

RISK MANAGEMENT FOR NONPROFIT ORGANIZATIONS

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ABSTRACT. We introduce a model for the surplus of nonprofit organizations (NPO). We assume two types of spending schemes for an NPO. Type I is a constant spending rate and Type II is a variable rate above and below a cut-off reserve level. Under steady state, we compute and compare the dysfunction probability, mean, and variance for these two spending schemes.

1. INTRODUCTION

The nonprofit sector is referred to by many names such as “nonprofit organizations (NPO)”, “non-governmental organization (NGO)”, “the charitable sector”, etc. NPOs often offer services to people who do not have the opportunity to pay for them. They also provide individuals volunteering opportunities or jobs as to help lower the unemployment rate. Moreover NPOs return a large amount of goods to the general public. They range from hospitals, day-care centers, community centers, schools and religious institutions.

NPOs play a crucial role in modern economy and exert a vital influence in our daily lives. The Stanford Project on the Evolution of Nonprofits reports that 183,769 NPOs in the U.S. expended \$686.5 billion and accounted for 6.9% of the U.S. GDP in 2000 (Gammal et al. 2005). The Comparative Nonprofit Project at Johns Hopkins University shows that the NPO sector contributed more to the U.S. GDP than

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industries such as construction, transportation, communication, electricity, gas and water supply in 2003 (Center for Civil Society Studies 2007).

The biggest distinction between NPOs and their for-profit counterparts is the distribution of “pure earnings” (Hansmann 1987), also referred to as “pure profits”. While profit accumulation is permissible for NPOs, unlike for-profits who aim at maximizing the earnings, the profit gains of NPOs must be devoted entirely to support future and further services or purposes based on the NPOs’ foundational missions. Therefore, profit maximization is not the ultimate or rational goal for NPOs. The connection between nonprofits and for-profits does exist even though their names seem quite disjoint. For-profit firms make charitable or private donations to NPOs. Furthermore, NPOs such as day-care centers or nursing homes have both for-profit and nonprofit providers.

Classifications of NPOs diverge among different economists. In this paper, we adopt the categorization by Henry B. Hansmann (1980). He grouped NPOs by their financing operations and controlling administrations. NPO finances are divided in two ways, donative and commercial. “Donative” finance refers to NPOs that receive the majority of the income from grants and donations; examples include the Red Cross. “Commercial” finance describes NPOs that obtain earnings from charging the services they give. Nursing homes and hospitals are of this type. In terms of controlling administrations, NPO are grouped as “mutual”, which means the NPOs are controlled by their patrons, and “entrepreneurial”, which means the NPOs are controlled by a self-perpetuating board of directors. The intersection of the divisions mentioned above yields four forms of NPO: donative and mutual, donative and entrepreneurial, commercial and mutual, commercial and entrepreneurial. In

this paper, we are particularly concerned about NPOs with donative financing with either controlling administrations.

During a tight economy, many NPOs that heavily depend on contributions to support their missions and programs see a big decrease in donations, which primarily come from corporate donors and sponsorships, individual households, and private foundations. The decline in income leads to many financial risks for NPOs. Consequently NPOs either try to fundraise more during economic down times, which normally turns out to be difficult, or cut back on benefiting services. However at the same time, the demand for nonprofit services becomes high as unemployment and other economic woes rise during tough cycles. Suffering such a dilemma, NPOs have sought risk assessment to set back the negative effects.

Nonetheless, analytical research regarding this aspect is relatively little. The need to establish a scientific model for NPO revenue management is only increasing. In this paper, the NPO model seeks to provide a framework to address the following key issues:

- (1) What is the chance that an NPO runs out of money to fund their programs?
- (2) What type of spending patterns apply better to NPOs? Should they prefix a spending amount or change spending rate based on a cut-off reserve level? What are the comparative features for both spending types?

We apply queueing theory and actuarial risk theory for the commercial companies to the nonprofit side of the world. In response to the first question, we introduce the notion of “dysfunction probability” to characterize, both quantitatively and practically, the probability that the NPO reserve equals to zero. We are able to determine

explicit forms of the dysfunction probability in continuous time. The results are related to donations and expenditures. Hence NPO management decision makers can target the risk of dysfunction by adjusting parameters from these two perspectives.

It should be noted that unlike the case of insurance companies that are subject to ruin when capital drops below zero, NPOs can rebound when the next donation arrives. Thus the dysfunction state for NPOs is not absorbing. As a result, it is possible to determine conditions for a steady state to exist, i.e. a stationary distribution of the NPO reserve. Our first task is to find such conditions.

As for types of spending schemes, we develop two expenditure scenarios noted as Type I and Type II in Section 3. They will be compared under steady state conditions. Upon equating the same risk measure of dysfunction, we compute the means and the variances of the respective two types of spending. We show the mean in Type I can be higher or lower than that in Type II in steady state given the same dysfunction probability. If given the same dysfunction probability and the same mean reserve in steady state, we show the variance in Type II is lower than in Type I if the cut-off level and the lower spending rate in Type II are adjusted within certain ranges. Therefore, if the risk of fluctuations in the reserve level is a concern, the NPO may prefer Type II spending strategy. However such preference will also depend on other spending objective and opportunities.

The remainder of the paper is organized as follows. Section 2 will explain the preliminaries needed from Markov process and stationary distribution. Section 3 will dive into the NPO model in continuous time, where we compute the stationary distribution of the mean reserve and compare the two methods of spending. Section 4 is a case study that applies the theorems derived from Section 3 to the data set of

expenditures by a real Chinese NPO. Finally, Section 5 investigates the possibilities of future research directions.

Below are the summarized results of this thesis:

- (1) A risk model on the surplus process of NPOs.
- (2) The Explicit form of dysfunction probability in continuous times.
- (3) A comparison of spending strategies under considerations of dysfunction risk measures, reserve levels and fluctuations in the stationary distribution.
- (4) A case study illustrating Type I strategy where we also compute the average amount of income per day needed for a given dysfunction probability.

2. PRELIMINARIES

This section introduces the basic definitions and propositions on Markov processes. Most of them are standard foundations. We will only sketch proofs and leave the details to text books, e.g. Bhattacharya and Waymire (2009).

Definition 2.1. Given an index set I , a stochastic process indexed by I is a collection of random variables $\{X_\lambda : \lambda \in I\}$ on the probability space (Ω, \mathcal{F}, P) taking values in a set S , where S is called the state space of the process.

Definition 2.2. A stochastic process $\{X_0, X_1, \dots, X_n, \dots\}$ having state space S equipped with a σ -field \mathcal{S} has the Markov property with regular transition probabilities if for each $n \geq 0$,

$$P(X_{n+1} \in B | \sigma\{X_0, X_1, \dots, X_n\}) = p_n(X_n, B)$$

where $B \in \mathcal{S}$, and

for each n

- (1) For each $B \in \mathcal{B}$, $x \rightarrow p_n(x, B)$ is a measurable function on \mathcal{S} .
- (2) For each $x \in \mathcal{S}$, $B \rightarrow p_n(x, B)$ is a probability measure on \mathcal{S} .

A stochastic process having the Markov property with regular transition probabilities is called a Markov process with transition probabilities $p_n(x, dy)$, $n \geq 0$.

We assume stationary transition probabilities, i.e., $p_n(x, dy) \equiv p(x, dy)$ do not depend on n . If $\{X_n\}_{n \geq 0}^\infty$ has the Markov property, then we can write the distribution at $m \geq 1$ time points into the future inductively for $B_1, B_2, \dots, B_m \in \mathcal{S}$, as

$$\begin{aligned} & P_\mu(X_{n+m} \in B_m, \dots, X_{n+1} \in B_1 | \sigma\{X_0, X_1, \dots, X_n\}) \\ &= \int_{B_1} \dots \int_{B_m} p(x_{m-1}, dx_m) \dots p(x_1, dx_2) p(X_n, dx_1). \end{aligned}$$

Definition 2.3. A probability π on S is said to be an invariant probability or steady state distribution for a Markov process $\{X_n\}_{n \geq 0}$ with transition probabilities $p(x, dy)$ if and only if

$$\int_S p(x, B) \pi(dx) = \pi(B)$$

for all $B \in \mathcal{S}$.

Next, we consider what we mean by a continuous time model. For this, we use the discrete structure defined above. We consider a process which, starting in a given state $x \in S$, holds for an exponentially distributed length of time T_0 with parameter $\lambda(x)$, depending on x , and then makes a transition to a new state Y_1 in \mathcal{S} , randomly selected from a probability $k(x, dy)$ on S , also depending on x . Conditionally, given the value of $Y_1 = y_1$, an independent exponential holding time (clock) with parameter $\lambda(y_1)$ is reset and, at the end of this time T_1 , a new state Y_2 is selected in S . The lifetime \mathcal{L} is defined by the possibly infinite random variable

$$\mathcal{L} = \sum_{n=0}^{\infty} T_n$$

Definition 2.4. Let $\lambda : S \rightarrow (0, \infty)$ be a measurable function and let $\{k(x, dy) : x \in S\}$ denote a collection of probabilities on S such that $x \rightarrow k(x, E)$ is measurable for each $E \in \mathcal{S}$ and such that S contains all singletons $\{x\}$ and $k(x, \{x\}) = 0$ for all $x \in S$. $\lambda(x)$ and $k(x, dy)$ are called infinitesimal parameters. Symbolically, one may write

$$p_{dt}(x, dy) := P(X(t + dt) \in dy | X(t) = x) \approx \lambda(x) k(x, dy) dt$$

Definition 2.5. Let $\lambda(x)$, $k(x, dy)$ be infinitesimal parameters defined by Definition 2.4. Let π be a probability on (S, \mathcal{S}) . Let $\{Y_n\}_{n=0}^{\infty}$ and $\{T_n\}_{n=0}^{\infty}$ be two sequences of

random variables defined on a probability space $(\Omega, \mathcal{F}, P_\pi)$ such that both sequences are jointly measurable and

- (1) $\{Y_n\}_{n=0}^\infty$ is a discrete parameter Markov process on S with initial distribution $\pi(dy)$ and homogenous transition probability $k(x, dy)$.
- (2) Conditionally given $\{Y_n\}_{n=0}^\infty$, T_0, T_1, T_2, \dots are independent exponentially distributed with parameters $\lambda(Y_0), \lambda(Y_1), \dots$, respectively.

Then the process $\{X(t) : 0 \leq t \leq \mathcal{L} := \sum_{n=0}^\infty T_n\}$ defined by

$$X(t) = \begin{cases} Y_0, & 0 \leq t \leq T_0 \\ Y_k, & T_0 + \dots + T_{k-1} \leq t \leq T_0 + \dots + T_k, k \geq 1, \end{cases}$$

$0 \leq t \leq \mathcal{L}$ is referred to as a jump process on S with initial distribution π and infinitesimal parameters $\lambda(x), k(x, dy)$. The random variable \mathcal{L} is the explosion time of the process $\{X(t) : 0 \leq t < \mathcal{L}\}$. The sequence $\{T_n\}_{n=0}^\infty$ is the holding time structure.

In the context of this paper, the parameter $\lambda(x) = \lambda$ for all x is constant. Thus one has the following:

Proposition 2.6. *For i.i.d. exponential random variables T_0, T_1, T_2, \dots with parameter $\lambda > 0$, the explosion time $\mathcal{L} = \sum_{n=0}^\infty T_n = \infty$ with probability one.*

Proof. $P(\mathcal{L} > t) = P(\sum_{j=1}^\infty T_j > t) \geq P(\sum_{j=1}^n T_j > t)$ for all n . We want to show that $P(\sum_{j=0}^n T_j > t) \rightarrow 1$. Since T_0, T_1, T_2, \dots are i.i.d. exponential random variable with parameter λ , the sum $\sum_{j=0}^n T_j$ has Gamma distribution with density

$\frac{\lambda}{(k-1)!}(\lambda x)^{k-1}e^{-\lambda x}$. Thus we rewrite the probability as

$$P\left(\sum_{j=1}^n T_j > t\right) = \int_t^\infty \frac{\lambda}{(k-1)!}(\lambda x)^{k-1}e^{-\lambda x} dx$$

By a change of variable $y = x - t$,

$$\begin{aligned} &= \int_0^\infty \frac{\lambda}{(k-1)!} \lambda^{k-1} (y+t)^{k-1} e^{-\lambda(y+t)} dy \\ &= \int_0^\infty \frac{\lambda}{(k-1)!} [\lambda(y+t)]^{k-1} e^{-\lambda(y+t)} dy \end{aligned}$$

By the Binomial Theorem,

$$\begin{aligned} &= \int_0^\infty \frac{\lambda}{(k-1)!} \sum_{j=0}^{k-1} \binom{k-1}{j} (\lambda y)^j (\lambda t)^{k-1-j} e^{-\lambda(y+t)} dy \\ &= \sum_{j=0}^{k-1} \left\{ \int_0^\infty \frac{\lambda(\lambda y)^j}{j!} e^{-\lambda y} dy \frac{(\lambda t)^{k-1-j}}{(k-1-j)!} \right\} e^{-\lambda t} \end{aligned}$$

Note that all the integrals integrate to 1 since each is the probability density function for the Gamma distribution. Thus, as $n \rightarrow \infty$,

$$P(\mathcal{L} > t) \geq \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \dots + \frac{(\lambda t)^n}{n!}\right) e^{-\lambda t} \rightarrow 1$$

□

Definition 2.7. A semi-group $\{T_t : t \geq 0\}$ for a Markov process $\{R_t : t \geq 0\}$ is defined as the operator such that

$$(2.1) \quad T_t f(u) = E_u f(R_t) = E(f(R_t) | R_0 = u)$$

for bounded and measurable functions f .

Definition 2.8. The infinitesimal generator of the semi-group $\{T_t : t \geq 0\}$ is an operator L with domain D_L , defined by

$$Lf(u) = \lim_{t \downarrow 0} \frac{T_t f(u) - f(u)}{t}$$

where $u \in S$ and D_L is defined for functions f on which the limit exists.

Proposition 2.9. *If a probability measure π is a stationary distribution for $\{R_t : t \geq 0\}$, then for any $f \in D_L$, we have*

$$\int_{S=[0,\infty)} Lf(u)\pi(du) = 0$$

Proof. (sketch of a proof) If $\pi(dx)$ is a stationary distribution

$$\int_{S=[0,\infty)} E_u f(R_t) \pi(du) = \int_S E_u f(R_0) \pi(du) = \int_S f(u) \pi(du)$$

then

$$\int_S E_u [f(R_t) - f(u)] \pi(du) = 0$$

for $t \geq 0$. Divide by t and let $t \rightarrow 0$.

$$\int_S Lf(u) \pi(du) = 0$$

□

3. THE NPO MODEL IN CONTINUOUS TIME

The main focus of the paper is to compare the features of two types of spending strategies.

3.1. The Model. We now introduce the NPO model in continuous time

$$(3.1) \quad R_t = u + S_t - \int_0^t r(R_s) ds$$

We assume the following:

- (1) u is the initial capital of the NPO.
- (2) $S_t = \sum_{i=1}^{N_t} D_i$ is a pure jump process where the counting process $\{N(t)\}_{t \geq 0}$ is a Poisson process with i.i.d. interarrival times T_1, T_2, \dots and with arrival intensity β .
- (3) The sequence $\{D_i\}_{i > 0}$ are i.i.d. exponentially distributed random variables with density function $p(d) = \delta e^{-\delta d}$, denoting donation sizes.
- (4) The sequence of random variables $\{D_i\}_{i > 0}$ and $\{T_i\}_{i \geq 1}$ are independent.
- (5) The rate of expenses is denoted by the function $r(R_t)$. This implies in between the upward jumps, $\{R_t : t \geq 0\}$ should satisfy the differential equation

$$\dot{R} = -r(R)$$

where \dot{R} denotes the left derivative. We assume further that $r(0) = 0$.

- (6) If the distribution of R_t is given by the invariant probability π , then assume the process is in steady state.

We define a new term dysfunction probability by first defining the time of dysfunction by

$$(3.2) \quad T = \min\{t : R_t = 0\}$$

and let π_0 denote the dysfunction probability for the process $\{R_t : t \geq 0\}$ in steady state.

Remark 3.1. Recall that in actuarial risk theory, the surplus model for insurance companies is

$$U(t) = u + \int_0^t c(s)ds - \sum_{n=1}^{N_t} C_n$$

where u is the initial fund of the company, $c(t)$ is the premium rate, N_t is a Poisson counting process, and $\{C_n : n \geq 1\}$ are i.i.d. random variables, denoting the claim sizes. This sequence of random variables is independent from $N(t)$. Claims arrive at random times $0 < T_1 < T_2 < \dots$. The probability of ruin, denoted by $\Psi(u)$ is the probability that the time of ruin $T_u := \inf\{t : U(t) \leq 0\}$ is finite, i.e. $\Psi(u) = P(T_u < \infty | U(0) = u)$. The notion of dysfunction probability is similar to that of ruin probability since both measure the probability when the reserve becomes zero. However, for dysfunction probability, we assume the surplus process to be in steady state. We call it the dysfunction probability for NPO because, unlike insurance companies who have to make claim payments and become bankrupt if they cannot, NPOs can stop functioning when the reserve is zero. They are able to refunction when the next donation arrives. We call the time when the NPO is not functioning the event of dysfunction and the probability of when this occurs the dysfunction probability.

We use the idea of applying semi-group property to compute the density in stationary distribution. Consider

$$(3.3) \quad \begin{aligned} E_u f(R_t) &= f(u - tr(u))e^{-\beta t} \\ &+ [\int_0^\infty f(u + v - r(u + v)t)b(v)dv]\beta te^{-\beta t} + o(t) \end{aligned}$$

The first term refers to the case when there is no donation, the second term refers to the case where there is a donation with pdf $b(v) = \delta e^{-\delta v}$. Then

$$\begin{aligned} E_u f(R_t) - f(u) \\ = E_u f(R_t) - (f(u)e^{-\beta t} + f(u)(1 - e^{-\beta t})) \end{aligned}$$

By plugging everything in Eq. (3.3) and the fact that $1 - e^{-\beta t} = \beta t e^{-\beta t} + o(t)$, we have

$$= [f(u - tr(u)) - f(u)]e^{-\beta t} + \int_0^\infty [f(u + v - r(u + v)t) - f(u)]b(v)dv(1 - e^{-\beta t}) + o(t)$$

Divide by t throughout,

$$\begin{aligned} & \frac{E_u f(R_t) - f(u)}{t} \\ = & -r(u) \frac{[f(u - tr(u)) - f(u)]e^{-\beta t}}{-tr(u)} + \frac{o(t)}{t} + \int_0^\infty \frac{[f(u + v - r(u + v)t) - f(u)]b(v)dv(1 - e^{-\beta t})}{t} \end{aligned}$$

As $t \rightarrow 0$, we get

$$Lf(u) = -r(u)f'(u) + \beta \int_0^\infty (f(u + v) - f(u))b(v)dv$$

for $u \geq 0$.

Theorem 3.2. *The infinitesimal generator for the continuous time NPO model is*

$$Lf(u) = -r(u)f'(u) + \beta \int_0^\infty (f(u + v) - f(u))b(v)dv$$

for $u \geq 0$.

Following Harrison and Resnick (1976), we will use the next lemma for computing the steady state distribution for the NPO model.

Lemma 3.3. *Let D_L be the domain such that $f \in D_L$ has a nonnegative density f' .*

For $f \in D_L$, we have

$$Lf(x) = \beta \int_x^\infty Q(y-x)f'(y)dy - r(x)f'(x)$$

where $x \geq 0$, $Q(x) = 1 - B(x)$, and $B(x)$ is the distribution of the donations.

Proof. We write $f(x+y) - f(x)$ as an integral of f' . Thus for any $x > 0$,

$$\begin{aligned} Lf(x) &= -r(x)f'(x) + \beta \int_0^\infty (f(x+y) - f(x))b(y)dy \\ &= -r(x)f'(x) + \beta \int_0^\infty \int_x^{x+y} f'(z)b(y)dzdy \\ &= -r(x)f'(x) + \beta \int_x^\infty f'(z) \int_{z-x}^\infty b(y)dy \\ (3.4) \quad &= -r(x)f'(x) + \beta \int_x^\infty f'(z)Q(z-x)dz \end{aligned}$$

Change z to y . Equation (3.4) becomes

$$Lf(x) = \beta \int_x^\infty Q(y-x)f'(y)dy - r(x)f'(x)$$

In particular since $r(0) = 0$, $Lf(0) = \beta \int_0^\infty Q(y)f'(y)dy$ holds at $x = 0$. \square

Proposition 3.4. *The stationary distribution π for $\{R_t : t \geq 0\}$ on the state space $S = [0, \infty)$ is given by*

$$\pi(dx) = \pi_0 \delta_{\{0\}}(dx) + g(x)dx$$

with density $g(x)$ of the form

$$g(x) = \frac{\beta}{r(x)} \left\{ \pi_0 Q(x) + \int_0^x Q(x-y)g(y)dy \right\}$$

where $x \geq 0$.

Proof. We will apply Proposition (2.9) and Lemma (3.3) .

$$\begin{aligned}
0 &= Lf(0)\pi_0 + \int_0^\infty Lf(x)\pi(dx) \\
&= \pi_0\beta \int_0^\infty Q(y)f'(y)dy + \int_0^\infty [\beta \int_x^\infty Q(y-x)f'(y)dy - r(x)f'(x)]\pi(dx) \\
&= \pi_0\beta \int_0^\infty Q(y)f'(y)dy + \int_0^\infty \beta \int_0^y Q(y-x)\pi(dx)f'(y)dy - \int_0^\infty r(x)f'(x)\pi(dx) \\
&= \beta \int_0^\infty [\int_0^y Q(y-x)\pi(dx)]f'(y)dy - \int_0^\infty r(x)f'(x)\pi(dx)
\end{aligned}$$

Then we have

$$\int_0^\infty r(x)f'(x)\pi(dx) = \int_0^\infty \left\{ \frac{\beta}{r(x)} \int_0^x Q(x-y)\pi(dy) \right\} r(x)f'(x)dx$$

Any π satisfying the above equation is a stationary distribution. And

$$g(x) = \frac{\beta}{r(x)} \left\{ \pi_0 Q(x) + \int_0^x Q(x-y)g(y)dy \right\}$$

□

3.2. Theorems Regarding the NPO Model.

Theorem 3.5. *Assume a NPO spends at a constant rate $c > \beta/\delta$. Let $\pi_0(c)$ be the atom at 0. The stationary distribution has density*

$$(3.5) \quad g(x) = \frac{\pi_0(c)\beta}{c} e^{-(\delta - \frac{\beta}{c})x}$$

Proof. By Proposition (3.4), $g(x) = \frac{\beta}{r(x)} \left\{ \pi_0 Q(x) + \int_0^x Q(x-y)g(y)dy \right\}$. Under the assumptions of the continuous-time NPO model, $Q(x) = e^{-\delta x}$, $Q(x-y) = e^{-\delta(x-y)}$,

$r(x) = c$. Then,

$$g(x) = \frac{\beta\pi_0}{c}e^{-\delta x} + \frac{\beta}{c} \int_0^x e^{-\delta(x-y)} g(y) dy$$

Multiply through by $e^{\delta x}$ and define $\tilde{g}(x) \equiv e^{\delta x} g(x)$. We have $\tilde{g}(x) = \pi_0\beta/c + \beta/c \int_0^x \tilde{g}(y) dy$. Or the ODE

$$(3.6) \quad \tilde{g}(x) = e^{\delta x} g(x)$$

$$\tilde{g}(0) = \frac{\beta\pi_0}{c}$$

Solving Equation (3.6) yields

$$\tilde{g}(x) = \frac{\beta\pi_0}{c} e^{\frac{\beta}{c}x}$$

Thus

$$g(x) = \frac{\beta\pi_0}{c} e^{-(\delta - \frac{\beta}{c})x}$$

The equilibrium condition is $c > \frac{\beta}{\delta}$; thus $\frac{\beta}{c} - \delta < 0$, otherwise $\int_0^\infty g(x) dx = \infty$. \square

Remark 3.6. The condition $c > \frac{\beta}{\delta}$ might seem surprising since people do not make ends meet when spending beyond their earnings. But if we examine the parameters more carefully, c and β/δ refer to the average spending and income per unit time. The condition $c > \beta/\delta$ means, in order to achieve a steady state, on average, rather than consistently, the nonprofits should spend more than what flows in on average. Otherwise, the reserve will keep accumulating, contradicting the process $\{R_t : t \geq 0\}$ being in a steady state. Further, the essence of NPOs supports the idea of the equilibrium condition. NPOs are established to help the needed instead of earning profits. This point of view also strengthens that $c > \beta/\delta$ is a valid assumption.

Theorem 3.7. *For a given constant spending rate $c > \frac{\beta}{\delta}$ and assuming β, δ are known, there is a unique dysfunction probability given by*

$$(3.7) \quad \pi_0(c) = 1 - \beta/(c\delta)$$

that assumes all values in $(0, 1]$, a unique corresponding steady mean given by

$$(3.8) \quad \mu(c) = \beta/(\delta(c\delta - \beta))$$

and a unique corresponding steady state variance given by

$$(3.9) \quad \sigma^2(c) = (-\beta^2 + 2c\beta\delta)/(\delta(-\beta + c\delta))$$

Proof. Since the stationary distribution is a mixture of the Dirac function $\delta_{\{0\}}$ and an absolute continuous part $g(x)$, we must have

$$(3.10) \quad \pi_0 + \int_0^\infty g(y)dy = 1$$

where

$$g(x) = \frac{\pi_0\beta}{c}e^{-(\delta - \frac{\beta}{c})x}$$

Thus,

$$\pi_0(c) = 1 - \beta/(c\delta)$$

We can see that as c increases from β/δ , π_0 is a continuous function of c and takes values from 0 as $c \rightarrow \infty$ and 1 as $c \downarrow \beta/\delta$.

Now use the definition of expectation and compute the mean reserve

$$\mu = \int_0^\infty xg(x)dx = \beta/(\delta(c\delta - \beta))$$

Similarly by the definition of variance, we have

$$\begin{aligned}
 \sigma^2 &= E(X^2) - \mu^2 \\
 &= \int_0^\infty x^2 g(x) dx - \mu^2 \\
 &= (-\beta^2 + 2c\beta\delta)/(\delta(-\beta + c\delta))
 \end{aligned}$$

Thus for a given spending rate c , and assuming the parameters β , δ are known, the dysfunction probability, mean reserve in the steady state and variance in the steady state are uniquely determined. \square

3.3. Management Decision: Two Types of Spending Strategies. As mentioned in Section 1, the motivation for this thesis comes from some real life NPOs' inability to function well during economic downturns. One approach on this issue concerns the spending strategies. Here lies the management question: How should an NPO plan its spending? Should the NPO always spend at a fixed rate or should it adjust the rate according to some certain level of its reserve. We first mathematically describe the two types of spending schemes for comparison.

- Type I: $r(R_t) = c$, i.e. the NPO spends money at an invariable rate. See Figure 3.1 for surplus plot.
- Type II: $r(R_t) = \begin{cases} c_0, & R_t \leq x_0 \\ c_1, & R_t > x_0 \end{cases}$, where $c_0 \leq c_1$, i.e. the NPO maintains at an expenditure rate c_1 if its reserve is about the level x_0 and lowers its spending rate to c_0 if the reserve drops below x_0 . See Figure 3.2 for surplus plot.

Under stationary distribution π , there are three aspects - dysfunction probability, mean reserve and variances to compare for the two types. Dysfunction probability

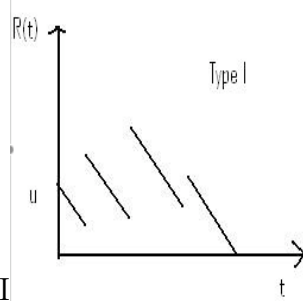


FIGURE 3.1. Surplus plot for Type I

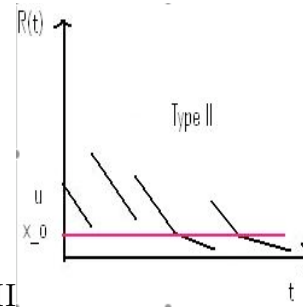


FIGURE 3.2. Surplus plot for Type II

measures the risk that an NPO runs out of money and the lower one is more desirable. Variances in stationary distribution, another kind of risks, measure the fluctuations of the mean reserve in stationary distribution and the less they diverge from the mean reserve, the more accurate the estimate of the mean is. The mean reserve itself, not regarded as a measure of risk, reports how much money in equilibrium the NPO will have.

Theorem 3.8. *Assume an NPO follows Type II spending with $\frac{\beta}{\delta} < c_0$ and let $\pi_0(c_0, c_1, x_0)$ denote the dysfunction probability. The stationary distribution has density*

$$(3.11) \quad g(x) = \begin{cases} \frac{\pi_0 \beta \exp(-(\delta - \beta/c_0)x)}{c_0}, & x \leq x_0 \\ \frac{\pi_0 \beta \exp(\beta(1/c_0 - 1/c_1)x_0) \exp(-(\delta - \beta/c_1)x)}{c_1}, & x > x_0 \end{cases}$$

Proof. If $x \leq x_0$, this is the same as spending at a constant rate c_0 . Thus we apply the result from Theorem (3.5). If $x > x_0$,

$$\begin{aligned} g(x) &= \frac{\pi_0 \beta}{c_1} e^{\int_0^{x_0} \frac{1}{c_0} dy + \beta (\int_{x_0}^x \frac{1}{c_1} dy - \delta x)} \\ &= \frac{\pi_0 \beta}{c_1} e^{\beta (\frac{1}{c_0} - \frac{1}{c_1}) x_0} e^{(\frac{\beta}{c_1} - \delta) x} \end{aligned}$$

□

Theorem 3.9. *Assume a NPO follows Type II spending with $c_0 > \frac{\beta}{\delta}$. Then the dysfunction probability $\pi_0(c_0, c_1, x_0)$ is given by*

$$(3.12) \quad \pi_0 = \frac{1}{(1 + \frac{a_0}{b_0} + \frac{a_1}{b_1} \exp(-b_1 x_0) - \frac{a_0}{b_0} \exp(-b_0 x_0))}$$

The mean reserve $\mu(c_0, c_1, x_0)$ in stationary distribution under Type II is given by

$$(3.13) \quad \mu(c_0, c_1, x_0) = \frac{a_0(1 - e^{-b_0 x_0}(1 + b_0 x_0)\pi_0)}{b_0^2} + \frac{a_1 e^{-b_1 x_0}(1 + b_1 x_0)\pi_0}{b_1^2}$$

and the variance $\sigma^2(c_0, c_1, x_0)$ is given by

$$\begin{aligned} \sigma^2(c_0, c_1, x_0) &= a_0 b \\ &+ a_0 e^{-b_0 x_0} (-2 + 2e^{b_0 x_0} - b_0 x_0 (2 + b_0 x_0)) \pi_0 \\ &+ a_1 e^{-b_1 x_0} (2 + b_1 x_0 (2 + b_1 x_0)) \pi_0 \\ &- \left(\frac{a_1 e^{-b_1 x_0} (1 + b_1 x_0) \pi_0}{b_1^2} + \frac{a_0 (1 - e^{-b_0 x_0} (1 + b_0 x_0)) \pi_0}{b_0^2} \right)^2 \end{aligned}$$

where

$$(3.14) \quad a_0 = \beta / c_0$$

$$(3.15) \quad b_0 = -(\beta/c_0 - \delta)$$

$$(3.16) \quad a_1 = \beta/c_1 \exp[\beta(\frac{1}{c_0} - \frac{1}{c_1})x_0]$$

$$(3.17) \quad b_1 = -(\beta/c_1 - \delta)$$

Proof. We use the equation

$$\pi_0 + \int_0^{x_0} h(x)dx + \int_{x_0}^{\infty} h(x)dx = 1$$

Then $1 - \pi_0 = (\frac{a_1}{b_1}e^{-b_1x_0} - \frac{a_0}{b_0}e^{-b_0x_0} + \frac{a_0}{b_0})\pi_0$. So

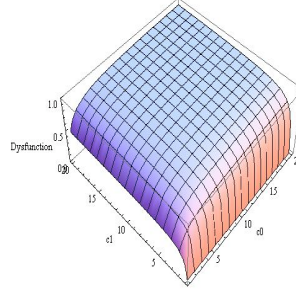
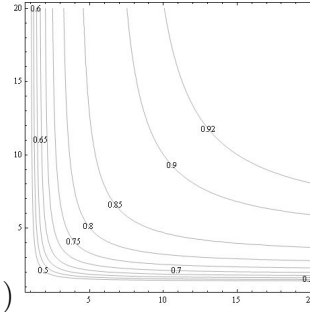
$$\pi_0 = \frac{1}{(1 + \frac{a_0}{b_0} + \frac{a_1}{b_1} \exp(-b_1x_0) - \frac{a_0}{b_0} \exp(-b_0x_0))}$$

We can check that when $c_0 = c_1$, we get $\pi_0 = 1 - \frac{\beta}{c_0\delta}$, i.e. Type II collapses to Type I. We use the definition of expectation and variance to compute μ and σ^2 . \square

3.3.1. Limiting Behaviors. Before comparing Type I and Type II spending strategies, we want to examine further the NPO model under Type II via looking at some limiting behaviors. Since the general expressions for the dysfunction probability, mean, and variance are relatively complicated, this will give some simple insight on their structures. We will also include some plots to demonstrate dependence on parameters. First we consider the two-dimensional case where we assume the cut-off reserve level $x_0 = 1$. To simplify the computation, we let $\beta = \delta = 1$.

Observation 1. The limiting behaviors of the dysfunction probability as one of the spending rates changes are the following:

$$(3.18) \quad \lim_{c_1 \rightarrow \infty} \pi_0(c_0, c_1) = \frac{(-1 + c_0)e}{c_0e - e^{1/c_0}}$$

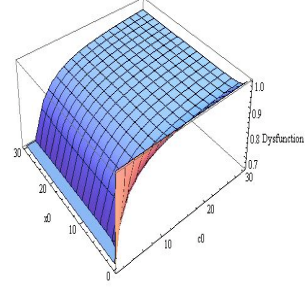
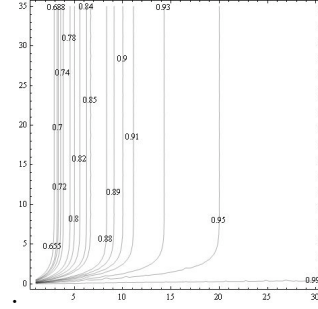
FIGURE 3.3. The 3D Plot of π_0 .FIGURE 3.4. The Contour plot of $\pi_0(c_0, c_1)$

$$(3.19) \quad \lim_{c_0 \rightarrow \infty} \lim_{c_1 \rightarrow \infty} \pi_0(c_0, c_1) = 1$$

$$(3.20) \quad \lim_{c_0 \rightarrow 1^+} \pi_0(c_0, c_1) = \frac{c_1 - 1}{c_1}$$

Remark 3.10. The contour plot of the dysfunction probability $\pi_0(c_0, c_1)$ shows the level curves are convex to the origin. (3.18) implies that the dysfunction probability when spending at a very large c_1 is bigger than the dysfunction probability spending at c_0 under Type I because $\frac{(-1+c_0)e}{c_0e-e^{1/c_0}} = \frac{-1+c_0}{c_0-e^{1/c_0}-1} > 1 - \frac{1}{c_0} = \pi_0(c_0)$. Moreover, as $c_0, c_1 \rightarrow \infty$, $\pi_0 \rightarrow 1$. In practice, if an NPO spends at high rates regardless of the cut-off reserve level, the possibility of running out of money is most likely to happen. (3.20) implies if $c_0 \downarrow \frac{\beta}{\delta} = 1$, Type II spending scheme becomes Type I with $c = c_1$.

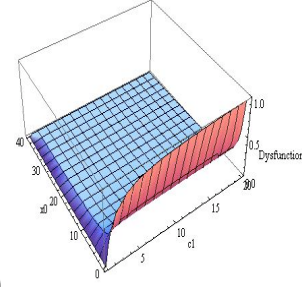
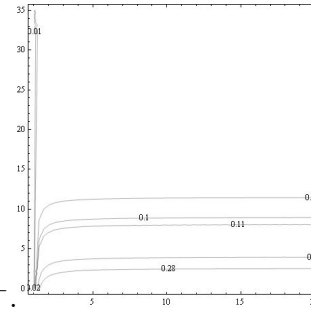
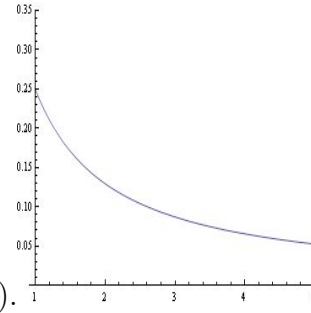
Now we consider the 3-D case of the dysfunction probability $\pi_0(c_0, c_1, x_0)$.

FIGURE 3.5. 3-D plot $\lim_{c_1 \rightarrow \infty} \pi_0(c_0, c_1, x_0)$.FIGURE 3.6. Contour plot $\lim_{c_1 \rightarrow \infty} \pi_0(c_0, c_1, x_0)$.

Observation 2. $\lim_{x_0 \rightarrow \infty} \pi_0(c_0, c_1, x_0) = 1 - \frac{1}{c_0}$

Remark 3.11. The dysfunction probability as $x_0 \rightarrow \infty$ coincides with the dysfunction probability in Type I. It makes sense because as we consider the cut-off level is set extremely high, the NPOs with less reserve money will hardly reach that cut-off level and consequently will not change spending to c_1 rate. Thus the NPOs will only spend at c_0 rate, which is Type I spending.

Observation 3. $\lim_{c_1 \rightarrow \infty} \pi_0(c_0, c_1, x_0) = \frac{(-1+c_0)e}{c_0 e - e^{1/c_0} x_0}$. The plot and contour plot of the limit are shown in Figure (3.5) and (3.6). We can see that as $c_0 \rightarrow \infty$, π_0 goes to 1. Furthermore, $\frac{(-1+c_0)e}{c_0 e - e^{1/c_0} x_0} = \frac{-1+c_0}{c_0 - e^{1/c_0 - 1} x_0} > \frac{-1+c_0}{c_0}$. Hence regardless of what the cut-off level x_0 is, Type II under $c_1 \rightarrow \infty$ has higher chance of dysfunction than at constant spending rate c_0 under Type I.

FIGURE 3.7. Contour Plot of $\lim_{c_0 \rightarrow 1^+} \pi_0(c_0, c_1, x_0)$ FIGURE 3.8. Contour Plot of π_0 as $c_0 \rightarrow 1^+$.FIGURE 3.9. $\lim_{c_1 \rightarrow \infty} \mu(c_0, c_1)$.

Observation 4. $\lim_{c_0 \rightarrow 1^+} \pi_0 = \frac{-1+c_1}{c_1-x_0+c_1x_0}$. The plot and contour plot of the limits are shown in Figure (3.7) and (3.8).

The limiting behaviors of the mean reserve are also interesting to look at. First we consider $x_0 = \beta = \delta = 1$

Observation 5. $\lim_{c_1 \rightarrow \infty} \mu(c_0, c_1) = \frac{e^{\frac{1}{c_0}} + c_0(e - 2e^{\frac{1}{c_0}})}{(-1+c_0)(c_0e - e^{\frac{1}{c_0}})}$. As shown in Figure (3.9), when $c_0 \rightarrow \infty$, $\mu \rightarrow 0$.

Observation 6. $\lim_{c_0 \rightarrow 1^+} \mu(c_0, c_1) = \frac{-1+2c_1+c_1^2}{2-6c_1+4c_1^2}$

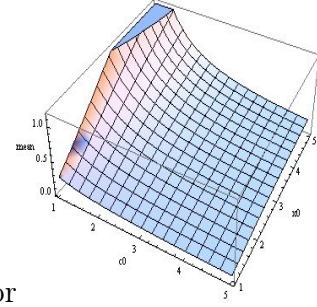
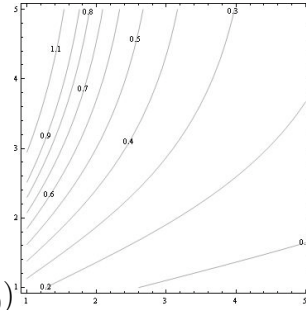


FIGURE 3.10. The 3D plot of the reserve limiting behavior

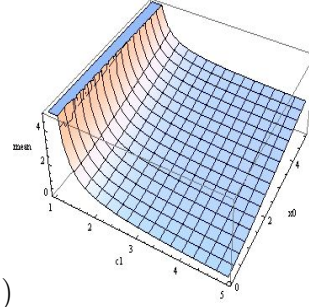
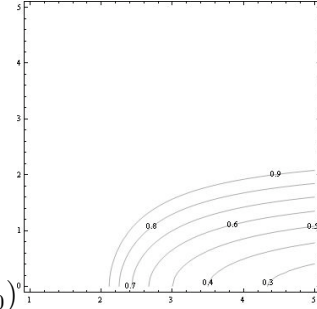
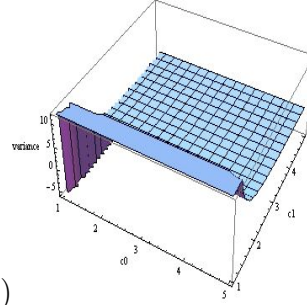
FIGURE 3.11. Contour of $\lim_{c_1 \rightarrow \infty} \mu(c_0, c_1, x_0)$

Observation 7. $\lim_{c_1 \rightarrow \infty} \mu(c_0, c_1, x_0) = \frac{\exp(x_0/c_0)x_0 + c_0(\exp(x_0) - \exp(x_0/c_0)(1+x_0))}{(-1+c_0)(c_0 \exp(x_0) - \exp(x_0/c_0))}$ and the 3-D plot and contour plot are shown in Figure (3.10) and Figure (3.11). We can see that the higher the cut-off level x_0 and the lower rate c_0 are set, the higher the reserve can be because it will be harder for the NPO to actually spend at c_1 rate.

Observation 8. $\lim_{c_0 \rightarrow 1^+} \lim_{c_1 \rightarrow \infty} \mu(c_0, c_1, x_0) =$

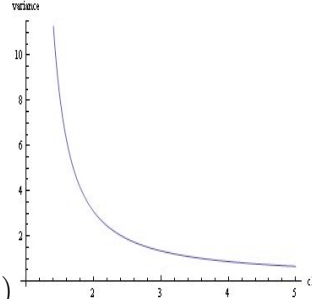
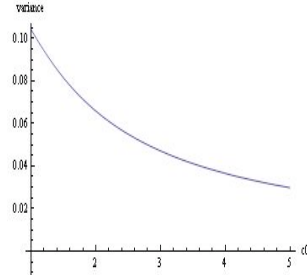
$$\begin{aligned} \lim_{c_0 \rightarrow 1^+} \frac{\exp(x_0/c_0) + c_0(\exp(x_0) - \exp(x_0/c_0)(1+x_0))}{(-1+c_0)(c_0 \exp(x_0) - \exp(x_0/c_0))} \\ = \frac{x_0^2}{2+2x_0} \end{aligned}$$

Observation 9. $\lim_{c_0 \rightarrow 1^+} \mu(c_0, c_1, x_0) = \frac{1}{c_1-1} + \frac{(-1+c_1)x_0^2}{2(c_1-x_0+c_1x_0)}$ and the plots are shown in Figure (3.12) and (3.13).

FIGURE 3.12. 3D plot of $\lim_{c_0 \rightarrow 1^+} \mu(c_0, c_1, x_0)$ FIGURE 3.13. Contour plot $\lim_{c_0 \rightarrow 1^+} \mu(c_0, c_1, x_0)$ FIGURE 3.14. Plot of $\sigma^2(c_0, c_1)$

Now we examine the limiting behavior of the variance. In the case of $x_0 = 1$, $\sigma^2(c_0, c_1)$ has the plot shown in Figure (3.14). It is interesting to note that if we spend at a higher rate when the surplus exceeds 1, there appears less fluctuation.

Observation 10. $\lim_{c_0 \rightarrow 1^+} \sigma^2 = \frac{-11+64c_1-126c_1^2+80c_1^3+5c_1^4}{12(1-3c_1+2c_{1,2})^2}$ and the plot of the limit is shown in Figure (3.15).

FIGURE 3.15. $\lim_{c_0 \rightarrow 1+} \sigma^2(c_0, c_1)$ FIGURE 3.16. Plot of $\lim_{c_1 \rightarrow \infty} \sigma^2(c_0, c_1)$

Observation 11.

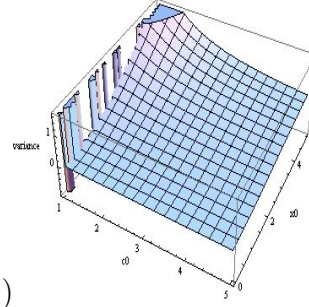
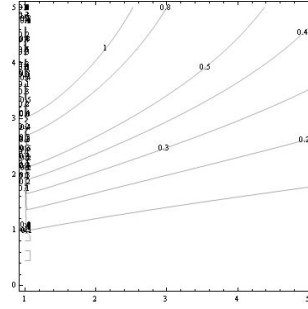
$$\begin{aligned} \lim_{c_1 \rightarrow \infty} \sigma^2(c_0, c_1) &= -\frac{c_0(3 \exp(1 + 1/c_0) + c_0^2 \exp(-2 \exp(1/c_0) + 5 \exp(1/c_0)))}{(-1 + c_0)^2(-c_0 e + \exp(1/c_0))^2} \\ &\quad + \frac{c_0(c_0(\exp(2) - 6 \exp(1 + 1/c_0) - \exp(2/c_0)))}{(-1 + c_0)^2(-c_0 e + \exp(1/c_0))^2} \end{aligned}$$

and Figure (3.16) shows that as c_0 gets bigger, the variance gets smaller. Recall that the mean under $c_1 \rightarrow \infty$ and $c_0 \rightarrow \infty$ goes to zero; hence it makes sense to see less fluctuations around the mean.

Now we discard the assumption $x_0 = 1$ and consider $\sigma^2(c_0, c_1, x_0)$.

$$\text{Observation 12. } \lim_{c_1 \rightarrow \infty} \sigma^2(c_0, c_1, x_0) = \frac{(c_0(-\exp((-1+c_0)x_0/c_0))x_0(2+x_0))}{(-1+c_0)^2(-1+c_0 \exp((-1+c_0)x_0/c_0))^2}$$

$$+ \frac{c_0^2 \exp((-1+c_0)x_0/c_0)(-2+2 \exp((-1+c_0)x_0/c_0) - 2x_0 - x_0^2)}{(-1+c_0)^2(-1+c_0 \exp((-1+c_0)x_0/c_0))^2}$$

FIGURE 3.17. 3-D plot $\lim_{c_1 \rightarrow \infty} \sigma^2(c_0, c_1, x_0)$ FIGURE 3.18. Contour Plot of $\lim_{c_1 \rightarrow \infty} \sigma^2$.

$$+ \frac{c_0(1 - \exp(2(-1 + c_0)x_0/c_0) + 2 \exp(-1 + c_0)x_0/c_0(2 + x_0))}{(-1 + c_0)^2(-1 + c_0 \exp((-1 + c_0)x_0/c_0)^2}$$

and the plots and contour plots are shown in Figure (3.17) and (3.18)

3.3.2. Management Decision: Type I or Type II. For an NPO, comparison of types of spending strategies may be useful in weighing which to choose.

Theorem 3.12. *Given the same dysfunction probability and assuming $c_0 > 1$ and $\beta = \delta = 1$, the expenditure rates must satisfy $1 < c_0 < c < c_1$, where c, c_0, c_1 are the rates defined in Type I and Type II.*

Proof. Let $\pi_0(c_0, c_1, x_0) = \pi_0(c)$. We solve for c_1 .

$$c_1 = \frac{-\beta(c \exp(x_0\delta)\beta + c_0 \exp(x_0\beta/c_0)\delta - cc_0 \exp(x_0\beta/c_0)\delta - c_0 \exp(x_0\delta)\delta)}{\delta(-\exp(x_0\beta/c_0) + c \exp(x_0\beta/c_0)\beta - c \exp(x_0\delta)\beta + c_0 \exp(x_0\delta)\delta)}$$

To simplify the computation, we take $\beta = \delta = 1$, i.e. the average income $\beta/\delta = 1$.

Thus we can simplify c_1 as the following

$$\begin{aligned} c_1 &= -\frac{c \exp(x_0) + c_0 \exp(x_0/c_0) - cc_0 \exp(x_0/c_0) - c_0 \exp(x_0)}{-\exp(x_0/c_0) + c \exp(x_0/c_0) - c \exp(x_0) + c_0 \exp(x_0)} \\ &= -\frac{\exp(x_0)(c - c_0) + c_0 \exp(x_0/c_0)(1 - c)}{(c - 1) \exp(x_0/c_0) + \exp(x_0)(c_0 - c)} \end{aligned}$$

Compute

$$\begin{aligned} c_1 - c &= -\frac{(-1 + c)(c - c_0)(\exp(x_0) - \exp(x_0/c_0))}{-c_0 \exp(x_0) + \exp(x_0/c_0) + c(\exp(x_0) - \exp(x_0/c_0))} \\ &= -\frac{(-1 + c)(c - c_0)(e^{x_0} - e^{\frac{x_0}{c_0}})}{(c - c_0)e^{x_0} + (1 - c)e^{\frac{x_0}{c_0}}} \end{aligned}$$

Since $c_0 > 1$, $\frac{1}{c_0} < 1$. Then $\frac{x_0}{c_0} < x_0$, which implies $\exp(x_0/c_0) < \exp(x_0)$. Also, $-1 + c > 0$ by the equilibrium condition.

If $c_0 > c$, that is $c - c_0 < 0$, we would have

$$c_1 - c < 0$$

which is a contradiction since by the definition of Type II, $c_0 \leq c_1$.

Thus we assume $c_0 < c$. Then the denominator is

$$\begin{aligned} &-c_0 \exp(x_0) + \exp(x_0/c_0) + c(\exp(x_0) - \exp(x_0/c_0)) > \\ &-c_0 \exp(x_0) + c_0 \exp(x_0/c_0) + c_0 \exp(x_0) - \exp(x_0/c_0) \\ &= (c_0 - 1) \exp(x_0/c_0) > 0 \end{aligned}$$

The signs in the denominator stay the same. Hence we have $c_1 - c > 0$. The range is $1 < c_0 < c < c_1$. □

Theorem 3.13. *Given the same dysfunction probability, the mean reserve in Type I can be adjusted to equal to the mean in Type II.*

Proof. Given the same dysfunction probability,

$$c_1 = -\frac{c \exp(x_0) + c_0 \exp(x_0/c_0) - cc_0 \exp(x_0/c_0) - c_0 \exp(x_0)}{-\exp(x_0/c_0) + c \exp(x_0/c_0) - c \exp(x_0) + c_0 \exp(x_0)}$$

Plug c_1 into the expression for mean reserve for Type II and take the difference between μ_1 and μ_2 . We get

$$(3.21) \quad -\frac{(c - c_0)e^{-\frac{x_0}{c_0}}(e^{\frac{x_0}{c_0}}x_0(e^{\frac{x_0}{c_0}}x_0 + c_0(e^{x_0} - e^{\frac{x_0}{c_0}}(1 + x_0)) + c(-e^{-x_0} + e^{\frac{x_0}{c_0}}(1 + (-1 + c_0x_0))))}{(-1 + c)c(-1 + c_0)^2}$$

Now we want to check the sign of the part

$$-ce^{x_0} + c_0e^{x_0} + ce^{\frac{x_0}{c_0}} - c_0e^{\frac{x_0}{c_0}} + e^{\frac{x_0}{c_0}}x_0 - ce^{\frac{x_0}{c_0}}x_0 - c_0e^{\frac{x_0}{c_0}}x_0 + cc_0e^{\frac{x_0}{c_0}}x_0$$

If this were zero, Type I and Type II will have the same mean. In other words, given a fixed c , c_0 and x_0 must also satisfy

$$c = \frac{c_0e^{x_0} - c_0e^{\frac{x_0}{c_0}} + e^{\frac{x_0}{c_0}}x_0 - c_0e^{\frac{x_0}{c_0}}x_0}{e^{x_0} - e^{\frac{x_0}{c_0}} + e^{\frac{x_0}{c_0}}x_0 - c_0e^{\frac{x_0}{c_0}}x_0}$$

□

Theorem 3.14. *Given the same dysfunction probability and the same mean reserve, the variance in Type II is lower than that in Type I for $x_0 > -\frac{\ln(1-\frac{2}{c_0})}{1-\frac{1}{c_0}}$ and $c_0 > 2$.*

Proof. Plug

$$c_1 = -\frac{c \exp(x_0) + c_0 \exp(x_0/c_0) - cc_0 \exp(x_0/c_0) - c_0 \exp(x_0)}{-\exp(x_0/c_0) + c \exp(x_0/c_0) - c \exp(x_0) + c_0 \exp(x_0)}$$

and

$$c = \frac{c_0 e^{x_0} - c_0 e^{\frac{x_0}{c_0}} + e^{\frac{x_0}{c_0}} x_0 - c_0 e^{\frac{x_0}{c_0}} x_0}{e^{x_0} - e^{\frac{x_0}{c_0}} + e^{\frac{x_0}{c_0}} x_0 - c_0 e^{\frac{x_0}{c_0}} x_0}$$

into the difference of the variances

$$(3.22) \quad \sigma_2^2 - \sigma_1^2 = -\frac{e^{\frac{x_0}{c_0}} x_0^2 + (-(e^{x_0} + e^{\frac{x_0}{c_0}})x_0 + c_0(e^{x_0}(-2 + x_0) + e^{\frac{x_0}{c_0}}(2 + x_0)))}{(e^{x_0} - e^{\frac{x_0}{c_0}})(e^{\frac{x_0}{c_0}} x_0 + c_0(e^{x_0} - e^{\frac{x_0}{c_0}}(1 + x_0)))}$$

Multiply by $\frac{e^{\frac{x_0}{c_0}} e^{-x_0}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}}$ through the expression

$$(3.23) \quad (-(e^{x_0} + e^{\frac{x_0}{c_0}})x_0 + c_0(e^{x_0}(-2 + x_0) + e^{\frac{x_0}{c_0}}(2 + x_0)))$$

Expression (3.23) becomes

$$(3.24) \quad \frac{e^{-x_0}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}}(2c_0 - x_0 + c_0 x_0) + \frac{e^{-\frac{x_0}{c_0}}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}}(-2c_0 - x_0 + c_0 x_0)$$

Let $p = \frac{e^{-x_0}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}}$ and $q = \frac{e^{-\frac{x_0}{c_0}}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}}$. Then $p + q = 1$. Moreover $p < q$, since we assume $c_0 > 1$.

We can rewrite Expression (3.24) as

$$p(2c_0 - x_0 + c_0 x_0) + q(-2c_0 - x_0 + c_0 x_0)$$

$$= 2c_0(p - q) - x_0 + c_0 x_0$$

$$= 2c_0(p - q) - (c_0 - 1)x_0$$

Similarly multiply

$$\frac{e^{\frac{x_0}{c_0}} e^{-x_0}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}}$$

through the expression

$$e^{\frac{x_0}{c_0}} x_0 + c_0(e^{x_0} - e^{\frac{x_0}{c_0}}(1 + x_0))$$

Then we have

$$\begin{aligned} & \frac{e^{-\frac{x_0}{c_0}}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}} c_0 + \frac{e^{-x_0}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}} (x_0 - c_0 - x_0 c_0) \\ &= q c_0 + p(x_0 - c_0 - x_0 c_0) \\ &= (q - p) c_0 + x_0 p(1 - c_0) \end{aligned}$$

If $2c_0(p - q) - (c_0 - 1)x_0 > 0$ and $(q - p)c_0 + x_0 p(1 - c_0) > 0$, i.e.

$$(3.25) \quad \frac{2(q - p)c_0}{c_0 - 1} < x_0 < \frac{(q - p)c_0}{p(c_0 - 1)}$$

Note this is valid because $p = \frac{e^{-x_0}}{e^{-\frac{x_0}{c_0}} + e^{-x_0}} < \frac{e^{-x_0}}{e^{-x_0} + e^{-x_0}} = \frac{1}{2}$. Then $\sigma_1^2 - \sigma_2^2 > 0$.

Now it remains to solve explicitly the inequality (3.25). Plug back p , q and gather x_0 -terms on one side.

$$(3.26) \quad \frac{e^{-\frac{x_0}{c_0}} - e^{-x_0}}{(e^{-\frac{x_0}{c_0}} + e^{-x_0})x_0} < \frac{c_0 - 1}{2c_0}$$

$$(3.27) \quad \frac{e^{-\frac{x_0}{c_0}} - e^{-x_0}}{e^{-x_0} x_0} > \frac{c_0 - 1}{c_0}$$

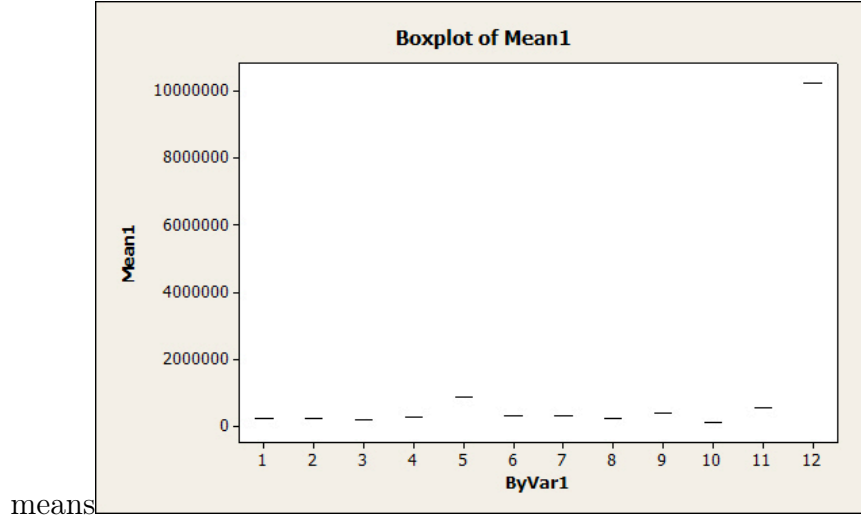
In order to solve (3.26), multiply by $\frac{e^{-x_0}}{e^{-x_0}}$ and notice the left hand side is the newton quotient. Let $k = 1 - \frac{1}{c_0}$. Note that $k > 0$. Then we have $ke^{x_0 k} > 1$, i.e., $x_0 > -\frac{\ln(1 - \frac{1}{c_0})}{1 - \frac{1}{c_0}}$.

To solve (3.27), we multiply the left hand side by $\frac{e^{x_0}}{e^{x_0}}$, then it becomes $\frac{1-e^{-kx_0}}{1+e^{-kx_0}} = \frac{e^{-kx_0}-e^0}{\frac{-kx_0}{1+e^{-kx_0}}}$. Solving for x_0 , we get $x_0 > -\frac{\ln(2k-1)}{k}$. Since $c_0 > 0$, $k-1 < 0$, then $-\ln(2k-1) > -\ln(k)$. Thus when $x_0 > -\frac{\ln(2k-1)}{k} = -\frac{\ln(1-\frac{2}{c_0})}{1-\frac{1}{c_0}}$ and $c_0 > 2$, the variance under Type II is smaller than that in Type I. \square

Remark 3.15. The Type II scheme allows the NPO the opportunity to spend on large projects. This is consistent with how current NPOs design their budgets. In an interview with a chairman of a local library¹, he said that if the library is going to spend lots of money on a big project, they will spend less before implementing the project and spend lot more for the project.

¹The short interview with Mr. Keane McGee took place after the Lonseth Lecture on May 11, 2010 at Oregon State University.

FIGURE 4.1. Plot on monthly expenditure



4. CASE STUDY ON A NONPROFIT ORGANIZATION IN CHINA

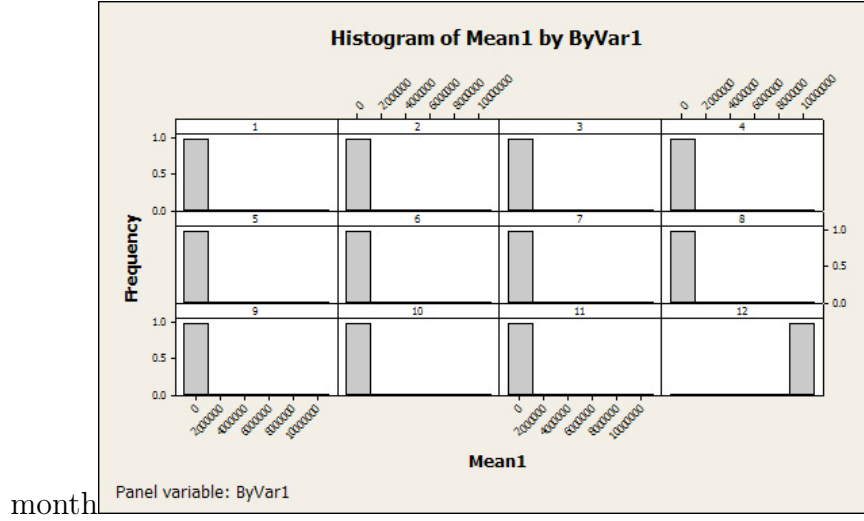
In this section, we use the data of expenditures² for disaster relief from a Chinese NPO *China Youth Foundation*. The expenditures were the rescue aid for the heavily damaged regions due to the 2008 Great Sichuan Earthquake³. We compute the average expenditures ct of each month and then by using the formula developed to compute the average income this NPO should have in order to stay active.

We only consider the first four months' expenditures and regard the spending as Type I. The expenditure rate c is approximately 8080 RMB per day. If a dysfunction risk measure is given, say 8%, i.e. the chance of running out of money is 8%, then by the formula $\pi_0 = 1 - \frac{\beta}{\delta c}$, we can compute the average amount of income rate $\frac{\beta}{\delta} = (1 - \pi_0)c = 92\% \cdot 8080 = 7433$. Then for the first four months, this NPO needs to raise 7433 RMB per day.

²Data from http://jiuzai.cctf.org.cn/sys/html/lm_33/2010-02-09/141453.htm

³The 8.0 Mc earthquake occurred on May 12, 2008 in Sichuan Province of China and killed at least 68,000 people.

FIGURE 4.2. Histogram on expenditures per



5. FUTURE RESEARCH

Under the NPO model $R_t = u + S_t - \int_0^t r(R_s)ds$, we only concerned the expenditure rates as constants when comparing spending strategies. We intend to expand the results to more general expenditure functions. We also want to investigate the behaviors of the variances under Type I and II if we set x_0 in a different range, for example $0 < x_0 < -\frac{\ln(1-\frac{1}{c_0})}{(1-\frac{1}{c_0})}$.

Our model in this paper is for NPOs' surplus with no investment. Nowadays more NPOs invest in both risk-free and risky assets. We develop the NPO model with risk-free investment at an interest rate r and a constant spending rate c as

$$(5.1) \quad R(t) = ue^{rt} + \sum_{j=1}^{N(t)} D_j \exp(r(t - T_j) - \frac{c}{r}(\exp(rt) - 1))$$

The first term means investing the initial capital u at time 0 and the money will accumulate to ue^{rt} at time t . The second term denotes the donation values arriving at different times and invested continuously at $(t - T_j)$, where T_j is the j^{th} donation

arrival time. The third term $c \int_0^t e^{rs} ds = c \frac{e^{rt}-1}{r}$ implies that expenditures will be continuously flowing out, thus the interest return will also be missing.

It is also natural to modify Equation (5.1) to model the surplus of an NPO, which invests all the donations continuously into a risky financial market whose price follows a nonnegative stochastic process.

REFERENCES

- [1] Asmussen, Soren. Ruin Probabilities expressed in Terms of Storage Process. *Advanced Applied Probability*. 1988: 913-916.
- [2] Asmussen, Soren. *Applied Probability and Queues*. Second Ed. Springer-Verlag, 2003.
- [3] Bhattacharya, R and E Waymire. *Stochastic Process with Applications*. SIAM Classics in Applied Mathematics. Vol 61. (Originally published by John Wiley and Sons, 1992).
- [4] Billingsley, Patrick. *Probability and Measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons. 1995.
- [5] Bowers, Newton, Hans Gerber, James Hickman, Donald Jones, Cecil Nesbitt. *Actuarial Mathematics*. The society of Actuaries. 1997.
- [6] Center for Civil Society Studies. *Measuring Civil Society and Volunteering: Initial Findings from Implementation of the UN Handbook on Nonprofit Institutions*. Johns Hopkins University. 2007.
- [7] Erhun, Feryal and Natalie Privett. *Efficient Funding: Auditing in the Nonprofit Sector*. 2008.
- [8] D. Gammal, C. Simard, H. Hwang and W. Powell. *Managing through challenges: A profile of San Francisco Bay Area nonprofits*. 2005.
- [9] Gerber, Hans. *Mathematical Fun with the Compound Binomial Process*. 1988.
- [10] Grimmett, G. R and D. R Stirzaker. *Probability and Random Process*. Second Ed. Clarendon Press, Oxford. 1992.
- [11] Harrison, Micheal J. and Sidney I. Resnick. The Stationary Distribution and First Exit Probabilities of a Storage Process with General Release Rule. *Mathematics of Operations Research*. Vol 1. No. 4. November, 1976.
- [12] Hansmann, Henry. The Role of Nonprofit Enterprise. *The Yale Law Journal*. Vol 89, number 5. April 1980.
- [13] Jalandoni, N., M. Weitzman, L. Lampkin and T. Pollak. *New Nonprofit Almanac & Desk Reference*. Jossey-Bass, 2002.
- [14] Kass, Rob, Marc Goureaerts and Jau Dhaene. *Modern Actuarial Risk Theory*. Kluwer Academic Publishers, New York, 2002.

- [15] Rose-Ackerman, Susan. Altruism, Nonprofits and Economic Theory. *Journal of Economic Literature*. Vol. XXXIV. June 1996: 701-723.
- [16] Valentinov, Vladislav. Nonprofit Organization and the Division of Labor: A theoretical Perspective. *Atlantic Economic Journal*. 2006: 435-447.