Normal modes in nonlinear nonsymmetrical systems with two degrees of freedom, and applications to air springs

The study of nonlinear vibrations of systems having two degrees of freedom has met considerable attention during the past few years. Nonlinear symmetrical systems have received most of it.

Normal mode motion of a system is defined as a periodic motion such that the masses of the system assume repeated displacements after some interval of time, called the period of oscillation. In this kind of motion there is a definite relation between the displacements of the masses which is called the modal relation. The modal line is defined as the locus of all points, in the plane representing the displacements, which set the system in normal mode motion when started from rest. This line passes through the origin.
of the plane. Systems vibrating in normal modes have also what is called the orthogonality property. That is, the modal relation curves intersect the total energy line orthogonally.

There exists a type of normal mode which has a straight modal relation. The linear systems belong to it. In this type of mode the orthogonality property is used to determine the modal relations and this in turn enables the equations of motion to be decoupled to form two separate systems, each with a single degree of freedom. This of course simplifies analysis of the system. Generally speaking the modal relations are not straight and they could be determined easily by numerical means. For this purpose an algorithm for determining the normal mode motions was developed. The application of the orthogonality property is also useful when applied to small displacements.

One striking phenomenon of nonlinear systems is that they may have more normal modes than the number of degrees of freedom. The existence of this excess of modes can be easily detected by using the orthogonality property.

The stability of systems oscillating in normal modes could be studied by using Liapounov's theorem of stability. The total energy equation of a dynamical system is a Liapounov function. The idea of the modal phase plane, which is a plot of the total energy line and
the modal line, is introduced. This helps in deducing the stability of normal modes. The concepts of singularities in this plane are defined.

The mathematical procedure for determining the normal modes was applied to an air spring system with two degrees of freedom. An experimental apparatus was constructed to compare the theoretical and the experimental results. This comparison showed fair agreement.

The forced motion of the air spring system was studied experimentally. Over some range of the exciting frequency two resonances appeared. Each one corresponds to one normal mode. The results showed that the system eventually oscillates essentially in normal modes when the exciting frequency is equal to a natural frequency of the system.
NORMAL MODES IN NONLINEAR NONSYMMETRICAL SYSTEMS WITH TWO DEGREES OF FREEDOM, AND APPLICATIONS TO AIR SPRINGS

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NORMAL MODES IN NONLINEAR NONSYMMETRICAL SYSTEMS WITH TWO DEGREES OF FREEDOM, AND APPLICATIONS TO AIR SPRINGS

INTRODUCTION

Normal mode motion is a property of all conservative systems having more than one degree of freedom. It is defined as a periodic motion for each mass in the system, with each mass having the same period. In other words the magnitude of the displacement of each mass is repeated after a period of time. If these displacements are designated by $x_i$, then $x_i(t) = x_i(t + T)$. When a system oscillates in a normal mode there is a definite relation between the displacements, called the modal relation. This means one displacement will determine the others. The nature of this relation depends mainly on the characteristics of the springs in the system. Systems oscillating in normal modes have an important property which is called the orthogonality property. This is according to the fact that the modal relation curve is orthogonal to the total energy line (or surface) at maximum displacements. For a given system it is possible to find initial displacements of different magnitudes which set the system in normal mode motion when it starts from rest. In the plane (or space)

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1 The modal relation curve is the curve which represents the modal relation in the plane (or space) of the displacements.
representing the displacements, the line passing through these points is called the modal line.

The characteristic of a spring is defined as the relation between the spring restoring force \( S \) and the spring extension \( q \). The spring is linear when \( S \) is proportional to \( q \). The spring is nonlinear when \( S \) is a more complicated function of \( q \), i.e. \( S = S(q) \). If this function is arbitrary then the spring is nonlinear nonsymmetrical. It is called nonlinear symmetrical if this function is odd, that is \( S(-q) = -S(q) \). A system is called linear, nonlinear symmetrical or nonlinear nonsymmetrical if all the springs in it have the corresponding characteristic. Some systems might have springs with different characteristics. In this case they are identified according to the springs.

For linear systems the concept of normal modes is well defined. A normal mode can be identified by an eigen vector in the Euclidian space, the number of coordinates of which depends on the number of degrees of freedom. For dual mode systems this vector can be represented in the planar coordinates and the relations between the two modes can always be interpreted geometrically. A general solution is formed as a linear combination of all the modes. This is a property of the linear systems which is not shared with the nonlinear system. The reason is that the superposition theorem cannot be applied to the latter.
The study of normal modes is of significant importance. Any system, linear or nonlinear, when subjected to exciting periodic external forces, exhibits amplitudes of oscillation which are large when the frequency of these forces is near one of the natural frequencies. Resonance occurs when they are the same. The system eventually oscillates essentially in the normal mode corresponding to the resonance frequency.

Nonlinear symmetrical systems, particularly those which have the duffing type characteristics, have been given considerable attention during the past years. Several approximation methods have been used for solving the equations of motion of this class. For instance, the method of one term sinusoidal approximation with graphical procedure for solving the amplitude equations (5), successive sinusoidal approximations (20), Ritz averaging method (1) and others are applied.

Among the normal modes of dynamical systems is a class which is characterized by straight modal relations. The linear system belongs to this class. This class is defined for the two-degree-of-freedom systems discussed in (11, p. 594-603) and recently by the work of Rosenberg (17) and (18). The definition of this class was extended to systems with several degrees of freedom by Rosenberg (19). In this case the modal lines are straight lines which coincide with the modal relation curves and intersect the surface (or line, for two
degrees of freedom) of total energy orthogonally. This class of normal modes is simple to analyze and the equations of motion can be decoupled to form separate systems with one degree of freedom.

In general the modal relations are not straight and the problem of finding the normal modes is difficult. In this situation numerical solutions are effective and yield satisfactory results, especially with the availability of high speed computers.

Nonlinear nonsymmetrical systems are an important class in the study of nonlinear vibrations. It is the object of this work to stimulate such study and to present a method for finding the normal mode motion. An algorithm has been developed and shown to be an effective tool for the study of nonlinear vibrations.

Concepts of stability of nonlinear nonsymmetrical systems are difficult to study by ordinary methods since numerical techniques are used for determining the solution. Useful information can be obtained by using Liapounov's theorem of stability. This gives rise to the development of the modal phase plane for the study of the stability of the normal modes of systems with two degrees of freedom.

The mathematical model outlined in the present work has been applied on a two-degree-of-freedom air spring system. One of the reasons for choosing this is that the air spring has nonsymmetrical characteristics. It was also possible to build an experimental apparatus to compare the theoretical results. Also, air springs
have some practical importance. They have met considerable attention during the past ten years as they are used for vibration isolators, as suspension system in the automotive industries and in many other fields. It has been shown that they can effect low natural frequencies and other advantages. The system has been constructed to represent the essential features of an actual mounting, for example, an automobile suspension.
LIST OF SYMBOLS

A  cross-sectional area of the cylinder
2a  the distance between the air spring units
2b  the distance between the two eccentricity sets
g  local gravitational acceleration
h  length of interval
h₁, h₂  increments in the initial displacements
I  mass moment of inertia
kᵢ  Lipschitz constants
k₁₁, k₁₂, k₂₂  spring stiffnesses (linear)
L, L₁, L₂  air column lengths at equilibrium
M  the eccentric mass
m, m₁, m₂, m₃  masses
n  number of intervals
P  absolute pressure at any time
P₀  absolute initial pressure
q  spring extension
R  the eccentricity radius
S, S₁, S₂, S₃  spring forces
T  period of oscillation, seconds
Tₑ  kinetic energy
\( T_0 \) period of the linearized system

\( \Delta T \) increment in the period

\( t \) time variable

\( U \) potential energy

\( U_0 \) total energy

\( V \) volume at any time

\( V_d \) Liapounov's function

\( V_o \) initial volume

\( x, y, z, z_1, z_2 \) displacements

\( x_0, y_0, x(1), y(1) \) initial displacements

\( \Delta x_0, \Delta y_0 \) change in the initial displacements

\( \gamma \) index of expansion and compression of air

\( \epsilon \) the error limit

\( \theta \) rotational angle of the mass on the air spring

\( \omega \) the exciting angular velocity

\( \omega_1, \omega_2 \) the natural frequencies
I. NONLINEAR NONSYMME TRICA L SYSTEMS

1. Equations of motion

The system, in general, consists of two masses whose magnitudes are arbitrary, coupled together with a coupling spring with characteristic $S_2(q)$ where $q$ denotes spring extension. Each mass is connected to an infinite mass, or a rigid boundary, by an anchor spring $S_1(q)$ and $S_3(q)$ as shown in Figure 1. $S_i(q)$, $(i = 1, 2, 3)$, are considered nonlinear in this text. They are symmetrical if $S_i(q) = -S_i(-q)$; otherwise they are nonsymmetrical.

The characteristics $S_i$ are continuous functions of the real variable $q$ and are defined over some closed interval. With this property, and according to Weierstrass' approximation theorem, these functions can be represented by polynomials. This is pursued in a later section.

\footnote{The theorem states that a continuous function of a real variable, defined in a closed interval, may be approximated arbitrarily closely by means of a polynomial (23, p. 414).} 

Figure 1. A two-degree-of-freedom system
The displacements of the masses at any time \( t \) are denoted by \( x(t) \) and \( y(t) \), positive in the direction to the right.

The kinetic energy of the system is given by:

\[
T_e = \frac{1}{2}(m_1 x^2 + m_2 y^2)
\]

The potential energy is given by

\[
U = \int_0^x S_1(q) \, dq + \int_{y-x}^y S_2(q) \, dq + \int_0^y S_3(q) \, dq.
\]

Lagrange's equation, with the absence of external forces, is

\[
\frac{d}{dt} \left( \frac{\partial T_e}{\partial \dot{x}} \right) + \frac{\partial U}{\partial x} = 0.
\]

But

\[
\frac{d}{dt} \left( \frac{\partial T_e}{\partial \dot{x}} \right) = m_1 \ddot{x}
\]

\[
\frac{d}{dt} \left( \frac{\partial T_e}{\partial \dot{y}} \right) = m_2 \ddot{y}
\]

\[
\frac{\partial U}{\partial x} = S_1(x) - S_2(y - x)
\]

\[
\frac{\partial U}{\partial y} = S_2(y - x) + S_3(y).
\]

The equations of motion then read
\[ m_1 \ddot{x} + S_1(x) - S_2(y-x) = 0 \]  \hspace{1cm} (1.1)

\[ m_2 \ddot{y} + S_2(y-x) + S_3(y) = 0. \]

A somewhat more general form of these equations is

\[ \ddot{x} = F(x, y) \]  \hspace{1cm} (1.2)

\[ \ddot{y} = G(x, y) \]

where in the above case

\[ F(x, y) = -\frac{1}{m_1} S_1(x) - S_2(y-x) = -\frac{1}{m_1} \frac{\partial U}{\partial x} \]

\[ G(x, y) = -\frac{1}{m_2} S_2(y-x) + S_3(y) = -\frac{1}{m_2} \frac{\partial U}{\partial y} \]

2. **Existence and uniqueness of the solutions**

The system of equations (1.2) can be put in the form of four first order differential equations. In general these equations read

\[ \dot{x}_i = f_i(x_1, x_2, x_3, x_4) \quad i = 1, 2, 3, 4. \]  \hspace{1cm} (2.1)

In detail, they are

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = F(x_1, x_3) \]

\[ \dot{x}_3 = x_4 \]
\[ \dot{x}_4 = G(x_1, x_3) \]

where

\[ x_1 = x \text{ and } x_3 = y. \]

If the functions \( f_1(x_1, x_2, x_3, x_4) \) in equations (2.1) satisfy a Lipschitz condition in some small domain \( D \) (i.e., there exist positive constants \( K_m \) and \( \mu \) such that for every pair of points \( (x_1^0, x_2^0, x_3^0, x_4^0) \) and \( (x_1, x_2, x_3, x_4) \) in \( D \)

\[ |f_1(x_1^0, x_2^0, x_3^0, x_4^0) - f_1(x_1, x_2, x_3, x_4)| < \sum_{m=1}^{4} K_m |x_m - x_m^0| \]

\( i = 1, 2, 3, 4 \)

whenever \( |x_m^0 - x_m| < \mu \) and if \( f_1(x_1, x_2, x_3, x_4) \) are real-valued continuous functions on

\[ |t - t_0| \leq a, \quad |x_m^0 - x_m| \leq \mu, \]

there exists one and only one set of solutions for a given initial condition in the interval

\[ |t - t_0| \leq h, \]

where \( h \) is equal to the smaller of \( a \) and \( \frac{\mu}{M} \), and \( M \) is the upper bound of

\[ |f_1(x_1, x_2, x_3, x_4)|. \]

A detailed discussion of this theorem could be found, for instance, in (25, p. 17).
It follows from the above that if the functions $f_i$ satisfy the two conditions of the preceding paragraph at every point of some region $R$, $D$ belonging to $R$, then it is possible to find a unique solution for this system in $R$ for a given initial condition. Thus, if the functions $f_i$ are of Class $C^1$ in $R$, there exists a unique solution for the system of equations (25, p. 3).

3. **Concept of normal modes**

   i. **General considerations.** A system is said to be oscillating in a normal mode when the masses assume the same displacements after some time $T$; $T$ is called the period of oscillation of the mode. By definition, then, a necessary and sufficient condition for the system given by equations (1.2) to oscillate in normal mode is that

   \[ x(t) = x(t + T) \]

   \[ y(t) = y(t + T). \]  

   (3.1)

   For one period there is a fixed relation between $x(t)$ and $y(t)$ which is a consequence of the above condition, that is

   \[ y = \phi(x), \quad \phi \text{ a single valued function}. \]

   This relation is called the modal relation. The nature of this relation depends upon whether the system is linear or nonlinear. For a
linear system it is well known that this relation is linear (10, p. 166); that is

\[ \frac{y}{x} = c. \]

For nonlinear systems this relation is not linear in general although there is, as will be pointed out later, what is called the straight modal relation (17). For this class of nonlinear systems the boundary condition \( \phi(0) = 0 \) is imposed. This means that the two masses will pass through the equilibrium position at the same time.

There is one more property which systems oscillating in normal modes possess. This property can be easily visualized after changing the variables in equations (1.1) by putting

\[ q_1 = \sqrt{m_1} x \]
\[ q_2 = \sqrt{m_2} y. \]

The modal relation is \( q_2 = \psi(q_1) \), and the equations of motion, then, read

\[ \ddot{q}_1 = \frac{1}{\sqrt{m_1}} \left[ -S_1 \left( \frac{q_1}{\sqrt{m_1}} \right) + S_2 \left( \frac{q_2}{\sqrt{m_2}} - \frac{q_1}{\sqrt{m_1}} \right) \right] \tag{3.2} \]
\[ \ddot{q}_2 = \frac{1}{\sqrt{m_2}} \left[ -S_2 \left( \frac{q_2}{\sqrt{m_2}} - \frac{q_1}{\sqrt{m_1}} \right) - S_3 \left( \frac{q_2}{\sqrt{m_2}} \right) \right] \]

and the potential energy is given by
\[ U = \int_{0}^{q_1} \frac{q_1}{\sqrt{m_1}} S_1(q) dq + \int_{0}^{q_2} \frac{q_2}{\sqrt{m_2}} S_2(q) dq + \int_{0}^{q_2} \frac{q_2}{\sqrt{m_2}} S_3(q) dq. \]

Obviously the right hand side of equations (3.2) are the partial derivatives of \( U \) with respect to \( q_1 \) and \( q_2 \) respectively, i.e.,

\[ \dot{q}_1 = -\frac{\partial U(q_1, q_2)}{\partial q_1} = -U_{q_1}(q_1, q_2) \]

(3.3)

\[ \dot{q}_2 = -\frac{\partial U(q_1, q_2)}{\partial q_2} = -U_{q_2}(q_1, q_2) \]

The total potential lines are represented in the \( q_1 - q_2 \) plane by

\[ U(q_1, q_2) = U_0 \quad (3.4) \]

where \( U_0 \) is an arbitrary constant. The slopes of these lines are given by

\[ -\frac{U_{q_1}}{U_{q_2}} \]

The slope of the modal relation curve \( q_2 = \psi(q_1) \) is given by

\[ \frac{dq_2}{dq_1} = \frac{\dot{q}_2}{\dot{q}_1}. \]

The extremes of the modal relation curve represent the initial
conditions which, with zero initial velocities, will set the system into the normal mode motion, i.e. $\dot{q}(0) = \ddot{q}(0) = 0$. This shows that $\frac{\dot{q}_2}{\dot{q}_1}$ at $t = 0$ is not defined. To obtain $\frac{dq_2}{dq_1}$ at $t = 0$, L'Hospital's rule is applied

$$\left. \frac{dq_2}{dq_1} \right|_{t=0} = \frac{\dot{q}_2(0)}{\dot{q}_1(0)} = \frac{\ddot{q}_2(0)}{\ddot{q}_1(0)}.$$

From (3.3)

$$\left. \frac{dq_2}{dq_1} \right|_{t=0} = \frac{U_{q_2}}{U_{q_1}}. \quad (3.5)$$

By comparing (3.5) and the slope of (3.4) it is seen that the modal relation curve $q_2 = \psi(q_1)$ intersects the line of total energy $U(q_1, q_2) = U_0$ orthogonally. This is called the orthogonality property of the normal modes and has quite important consequences. It can be used with linear systems as well as nonlinear systems having straight modal relations to find the relations between the two displacements. Once these relations are found, the equations of motion can be decoupled to form two separate equations, each representing a single-degree-of-freedom system. However, this advantage cannot be used in cases where the modal relations are not straight. For this situation the problem is rather difficult and other considerations should be used.
ii. Linear systems. As was stated by Rosenberg (17)

In the linear system the normal modes are given through the eigenvalues, and the latter are the roots of the characteristic equation. If the roots are distinct, one distinct eigen vector is attached uniquely to each eigenvalue, i.e. the former are defined by means of the latter. In fact, finding the normal modes is always preceded by finding the eigenvalues. The fact that the straight modal relation intersects the potential line orthogonally was used, and with the help of the maximum and minimum properties the modal relation is established.

The orthogonality property could be used in a different simpler way yielding the same results.

For a linear system with two degrees of freedom the potential energy is given in a general form by (10, p. 184)

\[ U = \frac{1}{2} (K_{11} q_1^2 + 2K_{12} q_1 q_2 + K_{22} q_2^2) \]

and the kinetic energy \( T_e \) for two equal masses is given by

\[ T_e = \frac{m}{2} (\dot{q}_1^2 + \dot{q}_2^2). \]

The variables \( q_1 \) and \( q_2 \) need not be changed because the two masses are equal.

The slope of the tangent to the potential line is given by

\[ -\frac{U}{U} \frac{q_1}{q_2} = -\frac{K_{11} q_1 + K_{12} q_2}{K_{12} q_1 + K_{22} q_2}. \]

The slope of the normal to this tangent is
The modal relations are straight lines passing through the origin and each has a slope given above. The equation of these lines is given by

\[
\frac{q_2 - 0}{q_1 - 0} = \frac{K_{12}q_1 + K_{22}q_2}{K_{11}q_1 + K_{12}q_2}
\]

or

\[
(K_{11} - K_{22})q_1 q_2 + K_{12}(q_2^2 - q_1^2) = 0. \quad (3.6)
\]

Solving this equation simultaneously with the total potential equation \( U = U_0 \) will yield four roots. These four points represent the initial displacements which, together with zero initial velocities, set the system in normal mode motion. Since the equation of the total energy is an ellipse, it is expected that each pair of points lies on one line, which is either the major or the minor axis of the ellipse. The modal relations could be found easily by setting

\[
q_1 = r \cos \theta \\
q_2 = r \sin \theta.
\]

Substituting in (3.7) then,

\[
(K_{11} - K_{22}) \sin \theta \cos \theta + K_{12}(\sin^2 \theta - \cos^2 \theta) = 0
\]

or
\[
\frac{1}{2}(K_{11} - K_{22}) \sin 2\theta - K_{12} \cos 2\theta = 0
\]

then

\[
\tan 2\theta = \frac{2K_{12}}{K_{11} - K_{22}}.
\]

(3.7)

(3.7) yields two values for \( \theta \) which differ by \( \frac{\pi}{2} \). Therefore the modal relations are given by

\[
q_2^{(1)} = \tan \theta_1 q_1^{(1)}
\]

\[
q_2^{(2)} = \tan (\theta_1 + \frac{\pi}{2}) q_1^{(2)}
\]

These are exactly the same results obtained in (10, p. 185).

The same idea can be applied to nonlinear systems which have straight modal relations. This is done in the next article (3-iii).

iii. Straight modal relations in nonlinear systems. It was mentioned before that there exists a class of normal modes which have a linear relation between the displacements. This class frequently occurs in symmetrical systems and in some special cases of non-symmetrical systems. In this case

\[
\frac{d^2 q_2}{dq_1^2} = 0.
\]

The equations of motion given by (3.3) are
\[
\ddot{q}_1 = \frac{U}{q_1}(q_1, q_2)
\]
\[
\ddot{q}_2 = \frac{U}{q_2}(q_1, q_2).
\]

After eliminating the time variables they could be written in the form (11, p. 594-603),
\[
2(U_0 - U) \frac{d^2 q_2}{dq_1^2} = \left[1 + \left(\frac{dq_2}{dq_1}\right)^2\right] \left[\frac{\partial U}{\partial q_1} \cdot \frac{dq_2}{dq_1} - \frac{\partial U}{\partial q_2}\right]. \tag{3.9}
\]

For the straight modal relations \(\frac{d^2 q_2}{dq_1^2} = 0\) and hence equation (3.9) becomes
\[
\frac{\partial U}{\partial q_1} \frac{dq_2}{dq_1} - \frac{\partial U}{\partial q_2} = 0
\]
or
\[
\frac{dq_2}{dq_1} = \frac{U_{q_2}}{U_{q_1}}.
\]

But since the slope of the potential lines is given by, (c.f. art 3-i),
\[
\frac{U_{q_1}}{U_{q_2}} - \frac{U_{q_1}}{U_{q_2}} \frac{U_{q_1}}{U_{q_2}}
\]
one concludes that the straight modal relation intersects all the total energy lines orthogonally. Each point on this line represents initial displacements which would cause the system to oscillate in a normal mode if it starts from rest. Any line which has this property is
called a modal line. It follows, in this case, that the modal line and the modal relation curve coincide.

For any nonlinear system, and for sufficiently small initial displacements the modal relation curves are arbitrarily close to straight lines passing through the origin. In this case the system is considered to have straight modal relations. Hence, as was pointed out recently, the modal lines and the modal relation curves coincide. Therefore the modal lines must pass through the origin.

Systems with straight modal relations, then, have the modal line and modal relation curve given by

\[ q_2 = c_1 q_1 \]

where \( c_1 \) is the slope of the line.

The application of the orthogonality property is important. It leads to the problem of solving a pair of simultaneous equations in the initial displacements. The number of roots satisfying these two equations depends on their order. For example, in the case of a linear system these equations are of the second order, and the number of roots is four. These roots are real, each pair lying on one modal relation curve. For nonlinear systems these two equations are in general of order higher than two. It is expected that for \( n \)th order equations that there will be \( 2n \) roots. It is a well-known fact that the complex roots occur in pairs. This means
the number of real roots is even. For a system with two degrees of freedom the minimum number of real roots is four. Then, according to the above discussion, there is a possibility that the real roots could exceed four and in this case the number of normal modes may exceed the number of degrees of freedom. This fact has been shown to be possible by the work of R. M. Rosenberg (18). In this respect the study of small oscillations of nonlinear systems is quite useful in giving an idea how the system might behave regarding normal modes.

By finding the modal relations the equations of motion can be decoupled to form separate single-degree-of-freedom systems. The amplitude frequency response for each normal mode can then be found by a single quadrature. If the characteristics of the springs are, for instance, polynomials of the third order the problem reduces to a complete elliptic integral of the first kind.

iv. **Curved modal relations.** For a general class of two-degree-of-freedom systems, the modal relations are, in general, not straight lines. In this case the modal relations must satisfy equation (3. 9) together with the boundary conditions which define normal modes. The solution of this differential equation is expected to be troublesome, even by numerical means, particularly in cases with high nonlinearity. In addition, the boundary conditions
are mixed. However, the problem could be approached with a much simpler way which leads to satisfactory results. It has been mentioned before that a necessary and sufficient condition for the system to oscillate in normal modes is that the displacements assume the same values after a period of time \( T \). If this condition is satisfied, then the other two conditions (the orthogonality property and the unique relation between \( x \) and \( y \)) are automatically satisfied. It is necessary of course, first of all, to check the existence of the solution in the region of interest \( R \) on the light of section 2.

4. **Algorithm for the determination of the normal mode.**

In this section an algorithm for determining the modal lines and modal relation curves for any nonlinear system is outlined.

Given a pair of simultaneous differential equations representing the motion of a dynamical system

\[
\begin{align*}
\ddot{x} &= F(x, y) \\
\ddot{y} &= G(x, y)
\end{align*}
\]  

(1.2)

and the initial conditions

\[
\begin{align*}
x(0) &= x_0 \\
y(0) &= y_0 \\
\dot{x}(0) &= 0 \\
\dot{y}(0) &= 0
\end{align*}
\]
it is required then to find a solution for this system subject to the boundary conditions

\[ \begin{align*}
  x(t) &= x(t + T) \\
  y(t) &= y(t + T).
\end{align*} \] (3.1)

Integrating equations (1.2) twice with respect to the time \( t \) one gets

\[ \begin{align*}
  x &= \int_0^t \int_0^\eta F[x(\xi), y(\xi)] \, d\xi + c_1 t + c_2 \\
  y &= \int_0^t \int_0^\eta G[x(\xi), y(\xi)] \, d\xi + c_3 t + c_4.
\end{align*} \] (4.1)

By applying the initial conditions, then

\[ \begin{align*}
  c_1 &= c_3 = 0 \\
  c_2 &= x_0 \\
  c_4 &= y_0
\end{align*} \]

and equations (4.1) read

\[ \begin{align*}
  x &= \int_0^t \int_0^\eta F[x(\xi), y(\xi)] \, d\xi + x_0 \\
  y &= \int_0^t \int_0^\eta G[x(\xi), y(\xi)] \, d\xi + y_0.
\end{align*} \]

Consider the integral
\[ J_1 = \int_0^t d\eta \int_0^\eta F[x(\xi), y(\xi)] \, d\xi. \]

Integrating the right hand side by parts

\[ J_1 = \eta \int_0^\eta F[x(\xi), y(\xi)] \, d\xi \bigg|_0^t - \int_0^t \frac{d}{d\eta} \int_0^\eta F[x(\xi), y(\xi)] \, d\xi \]

\[ = t \int_0^t F[x(\xi), y(\xi)] \, d\xi - \int_0^t \eta F[x(\eta), y(\eta)] \, d\eta \]

\[ = \int_0^t (t-\eta) F[x(\eta), y(\eta)] \, d\eta. \]

By the same way

\[ J_2 = \int_0^t d\eta \int_0^\eta G[x(\xi), y(\xi)] \, d\xi = \int_0^t (t-\eta) G[x(\eta), y(\eta)] \, d\eta. \]

Therefore equations (4.1) take the forms

\[
\begin{align*}
x(t) &= \int_0^t (t-\eta) F[x(\eta), y(\eta)] \, d\eta + x_0 \\
y(t) &= \int_0^t (t-\eta) G[x(\eta), y(\eta)] \, d\eta + y_0.
\end{align*}
\]

(4.2)

Upon applying the boundary conditions (3.1), there is no loss of generality if \( t \) is set to equal zero, that is

\[
\begin{align*}
x(0) &= x(T) \\
y(0) &= y(T).
\end{align*}
\]
Substituting these in equations (4.2), then

\[ x_o = \int_0^T (T - \eta) F[x(\eta), y(\eta)] \, d\eta + x_o \]

\[ y_o = \int_0^T (T - \eta) G[x(\eta), y(\eta)] \, d\eta + y_o \]

or

\[ J_1 = \int_0^T (T - \eta) F[x(\eta), y(\eta)] \, d\eta = 0 \] (4.3)

\[ J_2 = \int_0^T (T - \eta) G[x(\eta), y(\eta)] \, d\eta = 0. \]

For a given value of \( T \) equations (4.3) are two simultaneous homogeneous integral equations in \( x(t) \) and \( y(t) \).

The solution of these two integral equations is expected to be tedious. However the problem could be much simplified if the equations of motion (1.2) are used together with (4.3). The displacements \( x \) and \( y \) are functions of time and initial displacements as is seen from equations (4.2). In other words if \( x_o \) and \( y_o \) are defined, then the displacements of both masses are determined at any time.

Upon using the differential equations with (4.3) they reduce to two simultaneous equations in \( x_o, y_o \) and \( T \),

i.e.

\[ J_1 = J_1(x_o, y_o, T) = 0 \]

\[ J_2 = J_2(x_o, y_o, T) = 0. \]
Because of the complexity of the above two equations it appears that the simplest way to solve them is by numerical means.

From physical reasoning, if a normal mode motion exists, the equations \( J_1(x_o, y_o, T) = 0 \) and \( J_2(x_o, y_o, T) = 0 \) have a double root at the point \((x_o', y_o')\) for the period \(T'\). That is the curves \( J_1 = 0 \) and \( J_2 = 0 \) touch each other at that point. This can be shown formally as follows.

It is shown in (13, p. 49) that the two equations \( f(x, y) = 0 \) and \( g(x, y) = 0 \) have a multiple root at \((x_o, y_o)\) if

\[
\begin{vmatrix}
  f_x & f_y \\
  g_x & g_y \\
\end{vmatrix} = 0 \quad \text{at} \quad (x_o, y_o)
\]

where \( f_x \) denotes the partial derivative \( \frac{\partial f}{\partial x} \) etc. Now consider the equations (4.3). Suppose that for a given \( T' \) there exist \( x_o' \) and \( y_o' \) which satisfy (4.3). Expanding \( J_1 \) and \( J_2 \) in the neighborhood of \( x_o \) and \( y_o \), neglecting second and higher order derivatives

\[
J_1(x_o + h, y_o + k) = J_1(x_o', y_o') + h \frac{\partial J_1}{\partial x_o}
\bigg|_{(x_o', y_o')} + k \frac{\partial J_1}{\partial y_o}
\bigg|_{(x_o', y_o')}
\]

\[
J_2(x_o + h, y_o + k) = J_2(x_o', y_o') + h \frac{\partial J_2}{\partial x_o}
\bigg|_{(x_o', y_o')} + k \frac{\partial J_2}{\partial y_o}
\bigg|_{(x_o', y_o')}.
\]

It is clear that

\[
J_1(x_o', y_o') = J_2(x_o', y_o') = 0.
\]
$J_1$ and $J_2$ have a double or a multiple root if

$$
D = \begin{vmatrix}
\frac{\partial J_1}{\partial x_o} & \frac{\partial J_1}{\partial y_o} \\
\frac{\partial J_2}{\partial x_o} & \frac{\partial J_2}{\partial y_o}
\end{vmatrix} = 0 \text{ at } (x_o', y_o')
$$

The point $(x_o', y_o')$ lies on the modal line. For nonlinear systems the period of oscillation $T$ varies for different points on that line, that is

$$T = T(x_o', y_o').$$

So the directional derivative along the modal line is

$$\frac{\partial}{\partial T} \bigg|_{T = T'} = \frac{\partial}{\partial x_o} \frac{\partial x_o}{\partial T} (x_o', y_o') + \frac{\partial}{\partial y_o} \frac{\partial y_o}{\partial T} (x_o', y_o')$$

where $T'$ is the period that corresponds to the initial displacements $x_o'$ and $y_o'$.

Operating $\frac{\partial}{\partial T} \bigg|_{T = T'}$ on $J_1$ and $J_2$ one gets

$$\frac{\partial J_1}{\partial T} \bigg|_{T = T'} = \frac{\partial J_1}{\partial x_o} \frac{\partial x_o}{\partial T} (x_o', y_o') + \frac{\partial J_1}{\partial y_o} \frac{\partial y_o}{\partial T} (x_o', y_o')$$

$$\frac{\partial J_2}{\partial T} \bigg|_{T = T'} = \frac{\partial J_2}{\partial x_o} \frac{\partial x_o}{\partial T} (x_o', y_o') + \frac{\partial J_2}{\partial y_o} \frac{\partial y_o}{\partial T} (x_o', y_o')$$

(4.4)
\[
\frac{\partial J_1}{\partial T} \bigg|_{T = T'} = \frac{\partial}{\partial T} \int_0^T (T - \eta) \, F[x(\eta), y(\eta)] \, d\eta \bigg|_{T = T'} \\
= (T' - \eta) F[x(\eta), y(\eta)] \bigg|_{\eta = T'} + \int_0^{T'} F[x(\eta), y(\eta)] \, d\eta \\
= \int_0^{T'} F[x(\eta), y(\eta)] \, d\eta \\
= \dot{x}(T') = 0
\]

(since the initial velocity is zero).

By the same way it is seen that

\[
\frac{\partial J_2}{\partial T} \bigg|_{T = T'} = 0
\]

Therefore equations (4.4) take the form

\[
0 = \frac{\partial J_1}{\partial x_o} \frac{\partial x}{\partial T} \bigg|_{(x_o', y_o')} + \frac{\partial J_1}{\partial y_o} \frac{\partial y}{\partial T} \bigg|_{(x_o', y_o')}
\]

\[
0 = \frac{\partial J_2}{\partial x_o} \frac{\partial x}{\partial T} \bigg|_{(x_o', y_o')} + \frac{\partial J_2}{\partial y_o} \frac{\partial y}{\partial T} \bigg|_{(x_o', y_o')}.
\]

For nonlinear systems \( \frac{\partial x_o}{\partial T} \) and \( \frac{\partial y_o}{\partial T} \) are non-zero. Therefore the determinant
This implies that \( J_1 \) and \( J_2 \) have at least a double root at \((x_o', y_o')\). This fact has some practical importance. According to the errors involved in the computations, it is possible that \( J_1 = 0 \) and \( J_2 = 0 \) will either not intersect at all or overlap resulting in two points of intersections. Both cases will lead to false results.

To avoid this situation it is possible to make use of the following hypothesis:

If \( F \) and \( G \) are of class \( C^2 \) and a root \((x_o', y_o')\) of \( F \) and \( G \) [i.e. \( F(x_o', y_o') = G(x_o', y_o') = 0 \)] is a double root, then it is possible to find that root by solving for

\[
\frac{\partial F}{\partial T} = 0 \quad \text{and} \quad \frac{\partial G}{\partial T} = 0.
\]

where \( T = T(x, y) \), provided that at least one of

---

1 For every numerical procedure there are three kinds of errors involved, namely

i - The truncation error is the error in the formula that represents the exact solution. This error decreases with decreasing the interval length \( h \).

ii - The propagation error is that which is carried through every interval. It increases with increasing the number of intervals. That is increases with decreasing \( h \).

iii - Rounding off error.
\[
\frac{\partial^2 F}{\partial T^2} \quad \text{and} \quad \frac{\partial^2 G}{\partial T^2}
\]
does not vanish at the root. Otherwise there is a triple or a higher order root.

Proof

Consider the two equations

\[
F(x, y) = (x-x_o)^2 f_1(x, y) + (x-x_o)(y-y_o) f_2(x, y) + (y-y_o)^2 f_3(x, y) = 0
\]

\[
G(x, y) = (x-x_o)^2 g_1(x, y) + (x-x_o)(y-y_o) g_2(x, y) + (y-y_o)^2 g_3(x, y) = 0
\]

which have a double root at \((x_o, y_o)\).

Along the line \(T = T(x, y)\),

\[
\frac{\partial F}{\partial T} = \left[ (x-x_o)^2 \frac{\partial f_1}{\partial x} + 2(x-x_o) f_1 + (y-y_o) f_2 + (x-x_o)(y-y_o) \frac{\partial f_2}{\partial x} + \right.
\]

\[
(y-y_o)^2 \left( \frac{\partial f_3}{\partial x} \right) \frac{\partial x}{\partial T} + \left[ (x-x_o)^2 \frac{\partial f_1}{\partial y} + (x-x_o)(y-y_o) \frac{\partial f_2}{\partial y} + \right.
\]

\[
(x-x_o)f_2 + (y-y_o)^2 \frac{\partial f_3}{\partial y} + 2(y-y_o) f_3 \right] \frac{\partial y}{\partial T}
\]

\[
\frac{\partial G}{\partial T} = \left[ (x-x_o)^2 \frac{\partial g_1}{\partial x} + 2(x-x_o) g_1 + (y-y_o) g_2 + (x-x_o)(y-y_o) \frac{\partial g_2}{\partial x} + \right.
\]

\[
(y-y_o)^2 \left( \frac{\partial g_3}{\partial x} \right) \frac{\partial x}{\partial T} + \left[ (x-x_o)^2 \frac{\partial g_1}{\partial y} + (x-x_o) g_2 + \right.
\]

\[
(x-x_o)(y-y_o) \frac{\partial g_2}{\partial y} + (y-y_o)^2 \frac{\partial g_3}{\partial y} + 2(y-y_o) g_3 \right] \frac{\partial y}{\partial T}.
\]
It is clear that \( \frac{\partial F}{\partial T} \) and \( \frac{\partial G}{\partial T} \) have a common root at \((x_o, y_o)\) and it could be determined by solving the above two equations simultaneously.

Let

\[
\frac{\partial F}{\partial T} = P(x, y) = 0
\]

\[
\frac{\partial G}{\partial T} = Q(x, y) = 0.
\]

Expanding \( P(x, y) \) and \( Q(x, y) \) in the neighborhood of \((x_o, y_o)\)

\[
P(x_o + h, y_o + k) = P(x_o, y_o) + h \frac{\partial P}{\partial x}(x_o, y_o) + k \frac{\partial P}{\partial y}(x_o, y_o)
\]

\[
Q(x_o + h, y_o + k) = Q(x_o, y_o) + h \frac{\partial Q}{\partial x}(x_o, y_o) + k \frac{\partial Q}{\partial y}(x_o, y_o)
\]

\[
P(x_o, y_o) = Q(x_o, y_o) = 0.
\]

Now \( h \) and \( k \) exist if and only if

\[
D = \begin{vmatrix}
\frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\
\frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y}
\end{vmatrix} \neq 0 \text{ at } (x_o, y_o).
\]

To show that this last inequality is satisfied, note that

\[
\frac{\partial P}{\partial T} = \frac{\partial^2 F}{\partial T^2} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial T} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial T}
\]

\[
\frac{\partial Q}{\partial T} = \frac{\partial^2 G}{\partial T^2} = \frac{\partial Q}{\partial x} \frac{\partial x}{\partial T} + \frac{\partial Q}{\partial y} \frac{\partial y}{\partial T} \text{ at } (x_o, y_o).
\]
As long as \( \frac{\partial x}{\partial T} \) and \( \frac{\partial y}{\partial T} \) are non zero at \((x_o, y_o)\) it is clear that \( D \) does not vanish if one of

\[
\frac{\partial^2 F}{\partial T^2} \text{ and } \frac{\partial^2 G}{\partial T^2}
\]

does not vanish and the hypothesis is proved.

To apply this for the system under investigation, it was pointed out that the period \( T \) depends on the initial displacements, i.e. \( T = T(x_o, y_o) \).

It was also shown that

\[
\frac{\partial J_1}{\partial T} \bigg|_{(x'_o, y'_o)} = 0
\]

\[
\frac{\partial J_2}{\partial T} \bigg|_{(x'_o, y'_o)} = 0.
\]

Therefore

\[
\frac{\partial J_1}{\partial T} = 0
\]

and \( \frac{\partial J_2}{\partial T} = 0 \) have a root at \((x'_o, y'_o)\).

To find this root it is necessary to check the nonvanishing of either

\[
\frac{\partial^2 J_1}{\partial T^2} \text{ or } \frac{\partial^2 J_2}{\partial T^2}.
\]
\[ \frac{\partial J_1}{\partial T} = \int_0^T F[x(\eta), y(\eta)] \, d\eta \]
\[ \frac{\partial J_2}{\partial T} = \int_0^T G[x(\eta), y(\eta)] \, d\eta. \]

Hence
\[ \frac{\partial^2 J_1}{\partial T^2} \bigg|_{(x_o', y_o')} = F[x_o', y_o'] = \ddot{x}(0) \]
\[ \frac{\partial^2 J_2}{\partial T^2} \bigg|_{(x_o', y_o')} = G[x_o', y_o'] = \ddot{y}(0). \]

Obviously \( \ddot{x}(0) \) and \( \ddot{y}(0) \) do not vanish. Hence it is possible to find the root \((x_o', y_o')\) for the period \( T \) by solving simultaneously the two equations
\[ I_1 = \frac{\partial J_1}{\partial T} = \int_0^T F[x(\eta), y(\eta)] \, d\eta = 0 \] (4.5)
\[ I_2 = \frac{\partial J_2}{\partial T} = \int_0^T G[x(\eta), y(\eta)] \, d\eta = 0 \]

This is physically correct because the terms \( I_1 \) and \( I_2 \) represent the velocities at the end of one oscillation which must be zero.

The algorithm for solving these two equations is simple but involves lengthy calculations. But with the availability of modern high speed computers one can appreciate the effectiveness of this algorithm. An iteration technique is used and is described as follows:
a. The period $T$ (which has a given value) is divided into $n$ equal intervals each of length $h = \frac{T}{n}$. The end points of the intervals are designated by 1, 2, ..., $n+1$ and the displacements at these points are indicated by $x(j)$ and $y(j)$, $j = 1, 2, ..., n+1$. The choice of $h$ depends on the method used for the solution of the differential equations. The restrictions on $h$ are discussed in Appendix 1.

b. The initial displacements $x(1)$ and $y(1)$ are given some values as a first approximation.

c. With $x(1)$ and $y(1)$ the two differential equations (1.2) are solved numerically using the central difference method (6, p. 118-126). This method is divided into two parts,

i. Calculations of starting values

ii. The main calculations

For the starting values, according to the absence of the velocity exclusively from the equations of motion, it is sufficient to calculate only one point $x(2)$ and $y(2)$ using $x(1)$ and $y(1)$. The values $F[x(j), y(j)]$ and $G[x(j), y(j)]$ are designated by $F(j)$ and $G(j)$ respectively.

The formulas

$$x(2) = x(1) + \frac{h^2}{2} F(1)$$

$$y(2) = y(1) + \frac{h^2}{2} G(1)$$
are used as predictors, and the formulas

\[ x(2) = x(1) + h^2 \left[ \frac{1}{2} F(1) + \frac{1}{6} (F(2) - F(1)) \right] \]

\[ y(2) = y(1) + h^2 \left[ \frac{1}{2} G(1) + \frac{1}{6} (G(2) - G(1)) \right] \]

are correctors. The convergence of \( x(2) \) and \( y(2) \) is assured by the choice of \( h \) as is shown in appendix 1.

For the main calculations, the predictors have the forms

\[ x(j) = 2x(j-1) - x(j-2) + h^2 \left[ F(j-1) + \frac{1}{12} (F(j-2) - F(j-1)) \right] \]

\[ y(j) = 2y(j-1) - y(j-2) + h^2 \left[ G(j-1) + \frac{1}{12} (G(j-2) - G(j-1)) \right] \]

and the correctors are

\[ x(j) = 2x(j-1) - x(j-2) + h^2 \left[ F(j-1) + \frac{1}{12} (F(j-2) - 2F(j-1) + F(j-2)) \right] \]

\[ y(j) = 2y(j-1) - y(j-2) + h^2 \left[ G(j-1) + \frac{1}{12} (G(j-2) - 2G(j-1) + G(j-2)) \right] \]

\[ j = 3, 4, \ldots, n+1 \]

d. The values of the integrals \( I_1 \) and \( I_2 \) are evaluated by Simpson's rule (13, p. 120). Since Simpson's rule uses the ordinates of three points with two intervals, this requires that \( n \) be even. The values \( F(j) \) and \( G(j) \) are calculated at each point, thus,

\[ I_1' = \frac{h}{3} \sum_{i=1}^{n/2} [F(2i-1) + 4F(2i) + F(2i+1)] \]
The error in Simpson's rule is given in the interval \((i-1, i+1)\) by (13, p. 121),

\[
e_1 = -\frac{h^5 F^{(4)}}{90}
\]

\[
e_2 = -\frac{h^5 G^{(4)}}{90}
\]

where \(F^{(4)}\) and \(G^{(4)}\) are the fourth derivatives of \(F\) and \(G\) with respect to time at some point between \((i-1)\) and \((i+1)\).

The fourth derivatives can be calculated numerically at any point using the finite differences. From (13, p. 128)

\[
h^4 F^{(4)} = [F(j+4) - 4F(j+3) + 6F(j+2) - 4F(j+1) + F(j)]
\]

\[
h^4 G^{(4)} = [G(j+4) - 4G(j+3) + 6G(j+2) - 4G(j+1) + G(j)].
\]

Thus the values for \(I_1\) and \(I_2\) are given by

\[
I_1 = \frac{h}{3} \sum_{i=1}^{n/2} [F(2i-1) + 4F(2i) + F(2i+1)] - \frac{1}{30} [F(2i+3) - 4F(2i+3) + 6F(2i+1) - 4F(2i) + F(2i-1)]
\]
\[ I_2 = \frac{h}{3} \sum_{i=1}^{n/2} \left[ G(2i-1) + 4G(2i) + G(2i+1) \right] - \frac{1}{30} \left[ G(2i+3) - 4G(2i+2) + 6G(2i+1) - 4G(2i) + G(2i-1) \right]. \]

Now if the chosen values of \( x(1) \) and \( y(1) \) are the correct ones, the values of \( I_1 \) and \( I_2 \) are approximately zero. Obviously this will not happen, and to get better values \( I_1 \) and \( I_2 \) are expanded in the neighborhood of \( x(1) \) and \( y(1) \)

\[ I_1 \left[ x(1) + \Delta x_o, y(1) + \Delta y_o \right] = 0 = I_1[\dot{x}(1), y(1)] + \Delta x \frac{\partial I_1}{\partial x(1)} + \Delta y \frac{\partial I_1}{\partial y(1)} \]  

(4.6) \[ I_2 \left[ x(1) + \Delta x_o, y(1) + \Delta y_o \right] = 0 = I_2[\dot{x}(1), y(1)] + \Delta x \frac{\partial I_2}{\partial x(1)} + \Delta y \frac{\partial I_2}{\partial y(1)} . \]

It is required to find \( \Delta x_o \) and \( \Delta y_o \) which will give \( x(1) \) and \( y(1) \) better approximations. The values of \( I_1[x(1), y(1)] \) and \( I_2[x(1), y(1)] \) have already been calculated.

\[ \frac{\partial I_1}{\partial x(1)}, \frac{\partial I_1}{\partial y(1)}, \frac{\partial I_2}{\partial x(1)}, \frac{\partial I_2}{\partial y(1)} \]

can be easily calculated as follows:

Set
\[ \dot{x}(1) = x(1) + h_1 \]
\[ \dot{y}(1) = y(1) \]

and calculate \( I_1'' \) and \( I_2'' \) as before. Therefore
\[
\frac{\partial I_1}{\partial x(1)} = \frac{I'' - I_1}{h_1}
\]

\[
\frac{\partial I_2}{\partial x(1)} = \frac{I'' - I_2}{h_2}
\]

\[
\frac{\partial I_1}{\partial y(1)} \quad \text{and} \quad \frac{\partial I_2}{\partial y(1)}
\]
can be obtained with the same way by setting

\[
y''(1) = y(1) + h_2
\]

\[
x''(1) = x(1).
\]

An exact derivative could be obtained if \( h_1 \) and \( h_2 \) approach zero. But for the numerical estimations of these derivatives one should be very careful in selecting values for \( h_1 \) and \( h_2 \) in order to avoid the round off error.

Having determined the derivatives, the values of \( \Delta x_o \) and \( \Delta y_o \) can be calculated from equations (4.6).

The new values of the initial displacements, given by

\[
x^*(1) = x(1) + \Delta x_o
\]

\[
y^*(1) = y(1) + \Delta y_o
\]

are used again as an approximation to get closer values for the initial displacements. When the changes in \( x(1) \) and \( y(1) \) are within the permissible error \( \epsilon \), the procedure ends and the solutions
are obtained. The period $T$ is changed by $\Delta T$ and the values of $x(I)$ and $y(I)$ of the previous calculation are used as first approximation and so on.

5. **The general class system**

In section 1 the restoring forces of the springs in the dynamical system are assumed to be continuous functions in some closed interval. It was mentioned also, according to Weierstrass' approximation theorem, that these functions may be represented by polynomials. In this respect the spring characteristics can take the forms

$$S_1(q) = \sum_{i=1}^{n} a_i q^i$$

$$S_2(q) = \sum_{i=1}^{\ell} b_i q^i$$

$$S_3(q) = \sum_{i=1}^{s} c_i q^i$$

Where $a_i$, $b_i$, and $c_i$ are arbitrary constants and $n$, $\ell$ and $s$ are integers.

The equations of motion can, then, be written in the form
\[ m_1 \ddot{x} = - \sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{\ell} b_i (y-x)^i \] \hspace{1cm} (5.1) \\
\[ m_2 \ddot{y} = - \sum_{i=1}^{\ell} b_i (y-x)^i - \sum_{i=1}^{s} c_i y^i \]

The potential energy is also given by

\[ U = \sum_{i=1}^{n} \frac{a_i}{i+1} x^{i+1} + \sum_{i=1}^{\ell} \frac{b_i}{i+1} (y-x)^{i+1} + \sum_{i=1}^{s} \frac{c_i}{i+1} y^{i+1}. \] \hspace{1cm} (5.2)

Equations (5.1) may be put in the form

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = \frac{1}{m_1} \left[ - \sum_{i=1}^{n} a_i x_1^i + \sum_{i=1}^{\ell} b_i (x_3 - x_1)^i \right] \]
\[ \dot{x}_3 = x_4 \]
\[ \dot{x}_4 = \frac{1}{m_2} \left[ - \sum_{i=1}^{\ell} b_i (x_3 - x_1)^i - \sum_{i=1}^{s} c_i x_3^i \right] \]

or, more compactly, \[ \dot{x}_j = f_j(x_i) \quad i, j = 1, 2, 3, 4. \]

The right hand sides of these equations, \( f_j \), are polynomials in \( x_i \) in the region \( R \). It follows that

\[ \frac{\partial f_j}{\partial x_i} \]
are continuous in $R$. Therefore the Lipschitz condition is satisfied and the existence and uniqueness of the solution for the system is assured.

To apply the algorithm for finding the normal modes it will be helpful to examine more closely the structure of the system. In other words it is necessary to know where to begin and how to proceed.

It has been shown that the modal line must pass through the origin of the $x$-$y$ plane. Also, for small displacements, the system could be considered linear if the linear terms in the spring characteristic are present. From here it is possible to get useful information to start with.

For small displacements, by neglecting terms with higher degrees than the first, equations (5.1) may be written in the form

$$m_1 \ddot{x} = -(a_1 + b_1)x + b_1y$$
$$m_2 \ddot{y} = b_1 x - (b_1 + c_1)y.$$

After changing the variables the above equations read

\[1\] If the linear terms are not included, the orthogonality property is still used by assuming sufficiently small initial displacements. In this case the system is considered to have straight modal relations as was pointed out in article (3-iii).
\[
\ddot{q}_1 = -\frac{(a_1 + b_1)}{m_1} q_1 + \frac{b_1}{\sqrt{m_1 m_2}} q_2
\]

\[
\ddot{q}_2 = \frac{b_1}{\sqrt{m_1 m_2}} q_1 - \frac{(b_1 + c_1)}{m_2} q_2
\]

where

\[
q_1 = \sqrt{m_1} x
\]

\[
q_2 = \sqrt{m_2} y
\]

The equation of the straight modal lines which pass through the origin is

\[
\frac{q_2}{q_1} = \frac{\frac{b_1}{\sqrt{m_1 m_2}} q_1 - \frac{(b_1 + c_1)}{m_2} q_2}{-\frac{(a_1 + b_1)}{m_1} q_1 + \frac{b_1}{\sqrt{m_1 m_2}} q_2}
\]

or

\[
\frac{b_1}{\sqrt{m_1 m_2}} q_1 - \frac{(b_1 + c_1)}{m_2} q_2 = -\frac{(a_1 + b_1)}{m_1} q_1 q_2 + \frac{b_1}{\sqrt{m_1 m_2}} q_2^2
\]

or

\[
\frac{b_1}{\sqrt{m_1 m_2}} (q_1^2 - q_2^2) = \left[\frac{(b_1 + c_1)}{m_2} - \frac{(a_1 + b_1)}{m_1}\right] q_1 q_2
\]

Let

\[
q_1 = r \cos \theta
\]

\[
q_2 = r \sin \theta.
\]

Then,
\[
\frac{b_1}{\sqrt{m_1 m_2}} \left( \cos^2 \theta - \sin^2 \theta \right) = \frac{m_1 (b_1 + c_1) - m_2 (a_1 + b_1)}{m_1 m_2} \sin \theta \cos \theta
\]

and hence

\[
\tan 2\theta = \frac{2b_1 \sqrt{m_1 m_2}}{m_1 (b_1 + c_1) - m_2 (a_1 + b_1)}.
\]

The above equation yields two values for \( \theta \) each of which corresponds to a normal mode

\[
q_2 = \tan \frac{\theta_1}{2} q_1
\]

or

\[
y = \sqrt{\frac{m_1}{m_2}} \tan \frac{\theta_1}{2} x.
\]

The frequency of oscillation of each mode is given by (10, p. 167)

\[
f_k = \frac{\omega_k}{2\pi}, \quad k = 1, 2
\]

where

\[
\omega_1 = \sqrt{\frac{2}{2} \left( \frac{a_1 + b_1}{m_1} + \frac{b_1 + c_1}{m_2} \right)} \pm \frac{1}{\sqrt{2}} \left[ \left( \frac{a_1 + b_1}{m_1} - \frac{b_1 + c_1}{m_2} \right) + \frac{4b_1^2}{m_1 m_2} \right].
\]

This equation yields two roots resulting in two natural frequencies.

One corresponds to the masses moving in-phase and the other to the out-of-phase motion.

The period of oscillation is given by

\[
T_{ok} = \frac{1}{f_k}.
\]
Having determined $T_{ok}$ it is easy to apply the algorithm by substituting

$$F = \frac{1}{m_1} \left[ -\sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{l} b_i (y-x)^i \right]$$

$$G = \frac{1}{m_2} \left[ -\sum_{i=1}^{l} b_i (y-x)^i - \sum_{i=1}^{s} c_i y^i \right]$$

and solving for $x(1)$ and $y(1)$, the initial displacements that correspond to $T_{ok}$. As a first approximation $x(1)$ and $y(1)$ are chosen small and lie on the line

$$y = \sqrt{\frac{m_1}{m_2}} \tan \theta_k x.$$

The algorithm will yield the exact values of $x(1)$ and $y(1)$. To determine more points on the modal line it is necessary to know the relation between the initial displacements $x(1)$ and $y(1)$ and the period $T$, that is, how $T$ changes as the initial displacements increase. For nonlinear systems the period increases or decreases with the initial displacements according to whether the springs have a softening or a hardening effect. The terms hardening and softening are used as the spring stiffness increases or decreases with the extension. For symmetrical springs the general characteristic of the spring is either hardening or softening for extension and compression. But for a nonsymmetrical spring, it could have, for instance,
a hardening effect under extension and a softening effect under compression. If one complete cycle is to be considered the period of oscillation of a nonsymmetrical spring-mass system will either increase or decrease with the initial displacement depending on the spring characteristics. For example it was shown in (15, p. 21) that for the air spring, which is nonlinear nonsymmetrical, the period of oscillation increases with the initial displacements. That is, the air spring is equivalent to a spring with a softening effect. Generally speaking, it is not easy to predict which property a given system possesses. It is recommended then to plot the curves $I_1 = 0$ and $I_2 = 0$ for some points surrounding the origin in the x-y plane for a positive and a negative increment $\Delta T$ in the period $T_0$. One of these two periods will result in the intersection of the two curves near the origin. The increase or decrease of the period with the increase of the initial displacements can, then, be determined.

The origin, point (0, 0) in the x-y plane, is a solution for the dynamical system for any period of oscillation. For values of $T$ close to the period $T_0$ of the linearized system, the initial displacements $x(1)$ and $y(1)$, which provide the solution for the normal mode motion, are small. First approximations for these initial displacements are to be chosen to have values higher than the expected ones to insure convergence to the desired solution rather than to the trivial zero. Now when $T_0$ is changed by $\Delta T$, and the
previous initial displacements are used as an approximation for the new iteration, $\Delta T$ must be sufficiently small if the corresponding solution is to be obtained; otherwise one gets the trivial solution again.

To apply the algorithm efficiently, the curves $I_1 = 0$ and $I_2 = 0$ are plotted for different values of $T$. This has the advantage of giving an idea of how the system might behave. It also shows values of $T$ for which a normal mode motion does not exist. This can be easily seen when the two curves do not intersect. The plotting of the curves helps also to find out the possibility of the existence of more than two modal lines. This cannot be shown by the linearized system because it must have only two modes.

In doing this values of $T$ are selected and suitable first approximations for the initial displacements are used. $\Delta T$ is chosen to get smaller values for the initial displacements on the modal line. That is one starts with high points on the modal line and works to get the smaller values.

6. **Considerations on Stability**

The interest of the current discussion is concerned with systems oscillating in normal modes. Since this is a possible solution, it must follow the general considerations on stability. The different kinds of stability are stated briefly in (14, p. 118).
The application of the variational methods for the study of stability requires the knowledge of a solution, called the nonperturbed solution. Another solution which corresponds to some initial displacement is considered and is assumed to differ from the first solution by a small amount \( \xi(t) \), which is called the perturbation. The procedure leads to a system of variational equations for \( \xi(t) \). This system of equations will yield conditions for the different kinds of stability.

For nonlinear nonsymmetrical systems, solutions, in many cases, are obtained by numerical means, and the application of the variational methods cannot be easily performed. However, useful information concerning stability can be attained by following the method of Liapounov (14, p. 136).

Consider a dynamical system represented by the system of equations

\[
\dot{x}_i = X_i(x_1, x_2, \ldots, x_n)
\]

where \( X_i \) are Lipschitzian with a singular point at \( x_1 = x_2 = \ldots = x_n = 0 \).

Liapounov's first theorem states that the equilibrium for the above system is stable if it is possible to determine a definite function \( V' = V_d \) in some domain \( D \) whose Eulerian derivative \( \frac{dV'}{dt} \)

\(^1\) A function is called definite in \( D \) if it maintains the same sign and vanishes only at the origin.
is semi-definite\(^1\) with opposite sign to \(V_d\) or which vanishes identically in \(D\).

\[
\frac{dV'}{dt} = \sum_{i=1}^{n} \frac{\partial V'}{\partial x_i} \frac{\partial x_i}{\partial t} = \sum_{i=1}^{n} \frac{\partial V'}{\partial x_i} x_i.
\]

The equation \(V'(x_1, x_2, \ldots, x_n) = c\) represents a family of surfaces in the Euclidian space. The theorem always holds if these surfaces are closed.

The Liapounov function \(V_d\) can be interpreted physically as follows: If any trajectory \(S\) is issuing at \(t = t_0\) from a point then for \(t > t_0\) it is impossible for \(S\) to intersect \(V = c\) from inside to outside.

In fact the total potential function of a conservative system \(U(x_i)\) has the same property as the Liapounov function. The trajectories will never cross a closed total energy surface (the surface of constant energy level) from inside to outside. In the following section stability of normal modes are deduced from the configuration of the energy lines (for two-degrees-of-freedom systems) and the modal lines.

\(^1\)A function is called semi-definite if it has the same sign or zero in \(D\).
The idea of the modal phase plane

This section will be concerned with the stability of normal modes in the sense that the amplitudes remain bounded or become unbounded. In this case \( V_d(x, y) = c \) forms a family of curves in the \( x-y \) plane, where \( V_d \) is the Liapounov function.

Liapounov's method provides a criterion for stability when these curves are closed. But it does not tell what would happen if they are open.

Since the potential energy function \( U(x, y) \) is considered a Liapounov function, then the construction of the curves \( U(x, y) = c \) and the application of the fact that the system is conservative may be used to study the stability of the system. At points corresponding to stable equilibrium, \( U(x, y) = c \) form closed curves at least for sufficiently small \( c \). Consider regions bounded by closed lines of total energy. The existence theorem guarantees the existence and the uniqueness of the solutions in these regions. Among those solutions it might be possible to find some which satisfy the conditions for the normal mode motion. The modal lines can, then, be constructed in these regions. The normal mode motions are stable as long as the amplitudes are bounded by the closed total energy lines.

The question now rises, "what could happen to a normal mode motion in the region bounded by open curves?" The construction of the lines
of constant energy as well as the modal lines on the x-y plane easily answers this question. This construction helps in defining the singularities for systems with two degrees of freedom in parallel to the singular points defined by the phase plane for systems with one degree of freedom. For this reason this formation is given the name modal phase plane.

The shape of the energy lines depends on the characteristics of the springs in the system. There are two basic spring characteristics which lead to the concepts of singularity in the modal phase plane.

Case I.

Each spring force has a zero only at the origin, Figure 2.

In this case all the total energy lines are closed. There is no difficulty in applying the algorithm to determine the modal lines and the modal relation curves. The masses of a nonlinear nonsymmetrical system usually do not pass through the equilibrium position except for some special cases. For a particular oscillation the masses come to a point where the kinetic energy is maximum and the potential energy is minimum. This point can be considered as a center for the oscillation. It is determined as the tangent point of the modal relation curve and a total energy curve. Therefore the line of centers can be formed and it is the locus of the tangent points
Figure 2. Spring characteristics for Case I.

of the modal relation curves and the total energy lines. This is illustrated in Figure 3.

The total energy lines in Figure 3 are a plot for a system whose restoring forces are polynomial of the third degree. In the next chapter this configuration is shown for the air spring systems.

In the results from the air spring system (discussed later) the line of centers is issuing from the origin and is directed on one side of the modal line. This is because all the modal relation curves have curvature in essentially the same direction.
Figure 3. Modal phase plane for Case I
Case II.

Each spring force has zero at the origin and at some other point, Figure 4.

![Diagram](image_url)

Figure 4. Spring characteristics for Case II.

The total energy lines $U(x, y) = c$ are closed for small $c$ and are opened for larger values. The line separating both groups, $c = c_1$, can be called the separatrix.

Now consider displacements characterized by the point $C$ on the separatrix, Figure 5. Both regions, the one inside the loop and that limited by the line ACB, have lower energy levels than that of the point $C$ itself. In this case the masses have tendency to go to either region depending on the disturbance. That is the system is in unstable equilibrium. The total energy lines are not closed outside
Figure 5. Modal phase plane for Case II
the separatrix, i.e. for $c > c_1$. Any motion, which is not normal mode motion, and which starts on these lines may cause the displacements to increase indefinitely with the potential energy still decreasing. However, when the system is oscillating in a normal mode there is a definite relation between the displacements at any time. That is, the displacements of both masses are restricted within the modal relation. There is no chance for the displacements to take arbitrary values which might lead to instability as long as the system is in a normal mode motion. It follows that the motion might be stable for some open total energy lines unless the modal line passes through point $C$. In fact there is no assurance that this happens and the modal line might miss this point. There are, however, limiting amplitudes after which a normal mode motion ceases to exist.

The total energy lines outside the separatrix and close to it tend to follow it. They have peaks near the point $C$. A line passing through $C$ and normal to the total energy lines will pass through those peaks and is of some significance for the normal modes. Any motion that starts at any point on this line will cause the displacements to go either inside the loop or to the open region. It divides, then, the x-y plane to two regions namely stable and unstable. It might be given the name line of saddles according to this property. The points of intersection $M_i$ of the line of saddles with the modal
lines are unstable points for the normal mode motions. In other words modal lines vanish in the unstable region after these points.

Of course the line of centers is formed in the closed regions as in the previous case.

There are still other spring characteristics which lead to very interesting phenomena. The discussions of these follow closely those of the previously mentioned two cases.

Consider for example a system whose springs have the characteristic shown in Figure 6. Each spring force has three intersections with the displacement axis.

![Graph showing spring characteristics](image)

**Figure 6.** Spring characteristics for a particular system.
The modal phase plane is shown in Figure 7. It is seen that the potential lines have two closed families of curves separated by a loop (the separatrix). For higher energy levels they combine to a single closed family. The center of each group of the closed curves $O_1$ and $O_2$ is the point at which the potential lines shrink to a point. These points, then, have the lowest energy level among its group. The masses when located at the coordinates of these centers remain in stable equilibrium. The line of saddles can be formed from point $C$ on the separatrix as was outlined before.

By applying the algorithm for finding the normal modes it is possible to plot the modal lines issuing from each of the centers $O_1$ and $O_2$. These modal lines exist before they intersect the line of saddles at $M_i$. The points $M_i$ are unstable points for the normal mode motions and they do not exist behind these points unless the modal lines which belong to $O_1$ and $O_2$ meet at exactly the same point on the line of saddles. In this case normal mode motion can be obtained in either region if the motion starts at this point $M_i$. However, this situation is considered as a special case and the modal lines intersect the line of saddles independently at $M_i$. At any point $M_i$ the motion is in normal mode for a region and random for the other. This random motion takes place inside the closed total energy line which has an energy level equal to that of the point $M_i$. 
Figure 7. Modal phase plane for a particular case
II. NORMAL MODE MOTIONS FOR AN AIR SPRING SYSTEM

1. Characteristics of the Air Spring

The air spring in its simplest configuration consists of a cylinder and a piston carrying a mass. Air is confined in the cylinder by the piston. At the position of equilibrium the air pressure equalizes the weight applied, hence

\[ P_o A = mg + P_a A. \]

\( P_o \) is the absolute pressure at the equilibrium position.
\( A \) is the cross sectional area of the cylinder.
\( P_a \) is the atmospheric pressure.

![Figure 8. Air spring unit](image)

The pressure volume relation for the air is given by

\[ PV^Y = P_o V_o^Y. \]

\( P \) and \( V \) are the absolute pressure and the volume at any time.
\[ y \] is the index of compression and expansion.

\[ V_o = AL \]

\[ V = A(L + z) \]

where \( L \) denotes the air column length at equilibrium and \( z \) is the displacement measured from the equilibrium position. Substituting above, then

\[ PA^y (L + z)^y = P_o A^y L^y \]

or

\[ PA = P_o A \frac{L^y}{(L+z)^y} \]

\[ = (mg + Pa) \frac{L^y}{(L+z)^y} \]

\( PA \) is the force in the spring. The force necessary to hold the mass in equilibrium is

\[ S = PA - (mg + Pa) \]

or

\[ S = (mg + Pa) \frac{L^y}{(L+z)^y} - (mg + Pa) \cdot \]

Let

\[ x = \frac{z}{L} \]

then

\[ S = (mg + Pa) \left[ \frac{1}{(1+x)^y} - 1 \right] \]

This is the equation which gives the characteristics of the air spring.

The force displacement relation of the air spring is shown in Figure 9.
Obviously the air spring is nonsymmetrical and its characteristic is similar to that discussed in Case I, sec. 7 in the previous chapter.

2. The air spring system

Figure 9. The characteristics of the air spring

Figure 10. The air spring system
The air spring system consists of two air spring units each supporting the mass \( m_1 \) and \( m_3 \) respectively. The two units are connected together with a coupling beam of mass \( m_2 \). The special case where \( m_1 \) and \( m_3 \) are equal is studied. However, \( m_1 \) and \( m_3 \) could be unequal and this situation does not give particular difficulty. The moving mass of the whole system is given by

\[
m = m_1 + m_2 + m_3
\]
or in the studied particular case

\[
m = 2m_1 + m_2.
\]

The two air spring units are considered to have the same cross sectional area.

The system has two degrees of freedom, e.g., a translation of the center of gravity of the system in the vertical direction (the displacement is denoted by \( z \)) and a rotation around it. The angular rotation is denoted by \( \theta \). This is shown in Figure 11.

The masses \( m_1 \) and \( m_3 \) are considered to be concentrated at the end of the beam. The overall moment of inertia of the system is, then, given by

\[
I = 2m_1 a^2 + \frac{1}{3} m_2 a^2
\]

\[
= a^2 (2m_1 + \frac{1}{3} m_2)
\]

\[
= ma^2 \left[ \frac{2m_1 + \frac{1}{3} m_2}{m} \right]
\]
or finally

\[ I = ma^2 \]

where

\[ \lambda = \frac{2m_1 + \frac{1}{3}m_2}{m} \]

3. The equations of motion

![Diagram of displacements of the system]

Figure 11. Displacements of the system

The displacement \( z \) of the center of gravity is positive in the upward direction. The displacement of the mass on each unit is denoted by \( z_1 \) and \( z_2 \) respectively. The angle \( \theta \) is measured from the equilibrium position positive in the clockwise direction. For small \( \theta \), which shall be assumed,

\[ z_1 = z + a\theta \]

\[ z_2 = z - a\theta \]
or
\[
z = \frac{z_1 + z_2}{2}
\]
\[
\theta = \frac{z_1 + z_2}{2a}
\]

The kinetic energy is given by
\[
T_e = \frac{1}{2} m z^2 + \frac{1}{2} I \theta^2.
\]

The potential energy \( U \) has the form
\[
U = (m_1 g + P_a)z_1 + (m_2 g + P_a)z_2 - \int_0^{z_1} P \text{Ad}z - \int_0^{z_2} P \text{Ad}z.
\]

But
\[
\text{PA} = (m_1 g + P_a) \left( \frac{L_i}{(L_1 + z_1)^{\gamma}} \right)
\]

for \( i = 1, 2 \)

where \( m_i \) is the mass carried by each spring. If the whole mass \( m \) is considered to be supported equally on each spring unit, then
\[
m_i = \frac{m}{2}
\]

\[
(m_1 g + P_a)z_1 + (m_2 g + P_a)z_2 = (mg + 2P_a)z
\]

and hence
\[
U = (mg + 2P_a)z - \left( \frac{mg}{2} + P_a \right) \int_0^{z_1} \frac{L_1^\gamma}{(L_1 + z)^{\gamma}} \, dz + \left( \frac{mg}{2} + P_a \right) \int_0^{z_2} \frac{L_2^\gamma}{(L_2 + z)^{\gamma}} \, dz
\]
\[ U = (mg + 2P_a A)z + \frac{(mg + 2P_a A)}{2(\gamma - 1)} \left[ \frac{L_1}{(L_1 + z_1)^{\gamma - 1}} + \frac{L_2}{(L_2 + z_2)^{\gamma - 1}} \right] - L_1 - L_2 \]

\[ = (mg + 2P_a A)z - \frac{(mg + 2P_a A)}{2(\gamma - 1)} \left[ \frac{L_1}{(L_1 + z + a\theta)^{\gamma - 1}} + \frac{L_2}{(L_2 + z - a\theta)^{\gamma - 1}} \right] - L_1 - L_2. \]

Applying Lagrange's equation, then

\[ m\ddot{z} + (mg + 2P_a A) = \frac{(mg + 2P_a A)}{2} \left[ \frac{L_1}{(L_1 + z + a\theta)^{\gamma}} + \frac{L_2}{(L_2 + z - a\theta)^{\gamma}} \right] = 0 \]

\[ I\ddot{\theta} - \frac{(mg + 2P_a A)a}{2} \left[ \frac{L_1}{(L_1 + z + a\theta)^{\gamma}} - \frac{L_2}{(L_2 + z - a\theta)^{\gamma}} \right] = 0. \]

Setting

\[ x = \frac{z}{L_1} \]

\[ a = \frac{a}{L_2} \]

\[ \beta = \frac{a}{L_1} \]

\[ r = \frac{L_1}{L_2} \]

\[ \omega = \frac{2P_a A}{mg} \]

(3.1)
and substituting into the equations of motion using

\[ I = m a^2 \lambda, \text{ one gets} \]

\[ \ddot{y} = \frac{g}{2L_1} (1 + \sigma) \left[ \frac{1}{(1 + x + a \theta)^\gamma} + \frac{1}{(1 + rx - \beta \theta)^\gamma} \right] \]

\[ \theta = \frac{g}{2 \lambda a} (1 + \sigma) \left[ \frac{1}{(1 + x + a \theta)^\gamma} - \frac{1}{(1 + rx - \beta \theta)^\gamma} \right]. \]

To check the existence of the solution the equations of motion are put in the form

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = \frac{g}{2L_1} (1 + \sigma) \left[ \frac{1}{(1 + x_1 + a x_3)^\gamma} - \frac{1}{(1 + rx_1 - \beta x_3)^\gamma} \right] \]

\[ \dot{x}_3 = x_4 \]

\[ \dot{x}_4 = \frac{g}{2 \lambda a} (1 + \sigma) \left[ \frac{1}{(1 + x_1 + a x_3)^\gamma} - \frac{1}{(1 + rx_1 - \beta x_3)^\gamma} \right]. \]

For the second and fourth equations the partial derivative of the right hand sides with respect to \( x_1 \) and \( x_3 \) are not continuous at

\[ x_1 + a x_3 \leq -1 \]

and at

\[ rx_1 - \beta x_3 \leq -1. \]

This means that the solutions exists for any region not containing
points from the region defined by the above two inequalities.

4. **Theoretical analysis of the normal mode motions**

   The algorithm for determining the normal mode motions outlined in section 5, chapter I could be applied for the air spring system. As was recommended, the motion for small displacements is to be studied to get values to start with. In this case the equations of the linearized system are in the form

\[
x + \frac{gY}{2L_1} (1 + \sigma) \left[(1 + r)x + (a - \beta)\theta\right] = 0
\]

\[
y + \frac{gY}{2\lambda a} (1 + \sigma) \left[(1 - r)x + (a + \beta)\theta\right] = 0
\]

The period of oscillation is, then, given by

\[
T_{01} = \frac{2\pi}{\omega_{12}}
\]

where

\[
\omega_{12}^2 = \frac{gY}{2} (1 + \sigma) \left[\frac{(1+r)}{L_1} + \frac{(a+\beta)}{\lambda a} \pm \sqrt{\frac{(1+r)}{L_1} - \frac{(a+\beta)^2}{\lambda a} + \frac{4(1-r)(a-\beta)}{\lambda a L_1}}\right]
\]

As was mentioned before (p. 45), the air spring is equivalent to a spring with a softening effect. That is the period of oscillation increases with increasing initial displacement. Furthermore, the change in the period is small as the initial displacement changes (15, p. 21-24). According to these facts it is easy, then, to choose
suitable values for the period which correspond to initial displacements away from the origin. However, a program is made to plot the values of the integrals $I_1$ and $I_2$ given by equations (4.5) chapter 1. This will enable one to plot the curves $I_1 = 0$ and $I_2 = 0$ to determine their points of intersection. This program is written in Fortran language and is stated in Appendix 1. The algorithm for determining the normal mode motions is applied by substituting

\[
F(x, y) = \frac{g}{2L_1} (1+\sigma) \left[ \frac{1}{(1 + x + ay)^\gamma} + \frac{1}{(1 + rx - \beta y)^\gamma} - 2 \right]
\]

\[
G(x, y) = \frac{g}{2\lambda a} (1+\sigma) \left[ \frac{1}{(1 + x + ay)^\gamma} - \frac{1}{(1 + rx - \beta y)^\gamma} \right]
\]

where $\theta$ is replaced by $y$.

In Appendix 1 the condition necessary for the convergence of the central difference method is derived. The restriction on the length of the interval $h$ is given by

\[
h < \sqrt{\frac{6}{D + E}}
\]

where $D$ is the larger of $\left| \frac{\partial F}{\partial x} \right|$ and $\left| \frac{\partial G}{\partial x} \right|$ and $E$ is the larger of $\left| \frac{\partial F}{\partial y} \right|$ and $\left| \frac{\partial G}{\partial y} \right|$.
\[
\frac{\partial F}{\partial x} = \frac{gY}{2L_1} (1 + \sigma) \left[ -\frac{1}{(1 + x + ay)^{Y+1}} + \frac{r}{(1 + rx - \beta y)^{Y+1}} \right]
\]

\[
\frac{\partial G}{\partial x} = \frac{gY}{2\lambda a} (1 + \sigma) \left[ -\frac{1}{(1 + x + ay)^{Y+1}} + \frac{r}{(1 + rx - \beta y)^{Y+1}} \right]
\]

The denominators on the right hand sides of both equations are always positive. Also for practical configurations \(a\) is much greater than \(L_1\). Hence

\[
\left| \frac{\partial F}{\partial x} \right| > \left| \frac{\partial G}{\partial x} \right|
\]

and

\[
D = \frac{gY}{2L_1} (1 + \sigma) \left[ -\frac{1}{(1 + x + ay)^{Y+1}} + \frac{r}{(1 + rx - \beta y)^{Y+1}} \right]
\]

\[
\frac{\partial F}{\partial y} = \frac{gY}{2L_1} (1 + \sigma) \left[ -\frac{a}{(1 + x + ay)^{Y+1}} + \frac{\beta}{(1 + rx - \beta y)^{Y+1}} \right]
\]

\[
\frac{\partial G}{\partial y} = \frac{gY}{2\lambda a} (1 + \sigma) \left[ -\frac{a}{(1 + x + ay)^{Y+1}} - \frac{\beta}{(1 + ax + \beta y)^{Y+1}} \right].
\]

The choice of \(E\) depends on the values of \(L_1, \lambda a, \frac{1}{(1 + x + ay)^{Y+1}}\) and \(\frac{1}{(1 + rx - \beta y)^{Y+1}}\).

The last two are the most critical. They increase indefinitely as either \((x + ay)\) or \((rx - \beta y)\) approaches minus one. If the two masses are moving in-phase, \(E\) is given the value \(\left| \frac{\partial G}{\partial y} \right|\).
otherwise it is given the value $\left| \frac{\partial F}{\partial y} \right|$. It should be noted that both D and E must be given the maximum values in the region bounding the displacement in a particular motion.

The values of the parameters in the equations of motion are chosen to be the same as those of the experimental apparatus which is described later on. This makes it possible to compare between the theoretical and the experimental results.

$$A = \frac{\pi}{4} (4)^2 \text{ in}^2.$$  
$$a = 42.576 \text{ in.}$$  
$$m_1 g = m_3 g = 142 \text{ lbs.}$$  
$$m_2 g = 98 \text{ lbs.}$$  

In this case

$$\lambda = 0.827676.$$  

The atmospheric pressure $P_a$ is equal to $14.7 \text{ lb/in}^2$

hence

$$\sigma = 0.966.$$  

$L_1$ and $L_2$ are adjustable and, then, can assume different values.

The time for running one test is short and the amount of heat transferred, if any, is negligibly small. The expansion and compression is, then, considered to be adiabatic. $\gamma$ is given the value 1.4.
5. **Theoretical results**

The algorithm for determining the normal mode motion has been applied. The program is written in Fortran language and the IBM 7090 computer has been used. The output of the program gives points on the modal lines as well as the displacements time response. Tables 1 and 2 are examples of the outputs.

**Table 1. Time-displacements response for an in-phase motion.**

<table>
<thead>
<tr>
<th>J</th>
<th>x(J)</th>
<th>y(J)</th>
<th>J</th>
<th>x(J)</th>
<th>y(J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.536665</td>
<td>0.051823</td>
<td>17</td>
<td>-0.340212</td>
<td>-0.025520</td>
</tr>
<tr>
<td>2</td>
<td>0.529622</td>
<td>0.051367</td>
<td>18</td>
<td>-0.296080</td>
<td>-0.020027</td>
</tr>
<tr>
<td>3</td>
<td>0.508586</td>
<td>0.049998</td>
<td>19</td>
<td>-0.229608</td>
<td>-0.012254</td>
</tr>
<tr>
<td>4</td>
<td>0.473906</td>
<td>0.047708</td>
<td>20</td>
<td>-0.148001</td>
<td>-0.003419</td>
</tr>
<tr>
<td>5</td>
<td>0.426180</td>
<td>0.044485</td>
<td>21</td>
<td>-0.057821</td>
<td>0.005581</td>
</tr>
<tr>
<td>6</td>
<td>0.366288</td>
<td>0.040315</td>
<td>22</td>
<td>0.035500</td>
<td>0.014184</td>
</tr>
<tr>
<td>7</td>
<td>0.295436</td>
<td>0.035187</td>
<td>23</td>
<td>0.127623</td>
<td>0.022078</td>
</tr>
<tr>
<td>8</td>
<td>0.215221</td>
<td>0.029099</td>
<td>24</td>
<td>0.215133</td>
<td>0.029103</td>
</tr>
<tr>
<td>9</td>
<td>0.127718</td>
<td>0.022073</td>
<td>25</td>
<td>0.295357</td>
<td>0.035190</td>
</tr>
<tr>
<td>10</td>
<td>0.035598</td>
<td>0.014180</td>
<td>26</td>
<td>0.366221</td>
<td>0.040316</td>
</tr>
<tr>
<td>11</td>
<td>-0.057724</td>
<td>0.005576</td>
<td>27</td>
<td>0.426127</td>
<td>0.044485</td>
</tr>
<tr>
<td>12</td>
<td>-0.147911</td>
<td>-0.003423</td>
<td>28</td>
<td>0.473867</td>
<td>0.047706</td>
</tr>
<tr>
<td>13</td>
<td>-0.229531</td>
<td>-0.012257</td>
<td>29</td>
<td>0.508562</td>
<td>0.049994</td>
</tr>
<tr>
<td>14</td>
<td>-0.296024</td>
<td>-0.020029</td>
<td>30</td>
<td>0.529613</td>
<td>0.051361</td>
</tr>
<tr>
<td>15</td>
<td>-0.340181</td>
<td>-0.025521</td>
<td>31</td>
<td>0.536672</td>
<td>0.051815</td>
</tr>
<tr>
<td>16</td>
<td>-0.355772</td>
<td>-0.027529</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2. Time-displacements response for an out-of-phase motion.

<table>
<thead>
<tr>
<th>J</th>
<th>( x(J) )</th>
<th>( y(J) )</th>
<th>J</th>
<th>( x(J) )</th>
<th>( y(J) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.211627</td>
<td>-0.071158</td>
<td>17</td>
<td>-0.088412</td>
<td>0.046441</td>
</tr>
<tr>
<td>2</td>
<td>0.209647</td>
<td>-0.070230</td>
<td>18</td>
<td>-0.069597</td>
<td>0.040055</td>
</tr>
<tr>
<td>3</td>
<td>0.205708</td>
<td>-0.067460</td>
<td>19</td>
<td>-0.042441</td>
<td>0.030656</td>
</tr>
<tr>
<td>4</td>
<td>0.193805</td>
<td>-0.062900</td>
<td>20</td>
<td>-0.010692</td>
<td>0.019382</td>
</tr>
<tr>
<td>5</td>
<td>0.179945</td>
<td>-0.048782</td>
<td>21</td>
<td>0.022710</td>
<td>0.007164</td>
</tr>
<tr>
<td>6</td>
<td>0.162168</td>
<td>-0.048782</td>
<td>22</td>
<td>0.055702</td>
<td>-0.005297</td>
</tr>
<tr>
<td>7</td>
<td>0.140570</td>
<td>-0.039500</td>
<td>23</td>
<td>0.086898</td>
<td>-0.017477</td>
</tr>
<tr>
<td>8</td>
<td>0.115350</td>
<td>-0.028985</td>
<td>24</td>
<td>0.115394</td>
<td>-0.028979</td>
</tr>
<tr>
<td>9</td>
<td>0.086856</td>
<td>-0.017483</td>
<td>25</td>
<td>0.140615</td>
<td>-0.039493</td>
</tr>
<tr>
<td>10</td>
<td>0.055663</td>
<td>-0.005302</td>
<td>26</td>
<td>0.162212</td>
<td>-0.048775</td>
</tr>
<tr>
<td>11</td>
<td>0.022676</td>
<td>0.007160</td>
<td>27</td>
<td>0.179989</td>
<td>-0.056627</td>
</tr>
<tr>
<td>12</td>
<td>-0.010722</td>
<td>0.019379</td>
<td>28</td>
<td>0.193846</td>
<td>-0.062893</td>
</tr>
<tr>
<td>13</td>
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<td>0.030654</td>
<td>29</td>
<td>0.203746</td>
<td>-0.067454</td>
</tr>
<tr>
<td>14</td>
<td>-0.069614</td>
<td>0.040054</td>
<td>30</td>
<td>0.209680</td>
<td>-0.070225</td>
</tr>
<tr>
<td>15</td>
<td>-0.088420</td>
<td>0.046440</td>
<td>31</td>
<td>0.211654</td>
<td>-0.071154</td>
</tr>
<tr>
<td>16</td>
<td>-0.095228</td>
<td>0.048728</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Typical time displacements responses for \( r = 1.43 \) are illustrated in Figure 12 for the in-phase mode motion and in Figure 13 for the out-of-phase mode motion.
Figure 12. Time-displacement response for an in-phase mode "r = 1.43"
Figure 13. Time-displacement response for an out-of-phase mode "r = 1.43"
The velocities of the masses at the middle of a complete cycle of a system oscillating in a normal mode are zero. The displacements which the masses have at this time could be considered as initial displacements which set the system in normal mode when it starts from rest. The coordinates of these displacements fall on a modal line. These points have the values \( x\left(\frac{n}{2} + 1\right) \) and \( y\left(\frac{n}{2} + 1\right) \) if \( n \) is the number of intervals (\( n \) is even). The computed values of the initial displacement together with the period of oscillation are listed in Table 3 for different values of \( r \) and for the in-phase and the out-of-phase modes. The results are presented graphically in Figures 14, 15, 16 and 17. Figures 14 and 15 show the variation of the period of oscillation with the initial displacement for the two different modes. Figures 16 and 17 show the modal lines for the same two kinds of modes.

The modal phase planes are illustrated in Figures 18 and 19.
Table 3. The relations between initial displacement and the period of oscillation - Theoretical

<table>
<thead>
<tr>
<th>r</th>
<th>T</th>
<th>x(1)</th>
<th>y(1)</th>
<th>x((\frac{\pi}{2}))</th>
<th>y((\frac{\pi}{2}))</th>
<th>T</th>
<th>x(1)</th>
<th>y(1)</th>
<th>x((\frac{\pi}{2}))</th>
<th>y((\frac{\pi}{2}))</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>.649</td>
<td>1.183</td>
<td>0.0</td>
<td>-.602</td>
<td>0.0</td>
<td>.600</td>
<td>.329</td>
<td>.219</td>
<td>.331</td>
<td>.219</td>
</tr>
<tr>
<td>1.11</td>
<td>.635</td>
<td>.830</td>
<td>.107</td>
<td>-.445</td>
<td>-.044</td>
<td>.579</td>
<td>.359</td>
<td>.203</td>
<td>.184</td>
<td>.176</td>
</tr>
<tr>
<td>1.25</td>
<td>.628</td>
<td>.723</td>
<td>0.115</td>
<td>-.388</td>
<td>-.053</td>
<td>.565</td>
<td>.437</td>
<td>.207</td>
<td>.108</td>
<td>.165</td>
</tr>
<tr>
<td>1.43</td>
<td>.628</td>
<td>.708</td>
<td>0.128</td>
<td>-.364</td>
<td>-.061</td>
<td>.530</td>
<td>.413</td>
<td>.162</td>
<td>.037</td>
<td>.103</td>
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<tr>
<td>1.67</td>
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<td>.700</td>
<td>0.136</td>
<td>-.346</td>
<td>-.066</td>
<td>.488</td>
<td>.354</td>
<td>.119</td>
<td>.095</td>
<td>.070</td>
</tr>
<tr>
<td>2.00</td>
<td>.624</td>
<td>.655</td>
<td>0.137</td>
<td>-.332</td>
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<td>.460</td>
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<tr>
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<td>-.294</td>
<td>.068</td>
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<td></td>
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<td>.038</td>
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<td>.390</td>
<td>.185</td>
<td>-.054</td>
<td>.083</td>
<td>.033</td>
</tr>
</tbody>
</table>

Note: The values in the table represent the period of oscillation in seconds. The in-phase and out-of-phase modes are represented by the columns 'T', 'x(1)', 'y(1)', 'x(\(\frac{\pi}{2}\))', and 'y(\(\frac{\pi}{2}\))'.
Figure 14. The relations between the period $T$ and the initial displacements for the in-phase modes
Figure 15. The relations between the period $T$ and the initial displacements for the out-of-phase modes.
Figure 16. Modal lines for the in-phase modes
Figure 17. Modal lines for the out-of-phase modes
Figure 18. Modal phase plane for an in-phase motion "r = 1.25"
Figure 19. Modal phase plane for an out-of-phase mode "r = 1.25"
6. Experimental work

i. Apparatus and methods. The apparatus used in conducting the experimental work is shown in Figure 20. It consists of two air spring units connected together by a coupling beam. Each air spring unit is formed of a cylinder, a lower piston, an upper piston and a heavy mass. The lower piston is fixed in position while performing the experiments and is used for varying the air column length. The upper piston is connected to the heavy mass and is supported by the air under pressure. The upper piston is free to move in the vertical direction. The alignment of this piston is done by guiding the mass from four directions with eight ball bearings rolling on guiding channels. Each air spring unit is resting on a concrete slab.

To obtain a normal mode motion, the mass of each spring unit has to be released at the same time from certain positions. A mechanism was designed and constructed for this purpose, Figure 21. It consists of two pieces of channel section hinged together from one edge to form a clamp. A tension spring is used to hold this clamp in position. A selenoid is also used to open the clamp when an electric current is applied for releasing the mass.

The recording of motion time history is obtained by a movie camera. A pointer is extended from the mass of each spring unit. The ends of the pointers come close to within one foot from each
Figure 20. The apparatus
Figure 21. The releasing mechanism
other. Two scales are placed behind each pointer end and both are mounted on a white screen. The filming was made at a speed of 64 frames per second. Some shots were taken at the ordinary speed of 16 frames per second. The number of frames for a complete cycle is counted and the period of oscillation was computed.

ii. Experimental results. Experiments have been performed for different values of $r$ (air columns ratio). The masses of each air spring unit is set in motion by using initial displacements taken from theoretical results. The recorded motion showed that both masses come into stationary motion at the same time after one complete oscillation. This indicated that the system is oscillating in a normal mode.

The mass of each spring in the apparatus is restricted to a maximum displacement of 3.5 inches above and below the equilibrium position according to some construction limitation. For this reason the experiments were conducted for a maximum displacement of three inches for each value of $r$. The experimental data are tabulated in Table 4. $x_o$, $y_o$ denotes initial displacement.

Experimental results are plotted in Figures 22 and 23.
Table 4. Experimental results.

| r   | T     | z₁₀ | z₂₀ | x₀  | y₀  | T     | z₁₀ | z₂₀ | x₀  | y₀  |
|-----|-------|-----|-----|-----|-----|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---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-----
Figure 22. Experimental results for the variation of the period with the initial displacements "In-phase modes"
Figure 23. Experimental results for the variation of the period with the initial displacements "Out-of-phase modes"
The experimental results have accomplished a verification of the theory. By using values obtained from the theoretical analysis as initial displacements for the experiments, the experimental system oscillated in normal modes. However, there are differences between the natural frequencies obtained by both ways (theoretically and experimentally). This was expected because the apparatus has damping effect which results in diminishing the motion to zero after about five cycles. Comparison between the two results are shown in Figure 24 for the in-phase motion and in Figure 25 for the out-of-phase motion. The period of oscillations obtained experimentally are higher than those obtained theoretically. These differences are between five percent and 11 percent. The other deviation between the results can be seen by comparing the displacements at the middle of a complete cycle. The experiment shows lower absolute values for this mid-cycle displacements than those of the theory. This is, again, a cause of the damping in the experimental system. One might expect, then, that the mid-cycle displacements obtained experimentally would not lie on the other branch of the modal line. If this were true then the masses, at the end of the second half of the cycle, would not attain zero velocities at the same time. This is not what actually was observed. The mid-cycle displacements of the experiments do
lie on the other branch of the modal line, but their coordinates have smaller values than those obtained theoretically. This is illustrated in Figure 26. For this reason the modal relation curves are not the same as is predicted under the assumption of no damping.
Figure 24. Comparison between the experimental and the theoretical results for the in-phase modes.
Figure 25. Comparison between the experimental and the theoretical results for the out-of-phase modes
Figure 26. Modal relation curves for experiments and theory
"r = 1.25"
DISCUSSION

The application of the algorithm for determining the normal mode motion for the air spring system yielded reasonable results which came in fair agreement with the experimental results, see Figures 24 and 25. This shows that the algorithm is effective and can be used for the study of normal mode motion of nonlinear systems in general.

It was pointed out before that the study of small oscillations of nonlinear systems is important in determining points on the modal lines. It is easy, then, to determine the corresponding periods of oscillation. These periods and those points on the modal lines help in obtaining other points on the modal lines. The study of small oscillation also determines the minimum number of normal modes that the system possesses. For example if the springs characteristics are represented by polynomials and the linear terms are present, then, for small oscillations, the system is considered to be linear. The total energy lines are arbitrarily close to ellipses. Upon applying the orthogonality property there exist only four roots which correspond to two modes. This discussion leads to an important fact. For small oscillations the number of normal modes is dominated by the degree of the lowest power of the polynomial representing the spring characteristics. In other words if $k$ is the
degree of the lowest power of those polynomials then the number of normal modes (for small initial displacements) does not exceed $k+1$. However, it could happen that, for large initial displacements, there exist more than $k+1$ modes for the system. This is detected by plotting $I_1 = 0$ and $I_2 = 0$ for different values of the period as was pointed out early in the text. If this happens then at least each pair combine together as the modal lines approach the origin such that the net number of normal modes reduces to $k+1$.

There is one more important fact concerning normal modes. If a normal mode exists then the modal relation curve is unique according to the existence theorem. This implies that, for a given system, the modal lines do not intersect except at the origin. If, for example, two modal lines intersect at a point then their other branches must intersect at a point which has the same energy level as the other point, otherwise the existence theorem is contradicted.

The development of the modal phase plane is a new idea for the study of stability for systems with two degrees of freedom oscillating in normal modes. It could be useful in the study of automatic control. The process for obtaining this plane is simple. It needs the mapping of the total energy lines and the determination of some points on the modal lines. The stability of the system, then, can be easily predicted.

The lines of singularities are not completely developed. For
instance, the line of centers is determined after the solution is obtained. The line of saddles can be easily obtained and it is independent of the solutions. The determination of this line is important in locating the unstable regions for the normal mode motions.
III. FORCED OSCILLATIONS OF THE AIR SPRING SYSTEM

1. Introduction

It was pointed out earlier in this text that the study of normal modes is important for the study of forced motions of the dynamical systems. For nonlinear systems when the exciting frequency is equal to a natural frequency the system has a normal mode motion as that of the free motion of the system.

When a system, linear or nonlinear, is acted on by an external periodic exciting force the amplitudes depend on the exciting frequency as well as on the magnitude of the disturbing force. However, there is a significant difference between the forced motion of linear and nonlinear systems. This difference is a consequence of the relations between the amplitudes and the natural frequencies. For linear systems the natural frequencies are constant at any amplitude. The response curve is tangent to a vertical line whose abscissa is equal to the natural frequency (7, p. 87-91). For nonlinear systems the situation is different. The natural frequencies vary with the amplitudes. The response curves show what is called the back bone curves (22, p. 88).

Linear systems, nonlinear symmetrical systems and nonlinear nonsymmetrical systems all agree in that the phase relation between the exciting periodic forces and the amplitudes changes
when the disturbing frequency is changed passing through the natural frequencies of the systems. For linear and nonlinear symmetrical systems the negative amplitudes are equivalent to the positive ones. The response curves can, then, be presented using the absolute value of the amplitudes. The case is different for nonlinear nonsymmetrical systems. The negative and positive amplitude are not equivalent and the response curves must be presented by both positive and negative values of the amplitudes.

Except for analog computer results there are not many experiments carried out to study the forced motions of nonlinear systems having two degrees of freedom. It is, then, the purpose of this chapter to present some experimental results for a forced motion of a nonlinear nonsymmetrical system. The experiments were performed on the air spring system.

2. Apparatus

The air spring system is described in section 5, chapter 2. The releasing mechanisms were removed. An exciting system is added to the apparatus to produce the disturbing force. It consists of two sets of eccentric masses equally placed from the center of the coupling beam. They are three feet apart and rotating in opposite directions. Each set is formed of a shaft supported by two bearings and carrying a full length threaded rod at each end.
This rod is used to change the amount of eccentricity. Each eccentric unit is connected to the driving mechanism by a variable length shaft with a universal joint at each end. The double universal joints and the variable length shaft are used to transmit the rotary motion to the eccentric system while moving up and down with the beam.

The driving mechanism consists of a two horse power direct current motor, a gear, a pinion, two sprockets, and a chain. The gear is keyed on a shaft which is supported by two bearings and is connected to a universal joint. This gear takes its motion from the pinion which is keyed on the motor's shaft. The motion from the motor is transferred to the other universal joint by the two sprockets and a chain. This is illustrated in Figure 27. There is a reduction of 3.5 between the speed of the motor and the speed of the eccentric units.
Figure 27. The exciting force system
3. The Equations of Motion

Let the horizontal and the vertical components of the displacements of the eccentric masses be denoted by \( x_i \) and \( y_i \).

\[
\begin{align*}
x_1 &= R \cos \omega t \\
y_1 &= z + b\theta + R \sin \omega t \\
x_2 &= -R \cos \omega t \\
y_2 &= z - b\theta + R \sin \omega t
\end{align*}
\]

\( \omega t \) is measured from the horizontal position. Differentiating twice, then

\[
\begin{align*}
\ddot{x}_1 &= -\omega^2 R \cos \omega t \\
\ddot{y}_1 &= \ddot{z} + b\ddot{\theta} - \omega^2 R \sin \omega t \\
\ddot{x}_2 &= \omega^2 R \cos \omega t \\
\ddot{y}_2 &= \ddot{z} - b\ddot{\theta} - \omega^2 R \sin \omega t
\end{align*}
\]

Let the forces imposed on the system by the eccentric masses be denoted by \( X_i \) and \( Y_i \), then
\[ X_1 = MR \omega^2 \cos \omega t \]
\[ Y_1 = \mathcal{M} (-\ddot{z} - b\ddot{\theta} + R\omega^2 \sin \omega t) - Mg \]
\[ X_2 = MR \omega^2 \cos \omega t \]
\[ Y_2 = \mathcal{M} (-\ddot{z} + b\ddot{\theta} + R\omega^2 \sin \omega t) - Mg \]

The net vertical disturbing force is given by
\[ Y = \mathcal{M}(-2\ddot{z} + 2R\omega^2 \sin \omega t) - 2Mg \]

The net disturbing torque on the system is given by
\[ \text{Torque} = -2b^2 \mathcal{M}\ddot{\theta} \cos \theta + 2MRb\ddot{\theta} \cos \omega t. \]

According to previous assumption, \( \theta \) is small. Then \( \cos \theta \) is approximately one. Hence the equations of the forced motion of the air spring system are

\[
m\dddot{z} + \left( \frac{mg}{2} + P_a \right) \left[ 2 - \frac{L_1^Y}{(L_1'+z+a\theta)^Y} - \frac{L_2^Y}{(L_2'+z-a\theta)^Y} \right] = -2M\dddot{z} - 2Mg + 2MR\omega^2 \sin \omega t
\]
\[
I\dddot{\theta} + \left( \frac{mg}{2} + P_a A \right) a \left[ \frac{L_1^Y}{(L'_1+z+az)^Y} - \frac{L_2^Y}{(L'_2+z-a\theta)^Y} \right] = -2b^2 \mathcal{M}\dddot{\theta} + 2MRb\dddot{\theta} \cos \omega t
\]

Or after using the substitutions (3.1), chapter 2, these equations read
\[
\ddot{z} + \frac{g(1+\sigma)}{2L_1(1+\rho)} \left[ 2 - \frac{1}{(1+x+\alpha \theta)^\gamma} + \frac{1}{(1+rx-\beta \theta)^\gamma} \right] = \frac{2R \rho \omega^2}{(1+\rho)} \sin \omega t - \frac{2\rho}{(1+\rho)}
\]

\[
\left(1 + \frac{2b_2 \rho}{a^2 \lambda} \right) \ddot{\theta} + \frac{g(1+\sigma)}{2\lambda a} \left[ \frac{1}{(1+rx-\beta \theta)^\gamma} - \frac{1}{(1+x+\alpha \theta)^\gamma} \right] = 2 \frac{R \rho}{a^2} \theta \cos \omega t
\]

where
\[
\rho = \frac{M}{m}
\]

4. Experimental Results

Experiments have been conducted for different values of \( r \)

\[
(r = \frac{L_1}{L_2}).
\]

For each setting of \( r \) tests were run for different values of \( MR \).

The exciting frequency is changed from zero up to a speed higher than the second natural frequency of the system. The frequency is then decreased back to zero. For different values of the frequency the maximum and minimum amplitudes were recorded for the mass on each system. It was noticed that the displacement \( z_1 \) (\( z_1 \) is the displacement of the mass of the spring with larger air column length) is in-phase with the exciting forces at frequencies smaller than the first natural frequency and is out-of-phase with the force at higher frequencies. The displacement \( z_2 \) was in-phase with the disturbing
force at all speeds smaller than the second natural frequency and was
out-of-phase at higher frequencies.

In order to be consistent, the recorded data are transformed
to the translation and rotation coordinates $x$ and $\theta$. The phase
relations of $x$ and $\theta$ with the exciting force is determined from the
phase relation of the force and the displacements $z_1$ and $z_2$. When
both $z_1$ and $z_2$ are inphase with the force, then

$$x = \frac{z_{1 \text{ max}} + z_{2 \text{ max}}}{2L_1}$$

$$\theta = \frac{z_{1 \text{ max}} - z_{2 \text{ max}}}{2a}$$

When $z_1$ is out-of-phase and $z_2$ is in-phase with the force

$$x = \frac{z_{1 \text{ min}} + z_{2 \text{ max}}}{2L_1}$$

$$\theta = \frac{z_{1 \text{ min}} - z_{1 \text{ max}}}{2a}$$

When both displacements $z_1$ and $z_2$ are out of phase with the
force, then

$$x = \frac{z_{1 \text{ min}} + z_{2 \text{ min}}}{2L_1}$$

$$\theta = \frac{z_{1 \text{ min}} - z_{2 \text{ min}}}{2a}$$
Table 5. Results for the forced motion. MR = 7 lb inches. 
\( r = 1.67 \)

<table>
<thead>
<tr>
<th>Speed R.P.M.</th>
<th>( z_1 ) max inches</th>
<th>( z_1 ) min inches</th>
<th>( z_2 ) max inches</th>
<th>( z_2 ) min inches</th>
<th>( x )</th>
<th>( \theta )</th>
</tr>
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<td>.05</td>
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<td>.005</td>
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<td>.1</td>
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<td>-.065</td>
</tr>
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<td>*-1</td>
<td>.1</td>
<td>-.1</td>
<td>-.045</td>
</tr>
<tr>
<td>6</td>
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<td>.8</td>
<td>*-.8</td>
<td>.1</td>
<td>-.1</td>
<td>-.035</td>
</tr>
<tr>
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<td>109</td>
<td>.6</td>
<td>*-.65</td>
<td>.2</td>
<td>-.2</td>
<td>-.0225</td>
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<tr>
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<td>*-.2</td>
<td>.6</td>
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<td>*-.7</td>
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<td>*-.8</td>
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<td>.2</td>
<td>-.2</td>
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<td>-.03</td>
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<td>-.04</td>
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<td>.2</td>
<td>.05</td>
<td>-.05</td>
<td>.0175</td>
</tr>
</tbody>
</table>

* Denotes that the displacement is out of phase with the exciting force.
Table 6. Results for the forced motion. MR = 8 lb inches. 
\( r = 1.67 \)

<table>
<thead>
<tr>
<th>Speed R.P.M.</th>
<th>( z_1 ) max inches</th>
<th>( z_1 ) min inches</th>
<th>( z_2 ) max inches</th>
<th>( z_2 ) min inches</th>
<th>( x )</th>
<th>( \theta )</th>
</tr>
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<td>-.2</td>
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<td>.5 *</td>
<td>-.4</td>
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<td>-.2</td>
<td>.3 *</td>
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<td>.35 *</td>
<td>-.2</td>
<td>-.02</td>
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<td>.2 *</td>
<td>-.2</td>
<td>.4 *</td>
<td>-.35</td>
<td>-.028</td>
</tr>
<tr>
<td>15</td>
<td>114.2</td>
<td>.7 *</td>
<td>-.7</td>
<td>.3</td>
<td>-.25</td>
<td>-.02</td>
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<td>.1</td>
<td>-.25</td>
<td>.1</td>
<td>-.15</td>
<td>.01</td>
</tr>
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* Denotes that the displacement is out of phase with the exciting force.
Figure 28. The response curves for $x$
Figure 29. The response curves for $\theta$
Experimental results are listed in Tables 5 and 6 for the case $r = 1.67$. For Table 5, $MR = 7\text{ lb inches}$. For Table 6, $MR = 8\text{ lb inches}$. Figures 28 and 29 are a plot for the response curves.

5. **Discussion**

In chapter two the normal modes of the air spring system were determined theoretically and experimentally. The results showed that two modes exist for each value of $r$. Figures 28 and 29 are plots of the response curves of the air spring system for the case $r = 1.67$. The phase relation between the disturbing force and the amplitudes can be seen easily from the figures if the force is considered positive. The illustrations show that both amplitudes $x$ and $\theta$ have two peaks at two places. These peaks are located near the values of the natural frequencies. This indicates that resonance occurs when the exciting frequency is equal to a natural frequency of the system which is a well known fact. By comparing the amplitudes of the forced motion at the peaks with those of the free oscillations it is easy to see that they lie on the modal lines. This shows that at the resonance the system moves as if it is under free oscillation, which indicates the importance of the study of the normal modes of nonlinear systems. The first peak corresponds to the in-phase mode where $x$ and $\theta$ are of the same sign. The second peak corresponds to the out-of-phase mode.
When the exciting frequency is zero the amplitudes are identically zero. At small frequencies the magnitude of the exciting force is small and hence the amplitudes are small. The force is in-phase with both amplitudes. As the frequency is increased and approaches the natural frequency of the in-phase mode the amplitudes start to build up. As a result the first peak is obtained. At the peak the force and the amplitudes are 90 degrees out of phase. By increasing the disturbing frequency a little more the amplitudes decrease and the force is 180 degrees out of phase with the amplitudes. For some value of the exciting frequency between the two natural frequencies of the system the displacement $x$ becomes back in-phase with the force while $\theta$ remains out-of-phase with it. At this point the amplitudes attain their minimum values. Upon increasing the frequency more $x$ and $\theta$ start to increase until the second peak is obtained when the frequency is equal to the second natural frequency. The disturbing forces is again 90 degrees out of phase with the displacement. At high speeds, although the magnitude of the disturbing force is large, the amplitudes are small. A physical explanation for this is that the exciting force changes so fast that the masses have no time to follow. In this case the amplitude $x$ is out-of-phase and $\theta$ is in-phase with the disturbing forces.

When the value of MR increases the amplitudes change and have large absolute values except for frequencies less than the
second natural frequency where the phase of \( x \) changes. In this case they decrease. The amplitudes at the peaks still fall on the modal lines (Refer to Figures 16 and 17).

It was pointed out (p. 67) that the natural frequency of the air spring system has small variations with the amplitudes. This resulted in the lack of appearance of the well-known jump phenomenon which might otherwise be expected to appear.

From the results of chapter 2 the values of the natural frequencies change with \( r \). The in-phase mode has small changes whereas the out-of-phase mode has large changes. For large values of \( r \) the difference between the two natural frequencies is large. In this case the response curves show two distinct peaks for both \( x \) and \( \theta \) as is shown in Figures 28 and 29. As the value of \( r \) is decreased, this difference decreases and the peaks come closer. When \( r \) is equal to one, that is the two air spring units have the same air column lengths, the situation is different. The peaks depend on the nature of the exciting forces. If these forces are only in the direction of the \( x \) coordinate, then the masses on each spring have the same displacements, and the value of \( \theta \) is always zero, and hence there is no peak for it. The resonance occurs in the \( x \) direction only. If the exciting force is in the direction of \( \theta \), i.e. a periodic torque is applied or the system with zero net vertical force, the resonance occur at the out-of-phase mode. A peak will occur for both \( \theta \) and \( x \) (see Figure 17).
BIBLIOGRAPHY


APPENDIX 1

Convergence of the Central Difference Method

For the calculation of the starting values the formulas

\[ x_2^{[\nu+1]} = x_1^* + h^2 \left( \frac{1}{3} F_1 + \frac{1}{6} F_2^{[\nu]} \right) \]

\[ y_2^{[\nu+1]} = y_1^* + h^2 \left( \frac{1}{3} G_1 + \frac{1}{6} G_2^{[\nu]} \right) \]

are used for the iteration. \( \nu + 1 \) indicates the number of iterations.

\( x_1^* \) and \( y_1^* \) are actual values, and hence \( F_1 \) and \( G_1 \) are actual too, so that

\[ x_2^{[\nu+1]} - x_2^{[\nu]} = \xi^{[\nu]} = \frac{h^2}{6} (F_2^{[\nu]} - F_2^{[\nu-1]}) \]

\[ y_2^{[\nu+1]} - y_2^{[\nu]} = \eta^{[\nu]} = \frac{h^2}{6} (G_2^{[\nu]} - G_2^{[\nu-1]}) \]  

By using Lipschitz condition, then

\[ |F(x_2^{[\nu]}, y_2^{[\nu]}) - F(x_2^{[\nu-1]}, y_2^{[\nu-1]})| < k_1 |x_2^{[\nu]} - x_2^{[\nu-1]}| + k_2 |y_2^{[\nu]} - y_2^{[\nu-1]}| \]

where \( k_1, k_2 \) are Lipschitz constants. Substituting in
equation (1), then

\[ |\xi^{[\nu]}| < \frac{h^2}{6} (k_3 |\xi^{[\nu-1]}| + k_4 |\eta^{[\nu-1]}|) \]

\[ |\eta^{[\nu]}| < \frac{h^2}{6} (k_3 |\xi^{[\nu-1]}| + k_4 |\eta^{[\nu-1]}|) \]

Let \( D \) to be the larger of \( k_1 \) and \( k_3 \)

and \( E \) to be the larger of \( k_2 \) and \( k_4 \).

Then,

\[ |\xi^{[\nu]}| \text{ and } |\eta^{[\nu]}| < \frac{h^2}{6} (D |\xi^{[\nu-1]}| + E |\eta^{[\nu-1]}|) \]

let \( \nu = 1 \), then

\[ |\xi^{[1]}| < \frac{h^2}{6} (D |\xi^{[0]}| + E |\eta^{[0]}|) \]

also

\[ |\xi^{[2]}| < \frac{h^2}{6} (D |\xi^{[1]}| + E |\eta^{[0]}|) \]

\[ < \frac{h^2}{6} \left[ (D + E) \frac{h^2}{6} (D |\xi^{[0]}| + E |\eta^{[0]}|) \right] \]

In general

\[ |\xi^{[\nu]}| < \frac{h^2}{6} \left[ (D + E) \frac{h^2}{6} \right]^{\nu-1} (D |\xi^{[0]}| + E |\eta^{[0]}|) \]

and

\[ |\eta^{[\nu]}| < \frac{h^2}{6} \left[ (D + E) \frac{h^2}{6} \right]^{\nu-1} (D |\xi^{[0]}| + E |\eta^{[0]}|) \]

As \( \nu \to \infty \) it is required that \( |\xi^{[\nu]}| \) and \( |\eta^{[\nu]}| \) tend to zero. A necessary condition for this to happen is that

\[ \frac{h^2}{6} (D + E) < 1 \]
The same procedure could be carried for the iteration formulas of the main calculations to arrive at

$$\frac{h^2}{12} \ (D + E) < 1$$

$h$ is chosen to be the smaller of the two, that is

$$h^2 < \frac{6}{D + E}.$$
APPENDIX 2

Fortran Program for Plotting the Equations $I_1 = 0$ and $I_2 = 0$ for the Air Spring System

For some restrictions in the Fortran language the parameters in the program are given different symbols. The correspondence between these symbols is as follows:

- DIS stands for $a$
- AI " " $\lambda$
- T " " $\gamma$
- PTR " " $T$
- T1 " " $\Delta T$
- AL1 " " $L_1$
- AL2 " " $L_2$
- SA " " $S$

The program is arranged to calculate the values of $I_1$ and $I_2$ in any region in the x-y plane for any values of the period. This is done by adjusting the values:

- MP the number of steps to change the period
- T1 the change in the period
- MX the number of intervals in the x-axis
- HX the interval length in the x-axis
- MY the number of intervals on the y-axis
- HY the interval length on the y-axis
- XABC is equivalent to $x(1)$
- TFGH is equivalent to $y(1)$
Fortran Program Listing

```
DIMENSION X(90), Y(90), P(90), Q(90), F(90), G(90), U(90), V(90)
88 READ 10, DIS, A1, T, ER, N2, MP, MX, MY, HX, HY
    READ 80, PTR, T1, XABC, YFGH, AL1, AL2, BN, N, SA
    R=AL1/AL2
    C=DIS/AL2
    D=DIS/AL1
    DO 77 IP=1, MP
        PI=IP
        H=PER/BN
        DO 77 IX=1, MX
            XI=IX
            X(I)=XABC+XI*HX
            DO 36 IY=1, MY
                YI=IY
                Y(I)=YFGH+YI*HY
            42 J=1
                F(J)=(1./(1.+X(J)+D*Y(J))**T+1.)/(1.+R*X(J)-C*Y(J))**T-2.)*
                    193.2*(1+SA)/AL1
                G(J)=(1.)/(1.+X(J)+D*Y(J))**T-1.)/(1.+R*X(J)-C*Y(J))**T)*
                    193.2*(1+SA)/(A1*DIS)
        IF (J=2)1, 51, 52
1     F(2)=F(1)
        G(2)=G(1)
        X(2)=X(1)
        Y(2)=Y(1)
        J=2
        N1=1
3     P(2)=X(2)
        Q(2)=Y(2)
        X(2)=X(1)+H*H*(F(1)/3.+F(2)/6.)
        Y(2)=Y(1)+H*H*(G(1)/3.+G(2)/6.)
        GO TO 2
51    Z=ABSF(X(2)-P(2))+ABSF(Y(2)-Q(2))
        IF (Z-ER)6, 6, 4
        N1=N1+1
        IF (N2-N1)5, 5, 3
5    PRINT 20, J, X(J), P(J), Y(J), Q(J), X(1), Y(1)
        GO TO 36
6     J=3
7     F(J)=F(J-1)
        G(J)=G(J-1)
        X(J)=X(J-1)
        Y(J)=Y(J-1)
```

N1 = 1
8 P(J) = X(J)
   Q(J) = Y(J)
   X(J) = 2. * X(J - 1) - X(J - 2) + H*H*(5. * F(J - 1) / 6. + (F(J) + F(J - 2)) / 12.)
   Y(J) = 2. * Y(J - 1) - Y(J - 2) + H*H*(5. * G(J - 1) / 6. + (G(J) + G(J - 2)) / 12.)
   GO TO 2
52 Z = ABSF(X(J) - P(J)) + ABSF(Y(J) - Q(J))
   IF (Z - ER) 11, 11, 9
9 N1 = N1 + 1
   IF (N2 - N1) 5, 5, 8
11 J = J + 1
   M = N + 1
   BM = BN + 1.
   IF (M - J) 12, 7, 7
12 DO 13 J = 1, M
   BJ = J
   U(J) = F(J)
13 V(J) = G(J)
   U(N + 2) = U(N)
   U(N + 3) = U(N - 1)
   V(N + 2) = V(N)
   V(N + 3) = V(N - 1)
   AK1 = 0.
   AK2 = 0.
   R1 = 0.
   R2 = 0.
   L = N - 1
   DO 14 J = 1, L, 2
   AK1 = AK1 + H*(U(J) + 4. * U(J + 1) + U(J + 2)) / 3.
   AK2 = AK2 + H*(V(J) + 4. * V(J + 1) + V(J + 2)) / 3.
   R1 = R1 + H*(U(J + 4) - 4. * U(J + 3) + 6. * U(J + 2) - 4. * U(J + 1) + U(J)) / 90.
14 AK1 = AK1 - R1
   AK2 = AK2 - R2
   PRINT 50, AK1, AK2, X(1), Y(1), PER
36 CONTINUE
77 CONTINUE
   PRINT 30, AL1, AL2
   GO TO 88
10 FORMAT (F6.3, F8.6, F4.2, F7.6, 4I2, 2F6.3)
20 FORMAT (13, 6F9.5)
30 FORMAT (2F6.2)
50 FORMAT (5F12.5)
80 FORMAT (F6.4, F7.5, F8.5, F8.6, 2F6.3, F5.1, I3, F7.4)
END
APPENDIX 3

Fortran Program for Determining the Normal Mode Motion for the Air Spring System

The parameters are changed in the same way as mentioned in Appendix 2, except for the period $T$; it is given the symbol PER.

Fortran Program Listing

```
DIMENSION X(90), Y(90), P(90), Q(90), F(90), G(90), U(90), V(90)
DIMENSION A(90), B(90)
100 READ 10, DIS, A1, H1, H2, T, GE, ER, N2
   READ 80, PER, T1, X(1), Y(1), AL 1, AL 2, BN, N, SA
   R=AL 1/AL 2
   C=DIS/AL 2
   D=DIS/AL 1
19 I=1
   H=PER/BN
99 JF=0
   IG=0
42 J=1
2   F(J)=(1./(1.+X(J)+D*Y(J))**T+1. /(1.+R*X(J)-C*Y(J))**T-2.)*
1193.2*(1+SA)/AL1
   G(J)=(1./(1.+X(J)+D*Y(J))**T-1. /(1.+R*X(J)-C*Y(J))**T)*
1193.2*(1+SA)/(AL1*DIS)
   IF (J-2)1, 51, 52
1   F(2)=F(1)
   G(2)=G(1)
   X(2)=X(1)
   Y(2)=Y(1)
   J=2
   N1=1
3   P(2)=X(2)
   Q(2)=Y(2)
   X(2)=X(1)+H*H*(F(1)/3. +F(2)/6.)
   Y(2)=Y(1)+H*H*(G(1)/3. +G(2)/6.)
   GO TO 2
```
51 \( Z = \text{ABSF}(X(2) - P(2)) + \text{ABSF}(Y(2) - Q(2)) \)

\[ \text{IF } (Z - ER) 6, 6, 4 \]
4 \( N1 = N1 + 1 \)

\[ \text{IF } (N2 - N1) 5, 5, 3 \]
5 \( \text{PRINT } 20, J, X(J), P(J), Y(J), Q(J), X(1), Y(1) \)

\( \text{GO TO } 36 \)
6 \( J = 3 \)

7 \( F(J) = F(J - 1) \)

\( G(J) = G(J - 1) \)

\( X(J) = X(J - 1) \)

\( Y(J) = Y(J - 1) \)

\( N1 = 1 \)

8 \( P(J) = X(J) \)

\( Q(J) = Y(J) \)

\[ X(J) = 2.*X(J - 1) - X(J - 2) + H*H*(5.*F(J - 1)/6. + (F(J) + F(J - 2))/12.) \]

\[ Y(J) = 2.*Y(J - 1) - Y(J - 2) + H*H*(5.*G(J - 1)/6. + (G(J) + G(J - 2))/12.) \]

\( \text{GO TO } 2 \)

52 \( Z = \text{ABSF}(X(J) - P(J)) + \text{ABSF}(Y(J) - Q(J)) \)

\[ \text{IF } (Z - ER) 11, 11, 9 \]

9 \( N1 = N1 + 1 \)

\[ \text{IF } (N2 - N1) 5, 5, 8 \]

11 \( J = J + 1 \)

\( M = N + 1 \)

\( BM = BN + 1. \)

\[ \text{IF } (M - J) 12, 7, 7 \]

12 \( \text{DO } 13 \ J = 1, M \)

\( BJ = J \)

\( U(J) = F(J) \)

13 \( \text{V(J) = G(J)} \)

\( U(N + 2) = U(N) \)

\( U(N + 3) = U(N - 1) \)

\( V(N + 2) = V(N) \)

\( V(N + 3) = V(N - 1) \)

\( A1 = 0. \)

\( A2 = 0. \)

\( R1 = 0. \)

\( R2 = 0. \)

\( L = N - 1 \)

\( \text{DO } 14 \ J = 1, L, 2 \)

\( A1 = A1 + H*U(J + 4) - 4.*U(J + 1) + U(J + 2))/3. \)

\( A2 = A2 + H*V(J + 4) - 4.*V(J + 1) + V(J + 2))/3. \)

\( R1 = R1 + H*U(J + 4) - 4.*U(J + 1) + U(J + 2))/90. \)

\( R2 = R2 + H*V(J + 4) - 4.*V(J + 1) + V(J + 2))/90. \)

\( A1 = A1 - R1 \)

\( A2 = A2 - R2 \)
IF (JF) 15, 15, 16
15 UL = AK1
    VM = AK2
    DO 18 J = 1, M
        A(J) = X(J)
    18 B(J) = Y(J)
        JF = 1
        X(1) = X(1) + H1
        GO TO 42
16 IF (IG) 17, 17, 21
17 PS = (AK1 - UL) / H1
        IF (1000000. - ABSF(PS)) 37, 37, 22
22 QR = (AK2 - VM) / H1
        IF (1000000. - ABSF(QR)) 37, 37, 23
23 IG = 1
        Y(1) = Y(1) + H2
        X(1) = X(1) - H1
        GO TO 42
21 S = (AK1 - UL) / H2
        IF (1000000. - ABSF(S)) 37, 37, 24
24 W = (AK2 - VM) / H2
        IF (1000000. - ABSF(W)) 37, 37, 25
25 DEN = PS * W - QR * S
        IF (DEN) 26, 31, 26
26 CHX = (VM * S - UL * W) / DEN
        CHY = (UL * QR - VM * PS) / DEN
        Z = ABSF(CHX) + DIS * ABSF(CHY)
        IF (Z < GE) 35, 35, 27
27 X(1) = X(1) + CHX
        Y(1) = Y(1) + CHY - H2
        I = I + 1
        IF (N2 = I) 31, 31, 28
28 GO TO 99
31 PRINT 30, I, DEN, X(1), CHX, Y(1), CHY
        GO TO 36
37 PRINT 40, H1, H2
        GO TO 36
35 PRINT 50, (J, A(J), B(J), J = 1, M)
        PRINT 90, PER
        PER = PER + T1
        Z = DIS * ABSF(Y(1)) / AL1 + ABSF(X(1))
        IF (Z < .1) 36, 36, 19
36 PRINT 110, AL1, AL2
        GO TO 100
10 FORMAT (F9.6, F8.6, 2F6.5, F4.2, F5.4, F7.6, 12)
20 FORMAT (13, 6F10.6)
30 FORMAT (12, F20.5, 4F10.6)
40 FORMAT (2F5.2)
50 FORMAT (13, 2F10.6)
80 FORMAT (F6.4, F7.5, F8.5, F8.6, 2F6.3, F5.1, 13, F7.4)
90 FORMAT (F10.5)
110 FORMAT (2F7.2)
END