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<u>The processes of Electric Arc Steelmaking</u>

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Because of competition from abroad, the U. S. Steel industry has begun research into the processes involved with electric arc steelmaking. This paper addresses the mathematical background of these furnaces and the electro-magneto-hydrodynamical effects used to melt the large quantities of steel. A classical approach to the derivation of these equations, with particular interest in the electro-magneto effects of slowly moving non-magnetizable media and how this phenomenon couples with the classical fluid flow equations, is presented. The Navier-Stokes equations are tied to the Maxwell's equations for a set of nonlinear partial differential equations complimented by the equations of state given by Saha's equation and the gas law. A dimensional analysis shows that the equations can not be decoupled. A means to a solution to the system is discussed.

ELECTRO-MAGNETO-HYDRODYNAMICS:

The Processes of Electric Arc Steelmaking

by

Paul E. King

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Dean of Graduate School

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"Everything that can be invented has been invented"

John H. Duell

Director, U. S. Patent Office

1899

TABLE OF CONTENTS

LIST OF FIGURES

SYMBOLS

Here is a list of the symbols used within the following paper, in the order that they appear. This is a complete list with descriptions of what each symbol represents. Because this paper combines fluid dynamics with electrodynamics, some of the standard symbols have been changed to avoid confusion.

FLUID FLOW SYMBOLS

x = position in fixed coordinate

X = position of a fluid particle at time $t = 0$

$\psi = \psi(X,t)$ = transformation from moving frame to fixed frame

$\Psi = \Psi(x,t)$ = inverse transformation

$\rho = \rho(x,t)$ = fluid density

$u = u(x,t)$ = fluid particle velocity

F = arbitrary scalar function of x and t

$\mathcal{V}$ = arbitrary volume element

$\mathcal{S}$ = arbitrary surface element

dV = differential volume element

dS = differential surface element

n = outward unit normal

$\mathcal{M}$ = mass of a volume element

t = stress vector

f = extraneous forces per unit mass

$\mathcal{T}$ = stress tensor

$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x_3})$ = gradient operator

$r = x - x_0$ = position vector where $x_0 = x(0)$

$\mathcal{D}$ = deformation tensor

$\mathcal{T}$ = kinetic energy

Ω = vorticity tensor

P = fixed position in a vector field

$\omega = curl(u)$ = vorticity vector

$\mathcal{J}$ = Jacobian matrix

Θ = temperature

η = entropy

e = specific internal energy

Q = heat

p = pressure

c_v = heat capacity at constant volume

c_p = heat capacity at constant pressure

$\mathcal{C}$ = total entropy

$\mathcal{U}$ = internal energy

h = heat flux vector

$\mathcal{I}$ = identity matrix

$\mathcal{G} = \mathcal{T} + p\mathcal{I}$ = tensor

Υ = dissipation function

I, II, III = principle invariants of the deformation

$\zeta = div(u)$

λ, μ = scalar functions of the thermodynamic state

(viscosities)

ELECTRODYNAMIC SYMBOLS

q_i = source charge

Q = test charge

$\hat{r}$ = unit vector pointing from q_i to Q

F = force vector

E = electric field

ρ_c = $\rho_c(x,t)$ = charge distribution

Φ = flux

d = distance

i = $i(x,t)$ = current

B = magnetic field

dl = differential line segment

K = surface current density

J = volume current density

a, b = general vectors

v = $v(x,t)$ = charge velocity

σ = conductivity

f = force per unit charge

ϵ = electromotive force

C = closed path

δ = radius of a ball

p = dipole moment

α = polarizability

P = polarization

d = distance vector

ρ_p = polarization volume density

D = displacement vector field

A = magnetic vector potential

m = magnetic dipole moment

M = magnetization

H = magnetic field intensity

EQUATIONS OF STATE SYMBOLS

c_g = gas constant

e^- = free electron

k = Boltzman constant

Λ = ionization energy

h = Planck's constant

g_i = ground state degeneracy

Ξ = characteristic ionization temperature

V = volume

ϕ = degree of ionization

ELECTRO-MAGNETO-HYDRODYNAMICS:
The Processes of Electric Arc Steelmaking

INTRODUCTION

Electric arc furnaces have been used since the early 1900's for the efficient reduction of tons of scrap steel to a molten bath. The awesome power of these furnaces is used to quickly melt scraps for the production of high grade carbon alloys and stainless steels. This method is particularly well suited for these jobs because of the tremendous temperatures reached, and the availability of scraps rich in the valuable constituents.

Since the introduction of electric arc furnaces to the United States in 1906, little has changed to improve their efficiency. The electrodes have been made of a more durable material, and the power sources have become more powerful as the availability of inexpensive electrical energy has increased. But, with highly technical competition from overseas industrial nations, United States steel industry needs to make major modifications to the existing technology to increase the efficiency and production of the high grade steels.

With this in mind, the U. S. Bureau of Mines, Albany Research Center, Albany Oregon (ALRC) is conducting an ongoing investigation into the physical phenomenon of electric arcs and electric arc furnaces. The goal of this research project is to make recommendations on how todays

furnaces can become more efficient through design and implementation changes. Fundamental insight into the physical aspect of the problem is required in order for the techniques considered to be possible. A mathematical treatment became imperative and is now being actively pursued.

This paper is the beginning of the mathematical treatment. Due to the size and complexity of the problem, this portion is confined to the derivations of the governing equations giving a clear and concise classical treatment of the system, with some comments and recommendations on how a solution might be obtained.

The paper starts with a brief history of electric arcs and electric arc furnaces and how they emerged as a metallurgical tool, including how they work and what has been done in the past to improve them. The second part of the paper explains the experimental apparatus used by the ALRC with a description of data acquisition, analysis techniques and experimentation techniques. A brief description of the ultra-high powered furnace is presented and how the findings of the research at the ALRC correlates to them. Then, the governing equations are presented with some historical notes. The presentation of the fluid mechanics is based on the works of James Serrin, Sir Horace Lamb and the Russian physicists L. P. Landau and E. M. Lifshitz, who have done extensive work in this

field. The electrodynamics has been presented as a self contained, theoretical approach to electromagnetic phenomenon, with some historical arguments. A short section is then presented on the equations of state and how they interact with the field equations.

Finally, some comments are made on how the system of differential equations can be reduced to a form that may be solvable for the geometry of the experimental furnace at the ALRC, and how this may apply to the larger, more powerful furnaces used by the industry. The paper is concluded with some references to further reading and some closing comments on the problem.

HISTORY

With the advent of the discovery of the electric arc in 1800 by Sir Humphrey Davy came a metallurgical tool unparalleled in its ferocity and ability to reduce tons of scrap into a bath of molten steel. The method of Electric Arc Steelmaking has become so widely used today that, in 1983, more than 30% of the nations steel was produced in this manner. Electric furnaces are mainly used to make carbon steel, which accounts for 73.4% of the total amount of steel made in these furnaces. Since the scrap used in this process need not be preheated, it has become a widely used method. Other steels include stainless steel at 19.6% of the total, and alloys at 6.9%. The electric furnace produces high grades of the stainless and alloy steels that cannot be obtained by the other methods. These furnaces produce practically all of the stainless, constructional & special alloy steels used in the automobile, aviation and food producing industries today.

The first patent granted for an electric arc furnace was given on March 16, 1853 to a Frenchman, Pichon. Pichon claimed that he could economically melt minerals and metals, but the availability of sufficient amounts of electrical energy was rather limited. This lack of energy made the process very expensive and not very practical. For some 80 years after Sir Davy's discovery, little progress was made towards using the electric arc as a

metallurgical tool, but Pichon had shown that someday it might become a feasible tool.

In 1878, Wilhelm von Siemens was the first man to melt steel successfully using an electric current, which opened the doors to this powerful method of steel making. Von Siemens received a patent for his furnace in 1878. His first furnace used what is now known as an indirect arc to melt the steel. This method consisted of two electrodes suspended horizontally through the furnace walls opposite one another creating an electric arc between them and heating the metals by radiation.

A year later von Siemens received a patent for a direct arc furnace which consisted of a single electrode projecting through the roof of the furnace and another on the bottom of the furnace. In this manner he created an arc which jumped from the top electrode to the metals then out the bottom, melting the metals by directly using the electrical charges. Development of these furnaces for commercial use was slow due to the inadequate supplies of electrical energy and the high cost of producing it.

Dr. Paul Herault used electric arc furnaces for the production of aluminum, calcium carbide and ferroalloys. During the period of 1888-1894 he built a direct arc furnace which provided the fundamental features of furnaces today. In 1899 Dr. Herault commissioned the first successful direct arc furnace for the production of

steel in France. On December 28, 1900, the first shipment of "electrical steel", a carload of steel bars, was sent from Herault's plant in La Praz, France to Schneider and Company, Creusot, France. Thus was born the modern day electric arc furnace. Little has changed since these pioneers constructed the first few electric arc furnaces.

The first electric furnace used in the United States was at Halcomb Steel Company, Syracuse, New York. This was a Herault furnace, rectangular in shape, which had a steel capacity of 3 tons. It was a single phase, two electrode furnace with a 500 kilo Volt Ampere (kVA) low voltage, high current generator. It used amorphous carbon electrodes with a 15 3/4 inch quarter octagon shape, 80 inches long. The roof of this furnace was made of Silica bricks and the hearth sidewalls were made of basic materials. This furnace made its first run on April 5, 1906. The first few heats were made by melting a cold charge but the furnace had been installed to take on a molten charge supplied from a 20 ton open hearth furnace to cut the costs of operation. This was soon to be the adopted process of steelmaking of the time.

Not long after this, electric furnaces became the wave of the future. Firth Sterling Steel, McKeesport, Pennsylvania, installed a similar, but smaller furnace in 1908.

The next year South Works of the Illinois Steel

Company, South Chicago, Illinois, installed the first three phase furnace. This furnace had a 15 ton capacity, was the largest in the world, and was the first furnace with a round shell. It also operated with a hot charge supplied by a 15 ton Bessemer converter.

World War I caused a rapid growth of electric furnaces. By 1920, just 14 years after the first furnace came to the United States, 556,000 tons of steel were being made in electric arc furnaces annually. This accounted for just 1.2% of the total steel production, but the trend was set.

The second World War caused another large surge in electric furnace steelmaking, also. By 1945 3,457,000 tons of steel were made annually accounting for 4.3% of the total, and still rising.

The post war era was the turning point for the development of the electric arc furnaces. As the steel industry rebounded from full capacity production of steel alloys for the war effort, a great need was becoming apparent for carbon steels to be used for civilian purposes. It became necessary for research to be done to define a new melting practice for reaching a proper balance between the carbon and oxygen contents of rimming grades of steel. This task proved to be an extremely difficult, but surmountable one and carbon steel started being produced in large quantities soon after.

The furnaces have changed little since Herault received a patent for his, besides a few novel refinements, the same basic design is used today. The size of the furnaces have grown to meet demands. Along with the size requirements came electrode refinements. In 1915 the average electric furnace had a capacity of 6 tons with the shell being 11 ft. in diameter. It had an electrode made of amorphous carbon 17 inches in diameter. The furnace was powered by a steam generator operating at 1500 kVA. April of 1971 saw the Northwestern Steel and Wire Company, a pioneer in electric arc furnaces, install a 400 ton furnace with a 32 ft. diameter shell. This furnace, powered by a 162,000 kVA transformer had electrodes of 24 inches made of graphite, a much more durable conductor. These electrodes proved to be inadequate and soon were replaced by 28 inch ones to compensate. In May of 1976, this same company installed a furnace with a 38 ft. diameter shell and powered by the same type of transformer, with a 162,000 kVA rating.

The conductor or electrode that Sir Davy used was made of wood, charcoal and syrup of tar. It was molded together under what were then tremendous pressures. The arc was established by utilizing the current of a storage battery.

Dr. Herault's first furnaces used amorphous carbon electrodes which were made from petroleum coke or low ash

anthracite mixed with tar. After mixing, they were then baked and machined to specifications. These electrodes were very brittle and required extreme caution while in operation. Because they were so brittle, a new, stronger material was needed for the manufacture of a less costly type of electrode. Graphite emerged as the answer to this dilemma. The graphite electrodes, made in much the same way as their carbon predecessors, were petroleum coke and tar, mixed and then extruded. They were then baked at 4000 degrees Fahrenheit which graphitizes the mixture, tempering it to give it much more strength and enhancing its current carrying capabilities. A 6 ton furnace with a 2000 kVA generator required a 17 inch diameter carbon electrode while the same furnace only needed a 10 inch graphite electrode.

As the furnaces grew larger, and higher power ratings caused higher heats, the demand for improved refractories became imperative. The linings of these shells today are commonly zoned linings using high quality fusion-cast bricks. Recently the trend has become water cooled sidewalls to extend the life of the walls and roof as well as the refractory materials. As of yet, there are no refractories that can withstand the temperatures generated during a melt for any extended period of time. This causes the necessity of shutting down the furnaces periodically to have the wall linings repaired or

replaced, a very costly and time consuming process.

As the power requirements increased, so did the background noise levels, both acoustically as well as electrically. In a furnace with a power rating of 162,000 kVA the noise is deafening, a tremendous 110-115 decibels (db). If, at any time, some internal event causes a power surge, whole cities can be affected by a brown out or even a black out. Such power surges could be extremely hazardous to industries in and around the area as well as being very costly, and a major inconvenience.

Future refinements of the furnaces are headed towards dynamic control. With the computer age comes a very fast, powerful tool in controlling the furnaces. Utilizing signature analysis, one can theoretically predict events before they happen and adjust the furnace environment accordingly to prevent them. Such dynamic control would also cut down on the acoustical noise by controlling it at its source, prolong the life of the refractories and signal when the melt is complete, all appealing money and time saving advantages.

THE EXPERIMENTAL FURNACE

In the hope of being able to understand better the physical phenomenon taking place within an electric arc furnace (so that dynamic control may be possible), the United States Bureau of Mines, Albany Research Center is conducting an ongoing research project on electric arc furnaces. The attack of the problem is in two phases, simultaneously being approached. The first phase is the experimental furnace built on station which is used in studying the arc used in melting the scrap. The second phase is an ongoing agreement with Oregon Steel Mills and the Bonneville Power Administration to conduct experiments, gather data and take high speed films of a working furnace. The outcome of these experiments is to obtain enough information about the phenomenon to be able to categorize internal events and implement dynamic or predictive control of the furnace.

The first phase is being conducted on the experimental furnace installed at the ALRC. The furnace is a 200 lb. capacity single phase alternating current (AC) electric furnace. It is powered by two parallel AC welders with a rated current capacity of 1500 Amperes (A) at a low voltage of 40 volts (V) each. The primary rated current is 170 A single phase with the primary voltage of 440 V each. A water cooled shunt with a resistance rating of 1.009×10^{-4} ohms was installed on the connecting bus

for current measurements.

The experimental furnace has been modified for data acquisition purposes. The furnace shell was replaced by an airtight enclosure with two viewports giving orthogonal views of the arc (Fig. 1). The ports were installed for visual and photographic observations of the arc.

The atmosphere inside the chamber can be changed by injection of inert gases at three locations. The injection of the gases occurs at the two viewports, giving the extra advantage of cooling the pyrex glass windows while also keeping any soot or smoke away from them. The third location is at the electrode tip, which helps to control the arc and to cool the radiated heat.

The electrodes are made of graphite, with a one inch diameter tip machined to a 3 inch column (Fig. 2). Gas ports of 1/32 inch have been drilled through the electrode, surrounding the tip. Inert gases are injected through these ports to enhance the stability of the arc during experimental runs. The target block is a 16 inch diameter graphite block, 4 inches thick. One electrode has been threaded and screwed into the target block to maintain the same physical structure and theoretical path of the current in a single phase, two electrode furnace. This has the advantage of eliminating the averaging effects on the current and voltage data taken across the arcs by eliminating one while studying the other. Thus,

we are able to take better, more accurate data. Other target blocks have been experimented with, including copper and steel. Using the different materials for the target block gives the advantage of comparing wave form signatures.

Eight thermocouples have been placed on and inside the target block for calorimetry. These thermocouples are monitored by an Analog Devices μMAC-4000 microcomputer based measurement and control system. Heat transfer to the block, i.e. energy loss to the block, can thus be calculated.

The inert gases used are Helium and Argon. A mixture of 95% Helium and 5% Argon at 1.5 standard cubic feet per minute gives best results for the furnace. One half cubic feet per minute of the gas mixture is routed through the electrode tip while the rest is split between the viewports. The gas flow is monitored by thermal mass flow meters, allowing for changes of the flow rates. The furnace has a pressure release valve to keep the internal pressure at a nominal 1 atmosphere during an arc run. Sample gases are taken before, during and after each run to standardize the data and to keep track of materials burned.

Data acquisition takes several forms. High speed films are taken from the orthogonal views utilizing a system of mirrors installed for this purpose. The mirrors

are set so that the orthogonal views of the arc divide the film in half allowing the two views to appear side by side (Fig. 3). The camera has the high speed capability of 11,000 frames per second. With a system of prisms, a quarter frame head has been installed to reach an effective shutter speed of 44,000 frames per second. Video tapes are taken of each run to give real time images of the arc.

In conjunction with the camera is a Precision Instruments Data-6000 Digital Wave form Analyzer. The Data-6000 uses two channels with a 1 MHz aggregate digitizing rate for the storage of 64,000 data points. The voltage and current wave forms are digitized at 50 kHz per channel and stored to floppy disks for later analysis.

When the camera is operated, a triggering device signals the wave form analyzer to start taking data. Because the camera takes 200 feet of film to accelerate to optimum filming speed the necessity of such a triggering mechanism becomes apparent. Once the film reaches speed, the camera simultaneously marks the film by means of a pulse of light while signaling the wave form analyzer to start the data acquisition. In this manner high-speed film, and the current and voltage wave forms can be synchronized for the study of individual events, enhancing analysis.

Because of the magnetic and electric fields generated

by the transformers, an antimagnetic cage, or Faraday cage, has been installed to house all computer and computer assisted equipment (Fig. 4). This insures correct data acquisition and prolongs the life of the equipment under otherwise adverse conditions.

The second phase of the research is being conducted as a cooperative agreement between the Bonneville Power Administration (BPA), Oregon Steel Mills (OSM), and the Albany Research Center. Monitoring of Oregon Steel Mills furnace is fundamental to the understanding of the system and how one may gain dynamic control.

Commercial electric arc furnaces use high current and medium voltages for melting scrap steel. The instability of the arcs in the furnaces is a basic cause for feedback into the power grid and inefficiency in melting operations. The start-up or bore-down procedure is the most energy intensive and noisy portion of a heat due to the rapid fluctuations in the voltage and current. These fluctuations increase energy consumption and cause the power feedback.

Oregon Steel Mills is the worlds only mini-mill producer of carbon steel plate. OSM started in 1969 and by 1995 will be a company solely owned by the employees.

The furnace to be studied is a three phase Lectromelt furnace with a rated capacity of 90 tons and an average heat size of 83 tons. The furnace has a shell diameter of

17 feet and uses 22 inch diameter graphite electrodes (Fig. 5). The primary voltage of the transformer is 23 kV and secondary voltage is 648 V root mean square (RMS) with a current rating of 55,000 - 57,000 A. This furnace has a power consumption of 425 kilo Watt hours per ton (kWh/ton) of molten steel and takes approximately 23 minutes to reach a tap temperature of 3100° F. The furnace takes 50 minutes to run from tap to tap and OSM gets anywhere from 18 to 30 melts per day.

The objective of this research is to monitor a working ultra high powered (UHP) furnace during different times of a melt as well as during different times of the day. This data will then be analyzed by the methods pioneered for the small scale experimental furnace on station. This, in turn, allows for the identification of the events that precede the network disturbances, allowing for categorization of the events to be used in dynamic control. This will in turn increase efficiency while reducing the power feedback and the noise levels during bore-down and the rest of the run.

THE GOVERNING EQUATIONS OF MOTION

A. Hydrodynamics

The first step towards a greater understanding of the physical system is to derive the equations of motion. The equations governing the flow of the current and voltage in the system can be broken into two distinct parts each of which can be treated separately. The first part is to derive the equations for the plasma flow where the current and voltage paths are unknown due to the nature of the problem. We model this portion of the problem as a circuit element to be inserted into the circuit equations, for the second part of the derivation.

Traditionally plasma flow has been modelled by the equations of Magneto-Hydrodynamics (MHD). Since electrical forces as well as magnetic forces manifest themselves in a predominant fashion in the UHP furnaces, the term Electro-Magneto-Hydrodynamics (EMHD) is more appropriately used and this paper adopts this convention. Since the current ranges that these furnaces run at are typically 50,000 to 200,000 A and voltages of about 5000 V, this is a reasonable convention to make.

As a starting point, we consider the equations of fluid motion (or fluid dynamics). Some kinematical preliminaries are needed in order to set a standard for the derivations to follow: Since fluid mechanics concerns itself with the study of fluid flows, we treat the fluid

as a continuous medium. Thus, when we talk about infinitely small elements of volume we mean small in comparison with the bulk volume being considered but large with respect to molecular distances. These infinitely small volumes are the smallest elements that we can conceive and still retain the same physical properties as the bulk fluid. We assume that the fluid flow can be represented by a continuous transformation from Euclidean three space into itself with the parameter t, describing the elapsed time. We introduce a fixed coordinate system and denote the position of a particle in this system by $x = (x_1, x_2, x_3)$. At time t=0, we let $X = (X_1, X_2, X_3)$ be the position occupied by a fluid particle, then at time t the particle has moved to $x = (x_1, x_2, x_3)$, thus x is represented as a function of X and t, say $x = \psi(X, t)$ or $x_i = \psi_i(X, t)$. We note here that $\psi : \mathbf{R}^3 X \mathbf{R} \rightarrow \mathbf{R}^3 X \mathbf{R}$ and since we assume that distinct points remain distinct then the transformation has an inverse Ψ such that $X = \Psi(x, t)$.

If X is held constant while we let t vary then ψ specifies the path of a fluid particle with initial condition given by X. If, on the other hand, t is held constant then ψ determines the transformation of a region initially occupied by the fluid into a new one at time t.

We wish to consider the state of motion of a fluid element also. This is described by the functions giving the density and velocity of the element. The density of

the fluid is given as a function of position as well as time i.e. $\rho=\rho(x,t)$, specifying the density of the fluid element at position x and time t. Similarly the velocity u is given by $u = u(x,t)$.

We now introduce a two variable systems used in studying the field movements. The variables (x,t), called spatial or Eulerian variables, are used in field descriptions and the variables (X,t), called material or Lagrangian variables, are used to single out individual particles.[1] If F is an arbitrary function of say (x,t) then it is also a function of (X,t). With this in mind, we define

$$\frac{\partial F}{\partial t} \equiv \frac{\partial F(x,t)}{\partial t}$$

as the rate of change of F as observed from a stationary point x, and we define

$$\frac{dF}{dt} \equiv \frac{\partial F(X,t)}{\partial t}$$

as the material derivative which measures the rate of change of F following the particle. We can see directly that the material derivative and the spatial derivative are related by the chain rule, where the relation is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial F}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial F}{\partial x_3}\frac{dx_3}{dt}.$$

We can combine the last three terms in this relation, into a more compact form by realizing that we are just using

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[1] These two sets of variables were both due to Euler.

the dot product of the velocity and the gradient operator ∇, i.e.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (u \cdot \nabla)F.$$

This derivative operation is called the convective derivative and will be used extensively in this paper.

At the risk of seeming out of context, an important result related to the above equation needs to be presented. This result is the transport theorem and is obtained by considering a volume element flowing with the fluid. If we denote the Jacobian of a transformation by $\mathbf{J} = \mathbf{J}(X,t)$, then

$$\frac{d}{dt} \int_{\mathscr{V}(t)} F(x,t)\ dV_x = \frac{d}{dt} \int_{\mathscr{V}(0)} F(\psi(X,t),t)\ \mathbf{J}(X,t)\ dV_0 =$$

$$\int_{\mathscr{V}(0)} \left[\sum_{i=1}^{3} \frac{\partial F}{\partial x_i}\frac{\partial \psi_i}{\partial t} + \frac{\partial F}{\partial t} \right] \mathbf{J}\ dV_0 + \int_{V(0)} F\ \frac{\partial \mathbf{J}}{\partial t}\ dV_0 =$$

$$\int_{\mathscr{V}(t)} \left[\sum_{i=1}^{3} \frac{\partial F}{\partial x_i} u_i(x,t) + \frac{\partial F}{\partial t}(x,t) \right] dV_x + \int_{V(t)} F(x,t)\ \mathbf{J}_t\ \mathbf{J}^{-1}\ dV_x.$$

By the so called Euler formula relating the time rate of change of the Jacobian, $\mathbf{J}_t = \mathbf{J}\ (\nabla \cdot u)$, and realizing that the sum of the derivatives is just the gradient of $F(x,t)$, we get

$$\frac{d}{dt} \int_{\mathscr{V}(t)} F(x,t)\ dV = \int_{\mathscr{V}(t)} \left[\nabla F \cdot u + \frac{\partial F}{\partial t} + F\ (\nabla \cdot u) \right] dV =$$

$$\int_{\mathscr{V}(t)} \left[\frac{\partial F}{\partial t} + \nabla \cdot (Fu) \right] dV = \int_{\mathscr{V}(t)} \frac{\partial F}{\partial t}\ dV + \int_{\mathscr{S}(t)} F\ u \cdot n\ dS,$$

where n is the outward normal vector, the last term is

obtained by use of the divergence theorem and we have suppressed the dependence of the differential volume element on the position. This form of the transport theorem will become extremely useful in the derivations to come.

Now, we have given the velocity as a function of the spatial variables and we can then define the acceleration as the rate of change of the velocity experienced by a moving particle, i.e.

$$a = \frac{du}{dt}.$$

But, $u = u(x(t),t)$ (since $dx/dt = u(x,t)$) and by the work done previously, we get

$$a = \frac{\partial u}{\partial t} + (u \cdot \nabla)u.$$

Now, let us consider an arbitrary fixed volume Υ. The mass of the fluid in this volume is given by

$$\mathcal{M} = \int_\Upsilon \rho \ dV$$

where dV is a volume element. The mass of fluid flowing through the volume element is equal to the fluid flowing across the surface of the element, that is, the total fluid flow out of the volume Υ is

$$\int_\mathcal{S} \rho u \cdot n \ dS \qquad \text{(this is the mass flux out of } \Upsilon\text{)}$$

where $\mathcal{S}$ is the surface of the volume Υ, dS is a surface element, and n is the (outward) unit normal to the surface. The rate of change of mass in our volume Υ is equal to the mass flux out, i.e.

$$\frac{\partial}{\partial t} \int_\Upsilon \rho \ dV = -\int_\mathcal{S} \rho u \cdot n \ dS.$$

Applying the divergence theorem to the right hand side and taking the derivative under the integral sign we get

$$\int_{\mathcal{V}} (\frac{\partial \rho}{\partial t} + \text{div}(\rho u))dV = 0,$$

and since our volume was arbitrary, we get the well known relation

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0$$

which is the equation of continuity (or conservation of mass) in the Eulerian form. Another derivation can be realized from the principle of conservation of mass which states that the mass of a fluid in a moving material volume $\mathcal{V}(t)$, does not change as the volume moves with the fluid. That is,

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \ dV = 0.$$

Using the transport theorem, it easily follows that

$$\int_{\mathcal{V}} (\frac{d\rho}{dt} + \rho\,\text{div}(u))dV = 0.$$

Thus, since $\mathcal{V}(t)$ was arbitrary, it must be true that

$$\frac{d\rho}{dt} + \rho\,\text{div}(u) = 0.$$

We now consider the dynamics of fluid motion. The stress principle of Cauchy states that "upon any imagined closed surface $\mathcal{S}$ there exists a distribution of stress vectors t whose resultant and moment are equivalent to those of actual forces of material continuity exerted by the material outside $\mathcal{S}$ upon that inside."[2] We assume that the stress vectors, t, depend only on position and the

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[2]This statement of Cauchy's principle is due to Truesdell, J. Rational Mech. Anal, 1, 125 (1952).

orientation of the surface element dS at any given time. If n is the outward unit normal to the surface $\mathscr{S}$, then t = t(x,t;n). This gives rise to the fundamental principle of the dynamics of fluid motion, namely the principle of conservation of linear momentum which states that the time rate of change of the linear momentum of a material volume $\mathscr{V}(t)$ equals the resultant force on the volume, or

$$\frac{d}{dt}\int_{\mathscr{V}}\rho u \ dV = \int_{\mathscr{V}}\rho f \ dV + \int_{\mathscr{S}} t dS$$

where f is the extraneous force per unit mass and will be derived at a later time. By virtue of previous work, this may be written in the form

$$\int_{\mathscr{V}}\rho\frac{du}{dt} \ dV = \int_{\mathscr{V}}\rho f \ dV + \int_{\mathscr{S}} t \ dS.$$

It can be shown that the stress forces are in local equilibrium (i.e. that the stress forces vanish as the volume does) and that the stress vectors t can be expressed as a linear function of the components of n, that is $t^i=n_j\mathscr{T}^{ji}$ where $\mathscr{T}^{ji}$ are the matrix coefficients which form a tensor called the stress tensor, denoted by $\mathscr{T}$. Physically, $\mathscr{T}^{ij}$ is the j-component of the force acting on the surface element with outer normal in the i-direction showing that t = n $\cdot$ $\mathscr{T}$. Using this fact and the divergence theorem gives

$$\int_{\mathscr{V}}\rho\frac{du}{dt} \ dV = \int_{\mathscr{V}}(\rho f + \nabla \cdot \mathscr{T}) \ dV$$

where the divergence of a tensor means the divergence of the tensor elements, i.e.

$$(\nabla \cdot \mathscr{T})_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_i}\mathscr{T}^{ij}$$

is the i-th component. Again since Υ was arbitrary we get the equation of motion discovered by Cauchy, namely

$$\rho\frac{du}{dt} = \rho f + \text{div}(\mathcal{T}).$$

It is appropriate to note here that we have derived four equations that relate the four unknowns given by ρ and the components of u. These four equations being the equation of continuity and the components of the equation of motion.

The equation of motion can be transformed into the well known Euler's Equation of Motion by recalling that

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u$$

to get

$$\rho(\frac{\partial u}{\partial t} + (u \cdot \nabla)u) = \rho f + \text{div}(\mathcal{T}).$$

At this point we adopt Boltzmann's postulate[3], namely that the stress tensor $\mathcal{T}$ is symmetric i.e. $\mathcal{T}^{ij} = \mathcal{T}^{ji}$ and then give the theorem of conservation of angular momentum. If conservation of angular momentum is not assumed, one must find physical, extraneous forces which cancel the extraneous couples in the stress tensor. As of yet, no such forces have been found.

Theorem. Conservation of Angular Momentum:

Let $r(t) = x(t) - x(0)$, and for an arbitrary continuous medium satisfying the continuity equation, the equation of motion, and Boltzmann's postulate, then

$$\frac{d}{dt}\int_{\Upsilon}\rho(r \times u)\ dV = \int_{\Upsilon}\rho(r \times f)\ dV + \int_{\mathcal{S}} r \times t\ dS$$

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[3]This was originally due to Cauchy, but Boltzmann recognized that it could serve as an axiom.

where Υ is an arbitrary material volume.

Proof.

$$\frac{d}{dt}\int_{\Upsilon} \rho(r \times u)\ dV = \int_{\Upsilon} \rho\frac{d}{dt}(r \times u)\ dV = \int_{\Upsilon} \rho(r \times \frac{du}{dt})\ dV =$$

$$\int_{\Upsilon} (\rho(r \times f) + r \times div(\mathfrak{T}))\ dV = \int_{\Upsilon} \rho(r \times f)\ dV + \int_{\Upsilon} (r \times div(\mathfrak{T}))\ dV$$

and, applying the divergence theorem to the last term we

get

$$\int_{\Upsilon} \rho(r \times f)\ dV + \int_{\mathcal{S}} (r \times t)\ dS,$$

since $\mathfrak{T}$ is symmetric, thus the theorem is proved.

We next wish to look at the energy transfer equation and the momentum transfer equation to see in what manner the presence of external forces effect the fluid flow. Consider the kinetic energy $\mathfrak{T}$,

$$\mathfrak{T} = \frac{1}{2} \int_{\Upsilon} \rho u \cdot u\ dV = \frac{1}{2}\int_{\Upsilon} \rho q^2 dV\ .$$

Let $\mathfrak{D}$ be the deformation tensor defined by

$$\mathfrak{D}_{ij} = \frac{1}{2}(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}).$$

We then define the energy transfer equation as

$$\frac{d\mathfrak{T}}{dt} = \int_{\Upsilon} \rho\ f \cdot u\ dV + \oint_{\mathcal{S}} t \cdot u\ dS - \int_{\Upsilon} \mathfrak{T} : \mathfrak{D}\ dV$$

where $\mathfrak{T} : \mathfrak{D}$ is the scalar tensor product defined by $(\mathfrak{T} : \mathfrak{D})|_{ij} = \mathfrak{T}^{ij}\mathfrak{D}_{ij}$. That is, "the rate of change of kinetic energy of a moving volume is equal to the rate at which work is being done on the volume by external forces minus a dissipation term involving the interactions of stress and deformation."[4] This latter term represents the rate at which work must be done to change the volume and fluid

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[4] James Serrin, Handbuch der Physik, Vol. VIII/1, pg.138

element shape. It is instrumental to note that much of the power in this term must be accountable as heat loss.

We can derive the momentum transfer equation by utilizing the principle of conservation of linear momentum and applying the divergence theorem to get

$$\int_V \rho\ f\ dV + \oint_S t\ dS = \frac{d}{dt}\int_V \rho\ u\ dV = \frac{\partial}{\partial t}\int_V \rho\ u\ dV + \oint_S \rho u\ (u \cdot n)\ dS$$

by virtue of the transport theorem. Thus we get the momentum transfer equation

$$\frac{\partial}{\partial t}\int_V \rho\ u\ dV = \int_V \rho\ f\ dV + \oint_S [\ t - \rho u\ (\ u \cdot n\ )]\ dS$$

which expresses the rate of change of momentum of a fixed volume. (This equation is used to determine the force on an obstacle immersed in a steady flow).

We now turn our attention to the quantity grad(u). If grad(u) is decomposed into

$$grad(u) = \mathfrak{D} + \Omega$$

where $\mathfrak{D}$ is as before and $\Omega = \frac{1}{2}(\ \frac{\partial u^i}{\partial x_j} - \frac{\partial u^j}{\partial x_i}\ )$, then we have decomposed this term into its symmetric and skew symmetric parts. The symmetric part, $\mathfrak{D}$, is called the deformation tensor while the skew symmetric part, Ω, is called the vorticity tensor. These tensors have properties which are convenient to look at in a little more detail. First, the deformation tensor: Let $dx = (dx_1, dx_2, dx_3)$ denote a material element of arc, then its rate of change during the fluid motion is given by $\frac{d}{dt}(dx^i) = \frac{d}{dt}(\frac{\partial x^i}{\partial X^a}dX^a) = \frac{\partial u^i}{\partial X^a}dX^a = \frac{\partial u^i}{\partial x^j}dx^j$. So, we get $\frac{d}{dt}(dx) = dx \cdot grad(u)$. If $ds = |dx|$, then it is easy to see that $\frac{d}{dt}(ds^2) = 2\ dx \cdot \mathfrak{D} \cdot dx$, thus $\mathfrak{D}$

is a measure of the rate of change of the squared element of arc following a fluid motion. Consider rigid motion, ds = const., then a necessary and sufficient condition for motion to be locally and instantaneously rigid is that $\mathfrak{D}$ = 0. For this reason $\mathfrak{D}$ is called the deformation tensor, or the amount of deformation. A necessary and sufficient condition for motion locally preserving angles is that the tensor $\mathfrak{D} - \frac{1}{3}\text{trace}(\mathfrak{D})\ \mathfrak{I} = 0$. If $\mathfrak{D} = 0$ everywhere in the fluid then the motion is rigid.

We turn now to the general motion of a fluid. Consider the velocity field near a fixed point P. Then, near P we have u = u_p + r · (grad u)$_p$ + $O(r^2)$, where r denotes the radius vector from P. If we neglect terms of order r^2 we obtain the relation u = u_p + r · $\mathfrak{D}_p$ + r · Ω_p utilizing the decomposition of grad(u). Now, u_p represents a uniform translation with velocity u_p. If we let $\mathfrak{D} = r \cdot \mathfrak{D} \cdot r$ then [r·$\mathfrak{D}$] = grad($\frac{1}{2}\mathfrak{D}$) where grad($\frac{1}{2}\mathfrak{D}$) represents a velocity field normal at each point to the quadric surface $\mathfrak{D}$ = const. which passes through that point. There exists three mutually perpendicular directions with no instantaneous rotation, called the axes of strain. The eigenvalues of $\mathfrak{D}$ measure the rates of expansion per unit length of fluid element in these directions. The last term, r · Ω, can be written as $\frac{1}{2}\omega_p \times r$ where ω = curl(u) is the vorticity vector. We can see this since $\omega = (\text{grad}(u))_x = \Omega_x = 2(\Omega_{23}, \Omega_{31}, \Omega_{12})$ = the

components of $r \cdot \Omega_p$. This shows that $r \cdot \Omega_p$ is a rigid rotation with angular velocity $\frac{1}{2}\omega_p$. From this we see that for an arbitrary motion in a neighborhood of P, the velocity is, up to terms of order r^2, given by $u = u_p + \text{grad}(\frac{1}{2}\mathfrak{D}) + \frac{1}{2}\omega_p \times r$ where $\mathfrak{D} = r \cdot \mathfrak{S} \cdot r$ is the rate of strain quadric and $\omega = \text{curl}(u)$ is the vorticity vector. Thus, an arbitrary instantaneous flow is, at each point, a superposition of uniform velocity of translation, a dilation along the axes of strain and a rigid rotation about these axes. It is instrumental to note that if $\omega = \Omega = 0$ in a finite portion of fluid then the relative motion of any element of that portion consists of pure deformation and is called irrotational.

In order for a complete description of the fluid flow, we must address the thermodynamics of the fluid flow. This is necessary a total energy equation of a fluid element. Here, we outline some basic thermodynamical relationships which are relevant to the derivations. It must be noted that since our derivation of the EMHD equations at this point is confined to classical fluid flows, we consider a one-phase system only. We will attempt to treat the thermodynamics of the EMHD problem when it becomes appropriate to do so. The one phase system underlies all of the hydrodynamics of compressible fluids, thus this is not an unreasonable direction to take at this point.

In describing the single phase system, the state variables, volume V, entropy η, specific internal energy e, pressure p and absolute temperature Θ must be related in some fundamental form. We start with the fundamental state equation, $e = e(\eta, V)$, and define the state variables p and Θ by

$$p = -\frac{\partial e}{\partial V} > 0 \quad \text{and} \quad \Theta = -\frac{\partial e}{\partial \eta} > 0.$$

We can thus relate the state variables by forming the total differential of e, which gives

$$\Theta \, d\eta = de + p \, dV.$$

A single phase system is said to undergo a differential process if its state variables are differentiable functions of time. If the phase moves reversibly, that is, if it is in a state of equilibrium with its surroundings at each instant, then the heat, Q, supplied to the phase during the process is given by

$$dQ = de + p \, dV.$$

The heat capacities of the phases are given by

$$c_v = \Theta \left(\frac{\partial \eta}{\partial \Theta}\right)\Big|_v = \left(\frac{dQ}{d\Theta}\right)\Big|_v \quad \text{and} \quad c_p = \Theta \left(\frac{\partial \eta}{\partial \Theta}\right)\Big|_p = \left(\frac{dQ}{d\Theta}\right)\Big|_p$$

where c_v is the heat capacity at constant volume and c_p is the heat capacity at constant pressure. It is evident that these two quantities are also state variables. It is always found experimentally that $c_p > c_v > 0$.

The following theoretical relations will be needed in this treatment;

$$\left(\frac{\partial e}{\partial V}\right)\Big|_\Theta = \Theta \left(\frac{\partial p}{\partial \Theta}\right)\Big|_v - p$$

$$c_p - c_v = \Theta \left(\frac{\partial p}{\partial \Theta}\right)|_v \left(\frac{\partial V}{\partial \Theta}\right)|_p$$

$$\left(\frac{\partial p}{\partial \Theta}\right)|_v = -\left(\frac{\partial V}{\partial \Theta}\right)|_p \left(\frac{\partial p}{\partial V}\right)|_\Theta$$

$$\frac{c_p}{c_v} = \left(\frac{\partial p}{\partial V}\right)|_\eta \Big/ \left(\frac{\partial p}{\partial V}\right)|_\Theta$$

and from these it is easily deduced that

$$\left(\frac{\partial p}{\partial V}\right)|_\eta < \left(\frac{\partial p}{\partial V}\right)|_\Theta.$$

The first law of thermodynamics states that the amount of energy put into a control volume is equal to the amount of energy released by that control volume plus the energy used in changing the volume in any manor, that is dQ = de + p dV, which we have already noted.

The second law of thermodynamics states that the rate of change of entropy inside a control volume plus the rate of flow across the boundary of the volume is equal to the heat flow across the boundary plus heat energy lost within the volume plus the amount of work done to the volume. Since it is nearly impossible to define how much heat is lost within the volume or the work done to the volume, we use the differential form,

$$d\mathfrak{C} \geq \sum_a \frac{dQ_a^e}{\Theta_a} \quad \text{where,} \quad \mathfrak{C} = \sum_a \eta_a$$

is the total entropy in all phases a, Q_a^e is the amount of heat supplied to phase a from external sources and Θ_a is the absolute temperature at phase a. A process is called spontaneous if inequality holds. If the state variables are such that no spontaneous process is available then the system is said to be in equilibrium.

We define the total energy of a volume V as the sum

of its kinetic energy $\mathfrak{T}$ and its internal energy $\mathfrak{U}$ where

$$\mathfrak{T} = \tfrac{1}{2}\int \rho(u \cdot u)\, dV, \quad \mathfrak{U} = \int \rho e\, dV$$

and e is the specific internal energy. For a compressible fluid, e is a thermodynamic state variable satisfying $\Theta\, d\eta = de + d(\tfrac{1}{\rho})$, where the quantity $\tfrac{1}{\rho}$ is the specific volume. Since total energy is conserved within the fluid flow, we have that the rate of change of the total energy of a material volume is equal to the rate at which work is being done on the volume plus the rate at which heat is conducted into the volume, or

$$\frac{d}{dt}(\mathfrak{T} + \mathfrak{U}) = \int_{\mathscr{V}} \rho f \cdot u\, dV + \int_{\mathscr{S}} t \cdot u\, dS - \int_{\mathscr{S}} h \cdot n\, dS$$

where h is the heat flux vector and n is the outward unit normal vector to the surface $\mathscr{S}$. Recalling the energy transfer equation, we can rewrite the energy equation or total energy equation as

$$\rho\, \frac{de}{dt} = \mathfrak{T} : \mathfrak{D} - \mathrm{div}(h),$$

where $\mathfrak{T} : \mathfrak{D} = \mathrm{trace}(\mathfrak{T}\mathfrak{D}) = \sum_{ij=1}^{3} \mathfrak{T}_{ij}\mathfrak{D}_{ji}$. We see then that conservation of energy corresponds to the first law of thermodynamics. In analogy, we have the equation

$$\frac{d}{dt}\int_{\mathscr{V}} \rho\eta\, dV \geq -\oint_{\mathscr{S}} \frac{h \cdot n}{\Theta}\, dS$$

corresponding to the second law, which states that the rate of change of entropy within a control volume $\mathscr{V}$ is greater then or equal to the total amount of heat flow through the surface of the volume.

Consider the thermodynamics of deformation. When tangential stresses may not be neglected, we write the

stress tensor $\mathfrak{I}$ as $\mathfrak{I} = -p\mathfrak{I} + \mathfrak{g}$ (defining $\mathfrak{g}$) where p is the thermodynamic pressure for a compressible fluid. Then, substituting this into the total energy equation we get

$$\rho\frac{de}{dt} + p \text{ div } u = \Upsilon - \text{div } h$$

where $\Upsilon = \mathfrak{g} : \mathfrak{D}$. By using the continuity equation, this reduces to

$$\rho\Theta \frac{d\eta}{dt} = \Upsilon - \text{div } h$$

which expresses the rate of change of entropy following a particle. Thus, $\Upsilon - \text{div}(h)$ is the rate per unit volume at which heat is absorbed and Υ is the rate per unit volume at which heat is generated by deformation of the fluid elements. This is evident since $\text{div}(h)$ represents the conduction of heat from neighboring fluid elements. For this reason, Υ is called the dissipation function.

Now, if we divide the above equation by the temperature Θ and integrate over a volume moving with the fluid we get

$$\frac{d}{dt}\int_{\Upsilon} \rho\eta \text{ dV} = \int_{\Upsilon} (\frac{\Upsilon}{\Theta} - \frac{h \cdot \text{grad } \Theta}{\Theta^2}) \text{ dV} - \oint_{\mathcal{J}} \frac{h \cdot n}{\Theta} \text{ dS}$$

which implies that $\Upsilon - \frac{h \cdot \text{grad } \Theta}{\Theta} \geq 0$. This gives us $\Upsilon \geq 0$ and $h \cdot \text{grad } \Theta \leq 0$ since heat never flows against a temperature gradient and deformation absorbs energy, converting it to heat, but never releasing it.

Recall that up to now, much of what has been done pertains to nonviscous fluids. We turn then to the viscous case. A viscous fluid, put simply, is a fluid in which internal stresses cause an irreversible transfer of

momentum in a fluid volume. Recalling that the stress tensor governs the dynamic response of the medium, we attempt to write constitutive equations of viscous fluid flow.

Sir George Stokes postulated that: 1) $\mathcal{T}$ is a continuous function of the deformation tensor $\mathcal{D}$, and is independent of all other kinematic quantities. 2) $\mathcal{T}$ does not depend explicitly on the position, X (spatial homogeneity). 3) There is no preferred direction in space (isotropy). 4) When $\mathcal{D} = 0$ then $\mathcal{T} = - p\mathcal{J}$. Thus, we have that $\mathcal{T} = f(\mathcal{D})$ from numbers 1) and 2) above and $S\mathcal{T}S^{-1} = f(S\mathcal{T}S^{-1})$ for all orthogonal transformation matrices S, from number 3 above. The second equation implies that a given deformation produces the same intrinsic response regardless of its orientation. That is to say, it is invariant under coordinate transformations.

Together with the first equation above, it can be shown that Stokes' postulates leads to the stress tensor being a quadratic function of the deformation tensor. However, it is reasonable to assume that, since the deformation tensor is fairly small with respect to the ratio of some reference speed and reference length, then there exists a linear relationship between $\mathcal{T}$ and $\mathcal{D}$. This hypothesis leads to the classical constitutive equation, called the Cauchy-Poisson law,

$$\mathcal{T} = (-p + \lambda\zeta)I + 2\mu\mathcal{D}$$

where λ and μ are scalar functions of the thermodynamic state called the shear and bulk viscosities, $\zeta = \nabla \cdot u$ and I is a principle invariant of the deformation tensor.

The dissipation function, Υ, corresponding to the above stress tensor for compressible flows is thus given as $\Upsilon = \mathfrak{g} : \mathfrak{H} = \lambda\zeta^2 + 2\mu\mathfrak{H} : \mathfrak{H}$ where as before we have the condition that $\Psi \geq 0$. This last relation places restrictions upon the viscosities λ and μ, namely that for a compressible fluid we have that $3\lambda + 2\mu \geq 0$ and $\mu \geq 0$. Now, placing the Cauchy-Poisson law into the equation

$$\rho\frac{du}{dt} = \rho f + \text{div}(\mathfrak{T})$$

yields the well known Navier-Stokes equation

$$\rho\frac{du}{dt} = -\nabla p + \nabla (\lambda \, \text{div}(u)) + \text{div}(2\mu\mathfrak{H}) + \rho f.$$

The last step in the derivation of the equations of classical hydrodynamics is to express the heat conduction vector h in terms of mechanical and thermodynamic variables. In so doing, we assume that h is an isotropic function of the temperature gradient and thermodynamic state. Thus, h is parallel to $\nabla\Theta$ which gives rise to the classical Newton-Fourier law

$$h = -\kappa \, \nabla\Theta,$$

where κ is a scalar function of thermodynamic variables and $|\nabla\Theta|$. The thermodynamic condition

$$\rho\Theta\frac{d\eta}{dt} = \Upsilon - \text{div}(h)$$

implies that $\kappa \geq 0$ and has been seen by experimental observations that

$$\frac{\mu c_p}{\kappa} = P_r$$

where P_r is a constant commonly called the Prandtl number. Inserting this into the thermodynamic condition we get

$$\rho \Theta \frac{d\eta}{dt} = \Upsilon + \text{div}(\kappa \nabla\Theta).$$

With this last step, we have completed our quick discussion of classical fluid dynamics. We have found that the governing equations of fluid flow are:

1) $\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho u) = 0$ (the equation of continuity)

2) $\rho\frac{du}{dt} = -\nabla p + \nabla(\lambda(\nabla \cdot u) + \nabla \cdot (2\mu\mathfrak{I}) + \rho f$ (Navier-Stokes)

3) $\rho\Theta\frac{d\eta}{dt} = \Upsilon + \nabla \cdot (\kappa\nabla\Theta)$ (heat conduction)

and the thermodynamic equations of state. The only thing left is to say something about the external force term ρf in the Navier-Stokes equation. In our setting the extraneous forces are those which affect the fluid flow through transmission of energy to the fluid. These terms being the force due to gravity and the electromagnetic forces. Thus, the extraneous forces are given by

$$\rho f = -\rho g + \rho\frac{e^-}{m_e}(E + v \times B)$$

which is the force due to gravity and the Lorentz force, where e^- is the charge of an electron and m_e is the mass of an electron.

We can thus finally write the equations of fluid motion given these extraneous forces. The system of equations are:

1) $\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho u) = 0$ (the equation of continuity)

2) $\rho\frac{du}{dt} = - \nabla p + \nabla(\lambda(\nabla \cdot u)) +$

$$\nabla \cdot (2\mu \vartheta) - \rho g + \rho (E + v \times B) \qquad \text{(Navier-Stokes)}$$

3) $\rho\Theta\dfrac{d\eta}{dt} = \Upsilon + \nabla \cdot (\kappa \nabla\Theta)$ \qquad (heat conduction)

and the thermodynamic equations of state. We can make one more simplification by reducing the last two sets of equations, the heat conduction equation and the thermodynamic equations of state, yielding an energy equation. By realizing the thermodynamic relation $d\eta = \dfrac{c_p d\Theta}{\Theta}$ then the heat conduction equation becomes

$$\rho\Theta\frac{d\eta}{dt} = \rho\Theta\frac{c_p d\Theta}{\Theta \, dt} = \rho c_p \frac{d\Theta}{dt} = \Upsilon + \kappa\Delta\Theta + \nabla\Theta \cdot \nabla\kappa$$

and we get the new system of equations by the equation of continuity, Navier-Stokes equation and the energy equation

3) $\rho c_p \dfrac{d\Theta}{dt} = \Upsilon + \kappa\Delta\Theta + \nabla\Theta \cdot \nabla\kappa$

where the time derivatives are given by the convective derivative

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (u \cdot \nabla)F.$$

B. Electrodynamics

Here, we wish to derive Maxwell's equations for slowly moving nonmagnetizeable media, where by nonmagnetizeable we mean a media that cannot be effected macroscopically by the introduction of a magnetic field. We start with a treatment of stationary charged particles in free space. In 1785, Charles Coulomb was able to express the experimental observations seen by himself and his peers. He found that for a test charge Q, the force on Q due to a single source charge q_i is given by

$$\mathbf{F}_i = \frac{1}{4\pi\epsilon_o} q_i Q \, \hat{r}_i$$

where $\hat{r}_i$ is the unit vector pointing from the position of q_i to the position of Q, and the constant ϵ_o is the permitivity of free space. The constant of proportionality, $\frac{1}{4\pi\epsilon_o}$, depends on the system used and the factor of 4π is a normalization inserted to cancel a 4π that will arise from other sources. We also note here that this is a repulsive force if the charges have the same sign and attractive otherwise.

Since the force is linear in the source charge q_i, then for a distribution of point charges, the total force is given by the law of superposition which states that "the interaction between any two charges is completely unaffected by the presence of all other charges."[5] This

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[5]Griffiths, David G., Introduction to Electrodynamics, Prentice Hall, pg. 49.

gives the total force on a test charge Q due to a distribution of source charges q_i as

$$F = \sum_{i=1}^{n} F_i = \frac{Q}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i \hat{r}_i}{r_i^2}.$$

We define the quantity $E = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i \hat{r}_i}{r_i^2}$ as the electric field of the source charges and note that it depends on the configuration of the source charges in space. If we have a continuous charge distribution, then the point charges $q_i \rightarrow dq = \rho_c(\vec{r})d\nu$ with a charge density per unit volume $\rho_c(\vec{r})$ and the sum passes to the appropriate integral

$$E(P) = \frac{1}{4\pi\epsilon_0} \int_{\mathscr{V}} \rho_c(\vec{r}) \frac{\hat{r}}{r_i^2} dV.$$

Since this last integral is, in general, an extremely difficult one to handle, we wish to recast it into a more useful form. Consider the divergence of the electric field,

$$\nabla \cdot E = \frac{1}{4\pi\epsilon_0} \int_{\mathscr{V}} \nabla \cdot \frac{\vec{r}}{r^3} \rho_c \, dV.$$

Here, the differentials are with respect to the field coordinates and the charge density is a function of the source coordinates, thus we need only look at the term involving the position vector,

$$\nabla \cdot \frac{\vec{r}}{r^3} = (\nabla \cdot \vec{r}) \frac{1}{r^3} + \vec{r} \cdot \nabla(\frac{1}{r^3}).$$

Now, the first term is just $\frac{3}{r^3}$, while the second term is just the negative of this and we see that the divergence here is zero. But, what happens if $r \rightarrow 0$. Then we need to take a closer look at what is happening in the divergence.

Consider a ball of radius δ centered at (x',y',z').

If δ is small enough then ρ_c can be considered constant and brought outside of the integral to get

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{4\pi\epsilon_o}\int \nabla \cdot \frac{\vec{r}}{r^3}\, dV.$$

Since $\vec{r} = (x-x', y-y', z-z')$, the differentials with respect to the field coordinates (unprimed) can be replaced with differentials with respect to the source coordinates (primed) at the cost of a minus sign, $\nabla \cdot \frac{\vec{r}}{r^3} = -\nabla' \cdot \frac{\vec{r}}{r^3}$. Now, a simple application of the divergence theorem (realizing that $\hat{r}$ is an inward normal vector and the normal vector in the divergence theorem is an outward normal vector) gives

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{4\pi\epsilon_o}\int \frac{\hat{r}}{r^2} \cdot (r^2\sin(\theta)\, d\theta\, d\phi\, \hat{r}) = \frac{\rho_c}{\epsilon_o},$$

as $r \to 0$, where we have used spherical polar coordinates to evaluate the integral. Thus we obtain the first of Maxwell's equations, namely Gauss' Law

1) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_o}\rho_c(r)$

where $\rho_c(r)$ is the charge distribution.

There are two more experimental facts that need to be addressed at this point, the first being the Lorentz Force Law and the second the law of Biot-Savart. First, if we consider two very long wires hooked to a battery and separated by a distance d, then when a current i travels through the wires, there exists a force exerted upon them (attractive if the currents are running in the same direction and repulsive otherwise). What is this force? In 1820, Hans Christian Oersted discovered that a magnet

placed near a current carrying wire will align itself perpendicularly to the wire. That is, each particle of charge, q, which passes a given point in the magnetic field B experiences a force $F_m = q(v \times B)$. If we combine this with the electric force, $F_e = qE$, we get the Lorentz Force Law

$$F = q (E + v \times B).$$

But, what does this have to do with the two wires laying side by side? Since we had no magnet producing a magnetic field, how can there exist this force between the two wires? The answer was given by Ampere who showed that by replacing the magnet by a current carrying wire, the same effect was observed. Biot and Savart were able to quantify Ampere's measurements by showing that the magnetic field due to a line current is given by

$$B(P) = \frac{\mu_o}{4\pi} \int \frac{i \times \hat{r}}{r^2} dl = - \frac{\mu_o}{4\pi} i \int \frac{\hat{r} \times dl}{r^2}$$

where $\mu_o \equiv$ permeability of free space, $\vec{r} = x - x'$ is the vector pointing from the source to the test position in the field coordinates, the integral runs over the line carrying the current, and i and dl point in the same direction. We see then, that the magnetic field is proportional to the current. Similar expressions for surface currents and volume currents are given by

$$B(P) = \frac{\mu_o}{4\pi} \int_{\mathcal{J}} \frac{K \times \hat{r}}{r^2} dS \qquad \text{or} \qquad B(P) = \frac{\mu_o}{4\pi} \int_{\mathcal{V}} \frac{J \times \hat{r}}{r^2} dV,$$

where the integration is over the current distribution, or the source coordinate frame of reference as we will call

it.

Let us examine the divergence of the magnetic field:

$$\nabla \cdot B = \frac{\mu_0}{4\pi}\left[\nabla \cdot \int_V \frac{J(r) \times \hat{r}}{r^2}dV\right] = \frac{\mu_0}{4\pi}\int_V \nabla \cdot \frac{J(r) \times \hat{r}}{r^2} dV$$

We note two things here, first that

$$\nabla(\tfrac{1}{r}) = -\frac{\hat{r}}{r^2}$$

where $r = ((x-x')^2 + (y-y')^2 + (z-z')^2)^{1/2}$ and the vector identity

$$\nabla \cdot a \times b = b \cdot (\nabla \times a) - a \cdot (\nabla \times b)$$

gives

$$\nabla \cdot J \times \nabla(\tfrac{1}{r}) = \nabla(\tfrac{1}{r}) \cdot (\nabla \times J) - J \cdot (\nabla \times \nabla(\tfrac{1}{r})).$$

Now, J is the charge distribution, it is not a function of the field coordinates so that $\nabla \times J = 0$ and the last term is zero because the curl of a gradient is always zero. We see then that

$$\nabla \cdot B = \frac{\mu_0}{4\pi}\int_V \nabla \cdot \frac{J \times \hat{r}}{r^2} dV = 0$$

and we get the second of Maxwell's equations

2) $\nabla \cdot B = 0$.

which, incidentally has no name.

If we take a closer look at the moving charges, in particular, consider an arbitrary volume of charge density $\rho_c(r,t)$ at time t and $\rho_c(r,t+\Delta t)$ at a time $t+\Delta t$ later. Then, since the charge is moving, this gives rise to a current flowing through the surface of the volume V, that is

$$i = \int_V J \cdot dS$$

and, the total change of the amount of charge within the

volume is

$$\Delta Q = \int_V \left[\rho_c(r,t+\Delta t) - \rho_c(r,t)\right] dV = -\oint_S \rho_c \, v\Delta t \cdot dS = -\oint_S J\Delta t \cdot dS$$

thus

$$\int_V \frac{\rho_c(r,t+\Delta t) - \rho_c(r,t)}{\Delta t} \, dV + \oint_S J \cdot dS = 0$$

and, as $\Delta t \to 0$ we get

$$\int_V \frac{\partial \rho_c}{\partial t} \, dV + \oint_S J \cdot dS = 0,$$

applying the divergence theorem to the surface integral, and denoting differentiation with respect to the source coordinates by a prime, we get

$$0 = \int_V \frac{\partial \rho_c}{\partial t} \, dV + \int_V \nabla' \cdot J \, dV = \int_V \left[\frac{\partial \rho_c}{\partial t} + \nabla' \cdot J \right] dV,$$

Since the volume was arbitrary, then the integrand must be zero, which gives the equation of continuity

$$\frac{\partial \rho_c}{\partial t} + \nabla' \cdot J = 0,$$

stating that charge is conserved.

Suppose we want to make charges move within a conductor. Then, we must push them by some manner. How fast they move is proportional to how hard we push, that is, the current density is proportional to the force per unit charge exerted, i.e. $J = \sigma f$, where $\sigma \equiv$ conductivity of the conductor. But, the force per unit charge is given by the Lorentz Force Law as

$$f = \frac{F}{q} = E + v \times B.$$

Now, if we are trying to push a charge around a wire, then the net effect of the force is given as an electromotive force

$$\epsilon = \oint_C f \cdot dl.$$

We also are interested in how this relates to the electric and magnetic fields. The magnetic flux through a loop of wire is given by

$$\Phi = \oint_C B \cdot da,$$

but, if the wire loop is in motion then $da = (v \times dl)dt$, and the flux becomes

$$\Phi = \oint_C B \cdot da = \oint_C B \cdot (v \times dl) \, dt.$$

Then, the rate of change of flux is

$$\frac{d\Phi}{dt} = \oint_C B \cdot (v \times dl) = -\oint_C v \times B \cdot dl = -\oint_C f_m \cdot dl.$$

Recall that it was the motion of the conductor that we are considering and that since the electric field is a static one, then it cannot generate an electromotive force, i.e. $\oint_C E \cdot dl = 0$. Adding this term to the flux change, since it contributes nothing to the integral, we get

$$\frac{d\Phi}{dt} = -\oint_C v \times B \cdot dl = -\oint_C [E + v \times B] \cdot dl = -\epsilon$$

and we see that the electromotive force is just the negative of the time rate of change of the magnetic flux,

$$\epsilon = -\frac{d\Phi}{dt}.$$

What would happen if instead of the loop moving we fixed the loop and made the magnetic field change with time? Michael Faraday found in his experiments in 1831 that the result was the same. The changing magnetic field induced an electric field in the wire. We see that it is the relative movement of the loop and the magnetic field that generates the electromotive force in the wire and we

are led to conclude that

$$\oint_C E \cdot dl = \epsilon = - \frac{d\Phi}{dt} = - \frac{d}{dt} \int_\mathcal{S} B \cdot da = - \int_\mathcal{S} \frac{\partial}{\partial t} B \cdot da.$$

Applying Stokes Theorem to the term on the right we get

$$\oint_C E \cdot dl = \int_\mathcal{S} \nabla \times E \cdot da = - \int_\mathcal{S} \frac{\partial B}{\partial t} \cdot da.$$

Again, since the volume was arbitrary then it must be true

that

3) $\nabla \times E = - \frac{\partial B}{\partial t}$

which is the third equation of Maxwell, the well known

Faraday's Law.

Recall the law of Biot and Savart;

$$B(P) = \frac{\mu_o}{4\pi} \int_\mathcal{V} \frac{J(r) \times \hat{r}}{r^2} dV.$$

If we take the curl of this, motivated by the fact that we

have very similar equations for electric and magnetic

fields so far, we get

$$\nabla \times B = \frac{\mu_o}{4\pi} \nabla \times \int_\mathcal{V} \frac{J(r) \times \hat{r}}{r^2} dV = \frac{\mu_o}{4\pi} \int_\mathcal{V} \nabla \times \frac{J(r) \times \hat{r}}{r^2} dV.$$

Consider the integrand, the vector identity of the curl of

a vector cross-product gives

$$\nabla \times \frac{J(r) \times \hat{r}}{r^2} = \nabla \times \left[J \times \frac{\hat{r}}{r^2} \right] = \left[\frac{\hat{r}}{r^2} \cdot \nabla \right] J - \left[J \cdot \nabla \right] \frac{\hat{r}}{r^2}$$

$$+ J \left[\nabla \cdot \frac{\hat{r}}{r^2} \right] - \frac{\hat{r}}{r^2} \left[\nabla \cdot J \right].$$

Considering each term separately and by components, we

have for the first term that

$$\left[\frac{\hat{r}}{r^2} \cdot \nabla \right] J = \left[\frac{x - x'}{(x - x')^2} \frac{\partial}{\partial x} + \frac{y - y'}{(y - y')^2} \frac{\partial}{\partial y} + \frac{z - z'}{(z - z')^2} \frac{\partial}{\partial z} \right] J$$

but, the differentials here are with respect to the field

coordinates and the current density is a function of the

source coordinates, thus we have that

$$\frac{\partial}{\partial x} \ J = \frac{\partial}{\partial y} \ J = \frac{\partial}{\partial z} \ J = 0$$

which shows that the first term is identically zero. Similarly, we see that the last term $\nabla \cdot J = 0$ since this is also differentiation with respect to the field coordinates.

A look at the third term gives

$$J \left[\nabla \cdot \frac{r}{r^3} \right] = J \left[\frac{1}{r^3} (\nabla \cdot r) + r \cdot \left[\nabla \frac{1}{r^3} \right] \right] \cdot$$

Now it is clear that $\nabla \cdot r = 3$ since this is just the derivatives of the position vector. What about the gradient term?

$$\nabla \left[\frac{1}{r^3} \right] = \frac{\partial}{\partial x} \frac{1}{\left[(x - x')^2 + (y - y')^2 + (z - z')^2 \right]^{3/2}} + \cdots$$

$$= -\frac{3}{2} r^{-5} 2 \left[(x - x') + (y - y') + (z - z') \right] = -\frac{3r}{r^5}$$

and we get that

$$\nabla \cdot \frac{r}{r^3} = \frac{3}{r^3} - 3 \frac{r \cdot r}{r^5} = \frac{3}{r^3} - 3 \frac{r^2}{r^5} = 0 \ .$$

Now, what happens when the position vector approaches the field vector i.e. $r \to 0$? We need to investigate this in much more detail as we did for Gauss' law. Since we are dealing with real, physical quantities, then the current density must be at least continuous on a ball of radius δ centered at (x', y', z'). If we take δ small enough, J can be considered constant on that ball and can then be brought out of the integral to get

$$\frac{\mu_o}{4\pi} J \int_{\mathcal{B}} \left[\nabla \cdot \frac{r}{r^3} \right] dV = \frac{\mu_o}{4\pi} J \int_{\mathcal{J}} \frac{\hat{r}}{r^2} \cdot da = \frac{\mu_o}{4\pi} J \int_{\mathcal{J}} \frac{\hat{r}}{r^2} \cdot \hat{r} r^2 \sin(\theta) \ d\theta \ d\phi$$

$$= \frac{\mu_o}{4\pi} J \int_{\mathcal{J}} \sin(\theta) \ d\theta \ d\phi = \mu_o J$$

as r → 0, where here $\mathcal{S}$ is the surface of the ball and we have used spherical polar coordinates to evaluate the integral. So, we are left with

$$\nabla \times B = \mu_0 J + \frac{\mu_0}{4\pi} \int_{\mathcal{S}} - [\; J \cdot \nabla \;] \frac{r}{r^3} \, dV \; .$$

The integrand here is a term which involves derivatives of the position vector $\frac{r}{r^3}$. We can thus change differentiation with respect to the field coordinates to differentiation with respect to the source coordinates at the cost of a minus sign, i.e.

$$[\; J \cdot \nabla \;] \frac{r}{r^3} = - [\; J \cdot \nabla' \;] \frac{r}{r^3} \; ,$$

where the prime signifies differentiation with respect to the source coordinates. If we look at this component by component and use the vector identity which relates the divergence of a scalar function times a vector function, $(\nabla \cdot (Fa)\,) = F(\nabla \cdot a) + a \cdot (\nabla F)$, we get

$$(J \cdot \nabla') \left[\frac{x - x'}{r^3}\right] = \nabla' \cdot \left[J \frac{x - x'}{r^3}\right] - (\nabla' \cdot J) \left[\frac{x - x'}{r^3}\right] \; .$$

So, this term contributes an amount

$$\frac{\mu_0}{4\pi} \int_{\mathcal{S}} \left[\nabla' \cdot \left[J \frac{x - x'}{r^3}\right] - (\nabla' \cdot J)\left[\frac{x - x'}{r^3}\right]\right] \, dV \; .$$

The first term of this integral, by use of the divergence theorem, gives us

$$\int_{\mathcal{S}} \left[\nabla' \cdot \left[\frac{x - x'}{r^3}J\right]\right] \, dV = \int_{\mathcal{S}} \frac{x - x'}{r^3} J \cdot da.$$

The surface $\mathcal{S}$ is any surface which encloses the current, but, on the surface $\mathcal{S}$, J = 0 so this term contributes nothing to the integral. (If you don't like this argument, we can make the volume larger since the current density is zero out there any ways and the integral is

unchanged. We see then by the same argument that we must get zero and the result still holds.) So we are left with

$$\nabla \times B = \mu_o J - \frac{\mu_o}{4\pi} \int [\nabla' \cdot J] \frac{r}{r^3} dV \ ,$$

but by the continuity equation we know that $\nabla' \cdot J = - \frac{\partial \rho}{\partial t}$ and the integral becomes

$$- \frac{\mu_o}{4\pi} \int [\nabla' \cdot J] \frac{r}{r^3} dV = \frac{\mu_o}{4\pi} \int \frac{\partial \rho_c}{\partial t} \frac{r}{r^3} dV = \frac{\mu_o}{4\pi} \int \frac{\partial \rho_c}{\partial t} \frac{\hat{r}}{r^2} dV$$

$$= \frac{\partial}{\partial t} \frac{\mu_o}{4\pi} \int \rho_c(r) \frac{\hat{r}}{r^2} dV$$

and we see that this last integral is just $\mu_o \epsilon_o \frac{\partial}{\partial t} E$. From this we get the forth of Maxwell's equations known as Ampere's Law,

4) $\nabla \times B = \mu_o J + \mu_o \epsilon_o \frac{\partial E}{\partial t}$.

Up to this point we have considered stationary charges in free space, a pretty unrealistic but instructional assumption. We now wish to relax this condition somewhat, and later omit it. As a first step in this direction, we consider stationary charges again, but we will allow them to be in some sort of conducting medium. The first question which naturally arises is what happens to a neutral atom placed in an electric field? The answer to this lies in the fact that a neutral atom is made up of a positively charged nucleus and a negatively charged electron cloud surrounding the nucleus. This cloud acts in such a way as to cancel the net electric charge of the atom, hence it appears neutral on the macroscopic level.

The effect of placing this atom in an electric field causes the atoms microscopic elements to interact with the field, pulling the electrons and the positively charged nucleus apart. Provided that the field isn't so strong as to pull the electrons completely away from the nucleus (i.e. it's not strong enough to ionize the atom), a state of equilibrium is reached and the atom remains in a state of physical distortion, where the electrons are pulled one way while the nucleus is shifted in the opposite way, leaving the atom polarized. The net effect is then that the atom now has a dipole moment, p, pointing in the same direction as the field and, as long as the field is reasonably weak, the dipole moment is proportional to the field applied, $p = \alpha E$, where α is defined to be the atomic polarizability.

In general, for a distribution of molecules, each molecule undergoes this transformation and a net dipole moment per unit volume is set up. It should be noted that this net alignment is an average of all the microscopic dipole moments since the molecular collisions, thermal motions and microscopic field interactions destroy individual alignments. However, the net effect is the dipole moment per unit volume, denoted by P, and called the polarization of the medium.

What does this polarization do to change Maxwell's equations? To answer this, consider a large number of

dipoles within a dielectric medium. If we break the volume up into differential cubes with volume $\Delta x \Delta y \Delta z$ then all the dipoles completely enclosed in the volume element contribute no net charge within the volume, while those within a distance $d \cdot n$ of each of the surfaces contributes a net charge, where n is the outward unit normal and d is the distance from the negatively electrons to the positively charged nucleus, i.e. $qd = p$. Thus the polarization P of the volume is given by the number of dipoles per unit volume i.e. $P = Np = Nqd$.

If we consider each axis of the cube separately, we find that the net charge in the x-direction is given by

$$dq_{x_t} = dq_{x_1} + dq_{x_2} + dq_{x_3} =$$
$$(Nq\ dx)|_x\ \Delta y \Delta z - (Nq\ dx)|_{x+\Delta x} \Delta y \Delta z =$$
$$P_x(x) \Delta y \Delta z - P_x(x+\Delta x) \Delta y \Delta z.$$

Here we assume that Δy and Δz are sufficiently small so that the polarization is constant over the surface and the minus sign comes from the fact that near the surface at x a charge of one sign is present while near the surface at $x+\Delta x$ the oppositely charged part of the atom is present (since they are all aligned by the polarization). Similarly

$$dq_{y_t} = dq_{y_1} + dq_{y_2} + dq_{y_3} = \Delta x\ P_y(y)\ \Delta z - \Delta x\ P_y(y+\Delta y)\ \Delta z$$

and

$$dq_{z_t} = dq_{z_1} + dq_{z_2} + dq_{z_3} = \Delta x \Delta y\ P_z(z) - \Delta x \Delta y\ P_z(z+\Delta z).$$

The total net charge enclosed by the volume is thus the

sum of the individual components of the charge

$$dq = dq_x + dq_y + dq_z = \left[P_x(x) - P_x(x+\Delta x) \right] \Delta y \Delta z$$

$$+ \left[P_y(y) - P_y(y+\Delta y) \right] \Delta x \Delta z + \left[P_z(z) - P_z(z+\Delta z) \right] \Delta x \Delta y =$$

$$\left(\frac{P_x(x) - P_x(x+\Delta x)}{\Delta x} + \frac{P_y(y) - P_y(y+\Delta y)}{\Delta y} + \right.$$

$$\left. \frac{P_z(z) - P_z(z+\Delta z)}{\Delta z} \right) \Delta x \Delta y \Delta z .$$

So, the charge per unit volume is thus given by

$$\frac{dq}{\Delta x \Delta y \Delta z} = \frac{P_x(x) - P_x(x+\Delta x)}{\Delta x} + \frac{P_y(y) - P_y(y+\Delta y)}{\Delta y} +$$

$$\frac{P_z(z) - P_z(z+\Delta z)}{\Delta z}$$

and in the limit we get a polarization volume charge density given by

$$\rho_p = \lim \frac{dq}{\Delta x \Delta y \Delta z} =$$

$$\lim \left[\frac{P_x(x) - P_x(x+\Delta x)}{\Delta x} + \frac{P_y(y) - P_y(y+\Delta y)}{\Delta y} + \frac{P_z(z) - P_z(z+\Delta z)}{\Delta z} \right]$$

$$= - \left[\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right] .$$

But, this is just the negative of the divergence of the polarization $\rho_p = -\nabla \cdot P$. Recalling Gauss's law, we have $\nabla \cdot E = \frac{1}{\epsilon_0} \rho_c(r)$ where $\rho_c(r)$ is the total charge density, we can now break up the total charge density as a sum of two such charge densities, the polarization charge density ρ_p and everything that is left over. But, what is left over? Since we have taken care of the charge that is attributed to the medium, the left over charge is called the free charge and consists of electrons on a conductor or ions

embedded in the materials. Actually the term free charge is used for lack of anything better. So, $\rho_c(r) = \rho_f(r) + \rho_p(r)$ and Gauss's law reads $\epsilon_o(\nabla \cdot E) = \rho_c(r) = \rho_f + \rho_p = \rho_f - \nabla \cdot P$ or $\nabla \cdot (\epsilon_o E + P) = \rho_f$. We define the vector quantity $D = \epsilon_o E + P$ as the electric displacement vector and note that when considering dielectric mediums, it allows us to consider only the free charge, which is what we control. Thus, Gauss's law becomes

1') $\nabla \cdot D = \rho_f$

with the constitutive relation $D = \epsilon_o E + P$.

Recall that Maxwell's equations in free space give differential equations relating the magnetic field B and the electric field E. We then showed that when a dielectric material is placed within an electric field there is an induced alignment of the molecules charged constituents yielding a new field, the displacement field, D. So, one should not be surprised to learn that a similar occurrence happens when materials are placed in a magnetic field. In fact, since magnetic fields are produced by electrons in motion, one expects such an occurrence.

Since we know from Maxwell's equations that the magnetic field is divergenceless, we then can write the magnetic field as the curl of a vector potential. So, $\nabla \cdot B = 0 = \nabla \cdot (\nabla \times A)$ since the divergence of the curl of any vector function is zero (or at least any function that

is twice continuously differentiable). Thus, we have that B = ∇ x A. The usefulness of this relation can be realized by using Amperes law in integral form. If we consider the magnetic flux through a loop of wire we have that

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = \int_{\mathcal{S}} \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}.$$

So, we get

$$\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}.$$

If we take a step back and consider a particular problem in magnetostatics, that is where the fields do not vary in time, we see that Amperes law reads

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

Up to this point we have put no restrictions upon the vector potential, all that was required was that at the very least, the vector potential is twice continuously differentiable since it comes from a physical quantity. Without loss of generality we can stipulate that ∇ · A = 0 since, suppose A were not divergence free, then we can add to it the gradient of a scalar so that A = A' + ∇f. This gives us ∇ · A = ∇ · (A'+∇f) = ∇ · A' + ∇ · ∇f, and we can then take the quantity ∇ · ∇f = − ∇ · A', that is solve the system of differential equations given by ∇ · ∇λ = Δλ = − ∇ · A' where the operator Δ is the Laplacian operator. This is just Poisson's equation and with appropriate boundary conditions can be solved explicitly for λ. Thus, there is no loss of generality in the assumption that the

magnetic vector potential is divergenceless.

With this assumption, we have that

$$\nabla \times B = \nabla \times (\nabla \times A) = - \Delta A = \mu_o J$$

or that $\Delta A = - \mu_o J$ which is again Poisson's equation and has general solution

$$A(r) = \frac{\mu_o}{4\pi} \int_V \frac{J}{r} \, dV' \quad , \text{ where as usual } r = |r - r'| .$$

Then

$$\nabla \times A = \nabla \times \frac{\mu_o}{4\pi} \int_V \frac{J(r')}{r} dV' = \frac{\mu_o}{4\pi} \int_V \nabla \times \frac{J(r')}{r} \, dV'$$

$$= \frac{\mu_o}{4\pi} \int_V \left[\frac{1}{r}(\nabla \times J) - J \times \nabla(\frac{1}{r}) \right] dV'$$

$$= \frac{\mu_o}{4\pi} \int_V -J \times \nabla(\frac{1}{r}) dV'$$

$$= - \frac{\mu_o}{4\pi} \int_V J \times \left(\frac{-r}{r^3} \right) dV' = \frac{\mu_o}{4\pi} \int_V \frac{J \times r}{r^3} \, dV' = \frac{\mu_o}{4\pi} \int_V \frac{J \times \hat{r}}{r^2} \, dV'$$

($\nabla \times J = 0$ since J doesn't depend on x,y, and z, only upon on x',y', and z'). Solving for the curl we get that

$$A = \frac{\mu_o}{4\pi} \int_V \left[\left(\int_V \frac{J \times \hat{r}}{r^2} \, dV \right) \times \frac{\hat{r}}{r^2} \right] dV = \frac{\mu_o}{4\pi} \int_V B \times \frac{\hat{r}}{r^2} \, dV$$

since the inner integral is just the Biot and Savart law.

Suppose we want to approximate the vector potential at points far from the source, i.e. R >> r (or r>>0). We can use the fact that

$$\frac{1}{r} = \frac{1}{R} \sum_{n=0}^{\infty} \left[\frac{r}{R} \right]^n P_n(\cos(\theta))$$

where R is the position vector from the origin to the point in question, r is the position vector from the origin to the source and $P_n(x)$ are the Legendre polynomials of degree n. So, for a current loop,

$$A = \frac{\mu_o}{4\pi} i \oint_C \frac{1}{r} \, dl = \frac{\mu_o}{4\pi} i \left[\sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \oint_C r^n P_n(\cos(\theta)) dl \right]$$

$$= \frac{\mu_o}{4\pi} i \left[\frac{1}{R} \oint_C dl + \frac{1}{R^2} \oint_C r \cos(\theta) \, dl + \sum_{n=2}^{\infty} \frac{1}{R^{n+1}} \oint_C r^n P_n(\cos(\theta)) \, dl \right].$$

We know that any integral around a closed loop is zero and that in the limit $R \gg r$ then $\frac{r^n}{R^{n+1}} \to 0$, so that the dominant term in the brackets is the second one, called the dipole term. If we consider the dipole term only, we have

$$A_d(r) = \frac{\mu_o}{4\pi} \frac{i}{R^2} \oint_C r \cos(\theta) \, dl = \frac{\mu_o}{4\pi} \frac{i}{R^2} \oint_C \hat{R} \cdot r \, dl.$$

With the aid of some vector identities and some algebraic manipulations, it can be shown that

$$\oint_C \hat{R} \cdot r \, dl = -\frac{1}{2} \left[\hat{R} \times \oint_C r \times dl \right]$$

then

$$A_d(r) = \frac{\mu_o}{4\pi} \frac{i}{R^2} \left[-\frac{1}{2} \hat{R} \times \oint_C r \times dl \right] = \frac{\mu_o}{4\pi} I \left[\frac{1}{2} \oint_C r \times dl \times \frac{\hat{R}}{R^2} \right].$$

If we define a new quantity called the magnetic dipole moment by

$$m = \frac{i}{2} \oint_C r \times dl$$

we then have

$$A_d(r) = \frac{\mu_o}{4\pi} \frac{m \times \hat{R}}{R^2}$$

as the dipole term in the vector potential. The physical interpretation of this is in direct analogy with the electric dipole. It is instrumental to note here that if we are considering a flat current loop, then

$$\frac{1}{2} \oint_C r \times dl = a,$$

where a is the area of the current loop, and we get that $m = ia$.

We now define the magnetization, **M**, as the magnetic dipole densityy of a magnetic dipole distribution in some volume. If we consider this volume made up of tiny current loops, then **M** = Ni da. Looking at one side of a magnetic cube with N such current loops along each side, then the only current loops that contribute are those that have a dipole on either side of the edges. If the current loops make an angle β with the coordinate axis then the contribution from these current loops on the side x to the magnetization is

$$- \text{iN ds } \Delta y \cos(\beta) \mid_x = - M_y(x) \, \Delta y$$

and the contribution on the side x+Δx is

$$\text{iN ds } \Delta y \cos(\beta) \mid_{x+\Delta x} = M_y(x+\Delta x) \, \Delta y.$$

The change in sign comes from the fact that the loops pass through the plane in different directions.

Similarly, for the edges y and y+Δy we have

$$\text{iN ds } \Delta x \cos(\beta) \mid_y = M_x(y) \, \Delta x$$

$$- \text{iN ds } \Delta x \cos(\beta) \mid_{y+\Delta y} = M_x(y+\Delta y) \, \Delta x.$$

So, for this planar loop we get a net current density, call it J_z, as the sum of the current per unit volumes, i.e.

$$J_z = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{i}{\Delta x \Delta y} =$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{M_y(x+\Delta x) - M_y(x)}{\Delta x} - \frac{M_x(y+\Delta y) - M_x(y)}{\Delta y} \right]$$

$$= \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \ .$$

But, this is just the z-component of the curl of **M**. A similar argument for each of the coordinate faces of the cube yields the x and y components of the curl to get a magnetization current density given by $J_m = \nabla \times M$. Recalling Amperes law for magnetostatics, we have $\nabla \times B = \mu_o J$ where J is the total current density. We can now break this up into a magnetization current density and a "free" current density to get

$$\frac{1}{\mu_o}\nabla \times B = J = J_f + J_m = J_f + \nabla \times M.$$

Combining the curls we have

$$\nabla \times (\frac{1}{\mu_o}B - M) = J_f$$

and we define the quantity $H = \frac{1}{\mu_o}B - M$ and call **H** the magnetic field intensity. This reduces Amperes law for magnetostatics in matter to

$$\nabla \times H = J_f.$$

Let us consider the "free" current density J_f. If we are no longer in the static case, then the current moves as a function of time, thus the charge distribution also moves as a function of time and as we have already seen, this causes a change in the polarization of the material. In this case, at least some of this free current goes into changing the polarization, so why not break J_f up into two more parts, call them J_f and J_p, so that $J = J_f + J_m + J_p$.

Consider the polarization given by the sum of the dipole moments per unit volume, i.e.

$$P = \sum_{j=1}^{N} \Delta P_j = \sum_{j=1}^{N} \frac{1}{\Delta V} (q_j \Delta r_j).$$

But, these charges are "bound" to the atoms so, we see an effective current due to the linear motion of the charges when the electric polarization changes. Fixing the volume we get for each :

$$\frac{\Delta P_j}{\Delta t} = \frac{1}{\Delta V} \, q_j \, \frac{\Delta r_j}{\Delta t}$$

and, in the limit as $\Delta t \rightarrow 0$ we get

$$\frac{\partial P_j}{\partial t} = \frac{1}{\Delta V} \, q_j \, <v_j>$$

where $<v_j>$ is the average velocity of the charge q_j. Summing over j and allowing the volume elements to get arbitrarily small we get that

$$\frac{\partial P}{\partial t} = \rho_b \, <v(t)> = J_p,$$

since $\rho_b <v>$ is just the current density due to the movement of the charges, in this case the polarization current density. To see that this is consistent with the equation of continuity we take the divergence of the polarization current density to get

$$\nabla \cdot J_p = \nabla \cdot \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot P) = -\frac{\partial}{\partial t}(\rho_b)$$

which is just what we expect to get.

So, we see that the current density becomes the sum of three separate terms given by

$$J = J_f + \nabla \times M + \frac{\partial P}{\partial t}$$

which accounts for all of the charges. So Amperes law becomes

$$\nabla \times B = \mu_o J + \epsilon_o \mu_o \frac{\partial E}{\partial t} \quad \text{or}$$

$$\frac{1}{\mu_o} \nabla \times B = J_f + \nabla \times M + \frac{\partial P}{\partial t} + \epsilon_o \frac{\partial E}{\partial t}$$

$$\Rightarrow \nabla \times (\tfrac{1}{\mu_o}B - M) = J_f + \frac{\partial(\epsilon_o E + P)}{\partial t}$$

$$\Rightarrow \nabla \times H = J_f + \frac{\partial D}{\partial t}$$

and Maxwell's equations in media become

$$\nabla \cdot D = \rho_f \qquad\qquad\text{(Gauss's law)}$$

$$\nabla \times E = - \frac{\partial B}{\partial t} \qquad\qquad\text{(Faraday's law)}$$

$$\nabla \cdot B = 0$$

$$\nabla \times H = J_f + \frac{\partial D}{\partial t} \quad\text{(Ampere's law with Maxwell's correction)}$$

with the constitutive equations relating the fields

$$H = \tfrac{1}{\mu_o}B - M,$$

$$D = \epsilon_o E + P,$$

and the Lorentz force law given by

$$F = q(E + v \times B).$$

To transform Maxwell's equations into a moving frame of reference, i.e. for moving media, we first make the distinction between the charge velocity, v, and the fluid velocity, u. Then, notice that $\nabla \cdot B = 0$ will remain unchanged because it can not be effected by the motion of nonmagnetizeable media.

Consider Faraday's law, let Φ_m be the magnetic flux across a surface $\mathcal{S}$, then

$$\frac{d\Phi_m}{dt} = \frac{d}{dt}\int_{\mathcal{S}} B \cdot dS = \int_{\mathcal{S}}\frac{dB}{dt} \cdot dS = \int_{\mathcal{S}}\left[\frac{\partial B}{\partial t} + (v \cdot \nabla)B\right] \cdot dS$$

$$= \int_{\mathcal{S}}\frac{\partial B}{\partial t} \cdot dS + \int_{\mathcal{S}}(v \cdot \nabla)B \cdot dS = \int_{\mathcal{S}}\frac{\partial B}{\partial t} \cdot dS +$$

$$\int_{\mathcal{S}}\left[\nabla \times (B \times v) + (B \cdot \nabla)v - B(\nabla \cdot v) + v(\nabla \cdot B)\right] \cdot dS$$

$$= \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_{\mathcal{S}} \nabla \times (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{S} + \int_{\mathcal{S}} (\mathbf{B} \cdot \nabla)\mathbf{v} \cdot d\mathbf{S}$$

where we have used a vector identity and the fact that $\nabla \cdot \mathbf{B} = 0$ always, and that $\nabla \cdot \mathbf{v} = 0$ since velocities are not being created within any control volume. So,

$$\frac{d\Phi_m}{dt} = \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_{\mathcal{S}} \nabla \times (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{S} + \int_{\mathcal{S}} (\mathbf{B} \cdot \nabla)\mathbf{v} \cdot d\mathbf{S}$$

$$= \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_{\mathcal{S}} (\mathbf{B} \cdot \nabla)\mathbf{v} \cdot d\mathbf{S} + \oint_{\mathcal{C}} \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l},$$

by the use of Stokes theorem. Applying Faraday's law of induction we get

$$\oint_{\mathcal{C}} \mathbf{E'} \cdot d\mathbf{l} = -\frac{d\Phi_m}{dt} = -\int_{\mathcal{S}} \left[\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B} \cdot \nabla)\mathbf{v}\right] \cdot d\mathbf{S} + \oint_{\mathcal{C}} \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l}$$

where the term $\frac{\partial \mathbf{B}}{\partial t}$ represents the flux through the surface caused by the time variation of the field, $\nabla \times (\mathbf{B} \times \mathbf{v})$ represents the flux across the boundary of the surface and $(\mathbf{B} \cdot \nabla)\mathbf{v}$ represents the passage of the surface through an inhomogeneous field which generates flux lines. So,

$$\oint_{\mathcal{C}} (\mathbf{E'} - \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\int_{\mathcal{S}} \left[\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B} \cdot \nabla)\mathbf{v}\right] \cdot d\mathbf{S}.$$

The use of Stokes theorem once again yields

$$\int_{\mathcal{S}} \nabla \times (\mathbf{E'} - \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{S} = -\int_{\mathcal{S}} \left[\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B} \cdot \nabla)\mathbf{v}\right] \cdot d\mathbf{S}$$

or, since the surface element is the same we have

$$\nabla \times (\mathbf{E'} - \mathbf{v} \times \mathbf{B}) = -\left[\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B} \cdot \nabla)\mathbf{v}\right].$$

The quantity on the right of the equality is the field that is measured by an observer at rest since, if the observer carried a charge q through a field B with a velocity v, he would feel a force given by $q(\mathbf{v} \times \mathbf{B})$. So, in the motionless frame of reference we have that $\mathbf{E} = \mathbf{E'}$

− v x B. Now, since we are considering the instantaneous movements of the vector field, the velocity field is a fixed (constant) vector field and differentiation with respect to v yields zero i.e.

$$(B \cdot \nabla)v = B_x\frac{\partial}{\partial x}v + B_y\frac{\partial}{\partial y}v + B_z\frac{\partial}{\partial z}v = 0$$

and we get

$$\nabla \times E = - \frac{\partial B}{\partial t}$$

which shows that Faraday's law is independent of the motion of the medium inside the fields. We note that this is valid only if the velocity of the medium is much less than that of the speed of light, which we know to be true in our setting.

Now, since we are dealing with non-relativistic motion, we see that Gauss' law in a medium (i.e. $\nabla \cdot D = \rho_f$) also can not be effected by the motion of nonmagnetizeable matter. It is instrumental to note that $\nabla \cdot E = \frac{1}{\epsilon_o}\rho_e$ is effected by the motion of matter. Since we know that $E = E' - v \times B$, then for an observer moving with the matter $E' = E + v \times B$ and we get an effective current density given by Ohms law $J = \sigma E' = \sigma(E + v \times B)$ so $P = \chi_e E' = \chi_e(E + v \times B)$ and we again define the displacement vector $D = \epsilon_o E + P = \epsilon E + (\epsilon-1)v \times B$ and we see once again that $\nabla \cdot D = \epsilon_o\nabla \cdot E + \nabla \cdot P = \rho_e - \rho_b = \rho_f$.

Consider now Amperes law which is

$$\nabla \times B = \mu_o J + \mu_o\epsilon_o\frac{\partial E}{\partial t} = \mu_o(J_f + J_m + J_p) + \mu_o\epsilon_o\frac{\partial E}{\partial t}.$$

First, we break the polarization current J_p up into two

parts, the first part being the current relative to the moving matter as seen by an observer moving with the material and the second part being a current from all the charges in the moving matter. In the first case this is just $J_{p_1} = \rho_c u = (\rho_f - \nabla \cdot P)u$, where here we make the distinction between the velocity of the media, u, and the velocity of the charges, v. In the second case we must consider the change in coordinate systems. By use of the convective derivative we have that

$$\frac{\partial P^*}{\partial t} = \frac{\partial P}{\partial t} + (v \cdot \nabla)P$$

where the star indicates the polarization as seen from the rest frame of reference. By use of the vector identities already exploited we get

$$\frac{\partial P^*}{\partial t} = \frac{\partial P}{\partial t} + (v \cdot \nabla)P =$$

$$\frac{\partial P}{\partial t} + \nabla \times (P \times u) + (P \cdot \nabla)u - P(\nabla \cdot u) + u(\nabla \cdot P) =$$

$$\frac{\partial P}{\partial t} + \nabla \times (P \times u) + u(\nabla \cdot P)$$

since $\nabla \cdot u = 0$ and $(P \cdot \nabla)u = 0$ for the same reasons as before. So, the polarization current density is

$$J_p = J_{p_1} + J_{p_2} = \rho_f u - (\nabla \cdot P)u + \frac{\partial P}{\partial t} + u(\nabla \cdot P) + \nabla \times (P \times u)$$

$$= \rho_f u + \frac{\partial P}{\partial t} + \nabla \times (P \times u).$$

Since the media in question is nonmagnetizeable, then $\nabla \times M = 0$ which means that the magnetic current density $J_m = 0$ and we are left with

$$\nabla \times B = \mu_o \epsilon_o \frac{\partial E}{\partial t} + \mu_o \left[J_f + \rho_f u + \frac{\partial P}{\partial t} + \nabla \times (P \times u) \right]$$

$$= \mu_o \left[J_f + \rho_f u + \frac{\partial D}{\partial t} + \nabla \times (P \times u) \right],$$

where as usual $J_f = \rho_f v$ and v is the velocity of the charges.

So, we now have all of Maxwell's equations in the proper form for slowly moving nonmagnetizeable media. They are

1) $\nabla \cdot D = \rho_f$ (Gauss' law in dielectric mediums)

2) $\nabla \cdot B = 0$

3) $\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Faraday's law)

4) $\nabla \times B =$

$\mu_o \left[J_f + \rho_f u + \dfrac{\partial D}{\partial t} + \nabla \times (P \times u) \right]$ (Ampere's law)

together with the constitutive relations

5) $D = \epsilon_o E + P$

6) $H = \dfrac{1}{\mu_o} B - M$

and the Lorentz force law

7) $F = q(E + v \times B)$

these equations govern the electromagnetic effects of the problem.

C. Equations of State

We are left with the derivation of the equations of state. We need two equations of state, the first which relates the pressure, p, with the gas density ρ, and the temperature, Θ, and the second is an equation which relates the amount of ionization of the control gas to the amount of control gas and temperature.

To start with, as usual we take the ideal case and then expand upon that. There are three laws that govern the effects of pressure, temperature and volume in an ideal gas. The first law is Boyle's law, Robert Boyle found that at a constant temperature the volume of a fixed quantity of gas decreases as the pressure is increased. He formulated his experimental data into Boyle's law which states that "at a constant temperature, the volume occupied by a fixed quantity of gas is inversely proportional to the applied pressure". Or mathematically we have that $V \propto \frac{1}{p}$. Any gas that follows Boyle's law is called an ideal gas.

The second law is due to Jacques Charles who found that if a gas is heated in such a way that the pressure remains constant then the gas will expand. Charles found that the relationship between the volume of a gas and the absolute temperature of the same gas is a linear one and he summarized it by Charles's law which states that "at constant pressure, the volume of a given quantity of a gas

varies directly with its absolute temperature". That is V $\propto \Theta$.

Third is Dalton's law of partial pressures. John Dalton found that the sum of the pressures exerted by each gas in a mixture is just the total pressure of the gas. Dalton's law of partial pressure is $p_t = \sum_{i=1}^{N} p_i$, where p_t is the total pressure and p_i is the pressure of the i-th constituent of the gas.

These three laws, together with Avogadro's principle proposed by Amadeo Avogadro which states that "under conditions of constant temperature and pressure, equal volumes of gas contain equal numbers of molecules", that is V $\propto$ n where n is the number of moles of gas, yields the ideal gas law in the well known form of pV = nRT, where R is the constant of proportionality called the universal gas constant. To cast this in a more useful manner we divide both sides by the volume and then multiply both top and bottom by the molecular weight of the gas in question to get $p = \frac{MnR\Theta}{MV} = \frac{MnR\Theta}{V\,M} = \rho\frac{R}{M}\Theta = c_g\rho\Theta$ where the constant c_g $= \frac{R}{M}$. So, for this case of an ideal gas, we have an equation of state given by $p(\rho,\Theta) = c_g\rho\Theta$. To get away from the ideal case we notice that because of the molecular attractions between the constituent parts of the gas, a correction to the volume, V, and the pressure, p, must be made. According to J. D. van der Waals, the real volume measured is given by $V_r = V_i + nb$, where b is the

correction due to the excluded volume from the forces of attraction. Thus to replace this in the ideal gas law we must make the substitution $V_i = V_r - nb$. Similarly the ideal pressure is higher than the actual pressure due to the intermolecular attractions by an amount $p_i = p_r + \frac{an^2}{V_r^2}$, where a is a constant which depends on the strength of the intermolecular forces. Substitution of the actual pressure and volume into the ideal gas law gives $\left[p + \frac{an^2}{V^2} \right](V - nb) = nR\Theta$, which is known as van der Waals equation of state. If we multiply this out, grouping the appropriate terms and then divide by the volume term we get $p = \frac{nR\Theta}{V - nb} - \frac{an^2}{V^2}$. The denominator of the first term is the effective volume, where here V is the ideal gas volume, so we can recast this into a more useful form by using the substitution $V_r = V_i + nb$, and then multiplying and dividing by the molecular weight to get $p = c_s \rho \Theta - \frac{an^2}{(V + nb)^2}$. At very high temperatures, a real gas behaves in the same manner as does an ideal gas, thus in effect we can ignore the correction term in areas of very high temperatures.

We know move to Saha's equation relating the amount of ionized gas and the amount of host gas. Because this is a classical treatment of EMHD and the necessity of the knowledge of statistical thermodynamics in the complete derivation of Saha's equation, much will be left out. We first assume that we are in a state of local thermodynamic

equilibrium and proceed to calculate the chemical composition of a plasma as a function of temperature. Conservation of mass gives us that for a species A, if this species is ionized we have that $A = A^+ + e^-$, where e^- denotes a free electron. The ionization energy for species A is given by $\Lambda_i = k\Xi_i$ where Ξ_i is the characteristic ionization temperature for species A and k is the Boltzmann constant.

Assuming that the ionized gas resulted from $N_o = n_o N$ atoms of a neutral gas, and denoting the degree of ionization by ϕ, then the constituents of the ionized gas is given by $N_a = (1 - \phi)N_o$, $N_i = \phi N_o$, and $N_e = \phi N_o$. Thus, since the typical ionization temperature is high, we use the ideal gas law to see that $pV = (N_a + N_i + N_e)k\Theta = (1 + \phi)N_o k\Theta$. The fraction of ionized gas can be shown to be

$$\frac{\phi^2}{1 - \phi(1 + \phi)k\Theta}\frac{pV}{} = \frac{(2\pi m_e k\Theta)^{3/2}V}{h^3}\left[\frac{2g_i}{g_a}\right]\exp(-\Xi_i/T) ,$$

or, after some algebraic manipulations

$$\frac{\phi^2}{1 - \phi^2} = \left[\frac{2g_i}{g_a}\right]\frac{(2\pi m_e k\Theta)^{3/2}}{h^3}\exp(-\Lambda_i/k\Theta)$$

where h is Planck's constant, g_i is the ionization ground state degeneracy of species A, g_a is the ground state degeneracy of species A, and m_e is the mass of the electron.

With this last step completed, we are finally ready to write down the complete equations of motion of magneto-hydrodynamics. First we start with the equations of fluid motion

1) $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$ (the equation of continuity)

2) $\rho \frac{du}{dt} = -\nabla p + \nabla(\lambda(\nabla \cdot u)) +$

       $\nabla \cdot (2\mu \mathfrak{I}) - \rho g + \rho \frac{e^-}{m_e}(E+u\times B)$ (Navier-Stokes)

3) $\rho c_p \frac{d\Theta}{dt} = \kappa \Delta \Theta + \nabla \Theta \cdot \nabla \kappa + \Upsilon$ (energy equation)

Maxwell's equations

4) $\nabla \cdot D = \rho_c$ (Gauss' law)

5) $\nabla \cdot B = 0$

6) $\nabla \times E = - \frac{\partial B}{\partial t}$ (Faraday's law)

7) $\nabla \times B = \mu_o\left[J_f + \rho_c u + \frac{\partial D}{\partial t} + \nabla \times (P \times u)\right]$ (Ampere's law)

the equations of state

8) $p = c_g \rho \Theta$ (gas law)

9) $\frac{\rho_c}{\rho} = \left[\frac{2g_i}{g_a}\right]\frac{(2\pi m_e k\Theta)^{3/2}}{h^3}\exp(-\Lambda_i/k\Theta)$ (Saha's equation)

and the constitutive relations given by

10) $3\lambda + 2\mu \geq 0$ and $\mu \geq 0$

11) $D = \epsilon_o E + P$

12) $H = \frac{1}{\mu_o}B - M$

14) $F = q(E + v \times B)$ (Lorentz force law).

This is a set of 13 coupled differential equations in 13 unknowns, the unknowns being p, the pressure, E = (E_1, E_2, E_3), the electric field, H = (H_1, H_2, H_3), the magnetic field intensity which for us is the same as the magnetic field B since the media is non magnetizable, Θ, the temperature function, ρ, the gas density, ρ_c, the charge density, and the gas velocity u = (u_1, u_2, u_3).

A PATH TO THE SOLUTION

Given the system described by equations 1 - 14, a solution will be extremely difficult to obtain in general. Some reductions can be made to the equations as they pertain to the lower powered experimental furnace. From this reduced system, fundamental insight into the fluctuations within the arc can be gained and a control scheme devised.

In order to reduce the equations to a more manageable system, we nondimensionalize them to determine the relative importance of each of the terms. The ones that turn out to be small in magnitude can be neglected and the resulting system attacked. In so doing, we use data gathered by the Idaho National Engineering Laboratory (INEL) and the Albany Research Center. With this data, characteristic parameter values can be substituted and the relative magnitudes determined.

Before we start, we make several observations. First, on the basis of thermodynamic arguments, one always has $\mu \geq 0$ and moreover $3\lambda + 2\mu \geq 0$. It is sometimes experimentally even found that $\lambda > \mu$. Nevertheless, in the absence of additional data, it is customary to adopt Maxwell's kinetic theory for monatomic gases, $3\lambda + 2\mu = 0$, $\mu \geq 0$ in this type of setting, which gives $\lambda = -\frac{2}{3}\mu$. Also, it has been experimentally found by INEL that the viscosity, μ, is a linear function of temperature in the

temperature ranges we are considering. With this in mind, the Navier-Stokes equation becomes

$$\rho\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] = -\nabla p + \frac{1}{3}\mu\nabla(\nabla \cdot u) + \mu\Delta u +$$

$$\left[-\frac{2}{3}(\nabla \cdot u) + 2\mathfrak{D}\right]\frac{d\mu}{d\Theta}\nabla\Theta - \rho g + \rho\frac{e^-}{m_e}(E + u \times B).$$

Now, we introduce a new, nondimensional coordinate system. If we let L be a characteristic length, U_o a characteristic velocity, and $\tau = L/U_o$ a characteristic time then the dimensionless variables $x = Lx'$ and $u = U_o u'$ can be used to transform the equations into a set of dimensionless ones. In addition to the length and velocity, similar substitutions for the temperature, electric and magnetic fields and charge velocity can be used to get the system given by

1) $\frac{1}{\tau}\frac{\partial\rho}{\partial t'} + \frac{U_o}{L}\nabla' \cdot \rho u' = 0$

2) $\rho\left[\frac{U_o}{\tau}\frac{\partial u'}{\partial t'} + \frac{U_o^2}{L}(u' \cdot \nabla')u'\right] = -\frac{1}{L}\nabla'p + \frac{1}{3}\mu\frac{U_o^2}{L}\nabla'(\nabla' \cdot u')$

$$+ \mu\frac{U_o^2}{L}\Delta'u' + \left[-\frac{2U_o}{3L}(\nabla' \cdot u') + 2\frac{U_o}{L}\mathfrak{D}'\right]\frac{T}{L}\frac{d\mu}{d\Theta}\nabla'\Theta'$$

$$- \rho g + \rho\frac{e^-}{m_e}(E_o E + U_o B_o u \times B).$$

3) $\rho c_p\left[\frac{T_o}{\tau}\frac{\partial\Theta'}{\partial t'} + \frac{T_o U_o}{L}(u' \cdot \nabla')\Theta'\right] = \kappa\frac{T_o}{L^2}\Delta'\Theta' + \frac{T_o}{L^2}\nabla'\Theta' \cdot \nabla'\kappa'$

$$+ 2\mu\left[\frac{U_o^2}{L^2}\mathfrak{D}' : \mathfrak{D}' - \frac{U_o^2}{3L^2}(\nabla' \cdot u')^2\right]$$

4) $\frac{D_o}{L}\nabla' \cdot D' = \rho_c$

5) $\frac{B_o}{L}\nabla' \cdot B' = 0$

6) $\frac{E_o}{L}\nabla' \times E' = \frac{B_o}{\tau}\frac{\partial B'}{\partial t'}$

7) $\frac{B_o}{L}\nabla' \times B' = \mu_o\left[V_o\rho_c v' + U_o\rho_c u' + \frac{D_o}{\tau}\frac{\partial D'}{\partial t'} + \frac{U_oP_o}{L}\nabla' \times (P' \times u')\right]$

8) $p = Tc_g\rho\Theta'$

9) $\frac{\rho_c}{\rho} = \frac{2g_i}{g_a}\frac{(2\pi m_e kT\Theta')^{3/2}}{h^3}\exp(-\Lambda_i/kT\Theta')$,

with the appropriate constitutive relations.

If we now divide out the constants on the left hand side, giving terms of order one, then the right hand side becomes

1) $\frac{\partial \rho}{\partial t'} + \nabla' \cdot \rho u' = 0$

2) $\frac{\partial u'}{\partial t'} + (u' \cdot \nabla')u' = -\nabla'\Pi + \frac{1}{3}\frac{1}{Re}\nabla'(\nabla' \cdot u') + \frac{1}{Re}\Delta'u' +$

$-\frac{2}{3}\frac{T}{\rho U_o L}\frac{d\mu}{d\Theta}\nabla'\Theta' (\nabla' \cdot u') + \frac{2T}{\rho U_o L}\frac{d\mu}{d\Theta}\nabla'\Theta' -$

$\frac{L}{U_o^2}g + \frac{L}{U_o^2}\frac{e^-}{m_e}(E_oE + U_oB_o u \times B)$.

3) $\frac{\partial \Theta'}{\partial t'} + (u' \cdot \nabla')\Theta' = \frac{\kappa}{U_o\rho c_p L}\Delta'\Theta' + \frac{1}{U_o\rho c_p L}\nabla'\Theta' \cdot \nabla'\kappa'$

$+ \left[\frac{2\mu U_o}{\rho c_p T_o L}\mathcal{D}' : \mathcal{D}' - \frac{2\mu U_o}{3\rho c_p T_o L}(\nabla' \cdot u')^2\right]$

4) $\nabla' \cdot D' = \frac{L}{D_o}\rho_c$

5) $\nabla' \cdot B' = 0$

6) $\nabla' \times E' = \frac{U_o B_o}{E_o}\frac{\partial B'}{\partial t'}$

7) $\nabla' \times B' = \mu_o\left[\frac{LV_o}{B_o}\rho_c v' + \frac{LU_o}{B_o}\rho_c u' + \frac{D_oU_o}{B_o}\frac{\partial D'}{\partial t'} + \frac{U_oP_o}{B_o}\nabla' \times (P' \times u')\right]$

8) $p = Tc_g\rho\Theta'$

9) $\dfrac{\rho_c}{\rho} = \dfrac{2g_i}{g_a} \dfrac{(2\pi m_e kT\Theta')^{3/2}}{h^3} \exp(-\Lambda_i/kT\Theta')$,

where we have defined the Reynolds number as $Re=\dfrac{\rho U_o L}{\mu}$ and

used $\pi=\dfrac{p}{\rho U_o^2}$.

Quantities for these characteristic values have been measured by INEL and the ALRC. A characteristic length is given by the electrode gap which the arc must jump. Typically this gap length is about L=0.005 m. Characteristic velocities of the gas mixture have been calculated as the rate at which the arc "walks" out to the side and extinguishes itself. This, of course, is not the actual velocity that the gas molecules are travelling at, but is a good indication of the velocity that the bulk of the gas obtains. These walking arcs have been measured at speeds of around U_o=900 m/s. We have a characteristic temperature given by T_o=15,000 °K, the magnitude of the electric field is E_o=10000 V/m and the magnitude of the magnetic field is $B_o=E_o/U_o$=11.1 V-s/m². The last three quantities represent maximum obtainable values and have been obtained by measurements taken at the ALRC.

The quantities ϵ_o, μ_o, e^-, m_e and g=|g| are universal constants. For the quantities given by the density ρ, specific heat c_p, thermal conductivity κ, and the viscosity μ, a much more subjective means is used to find the characteristic quantities. For the specific heat and thermal conductivity, we use the average of these quantities over the temperature region in question. The

data from INEL shows this to be, for the specific heat, c_p=7500 J/Kg-°K and the thermal conductivity κ=13 J/°K-m-s. This brings us to the density ρ, and the viscosity μ. For these quantities we take the maximum values over the temperature range. Again INEL has supplied the information needed to obtain a value. We use as the gas density ρ=0.075 Kg/m^3 and the viscosity μ=1.5x10^{-3} Kg/m-s. Finally, the term $\frac{d\mu}{d\theta}$ is experimentally found to be on the order of 2x10^{7}Kg/m-s-°K.

With these values, we are able to identify the terms which become important in our analysis of the system. If we consider Saha's equation first, then for Helium it is easy to show that the extent of ionization is about 4%, and remains around this value over the temperature ranges we are dealing with. With this fact, we can replace Saha's equation by $\rho_c = 0.04 \frac{e^-}{m_e}\rho$.

Next, consider the Navier-Stokes equation, we see that 1/Re = 4.44x10^{-3} and 1/(3Re) = 1.48x10^{-3}. The gravity term is of magnitude L/U$_o$|g| = 6.06x10^{-8}, the terms multiplying the electromagnetic forces are

$$\frac{L\ E_o\ e^-}{U_o{}^2 m_e} = 1.084 \times 10^7 \quad \text{and} \quad \frac{L\ U_o\ B_o}{U_o{}^2 m_e} = 1.083 \times 10^7,$$

and the terms involving the temperature gradient have magnitude

$$\frac{2T}{3U_o\rho L}\frac{d\mu}{d\theta} = 5.93 \times 10^{11} \quad \text{and} \quad \frac{2T}{U_o\rho L}\frac{d\mu}{d\theta} = 1.78 \times 10^{12}.$$

From these values, we see that the gravitational term can be neglected but the electromagnetic and deformation terms

cannot. This is unfortunate in that this means that the fluid equations and the electromagnetic equations can't be decoupled. Because the Reynolds number terms are approaching order one, it is unclear whether they can be neglected. As a first attempt, one might neglect them, but it is recommended that they remain in the problem when the true solution is solved for.

If we take another look at the Newton-Fourier law, the INEL data shows that the function, κ, is a linear function of temperature. The energy equation then reduces to

$$\frac{\partial \Theta'}{\partial t'} + (u' \cdot \nabla')\Theta' = \frac{\kappa}{U_o \rho c_p L}\Delta'\Theta' + \frac{k_1}{U_o \rho c_p L}\nabla'\Theta' \cdot \nabla'\Theta'$$

$$+ \left[\frac{2\mu U_o}{\rho c_p T_o L}\mathfrak{D}' : \mathfrak{D}' - \frac{2\mu U_o}{3\rho c_p T_o L}(\nabla' \cdot u')^2\right].$$

With the data given, we see that the term $k_1 = 8.0 \times 10^{-4}$ and hence the constants are of the order

$$\frac{\kappa}{\rho c_p \ U_o \ L} = 5.14 \times 10^{-3}, \quad \frac{k_1 \ T_o}{\rho \ c_p \ U_o \ L} = 4.74 \times 10^{-3},$$

$$\frac{2 \ U_o \ \mu}{\rho \ c_p \ T_o \ L} = 6.4 \times 10^{-5}, \quad \text{and} \quad \frac{1}{3} \frac{2 \ U_o \ \mu}{\rho \ c_p \ T_o \ L} = 2.13 \times 10^{-5}.$$

Because of the relative magnitudes of these last four values, several approaches can be taken. The recommended approach is to consider the last two terms to be negligible. Another approach is to consider all terms important and solve the full energy equation.

We do a similar analysis of the Maxwell's equations.

We know that for Helium, the polarization and hence, the displacement current is proportional to the electric field. Data from the Handbook of Chemistry and Physics, 59th ed., Cleveland CRC Press, Inc. shows that the dielectric constant for Helium is 1.00 so that $D = \epsilon_0 E$. This gives the constant of proportionality between P and E to be very near zero, allowing us to neglect the polarization terms. It must be stressed that this is true because we are using Helium as our test gas. Because the charge velocity is so much higher than the gas velocity we are able to combine these terms into an effective free current. This is because the gas velocity will not affect the charge carriers enough to cause any significant difference. Thus, Maxwell's equations are reduced to the equations in free space in this instance and the analysis reveals that they must be left unchanged from this point on. It does become apparent that the magnetic field terms play less of a role than the electric field terms, but all figure prominently in the analysis.

With this in mind, several reduced systems can be studied. The first system is given by 1) the equation of continuity, 2) the Navier-Stokes equation, neglecting all terms of order less than one, 3) the energy equation neglecting the dissipative terms, and 4) the Maxwell's equations in free space:

1) $\frac{\partial \rho}{\partial t'} + \nabla' \cdot \rho u' = 0$

2) $\frac{\partial u'}{\partial t'} + (u' \cdot \nabla')u' = -\nabla'\Pi + -\frac{2}{3}\frac{T}{\rho U_o L}\frac{d\mu}{d\Theta}\nabla'\Theta' \; (\nabla' \cdot u')$

$+ \frac{2T}{\rho U_o L}\frac{d\mu}{d\Theta}\nabla'\Theta' + \frac{L}{U_o^2}\frac{e^-}{m_e}(E_o E + U_o B_o u \times B).$

3) $\frac{\partial\Theta'}{\partial t'} + (u' \cdot \nabla')\Theta' = \frac{\kappa}{U_o\rho c_p L}\Delta'\Theta' + \frac{k_1}{U_o\rho c_p L}\nabla'\Theta' \cdot \nabla'\Theta'$

4) $\nabla' \cdot E' = \frac{e^- L}{\epsilon_o E_o m_e}\rho$

5) $\nabla' \cdot B' = 0$

6) $\nabla' \times E' = \frac{U_o B_o}{E_o}\frac{\partial B'}{\partial t'}$

7) $\nabla' \times B' = \mu_o\frac{LV_o}{B_o}J' + \mu_o\epsilon_o\frac{E_o U_o}{B_o}\frac{\partial E'}{\partial t'}$

8) $\Pi = \frac{Tc_g}{U_o^2}\Theta'.$

This system represents the simplest case that we can consider and yet maintain some assemblence of reality. It will be easier to numerically solve, and may be a good test of the method used.

A more realistic system is given by

1) $\frac{\partial\rho}{\partial t'} + \nabla' \cdot \rho u' = 0$

2) $\frac{\partial u'}{\partial t'} + (u' \cdot \nabla')u' = -\nabla'\Pi + \frac{1}{3}\frac{1}{Re}\nabla'(\nabla' \cdot u') + \frac{1}{Re}\Delta'u' +$

$-\frac{2}{3}\frac{T}{\rho U_o L}\frac{d\mu}{d\Theta}\nabla'\Theta' \; (\nabla' \cdot u') + \frac{2T}{\rho U_o L}\frac{d\mu}{d\Theta}\nabla'\Theta' -$

$+ \frac{L}{U_o^2}\frac{e^-}{m_e}(E_o E + U_o B_o u \times B).$

3) $\frac{\partial\Theta'}{\partial t'} + (u' \cdot \nabla')\Theta' = \frac{\kappa}{U_o\rho c_p L}\Delta'\Theta' + \frac{k_1}{U_o\rho c_p L}\nabla'\Theta' \cdot \nabla'\Theta'$

4) $\nabla' \cdot E' = \frac{e^- L}{\epsilon_o m_e E_o}\rho$

5) $\nabla' \cdot B' = 0$

6) $\nabla' \times E' = \frac{U_o B_o}{E_o}\frac{\partial B'}{\partial t'}$

7) $\nabla' \times B' = \mu_o\frac{LV_o}{B_o}J' + \mu_o\epsilon_o\frac{E_o U_o}{B_o}\frac{\partial E'}{\partial t'}$

8) $\Pi = \frac{Tc_g}{U_o^2}\Theta'$

with the appropriate constitutive equations. The

important fact to remember in both these cases is that these equations cannot be decoupled and hence must be solved simultaneously.

With this in mind, the path can be generated as the solution to a generalized variational problem. If we assume that the conduction column takes a path that minimizes the time of conduction across the arc gap, we can then solve the generalized variational problem

$$\frac{ds}{dt} = u(x,t)$$

where $s(x(t),t)$ is the arclength in the vector calculus sense. The solution to this gives a differential equation for which boundary value problem techniques can be used, in an iterative fashion, to successively approximate the path of conduction as this path fluctuates. That is, we assume an initial conduction column and then successively solve for the new column as it undergoes the effects of the external forces given by the coupled, nonlinear differential equations. Numerical routines for this type of process are abundant and in most cases are readily available in canned type program packages.

Once the evolution of the conducting column can is solved for, it can be placed into the telegraphers equation. That is, once $i(x,t)$ is known across the arc, then the equation

$$\Delta i(x,t) = S_c R\ i(x,t) + (S_c L + RC)\ \frac{\partial i(x,t)}{\partial t} + LC \frac{\partial^2 i(x,t)}{\partial t^2}$$

(where S_c is the shunt resistance, R is the circuit resistance, L is the circuit inductance and C is the circuit capacitance) will give the complete solution of the current as it travels through the circuit.

Knowing the solution of the circuit equations will allow the furnace controller to adjust the furnace environment parameters when the precursors to electrical fluctuations are observed. In this manner, dynamic control is possible, enabling a much smoother, more efficient melt. It must be noted that some of these equations of motion are highly nonlinear and in particular the Navier-Stokes equation has been shown to be chaotic. Because of this fact, other techniques for dynamic control are being investigated at the ALRC and some progress is being made towards this end.

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CONCLUSION

The future of competitive steel making in the United States is at an impasse today. Without more efficient use of the available resources, the highly technological overseas countries will corner the steel market and force the U. S. steels out as an export commodity, as well as under sell them internally. In order to at least delay this scenario, electric arc furnaces are being studied extensively throughout the industry and at the U. S. Bureau of Mines, Albany Research Center.

Research into this physical problem at the ALRC is directed towards a better understanding of arc behavior, with dynamic control of electric arc furnaces the main goal. The ability to take large amounts of data at a very rapid pace and to correlate this data with high speed photography has allowed the researchers to view the discrete fluctuations in the arc and identify electrical signatures corresponding to them. In this manner arc behavior has lost some of its mystery. Researchers are slowly getting away from the idea that randomness is inherent in the arcs and any type of predictability is impossible.

A mathematical model will give researchers a means at obtaining information which is not available, or hidden in the discrete events observed in the wave forms and films. The system of coupled nonlinear differential equations

presented constitute the classical model for electro-magneto-hydrodynamics, where classical magneto-hydrodynamics can be obtained by neglecting electric field effects. The latter case applies to the experimental furnace at the ALRC where the voltages and currents are sufficiently low to allow for the neglect of these effects.

After appropriate reduction of the equations, an iterative numerical scheme can then be utilized to solve the system for the current path. Once knowledge of the path is obtained, it can then be used in the circuit equations to solve for the current as it evolves in the circuit. With the knowledge of the evolution of the current, a control scheme can be devised and implemented by monitoring of the current and voltages and then continuously changing operating parameters to force the wave forms to behave in a desired manner.

This type of control allows the system to operate in a more efficient manner by damping out or omitting electrical fluctuations, causing a decreased use of electrical energy, stopping power grid fluctuations, reducing electrical and acoustical noise and shortening tap times, all very appealing monetary benefits.
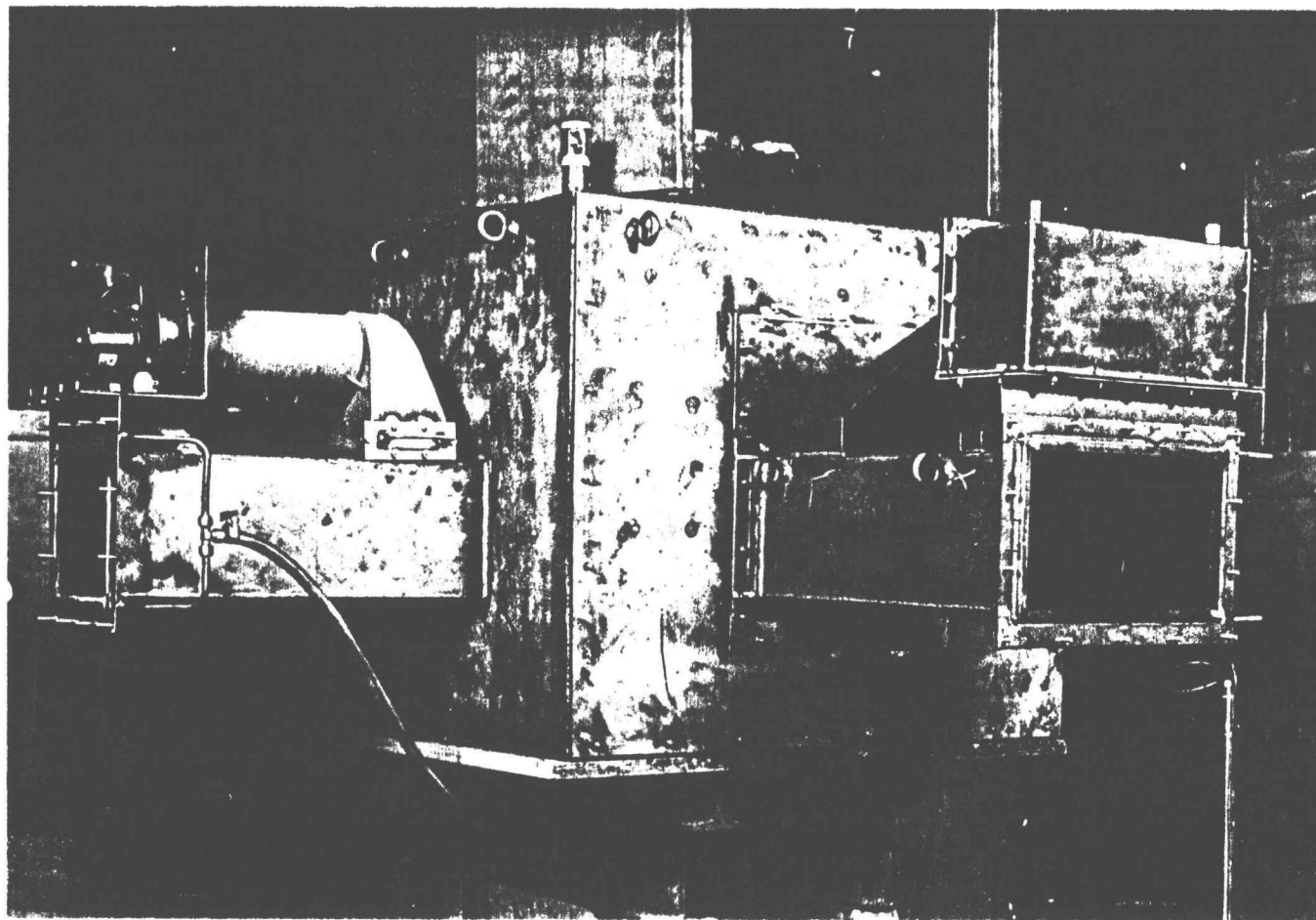
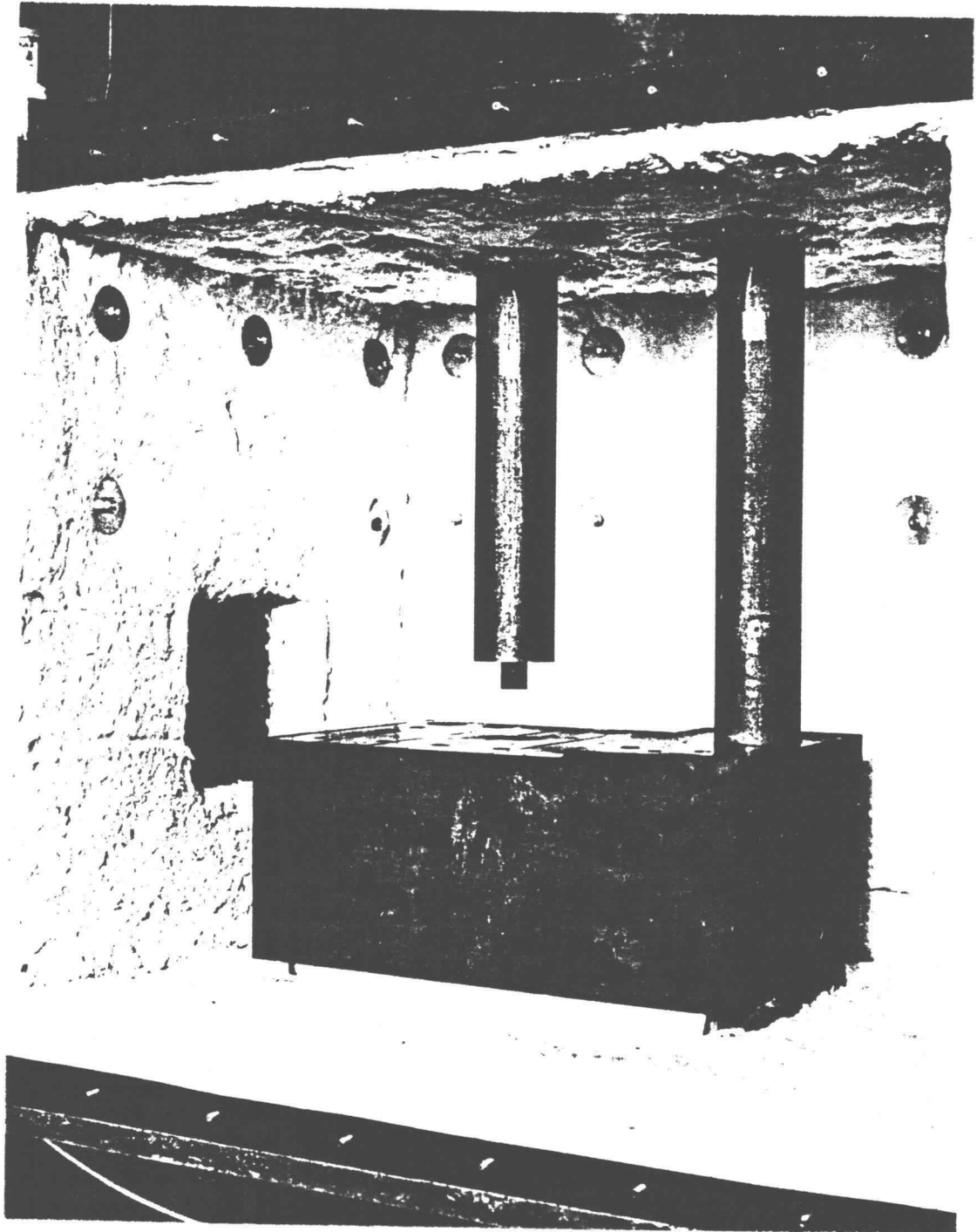Figure 1.   The experiemntal furnace at ALRC.
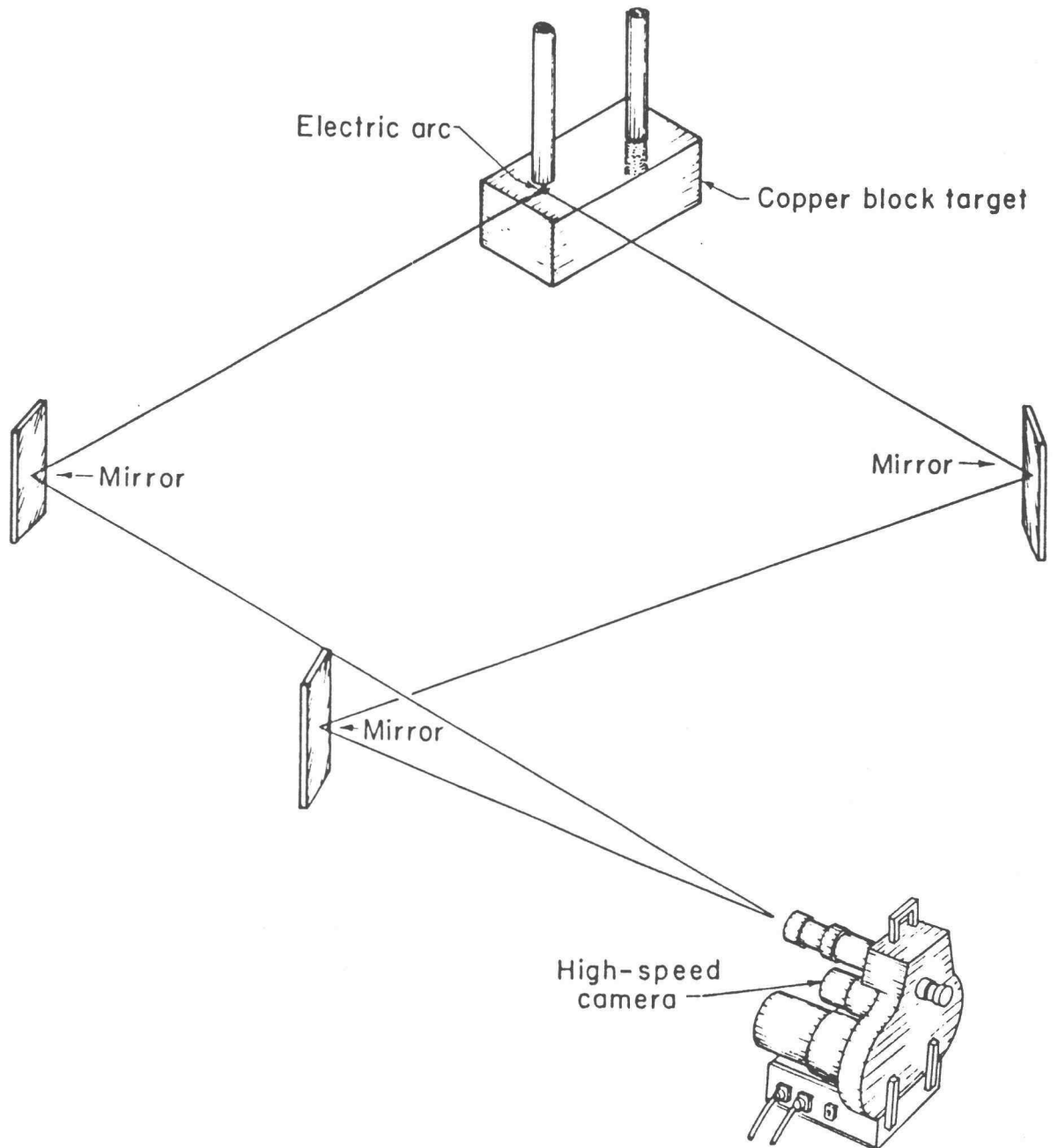
Figure 2.   Inside the experimental furnace.

Figure 3. The camera setup for data acquisition.

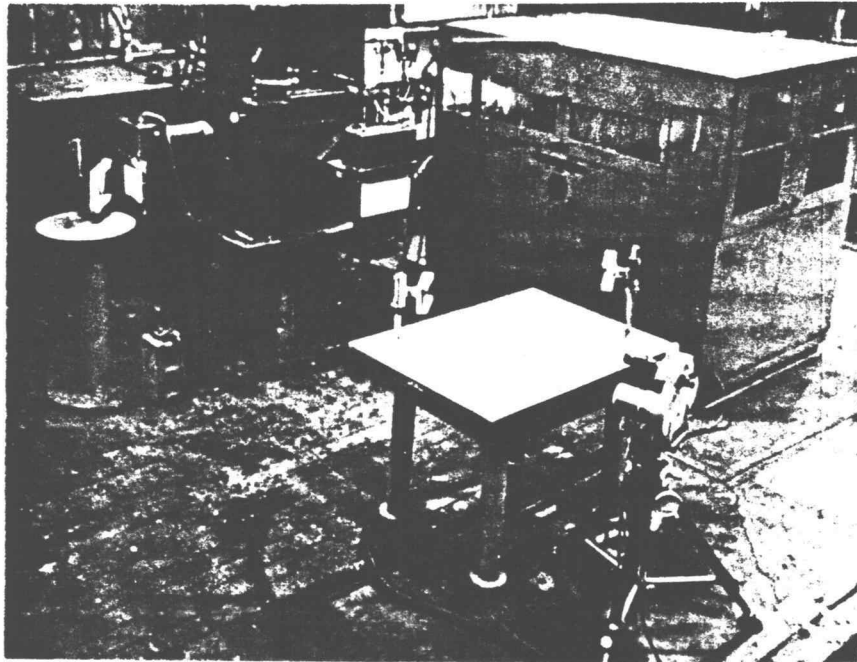Figure 4.   The Faraday cage to protect the electronics.
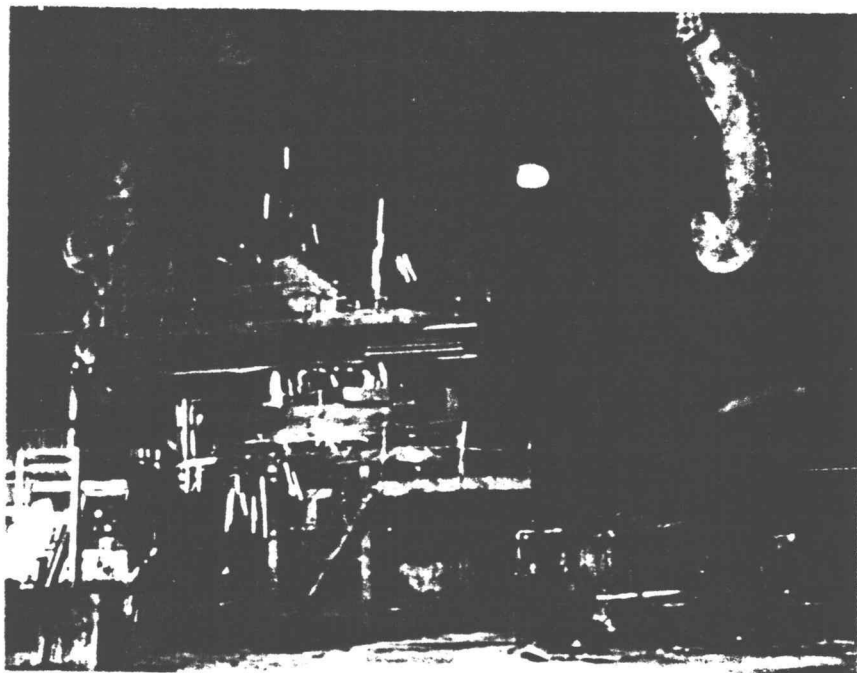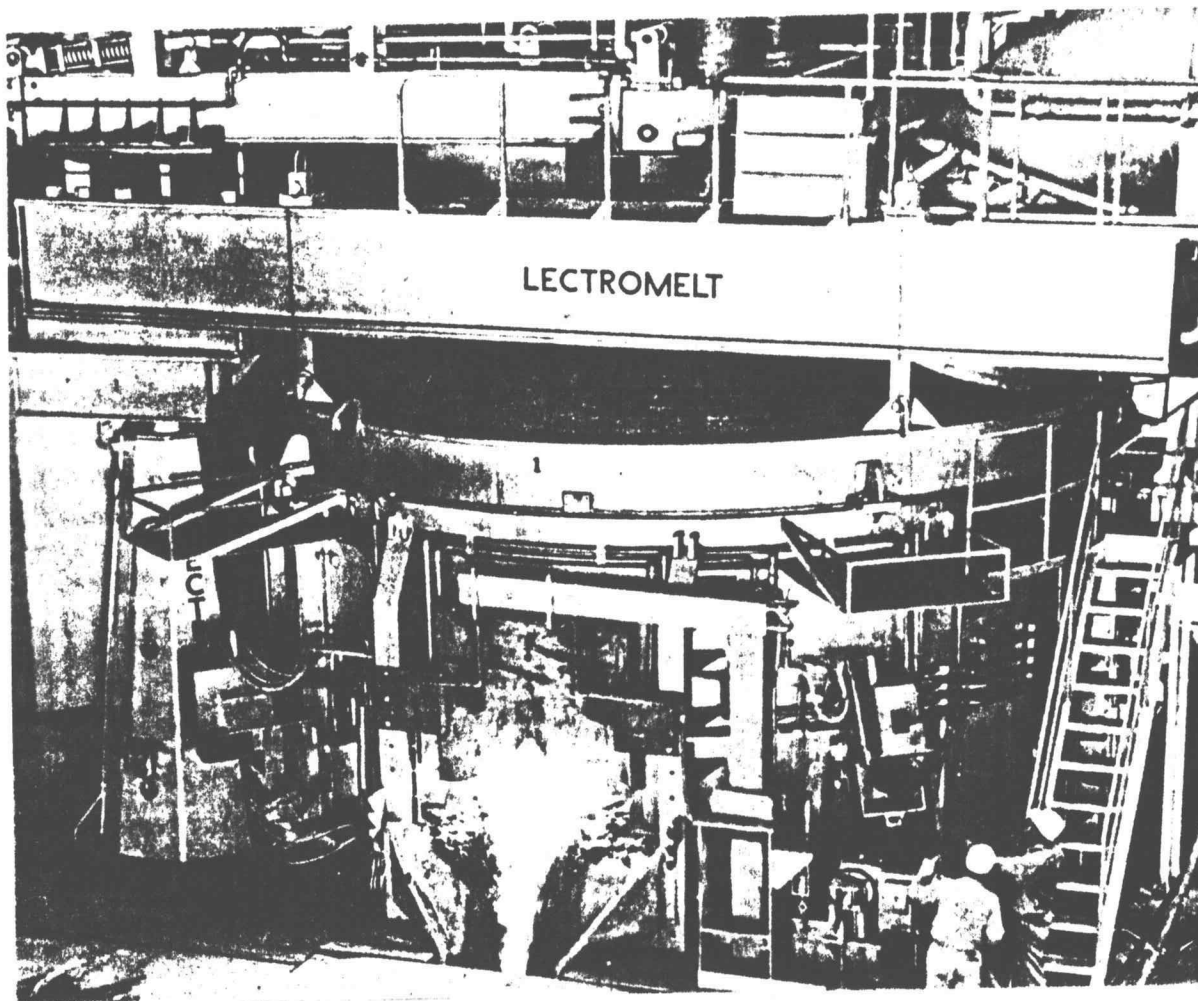

Figure 5.   OSM's 90 ton furnace.
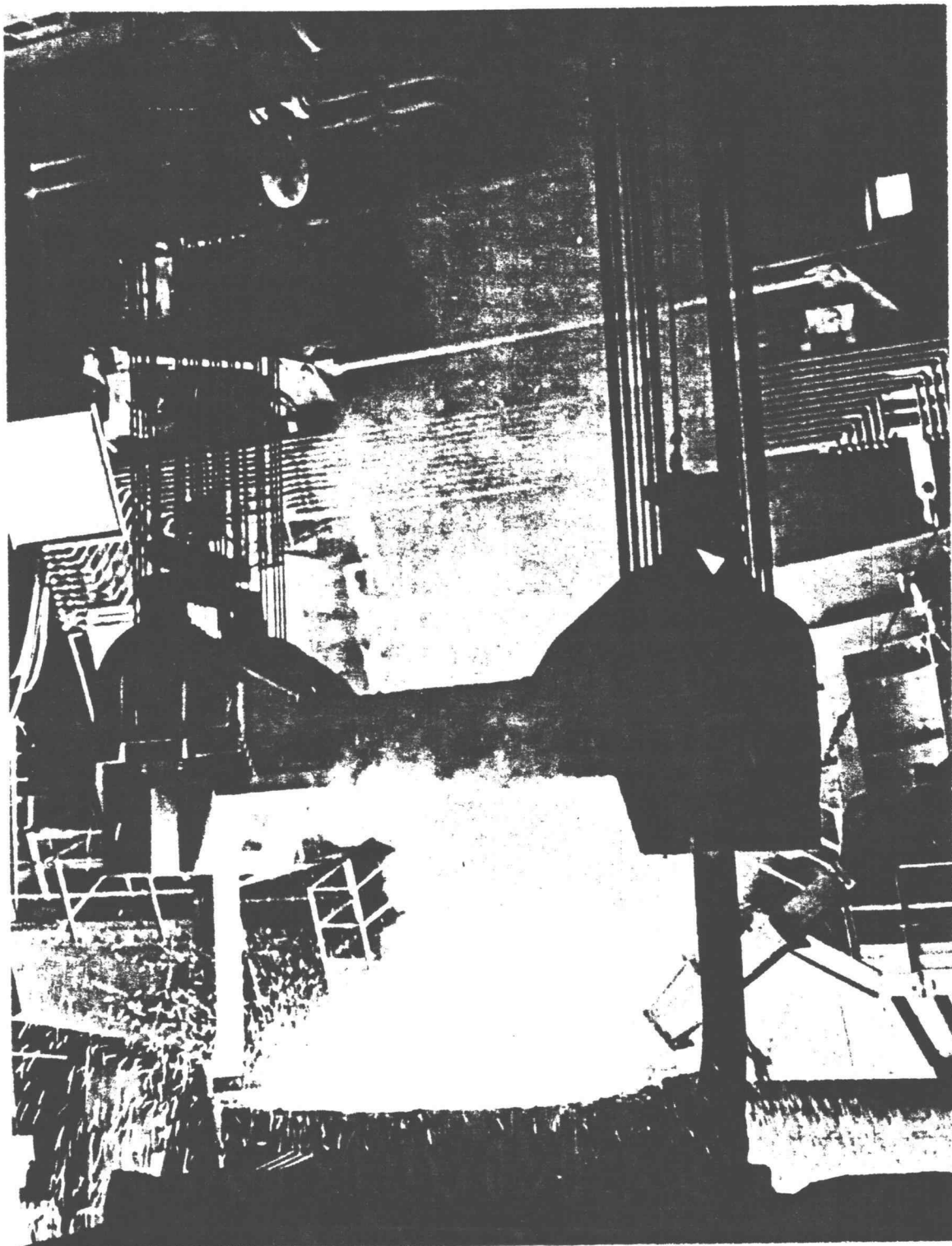
Figure 6.   The LECTROMELT "FT" furnace.

Figure 7.   The LECTROMELT 400 ton furnace.

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