the empirical research studies included in this chapter. However, some studies that have explored elementary and college students' understanding of mathematical proof were included based on their relevancy to the issues surrounding mathematical proof in schools. A review of the literature included in this chapter explores student's understanding of mathematical proof through Skemp's (1979) conceptualization for understanding mathematics: instrumental, relational, and formal.

High school students' lack of ability to produce algebraic and geometric proofs has been documented (Healy & Hoyles, 2000; Kahan, 1999; Senk, 1989) as well as their lack of understanding of the domain of validity of a proven generalization (Fischbein & Kedem, 1982; Healy & Hoyles, 2000; Kahan, 1999; Porteous, 1991; Vinner, 1983). Students' formal understanding of mathematical proof has also been explored (Balacheff, 1988; Bell, 1976; Chazen, 1993; Galbraith, 1981; Healy & Hoyles, 2000; Martin & Harel, 1989; Schoenfeld, 1983). The utility of the van Hiele theory in describing students' thinking relating to geometric proof has also been documented (Burger & Shaughnessy, 1986; Fuys, Geddes, & Tishler, 1988; Mayberry, 1983; Senk, 1989; Usiskin, 1982).

Definitions

Attention to the prevailing definitions of common terms used in this chapter is crucial. In *Principles and Standards for School Mathematics* (NCTM, 2000), argumentation, justification, and proof are used. Yet, only mathematical proof has been formally defined. All of these terms hold distinctive meanings in the research literature.

Argumentation and Proof

In terms of school mathematics, mathematical proof is defined as an argument "consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 356). There are many types of arguments used by people in various professions (see Locke, 1959). Scientists use inductive arguments based on empirical evidence, while lawyers use 'presumptively plausible' arguments where the burden of proof is on the accuser. Teenagers often use 'argumentum ad populum' when trying to convince their parents. Douek (2000) explained that mathematical proof is a type of argumentation. This type of argumentation involves evidence gathered through inductive reasoning, diagrams, definitions, axioms, and previously proven theorems bound within an axiomatic system, also known as systematization. Researchers in mathematics education have viewed systematization as an important aspect of proof (de Villiers, 1999; Hanna, 2000; Healy & Hoyles, 2000). De Villiers has provided some interesting perspectives on systematization:

1. It helps identify inconsistencies, circular arguments, and hidden or not explicitly stated assumptions.

2. It unifies and simplifies mathematical theories by integrating unrelated statements, theorems, and concepts, with one another, thus leading to an economical presentation of results.

3. It provides a useful global perspective or bird's eye view of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived.

4. It is helpful for the application both within and outside mathematics, since it makes it possible to check the applicability of a whole complex structure or theory by simply evaluating the suitability of its axioms and definitions.

5. It often leads to alternative deductive systems that provide new perspectives and/or are more economical, elegant, and powerful than existing ones. (p. 7)

While there are various kinds of proofs that are used in the mathematics classrooms, all mathematical proofs are deductive. A *direct proof* is an argument that uses rules of logic to join premise statements in order to produce a conclusion. *Proof by induction* is a also a deductive proof that makes use of assuming the statement true for the k th element, P(k), and demonstrating deductively that the statement is true for the (k+1) th element, P(k+1). Proof by induction should not be confused with the term induction, which is a "process of reasoning by which a generalization is reached from a study of particular facts" (Williams, 1979, p.12).

The following example illustrates the difference between an inductive argument and deductive proof: Suppose that a teacher wants her students to prove that the sum of any three consecutive numbers is divisible by 3. A student who relies on induction to prove this conjecture may pick a series of triplets ($1 + 2 + 3 = 6$, $3 + 4 + 5 = 12$, $6 + 7 + 8 = 21$) to demonstrate the validity. On the other hand, a student using a deductive proof may present the following: Let the first unknown number be called "n." The next two consecutive numbers must be "n+1" and "n+2." When we add all three of these numbers, we get $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$. Because 3 is a factor of the sum $3 (n + 1)$, the sum of the three consecutive numbers is divisible by 3.

An *indirect proof*, or proof by contradiction, is also a deductive proof. The basic process for an indirect proof is to assume the proposition is false and then demonstrate that a contradiction to the proposition exists. In college mathematics courses, indirect proofs are typically introduced to students as a way to demonstrate uniqueness (such as there is one and only one identity element over addition). The notion that indirect proof should be reserved for college students has been diminished in light of Maher's and Martino's (1996) documentation of a fifth grader's use of indirect proof in a problem involving combinations. However, it should be noted that this study highlighted the proof activities of one child rather than the norm of the classroom using experimental teaching activities.

Reasoning and Justification

NCTM also stated that mathematical proof is a "formal way of expressing particular kinds of reasoning and justification" (p. 56). Reasoning is a way of thinking about concepts, and there are many kinds of reasoning used in the mathematics classroom (quantitative, inductive, deductive, probabilistic, geometric, etc…). Recently, transformational reasoning, which is neither inductive nor deductive, has received some attention in the research literature for its applicability to dynamic geometry systems and proof construction:

> Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated. (Simon, 1996, p. 201)

Fischbein (1982) called this type of reasoning 'intuitive' reasoning. In reasoning intuitively about the sum of the angles in a triangle, Fischbein described that one may draw two perpendicular lines to a base line, noting that two right angles have been formed whose sum is 180 degrees. As the perpendicular lines are rotated toward each other, a triangle is formed. In addition, the vertex angle appears to be getting larger, yet the base angles (whose sum was originally 180 degrees) appear to be getting smaller, thus creating equilibrium. The movement of the perpendicular lines toward each other might be used to evoke insight about potential relationships within the situation. The potential for developing transformational reasoning skills within dynamic geometry software exists. As Simon pointed out, transformational reasoning may be a useful skill in both an algebraic and geometric context.

Essentially all types of reasoning are used in the process of justification. Wolfe (1970) described justification as a validation (of an assertion) or vindication (of an action), and observed six different means of justification in algebra and geometry classes:

1. A universal generalization is quoted of which the given case, either a generalization or a singular statement, is either an instance or a consequence;

2. A true instance of a generalization is noted, making the generalization seem more plausible;

3. A counterexample which produces a counter-instance of a universal generalization is noted or discussed, or the absence of a counterexample is noted or discussed;

4. A deductive proof of a generalization or a singular statement is presented or discussed. The proof may vary in method, such as direct or indirect, as well as degree of formality and logical completeness;

5. A pragmatic reason in defense of a proposed or completed action is noted or discussed. This reason generally is in the form of show that following the prescription will lead to the correct answer, make possible a poof, or simplify or reduce the required amount of work;

6. An algorithm expressing verbally a performed or proposed action or course of action is justified mathematically. (p. 337)

What is acceptable justification for students engaged in school tasks should be determined by the age of the student and the particular task (NCTM, 2000). However, Dreyfus (1999) raised a valid question about the possible difficulties students may encounter when having to select the type of justification desired by the classroom teacher.

The Role of Proof in the Mathematics Classroom

In 1938, Fawcett addressed the issue of the role of proof in the mathematics classroom by defending the inclusion of Euclidean geometry in the mathematics curriculum. Fawcett's main defense stemmed from issues of transfer of logical deductive thinking to other academic areas and issues students would meet outside school. He contended that teaching proof in a manner where students were allowed to develop a theory about the space around them through careful thought about mathematical definitions was both beneficial to learning geometry content and logical thought in general. Mathematical proof held prominence in the mathematics curriculum for many decades until the 1960s. Curriculum reform efforts began to take shape during that time, and included a broad range of mathematical topics such as mathematical problem solving and justification. Recently, research literature in mathematics education has focused on all forms of mathematical justification and pedagogical changes toward establishing sociomathematical norms in the classroom. Yackel & Cobb (1996) defined sociomathematical norms as "normative understandings" in the classroom, including understandings that become "acceptable" practices in classroom mathematical explanation and justification. Lampert (1990) used sociomathematical norms and suggested that students should come to know mathematical argumentation as a process of the discipline of mathematics as well as a product. Lampert's view of the process of establishing the validity in the mathematics classroom included a "zig-zag" approach to proving with conjecturing, counterexamples, and revisions. These views are also consistent with those of Lakatos (1976) in that students generally only see a presentation of an explanation of a rule as a finished product rather than the path leading to the explanation. Lampert went on to explain that revision of ideas should be acceptable in the classroom only after careful consideration has been taken. In this study, Lampert described the events of her fifth grade classroom as students explored the last digit of expressions involving exponents such as 5^4. She contended that by establishing social norms and discourse, these students "learned to do mathematics in a way that is congruent with disciplinary discourse" (p. 58).

Cobb, Boufi, McClain, and Whitenack (1997) supported Lampert's attempt to encapsulate mathematical discourse with attention to Polya's (1954) mantra for mathematical argumentation with *intellectual courage* and *intellectual honesty.* Using an elementary combinatorics problem, Cobb et al. investigated a construct called "reflective discourse." In this study, the classroom teacher asked students to find as many ways as they could that five monkeys could be in two trees (a small tree and a big tree). After the classroom teacher listed the various possibilities in a chart, the classroom teacher asked students whether they thought there were any more possibilities. Cobb et al. described this transition in the discourse as a reflection on the mathematical activity itself. The discourse that followed consisted of an argument offered by one student who discussed the chart generated in class, characterized as an example of collective reflection.

Through research on sociomathematical norms, mathematical investigation and experimentation received a great deal of attention in the research literature. As a result some mathematicians and mathematics educators perceived a de-emphasis on mathematical proof in the high school curriculum. After demonstrating the futility of experimentation in establishing the validity of a conjecture involving number theory, Wu (1996) stated:

> Now is not the time to belittle the importance of experimentation, because experimentation is essential in mathematics. What I am trying to do is point out the folly of educating students to rely solely on experimentation as a way of doing mathematics. Mathematics is concerned with statements

> that are true, forever and ever and without exceptions. and there is no other way of arriving at such statements except through the construction of proofs. (p. 224)

Wu also contended that two-column proofs, which aim to make logical connections, made in arriving at a conclusion clearer to students, are essential in the classroom. Other mathematicians have also pleaded for a balance between conjecture and experimentation. Haimo (1995), stated:

> The stress on problem-solving, where students are encouraged to look for patterns and draw conclusions, merits our applause. That is the nature of mathematics; that is when experimentation and conjecture occur. It is, however, only a beginning, and this must be made unmistakably clear if we are serious about educating our students fully. (p. 103)

Recently, there has been an effort to join investigation and experimentation with mathematical proof. This holistic approach to proof, called "cognitive unity of theorems," involves a change in the way students are engaged in problem solving activities and proof (Garuti, Boreo, Lemut, & Mariotti, 1996). Cognitive unity of theorems is a theoretical construct that describes a continuity that exists between problem solving activities and the production of proof. The task of asking students to prove a mathematical conjecture is devoid of the production of the conjecture itself, and thus, the continuity is broken. This perspective is not new as Lakatos (1976) presented a similar paradigm of proof and refutation. It has been proposed that students explore conjectures through dynamic geometry software first and then reflect on explaining why conjectures are valid (de Villiers, 1999). In his book *Rethinking Proof*, de Villiers asks students to follow a series of constructions, tells them to drag a particular point on the construction, asks them to make a conjecture, and then asks them to submit an explanation for the proposed observance.

Developmental Aspects of Justification

NCTM (2000) has made clear the expectations for pre-K –12 students regarding mathematical justification. The expectations are supported by research that attests to what is possible in students' thinking regarding successful writing and understanding of mathematical proofs. In some recommendations, isolated cases, rather than classroom norms, have served as the basis for NCTM recommendations. However, there is no doubt that very young children possess deductive reasoning skills needed to reason effectively in the mathematics classroom (Lester, 1973; Maher & Martino, 2000; Zack, 1999). NCTM has recommended that as early as third grade, students should be taught the role of a counterexample and be expected to use counterexamples in justification. Beginning in sixth grade, students should use logical chains of reasoning and be able to present informal mathematical proofs. It has also been recommended that high school students understand and be able to produce formal logical deductive mathematical proofs that could be "accepted by the standards of a mathematician" (p. 312). By the completion of high school, it is also recommended that students use indirect proofs and proof by induction.

One of the most important and influential ideas about the developmental aspects of justification leading to geometric proof has been the van Hiele theory of geometric thinking. In the 1950s, Dina van Hiele-Geldof and Pierre van Hiele formed a theory of geometric thinking after becoming acquainted with Piaget's theory of cognitive development and from reflecting on the problems their own students encountered while learning geometry in the Netherlands.

The van Hieles created distinct levels of geometric thinking that were very similar to Piaget's stages. In addition to the existence of levels of thinking, both Piaget and the van Hieles believed in the hierarchical nature of the levels or stages. That is, it would not be possible for students to think at a higher level without having passed through the lower levels first. Hoffer (1983) described the van Hiele levels:

> Level 0: Students recognize figures by their global appearance. They can say triangle, square, cube, and so forth, but they do not explicitly identify properties of figures.
>
> Level 1: Students analyze properties of figures; "rectangles have equal diagonals" and "a rhombus has all sides equal," but they do not explicitly interrelate figures or properties.
>
> Level 2: Students relate figures and their properties; "every square is a rectangle," but they do not organize sequences of statements to justify observations.
>
> Level 3: Students develop sequences of statements to deduce one statement from another, such as showing how the parallel postulate implies that the angle sum of a triangle is equal to 180°. However, they do not recognize the need for rigor nor do they understand relationships between other deductive systems.
>
> Level 4: Students analyze various deductive systems with a high degree of rigor comparable to Hilbert's approach to the foundations of geometry. They understand such properties of a deductive system as consistency, independence, and completeness of postulates.

The two theories also shared the notion of widespread applicability. That is, Piaget believed that once a student was able to think at a particular level, the students could do so in other tasks across various domains. While the van Hiele theory is specific to the domain of geometry, the van Hieles believed in transfer of thinking to other tasks within geometry. For example, if a student is able to understand and present a proof involving congruent triangles, the student should be able to understand and present a proof involving similar triangles. Unfortunately, research on Piagetian and van Hiele tasks has indicated that children do not always demonstrate mastery of levels across tasks (Burger & Shaughnessy, 1986; Mayberry, 1983).

The van Hieles distinguished their theory from Piaget's in two other respects. First, the van Hieles believed that instruction was the key element in moving students to higher levels. Piaget thought that students progressed through stages according to natural biological maturation. Second, the van Hieles believed that each level has its own linguistic symbols and its own syntax. Thus, when the classroom teacher and students are at different levels, they are not able to reason because they are essentially speaking different languages. Piaget underestimated the importance of language in cognitive development. While several research studies conducted in the 1980s on the van Hiele theory indicated some difficulties with the hierarchy and discreteness of the levels, researchers agreed that the van Hiele levels played a role in the development of geometric thinking leading to proof in geometry (Burger & Shaughnessy, 1986; Mayberry, 1983; Senk, 1989).

Aspects of Understanding Mathematical Proof

Throughout the past several decades, educational researchers have attempted to model what it means to understand in mathematics (Davis, 1979; Herscovics & Bergeron, 1983; Skemp, 1971, 1979). Among the various models proposed, understanding has generally been portrayed as a dynamic element in the learning process as demonstrated in Skemp's (1971) definition: "to understand something is to assimilate it into an appropriate schema." Thus, the student's level of understanding is dependent on whatever schema or "conceptual structure" is deemed appropriate during learning. Skemp (1979) introduced three kinds of understanding to account for various schemas: instrumental, relational, and formal. In recent reform documents, all of these levels of understanding have been established as important components for teaching and learning content, process skills, and the nature of mathematics (NCTM, 2000).

Instrumental Understanding

Instrumental understanding refers to the kind of understanding achieved when students focus solely on the procedures used to produce correct answers. This does not necessarily mean that students have learned the procedures through rote memorization. Students may develop specific conceptual links between the procedures and the algorithm used to solve the problem. For example, students may be able to solve problems involving the geometric mean by linking the structure or form of the diagrams to equations used to find a missing side of a particular triangle within diagrams. However, the defining feature of instrumental understanding is that the procedures used to solve these kinds of problems would not necessarily be linked to the geometric concept of similarity.

In terms of instrumental understanding of proof, high school students appear to lack the ability to produce standard textbook geometry proofs, which has been documented in past and current research studies. Senk (1989) reported that only 30% of high school students reached a mastery level of proof writing in geometry. Senk attributed these findings to the students' lack of content knowledge or their van Hiele level of geometric understanding. Recent investigations have shown that writing a geometry proof has not improved over the past ten years. Kahan (1999) investigated high school students' ability to construct proofs of students enrolled in a traditional mathematics classroom and those in the Core Plus Mathematics Program, a reform curriculum designed to foster investigation and real-world modeling. Using the same rating criteria established by Senk, less than 20% of the students in both the traditional and reform classrooms in the study demonstrated proof mastery.

A glimpse in the typical classroom suggested that content knowledge and van Hiele levels are probably not the only factors in proof writing ability. Schoenfeld (1988) raised the issue of form, in particular the two-column proof, when he observed that high school students identified the form of the proof was one of the most important factors in writing a good geometry proof. Healy and Hoyles (2000) found that a higher percentage of high school students in England attempting a narrative proof (42%) were able to construct a complete algebraic proof than students attempting formal proofs (17%).

Relational Understanding

When students focus on establishing connections between the procedures and the concepts involved, then they develop relational understanding. For example, students who possess a relational understanding of the concept of similarity would be able to understand why the concept of similarity allows for certain equalities to be established from a given diagram involving similar triangles. However, students who possess a relational understanding of a concept may not be able to carry out the procedures (instrumental understanding) to produce correct answers. Skemp (1979) suggested it is easy to determine whether students have achieved an instrumental understanding, but may be difficult to determine whether students have actually achieved a relational understanding. The connections the student makes between the procedures and the concepts are often only known to that student and may not be easily articulated by that student.

The domain of validity of a generalization, which means that once a proof has been given, no further checks of specific examples of the conjecture are necessary, is one aspect of relational understanding. Fischbein and Kedem (1982) proposed that high school students generally do not understand the concept of domain of validity of a mathematical proof. The authors hypothesized that students intuitively rely on the production of empirical evidence to establish the validity of everything they encounter in life. The participants in this study were 397 high school students from three schools in Tel Aviv, Israel. The participants were selected from intact classrooms (six tenth grade classes, six eleventh grade classes, and four twelfth grade classes).

The students were randomly given one of two forms of a questionnaire (algebra or geometry), which consisted of a theorem, its proof, and a passage explaining that a fictitious student wanted to check the proof with specific examples. The results indicated that approximately 40% of the students believed the proofs to be correct and valid, but only 10% of the participants assessing the geometry questions and 15% of the participants assessing the algebra questions believed that no further empirical checks were necessary. While the results of this study indicated that these students did not have an adequate understanding of domain of validity, the fact that roughly 40% of the participants gave irrelevant answers or incomplete answers makes these findings somewhat speculative.

Vinner (1983) widened the scope of Fischbein's and Kedem's (1982) study about the domain of validity of a mathematical proof by asking students to choose a preferred proof among the following: a proof by computation, a proof involving a particular case, and a proof involving the general case. The participants for the study were 365 high school students of mixed ability (227 in grade 10 and 138 in grade 11) in Israel. In a questionnaire, Vinner asked students to read a proof of a numerical theorem given by Fischbein and Kedem and then to mark as their preference one of three different arguments on the proof that $59^3 - 59$ was divisible by 6. The first argument involved computation, the second argument showed a deductive-style particular proof using $n = 59$, and the third simply stated that the situation could be proven by deduction given by the proof on the questionnaire (an application of the proof).

Percentages from Group A (those who said they understood the proof) and Group B (those who said they did not fully understand the proof and denoted by parentheses) were calculated. The results indicated that 9% (40%) preferred the proof by computation, 37% (24%) preferred the proof of the particular proof, 46% (25%) preferred the application of the general proof and 7% (11%) did not prefer any of the answers.

Almost 50% of the participants chose the computation for clarity and brevity, while 30% chose the computation because it did not rely on any formulae. A small number of students, 6%, chose the computation because they believed it showed that the formula was correct. Approximately 65% of the participants preferred the proof using the particular case because it was a reconstruction of the general proof. The author paraphrased one student's response, "By this method one can easily solve many similar exercises. It shows how the student got the answer. It contains the method of the proof. It has steps. This was an exercise and in exercises it is desirable to repeat the general procedure" (p. 293). Approximately 75% of the participants who preferred the application of the general proof claimed that the general proof was sufficient.

Porteous (1991) questioned the intuitive perspective taken by Fischbein and Kedem (1982) and Vinner (1983) in determining whether students were personally convinced of mathematical generalization, either by inductive reasoning or deductive logical proof. Porteous stated, "establishing that a person genuinely believes something should simply be that she would behave as if that thing were true" (p. 589). Porteous observed that an overwhelming majority of students, who said they believed a numerical generalization based on inductive reasoning, checked a particular case of the generalization rather than deduced it. More than half of the students who used a deductive argument to establish the validity of the generalization also chose to deduce the particular case of the generalization. A chi-square statistic of 67 indicated that a significantly higher percentage of students used inductive arguments for both the generalization and the particular case than students who used a deductive argument for the generalization and an inductive argument for the particular case. Thus, it appeared that an overwhelming majority of the students using inductive arguments for testing the validity of the generalization did not genuinely believe the generalization was true.

Approximately one-fourth of the students re-enacted a deductive argument using the numbers in the problem, similar to the style provided to students in the study conducted by Vinner (1983). Both Porteous (1991) and Vinner believed that students were using the deductive proof with numbers to check the logical structure of their proof rather than to establish the validity of the particular case. Based on the second part of the study, Porteous concluded that "children genuinely believe something which they have proved, but only provisionally accept something for which the evidence is purely empirical" (p. 597).

More recently, in reform classrooms in England, Healy and Hoyles (2000) reported that approximately 60% of the nearly 2500 high school students realized that once a valid proof had been constructed no further proof was necessary for subsets of cases in that domain of validity. In this study, students were told that a proof had already been given for a generalization about numbers and were asked whether a new one needed to be constructed for a particular case that was given. In addition, Kahan (1999) found that high school students from both traditional and reform-minded classrooms in the U.S. generally understood the domain of validity of a mathematical proof. This finding was evidenced during the interviews with students who attempted to prove the numerical problems involving a pattern. Kahan attributed this finding to the fact that students were given a problem that was familiar to them and one the students had tried to prove for themselves. Fischbein and Kedem (1982) used a problem that was unfamiliar to the students and asked students to evaluate a proof written by the researchers.

Formal Understanding

When students understand the value of proof as a means of justification in a public forum, then they possess a formal understanding of proof. In the mathematics classroom, students use proof to *convince* others and to *explain* their reasoning. Mathematical proof is based upon selected socially agreed upon axioms and theorems about non-empirical mathematical objects. In mathematics classrooms, students may test their conjectures by drawing inferences on a case-by-case basis but will ultimately try to convince others through mathematical proof. Thus, students have achieved a formal understanding when they realize what it means to justify knowledge within the discipline of mathematics. Hanna (2000) has provided a comprehensive list of the varied purposes of mathematical proof that pull together various researchers perspectives on the purposes of proof:

1. verification (concerned with the truth of a statement);
2. explanation (providing insight into why it is true);
3. systematization (the organization of various results into a deductive system of axioms, major concepts and theorems);
4. discovery (the discovery or invention of new results);
5. communication (the transmission of mathematical knowledge);
6. construction of an empirical theory;
7. exploration of the meaning of a definition or the consequences of an assumption;
8. incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective. (p. 9)

Establishing Validity (Truth) in Mathematics. The only way to establish truth of a generalization in mathematics is to present a deductive proof. Inductive arguments are limited because one cannot establish the validity of the generalization for *every* particular case that exists by showing that the generalization is true for just a few cases. A body of research has documented students' desires to rely on inductive arguments to establish the validity (or truth) of mathematical generalization while understanding the power of deductive proof seemed illusive for many students.

Many researchers have demonstrated students' reliance on inductive arguments when asked to engage in tasks involving number patterns or geometry concepts. Bell (1976) found that 40 high school freshman in Australia demonstrated various levels of inductive and deductive arguments to establish the validity of mathematical conjectures. One conjecture given to the students, called *One and the Next*, involved the numerical concept of multiplicity. Students were asked to write down any number less than 15, the next consecutive number, and the sum of these two numbers. Then students were asked to evaluate the validity of a conjecture made by a fictitious student (Gail): One and only one of these three numbers is a number from the list 3, 6, 9, 12, 15, 18, 21, 24, 27, and 30. Approximately 30% of the students presented some level of deductive argument for this problem while approximately 20% presented a full check of cases. Nearly 20% thought the situation stated that the sum of any two consecutive numbers was divisible by three and correctly presented a counterexample.

For another problem, called *Triangles*, students were presented with a diagram of a triangle plotted on isometric paper. Students were directed to identify as many equilateral triangles that could be drawn on the sides of the original triangle. For this problem, nearly one-third of the students misinterpreted the *Triangles* problem. Some thought that equilateral meant isosceles while others failed to draw triangles with vertices on the side of the original triangle. Less than 25% of the students used some level of a deductive argument for this problem.

The results of this study were difficult to interpret due to the fact that so many students failed to understand the problem before producing an argument. Recognizing the problem encountered in that study, Galbraith (1981) employed clinical interviews to investigate high school students' methods of proof. The participants in this study were 170 students from the Brisbane schools ranging in ages from 12 to 17 across all ability levels. The participants worked through each activity during an individual interview with their student teacher to ensure student understanding.

In the first activity, *Game of 25*, the first "player" chooses a number between 1 and 6, inclusive. The second player also chooses a number from the same list and adds it to the number chosen by the first player. The game continues in this manner with each player taking turns. The winner is the player who reaches 25 first. Students were asked to prove that the player who reaches 18 first would always win the game. Then, students were asked whether a player could win by reaching a number other than 18 first. Finally, students were asked whether it was possible for the first player to always win the game.

The results indicated that 50% of the students achieved a proof by exhaustion from the number 18. Approximately 15% of the participants checked some numbers from 18 and then assumed that a player reaching 18 would win. The most common method involved checking for the extreme values only. Students who did not make any progress ignored the rules of the game by either subtracting or exceeding 25. Approximately 20% of the participants recognized the mathematical principle involved and found the other two winning numbers while 30% of the students were able to do the same with prompting while about 10% of the participants made no progress at all. While the results indicated that nearly 15% of students used an inductive argument as proof, more students might have written an inductive proof had proof by exhaustion not been an option.

In another study, Balacheff (1988) presented a conjecture to fourteen pairs of 13-14 year olds in France that could not be proven through exhaustion. The students were asked to explain to their other classmates how many diagonals of a polygon existed when the number of vertices is known. The students were also asked to express their answer in a way that other students their age could understand. Four basic categories of proofs were observed.

Naïve Empiricism. This type of justification was based on observation only as the method of validation. The students were unable to reach an agreement or they so genuinely believed in their conjecture that they believed no further proof was necessary. Two students, Pierre and Philippe, demonstrated the latter by presenting the following conjecture: the number of diagonals in a polygon can be determined by multiplying the number of diagonals from one of the vertices by the number of vertices in the polygon. Interestingly, their accompanying diagram was inconsistent with their conjecture. These students seemed convinced of their conjecture but insisted that they did not know how to explain it. The students obviously failed to recognize that their method did not account for duplicate diagonals.

Crucial Experiment. Some of the students deliberately tested their conjectures with a polygon, usually one with several vertices. Nadine and Elisabeth developed a recursive type solution: a) Starting with the first vertex, the number of diagonals was equal to the number of vertices minus 3, b) The second vertex had the same number of diagonals as the first vertex, c) The third vertex had one less diagonal than the previous vertex, and so on. The students verified their conjecture by applying it to a polygon with 15 vertices.

Generic Experiment. This type of justification was described as a proof that involved "making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class" (Balacheff, 1988, p. 219). Three pairs of students made this type of justification. In the discussion, one student, Georges, explained to his partner that the formula $f(n) = (n-3) + (n-3) + (n-2) + \ldots + 2 + 1$ held for a polygon with six vertices because choosing any vertex, the vertices on either side cannot be used to form diagonals, which verified the first part of the formula. Since the student provided an underlying reason for the part of the conjecture, Georges viewed the six-sided polygon as a generic example rather than a specific case.

Thought Experiment. This type of justification was an extension of the generic experiment since the reasons for the conjecture were given in a more explicit manner. The influence of language and the ability to decontextualize (generalize) the polygons appeared to distinguish this type of justification from the others. The use of variables indicated a language involving a generalization. These students originally wrote an assertion involving a six-sided polygon (generic experiment) and then substituted variables for most of the numbers in the previous assertion, possibly marking a transition from a generic to thought experiment.

Martin and Harel (1989) approached the issue of inductive versus deductive argumentation by asking elementary preservice teachers to evaluate the validity of various types of inductive and deductive arguments to mathematical conjectures. The researchers proposed four typical types of inductive arguments found in the research literature: showing an example using a small number, showing an example using a large number, establishing a pattern from several examples, and showing an example and non-example within the same argument. In this study with elementary preservice teachers enrolled in mathematics course focusing on the concept of mathematical proof, Martin and Harel found that most of the various types of inductive arguments were equally appealing to many of the participants for both familiar and unfamiliar mathematical content. In addition, given three deductive-style proofs, the participants rated the deductive arguments equally convincing. The researchers called this phenomenon "proof frames," whereby inductive and deductive arguments were considered equally acceptable. They attributed this phenomenon to an overlapping of the inductive reasoning the participants had relied on as children and the deductive reasoning the participants had learned as adults.

The deductive-style proofs used in this study were of three types: a correct general deductive proof, an incorrect general deductive proof, and an argument written in the style of the correct deductive proof with the substitution of numbers rather than variables (called the particular proof). In rating these arguments, a majority of the preservice teachers who rated the correct general deductive proof high also rated the incorrect general deductive proof high for a conjecture they had not discussed in class. Thus, Martin & Harel (1989) contended that the participants were exhibiting "ritualistic" behavior. Also, the same group of participants who rated the correct general deductive proof high also rated the particular proof high for conjecture the participants had discussed in class and seemed to have based their opinion of proofs on appearance. Interestingly, this was the first research study in this line of investigation in which it was known that participants had the opportunity to learn about various types of arguments. Similar results were found among high school students in reform classrooms in England. Healy and Hoyles (2000) reported that high school students generally chose arguments they thought would receive the best grade based on its formal appearance (and that they were hard to follow). At the same time, their teachers thought the students would choose arguments for receiving the best grade based on the logical content of the argument.

Empiricism. The use of inductive reasoning in mathematics often leads students to empiricism, the belief that mathematical shapes are real-world objects. Students who are empiricist do not seem to view a rectangle as a generalization of all rectangles but as a real and physical object only. Thus, a student might want to measure the diagonals of a rectangle (or several rectangles) drawn on a piece of paper to establish that the diagonals are congruent rather than deduce it from the properties of an abstraction. Two studies have investigated high school students' use of empirical methods for establishing the validity of a conjecture (Chazen, 1993; Schoenfeld, 1983).

In the study conducted by Schoenfeld (1983), two college freshmen, enrolled in a course on problem solving were asked to solve two geometric construction problems. Analysis of this study was based on the researcher's interpretations of the students' conversations compared to his interpretations of a cognitive paradigm composed of three qualitatively different categories: resources (facts and algorithms), control (decision-making ability), and belief systems (about self and the subject matter). In the study, the students were asked to construct a circle that was tangent to two intersecting lines, passing through a designated point on one of the lines. The students seemed to believe that an accurate hand drawing of the outcome would provide insight into how to make the construction. Using the features of the hand drawn diagram, possible solutions were discussed and ranked. The students tested the possible solutions and then resorted to purely empirical methods of verification. The students did not seem to view mathematical proof as a necessary activity for personal verification or discovery of new mathematics. The students recognized several facts about circles, such as the diameter and radius, but lack decision-making ability to use them. The students also approached the problem empirically, demonstrating their belief that mathematical objects are concrete. Schoenfeld (1983) also believed that the students were not aware of their empirical views of mathematics and that the students' lack of awareness prevented them from solving the problem. A mathematician (serving as the expert in a novice-expert paradigm) was also observed solving a similar construction problem and was characterized as having: a) better control behavior, b) more reliable recall of relevant facts, and c) more confidence. The mathematician also approached construction problems from the perspective of proof rather than empiricism.

The other study, conducted by Chazen (1993), examined high school students' perceptions of mathematical proof in a classroom that used computer applications to gather empirical evidence prior to deductive proof. Students from five geometry classes from one urban school and one suburban school were selected for this study. All of the classes possessed three basic characteristics. First, students in these classes were encouraged to explore geometric concepts using Geometric Supposer. Second, the topic of formal deductive logical proof was emphasized in the classroom. Third, the teachers in these classes were willing to use a teaching unit written by the author that was intended to be explicit about mathematical argumentation. Students in these classes completed a questionnaire on various arguments developed for the study.

The instructional unit on argumentation was delivered after the teachers presented their usual introduction to logical proof. The focus of the instruction unit on argumentation was to differentiate between arguments based on measurement and arguments based on deductive logic and to ensure students were familiar with the topic of inductive and deductive argumentation. The unit was intended to highlight the problems with arguments based on measurement and the benefits of deductive proof. Specifically, the limitations of accuracy and precision were targeted. For example, the measurement of an example might erroneously support a false conjecture or contradict a true conjecture. The problem of missing a counterexample was also made explicit. Another goal of the course was to show that deductive proofs were beneficial because of their power to explain the mathematical situation.

Based on diversity of responses on the questionnaire, 17 students were chosen for interviews. The interview consisted of two parts. The first part focused on comparing two arguments (deductive and inductive). The second part of the interview focused on the aspects of the deductive proof. The deductive proof chosen for the interviews had been discussed in class prior to the study. The inductive measurement argument dealt with the following conjecture: In any triangle, a line connecting the midpoints of its sides is parallel to the third side. Four diagrams of various size triangles illustrating the conjecture were presented along with an inductive argument describing how each triangle had been measured to ensure the conjecture was true. This inductive measurement argument was chosen because most students indicated on the questionnaire that this argument was a proof. After rapport had been established, students were shown the deductive proof and asked to explain why they thought these kinds of proof were done in class. The question seemed too abstract for many so students were asked what other ways there were to prove things. Students were shown the inductive measurement argument and asked questions similar to the ones posed for the deductive argument.

Two students from the urban school and three students from the suburban school firmly believed the inductive argument. Roughly an equal number of students were unsure about measuring as a method of proof, with three students finally ending up committing to this view. Students believed that every different kind of triangle had to be measured before believing that the conjecture worked for all triangles. This belief included "special" cases of certain kinds of triangles. For example, if the conjecture involved all kinds of obtuse triangles, some students said they would want to measure obtuse isosceles triangles as well as obtuse scalene triangles. Students were also aware that they held contradictory beliefs. For example, one student discussed her struggle believing that measuring each kind of triangle ensures proof even though the possibility exists of missing a counterexample for one type of triangle.

Some students believed that measurement argumentation did not constitute mathematical proof. Students said that measurement arguments might ignore possible counterexamples and that measurements are not always exact. Students also viewed deductive proof as a type of "evidence." These students believed that deductive proofs did not guard against the existence of counterexamples. Students also believed that the deductive proof was meant to show the conjecture was true for the accompanying diagram only. One student thought that the way the assumptions were worded in the proof indicated to him that the deductive proof was designated for a single case.

Another student had a slightly different problem in understanding what was "given." He stated in the interview that the proof was not true for all cases, but claimed that it would be true under other circumstances. However, these "other" circumstances were a paraphrasing of what was originally given in the proof. When a different student was asked to present a counterexample for the deductive proof, the student realized that one could not be drawn. However, he insisted he would always check any deductive proof because he is a natural doubter. The author suggested that other participants who also checked deductive proofs for counterexamples might have done so because they were taught to look for counterexamples within the inductive arguments.

Students also expressed difficulty understanding the general need for proofs. One participant remarked, "I can understand everything you say equal or this is not equal, but I find no reason to do [a deductive proof]" (Chazen, 1993, p. 382). Two students from the urban school and five students from the suburban school held adequate views of deductive proofs.

Two additional factors seemed to influence the students' preferences for using the two types of arguments. First, students said inductive measurement arguments were preferable because these arguments were concrete. Other students said they preferred deductive proofs because they help explain the mathematical situation. Chazen (1993) concluded that several students were able to see the limitations of the measurement arguments that could be done through Geometric Supposer.

Explanation. Hersh (1993) has argued that the main role of proof in schools is to explain and provide insight as students are so readily convinced by the conjectures that are presented to them. While one may gain conviction in the quasi-empirical methods used to establish validity of conjectures, say through dynamic geometry software, students should realize these methods offer little to "the understanding into how the conjecture is the consequence of other familiar results" (de Villiers, 1999, p. 5). Healy and Hoyles (2000) investigated students' understanding of the explanatory power of proof. They provided various arguments (empirical, generic example, deductive – correct and incorrect) to high school students and asked to rate the explanatory power of the arguments. The results indicated that only 25% of high school students realized that empirical (inductive) arguments possess no explanatory power. Interviews with the students indicated that students referring to the purposes of a mathematical proof were: 50% (verification), 35% (explanation), 1% (discovery or systematization), and 28% (no response or demonstrated no understanding of mathematical proof). The researchers were disappointed that students generally did not view discovery or systematization as a purpose of mathematical proofs and attributed this observation to the fact that proof is taught separate from content, such as geometry.

Systematization and Discovery. The utilization of definitions, axioms, and theorems is an important aspect of mathematical proof. Once a theorem has been proven with a mathematical system, such as Euclidean geometry, it can be used to discover other relationships within the system. This integration of the system should show students that mathematical concepts are connected to each other through logical reasoning. The inability to recognize the value of a previously proven theorem in discovery new mathematics was observed by Schoenfeld (1989). After proving two theorems regarding tangent lines to a circle, students failed to use the theorems in proving other conjectures involving the concepts. In addition, the students developed new conjectures that violated the previously proven theorems.

The van Hiele theory of geometric thinking included systematization as the highest level of thinking. Unfortunately, systematization was de-emphasize within the theory because it was viewed that students rarely reached this level (van Hiele, 1986).

Researchers assessed systematization in blunt way, testing to see if students could use previously defined mathematical terms defined differently in other systems. For example, students were given an alternative definition for the term "parallel" for a completely different geometric system and asked to apply the new definition to various problems (Burger & Shaughnessy, 1986; Mayberry, 1983; Senk, 1985, 1989; Usiskin, 1982). The participants in these studies, high school students and preservice teachers, experienced great difficulty in achieving systematization tasks. A more descriptive view of systematization in geometry, provided by de Villiers (1999), currently incorporates several intermediary steps that should be included in further investigations and in instructional teaching methods.

Conclusions

The purpose of the proposed study is to examine high school students' views of the purposes of geometric proof in the context of learning geometry proof. In order to approach this problem with insight, an examination regarding the prior studies, the theoretical perspectives taken, and the research methods used was necessary. As a result of this review, several issues regarding mathematical proof in schools have emerged from the literature presented by scholars in this field.

Mathematical proof is more than a product from logical deductions. Mathematical proof is a convincing argument that is used to communicate the validity of mathematical conjectures in a public way. Mathematical proof is also embedded in an axiomatic system (Bell, 1976; de Villiers, 1999; Douek, 2000; Hanna & Jenke, 1993), and students should understand that systematization is essential for further discovery of mathematics. In high school mathematics, students in the United States are exposed to systematization through traditional Euclidean geometry. Systematization is rarely addressed in algebra and is typically reserved for advanced high school mathematics, such as a pre-calculus course or college mathematics.

Skemp's (1979) conceptualization of understanding was utilized in order to gain one perspective of students' understanding of mathematical proof. Skemp's framework included three kinds of understanding (instrumental, relational, and formal) that can be obtained independently from each other. In terms of instrumental understanding, content knowledge, van Hiele levels, and form appear to be relevant factors determining a students' ability to produce a mathematical proof. Senk's (1989) findings indicated a general lack of ability in high school geometry students to be able to produce a deductive proof. Proof form, such as narrative versus algebraic, has also appeared to be a relevant factor in proof writing ability (Healy & Hoyles, 2000).

Several studies have been conducted to investigate students' relational understanding of mathematical proof. Students have been found to question the validity of a conjecture even after its proof has been given and understood. Fischbein (1982) attributed this phenomenon to students' intuitive responses. He contended that it was a natural response for the students to check a particular example of a generalization, rather than apply the proven generalization to the particular example. On the other hand, Vinner (1983) believed that students were reconstructing a proof of a generalization in order to check the logic used in the original proof. Regardless of the reason, approximately one-fourth of the students in both studies demonstrated this behavior.

Studies investigating students' formal understanding of mathematics have generally indicated that high school students and preservice elementary teachers are more readily convinced by inductive arguments and empiricism (Balacheff, 1988; Bell, 1976; Chazen, 1993; Galbraith, 1981; Healy & Hoyles, 2000; Martin & Harel, 1989; Schoenfeld, 1983). Whether students in the study had an opportunity to learn about the limitations of inductive arguments is an issue of concern among these studies. While some researchers contended that the students had been instructed on the limitations of inductive arguments (Healy & Hoyles, 2000; Martin & Harel, 1989), more details of the instruction are needed to completely assess the findings.

Examining students' formal understanding in context is an important step toward understanding how or whether students come to understand mathematical proof. The proposed study will investigate a group of high school students' views of the purposes of proof in the context of learning geometry proof. Instruction, curriculum and assessment have all been identified as important aspects in the on-going development of students' understanding of mathematical concepts. In the next chapter, the design and methods for the proposed study will explicate how a single classroom was used to investigate high school students' views of the purposes of geometric proof.

CHAPTER III

METHODOLOGY

The design and method for this study were selected to examine the formal understanding of geometric proof of high school students from a single classroom. The primary research question for this study was: What are the views of tenth graders regarding the purposes of geometric proof in the context of their classroom experiences as they learn geometry proof? The purposes of proof, as defined by Hanna (2000), have been used in the previous chapter to represent specific aspects of Skemp's (1979) conceptualization of a formal understanding of proof.

This chapter provides the details of the design and methods used in this study. This study used an integration of a naturalistic paradigm and cognitive psychology to investigate students' understanding of the purposes of geometric proof. This approach was unique to the problem since prior studies focusing on high school students' views of the purposes of proof have not yet provided details of their classroom experiences including instruction on proof, curriculum materials, and assessment. A description of the process used in locating the classroom for the study details the criteria used and the constraints imposed by the researcher, which documents some of the difficulties in locating a classroom. A short narrative of the community, school, and classroom is included to provide additional context regarding the setting in which the study took place as well as a description of how entry to the fieldwork was gained. Data collection methods, such as participant observation, individual interviews, and documents are also described as well as the data analysis used. Finally, the role of the researcher is described.

Design of the Study

The design chosen for this study was an integration of a naturalistic paradigm into an investigation of the students' cognitive views of the purposes of geometric proof. In a naturalistic paradigm, a holistic view is taken regarding the participants' (and researcher's) constructions within the context of the learning situation (Lincoln & Guba, 1985). Moschkovich and Brenner (2000) recommended that an integration of a naturalistic and cognitive study in a spiraling design leads the researcher to examine the meanings that participants in the study give to socially constructed knowledge. An integrative approach to investigating students' cognition provided multiple data sources for understanding the students' views. This spiraling design also enabled the researcher to pull together the data sources throughout the study rather than at the end (Spradley, 1980).

The study was conducted over a period of approximately four months. The length of time for data collection may seem short in terms of typical ethnographic studies conducted by anthropologists. However, researchers experienced in ethnographic studies generally maintain that the time in the field should be long enough for the investigator to see a completed cycle (LeCompte & Preissle, 1993; Strauss & Corbin, 1990; Wolcott, 1995). In this study, one cycle was defined by all of the classroom activities involving two units on proofs involving geometry occurring over four weeks of instruction. Specific classroom observations began when the teacher first introduced the concept of coordinate geometry proof and continued throughout the unit on two-column geometric proofs. Classroom observations ended once the teacher began to teach the following unit on linear regression.

Only one classroom was selected since a considerable amount of time was devoted to observing the classroom, establishing rapport with the participants, and collecting documents. In this study, multiple sites would only reduce the total attention that could be given to any one classroom (Krathwohl, 1998; Wolcott, 1995). As a case study, comparability and translatability were important aspects (LeCompte & Preissle, 1993). Descriptions of the community, school, and classroom are provided for comparability. Details of the selection of the classroom as well as explicit details about the design and method used in this study are provided to carry out comparable studies across various groups.

Selection of the Classroom

The process for selecting the classroom began six months prior to data collection. The selection process involved three criteria necessary for the research project as well as two constraints invoked by the researcher. Since the focus of the study was mainly driven by the content of geometric proof, the foremost criterion was to locate a group of tenth grade students who would be learning geometric proof in the upcoming school year. In attending to this first criterion, several classroom teachers were contacted by a network of professionals, including co-workers and school administrators (LeCompte & Preissle, 1993).

With much surprise, some of the mathematics teachers in area schools were not going to be teaching geometric proof. One teacher who was contacted explained that even though geometric proof is a topic on the state assessment, she was not going to teach proof because students could still get a passing grade on the state assessment even if they were to skip questions involving geometric proof. In addition, she contended that this decision was made by the entire mathematics department at this particular school site. From a different school site, another teacher explained that he did not believe in teaching geometric proofs because he believed most students never have to produce proofs in their future careers.

A second criterion was to locate an experienced classroom teacher who was willing to participate in the study. An experienced classroom teacher, as suggested by Berliner (1987), is one who feels comfortable with the school subject matter, the students, and the school district. According to Berliner, experienced teachers have typically taught for at least five years in the setting and usually demonstrate consistent instructional strategies.

A third criterion was to locate a classroom teacher who intended to teach at least some of the purposes of proofs as well as the procedures of writing a proof. It was not necessary to select a teacher whose views of the purposes of proof were consistent with the research literature nor was it necessary to select a teacher who was explicitly or implicitly going to instruct all of the purposes of proof described in the research literature. However, it was necessary to select a classroom teacher whose instructional goals included a formal understanding of geometric proof as well as instrumental and relational understandings.

The two constraints made by the researcher in selecting a location were the geographic location of the project site and class meeting times. Since the researcher was teaching full-time at a two-year college, the location of the classroom had to be in close proximity to the researcher's workplace. In this rural area, seven high schools were within a 30-minute drive from the researcher's workplace. In addition, class times at the college and at the project site could not conflict on any day of the week during the time of the study.

The selection process began by identifying a list of experienced teachers within the specified geographic location. Three teachers from the largest school district were initially contacted, but none of their classrooms met the criteria. In a second school district, the teacher expressed no interest in participating. Mrs. Kelly, the classroom teacher selected for the study, was the fifth teacher who was contacted. Mrs. Kelly worked briefly with the researcher several years prior to the study as an adjunct instructor, yet quickly remembered the researcher upon the initial contact. Upon learning that she would be teaching geometric proof to two groups of students in the upcoming school year, a face-to-face interview was arranged.

In the interview, the purpose of the study was described as an investigation about her students' understanding of geometric proofs. It was stressed that the students were the focus of the study rather than the classroom teacher's ability to instruct the students, yet classroom observations would need to be tape recorded for context. Mrs. Kelly seemed excited about the possibility of learning more about why students struggle with geometry proofs. Since many of her comments during the beginning part of the interview focused on the instrumental understanding of geometric proof, a clarification of the purpose of the study was made and included asking students what they thought doing a geometry proof accomplishes. Mrs. Kelly described the kinds of proof problems her students would be doing and that she intended to also teach students the reasons people write geometry proofs. She did not reveal her views of the purposes of proof at that time, nor were her views probed by the researcher for fear of introducing bias. In addition, Mrs. Kelly noted that she was not intending to use dynamic geometry software, even though it was available at the school site, due to the time restrictions of the course. However, she felt there would be plenty of opportunities for observations of the students working in small groups.

During the interview, Mrs. Kelly mentioned her desire to change the way she normally taught geometry proofs. She expressed her dissatisfaction with the results she typically sees year after year. She thought that if students could generate a list of theorems and properties that were related to the problem first and then cross off anything that was used in the proof, they might experience more success. Mrs. Kelly also indicated that teaching geometry proofs was not her favorite teaching activity.

While Mrs. Kelly appeared to be willing to participate, it was important to establish whether she would have adequate time to be involved with such a project.

Suspecting that she might have been a cooperating teacher, it was made clear to Mrs. Kelly that she could not mentor any student teachers during the study. It was also necessary to confirm that class time would need to be made available for two questionnaires and several journal questions, and that interviews could be arranged during the students' free time. At that time, Mrs. Kelly assured the researcher that all requests could be accommodated. The researcher outlined the potential benefits in terms of professional growth as well as the potential for informing state curriculum developers. At the end of the interview, Mrs. Kelly expressed a genuine desire to assist in the study and agreed to participate.

The Classroom Teacher and Her Students

The classroom consisted of 19 students and their teacher; 15 were participants and 4 were non-participants. The 15 participants in the study were 5 males and 10 females. Twelve of the participants were sophomores. One participant was a freshman (Kim) and two were juniors (Cathy and Eric). All of the participants had just passed a state assessment called the Math A Regents in January 2003. Thus, students in the study had been exposed to basic algebra, basic geometry concepts, right triangle trigonometry, probability and statistics, and elementary logic (such as basic truth tables). Six students (participants and non-participants) in the classroom had failed the Math A Regents at least one time. In addition, one of the non-participants was a special needs student who left frequently to receive individualized instruction. Only one student participant in the class was academically advanced in mathematics. Another notable feature of the students in this class was absenteeism. On any given day, two or three students would be absent. Typically, students who were absent would be absent for more than one day. Most of the absenteeism was attributed to the flu and cold season; however, on occasion, students were excused for other reasons such as band lessons.

Mrs. Kelly had taught high school mathematics for eight years at the same public school. She received her B.A. in Mathematics from a four-year public college during the 1989 and was certified to teach high school mathematics at the time of her college graduation. Her coursework in mathematics included a three-semester sequence in calculus, linear algebra, group theory, and analysis. Mrs. Kelly described her coursework as theoretical rather than applied and that mathematical proof received constant attention during instruction and assessment throughout each course.

Mrs. Kelly received her M.S. in Mathematics Education in 1993 from the same college. Her coursework for her master's degree included pedagogy, curriculum, and assessment. Mrs. Kelly had been a cooperating teacher mentoring several student teachers throughout her teaching career. Her knowledge of pedagogy seemed current, yet she had not directly read any of the educational research literature on mathematics education.

A Description of the Community, School, and Classroom

The Community

The small, rural community in upstate New York, where the study was conducted, was located approximately 150 miles from a major metropolitan city. Approximately one-third of the 300 families living in this community were Amish. While no Amish children attended the public school, an Amish presence in the community was notable. After remarking to Mrs. Kelly one day about seeing several carriages on the road to the school, she explained that the Amish community had grown tremendously over the past few years.

The School and Classroom

The school district consisted of approximately 700 pre-K – 12 students and 50 faculty and staff, which was a mid-sized district compared to other districts in the county. The school district received lunch aid for approximately one-third of the students in the building. The hallways were clean and free of graffiti, as were the individual classrooms. All of the students attended classes in this centralized building. Because of the centralized building, the three teachers who taught mathematics above the sixth grade level were all required to teach both middle and high school mathematics courses throughout the day. Mrs. Kelly's classroom was located on the second floor. The individual classroom desks were arranged in groups of three scattered throughout the room so that the teacher was able to move from group to group with relative ease (see Figure 1, where NP identifies non-participants).

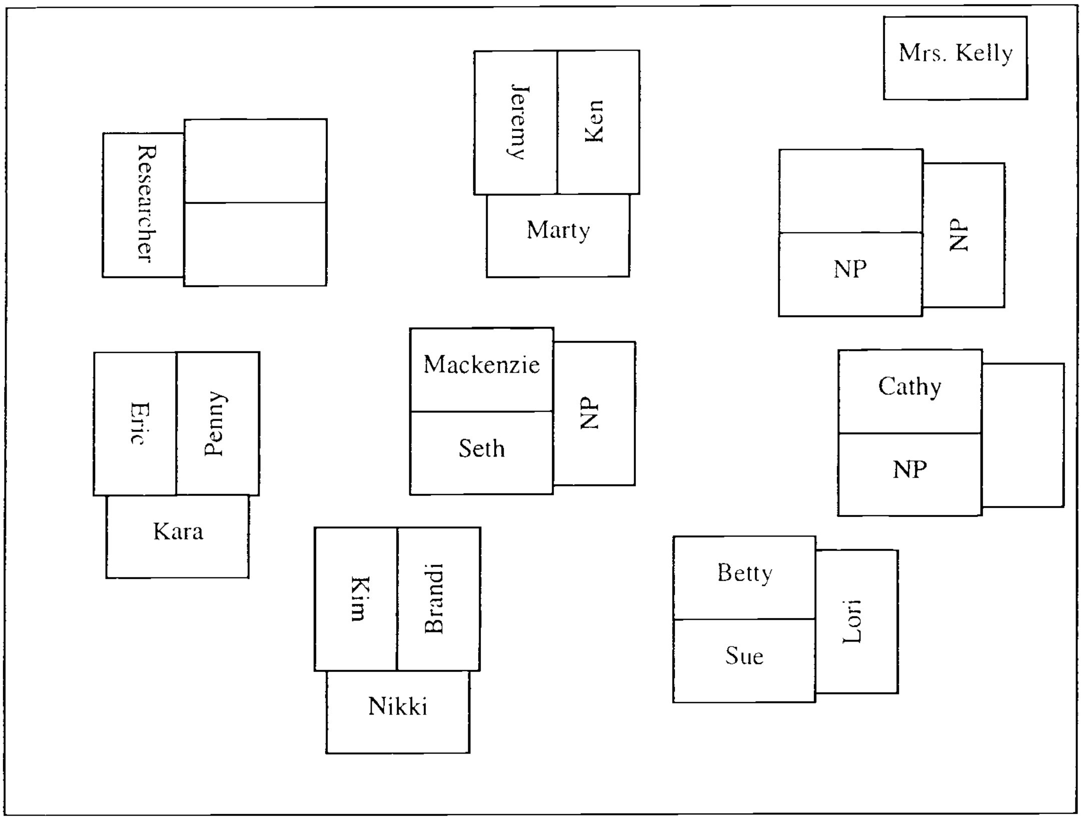


Figure 1. Physical arrangement of groups of students in the classroom.

Mrs. Kelly's desk was located in front right of the classroom near the chalkboard. Several encouraging posters were hung on the walls along with a calculator caddy that was sometimes filled with TI-82 graphing calculators. There was one laptop computer and one desktop computer located by the teacher's desk and both computers seemed to be reserved for the classroom teacher. An overhead and TI-82 overhead display device resided on a cart near the front of the classroom. Rulers, protractors, and compasses were located on the countertop in the back of the room near the windows. Manipulative objects and construction paper were also stored in the back of the room. One small traditional clock hung near the doorway consistently marked the end of the 40- minute class period. The class met everyday of the week, with the exception of typical vacation days in February and April, an in-house professional development day, and a parent-teacher conference day.

Gaining Entry: The First Day of Fieldwork

Soon after the research proposal was approved, a meeting involving the high school principal, the classroom teacher, and the researcher was arranged to obtain official permission to conduct this study. The principal was given a packet including: a brief description of the study, the script to be used for soliciting the students to volunteer, the letter of introduction, an information sheet describing the study, and the consent form. Having previously discussed the study with the principal, Mrs. Kelly was confident that the principal would be receptive. The principal was extremely pleasant and cooperative. His only concern was that a formal letter be written to him explaining the length of time that the researcher would be physically present at the school site. At the end of the meeting, he mentioned that the study would be a good way to maintain the relationship between his school and the local two-year college where the researcher was employed, which had already been established through a distance learning partnership.

In mid-January, students who were considered potential participants (any student taking the Math A Regents in late January) were verbally solicited during their regular class meeting times. None of these students had any questions at that time. Students were asked to discuss the information with their parent or guardian and return the consent form to the classroom teacher within one week. In the last week of January, students took the Math A Regents were reassigned to new classes. The second period class consisted of mostly tenth graders while the third period class consisted of mostly advanced ninth graders. A sixth period class consisted of students who did not pass the Math A Regents. Unfortunately, entry into a classroom was not secured at this time because some students had not returned their consent form. The researcher contacted Mrs. Kelly during the first week of February to inquire about the status of returned consent forms. Knowing that Mrs. Kelly was under stress in getting these three new courses underway, the researcher only encouraged her to keep reminding the students about the study. While waiting for the return of the consent forms, Mrs. Kelly expressed her concern for the selection of the second period class for the study, which consisted of mainly regular tenth graders. Mrs. Kelly thought that the advanced ninth graders would make better participants for the study. At first, the researcher thought her concerns were based on classroom management issues. It was not until Mrs. Kelly read the students' responses of the first journal question when Mrs. Kelly expressed her fears that the regular tenth graders would not say anything interesting and that the study would be a failure.

After another week had passed, Mrs. Kelly was contacted a second time. This time, she seemed rather indifferent, mainly because she thought only a few days was needed for the researcher to establish rapport with the students prior to the start of the first unit. Having realized that no official starting date had actually been established, a date was chosen, and approximately one week later, all of the consent forms had been signed and returned for the second period class. Participants were asked to choose a pseudonym in order to maintain anonymity during data collection (Spradley, 1980). Mrs. Kelly was also asked to sign a consent form, which delineated the researcher's expectations of the teacher's participation.

Method

The method for this study was selected to examine the views of tenth grade students from a single classroom regarding the purposes of geometric proof. The students' views were contextualized through classroom observations, conducted in the style of participant-as-observer (Spradley, 1980). The classroom observations were mainly used to develop instruments for data collection such as journal questions and a post-instruction questionnaire. In addition, the classroom observations were used to capture the subtleties and nuances of the meanings expressed by the classroom teacher and students. The classroom observations were also used to collect data on the students' views through direct observations as they worked in small groups. Participant observation was an important part of the method for collecting data since prior studies only provided, at best, general descriptions of the students' learning environment.

Data Sources

The study utilized multiple data sources to inform the research question: field notes and transcripts of classroom observations from participant observation, the Preconceptions Questionnaire (Appendix A), the Post-instruction Questionnaire (Appendix B), journal questions (Appendix C), informal and formal interviews with the students, Document Summary Forms (Appendix D) for curriculum materials, homework assignments, and assessment tools, and a researcher journal. The primary data sources for investigating the students' views of the purposes of proof were the transcripts from classroom observations as students worked in groups, Post-instruction Questionnaire, journal questions and interviews. Secondary sources included the Preconception Questionnaire, documents, and the researcher journal, which served as sources for the context of the students' views of the purposes of geometric proof. Field notes and transcripts from classroom observations also served as a secondary data source.

Participant Observation

Participant observation was conducted to examine the nature of the classroom instruction as well as the emerging views of the students. As Spradley (1980) suggested these observations generally begin with observing the classroom, participants, and activities through a wide-angle lens, followed by more focused and narrow observations. The classroom observations were guided by the following questions:

1. What purposes of proof are discussed in the classroom?

2. How are discussions about the purposes of geometric proof initiated in the classroom?

3. What is the nature of the tasks involving the purposes of geometric proof? Was exploration involved?

During the study, the researcher assumed the role of "participant-as-observer," indicating that the researcher mainly observed the classroom events. The researcher attempted to take a "one down" position in the classroom by acting as a subordinate rather than a controller in the study in the hopes of learning from the participants rather than testing the participants' knowledge (Agar, 1980). However, even though the "one down" approach was taken, students knew that the researcher was a professor at a nearby college and seemed to treat the researcher as an authority figure. Contact with the students was made during class in order to establish rapport with the students and to gain an understanding of their views while they learned to write geometric proofs. The researcher assisted the students with writing proofs during times when the students were assigned seatwork. The assistance was mainly in the form of asking questions, rather than direct instruction. This type of assistance was used to reduce the possibility of a treatment effect by the researcher. The researcher also made an effort to assist all of the students participating in the study on a rotating basis. Students showed trust in the researcher quickly by directly asking for help on the second day of fieldwork. However, the students rarely called the researcher by name, even near the end of the study, which demonstrated a lower level of rapport than hoped.

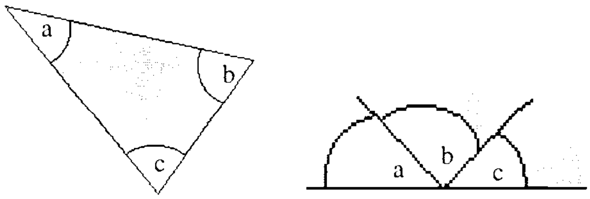
Classroom observations were conducted on a daily basis. All 15 classroom sessions during the units were audio taped and transcribed on a daily basis (LeCompte & Preissle. 1993). Condensed field notes were written during and after each classroom observation and included anything written on the overhead by the classroom teacher. The condensed field notes were expanded and included during the audio tape transcription process. Since classroom events were audio taped, there was little concern for an "amalgamated" language that might have emerged from combining the researcher's and participant's expressions (Spradley, 1980). Two groups of students agreed to be audio taped during seatwork. These audio tapes were also transcribed and used to provide context and data sources for the students' views on the purposes of geometric proof.

Preconception Questionnaire

An open-ended questionnaire was given to all of the participants at the beginning of the study by the researcher and the classroom teacher during regular class time. At the same time participants were responding to the questionnaire, non-participants were given an activity unrelated to the material currently being studied. Neither the activity nor the Preconceptions Questionnaire was graded by the classroom teacher. The purpose of the questionnaire was to determine the students' preconceptions about various topics related to mathematical justification, such as making conjectures, empiricism, use of deductive logic, and the role of a counterexample. Not knowing whether students had actually been taught any of these concepts was not important since research studies have indicated that students typically form conceptions of mathematical concepts prior to formal instruction from everyday experiences or other classroom experiences (Nunes, 1994). The items were developed prior to entry into the field. Items were validated by two professors of mathematics education and one professor of mathematics. Modifications to the original questionnaire were made in the form of wording for clearer understanding of the problem. The Preconceptions Questionnaire was given during the first few days of fieldwork prior to instruction on coordinate geometry proofs.

Item 1 (The Cardboard Triangle Activity Problem). The purpose of Item 1 was to explore participants' understanding of proof as verification versus proof as explanation (see Figure 2). While this common classroom activity is often used to convince students that the sum of the angles of a triangle is 180, it does not provide explanation.

Suppose your math teacher held up a huge triangle made from cardboard. Then, she tore off the angles and put them together in a straight line.



Then she said that there are 180° in a straight line and when you add the interior angles of any triangle your answer is always 180°.Do you think your teacher proved that the sum of the interior angles of any triangle is 180°? Why or why not?

Figure 2. Item 1 (Preconceptions Questionnaire).

Item 2 (The Isosceles Triangle Problem). The purpose of Item 2 was to explore students' use of deductive logic on a particular example (see Figure 3). This question originally appeared on a past Third International Mathematics and Science Study [TIMSS] assessment and in the study conducted by Kahan (1999). In this item, Triangle ABC is an isosceles triangle and all of the angles can be found using various properties of angles.

In the triangle ABC shown below, the altitudes BN and CM intersect at point S. The measure of ∠MSB is 40 degrees and the measure of ∠SBC is 20 degrees. Is Triangle ABC an isosceles triangle? Explain reasoning.

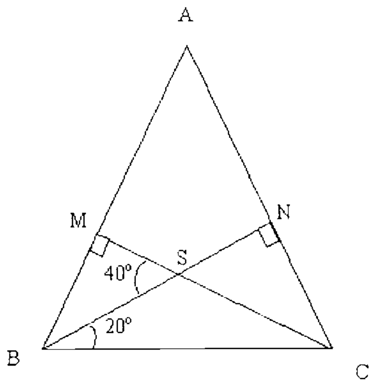
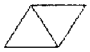
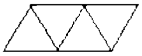


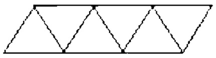
Figure 3. Item 2 (Preconceptions Questionnaire). Reprinted with permission from the Third International Mathematics and Science Study and the International Association for the Evaluation of Educational Achievement.

Item 3 [The Toothpick Problem]. The purpose of Item 3 was to explore participants' methods of verification (see Figure 4). The item, a modification of a problem presented on a sample state assessment for Grade 8, was also used to examine whether students would notice a pattern and construct a formula to find the desired answer. Thus, participants could also make a conjecture (develop a formula) for finding the number of toothpicks for the nth design. The number of toothpicks needed for the 40th design was 161.

Tina is making designs with toothpicks.

She used five toothpicks to make the first design: 

She used nine toothpicks to make the second design: 

She used thirteen toothpicks to make the third design: 

How many toothpicks does she need to make the 40^{th} design? Explain how you arrived at your answer.

Figure 4. Item 3 (Preconceptions Questionnaire).

Item 4 (The Prime Number Problem). The purpose of Item 4 was to examine students' understanding of the role of a counterexample in mathematics (see Figure 5). Participants who agreed with Amy understand that a conjecture can be refuted by a single example that supports the hypothesis of the conjecture, but not its conclusion.

In a math class, the teacher asked the students to discuss a conjecture about a formula that always produces prime numbers. The teacher reminded the students that prime numbers are numbers whose only factors are 1 and itself like

1x 2 = 2

1 x 7 = 7

1 x 11 = 11

1 x 13 = 13

1 x 17 = 17

The conjecture was: The formula $n^2 - n + 17$ always produces a prime number for all positive whole numbers n.

While the students were examining the formula, Amy said that the formula was false because when n = 17, the number is $17^2 - 17 + 17 = 289$ and that 289 was not a prime number because it can be factored as 17 x 17.

John said that the formula was true because the formula works for almost all numbers and because there are always "exceptions to the rule."

Do you agree with Amy or John? Why?

Figure 5. Item 4 (Preconceptions Questionnaire).

Item 5 (The Even/Odd Problem). The purpose of Item 5 was to explore whether participants valued an argument classified as *crucial experimentation*, as described by Balacheff (1988). In the argument presented in Figure 6, two fictitious students used crucial experimentation by adding two large odd numbers to verify that the sum of any two odd numbers is an even number.

> In a math class, the teacher promised ten extra bonus points on the next test if anyone could show him a convincing argument that the sum of any two odd numbers is an even number. Students were allowed to work on the problem together before submitting their answer.
>
> Michelle and Jim worked on the problem together and this is what they said:
>
> Michelle: " This is easy. 1 + 1 = 2, 1 + 3 = 4, 1 + 5 = 6. You can see a pattern."
>
> Jim: "Right, but I think we should pick some really big numbers, just to make sure."
>
> Michelle: " Okay, 245 + 347 = 592, 1923 + 2237 = 4160."
>
> Do you think their teacher should give them the bonus points? Explain why or why not?

Figure 6. Item 5 (Preconceptions Questionnaire).

Journal Questions

Throughout the unit on coordinate geometry proofs and two-column deductive geometric proofs, participants were given journal questions written by the researcher and validated by the classroom teacher. Non-participants were not asked to respond to journal questions. A total of five journal questions were developed and based on the coded transcriptions of the classroom observations made during the previous week.

All of the journal questions were given at the beginning of the class. Participants typically completed journal questions in approximately five minutes. Students were allowed to use their notebooks but not their textbook for all of the journal questions.

Mrs. Kelly was asked to comment on the participants' responses. However, her initial attempts focused on writing comments to the researcher rather than the students. When it was explained to her that it was desirable to see her responses to the students, Mrs. Kelly said that she felt uneasy about doing so. She explained that she had given journal questions before to students, but that they were much shorter and required little feedback. As a result, Mrs. Kelly only read the responses to the journal questions, but did not continue to write comments.

Post-instruction Questionnaire

A second open-ended questionnaire, called the Post-instruction Questionnaire, was given to the participants at the end of the initial instruction on geometric proof by the researcher, approximately five weeks after entry into the field (see Appendix B). The questionnaire was developed from the analyses of the classroom observations, the curriculum materials, and assessment tools used over the course of the study. Using the various data sources, three of the items for the questionnaire were written in the context of the students' learning experiences. Two mathematics educators and one mathematician provided feedback on the items prior to the administration of the questionnaire and suggested changes in the wording of some of the original Post-instruction Questionnaire items. Participants were given one full class period to complete the questionnaire, yet most participants took approximately 20 minutes to complete the questionnaire. Participants completed the questionnaire on their own with the exception of Item 3. One participant raised his hand and asked for a definition of intuition. Two other participants nearby looked up and whispered that they did not know what the word meant either. The participants were asked what they thought it meant, but none had any idea. So, three of the participants were told that one meaning might be that of a "feeling" or "hunch." One of the students, who called himself Jeremy (McNeil), responded, "Like, when I knew Syracuse was going to win it all."

The questionnaire was essential in building a framework of the students' views from their own perspective. As seen in prior research (Williams, 1979), students have multiple meanings for terms such as verify, convince, and proof. The questionnaire also provided an opportunity for the students to express their views about the purposes of geometric proof that may not have been verbally expressed during classroom observations.

Interviews

This study utilized two types of interviews: informal and formal. Agar (1980) suggested that researchers use a range of techniques in informal interviews such as nodding to indicate interest, repeating what the participant said, and probing. Students were informally interviewed before class started or during times in the classroom when the researcher was assisting the students during seatwork. All of the participants were formally interviewed. The length of the interviews ranged from 10 to 30 minutes. Formal interviews were audio recorded and later transcribed. The interviews were conducted at the school site during regular school hours in a room located away from heavy school traffic. The classroom teacher did not participate in the interviews so that the teacher's presence could not influence the participants' views. Notes from the interviews were recorded in the researcher journal and examined prior to the next individual interview.

A general interview protocol was written for similar responses on the Preconceptions and Post-instruction Questionnaires. The general interview protocol consisted of probes that a) allowed students to clarify the meanings of the terms they use, and b) asked students to back up their statements with relevant examples. Then, an individual interview protocol was written for each participant based on individual responses. During the interviews, participants were given an opportunity to look at their responses to questionnaires. At the end of each interview, participants were given a copy of their responses from the journal questions and the questionnaires.

Documents

Three types of documents were examined throughout the study: the textbook and supplemental materials used by the teacher, homework, and classroom assessments completed by the students during instruction on geometric proof. Each section of the textbook used by the classroom teacher during data collection was examined and recorded on a document summary form (see Appendix D). The document summary form was used for supplemental material as well. Copies of samples of the participants' responses to homework as well as classroom assessment items relating to the purposes of geometric proof were made and then summarized on a document summary form. The examination of the documents was guided by the following questions:

1. What purposes of geometric proof are explicitly stated in the curriculum materials?

2. Were students assigned any homework relating to the purposes of geometric proof? If so, what was the nature of the assignments?

3. Were there any assessment tools that involved the purposes of geometric proof? If so, what was the nature of the assessment?

Researcher Journal

In order to reduce bias, the researcher kept a journal. Journal entries were written to describe contact with participants (i.e., during seatwork). Entries consisted of the pseudonyms of the participants who were assisted during seatwork, the nature of the problem, questions asked by the students, and questions or prompts asked by the researcher. The entries were crucial in making sure that the researcher provided the same number of opportunities for each participant in establishing rapport and in gathering students' views informally. All other significant points of contact with the participants and the classroom teacher were recorded in the researcher journal. The entries were made during class as well as from memory immediately after class.

The Researcher

The researcher's background was an important aspect of this study since the background served as the "initial framework" for making sense of the events in the study (Agar, 1980). The researcher had thirteen years teaching experience, including five years teaching middle and high school mathematics in a small school in the same county where the study was conducted. The researcher currently teaches developmental and undergraduate mathematics at a two-year college.

The researcher's school teaching experience was unique since it provided experience teaching all of the high school mathematics courses each year. In other words, the researcher taught all of the high school mathematics courses, including geometric proof, throughout the five years of employment. This experience provided the researcher the opportunity to observe the students' progression year after year. A limited staff resulted in the researcher teaching students of varying ability as students seeking Regents and non-Regents diplomas were enrolled in the same mathematics classroom. The researcher had additional experience teaching geometric proof to high school students in summer school while earning a Master's degree in Mathematics. The experience helped to broaden the researcher's views of students from various social and economic backgrounds.

As a high school teacher, the researcher recalled wishing that the state assessment would focus less attention on the instrumental understanding of proof and more on its purposes. While the researcher recalled classroom discussions about some of the purposes of proof, the researcher did little to assess her own students' views of the purposes of proof through classroom assessment.

All researchers' backgrounds possess biases that are inherently problematic to their study. In this study, some of the researcher's bias resided with her past experiences in teaching high school. This study employed two main methods for reducing researcher bias. First, the researcher use audio recordings of classroom accounts. The use of audio tapes prevented the researcher from using her past experiences as a reference to the classroom events. Second, the researcher made a conscious effort to recognize her bias about teaching mathematics and about what high school students should know about mathematical proof.

During data collection, the researcher assumed many roles: observer, confidant, stranger, and interviewer. Since the researcher had no connection with the school district, the researcher's presence was somewhat obtrusive to the students. However, the researcher's substantial classroom teaching experience helped to establish some rapport with the students during the course of the study. The researcher also assumed the role of interpreter and was instrumental in determining what the students' views were about the purposes of geometric proof. As an interpreter of the students' views, the researcher used her awareness of the purposes of mathematical proof obtained through the review of the literature and past teaching experience.

Data Collection

As suggested by Moschkovich and Brenner (2000), naturalistic and cognitive approaches were alternated and intertwined throughout the study. While it was suggested that the naturalistic component be conducted prior to all other data collection, it was essential to examine students' views of mathematical proof prior to instruction. Thus, the Preconceptions Questionnaire was given prior to participant observation. However, the Preconceptions Questionnaire was not analyzed until two weeks into participant observation in order to reduce researcher bias. Throughout the study, documents such as curriculum materials, homework assignments, and assessment documents were collected and analyzed.

Data Analysis

A combination of analyses was conducted: analytic induction, constant comparison, and typology. Analyses were conducted throughout and after collection of data in both the initial naturalistic and cognitive cycle. On-going analyses of the field notes obtained from the classroom observations and documents were used to inform the interviews. As new data were collected, changes in categories and perceptions were made.

Miles and Huberman (1994) stated that data analysis consists of three major components: data reduction, data display, and conclusion drawing. Data reduction is a necessary process in collecting qualitative data, which involves selective data collecting techniques based on the theoretical framework and the research questions. In this study, the initial theoretical framework consisted of the instruction, curriculum, and assessment with regards to the students' views of the purposes of geometric proof. Thus, "anticipatory data reduction" was employed from these three perspectives in analyzing the field notes obtained from data sources. However, coding was not limited to the five purposes of geometric proof provided by de Villiers (1999). Field notes, responses from the open-ended questionnaires, and interviews were word-processed.

ATLAS.ti was used for coding and sorting the data. There were two major components of the main research question that were analyzed. The first component focused on the students' views of the purposes of geometric proof, and the second component provided the context of the students' in terms of the classroom instruction, curriculum materials, and classroom assessment.

Journal Questions

Each journal question was analyzed through coding and analytic induction. Journal questions were read the day after administration. Codes were assigned prior to the administration of the next journal question.

Preconceptions Questionnaire

Each item on the Preconceptions Questionnaire was analyzed through coding and analytic induction. The questionnaire was analyzed during the second week into the fieldwork through coding and analytic induction. The codes for each item were developed at the time the analysis was made. From the codes, emerging categories of students' preconceptions of justification was formed, followed by enumeration of the categories. Some of the items on the Preconceptions Questionnaire were discussed during individual interviews. In some cases, the responses were re-assigned to different categories once clarification from the students' had been made.

Documents

Chapters and sections that were assigned to the students prior to and during the study were identified by Mrs. Kelly. Document summary forms (see Appendix D) were used for each page noting a purpose of proofs involving geometry and sorted according to the various purposes of geometric proof that were addressed. Assessment tools, including quizzes, tests, homework, and projects were analyzed in a similar way.

Researcher Journal

The researcher journal was analyzed on an on-going basis. Entries were read prior to the following classroom observation. At the end of each week, the researcher journal was reviewed to ensure contact has been made with all of the participants at least once. At the end of the study, the researcher journal was scanned to identify any possible researcher bias that might have emerged from the students' responses from the Post-instruction Questionnaire or interviews. Notes of the conversations that took place between the participants and the researcher were word-processed and reviewed throughout the analysis.

Classroom Observation

All classroom observations were audio taped. The tapes were transcribed and coded on a daily basis. Researcher comments, which consisted of questions, reflective thoughts, and evolving hypotheses about the data were added to the codes. On the days when groups of students were tape recorded, audio tapes were reviewed by the researcher on the evening or morning before the next taping, but not necessarily transcribed and coded. However, all audio tapes were transcribed and coded within a week from the initial taping.

The coded field notes were analyzed to identify a pool of questions for the Post-instruction Questionnaire. Themes regarding specific comments made by the classroom teacher or particular tasks or activities that were discussed in the classroom involving various purposes of geometric proof were identified through constant comparison. Occurrences of each instance when the classroom teacher discussed a purpose of proof were tallied. Enumeration of the frequencies for each instructional day was used to identify the ebb and flow of the various purposes of proof discussed throughout the two units. Eleven classroom episodes, which included descriptions of the instruction and seatwork, were summarized to provide the context of the students' learning.

Post-instruction Questionnaire

The Post-instruction Questionnaire was coded and constant comparison was used. Emerging categories of students' views of the purposes of geometric proof that were established from the analysis of the field notes were initially used, and then, modified. An interview protocol was established for each participant based on the responses from this questionnaire.

Interviews

Transcripts of the formal interviews were made as soon as possible after the interview. The transcripts were coded using codes from the field notes and the Post-instruction questionnaire. Constant comparison was used to add or modify existing categories of the students' views of the purposes of geometric proof.

CHAPTER IV

CONTEXT

An examination of the students' views about the purposes of proof would be limited without the context surrounding the students' classroom learning about geometric proofs. In this study, the curriculum, the assessment tools, responses from the preconceptions questionnaire, and daily classroom learning experiences formed the basis of the context for the findings reported in the next chapter.

Curriculum, Textbook, and Assessment

Curriculum

The mathematics curriculum in New York State is guided by the *Learning Standards for Mathematics, Science and Technology* (New York State Education Department [NYSED], 1996). These standards provide specific benchmarks in the areas of arithmetic, algebra, geometry, trigonometry, probability and statistics, and discrete mathematics. NYSED mandates mathematics assessments to students in fourth and eighth grade as well as two different assessments, called Math A and Math B, to those in high school. Students are required to pass the Math A assessment graduate from high school. Students who pass both Math A and Math B receive an advanced Regents diploma from the State of New York.

According to the *Mathematics Resource Guide with Core Curriculum* (NYSED, 1999) mathematical reasoning is just one of the seven key ideas (which also includes number and numeration, operations, modeling and multiple representations, measurement, uncertainty, and patterns and functions). Mathematical reasoning includes analyzing mathematical situations, making conjectures, gathering evidence, and constructing arguments. Performance indicators for each grade level are given.

For the pre-K through kindergarten level, mathematical reasoning includes sorting and classifying activities using concrete objects through observation of likenesses and differences such as color and shape, recognizing number sequences and number patterns, and being able to give reasons why they sorted or classified objects in a particular manner. In Grades 1 – 2, mathematical reasoning is extended to classifying at least two categories at a time, finding patterns for numbers that add up to specific sums, explaining to others how to solve a numerical problem, using pictures or tables to represent solutions. In Grades 3 – 4, mathematical reasoning is using patterns with factors, statements that use *and*, *or*, and *not*, symmetry in patterns, patterns involving triangular or square numbers, verifying an answer to a problem and checks to justify an answer, and using open sentences to solve problems. In Grades 5 – 6, mathematical reasoning is the application of various reasoning strategies, making and evaluating conjectures, making conclusions based on inductive reasoning, and justifying conclusions through Venn diagrams. In Grades 7 – 8, mathematical reasoning involves guessing, working backwards, identifying similarity and differences among types of problems, deciding relevant from irrelevant information, general solutions, using correct notation, understanding that there is no one right way to solve a problem, identifying patterns for sequences with integral terms, devise formulas, and applying strategies used on simpler problems to more complex problems.

In high school, mathematical reasoning includes constructing valid arguments with truth tables (conjunction, disjunction, conditional, converse, contrapositive, inverse, and biconditional) and finding the truth value of simple and compound statements (Math A) as well as constructing Euclidean and analytic direct proofs and Euclidean indirect proofs (Math B).

Textbook

The textbook, *Houghton Mifflin Unified Mathematics, Book 2* (Rising, Graham, Bailey, King, & Brown, 1991), was used primarily for assigning proof problems rather than for reading its contents. Mrs. Kelly kept several recent textbooks in a shelf by her desk, yet rarely used them. She expressed dissatisfaction with the recent textbooks due to their "action-packed, game-like" characteristic. In discussing her choice of textbooks, Mrs. Kelly showed a page of the newest textbook specifically designed for the Math B assessment and said, "Look at this. Don't you feel like they are throwing all this stuff at you, like an advertisement or a movie, or something?"

In total, four purposes of proof were mentioned: systematization, verification, explanation, and communication. Table 1 shows the purpose of geometric proof explicitly addressed.

Table 1

Quotes of Purposes of Geometric Proof from the Classroom Textbook

Purpose	Quotation
Verification	If at each stage of our progress we insist on proving each new fact, it is because we want to be certain that each step, however obvious, will remain true mathematically no matter what scientific discoveries future ages may bring. (Chapter 3 Introduction, p. 99) We try to assert as postulates only those facts about which there can be no doubt and we prove everything else. (Section 3-2, p. 104) Proofs will be used to show the validity of theorems, all of which can be written in *If...then...*form. (Section 2-8, p. 82)
Explanation	Every step in a proof must be justified by a previous definition or principle that you have accepted as (or proved to be) true. (Section 2-8, p. 82)
Communication	In the final version of your proof, you should show enough steps so that the reader will be able to follow your argument and see why the theorem is true (or so that your teacher will see that you know what you are talking about). (Section 2-8, p. 82) A theorem is a statement that can be proved using postulates, properties, definitions and previously proved theorems. (Section 3-2, p. 104)
Systematization	Over 2000 years ago a Greek mathematician named Euclid first set down principles of geometry in a systematic way. This system, now called *Euclidean geometry*, is the one we will study in this book. (Chapter 3 Introduction, p. 99)

Table 1 (continued)

Quotes of Purposes of Geometric Proof from the Classroom Textbook

Purpose	Quotation
Systematization	If you assume no knowledge of any words, you will simply go around in circles trying to define one unknown word with another. (Section 3-1, p. 100)
	In your study of geometry, there are certain things you may conclude from a diagram and other things you may not. (Section 3-1, p. 101)
	A postulate, or axiom, is accepted without proof. The chief requirement of a postulate is that it should not contradict another postulate. (Section 3-2, p. 104)

Note. From *Houghton Mifflin Unified Mathematics, Book 2* by Gerald R. Rising, John A. Graham, William T. Bailey, Alice M. King, and Stephen I. Brown.

Assessment

Mrs. Kelly chose traditional assessment techniques for both units on proofs involving geometry, which focused solely on an instrumental understanding of proofs. There were two assessment tools used throughout the unit: homework and an exam at the end of each unit. Homework was collected and graded on a daily basis, but only when students felt comfortable with the work they had shown. Since all of the homework problems on proofs modeled state assessment items, Mrs. Kelly graded the homework assignments using New York State guidelines. Students were allowed to re-submit graded homework assignments until a 100% was achieved. On average, students were given approximately three proof problems each night as homework.

No in-class quizzes were given, nor were any projects assigned. Journal questions written specifically for the research study were given to the participants only. The four non-participants were exempt from this activity. Since the journal questions were not given to all of the students in the class, they could not be used as graded assignments. As a result, the validity of this data source should be questioned. At the end of each unit, Mrs. Kelly administered a unit exam, which she call "unit quiz," consisting of proofs. Both unit quizzes consisted of questions taken from the textbook. All of the questions on each unit quiz consisted of given information and a prove statement. On both of these assessments, students were allowed to choose a specified portion of the proof problems. On the day prior to the exams, Mrs. Kelly administered a practice exam. Students were given 30 minutes to complete the practice exam and were not allowed to use any resources. For the remaining 15 minutes, Mrs. Kelly showed students possible ways to prove the statements. The practice exams were not graded.

Preconceptions Questionnaire

Fourteen of the fifteen participants took the preconceptions questionnaire. Due to frequent absenteeism, this questionnaire was not administered to one of the participants, Brandi. The students' responses revealed similar but subtle differences in the students' thinking about the concepts explored through this data source. Because the desks were arranged in groups, sporadic conversations broke out during the administration of this questionnaire. As a result, the participants were reminded periodically to fill out the questionnaire individually.

Item 1 (The Cardboard Triangle Activity Problem)

For this item, participants were asked to evaluate a fictitious teacher's argument about the sum of the angles of a triangle (see Appendix A). Initially, eleven students accepted the fictitious teacher's argument as proof. Three of these students (Lori, Nikki, and Sue) did not give a reason for why the activity was a proof. Three other students, who did not accept the argument, were concerned that the arcs were not aligned. They all stated later during individual interviews that the argument would have been a proof had it not been for the mismatched arcs in the diagram. Regardless of their decision, the reasons the students gave were sorted into four main categories: a) Fact or Rule-Based, b) Particular Case, c) Reasoning Connected Lines, and d) Judgment of Diagram.

For the first category, *Fact or Rule-Based*, three students (Kara, Penny, and Seth) accepted the argument as proof, but seemed to focus solely on the fact that, in Euclidean geometry, all triangles contain 180 degrees. These students did not try to relate the number of degrees in a straight line to the number of degrees in a triangle. For example:

> Yes, because she said when you add interior angles of any triangle it is always 180 degrees. (Penny, Item 1, Preconceptions Questionnaire)

For the second category, *Particular Case*, two students (Eric and Betty) focused their attention to finding specific angle measurements in the diagram. For example:

> Yes, because 180 divided by 3 is 60. (Eric, Item 1, Preconceptions Questionnaire)

For the third category, *Reasoning Connected to Lines*, four students (Cathy, Kim, Mackenzie, and Betty) referred to the concept of a straight angle in their reasoning. For example:

> Yes, because she put them on a straight line which equals 180 degrees. (Cathy, Item 1, Preconceptions Questionnaire)

For the fourth category, *Judgment of Diagram*, three students (Ken, Marty, and Jeremy) did not accept the argument as proof because they were not convinced by the diagram. These responses seemed to indicate a group decision based on the fact that the arcs representing the angles in the second diagram did not match to form a semicircle. Marty stated, "No, A, B, and C don't match up, but the sum of the interior angles is 180." However, during individual interviews, all three students stated that if the arcs matched, the argument would have been a proof.

Item 2 (The Isosceles Triangle Problem)

For this item, participants were asked to determine whether a triangle was isosceles given certain angle measurements (see Appendix A). Three main categories were established in order to describe the students' responses: a) Use of Deductive Reasoning, b) Use of Partial Deductive, c) Use of a Property or Definition.

In the first category, *Use of Deductive Reasoning*, six students (Penny, Kim, Nikki, Eric, Seth, and Cathy) correctly deduced the angle measurements needed to establish that the base angles of the triangle were congruent. However, two of these students either failed to make a final judgment (Cathy) or failed to make the correct judgment (Seth) about the Triangle ABC. Cathy wrote, "I don't know exactly how to solve this," even though she was successful at finding the measurements of all of the angles in the triangle. Seth also found all of the angles in the diagram except $\angle$A and said that the triangle was isosceles (see Figure 7). Then, he changed his affirmative response and wrote, "No, because the angle and the altitude do not equal up." Seth, who was sitting with Mackenzie, changed his response and reasoning to match Mackenzie's response. In her explanation, Mackenzie referred to both altitudes and both angles being the same in an isosceles triangle.

In the second category, *Use of Partial Deductive Reasoning*, two students (Kara and Marty) used deductive reasoning to establish that vertical angles $\angle$MSB and $\angle$NSC were congruent. Both students believed that the triangle was isosceles, but failed to find the other angles in the triangle needed to establish congruent base angles.

In the third category, *Use of Property or Definition*, six students (Ken, Jeremy, Lori, Sue, Betty, and Mackenzie) attempted to justify their responses by stating a property or a theorem about isosceles triangles. For example, Jeremy said that the triangle was isosceles because "the base angles are equal," yet failed to show any angle measurements. Of the six responses in this category, two of the responses (Betty and Mackenzie) indicated that the triangle was not isosceles. Betty, who incorrectly stated that ∠SBM was 20 degrees, said that the triangle was not isosceles because the sides were not equal. Overall, only four of the fourteen students were able to make a correct judgment about the triangle and have correct reasons to justify their responses.

2. In the triangle ABC shown below, the altitudes BN and CM intersect at point S. The measure of ∠MSB is 40 degrees and the measure of ∠SBC is 20 degrees.

   Is Triangle ABC an isosceles triangle?

   ~~Yes~~ NO

   Explain your reasoning.

   No, b/c the angle and the altitude do not equal up.

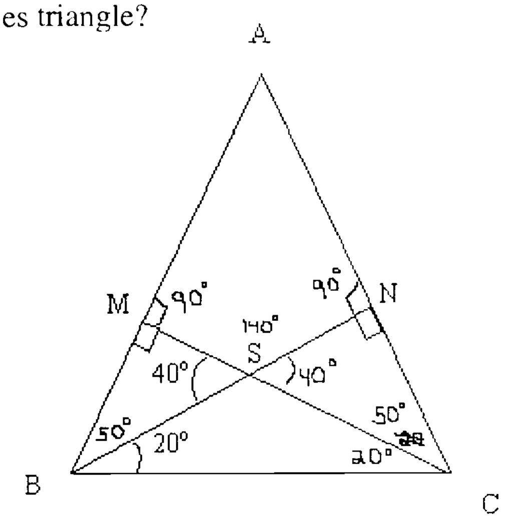


Figure 7. Seth's response to Preconceptions Questionnaire Item 2.

Item 3 (The Toothpick Problem)

For this item, participants were given the first three designs in a series of repeated designs supposedly made from toothpicks (see Appendix A). The number of toothpicks needed for each of the three designs was given and the students were asked to find the number of toothpicks needed for the 40th design. None of the fourteen students developed an algebraic formula for the nth design. Six of the fourteen students (Sue, Lori, Ken, Kim, Mackenzie, and Betty) arrived at the correct answer, 161. Mackenzie made two columns, one column for the design number and the other column for the number of toothpicks needed. At first, Mackenzie found the second column by adding three to each previous result and arrived at 127 toothpicks. She then crossed out the whole column and wrote the correct series of numbers by adding four to each previous result. Ken, Betty, and Lori also used repeated addition, but did not display their work. Sue stated that she started at 13 and then "added 4 to it 37 times."

Other students who used similar methods as those who answered correctly, made minor errors. Kim, who arrived at 162 toothpicks, used the same method as Sue, yet added 1 to the final result. She obviously forgot that, by using 13 as her starting point, she already incorporated the extra toothpick from the first design. Cathy also multiplied 4 and 37, but forgot to add the result of 148 to 13. Seth, who arrived at 157 toothpicks, used Mackenzie's method of writing two columns. However, he forgot to write the 12th design and fell short in his final answer.

The remaining six students (Penny, Kara, Marty, Jeremy, Nikki, and Eric) used only multiplication to arrive at an answer. Jeremy and Marty multiplied 4 and 40 to arrive at 160 toothpicks. Penny and Kara also arrived at 160. Eric multiplied 4 and 41 while Nikki multiplied 5 and 40.

Item 4 (The Prime Number Problem)

For this item, participants were asked to evaluate arguments made by fictitious students who were arguing over a conjecture about a formula that purported to always produce prime numbers for all natural numbers (see Appendix A). For this problem, one of the fictitious students, Amy, stated that the generalization was false because a counterexample was produced. The other fictitious student, John, stated that the generalization was true even though a counterexample existed. Ten participants believed that Amy was correct in stating that the conjecture was false. However, their reasons varied. Six students (Cathy, Jeremy, Kim, Mackenzie, Lori, and Seth), who were correct in their reasoning, stated that the formula did not always produce a prime number. For example:

> Amy, because the teacher said a formula that always produces a prime number and it doesn't always produce a prime number. (Mackenzie, Item 4, Preconceptions Questionnaire)

However, Betty provided her conception of a "rule" for her reasoning that seemed to demonstrate a colloquial view of "rules" in everyday life. For example:

> Amy, because it is a rule then it should always be true and if there are exceptions it can't be a rule. (Betty, Item 4, Preconceptions Questionnaire)

Eric also produced a correct response along with reasoning that was different from Mackenzie's response, yet similar to Betty's response. He seemed concerned about having exceptions to a rule with regard to prime numbers. For example:

> Amy, because her theory is right, and I don't think there are any exceptions to rules dealing with prime numbers. If it's prime, then it's prime. If it's not, then it's not. (Eric, Item 4, Preconceptions Questionnaire)

Two students (Nikki and Kara) stated that John was correct. Interestingly, these students did realize that there could be exceptions to rules in mathematics, yet failed to recognize that this particular formula was supposed to work for all natural numbers. Kara stated "With John, because there is exceptions to most mathematical rules."

Two students (Sue and Marty) misinterpreted the item. Sue stated, "I agree with Amy because if you plugged in different prime numbers like '2' it would come out to be factored to 2 X 2." Marty said he agreed with Amy because the example had shown the conjecture to be true, which was the opposite of Amy's stance.

Item 5 (The Even/Odd Problem)

In Item 5, participants were shown an argument constructed by two fictitious students who used extreme values to prove that the sum of two odd numbers is even (see Appendix A). The responses were sorted into three main categories: a) Fact or Rule-Based, b) Naïve Empiricism, and c) Crucial Experiment.

In the *Fact or Rule-Based* category, five students (Seth, Ken, Nikki, Mackenzie, and Eric) recited the conjecture as a fact or rule. For example:

> Yes, because any two odd numbers added together equals an even number. (Ken, Item 5, Preconceptions Questionnaire)

In the *Naïve Empiricism* category, five participants (Marty, Betty, Kara, Penny, and Jeremy) thought that the fictitious students' use of examples was enough for proof. For example:

> Yes, because they used two odd numbers, and got an even number for an answer. They did it a few times to make sure. (Penny, Item 5, Preconceptions Questionnaire)

Marty, who also displayed similar reasoning, also remarked on the authority of the classroom teacher in establishing truth. He stated, "Yes, unless the teacher proves them wrong, their example is true."

In the *Crucial Experiment* category, three participants (Sue, Kim, and Cathy) referred to the size of the numbers used in the examples. For example:

> Yes, because they used big odd numbers and small odd numbers and both tries the answers were even. (Sue, Item 5, Preconceptions Questionnaire)

Lori was the only participant who did not provide a description of the argument and simply stated that the fictitious students had proved the statement about the sum of odd numbers.

Mrs. Kelly was asked to read all of the responses for the Preconceptions Questionnaire approximately two days after the administration of the questionnaire. She stated that she was not surprised with the responses but that she was concerned about how to change the students' focus on empirical thinking to generalization for the impending proof units.

Classroom Instruction on Proofs Involving Geometry

This section presents an overview and details of the instruction given by the classroom teacher spanning the two units on proofs involving geometry. The first unit on coordinate geometry proof was conducted over a two-week period, and was followed by a two-week unit on two-column geometric proofs. A description of the attention given to a variety of themes relating to the various purposes of geometric proof within the instruction is described and supported by teaching episodes. Within the two units, Mrs. Kelly either implicitly or explicitly referred to proof as explanation, verification, systematization, and communication. At some level, Mrs. Kelly mentioned explanation, verification, and communication on the first day of the coordinate geometry unit. References to proof as a means of systematization did not appear until the first day of the unit on two-column geometric proofs.

Episode 1: Day 1 of Coordinate Geometry Proofs

On this first day of the coordinate geometry unit, Mrs. Kelly began the lesson by writing the topic "Coordinate Geometry Proofs" on the overhead. Her introductory statement about the topic focused on communication:

> We are going to raise the bar, raise the *standard* on your communication skills and your thought processes, and we are going to ask you to think a little bit deeper, and use the knowledge you already have. (Classroom Observation, 3/6/03)

Without further explanation about standards of communication, she then referred to coordinate geometry proof as a task to be completed by the students. Three formulas (midpoint, distance, and slope) and one theorem (the Pythagorean Theorem) were written on the overhead, and it was explained to the students that only these three formulas and the one theorem would be involved in the tasks for this unit. Then, Mrs. Kelly reminded the students about a similar task in Math A that dealt with plotting graphs of various polygons. Mrs. Kelly focused once again on communication skills, mentioning only briefly explanation:

> I mentioned communication skills. You are going to have to do some writing, some explaining. We aren't talking about three page papers here; we are talking about little paragraphs or sentences. But it will be in the English language and fairly grammatically correct. I won't take off points for that but it shouldn't be hard to understand what you are trying to say. (Classroom Observation, 3/6/03)

Implicitly, it was here that Mrs. Kelly first introduced two standards of communication for proofs involving geometry: the length of an explanation and the clarity of the explanation. Mrs. Kelly went on to explicitly discuss a third standard of communication, the structure of a coordinate geometry proof:

> Basically we are going to ask you to complete a task and you are going to do that in two steps. First, you are going to show the work that's necessary for the tasks. That will be the formula, the theorems and calculating that out. Then, the second step you are going to write a summary statement that basically explains how your work completes the task. (Classroom Observation, 3/6/03)

Directly following this discussion on the structure of coordinate geometry proof, Mrs. Kelly introduced a fourth standard of communication, the audience. Mrs. Kelly wanted her students to imagine that they were communicating their explanation to a fictitious audience. The audience she described was that of a younger student who knew some mathematics:

> Pretend that you are explaining to, not really to someone in your class, but explaining to someone a couple of years younger than you. Not someone in kindergarten, but someone who has a basic understanding of the math that you are doing, but maybe not all of it, maybe not as much as you know. (Classroom Observation, 3/6/03)

Then, Mrs. Kelly distributed graph paper to the students and wrote the first "task" on the overhead:

> Quadrilateral ABCD. A(0, 2), B(8, 0), C(9, 4), D(1, 6). Prove ABCD is a rectangle.

Students were encouraged to use a reference sheet in the form of a flow chart on the properties of quadrilaterals that they filled out in Math A (see Appendix E). As she plotted the points on the overhead graph paper, Mrs. Kelly asked a student, Sue, whether she thought the figure looked like a rectangle. Sue did not respond. Mrs. Kelly attempted to encourage Sue by saying, "It's just an opinion question. You know my art skills leave something to be desired." This prompt indicated that, implicitly, it was acceptable for students to make a judgment about a geometric figure. After a few seconds, Sue thought the figure looked like a rectangle, and Mrs. Kelly agreed with her. The discussion continued as Mrs. Kelly asked another student, Ken, whether he thought the figure looked like a rectangle. After his affirmative response, Mrs. Kelly grabbed a folding ruler and made a rectangle. She then moved the hinges to make a parallelogram and asked, "What if someone came in here and tilted this (angle) just one-millionth of an inch? Would it be a rectangle? Are we going to be able to tell by looking at the picture?" Ken said that it would not be enough. Mrs. Kelly emphasized the fact that looking at a picture was not enough for determining whether the lengths of a figure were congruent, yet did not return to the role of making initial judgments from the diagram. Thus, students may have disregarded the role of initial judgment in the verification process.

Mrs. Kelly also explained that there were several ways to complete the task. But before she prompted the students to think of a plan, she reviewed the three formulas and the Pythagorean Theorem. Mrs. Kelly directed the students to show that the figure was a parallelogram with one right angle. Mrs. Kelly reminded the students that a parallelogram was a figure with opposite sides parallel and congruent and called on another student, Cathy, to decide how the class should show the figure is a parallelogram. Cathy chose to show that the figure was a parallelogram by showing both pairs of opposite sides parallel. Mrs. Kelly praised Cathy for her choice by remarking that slopes could be used to show the figure was parallel and that it contained a right angle. Mrs. Kelly reminded the students to be clear when writing the summary so that the "reader" would not have to guess what was happening. The students did not seem to be bothered by the use of the word "reader" and did not ask Mrs. Kelly who the reader was.

After all of the slopes were calculated, Mrs. Kelly explained that the summary portion of the proof was to occur next. She reminded her students of their fictitious audience and said, "Here's the part where we have to explain. You have to explain to a seventh or eighth grader." Mrs. Kelly led the students through the summary portion, and explained how the calculations from the slope formula justified parallel lines and right angles in ABCD. After the completion of the proof, Mrs. Kelly explicitly referred to the length of an explanation by mentioning that the length of an explanation depended on the task. The explanation for proving ABCD was a rectangle was longer due to the various ways the slope formula could be used in these types of tasks:

> Sometimes, depending on the task, sometimes your explanations can be done very well in just a sentence. This one was a little longer because it dealt with slopes. When you are dealing with slopes, because it is used in so many ways, you are going to have to explain a little bit longer. But our task was to show that this was a rectangle. (Classroom Observation, 3/6/03)

During the last five minutes of class, students were given their homework:

> Quadrilateral MATH has vertices M (-1, 4), A (4, 7), T (7, 2), and H (2, -1). Prove that MATH is a square.

Using a folding rule, Mrs. Kelly demonstrated that it was the angles of the figure rather than its sides that determined whether the figure was a square.

Episode 2: Day 2 of Coordinate Geometry Proofs

Mrs. Kelly started by reviewing the task students were given at the end of the previous class. She stressed that there were multiple ways to complete the task, but that one of the ways was perhaps the most direct. As Mrs. Kelly prompted the students, several students seemed to have problems identifying correct formulas in completing the task (Classroom Observation, 3/7/03):

> Mrs. Kelly: Sue, is it distance, slope or midpoint that is going to tell you whether the sides are the same?
>
> Sue: Midpoint? (pause) Distance?
>
> Mrs. Kelly: Distance. So you should have four distance calculations if you are trying to show that it has four congruent sides. (Mrs. Kelly draws a sketch of MATH without plotting the points). Let's suppose you pick Angle A, up here. If you chose Angle A, out of these three formulas that we have, distance, slope or midpoint, Seth, which one in your notes did I tell you that you would have to use to show a right angle?
>
> Seth: Midpoint?
>
> Mrs. Kelly: No. Cathy, which one in our notes?
>
> Cathy: Slope.

After completing the calculations, Mrs. Kelly asked the students to hand in the assignment if they felt comfortable with the summary they wrote. She reminded the students that they would be graded on a scale of 0 to 10 and that they could redo a proof until they achieved a perfect score.

Next, Mrs. Kelly showed the students an alternative proof written by a former student. In this proof, the student calculated the lengths of the sides and then drew in the diagonals and calculated their lengths. Mrs. Kelly emphasized the clarity of the former student's summary and stated:

> So she [the former student] said, "and they both came out the same," the square root of 68. So then she said, "and since the diagonals are congruent, that makes it a rectangle. A rectangle and a rhombus, make a square." That's kind of what we are looking for from you. (Classroom Observation, 3/07/03)

She then pointed out the favorable grade that was given to the student. Mrs. Kelly appeared to use the former student's proof as a model and to reinforce one of the standards of communication, clarity, which was established on the previous day.

Since several students did not have their flow chart of the properties of quadrilaterals, Mrs. Kelly spent the rest of the class period having the students fill out the reference sheet in a whole group setting. Occasionally, Mrs. Kelly pointed out certain properties that the students might use while completing a task and those they were not likely to use. Mrs. Kelly also reminded students when it was necessary to use a combination of properties for certain figures. For example, students were reminded that in order to show that a figure was a trapezoid, they had to show that two opposite sides were parallel and that two opposite sides were not parallel. After the completion of the reference sheet students were given another task:

> Quadrilateral ABCD, A (-3, 6), B (6, 0), C (9, -9), D (0, -3).
> Prove a) ABCD is a parallelogram, and b) ABCD is not a rhombus.

Mrs. Kelly told the students that they could write one summary for the first part and another summary for second part, or that they could combine their summaries. After the students plotted the points, Mrs. Kelly told the students that their diagram should look like a parallelogram because "they" were asking them to prove the figure was a parallelogram.

Implicitly, this statement indicated two consequences. First, the figure purported in the task must be that figure. Second, visually (or intuitively) students should be able to use this information to initially judge congruent or parallel sides from a diagram. Mrs. Kelly assisted the students in determining the appropriate formulas to complete the task. This time, all of the students who were called upon, Kim, Amy, and Seth knew which formulas were appropriate for showing parallelism and congruency. Mrs. Kelly began with the distance formula and followed with a gentle reminder that one of the purposes of proof was communication, "Communicate to your reader. I'm going to start with side AB, so I'm going to tell my reader that."

Throughout this lesson, as well as the first, none of the students initiated questions. Mrs. Kelly was thorough in explaining how certain properties aided in completing the task and why various calculations were needed. At the end of the class, students were given their homework assignment:

> The coordinates of triangle ABC are A (0, 0), B (2, 6), and C (4, 2). Using coordinate geometry, prove that if the midpoints of sides AB and AC are joined, the segment formed is parallel to the third side.

Mrs. Kelly outlined the formulas students would need to complete the task and the specific number of calculations needed along with a summary.

<u>Episode 3: Day 3 of Coordinate Geometry Proofs</u>

The class began with the collection of homework followed by a student work sample completed by a former student. This student sample was shown as an example of a clearly written summary. After reading aloud the former student's summary, Mrs. Kelly remarked, "Simple, direct, very straightforward." She also remarked about the labels, M and N, this student created, but told the students it was not a requirement for them to label points that they find through calculations. At this point, Mrs. Kelly praised the students for their efforts in writing clear summaries on homework tasks. Then, students were given the following task to work on in class in small groups:

> Quadrilateral DRAW has vertices at D (-3, 6), R (6, 3), A (6, -2), and W (-6, 2). Prove that DRAW is an isosceles trapezoid.

Students were reminded of the three formulas and encouraged to use their reference sheet on the properties of quadrilaterals. Students were also reminded to examine their diagrams once they plotted their points to make sure that it looked like an isosceles trapezoid.

In the following dialog of one groups' struggle to complete the task, Marty, Jeremy, and Ken failed to use the diagram in deciding which formulas were appropriate for the coordinates. First, Jeremy failed to see if his diagram looked like a trapezoid and realized a mistake in plotting a coordinate only after his expectations for a distance calculation was not realized. Secondly, even though Ken knew that he had to apply two distance calculations and two slope calculations, he did not use his diagram to decide which formulas to use with the coordinates (Classroom Observation, 3/10/03):

> Mrs. Kelly: Make sure your picture looks about like that. Remember, if they are telling you it is an isosceles trapezoid, it really should look like an isosceles trapezoid. Get out your green sheet.

Marty: I got it.

Mrs. Kelly: Okay. It's not even open. Well, what do you need? You're trying to show that it is an isosceles trapezoid. So you're probably going to have to show that it is a trapezoid first. So figure out what you are going to do there.

Ken: Distance.

Marty: That's what I said. For D and R and W and midpoint.

Jeremy: Nope.

Marty: Two distances and two slopes.

Ken: Two? Wouldn't that be four?

Marty: I don't know.

Ken: (Checking the calculations) Close!

Marty: It's not even close to what the other one is. It can't be that. It can't be. Ahhh.

Ken: We screwed something up.

Marty: It's not distance.

Ken: Obviously not.

Jeremy: Okay, let's try something else.

Marty: Unless it is distance and its DW and RA. Do you want to try that one? Is it distance of DW and RA?

Mrs. Kelly: Those are important.

Marty: Do we need a midpoint for this?

Mrs. Kelly: Are you going through the middle of anything?

Marty: That's what I said. I was just making sure.

Mrs. Kelly: Ken, what's Ken calculating here, because you are not being a good communicator. I have no idea, I can see distance, but I have no idea what side. What side did we calculate the distance of?

Marty: DR and WA.

Mrs. Kelly: DR and WA? We want the lengths of those sides?

Marty: That's why I said, DW and RA.

Ken: They are supposed to be the same.

Mrs. Kelly: Which ones are supposed to be the same? The legs are supposed to be the same. So you don't want to calculate the lengths of the bases.

Marty: Those are slope.

Mrs. Kelly: Those need to be parallel. Right. Basically you are going to have to, um, probably do number one, the isosceles trapezoid or three. But you can't just say that the legs are congruent, unless you also make sure it is a trapezoid. And remember we had circled one and two.

Marty: I don't know what to do.

Jeremy: We gotta do the distances, WD and RA. And then we have to do the slope of DR and WA.

Marty: Are you sure?

Jeremy: Yeah, that's what she just said.

Marty: That's what I just said and she said, "no." She said don't do the distances of DW and RA.

Jeremy: No, we are doing the distances of these two.

Marty: DW and RA!

Jeremy: Yeah, oh my goodness!

Marty: Jeremy is confusing me (pause). Okay why don't you try it.

Mrs. Kelly: Finding the two legs congruent is going to make it isosceles. But you've got to make it a trapezoid also. You can't just find the legs and quit.

After Mrs. Kelly pulled the whole class back together, she reiterated the fact that showing two legs congruent was necessary, but not sufficient. She also emphasized the importance of being "good communicators" and reminded students to label the calculation with the letters of the side whose distance is being calculated. Mrs. Kelly continued to show students the appropriate distance and slope calculations and finalized the process through a written summary. She reminded students to be considerate of the "reader" and even changed her voice as if she was pretending to be an anonymous reader stating, "So now a reader would go over and look, 'yeah, they did have equal lengths,' they are congruent, making DRAW isosceles. A little longer summary than we've had, but there were three different parts we had to talk about too." Again, Mrs. Kelly explained that some summaries would be longer than others depending on the task. Finally, students were given another task for a homework assignment.

Episode 4: Day 4 of Coordinate Geometry Proofs

Mrs. Kelly began class by collecting homework and discussing briefly a unique way to prove the homework problem. She announced the homework policy and urged students to correct their proofs until they received a perfect grade. Students were not pressured to hand in the assignment she was collecting. Once the homework was collected, Mrs. Kelly told the students to work on the following proof in small groups:

> Triangle ABC has vertices A (-2, -1), B (4, 7), C (-4, 3). Prove that triangle ABC is a right triangle.

A second part, prove that the median to the hypotenuse is equal to one-half the hypotenuse, was revealed later in the class period.

The task required students to either apply the Pythagorean Theorem by finding the lengths of all three sides or showing that CB and CA were perpendicular to each other. In either case, students could have used their judgment in deciding which side was the hypotenuse and which two sides formed the right angle. In the following dialog, one member of the group wanted to change his plan from using the Pythagorean Theorem to finding two lines perpendicular. However, none of the students in the group checked the diagram to make a judgment about which two lines were perpendicular (Classroom Observation, 3/11/03):

Jeremy: Now what should we do, should we do slope?

Ken: Yeah, but why do you do slope then?

Jeremy: To show right angles? That would prove it was a right angle.

Ken: But you got to prove it's perpendicular though.

Jeremy: But if it's a right angle, it's got to be perpendicular.

Ken: Yeah, I know.

Jeremy: So wouldn't we use slope?

Marty: How about the Pythagorean Theorem?

Jeremy: 64 plus 36, that will give us (interrupted by Marty).

Marty: It's 116.

Jeremy: All right, the square root of that is what?

Marty: Oh my God, it's all screwed up.

Jeremy: Wait, no.

Marty: That's a hundred.

Jeremy: That's 10. (pause) The square root of 100 is 10.

Marty: Yeah. (pause) Slope. Do slope of CA and BA.

Jeremy: You gotta do CA and BA.

Ken: CA and BA?

After the students abandoned the use of the slope formula, students were asked to point to the hypotenuse in their diagram. The students correctly identified the hypotenuse from both a visual standpoint and an algebraic one:

Marty: How do you know that's the hypotenuse?

Jeremy: It's the longest, isn't it?

Marty: No, what's that word? From the angle.

Researcher: How do you know that's your right angle though?

Ken: Because it looks like it.

Researcher: It looks like it. It's also the longest side. So that would make it the hypotenuse. So where is your longest side?

Jeremy: CB. No AB. (pause) Because if you square rooted 80.

Marty: All right, I see what you're saying.

On the other hand, another group of students (Lori, Sue, and Betty), used their visual judgment when they began the problem:

> Lori: Oh, a squared plus b squared...The theorem is for right triangles, a squared plus b squared equals c squared, right? (pause) Looks like, it doesn't even look like a right triangle!
>
> Sue: This part does.
>
> Lori: Oh yeah, it does!

Once Mrs. Kelly saw that most of the groups were almost finished with the first part of the problem, she addressed the whole class and completed the problem with the Pythagorean Theorem. She reminded the students once again to be good communicators by labeling the distance calculation with the corresponding side. Most of the students were able to complete the second part of the problem, once they located the midpoint on the hypotenuse. Since the midpoint and distance were relatively easy to find, Mrs. Kelly spent the remaining time discussing the summary rather than the calculations. Once again, she focused on communicating to the reader:

> You can't just leave it up to the reader to say, "Oh yes, five is half of ten." We're just going to write a little equation [Median = ½ (Hypotenuse), 5 = ½(10), 5 = 5]. (Classroom Observation, 3/11/03)

In an informal interview with Mrs. Kelly at the end of class, the researcher asked whether the students would be proving the generalization of this theorem. She thought that students would probably do those types of proof later in Math B, but that she would not be teaching those kinds of proofs in the remaining school year.

Episode 5: Day 5 and 6 of Coordinate Geometry Proofs

After discussing the main points of the homework problem for the previous night, Mrs. Kelly gave the students another problem:

> The vertices of Triangle ABC are A (-2, 3), B (0, 3), C (4, 1).
> Prove that triangle ABC is isosceles and that the median to side BC is also the altitude to side BC.

After students had a chance to try the problem on their own, Mrs. Kelly reviewed the first part of the problem. One of the first things that Mrs. Kelly mentioned about the problem was the task of finding which two sides of the triangle were congruent. She reminded students that it was acceptable to judge the diagram to see which two sides were congruent (Classroom Observation, 3/12/03):

> Mrs. Kelly: Here's our triangle. The first part, I thought, was pretty easy for you, to show that a triangle is isosceles. Cathy, what did you need to do?
>
> Cathy: Distance formula.
>
> Mrs. Kelly: Distance formula. AB and BC, yeah, or, AB and AC probably. Some of you probably did all three sides, okay, and then figured if two were the same. Some of you might have just judged by the picture which two to try. Doesn't really matter if you did all three or just the two.

Shortly after, she reminded students to be careful in communicating to the reader, stating, "And I am going to communicate to the reader which side I am calculating. Some of you are a little lax in that area." When reviewing the second part of the problem with the students, Mrs. Kelly once again told students that judging a picture was a worthwhile task. Referring to the median, Mrs. Kelly stated, "It's always a good idea to make sure that it really does look right. And that looks right." At the end of class, students were reminded of their practice quiz.

On the next day, students took the practice quiz, which consisted of two tasks. They were given 20 minutes to complete the quiz without the use of their reference sheet on the properties of quadrilaterals. Mrs. Kelly collected the practice quiz after the 20 minutes and made comments about the homework problems given the previous day. She pointed out a common mistake students had made regarding a proof involving a rhombus. Several students attempted to show that the figure was a rhombus by only showing perpendicular diagonals. Mrs. Kelly drew a kite on the chalkboard to demonstrate that proving perpendicular diagonals was not sufficient to show a figure was a rhombus. Mrs. Kelly concluded the class by pointing out some of the salient features of each proof on the practice quiz.

Episode 6: Day 1 of Two-Column Geometric Proofs

On this first day of the unit on two-column geometric proofs, Mrs. Kelly gave the students the following logic problem to work on as they entered the classroom:

> Given: If Johnny likes soccer, then he's a smart guy and Johnny likes soccer are both true statements, what else can you say is true?

Mrs. Kelly told the students that this problem was "mysteriously connected" to the lecture she was going to give that day. As Mrs. Kelly circulated, she acknowledged several correct responses, that Johnny was a smart guy. She reminded the students that the *if..., then...* statement was a conditional statement, and if the first part was true, then the second part would be true.

Next, Mrs. Kelly informed the students that this new unit would be about Euclidean geometry proofs. Since the students supposedly learned about postulates and theorems in the beginning of the school year, her references to systematization were brief:

> We're starting our new unit, which will be all about Euclidean geometry proofs. Well, all that means is that it's named after the mathematician Euclid. Euclid was an important mathematician because he made what we called an axiomatic system. It happened over 2000 years ago, which means that stuff that we talked about the first semester. Remember, theorems that were statements that had to be proven. Postulates, and other name for postulates are your axioms. Those are your statements that you accepted without proof. And then we had corollaries. He's the one who developed that sort of a building system for geometry. So, they named these proofs after him. (Classroom Observation, 3/17/03)

Mrs. Kelly went on to explain the difference between coordinate geometry proofs and Euclidean proofs. She explained that the figures in Euclidean proofs would not have any coordinates and that they were "just going to be hanging out there in space."

Then, Mrs. Kelly compared the structure of the two kinds of proofs. She began by explaining that two-column proofs would be more organized and also explained the purpose of each column. On the chalkboard, she wrote and underlined the words *Statements* and *Reasons*. Underneath these two titles, she put "what you know is true" and "how you know it is true," respectively. She informed the students that each of the items in the proof were going to be "itemized and numbered," and that skipping any of the items was not allowed. Mrs. Kelly attempted to reinforce the idea of the *Reasons* column by relating the concept to the students' every day life:

> I'm sure we've all been caught in those situations where you were caught doing something or you get in trouble and they say, why on Earth did you do that? Did that ever happen to you Eric? No? No one ever questioned

> anything that you ever did (laughter from the class). And you say, I don't know. So if you can think about everything that you do, and put a reason to it sometimes, maybe you'll make some better decisions. So we're going to try to get you real organized and nice and neat and tidy and all that kind of stuff.

In addition, students were told that almost all of the statements in the *Reasons* column were conditional statements, just like the example given in the beginning of the class. In an informal conversation, Mrs. Kelly explained why she insisted on having students write statements in conditional form. She explained:

> Well, I went to a conference and in one of the presentations the speaker was telling us that she had better results when she forced her students to write in this style, using *if…, then…* statements. I've seen a huge difference since I started having my students do this. (Researcher Journal, 3/19/03)

Thus, Mrs. Kelly had explicitly imposed this set of standards of communication for two-column proofs:

1. Two-column proofs consist of two columns, *Statements* and *Reasons*.
2. Skipping steps within the proofs is not allowed.
3. Use conditional statements in the *Reasons* column whenever possible.

Next, Mrs. Kelly explained to the students that they needed "tools" to use in their proofs. She presented the Betweeness Property (Point B is between A and C if and only if AB + BC = AC) and then used the property as a reason in the following simple two-column proof exercise from page 101 in the textbook as shown in Figure 2. As Mrs. Kelly started to explain how to begin the proof, she focused the students' attention to the diagram. In her explanation, she told the students to use the diagram in two different ways. First, the diagram provided hidden assumptions about the points X and Y, which allowed the use of the Betweenness Property:

> You are given some information and a picture. Now, the picture is just as important as this information. If this is told to you as given, then you have to assume that those are true statements. Here's the picture that goes along with it. So without even telling you what we're going to prove yet, let's take a look. I see from the picture some points that are between. I see that the Y is between B and C. I see that the X is between A and C. (Classroom Observation, 3/17/03)

Second, the diagram provided a sense of "believability" in the statement prior to formal verification:

> First of all, let's see if it makes sense in our head. If these two pieces are exactly the same, and these two pieces are exactly the same, is it a stretch for you imagination to believe that this whole and this whole are the same? No, we understand that. (Classroom Observation, 3/17/03)

Yet, in the same breath, Mrs. Kelly also justified the need for proof, telling students, "Your mind flips forward so fast that you're actually skipping over several steps, like how you know that."

Given: AX = BY
XC = YC

Prove: AC = BC

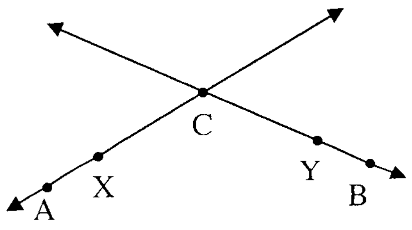


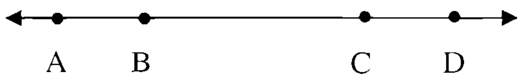
Figure 8. Simple two-column proof exercise. From Houghton Mifflin, *Unified Mathematics, Book 2* by Gerald R. Rising, John A. Graham, William T. Bailey, Alice M. King, and Stephen I. Brown. Copyright © 1991, 1989, 1985, 1982 by Houghton Mifflin Company. All rights reserved. Reprinted by permission of McDougal Littell, a division of Houghton Mifflin Company.

After completing the proof, Mrs. Kelly stated several algebraic reasons that students could use in their proofs: Substitution, Addition Property, Subtraction Property, Multiplication Property, and Division Property. Next, Mrs. Kelly introduced the definition of a midpoint and Theorem 3-2, which stated that a midpoint divides a segment into two segments that are half the length of the original segment. Then, Mrs. Kelly proved the theorem for the students. She warned the students that just because a point in a diagram looks like a midpoint, the point was only a midpoint if the information was given directly to the students. At the end of class, Mrs. Kelly provided the definition of a bisector of a line to add to their list of "tools." Students were assigned two proofs that were similar to the proofs completed in class.

Episode 7: Day 2 of Two-Column Geometric Proofs

On the second day of the unit, Mrs. Kelly introduced three new properties: Reflexive, Symmetric, and Transitive. She drew several diagrams and explained how each property could potentially be used in a proof. For example, when discussing the Reflexive Property, Mrs. Kelly drew an isosceles triangle with a line down the middle forming two congruent triangles. She reminded the students how they used "shortcuts" like SSS in the previous semester and that the Reflexive Property would establish one of the corresponding sides needed. Later, the researcher asked Mrs. Kelly about the extent of instruction students had received about proving congruent triangles prior to this class. She explained that the only thing students knew about this topic was how to identify congruent triangles given specific lengths and angle measurements.

Mrs. Kelly then wrote down all of the properties and definitions that had been discussed up to that point and indicated those required to be written in the *if..., then...* form. Students were told that the Betweenness Property, the definition of a midpoint and Theorem 3-2, the definition of a bisector, the Symmetric Property, and the Transitive Property all required the *if..., then...* form. None of the students asked Mrs. Kelly why some reasons had to be written in *if..., then...* form. With approximately 10 minutes left in the period, the students were given a worksheet for homework. In one of the problems students were asked to write appropriate reasons for each statement in the proof (see Figure 9).



Given: AC = BD

Prove: AB = CD

Proof: AC = BD
AC = AB + BC
BD = BC + CD
AB + BC = BC + CD
AB + BC − BC = BC + CD − BC
AB = CD

Figure 9. Homework problem on first proof. From *Extra Practice Copymasters* by David C. Falvo. Copyright © 1995, by Houghton Mifflin Company. All rights reserved. Reprinted by permission of McDougal Littell, a division of Houghton Mifflin Company.

The group experienced difficulty in completing the reason column for this proof. Before they began, Jeremy stated, "I don't know if I can explain this," which marked the first time the students appeared to think of geometric proof as an explanation. Ken replied, "These are going to be the *if..., then...* ones now." Jeremy agreed, while Marty, who was absent the previous day, was uncertain. The students started by identifying the statement that need to be proven. While Jeremy correctly wrote the first reason as the given, Marty asked Jeremy to explain what he wrote and why. Neither Jeremy nor Ken were able to explain (Classroom Observation, 3/18/03):

> Marty: Isn't that the transitive one we just did? What'd you write, Jeremy?
>
> Jeremy: One's given.
>
> Marty: What's this given stuff?
>
> Jeremy: The one's that given?
>
> Marty: I don't get it.
>
> Jeremy: AC and BD.
>
> Ken: I don't know. That's what we're supposed to do.
>
> Jeremy: That's what we're supposed to do.

Since the next statement contained an addition sign, Marty incorrectly offered the Addition Property as a reason. Both Jeremy and Ken agreed, but then Ken said jokingly, "They're all addition." Skipping to the fourth step, the students guessed the Transitive Property because they saw BC appear twice in the statement, then the Reflexive Property for the same reason. Finally, the students asked Mrs. Kelly for assistance and she immediately saw their problem. She then focused their attention on the diagram (Classroom Observation, 3/18/03):

Ken: I'm confused now.

Mrs. Kelly: You subtracted equal values from both sides of the equation. But that's not what you do in step two. That's not where this guy came from. Did you look at your picture?

Marty: Oh, there's a picture.

Mrs. Kelly: How do you know that AB + BC is equal to AC? Why does that work, with the B twice? How come the B shows up twice and the A and C only show up once?

Ken: Because it's in both lines?

Mrs. Kelly: Yeah.

Marty: It's in the middle of the line?

Mrs. Kelly: Yeah. It's in the middle.

Marty: The midpoint one.

Mrs. Kelly: It's not the midpoint. It's not in the middle.

Jeremy: Betweenness.

Episode 8: Day 3 of Two-Column Geometric Proofs

Mrs. Kelly began class by reviewing the concepts of adjacent angles and angle bisectors. She drew a diagram of adjacent angles and told the students how she purposively drew the common side off center to reinforce the concept of betweenness for angles. She reminded the students that the algebra of angle addition could be used with adjacent angles but that an *if..., then...* statement would be required. Mrs. Kelly also told students that a statement about adjacent angles would typically come from the diagram. Implicitly, Mrs. Kelly was showing her students how to identify hidden assumptions in a geometric diagram. However, she did not explicitly state that one of the purposes of geometric proof was to reveal hidden assumptions. Mrs. Kelly changed the diagram to reflect an angle bisector and stated that information about bisectors must be given along with the diagram before concluding that the line segment was an angle bisector. Mrs. Kelly also showed students a circular argument:

> So if you need a reason, make sure you start by saying, if BD is a bisector, and that would have to be given somewhere, or has to be in that information already. If I see this as a reason, I better see this already in your proof, not in the statement your making then. Now by the equation that you're putting in as reasons, I would expect you to fit it to whatever you're doing in that proof. You know, if you're using angle A and angle B, then you're going to say, the measure of angle A equals the measure of angle B. This is going to be the statement that you are creating at that moment. (Classroom Observation, 3/19/03)

Next, Mrs. Kelly presented a theorem about angle bisectors that was similar to the theorem about line segment bisectors. Prior to the proof, she referred briefly to systematization:

> We have a theorem that we can do. And, we're also going to prove that theorem. And you won't need to prove it again and you'll be able to use it.

This was the first time Mrs. Kelly discussed the notion of proving theorems prior to using them as newly created knowledge in a proof. Then, Mrs. Kelly reminded the students to look at the diagram as part of the given information. After proving the theorem Mrs. Kelly reiterated the notion of systematization:

> So now, when you're doing your proofs, if you are told you have a bisector, you have two choices. You could get this equation from the definition, or you could get these two equations because now we've proved that theorem true. (Classroom Observation, 3/19/03)

Episode 9: Day 4 of Two-Column Geometry Proofs

Mrs. Kelly began instruction by explaining the concept of linear pairs. She presented the definition as another tool for developing proofs that would help students to "change geometry into algebra." Mrs. Kelly presented the next theorem, which stated that two angles supplementary to the same angle (or to congruent angles) are congruent to each other, along with her definition of a theorem:

> We have a theorem. A theorem is a statement that must be proven in order to be used. We're not going to prove this one, but we have proved several, which is a very important part of an axiomatic system, from "a Euclid." (Classroom Observation, 3/24/03)

Mrs. Kelly felt the need to give a definition of a theorem because as she collected responses from the third journal question prior to instruction, she noticed that most of the students were unable to explain an axiomatic system. She chuckled as she finished her definition of a theorem because she remembered Ken's response to the journal question which was that an axiomatic system was made from "a Euclid." Nevertheless, the notion of proving a theorem before it could be used in a proof was presented, but not fully explained at this time.

At this point, Mrs. Kelly presented two algebraic situations that represented one of the hypotheses of the theorem and the conclusion that could be made by applying the theorem. For example, Mrs. Kelly wrote the theorem using the first hypothesis, that two angles are supplementary to the same angle:

$$\begin{array}{l} m\angle 1 + m\angle 2 = 180 \\ \underline{m\angle 3 + m\angle 2 = 180} \\ m\angle 1 = m\angle 3 \end{array}$$

Mrs. Kelly chose to show the situations without an accompanying diagram. She also explained that time could be saved in a proof if students could remember the theorem, but that memorization was not necessary if students could figure out the algebra involved. Once again, efficiency did not appear to be a necessity in proof writing.

Next, Mrs. Kelly led a discussion about perpendicular lines in order to introduce another theorem. One student seemed to have difficulty expressing the concept of perpendicular lines while the other student appeared to have exhibited the lowest van Hiele level thinking. Throughout this discussion Mrs. Kelly drew three diagrams of perpendicular and non-perpendicular lines as shown in Figure 10 (Classroom Observation, 3/24/03):

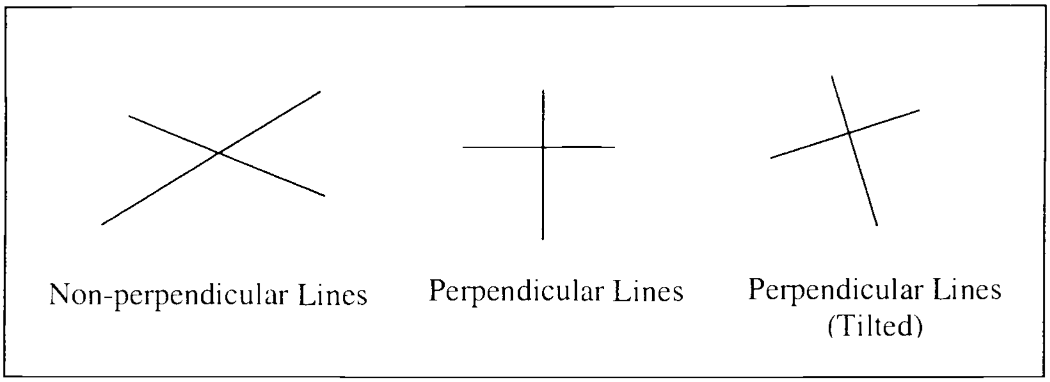


Figure 10. Mrs. Kelly's drawing of non-perpendicular and perpendicular lines.

Mrs. Kelly: Okay. Perpendicular lines. Cathy, what do you remember about perpendicular lines? What makes two lines perpendicular and other lines not?

Cathy: (no response)

Mrs. Kelly: Want me to draw you a picture? (Mrs. Kelly drew a pair of non-perpendicular lines). You tell me whether you think they are perpendicular (see Figure 10). Nice straight line with my beautiful artistic skills that I never developed, and another straight line. Perpendicular, Cathy?

Cathy: No.

Mrs. Kelly: No, okay, well you seem pretty confident so you must know.

Cathy: It's where there's a horizontal line and a vertical line.

Mrs. Kelly: I would agree with you part way. (Mrs. Kelly drew intersecting horizontal and perpendicular lines). A horizontal line and a vertical line are perpendicular (see Figure 10). However, if I were to take that picture and in my mind just turn it a little bit, could those two lines be perpendicular (Mrs. Kelly drew another set of perpendicular lines, titling them, as shown in Figure 10)?

Cathy: No response

Mrs. Kelly: What's special about it? What's the piece she's missing?

Seth: You're not supposed to have that bottom part hanging down?

Mrs. Kelly: Nope. But you're getting that from the symbol of perpendicular. It looks like an upside down capital T. What's the key? What are these two sets of lines doing that this set is not?

Mackenzie: Angles are 90 degrees.

Mrs. Kelly then led a discussion about other theorems involving complementary and supplementary angles as more algebraic tools. She also presented two intersecting lines that did not appear perpendicular to each other. Jeremy was asked to make a judgment regarding the perpendicularity of the diagram, and then others were prompted to explain the relationship between two marked vertical angles. Implicitly, Mrs. Kelly pointed out potential hidden assumptions about vertical angles from a diagram:

> So you can use this reason. This is actually a theorem, which just means that somebody worked very, very hard during their life to prove this to be true. But you can say that if two angles are vertical, and that will be

> something I could see in the diagram, it won't be listed as a statement in your proof, but I will see it in the picture, then they are equal. (Classroom Observation, 3/24/03)

In the next portion of the lesson, Mrs. Kelly reviewed theorems established last semester that could be used to prove congruent triangles without measurement. These theorems include showing that three corresponding sides of two triangles are congruent (SSS), two corresponding sides and the included angle of two triangles are congruent (SAS), two corresponding angles and the included side (ASA), two corresponding angles and a side not included (AAS) and corresponding hypotenuses and leg (HL). Mrs. Kelly referred to these theorems as "shortcuts," meaning that not all sides and angles of two triangles needed to be found congruent in order for two triangles to be congruent. Then, a proof exercise from page 121 of the textbook, which used one of the theorems, was presented (see Figure 11).

Given: $\angle A \cong \angle C$
X is a midpoint of $\overline{AC}$

Prove: $\Delta ABX \cong \Delta CDX$

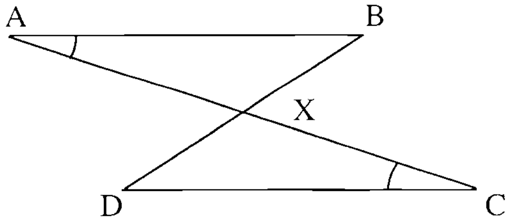


Figure 11. Congruent triangles problem. From *Houghton Mifflin, Unified Mathematics, Book 2* by Gerald R. Rising, John A. Graham, William T. Bailey, Alice M. King, and Stephen I. Brown. Copyright © 1991, 1989, 1985, 1982 by Houghton Mifflin Company. All rights reserved. Reprinted by permission of McDougal Littell, a division of Houghton Mifflin Company.

In working through the proof, Ken was not able to correctly identify a hidden assumption in the diagram (Classroom Observation, 3/24/03):

Mrs. Kelly: When I look at the picture, Ken, are there any angles or sides that I automatically see congruent, besides the stuff I had over here, (pause) any new ones.

Ken: Angle B and D

Mrs. Kelly: Angle B and D? How do I know they are congruent by looking at the picture? I don't think they do, but they're probably going to end up congruent, but I think I can see it. What can I see just by looking at the picture? Think of some of those theorems that we just did.

Ken: No, yes, (pause) I don't know.

Mrs. Kelly: Yes, no, what did Theorem 3-12 say?

Ken: If two angles are vertical, they are congruent.

Mrs. Kelly: Do you have any vertical angles there?

Ken: Yes.

After Mrs. Kelly decided that no other information could be taken from the diagram, students were directed to look at the given information.

Episode 10: Day 5 of Two-Column Geometric Proofs

At the beginning of class, Mrs. Kelly offered to work through one of the problems chosen by the students. She encouraged students to look at the diagram first, to identify the hidden assumptions. It seemed that an examination of a diagram was no longer encouraged to establish believability unlike coordinate geometry proofs. After creating a few algebraic sentences, students were encouraged to use common English if they could not remember the theorems. Yet, Mrs. Kelly was relating proof as explanation stating, "And, when you are stuck for a reason, just explain. Use your English language." Next, the students worked on the following proof from a worksheet in small groups (see Figure 12). Marty, Ken, and Jeremy started the proof by correctly establishing parallel line segments from the given. The students knew that they had to apply one of the "shortcuts," so they quickly dropped the given information and decided to pursue congruent sides.

Given: $\overline{AB} \parallel \overline{CD}, \overline{AC} \parallel \overline{BD}$
Prove: $\Delta ABC \cong \Delta DCB$

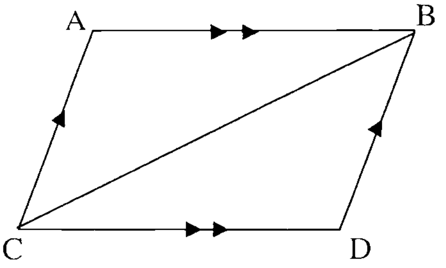


Figure 12. Homework problem of proof involving congruent triangles. From *Houghton Mifflin Unified Mathematics, Book 2* by Gerald R. Rising, John A. Graham, William T. Bailey, Alice M. King, and Stephen I. Brown. Copyright © 1991, 1989, 1985, 1982, by Houghton Mifflin Company. All rights reserved. Reprinted by permission of McDougal Littell, a division of Houghton Mifflin Company.

In this passage, Marty seemed persistent in stating that BC was an angle bisector without deducing it first (Classroom Observation, 3/25/03):

Marty: AB is congruent to DC. AC is congruent to BD.

Ken: Yeah, but how do you do all of that?

Marty: Huh?

Ken: How would you prove it then?

Marty: It's an angle bisector. BC is an angle bisector.

Ken: Okay.

Marty: Can we write that though?

Ken: Oh, you want a reason for it?

Marty: Because it's there!

Ken: Huh?

Marty: It's given. Look at it.

Ken: You just can't write that though.

Marty: Why?

Ken: Because, you can't just write given.

Marty: Giving us…

Ken: I don't know.

Students were reminded to be careful about assuming information from the diagram regarding midpoints and bisectors (Classroom Observation, 3/25/03):

Researcher: You've got the given, all right. What's the next thing? What did she say to do? What does she [Mrs. Kelly] usually do?

Ken: Look at the picture?

Researcher: Look at the picture, right!

Marty: Yeah.

Researcher: Is there anything you can get from the picture?

Marty: I put angle bisector.

Researcher: Um. Does it say it's a bisector?

Marty: No.

Researcher: Okay. You've got to be careful with things like midpoints and bisectors if they don't actually tell you. Because you can't tell angle bisectors from the picture.

Later, Mrs. Kelly tried to direct the students to use the diagram in a different way, to make a judgment about the shape of the diagram (Classroom Observation, 3/25/03):

Mrs. Kelly: So anytime you read parallel, you should start looking for alternate interior or corresponding angles. The other idea is since they gave you two pairs, what kind of quadrilateral is that? (pause) Two pairs of parallel sides. What's it look like?

Marty: I don't know. Square?

Mrs. Kelly: I'd say that's longer than that. (pause) Parallel sides. Par-a-llel...

Marty: Parallelogram.

Once the students voiced that the figure had the shape of a parallelogram, the students argued about the reason why it was a parallelogram (Classroom Observation, 3/25/03):

Ken: How would we write it's a parallelogram?

Marty: You can write that and say that it's given.

Ken: We can't write that it's given.

Marty: Why not??

Ken: Because it says that's the only thing that's given.

Marty: So.

Ken: We could write everything is given but that would be wrong.

Marty: If you say given, you can just look at the picture.

Ken: We could write parallelogram. Look at the picture. (laughter)

Marty: I don't know how to do these. I have about 40 theorems over here and I don't even know which ones to use.

Ken: Oh, it's because these two. Yup, that's why.

Marty: Alternate interior angles?

Ken: No, because this is a triangle.

Marty: Parallel.

Mrs. Kelly: So you know it's a parallelogram because you have two pairs of parallel sides, because that's what they gave you.

Marty: So you can write given?

Ken: No. You got to write two parallel sides.

The students were finally able to establish that the figure was a parallelogram, and then turned their attention to opposite angles and opposite sides of a parallelogram (Classroom Observation, 3/25/03):

Ken: Opposite angles are congruent. So angle A and angle D are congruent.

Marty: ACD is congruent to CAD.

Ken: No. CBD. No, and that's not right either.

Marty: Why not?

Ken: ABCD. It's an exact line. ABDC. Something.

Marty: Angle CDB is congruent to BAD. No wait. ACD is congruent to ABD.

Ken: Okay.

Marty: Now how are you going to reason that.

Ken: Because, in a parallelogram, opposite angles are congruent.

Marty: (whispers) Opposite sides are congruent too, Ken!

Ken: AD is congruent to C… No, we don't got, we don't got AC congruent to BD. Or AB to CD.

Marty: What?

Ken: Look. AC is congruent to BD, AB congruent to CD and then put for the reason, in a parallelogram, opposite sides are congruent.

Finally, the students decided to prove the two triangles using SAS, but struggled to find the included angle.

Episode 11: Day 6 of Two-Column Geometric Proofs

At the beginning of this class, Mrs. Kelly posed a problem (see Figure 13) that required students to determine hidden assumptions. As Mrs. Kelly probed individual students, the only difficulty exposed was the concept of bisection. One student thought that it meant BD and AE were congruent and another student thought that AE bisected BD. At the end of the proof, Mrs. Kelly reminded students that they would not be told vertical angles are congruent, but that they could get that information from the diagram.

Is there enough information to determine whether the two triangles are congruent? You can take the information from the diagram, see the two green arcs plus I told you that BD bisects AE and anything else you know.

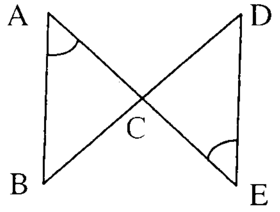


Figure 13. Diagram used to examine hidden assumptions.

Next, students were asked to prove two line segments of two triangles congruent in a completely different problem. This type of problem required students to show two triangles congruent first followed by the application of a theorem about corresponding parts of congruent triangles. Metaphorically, Mrs. Kelly referred to corresponding parts as "stealing parts" rather than the application of another theorem. While constructing the proof, Mrs. Kelly reminded students to use their diagrams in deciding the appropriate triangle congruency postulate. The correct postulate was ASA even though, chronologically, the proof was constructed by showing a pair of congruent angles, followed by another pair of congruent angles, and then a pair of congruent sides using the reflexive property. Since Mrs. Kelly encouraged student to write in the margin on the proof "A" and "S" whenever pair was found to be congruent, she was concerned that students would use the initials in the margin as the postulate.

Finally, students discussed a third problem in groups. In reviewing the problem with the students before they began, Mrs. Kelly judged the picture by stating, "Okay, this time our diagram starts with one big triangle that's split possibly down the middle, we don't know if it's down the middle," implicitly reminding students that it was acceptable to make judgments from diagrams. Mrs. Kelly also reminded students to look for given information from within the diagram. Marty, still confused about given information, raised his hand and asked. "So, if it's in the picture, then it's given?" Mrs. Kelly attempted to clarify his thinking, but Marty still appeared to struggle with this concept. After students worked on the proof for 10 minutes, Mrs.

Kelly showed a proof using HL (Hypotenuse and Leg) postulate. However, while circulating. Mrs. Kelly noticed that Seth used a much simpler method by showing that the big triangle was isosceles and thus the base angles would be congruent.

Summary of the Classroom Observations

The classroom observations revealed that several of the purposes of proofs as defined by de Villiers (1999) were not fully realized during instruction. In most instances, Mrs. Kelly presented *aspects* of the five purposes of proof without explicitly defining the purpose of proof itself. For example, several standards of communication were presented throughout the two units. Students were reminded throughout the coordinate geometry unit of three communication standards: clarity, length, and the audience. Mrs. Kelly insisted that students use conditional statements as reasons for geometric proofs. While these communication standards were reinforced in the two units, Mrs. Kelly did not state that proof is used to communicate ideas within a mathematical community.

Explanation was equated with the summary portion of a coordinate geometry proof as well as the *Reasons* column of a geometric proof. Similarly, verification was equated with the use of the three formulas in coordinate geometry as well as the *Statements* column in geometric proofs. Conviction through intuition or the judgment of diagrams was encouraged mostly during the coordinate geometry proof unit. The visual judgment of figures in the Euclidean geometry unit was often overshadowed by the examination of hidden assumptions. While Mrs. Kelly often modeled the use of visual judgment, students were also warned about assuming too much as with diagrams that appear to contain bisectors and midpoints.

In terms of systematization, students were shown how to identify assumptions and hidden assumptions, but it was never explicitly stated that one of the purposes of proof is to identify the assumptions of a situation. Two examples of circular arguments were mentioned. However, Mrs. Kelly never discussed that one of the purposes of geometric proof is to identify flaws in reasoning, such as assuming a statement attempted to be proven. Mrs. Kelly also mentioned that theorems needed to be proven to be used, but she did not relate this concept directly to systematization.

Aspects of discovery were stifled by the nature of the activities that were provided to the students. (Students were told that all of the statements given to them would be true.) Mrs. Kelly discussed the discovery aspect of geometric proofs by reviewing that additional statements were true from the application of the corresponding parts theorem in a geometric proof.

The classroom observation revealed that various levels of attention were given to aspects of five purposes of proof throughout the two units. The most consistently discussed purpose of proof was communication. Explanation and verification were sporadic, yet were attended throughout the two units in some form. Systematization was not introduced until the unit on two-column geometric proofs. Discovery was virtually non-existent, appearing only during the last few days of the two-column geometric proofs. In terms of two-column geometric proofs, systematization was the most frequently mentioned purpose of proof. Frequency counts of each reference to the purposes of proof for each unit made by Mrs. Kelly are displayed in Table 2.

Table 2

Frequency of References to the Purposes of Proofs Involving Geometry

Purpose of Proof	Coordinate Proofs	Two-column Proofs
Explanation	4	2
Verification	2	9
Communication		
Communication to Others (Reader)	7	1
Standards of Communication	2	9
Systematization		
Exhaust Assumptions	0	6
Check for Inconsistencies	0	10
Use Proven Theorems	0	6
Discovery	0	2

CHAPTER V

RESULTS

The purpose of this study was to describe tenth grade students' views of the purposes of geometric proof in the context of learning geometric proofs. An examination of the instruction provided by the classroom teacher, recorded conversations of students as they worked in small groups, the curriculum, assessment tools have been used to provide the context for the students' views regarding the purposes of geometric proof. Responses from the Preconceptions Questionnaire were used to describe the participants' levels of justification, which included the use of counterexamples, conjecturing, and two of Balacheff's (1988) levels of justification. Eleven instructional episodes based on the observation of two consecutive units, one unit on coordinate geometry proofs and the other unit on Euclidean geometry proofs, were provided.

In this chapter, the students' views regarding the purposes of geometric proof are provided. The students' views were taken from recorded conversations of students, responses from the journal questions, responses from the Post-instruction Questionnaire, and selected transcripts from individual interviews. The results of each journal question and item on the Post-instruction Questionnaire are presented along with the responses from individual interviews. A summary of the students' views of the purpose of geometric proof follows.

Journal Question

Journal Question 1

For Journal Question 1, students were asked to describe the process of writing a coordinate geometry proof (see Appendix C). Two students, Eric and Sue, were absent on the day this journal question was given. Three codes were created to describe the students' responses to Journal Question 1: a) Proofs Involve Explanations, b) Proofs Involve Verification, and c) Proofs Involve Formulas.

The first code, *Proofs Involve Explanation*, was created to describe the responses of five students (Cathy, Kara, Mackenzie, Marty, and Penny) who mentioned the explanatory feature of coordinate geometry proofs. For example:

> You are given things you have to prove. You take formulas, like the distance, slope, and midpoint formulas and you plug in the numbers for the coordinates in. Then you write a summary using ∴ explaining how you came to your conclusion. (Kara, Journal Question 1)

The second code, *Proofs Involve Verification*, was created to describe the responses of six students (Brandi, Cathy, Ken, Marty, Nikki, and Penny) who mentioned the truth of statements. For example:

> You have to prove the statement true or false by using distance or slope. (Brandi, Journal Question 1)

All of the students wrote that formulas were involved in the process of writing a coordinate geometry proof. However, five students (Betty, Jeremy, Kim, Lori, and Seth) referred only to formulas. For example:

> You could figure the slope of lines by using (wrote slope formula) and you could figure the distance of lines by using (wrote distance formula) or you

> could figure the midpoint by using (wrote midpoint formula). (Jeremy, Journal Question 1)

These responses were coded as *Proofs Involve Formulas*, which was the third code assigned to the responses in Journal Question 1.

Journal Question 2

For Journal Question 2, students were asked to discuss the purpose for each part of a coordinate geometry proof (see Appendix C). Two students, Jeremy and Lori, were absent on the day the journal question was given. Five codes were created to describe the students' responses: a) Summary as Explanation, b) Summary as Understanding for Others, c) Algebra as Support of Summary, d) Algebra as Verification, and e) Proof as Answer.

The first code, *Summary as Explanation*, was used to describe responses that referred to the summary part of a coordinate geometry proof as the main vehicle for showing how a statement had been proven. There were six students (Betty, Eric, Kara, Marty, Penny, and Seth) who described the summary portion as explanation. For example:

> The algebra finds out if the proof is true and the summary is the explanation of the proof. (Betty, Journal Question 2)

Initially, any response that contained a remark about the explanatory feature of a coordinate geometry proof was given this first code. However, four of the students (Cathy, Ken, Kim, and Mackenzie) focused on the explanation for the benefit of others, such as a reader. These responses were re-coded to indicate a slightly different view of explanation. For example:

> It is necessary to do both parts because the algebra shows how the math works out and the summary explains your work in words so the reader understands. (Ken, Journal Question 2)

Hence, the second code, *Summary as Understanding*, was used to describe responses that indicated the students' concern for having others understand the explanation provided in the proof. The third code, *Algebra as Support of Summary*, was used to describe responses that contained statements indicating the algebra or work that supported the summary. For example:

> It's necessary to do both the math part and summary part because the math part supports your summary and what you are proving. (Sue, Journal Question 2)

In a way, this type of response could have been interpreted as "algebra as explanation." However, since none of these students (Brandi, Cathy, and Sue) wrote about the summary portion as an explanation, the code did not seem to reflect "algebra as explanation." The fourth code, *Algebra as Verification,* was used to describe the responses that referred to the use of algebra in proving statements true. Through informal conversations with students after the journal question was read and from the Preconceptions Questionnaire, it was determined that these students were using the word "prove" to mean "verify." Thus, responses using any form of the word "prove" was given this fourth code. Five students (Betty, Kara, Marty, Penny, and Sue) wrote about verification. The fifth code, *Proof as Answer*, was used to describe responses that focused on proof as simply being an answer to a question asked in class. For example:

> You need equations and you must know some stuff on the shapes you are doing so you can figure out what the question is asking. (Nikki, Journal Question 2)

Six students (Brandi, Eric, Ken, Kim, Marty, and Nikki) expressed this perspective of coordinate geometry proofs.

Journal Question 3

Journal Question 3, which asked students to explain the meaning of an axiomatic system for geometry, was given on the fourth day of the Euclidean Geometry Unit (see Appendix C). On this day, Lori was absent again along with Mackenzie. Since several students experienced difficulty with this journal question and were unable to respond, it was decided that students would be given another opportunity on the following day to respond. Codes were assigned as: a) Statements and Reasons, b) Theorems and Postulates, and c) Other.

The first code, *Statements and Reasons*, was used to describe responses of six students (Brandi, Kara, Kim, Nikki, Penny, and Seth) that referred to statements and reasons in a proof. For example:

> It's the statements and reason thing. Where you put a statement that's true and give a reason supporting it. (Brandi, Journal Question 3)

The second code, *Postulates and Theorems*, was used to describe the one response made by Kim who referred to postulates and theorems. For example:

> An axiomatic system uses theorems and postulates. You make statements and reasons. (Kim, Journal Question 3)

The third code, *Other*, was used to describe responses that indicated no understanding of the topic or responses that indicated uncertainty. The five remaining students stated that they did not know or that they could not remember the meaning of the phrase. Two other students, Ken and Jeremy, tried to relate Euclid with axiomatic systematization. For example, Ken wrote, "It is made from a Euclid," not realizing that Euclid was a person.

Journal Question 4

In Journal Question 4, students were asked to list as many ways as possible for getting information for a two-column geometry proof (see Appendix C). The responses were coded as: a) Use of Given Information, b) Use of Diagrams, c) Use of Theorems or Properties, d) Use of Statements, and e) Notes or Other Examples.

The first code, *Use of Given Information*, was used to describe responses from seven students (Brandi, Cathy, Ken, Lori, Nikki, Seth and Sue) who listed "the given" as a source of information. The second code, *Use of Diagrams*, was used to describe responses from six students (Betty, Brandi, Ken, Kim, Nikki, and Sue) who referred to the diagram or picture as a source of information. The third code, *Use of Theorems or Properties*, was used to describe the responses that indicated theorems or properties as a source of information in geometric proofs. With the exception of Marty and Ken, all of the other students listed theorems or properties as sources of information in geometric proofs. The fifth code, *Use of Notes or Other Examples*, was used to describe the responses from four students (Betty, Ken, Marty, and Sue) that referred to the textbook, notes or other examples. Because some of the students also listed properties or theorems as a source, it was possible that students who listed notes and textbooks were possibly referring to these as sources for the "process" of writing proofs through example rather than specific properties or theorems.

Journal Question 5

For Journal Question 5, students were asked to compare coordinate geometry proofs with geometric proofs (see Appendix C). The same set of codes was used for similarities and differences between the two types of proofs: a) Verification, b) Explanation, c) Diagrams, and d) Other. Seven students (Brandi, Betty, Cathy, Kara, Lori, Seth, and Sue) expressed the view that both types of proofs show statements true (verification). Seven students (Betty, Cathy, Kara, Ken, Kim, Seth, and Sue) expressed the view that both types of proofs show why a statement is true (explanation). While many students were not descriptive in telling why both types of proofs were explanatory, one student made the following comparison. For example:

> You have to prove something, like if it's a square. A summary for coordinate geometry proofs is like writing one big reason for two-column geometry proof. (Cathy, Journal Question 5)

Five students (Betty, Cathy, Kara, Seth, and Sue) referred to both explanation and verification, and four other students (Eric, Mackenzie, Marty, and Nikki) either did not think the two types of proofs were similar or did not respond.

Eight students (Brandi, Betty, Ken, Lori, Mackenzie, Marty, Nikki, and Seth) listed the diagram, which became the most common difference between the two types of proofs. Only two of these students (Nikki and Seth) seemed to indicate, at a naïve level, that the diagrams for coordinate geometry proofs were particular cases and that the diagrams for two-column proofs were general cases. For example:

> One uses a graph and the other is just a picture. One you create statements the other there are no statements. (Seth, Journal Question 5)

The responses from listing the differences between the two types of proofs also yielded views on the form of proofs. For example:

> In two-column geometry proofs you have to make statements and reasons but you don't in coordinate geometry proofs. In coordinate geometry proofs you have to make an overall statement but not in two-column geometry proofs. (Kim, Journal Question 5)

Post-instruction Questionnaire

Item 1(Post-instruction Questionnaire)

For Item 1, students were given a completed proof written by a fictitious student and asked to identify a) what was assumed, and b) what was not assumed in the proof (see Appendix B).

Responses for Item 1A were sorted into three categories: a) Proven Statements, b) Underlying Mechanism, c) Other. For the first category, *Proven Statements*, three students (Cathy, Eric, Ken, and Seth) stated the conclusion as the statement that was assumed in the proof. Cathy seemed to be the only student who identified a "given" statement as an assumption of the proof by stating, "They assumed that since ABC is isosceles, the legs are congruent." However, through an individual interview, Cathy clarified her meaning, and insisted that it was only the congruency of the legs that was assumed and not the isosceles triangle. For the second category, *Underlying Mechanism*, eight students (Betty, Brandi, Kara, Kim, Lori, Mackenzie, Penny, and

Sue) stated that the fictitious students assumed that the two triangles were congruent. These students identified a statement that was related to the underlying mechanism of the geometric situation. The third category, *Other*, consisted of three responses that indicated no assumptions were made (Nikki) or no response was given (Marty and Jeremy).

For Item 1B, students were asked to state all that was *not* assumed in the proof. Four students (Betty, Cathy, Ken, and Nikki) stated one or more of the given statements, which further substantiated that students believed that "assuming" in geometry meant "proving." Six students (Brandi, Kim, Mackenzie, Penny, Seth, and Sue) referred to the fact that $\angle$B and $\angle$C were not stated as being congruent. The remaining five students did not respond to Item 1B.

Other individual interviews that were conducted revealed a more detailed description of the students' understanding of assumptions in geometric proofs. Students were first asked to explain their interpretation of the question and then asked to identify specific statements in the proof that the fictitious students assumed *true*. Three students (Betty, Lori, and Seth) correctly believed that assuming a statement in geometry meant that the statement was true without proof. Betty described her definition in the following way:

Researcher: What do you think I meant by assumed?

Betty: Like they know that it's true, but they don't, like they just have to solve it? I don't know.

Researcher: Okay. What if I change the question and said, in one through six, what was assumed true?

Betty: The given statements?

Researcher: Okay. Do you know where the given comes from?

Betty: Just the properties of the givens. Because they're given with those, you can prove (pause) the other parts.

Five students (Eric, Kara, Kim, Sue, and Penny) characterized "assuming" as devising a plan in advance and then judging the truth of the final statement. Penny described the process of "assuming" as looking at the diagram, thinking about it, and then "just thinking what they just figured would be the answer after doing the proof." Similarly, Marty and Ken believed that making a guess from a diagram was equivalent to "assuming." However, their definitions did not extend to devising a plan. Another student, Nikki, contradicted her statements about mathematical assumptions. During her individual interview, Nikki first believed that making an assumption was thinking that a statement was true when it was really false. When asked whether Step 4 was assumed, Nikki correctly said that it was assumed. Next, Nikki was asked to evaluate the given statements and also compare the given and Step 4 (the hidden assumption) with Step 3 (a proven statement):

Researcher: Would you assume the given?

Nikki: No.

Researcher: What would be the difference between the given and Number 3? Is there a difference between those types of statements, those types of reasons?

Nikki: Yeah, because you are told that those are true. You are told that. And then the other ones you are like saying I think this is it because of the different theories and stuff.

Researcher: Okay. So, Number 4, is it more like a given or more like Number 3?

Nikki: It's more like Number 3.

Researcher: And why is that?

Nikki: Because it's a..., you're doing a property like. It's something that you are told happens but you don't really know it for sure.

Even though students were unable to verbalize the given statements as assumptions in a geometric situation, a majority of the students eventually categorized "the given" with Step 3, which was a hidden assumption.

<u>Item 2 (Post-instruction Questionnaire)</u>

For Item 2, students were asked to list as many purposes of geometric proofs as possible (see Appendix B). The responses were coded as: a) Explanation, b) Verification, c) Systematization, and d) Other. Only three out of fifteen students (Betty, Kim, and Brandi) listed explanation as a purpose for writing geometric proofs. Brandi was the only students who mentioned explicitly explanation *and* verification: For example:

> To show what you think and why you think that it's true. To prove a question and give an answer. (Brandi, Item 2, Post-instruction Questionnaire)

Nikki and Betty were the only students who referred explicitly to the truth of statements only, which indicated verification as a purpose of two-column geometric proofs. For example:

> To state if theorems are true and other mathematical equations.
> (Nikki, Item 2, Post-instruction Questionnaire)

However, from individual interviews, it was determined that four other students (Kara, Ken, Penny, and Sue) also meant verification when they used the word "prove" in their responses.

In naïve ways, two students mentioned systematization, but in a less global sense than described by de Villiers (1999). In Cathy's individual interview, the notion of organization was discussed:

> Researcher: Let's look at the second one. Two column proofs serve many different purposes. List as many as you can. You stated, "Organized." What do you mean by that?
>
> Cathy: You have a list and you say, "Okay, I have to have this before I can say I have an isosceles triangle before I can say that the legs are congruent."
>
> Researcher: Right.
>
> Cathy: Like it goes kind of in order. It's organized and you get to see where (pause) I don't know.
>
> Researcher: Where it goes?
>
> Cathy: Yeah.

Kim also mentioned systematization:

> They tell you what to do and how to do it. They tell you what order to do problems in. They make you think about some things that your mind just normally skips over. (Kim, Item 2, Post-instruction Questionnaire)

During the individual interview, Kim explained why one should not skip over steps:

> Researcher: Why can't you just skip over things?
>
> Kim: It makes you think about how your mind works. You might need some information later in the proof.

Kim's response was interesting because, even though Mrs. Kelly mentioned in class that steps should not be skipped, she did not announce to the class any particular reason. However, during an informal interview, Mrs. Kelly mentioned that she had warned some students about making "half-circular" arguments on homework problems. Mrs. Kelly described "half-circular" arguments as proofs that contained reasons that have not yet been derived from the situation. For example, on a homework problem, Ken stated that P was the midpoint of CD. For the reason, he wrote; "If P is the midpoint of CD, then the measure of CP is the same as PD." Thus, he was assuming that which he was stating was true. Thus, it was possible that Mrs. Kelly spoke with Kim about skipping steps through feedback on homework.

Responses that were coded as *Other*, were diverse. Two students (Eric and Marty) listed a few properties and theorems, while Penny stated that proving in geometry is the same as proving in a court of law. During the individual interview, Penny could not elaborate on the similarities and differences between proving in geometry and proving in a court of law. Another isolated response was from Seth who indicated his concern with proof as a tool for assessment in the classroom by stating, "They test how well you listen in class and how well you can assess the problem to figure it out." Jeremy was the only students who did not respond to this item.

Since only three students had commented on the explanatory feature of two-column geometric proofs in Item 3, it had appeared that students were less likely to view two-column proofs as explanations than coordinate geometry proofs. However, further investigation was conducted through individual interviews. During individual interviews, eight students who did not list explanation as a purpose of two-column proofs were asked to describe the purpose of the *Statements* column and *Reasons* column. This alternative form of questioning was similar to Journal Question 2, which asked students to consider the purposes of the parts of a proof. Five students (Jeremy, Kara, Nikki, Seth, and Sue) mentioned the explanatory nature of the *Reasons* column. Two other students (Ken and Penny) referred to the *Reasons* column as the statements that "back up" the proof, which also seemed to indicate explanation. However, Cathy explicitly stated that the *Reasons* column was used to prove statements true, meaning that the reasons supplied in a proof helped to establish the validity of the statement.

Following the discussion on the purposes of the two columns during individual interviews, some of the students were asked to re-evaluate the argument in the Cardboard Triangle Activity problem on the Preconceptions Questionnaire. When asked whether the fictitious teacher's proof explained why there were 180 degrees in a triangle, four students (Eric, Jeremy, Marty, and Betty) maintained that it was a good explanation. However, the remaining three students (Cathy, Ken, and Lori) believed that the argument presented by the fictitious teacher did not explain why the angles in a triangle added up to 180 degrees. Ken was not sure why he thought the argument was not a good explanation while Cathy and Lori thought the fictitious teacher should have offered more mathematical explanations:

> Researcher: Now, everything you know about this word [proved] now, after having done two column proofs and you are going to re-evaluate this situation, would you stay with what you wrote or would you switch it?

Cathy: I would switch it because it's not really proven. When you look at that, it may look like 180 degrees but it might not be exactly. Like, it's not proven.

Researcher: So why doesn't it meet the qualifications for a proof?

Cathy: Because, um, (pause). It doesn't have any statements or reasons?

Researcher: It doesn't have any statements or reason? Okay, well, I could argue that it does have reasons because the reason was that she put them in a straight line. Is that a reason?

Cathy: Yeah, (pause). Basically she's just drawing a picture and playing with it. There, it's a 180 degrees but it (pause) I don't how to (pause)

Researcher: You don't know how to word it?

Cathy: Yeah! (expressing a sigh of relief)

Researcher: When you originally did this, what did you think that word meant? In what sense were you using the word proof? (pause) Can you think back and think why you said yes?

Cathy: Because, um, she drew a picture and, or not drew a picture, but did that, and um, she was like showing how it would be, but she didn't really do it, (pause) mathematically.

Since Cathy seemed frustrated by not being able to put into words what she was thinking, she was not asked to explain further what she meant by "mathematically." Lori also used the phrase "mathematical" in describing explanations in proofs. Lori, who did not seemed distressed during this part of the interview, was asked to explain what she meant by "mathematical," but she said that she could not explain it further.

Item 3 (Post-instruction Questionnaire)

On Item 3, students were asked to evaluate a statement made by two fictitious students about the value of developing a geometric proof (see Appendix B). The responses were sorted into four categories: a) Shows Understanding, b) Lack of Conviction, c) Discovery, and d) No Proof Needed. In the first category, *Shows Understanding*, three students (Cathy, Lori, and Kim) believed that a better understanding about the situation is provided when a proof has been developed. For example:

> Disagree. Because with developing a proof, I seem to understand more about exactly why something is what it is. Meaning why a triangle is a right triangle. (Cathy, Item 3, Post-instruction Questionnaire)

In the second category, *Lack of Conviction*, seven students (Brandi, Kara, Ken, Marty, Nikki, Penny, and Sue) remarked about the role of a diagram in geometry classes. These students believed that diagrams may be misleading and that a proof is the only way to establish the validity of the statement. For example,

> Disagree because even though something may look a certain way it might not be, so you have to prove it. You can't just look at it. (Penny, Item 3, Post-instruction Questionnaire)

In the third category, *Discovery*, one student, Eric, hinted about the benefit of discovery, having just been exposed to this concept through corresponding parts of congruent figures congruent. In the fourth category, *No Proof Needed*, two students agreed with the fictitious students, yet offered different reasons. Seth re-iterated his concern about the value of proofs as an assessment tool for the teacher while Jeremy expressed his view that proofs had not been used to discover any interesting or important knowledge about geometry.

Item 4 (Post-instruction Questionnaire)

For Item 4, students were asked about intuition in proving geometric statements (see Appendix B). Four students (Lori, Seth, Mackenzie, and Nikki) were sure that intuition was not used in proving statements in geometry:

> No, because the lines may look like they're the same but you have to prove it. Nothing is proven from intuition. (Seth, Item 4, Post-instruction Questionnaire)

To make sure that students considered the whole process of proving, these students were asked to consider the use of intuition from beginning to end. However, all three students stood by their original answer:

> Researcher: Okay. Do you think that proving a statement in geometry involves intuition? Why or why not? And you said, "No because you're proving it you don't need a hunch." How about anywhere in a proof? From when you first read a proof problem to the end.
>
> Nikki: No. Not really.
>
> Researcher: So, how do you do it?
>
> Nikki: I just use the math equations. What we were given.
>
> Researcher: Okay. When you see a 'prove' statement, do you see it as it is true and I just need to figure out how to get there, or it may be true or it may be false?
>
> Nikki: It may be true or it may be false.
>
> Researcher: Oh, okay. How do you do something like that if you don't know if it is false?

Nikki: The equations that we have to use don't work out.

Researcher: Okay. Equations like midpoint?

Nikki: No. The equations we used like (points to the statements in the first proof on the questionnaire).

Researcher: Oh, you mean the statements. So if they don't work out, then you know it's not going to work.

Nikki: Right.

Lori also seemed sure that intuition did not play a role in proving statements in geometry:

Researcher: Okay. Do you think that proving a statement in geometry involves intuition? And you said "No, because there's a reason behind everything that you've done. You can't just say that it is because you feel that it is." What is your definition of intuition?

Lori: I don't know. I just thought it was like something that you suspect something. Like your morals or something?

Researcher: Well, let's go with suspecting something. How about anywhere in a proof? Maybe even before you actually start writing a proof? Do you think you use intuition?

Lori: (shakes her head no)

Researcher: Why?

Lori: No. Because if you do, you'll just get it marked wrong.

Researcher: Okay.

Lori: You have to prove it.

Researcher: Even when you're thinking about it?

Lori: Well, you can. But, it really doesn't matter because you need proof.

Seth also believed that intuition should not be used in proving statements in geometry. During his interview, Seth denied using intuition in proofs altogether. His main contention with using intuition was the fact that using intuition may sometimes lead to the incorrect answer. Thus, Seth believed that there was an equal likelihood of a Type I and Type II error when attempting to use his judgment about the validity of geometric statements and diagrams.

Researcher: Do you think proving a statement in geometry involves intuition? Why or why not? And you said no because the lines may look like they are the same, you have to prove it. Nothing is proven from intuition. Well, what is your definition of intuition?

Seth: That you can feel that it is right.

Researcher: Do you ever have that feeling when you are doing a proof?

Seth: No.

Researcher: So when you look at a proof and it says 'given' and 'prove', did you think that it might not be true, the prove statement?

Seth: No.

Researcher: Did you think they were always true?

Seth: No, but that's not intuition. They're either true or they're not true. You don't get feelings about whether you think it's right or not. Because you can either look at it and it looks right but it could still be wrong. See? Or, it could look wrong and actually be right. Because lines mean absolutely nothing, it's just whether or not it's been proven.

Researcher: Okay. Do you think anyone uses intuition?

Seth: I'm sure people do, but (pause)

Researcher: But they shouldn't?

Seth: They're probably not always right.

Two students, Ken and Penny, expressed uncertainty about using intuition in proof writing, but made sure to explain that intuition did not prove anything:

> Maybe, not sure, probably not because you have to prove it either way. (Ken, Item 4, Post-instruction Questionnaire)

> Kind of. But you have to actually know how to prove it. You can't always write something in a proof just because you think it's right, you have to know it. (Penny, Item 4, Post-instruction Questionnaire)

Nine of the remaining students believed that intuition is used in writing geometry proofs. However, their descriptions of how intuition is used in the proving process indicated two different processes: judgment and "preformal" proving (Blum & Kirsh, 1991). Blum and Kirsh defined preformal proving as "a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises" (p. 187). Five of these students (Betty, Brandi, Cathy, Kara, and Marty) specifically mentioned that judgment of a diagram played a role in proving geometry. For example:

> Yes, because you have to look at the picture and have a feeling about something that would help you to solve the problem. Then you can get to where you need to be. (Kim, Item 4, Post-instruction Questionnaire)

When probed during individual interviews, students specifically mentioned judgment of diagrams in using intuition:

Researcher: Okay, can you give me an example of how you use intuition in a proof? Do you remember, thinking back on a certain proof, could you elaborate on that and say, yeah, I used intuition because I did such and such?

Cathy: Um. If I looked at two angles and I said well they must be congruent and then I just got to the point where I could prove they were congruent. And I just kind of followed my intuition by saying in my head that they were congruent and then I went through and proved it. Like I got up to the point where I could prove that they were congruent.

Researcher: Okay. So you basically used it in the middle of the proof. Have you ever used it in the beginning of a proof?

Cathy: Yeah. I think I used it all the way through. Like when I first look at something that has got to be proven, I say well okay, those two triangles, they must be congruent and then find out why.

At the end this discussion, Cathy seemed to include preformal proving as part of her description of intuition. Four other students (Betty, Brandi, Kara, and Marty) also seemed to indicate a preformal proving process. For example:

> Yes because you need to have some idea of what you have to do in order to complete the task of solving the proof. (Betty, Item 4, Post-instruction Questionnaire)

Brandi also seemed to indicate using a mixture of visual judgment along with preformal proving. For example, Brandi also stated, "If you think something is equal in length. You work on that. You use it throughout the proof."

Item 5 (Post-instruction Questionnaire)

For Item 5, students were asked to write about proof as a means of communication (see Appendix B). The responses formed three categories: a) Demonstration of Ideas, b) Exchange of Ideas, and c) Other.

For first category, *Demonstration of Ideas*, eight students (Cathy, Jeremy, Ken, Kim, Marty, Penny, Seth, and Sue) seemed to view proof as a demonstration of knowledge about geometry. For example:

> You're showing what you're thinking and why you're thinking it so when people look at them they know what was going through your mind when you were trying to solve it. (Kim, Item 5, Post-instruction Questionnaire)

For the second category, *Exchange of Ideas*, four students (Betty, Brandi, Kara, and Nikki) seemed to view proof as an exchange of ideas between people. For example:

> You have to talk and inquire if it's all of what you think or part of it. So you have to get what other people are thinking to get the full statement. (Brandi, Item 5, Post-instruction Questionnaire)

The group work that occurred during several of the classes throughout the two units seemed to instill in these students that proof writing was an activity that consisted of "helping" rather than "arguing." This view ran somewhat contradictory to the communication that was observed from the group work produced by the Ken, Marty, and Jeremy, but consistent with the observations of Betty, Lori, and Sue. When working in small groups, Ken, Marty, and Jeremy argued frequently about the conclusions that could be drawn as well as the meanings of the symbols used in the diagrams. For the third category, *Other*, Lori and Mackenzie simply stated that proofs show (and explain) how statements are true. Eric did not respond.

Item 6 (Post-instruction Questionnaire)

For Item 6, students were asked to describe how proof is used for discovery (see Appendix B). Nine of the students described discovery in terms of new knowledge and were generally thinking about the "prove" statement (conjecture). For example:

> Different theorems are results of someone's life work to prove something. Everything you do in math could form a new result. (Lori, Item 6, Post-instruction Questionnaire)

Four of the students also mentioned the discovery of new methods of proof. For example:

> You could find a new way in discovering steps for geometry proofs. (Marty, Item 6, Post-instruction Questionnaire)

Two other students did not provide meaningful answers. Seth simply stated that statements are true when they are proven, and Brandi said that she did not know how to answer the question. Since students mainly used the triangle congruency method in constructing proofs, it was not unusual to find that none of the students mentioned discovery in the context of discovering the underlying mechanism of a geometry situation. Three students (Cathy, Seth, and Nikki) were asked during their interview whether they believed they had discovered anything by constructing geometry proofs. All three students said that they did not feel like they discovered anything during the two units on proof. Seth's reason was that one could not discover what the teacher already knew. He seemed to be thinking in terms of public rather than private discovery. Nikki felt that because the same underlying principle had been applied to all of the problems that were assigned, no discovery occurred. Nikki also thought that specific information ascertained from constructing proofs was not useful:

> Researcher: Do you feel like you are discovering anything when you are doing a proof?
>
> Nikki: Not really.
>
> Researcher: Why?

Nikki: Because usually they are things we have done over and over and over again. Usually just different things that we've done over and over again.

Researcher: The stuff that you have proven, do you feel that it is useful knowledge?

Nikki: I don't think so. Not really.

Researcher: There's nothing that's been, like, wow, that's pretty interesting to know about that because I might be able to use that somewhere else?

Nikki: Not really.

Summary of the Results

The results indicate three main findings. First, several students experienced difficulty in expressing their views of the purposes of geometric proof when asked directly. One-third of the students could only list properties or theorems they encountered during the unit on geometric proof. However, when these students were asked to describe the purpose for each column, all of the students listed both explanation and verification. Second, the students expressed limited views of the purposes of proof, referring mainly to verification. Only a few students mentioned explanation, systematization, and communication. However, students generally referred to at least two purposes of proof such as explanation and verification when describing the proving process involved in coordinate geometry. Third, the students' views of various purposes of geometric proof were diverse.

As expected, students typically equated the word "proof" with verification or the establishment of the validity of a statement. The equality between proof and verification existed prior to the units on proof as evidenced by the Cardboard Triangle Activity. Throughout the units on proof, students routinely interchanged proof and verification on journal questions as well, even after referring to other purposes of proof on journal questions. For a few students, personal conviction about the truth of statements could be obtained through intuition. Intuition was described as the judgment of the lengths of line segments or measures of angles, a process of constructing a preformal proof, or a guess about the validity of a statement. A few students also mentioned that an explanation of a proof could be used to establish personal conviction, even for situations suspected true.

Explanation was also expressed as a purpose of geometric proof, but only after students had been asked to describe the purposes for each column in a geometric proof. Almost one-third of the students seemed to value the explanatory nature of geometric proofs for situations that seemed obviously true. These students seemed to indicate a belief that understanding why something was true was just as important as knowing that it was true. Only three students changed the status of the argument described in the Cardboard Triangle Activity from proof. One student could not describe why he changed his mind while the other two students stated that explanation (the reason) was not "mathematical."

Generally, students only referred to proof as communication on their own when referring to coordinate geometry proofs in Journal Question 2. However, on

Post-instruction Questionnaire Item 2, one student expressed the view that one of the purposes of geometric proof is to communicate by means of a demonstration of his own understanding of the geometric situation. The students' views about proof as communication were explored through Post-instruction Questionnaire Item 5. While none of the students described the communication through proofs as a form of argumentation, four students referred to proof as an exchange of ideas. Eight students described the communication that occurred through proofs as a demonstration of knowledge.

An overwhelming majority of students did not view proof as a means of systematization. Only two out of fifteen students seemed to indicate naïve views about systematization of geometric proofs in terms of the organization of statements in a proof and skipping steps in a proof. None of the students were able to discuss systematization when given the opportunity in Journal Question 3. Only one student mentioned theorems and postulates, yet failed to describe how these concepts are related to the axiomatic system of geometry. For a majority of the students, the word "assumed" meant "proved." When students were asked to identify statements in a proof that were not assumed, almost all of the students stated the given information, which provided more evidence that the students thought that "assumed" meant "proved." Thus, students seemed to be unfamiliar with the term "assume" in spite of Mrs. Kelly's use of the term throughout the two units on proof. However, several of the students were able to categorize hidden assumptions as "given" statements.

Similarly, the students also did not seem to view proof as a means of discovery. The students' views of proof as a means of discovery generally extended only to the discovery of new methods in mathematics and new mathematical statements, not to systematization. Due to the structure of the homework exercises on geometric proofs, none of the students expressed the view that geometric proof is useful in discovering the underlying mechanism for a geometric situation.

In conclusion, the diversity of the students' views of the purposes of two-column geometric proofs included the following:

1. Showing that a statement was valid was regarded as a purpose of a geometric proof. (Verification)

2. Showing why a statement is true was regarded as a purpose of geometric proofs. (Explanation)

3. Geometric proof was viewed as a demonstration of personal understanding and as an exchange of ideas. (Communication)

4. Explanation was also viewed as lending conviction to the validity of the statements. (Explanation/Verification)

5. Intuition was viewed as the judgment of a diagram and preformal proving. (Verification)

6. Explanations in proofs were characterized as "mathematical" by two students. (Explanation/Systematization)

7. Discovery meant the creation of new statements, formulas, or methods rather than the underlying mechanism of a geometric situation. (Discovery)

8. Discovery of new statements was not extended to the process of systematization for use with other proofs. (Discovery/Systematization)

CHAPTER VI

DISCUSSION AND CONCLUSION

The goal of this investigation was to describe tenth grade students' views about the purposes of geometric proof within the context of their learning. Classroom observations, the curriculum, assessment tools, journal questions, and a preconceptions questionnaire were used to provide context for the views expressed by students from a single classroom. Eleven classroom episodes selected from the classroom observations were used to describe the instructional context as well as discourse among the students during group work. The episodes provided specific details about how and when the classroom teacher addressed various purposes of proofs involving geometry concepts throughout two instructional units on coordinate geometry proofs and geometric proofs. The episodes also consisted of student discourse relating to the purposes of geometric proof as students worked on assigned proof problems. The students' views were examined through journal questions given at the beginning of selected days and through the Post-instruction Questionnaire. Individual interviews were conducted to validate the coding and to provide a more in-depth description of the students' views. This chapter examines how the results obtained from this study contribute to existing knowledgebase on students' formal understanding of geometric proof, with its limitations considered. In addition, implications and future research questions on this topic will be addressed.

Discussion of the Main Findings

In terms of the students' views of the purposes of geometric proof, there were three main findings. First, several students experienced difficulty in expressing their views of the purposes of geometric proof when asked directly. One-third of the students could only list properties or theorems they encountered during the unit on geometric proof. However, when these students were asked to describe the purpose for each column, all of the students listed both explanation and verification. Second, the students expressed limited views of the purposes of proof, referring mainly to verification. Only a few students mentioned explanation, systematization, and communication. However, students generally referred to at least two purposes of proof (explanation, verification, and communication) when describing the proving process involved in coordinate geometry. Third, the students' views of various purposes of geometric proof were diverse. The diversity of the students' views of the purposes of two-column geometric proofs included the following:

1. Showing that a statement was valid was regarded as a purpose of a geometric proof. (Verification)

2. Showing why a statement is true was regarded as a purpose of geometric proofs. (Explanation)

3. Geometric proof was viewed as a demonstration of personal understanding and as an exchange of ideas. (Communication)

4. Explanation was also viewed as lending conviction to the validity of the statements. (Explanation/Verification)

5. Intuition was viewed as the judgment of a diagram and preformal proving. (Verification)

6. Explanations in proofs were characterized as "mathematical" by two students. (Explanation/Systematization)

7. Discovery meant the creation of new statements, formulas, or methods rather than the underlying mechanism of a geometric situation. (Discovery)

8. Discovery of new statements was not extended to the process of systematization for use with other proofs. (Discovery/Systematization)

These diverse views are discussed within the following categories: verification, explanation, communication, systematization, and discovery.

Verification

Verification is the process for establishing the validity, or truth, of a geometric statement. One aspect of the verification process is conviction in the validity of a statement established through experimentation or intuition. The judgment of the lengths of line segments prior to geometric proof is one manifestation of intuition (Burton, 1999; de Villiers, 1999; Hersh, 1993). A second aspect of the verification process is to establish the truth of a geometric statement through a presentation of a proof. In coordinate geometry, proofs generally consist of calculations and the knowledge of properties for various quadrilaterals. In Euclidean geometry, proofs consist of a chain of statements joined together through logic. Both aspects of the verification process were revealed during classroom observations and explored in this study.

In terms of verification as a process for establishing the validity of a statement, approximately one-half the students in the study reported verification as the main purpose of two-column geometric proof. This finding was not surprising considering that, even at the end of the study, several students still considered the Cardboard Triangle Activity to represent a proof because they believed this well-known theorem about triangles to be true. The truth of the statement was the only aspect several of the students seemed to be using to qualify the argument as a proof. Classroom observations also indicated that establishing the validity of a statement in geometry through proofs was a consistent theme supported by Mrs. Kelly throughout the two units. During instruction on coordinate geometry proofs, Mrs. Kelly established a direct relationship between the algebraic portion of a proof and verification. Similarly, the *Statements* column of a geometric proof was the designated "area" for showing all valid statements in a proof. In addition, responses from the journal questions indicated that the students' views mirrored the same relationships that were established through the instruction regarding this aspect.

The second aspect, conviction, appeared to divide the students. The investigation of the students' views on the relationship between conviction and verification was difficult because it was understood by most of the students that all statements they were attempting to prove were true. The classroom activities lacked "cognitive unity of theorem" meaning that no experimentation or conjecturing was required by the student (Garuti et. al., 1996). Thus, the investigation of the students' views regarding the relationship between gaining conviction and establishing validity was examined by asking students about using intuition in the proving process. Fischbein (1982) would have called this form of intuition "intuitive acceptability." Five students believed that intuition is used in the proving process, intuition was also defined as a hunch or feeling, but that the feeling or hunch was based on the judging or examining diagrams. This view of intuition, visual judgment, was encouraged by the classroom teacher throughout both units. Four students also expressed a view of intuition that seemed to indicate "preformal" proving. In preformal proving, the students seemed to indicate that the desire to find the reasons something could be true before they actually started to write the proof. For the students who believed that intuition is not used in the proving process, intuition was defined as a hunch, feeling, or guess without specific reference to visual judgment. These students also seemed to be heavily entrenched in the end product of the verification process, the establishment of the validity of a geometric statement by deductive proof. For these students, only analytical thinking was allowed in the mathematics classroom. These students seemed to reject anything that was not founded on fact or could not be proven.

Explanation

Explanations of a geometric proof reveal the underlying mechanism why a statement in geometry is true. The properties of geometric figures were the underlying mechanism for the coordinate geometry proofs while the congruent triangle theorems were the underlying mechanisms for two-column geometric proofs in this study. In explaining the process of constructing a coordinate geometry proof, five students described the process of proving as explaining. When asked to describe the purpose of the summary portion of the coordinate geometry proof, almost all of the students listed explanation. In terms of two-column geometric proofs, a majority of the students also viewed explanation as a purpose of two-column proof, but only when the alternative question was asked regarding the purpose of the *Reasons* column. This finding seemed consistent with the classroom observations, which showed that proof as explanation was a theme throughout both units. Seven students also referred to the explanatory feature of proofs when comparing coordinate and two-column geometric proofs.

After the units on proof, the Cardboard Triangle Activity problem was used to probe the students' views with regards to explanation. Of the seven students asked to remark on the argument presented in the problem, four stated that the argument was a "good" proof. Three students said that the argument was not a good proof, but were unable to discuss why the proof was inadequate. However, two of the three students characterized explanation in geometric proofs as being "mathematical." Both students struggled to define "mathematical" in this context, most likely because they were not able to see the hidden underlying mechanism of parallel lines in the problem.

Hersh (1993) once contended that explanation should be the main reason for constructing a proof because students typically believe statements in the classroom. In this study, four students defended the development of a proof in showing why a statement is true rather than in showing the validity of the statement.

Communication

Proof as a means of communication was introduced on the first day of the coordinate geometry unit. Four students referred to communication when writing about the purpose of the summary part of a coordinate geometry proof. Students either referred to an external reader or simply stated that coordinate geometry proofs are written so that others may understand it. This view was, in general, not expressed by the students when discussing the purpose of two-column proofs. Two factors may have contributed to the differences in views regarding communication between coordinate geometry proofs and two-column proofs. First, proof as a means of communication to others was consistently addressed by Mrs. Kelly during the coordinate geometry unit, yet mentioned only once at the end of the unit on two-column proofs. Second, coordinate geometry proofs were written in paragraph form while two-column proofs were written as steps.

In both units, Mrs. Kelly established standards of communication. The standards of communication for coordinate geometry proofs and two-column proofs were different. During coordinate geometry proofs, students were asked to consider the audience or the reader. Yet, consideration for the audience was not discussed during instruction on geometric proofs. Clarity was another standard of communication that was addressed during the unit on coordinate geometry proofs, but mentioned only once at the end of the unit on geometric proofs. The use of *if..., then* statements was the only other standard of communication that students were given during the Euclidean geometry unit. Even though students mentioned that *if..., then* statements were required, none of the students seemed to be concerned with knowing why they were required. However, in some instances, students seemed concerned about knowing when *if..., then* statements were required. Even though standards of communication were given, it was never stated to the students that mathematical proof is a form of communication as a transmission of knowledge with the mathematical community (NCTM, 2000).

When students were asked directly to explain how geometric proof is a form of communication, eight students seemed to view the communicative aspect of geometric proofs as a demonstration of their own understanding. Four students characterized the communicative aspect of geometric proof as an exchange of ideas between two people. Since students were allowed to work on proofs together and also received feedback on all of the proofs they wrote for homework, it was surprising that more students did not view communication in this manner. Transcripts of the students working in small groups indicated that argumentation did occur during group work in the form of discussions regarding the appropriate statements or reasons to be written in the proof. Thus, at a rudimentary level, some meanings were negotiated, but few students described communication as an exchange of ideas at this level. Since it was understood that only true statements would be given to the students for homework and assessments, it was expected that none of the students would characterize communication through proofs as a form of argumentation. In addition, the transition to the units on proofs revealed that students were given little reference to other types of justification they might have encountered in prior units. Thus, geometric proof was not explicitly stated as a form of justification, as expressed by NCTM (2000) and Wolfe (1970).

Systematization

Proof as a means of systematization refers to the process of recognizing assumptions in a geometric situation and using previously proven statements as reasons why a geometric statement is true. Assumptions may be drawn directly from given information or from hidden assumptions in a diagram, such as the congruency of vertical angles. The ultimate reward of systematization is the reflection of the components of a system to create new statements in a particular mathematical system, such as Euclidean geometry, or to create new systems altogether. In the van Hiele theory, systematization seemed to be out of the reach of most students' understanding and was later eliminated from the theory of geometric thinking.

In this study, systematization was introduced on the first day of the unit on two-column geometric proofs, after the completion of coordinate geometry proofs. In this spiral curriculum, systematization was first introduced in the latter part of Math A prior to the inception of this study. Due to time constraints in the curriculum, Mrs. Kelly only mentioned systematization. The concept of mathematical definitions was not reviewed during the unit on two-column geometric proofs, and only one statement was made during the introduction of proofs about axioms and postulates as assumptions. Mrs. Kelly mentioned only a few times that theorems had to be proven before they could be used as reasons in another proof, although some theorems were used without proof.

Mrs. Kelly used the word "assume" often during Euclidean geometry proofs, yet never defined its meaning to the students. Since systematization as a purpose of proof was not mentioned during classroom instruction, students were only asked about their understanding of mathematical assumptions. While most students were unable to find the assumptions in a completed proof, several students were able to categorize hidden assumptions found from a diagram with given statements. From individual interviews, it became apparent that the main reason for the students' lack of ability to identify assumptions in a completed proof was the students' alternative definition for the word "assumed" as "proved."

Due to the lack of attention to systematization during classroom instruction, it was not surprising that only one of the students mentioned systematization as a purpose of proof when asked to list the purposes of geometric proof. These results were similar to the results reported by Healy and Hoyles (2000) of tenth grade students in England and Wales. Healy and Hoyles speculated that the students' lack of understanding of systematization as a purpose of mathematical proofs was attributed, in part, to a reform mathematics curriculum, in which mathematical proof was taught as a goal across various mathematical concepts, not just during the study of geometry. The findings in the present study suggest instructional influences as well as curricular influences.

Only two students held naïve views of systematization by briefly mentioning the ideas of organization in a two-column geometric proof. One student briefly mentioned that steps should not be skipped while the other student focused on making logically correct connections within the reasons of a two-column geometric proof.

Discovery

Proof as a means for discovery received little attention during instruction on either type of proof. In fact, proof as a means of discovery was portrayed only in terms of corresponding parts of congruent triangles by the classroom teacher. Theorems that were proven in class and then applied to other proofs, such as theorems involving midpoints and bisectors, did not seem to be regarded as discoveries in the classroom. As expected, none of the students cited discovery as a purpose of geometric proof on Item 2 of the Post-instruction Questionnaire. These findings are consistent with the results reported by Healy and Hoyles (2000), in which only 1% of high school students in a study of reformed mathematics classroom in England and Wales specifically stated that one of the purposes of mathematical proof was to discover new results in mathematics.

When students were asked to remark on the discovery feature of geometric proofs on Item 6 of the Post-instruction questionnaire, students cited two main responses, the discovery of "new results" and the discovery of "new methods." When probed during individual interviews, none of the students seemed to distinguish between new results and new results that could be used to prove other statements in geometry. These findings do not seem to be much different from those reported by Schoenfeld (1989) regarding the behavior of high school seniors some 20 years earlier.

Limitations of the Study

While this study has provided a rich description of the events from a single high school mathematics classroom, the study has several limitations. Leading directly from the goal of this study, a qualitative method and design were utilized to investigate the views of tenth grade students from a single classroom on the purposes of geometric proof. The context provided within the teaching episodes, curriculum, assessment and preconceptions of justification, along with quotes from the students, was, in part, intended for the purposes of translatability and comparability for future researchers. Thus, generalizability of the results was not a goal of this study, nor was it justified.

Further, it should be understood that it was not intended that the selection of the school site for this study be representative of all public schools in upstate New York. As with each school, the economic status and social aspects of the community has some influence on the culture and basic activities of the school. In addition, each school in New York State was adjusting to the new Regents assessment, which seemed to create a wide variation in the mathematics curriculum created individually by mathematics departments across New York State. At the classroom level, the emotional status of Mrs. Kelly regarding the placement of students in different classrooms should be considered as a factor in the study. While Mrs. Kelly showed angst about the shifting of students in the middle of the school year and the development of new courses for repeating students, other teachers may not have been so concerned with this matter.

Aspects of the data collection process should also be considered as a limiting factor of the findings in this study. First, even though a completed cycle of the instructional units on proofs involving geometry was observed, a visit with Mrs. Kelly in her classroom one month after the study was completed revealing that instruction on justification related to geometric proof was given as part of the students' review for the final exam. Perhaps further observations, including review classes for the final exam, might have added to the context of this study. Second, due to a somewhat protective classroom teacher, rapport with the students seemed only acceptable, rather than exceptional. As a mentor for student teachers, Mrs. Kelly seemed cautious about visitors in the classroom and even remarked one day about her protectiveness of "her children." Another factor in the difficulty establishing rapport with the students was that there seemed to be little time for personal conversations with the students as Mrs. Kelly was intent on keeping students on task most of the time. The extended holiday for Spring Break also added pressure on Mrs. Kelly to finish the unit on geometric proofs. Third, the pursuit of a one-down approach (Agar, 1980) seemed futile since the students displayed little confidence in their knowledge of geometric proof during the study. Students remarked quite often about the difficulties they were having with the geometric proofs. Even though students were reminded several times that they were the experts, they seemed to view the researcher as the expert. A fourth aspect of the data collection process itself was the implementation of the journal questions.

Even though Mrs. Kelly felt confident that the students would feel comfortable with this assessment tool, practice writing journal questions prior to the study might have revealed more in-depth responses. Mrs. Kelly would also have had more practice in writing responses that may have invoked different probes for interviews with the students. One final limitation of the study was the level of examination of the teacher's conception of geometric proof and its purposes. Since Mrs. Kelly was not directly asked to discuss her understanding of geometric proof, only inferences from the statements she made during classroom observations and informal conversations can be made.

As a qualitative study, this investigation presented the findings of a researcher who served as the main instrument in the study from the data collection process to the interpretations of the findings. As such, researcher bias was inevitable. Even though an interview protocol was established, some follow-up questions were missed by the researcher, while some follow-up questions were awkwardly worded or were leading. These factors were attributed to the lack of experience of the researcher with this particular research tradition, the students' lack of understanding of the subject matter, and the students' social development in terms of conversing with adults. Leading questions and corresponding responses were deleted from the transcripts and not used in the data analysis. A researcher journal that was kept and scanned for bias revealed that while contact with all of the students was made each day, more time was spent with the group of boys being recorded, due mainly to proximity. However, during the interviews, the boys did not appear to be more comfortable explaining their views than any other student in the study.

The researchers' evolving perceptions about the purposes of proof may have also influenced the findings reported in this study. At the beginning of the study, codes for the classroom observation were influenced by readings of prior research studies that occurred months before the study began. However, as the study progressed, the development of the Post-instruction Questionnaire led to a re-reading of de Villiers' (1999) *Re-thinking Proof*, which sparked more vigilant attention to the intuitive aspect of verification. In addition, believability was an emerging code from the beginning of the study, but was not viewed by the researcher as an aspect of verification until this turning event in the data analysis. Since the researcher' evolving perceptions of the data are natural and serve to inform the study in this type of researcher paradigm (Lesh & Kelly, 2000), only the fact that less attention was given to the researcher's evolving perceptions than to the researcher's perceptions of the participants in this study should be considered a limitation.

Implications and Recommendations for Future Research

The descriptions of the students' views, along with the context of their learning, lead to several implications relating to teaching and learning. These implications provide recommendations regarding the focus for future research in the area of students' formal understanding. The following implications and recommendations for teaching and learning are made.

First, several students experienced difficulty in expressing their views of the purposes of geometric proof when asked directly. One-third of the students could only list properties or theorems they encountered during the unit on geometric proof. However, when these students were asked to describe the purpose for each column, all of the students listed both explanation and verification. In addition, students who referred to verification when asked directly added explanation to their responses when asked to describe the purpose for each column. When describing the process of developing a coordinate geometry proof, twice as many students mentioned verification and explanation, even though coordinate geometry proofs were also taught as two components. Thus, the students' representation of the mathematical object (Dubinsky, 1991; Sfard, 1991) of geometric proof has been formed in a limited manner. It is possible that the students' representation of coordinate geometry proofs, that were written in narrative form, were different from their representation of Euclidean geometry proofs. Thus, future research should explore the possible differences in students' views about the purposes of geometric proof when using paragraph form and when using two columns.

Second, the students' views of two-column geometric proof consisted mainly of verification, which indicated a limited view of the purpose of geometric proof. Prior to instruction of the two units, students related proof as a means of verification. During instruction, the students often interchanged proof and verification, which seemed to indicate that proof as verification had been deeply-rooted in the students' conception of proof. Since most students are exposed to the word "prove" at an early age in everyday situations prior to learning formal mathematical proof in the mathematics classroom, more research is needed to learn about how students can learn the distinctions between everyday proof and mathematical proof. Instruction to broaden the students' views of the purposes of geometric proof probably requires more than simply stating in the classroom other purposes of proof. While Mrs. Kelly stated that the purpose of the second column was to show the reasons a statement was true, proof as explanation was not continued throughout the unit. Instead, Mrs. Kelly focused on the production of a sequence of logically connected statements, which was the main objective for future state assessment. Examinations of students' views of the purposes of geometric proof in classrooms using dynamic geometry software in conjunction with "cognitive unity of theorem" should be conducted to investigate whether this form of instruction will broaden students' views.

Third, the findings indicated that students typically did not view the reasons for a proof as members of an axiomatic system. The students' first exposure to the explanatory feature of geometric proof was a colloquial view, telling someone "why you did what you did if you got into trouble." Since the colloquial definition of explanation failed to mention that the reasons for an explanation must pertain to a "corpus of reference" (Douek, 2000), students were not given the opportunity to view explanation in mathematical proof separately from explanation in everyday argumentation. The students' acceptance of the argument shown in the Cardboard Triangle Activity problem after the two units on proof indicated that the students were followed the sociomathematical norms (Yackel & Cobb, 1996) that had been established by the classroom teacher regarding explanation. Thus, future investigations should examine how teachers can use problems like the Cardboard Triangle Activity as a way for students to explore the difference between explanation in everyday argumentation and mathematical explanation.

Fourth, future studies should explore various relationships among various purposes of geometric proof. Even though students did not typically express relationships between purposes of geometric proof, one student expressed two distinct relationships. Cathy related explanation and systematization in her response about the Cardboard Triangle Activity problem. She also indicated a relationship between explanation and verification by stating that she was more convinced that a statement was true after she saw the explanation. Another relationship between two purposes of proof that exist but was not expressed by the students was that of discovery and systematization. Future investigations should explore whether more mature mathematics students or mathematics teachers view the purposes as related concepts.

Fifth, the surprising finding that almost half of the students in the study had indicated that intuition was involved in the proving process was encouraging. Students described intuition as visual judgment and preformal proving. Both areas should be further investigated to explore the concepts in more detail. More research is needed to explore instruction regarding visual judgment in geometry. In the area of preformal proving, to what extent do students use preformal proving in geometry? How can instruction enhance the use of preformal proving? Do students use preformal proving in proofs involving algebraic concepts?

For the students who indicated that intuition is not used in developing a proof, more explicit instruction on the relationship between conviction and verification might be appropriate. For example, explicit instruction might entail the use of Ewy's (2003) "unit visual framework," in which the teacher and students co-create visual structures induced by classroom artifacts on the relationship between conviction and establishing validity. Ewy explained that unit visual frameworks differ from graphic organizers because "visual anchors" are used to assist in the learning experience. For example, a classroom teacher and her students might include a drawing of the development of a geometric proof, in which judgment of the lengths of sides were made by the students, prior to establishing the validity of a statement. Verbal reminders made by the classroom that geometric proof was the goal of the exercise sometimes overshadowed the importance of visual judgment in the development of a geometric proof. A diagram as described by Ewy might also reinforce the concept that geometric proof is a process rather than simply a sequence of connected statements or a final product of one's own thinking. Thus, further research on the use of unit visual frameworks in teaching geometric proof and its possible influence on students' views of geometric proof as a process should be conducted.

Sixth, another finding indicated that students viewed the communicative feature of geometric proof as a demonstration of understanding and an exchange of ideas rather than a forum for argumentation. All of statements that were given to the students were true statements, so students were not expected to argue about the validity of the statements. In addition, students were typically not expected to discuss or negotiate the meanings of terms during the two units. Properties of the quadrilaterals used for the coordinate geometry unit were given to the students. In only one instance, the teacher probe students to arrive at a definition for perpendicular. On the other hand, variations in proofs were discussed and encouraged. Variations in proofs seemed to impress five students who, when asked directly about discovery, mentioned the discovery of new ways to "solve" proofs. However, none of the students seemed to be concerned with how new proofs are accepted by others. Perhaps students need more exposure to proofs in order to develop the awareness for this connection. Thus, future studies should also include the development of students' views of the purposes of geometric proof.

Finally, in light of these results, more attention to the purposes of proof should be included in current reform documents. For example, *Principles and Standards for School Mathematics* (NCTM, 2000) explicated the importance of explanation, verification, and communication as purposes as geometric proof at the high school level. Explanation and verification are included in the discussion on reasoning while it has been recommended that proof be considered an "accepted method of communication" in the classroom. However, some improvements could help teachers and teacher educators in refining the purposes of proof in the high school classroom and teacher preparation programs. First, the expansion of the purposes of proof to include systematization and discovery is recommended. Systematization should be explained not only in terms of definitions, axioms, postulates, and theorems, but also in terms of its usefulness in recognizing inconsistencies, circular arguments, and missing links in reasoning. Discussion about proof as discovery should focus on how conclusions of proofs can be used in other proofs, which was clearly not realized by the students in this study. Second, teaching episodes should be developed and included that show connections between various purposes of proofs. For example, the Cardboard Triangle Activity could be used to show students specific criteria for explanation, in a mathematical sense. In addition, the difference between argumentation, justification, and explanation in mathematics should be more clearly defined. Teachers should pay special attention to the differences among the various terms expressed in *Principles and Standards for School Mathematics* (NCTM, 2000) as well as everyday meanings and mathematical meanings in the classroom.

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APPENDICES

APPENDIX A

PRECONCEPTIONS QUESTIONNAIRE

Name (psuedonym)________________________________

Date ______________________________

Answer as many questions as you can and write as many of your thoughts down as you can while you are thinking about the problem.

Please do not erase anything if you change your mind. Just cross out things if you change direction in your thinking.

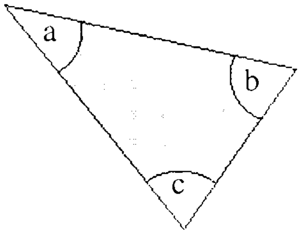
This questionnaire will not be graded and will have no affect on your grade for this class. I am interested in finding out about how you think about various mathematical concepts.

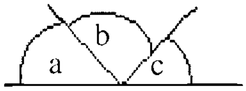
You may write on the back of the sheets if you need more space.

Thanks again for participating in this study.

Ms. Gfeller

1. Suppose your math teacher held up a huge triangle made from cardboard. Then, she tore off the angles and put them together in a straight line.





Then she said that there are 180° in a straight line and when you add the interior angles of any triangle your answer is always 180°.

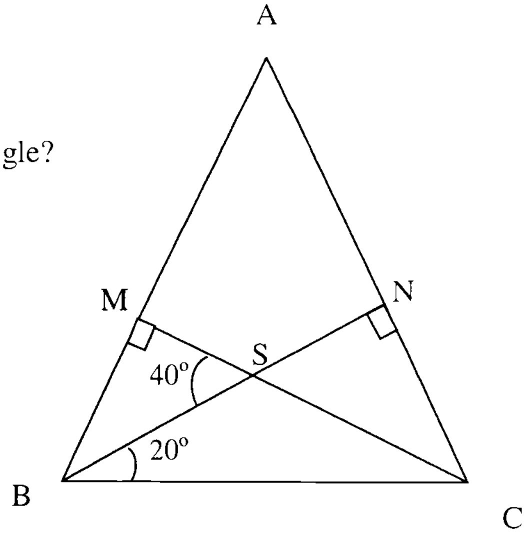
Do you think your teacher proved that the sum of the interior angles of any triangle is 180°? Why or why not?

[Additional space was provided]

2. In the triangle ABC shown below, the altitudes BN and CM intersect at point S. The measure of $\angle MSB$ is 40 degrees and the measure of $\angle SBC$ is 20 degrees.

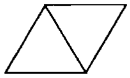
Is Triangle ABC an isosceles triangle?

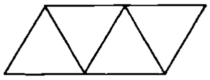
Explain your reasoning.

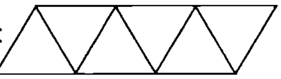


[Additional space was provided]

3. Tina is making designs with toothpicks.

   She used five toothpicks to make the first design: 

   She used nine toothpicks to make the second design: 

   She used thirteen toothpicks to make the third design: 

   How many toothpicks does she need to make the 40^{th} design? Explain how you arrived at your answer.

   [Additional space was provided]

4. In a math class, the teacher asked the students to discuss a conjecture about a formula that always produces prime numbers. The teacher reminded the students that prime numbers are numbers whose only factors are 1 and itself like

   $1x\ 2 = 2$
   $1 \times 7 = 7$
   $1 \times 11 = 11$
   $1 \times 13 = 13$
   $1 \times 17 = 17$

   The conjecture was: The formula $n^2 - n + 17$ always produce a prime number for all positive whole numbers n.

   While the students were examining the formula, Amy said that the formula was false because when $n = 17$, the number is $17^2 - 17 + 17 = 289$ and that 289 was not a prime number because it can be factored as 17×17.

John said that the formula was true because the formula works for almost all numbers and because there are always "exceptions to the rule."

   Do you agree with Amy or John? Why?

   [Additional space was provided]

5. In a math class, the teacher promised ten extra bonus points on the next test if anyone could show him a convincing argument that the sum of any two odd numbers is an even number. Students were allowed to work on the problem together before submitting their answer.

Michelle and Jim worked on the problem together and this is what they said:

Michelle: " This is easy. 1 + 1 = 2, 1 + 3 = 4, 1 + 5 = 6. You can see a pattern."

Jim: "Right, but I think we should pick some really big numbers, just to make sure"

Michelle: " Okay, 245 + 347 = 592, 1923 + 2237 = 4160"

Do you think their teacher should give them the bonus points? Explain why or why not?

APPENDIX B

POST-INSTRUCTION QUESTIONNAIRE

Name (pseudonym)____

Date_______________

There are six questions on this questionnaire. Please answer as many questions as you can.

Please do not erase anything if you change your mind. Just cross out things if you change direction in your thinking.

This questionnaire will not be graded and will have no affect on your grade for this class. I am interested in finding out how you think about various mathematical concepts.

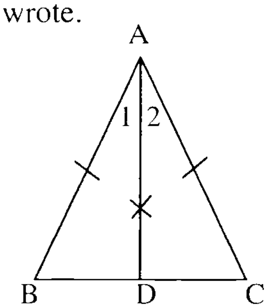
You may write on the back of the sheets if you need more space.

Thanks again for participating in this study.

Ms. Gfeller

1. Below is a proof that a student wrote.

Given: ΔABC is isosceles.
$\angle 1 \cong \angle 2$
Prove: $\overline{BD} \cong \overline{DC}$



1. ΔABC is isosceles.	1. Given
2. $\angle 1 \cong \angle 2$	2. Given
3. $\overline{AB} \cong \overline{AC}$	3. If ΔABC is isosceles, then the legs are congruent.
4. $\overline{AD} \cong \overline{AD}$	4. Reflexive Property
5. $\Delta ABD \cong \Delta ACD$	5. By SAS
6. $\overline{BD} \cong \overline{DC}$	6. If two triangles are congruent, then their corresponding parts are congruent.

a) What did the student assume in this situation?

[Additional space was provided]

b) What was not assumed?

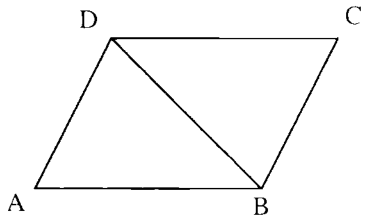
[Additional space was provided]

2. Two-column geometry proofs serve many different purposes. List as many of them as you can.

[Additional space was provided]

3. Two students were talking about the following homework problem.

   Given: Parallelogram ABCD and diagonal BD.
   Prove: $\Delta ABD \cong \Delta CDB$



   Jeff: You can just look at the diagram and you know that the two triangles are congruent. Why do we have to prove it?

   Barb: I know what you mean!!

   These students seem to think that nothing is really gained by developing a proof. Do you agree or disagree? Why?

   [Additional space was provided]

4. Do you think that proving a statement in geometry involves intuition? Why or why not?

   [Additional space was provided]

5. In the mathematics classroom, students communicate with each other and with their teacher all the time about mathematics in many different ways. How is proving a statement in geometry a way of communicating?

   [Additional space was provided]

6. How can developing a geometry proof be used to discover new results in geometry?

   [Additional space was provided]

APPENDIX C

JOURNAL QUESTIONS

Journal Question 1: Describe, in general, how to write a coordinate geometry proof. In other words, give an overview that could apply to any of the proofs involving coordinate geometry you did in class or for homework.

Journal Question 2: In this unit, your teacher said that there were two parts to doing coordinate geometry proofs. The first part is the algebra and the second part is the summary. Why is it necessary to do both parts?

Journal Question 3: What is an axiomatic system of geometry? Explain it to me like I was a younger math student. You may use your notebooks but not your textbook.

Journal Question 4: In writing two-column geometry proofs, your teacher said that there are many ways to get information so that you can make a proof. List as many as you can.

Journal Question 5: How are two-column geometry proofs similar to coordinate geometry proofs? How are they different?

Similarities

Differences

APPENDIX D

DOCUMENT SUMMARY FORM

Date Reviewed: __________

Document Type — *Function of Proof*

- ❑ Textbook/Supplement
  ____

- ❑ Homework/Project

- ❑ Classroom Assessment

  Quiz _________
  Test _________
  Project _________

Name or description of document

Contents (description)

APPENDIX E

PROPERTIES OF QUADRILATERAL FLOW CHART