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Abstract approved
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In this paper we present a model of Non-Euclidean geometry in three dimensions. This will show that the axioms of Non-Euclidean geometry are consistent if Euclidean geometry and, hence, arithmetic is consistent. However, the model is incomplete for we have not included the topic of congruence, the axiom of Archimedes, nor the axiom of completeness.

A MODEL OF NON-EUCLIDEAN GEOMETRY IN THREE DIMENSIONS

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TABLE OF CONTENTS

A MODEL OF NON-EUCLIDEAN GEOMETRY IN THREE DIMENSIONS

INTRODUCTION

The object of this work is to show that the axioms of Non-Euclidean geometry are consistent (i. e., do not lead to a contradiction) if the axioms of arithmetic are consistent. Since the axioms of Euclidean geometry are consistent if the axioms of arithmetic are consistent, these are also taken as axioms.

For the axioms of arithmetic and Euclidean geometry we shall take the well known axioms of Peano and Hilbert respectively. For the axioms of our Non-Euclidean geometry we shall take the axioms of Hilbert with his axiom of parallels replaced by the following axiom:

Axiom of Parallels. Let ℓ be a line and P a point not on ℓ. Then in the plane determined by P and ℓ there is more than one line passing through P parallel to ℓ.

To show the consistency of our Non-Euclidean geometry we exhibit a model based upon the algebra of the real number and Euclidean geometry. Once a model is constructed then any contradiction in our Non-Euclidean geometry will necessarily involve a contradiction in Euclidean geometry, assuming of course, that algebra is consistent. Since models of Euclidean geometry have been made (see [1, 3]), the creation of a model for Non-Euclidean geometry of the above type will show the equivalence of the problem of consistency for Euclidean geometry. The uniqueness of a proof of the consistency of the axioms of Non-Euclidean geometry is not included. The statement is probably false. However, the virtue of the present development is that it is applicable directly to the three-dimensional case. It should be expected that the ideas and concepts contained in this work will be understood in part, if not in whole, by a student with a modern high school education in mathematics.

In our model we shall assume familiarity with the usual three-dimensional rectangular coordinate system of Euclidean analytic geometry. We shall refer to this system as the xyz system. It shall be understood that the terms point, line and plane will refer to Non-Euclidean points, lines and planes. When we wish to speak of these concepts in the usual Euclidean sense then we shall prefix these terms by the word Euclidean.

A point in our model is an Euclidean point contained in an Euclidean sphere of unit radius with center at the origin. A plane in our model is the set of Euclidean points of an Euclidean sphere orthogonal to the Euclidean unit sphere and contained in the Euclidean unit sphere. A line in our model is the set of points of the intersection of two Euclidean spheres orthogonal to the Euclidean unit sphere and contained in the Euclidean unit sphere.

Because of lack of time, the discussion does not include the topic of congruence and the two undefined terms associated therewith. The axiom of Archimedes and the axiom of completeness also are not discussed. These omissions should be completed to make a thorough study.

I. AXIOMS OF CONNECTION

Metadef. 1. A <u>point</u> is an ordered triple of real numbers (x, y, z) such that $0 \leq x^2+y^2+z^2 < 1$, where x, y and z are real variables.

Metadef. 2. A <u>plane</u> is an equivalence class of equations of the form $D(x^2+y^2+z^2+1)+Ax+By+Cz = 0$ such that $A^2 + B^2 + C^2 > 4D^2$ where D, A, B and C are real constants and x, y and z are real variables. By analogy with the practice in elementary analytic geometry, we shall refer to any equation of an equivalence class as "the" equation of that equivalence class, although admittedly this practice is grammatically reprehensible.

Lemma 1. There exists a point that satisfies the equation of a plane.

Proof. Let a plane Π be represented by the equation $D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$. Then if $D = 0$, the point $(0, 0, 0)$ satisfies the equation of Π. If $D \neq 0$, then the point (x', y', z') satisfies the equation of the plane where

$$x' = \frac{-A}{2D} + AK,$$

$$y' = \frac{-B}{2D} + BK,$$

$$z' = \frac{-C}{2D} + CK$$

and

$$K = \frac{[(A^2+B^2+C^2)(A^2+B^2+C^2-4D^2)]^{\frac{1}{2}}}{2D(A^2+B^2+C^2)} .$$

Metadef. 3. A <u>line</u> is the equivalence class of pairs of equations of the form,

$$(1) \quad \begin{Bmatrix} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 \\ D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 \end{Bmatrix},$$

where D, A, B, C, D', A', B' and C' are real constants and x, y and z are real variables such that the following is true:

(1) There exists at least one point that satisfies the above system (1) of equations,

(2) $A^2+B^2+C^2 > 4D^2$ and $A'^2 + B'^2 + C'^2 > 4D'^2$,

(3) The rank of the following matrix is two:

$$\begin{vmatrix} D & A & B & C \\ D' & A' & B' & C' \end{vmatrix} .$$

As above we shall use the grammatically incorrect terminology of "the" equation of a line.

Lemma 2. If $P_1:(x_1, y_1, z_1)$, $P_2:(x_2, y_2, z_2)$ and $P_3:(x_3, y_3, z_3)$ are three distinct points, then the rank of

$$M = \begin{pmatrix} x_1^2+y_1^2+z_1^2+1 & x_1 & y_1 & z_1 \\ x_2^2+y_2^2+z_2^2+1 & x_2 & y_2 & z_2 \\ x_3^2+y_3^2+z_3^2+1 & x_3 & y_3 & z_3 \end{pmatrix}$$

is greater than one.

Proof. Clearly the rank of M is not zero. If the rank of M is one, then one of the rows is a linear combination of another row. We may assume, without loss of generality, that $x_1 = kx_2$, $y_1 = ky_2$, $z_1 = kz_2$ and $x_1^2+y_1^2+z_1^2+1 = k(x_2^2+y_2^2+z_2^2+1)$. By substitution, $k^2(x_2^2+y_2^2+z_2^2) + 1 = k(x_2^2+y_2^2+z_2^2+1)$. Simplifying we have, $rk^2 - k(r+1) + 1 = 0$, where $r = x_2^2+y_2^2+z_2^2$. Solving for k we find that $k = 1$ or $k = \frac{1}{r}$. But $k = 1$ implies $P_1 = P_2$, which implies a contradiction. If $k = \frac{1}{r}$, then we have $x_1^2+y_1^2+z_1^2 > 1$, which also implies a contradiction. Hence the rank of M is greater than one.

Metadef. 4. A line ℓ,

$$\ell: \begin{cases} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 \\ D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 \end{cases},$$

is <u>incident upon a pair of distinct points</u> $P_1:(x_1, y_1, z_1)$ and $P_2:(x_2, y_2, z_2)$ iff

$$D(x_1^2+y_1^2+z_1^2+1) + Ax_1 + By_1 + Cz_1 = 0,$$

$$D(x_2^2+y_2^2+z_2^2+1) + Ax_2 + By_2 + Cz_2 = 0,$$

$$D'(x_1^2+y_1^2+z_1^2+1) + A'x_1 + B'y_1 + C'z_1 = 0,$$

$$D'(x_2^2+y_2^2+z_2^2+1) + A'x_2 + B'y_2 + C'z_2 = 0 .$$

Theorem 1. There is one and only one line incident upon two distinct points $P_1:(x_1, y_1, z_1)$ and $P_2:(x_2, y_2, z_2)$.

Proof. We shall first show that if such a line exists, it is unique. Suppose, therefore, that a line ℓ represented by the pair of equations,

$$D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 ,$$

$$D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 ,$$

is incident upon P_1 and P_2. Then we have the following system of equations on the coefficients:

$$(1) \quad \cdots \quad D(x_1^2+y_1^2+z_1^2+1) + Ax_1 + By_1 + Cz_1 = 0$$

$$(2) \quad \cdots \quad D(x_2^2+y_2^2+z_2^2+1) + Ax_2 + By_2 + Cz_2 = 0$$

$$(3) \quad \cdots \quad D'(x_1^2+y_1^2+z_1^2+1) + A'x_1 + B'y_1 + C'z_1 = 0$$

$$(4) \quad \cdots \quad D'(x_2^2+y_2^2+z_2^2+1) + A'x_2 + B'y_2 + C'z_2 = 0 \ .$$

In lemma 2 we showed that no two rows in the matrix M were proportional provided the points were distinct. Therefore, in the above system, for each equation in the first pair we find that the coefficients of the unknowns D, A, B and C are not proportional. Hence the rank of the coefficient matrix is two. The same is true of the coefficient matrix of the second pair of equations. Therefore, we may, by using Cramer's Rule, solve for two of the variables in the first and second pair of equations above in terms of the remaining two. Once values are assigned and the variables determined, a pair of equations in the equivalence class of equations defining ℓ, will be uniquely determined.

To show that a line ℓ exists we write a pair of equations satisfied by P_1 and P_2. Let ℓ be defined as follows:

$$\ell : \begin{cases} (y_2x_1 - y_1x_2)(x^2+y^2+z^2+1) + [y_1(x_2^2+y_2^2+z_2^2+1) - y_2(x_1^2+y_1^2+z_1^2+1)]x \\ \quad + [x_2(x_1^2+y_1^2+z_1^2+1) - x_1(x_2^2+y_2^2+z_2^2+1)]y = 0, \\ (z_2x_1 - z_1x_2)(x^2+y^2+z^2+1) + [z_1(x_2^2+y_2^2+z_2^2+1) - z_2(x_1^2+y_1^2+z_1^2+1)]x \\ \quad + [x_2(x_1^2+y_1^2+z_1^2+1) - x_1(x_2^2+y_2^2+z_2^2+1)]z = 0 \end{cases}$$

Definition 1. A point P is a point _of_ a line iff there exists a point Q such that the line is incident upon P and Q. We shall also say P is a point _on_ a line or _in_ the line, _et cetera_.

Theorem 2. Two distinct points $P_1:(x_1, y_1, z_1)$ and $P_2:(x_2, y_2, z_2)$ of a line ℓ, have ℓ incident upon them.

Proof. It is unique by theorem 1. But then P_1 and P_2 satisfy independently the equations of this line and they also satisfy the equations of ℓ since ℓ is this line.

Theorem 3a. There exist at least two distinct points on a line.

Proof. Let a line ℓ be defined as follows:

$$\ell : \quad \left\{ \begin{array}{l} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 \\ \\ D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 \end{array} \right\} .$$

We shall consider two cases.

Case 1. $D = D' = 0$. Therefore the equations defining ℓ are the pair

$$Ax + By + Cz = 0,$$

$$A'x + B'y + C'z = 0$$

with the rank of the coefficient matrix equal to two.

$$\begin{pmatrix} A & B & C \\ & & \\ A' & B' & C' \end{pmatrix}$$

Clearly the point $(0,0,0)$ is on ℓ. Another point on ℓ, is (x', y', z') where

$$x' = \frac{\begin{vmatrix} B & C \\ B' & C' \end{vmatrix}}{2[(BC'-B'C)^2+(CA'-AC')^2+(AB'-BA')^2]^{\frac{1}{2}}}$$

$$y' = \frac{\begin{vmatrix} C & A \\ C' & A' \end{vmatrix}}{2[(BC'-B'C)^2+(CA'-AC')^2+(AB'-BA')^2]^{\frac{1}{2}}}$$

$$z' = \frac{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}{2[(BC'-B'C)^2+(CA'-AC')^2+(AB'-BA')^2]^{\frac{1}{2}}}$$

Case 2. At least one of D or D' is non-zero. Assume $D \neq 0$. Then the equations defining ℓ are equivalent to

$$x^2+y^2+z^2+1 + 2ax + 2by + 2cz = 0$$

$$a'x + b'y + c'z = 0$$

where $2a = \frac{A}{D}$, $2b = \frac{B}{D}$, $2c = \frac{C}{D}$ and if $D' = 0$, then $A' = a'$, $B' = b'$ and $C' = c'$. If $D' \neq 0$, then $a' = \frac{A}{D} - \frac{A'}{D'}$, $b' = \frac{B}{D} - \frac{B'}{D'}$ and $c' = \frac{C}{D} - \frac{C'}{D'}$. A point (x_1, y_1, z_1) is on ℓ, where,

$$x_1 = k - \frac{k(\Phi^2 - \Phi)^{\frac{1}{2}}}{\Phi} ,$$

$$y_1 = m - \frac{m(\Phi^2 - \Phi)^{\frac{1}{2}}}{\Phi} ,$$

$$z_1 = n - \frac{n(\Phi^2 - \Phi)^{\frac{1}{2}}}{\Phi} ,$$

in which $u_0 = \dfrac{aa' + bb' + cc'}{a'^2 + b'^2 + c'^2}$, $k = -a + a'u_0$, $m = -b+b'u_0$, $n = -c+c'u_0$ and $\Phi = k^2+m^2+n^2$.

The point (x_2, y_2, z_2) is also on ℓ in which

$$x_2 = \begin{cases} k + \dfrac{(p-k)(\Phi^2-1)^{\frac{1}{2}}}{1+\Phi} & \text{if } k(p-k) \le 0 \\[2ex] k - \dfrac{(p-k)(\Phi^2-1)^{\frac{1}{2}}}{1+\Phi} & \text{if } k(p-k) > 0, \end{cases}$$

$$y_2 = \begin{cases} m + \dfrac{(q-m)(\Phi^2-1)^{\frac{1}{2}}}{1+\Phi} & \text{if } m(q-m) \le 0 \\[2ex] m - \dfrac{(q-m)(\Phi^2-1)^{\frac{1}{2}}}{1+\Phi} & \text{if } m(q-m) > 0, \end{cases}$$

$$z_2 = \begin{cases} n + \dfrac{(r-n)(\Phi^2-1)^{\frac{1}{2}}}{1+\Phi} & \text{if } n(r-n) \le 0 \\[2ex] n - \dfrac{(r-n)(\Phi^2-1)^{\frac{1}{2}}}{1+\Phi} & \text{if } n(r-n) > 0, \end{cases}$$

with

$$p = \frac{(c'b-b'c)}{[(c'b-b'c)^2+(a'c-c'a)^2+(ab'-a'b)^2]^{\frac{1}{2}}},$$

$$q = \frac{(a'c-c'a)}{[(c'b-b'c)^2+(a'c-c'a)^2+(ab'-a'b)^2]^{\frac{1}{2}}},$$

$$r = \frac{(ab'-a'b)}{[(c'b-b'c)^2+(a'c-c'a)^2+(ab'-a'b)^2]^{\frac{1}{2}}}.$$

Figure 1 is a picture of how the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) of theorem 3a were constructed. We take the surface of the paper as the Euclidean plane $a'x + b'y + c'z = 0$. This Euclidean plane contains, as a subset, the points interior to the Euclidean unit sphere on it. The point O is the center or origin of our model, and hence the center of the Euclidean unit sphere, whose intersection with the Euclidean plane with the equation $a'x + b'y + c'z = 0$ is shown. The center of the Euclidean sphere $x^2+y^2+z^2+1 + 2ax + 2by + 2cz = 0$ is the Euclidean point C, which is on the Euclidean plane $a'x + b'y + c'z = 0$. The point P_1 corresponds to the point (x_1, y_1, z_1) of the theorem. P_1 is the intersection of the Euclidean line from O to C and the Euclidean sphere with center at C. The Euclidean line ℓ' is perpendicular to the Euclidean line from O to C and lies in the Euclidean plane $a'x + b'y + c'z = 0$. The intersection of ℓ' with the Euclidean unit sphere determines the Euclidean points Q_1 and Q_2. The Euclidean line from C to either Q_1 or Q_2 intersects the Euclidean sphere whose center is C at the points P_2 or P_2' respectively. One of these points corresponds to the point (x_2, y_2, z_2) of the theorem.

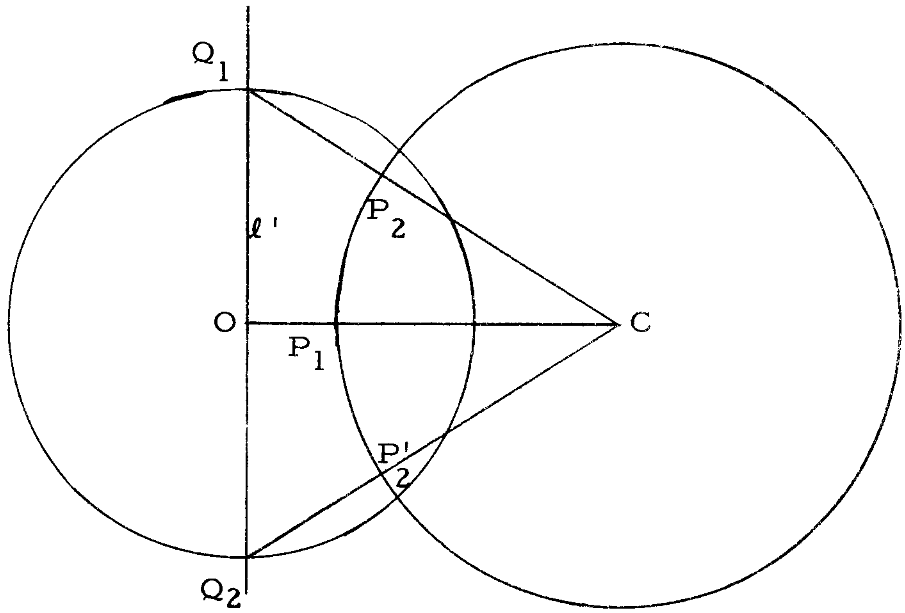

Figure 1

Definition 2. A point P is a point of a plane Π iff there exist two other points Q and R such that P, Q and R are not on the same line and such that the plane Π is incident upon the three points. We shall also say that the point is on or in the plane, et cetera. A line is a line of or in a plane, et cetera, iff every point of the line is a point of the plane.

Theorem 3b. There exist three distinct points on every plane such that the three points are not on any one line.

Proof. Let $D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$ be an equation of the plane. Since $A^2 + B^2 + C^2 > 4D^2$, at least one of A, B or C must be different from zero. Assume $A \neq 0$. We then define two lines, ℓ_1 and ℓ_2 of the plane as follows:

$$\ell_1: \left\{ \begin{aligned} D(x^2+y^2+z^2+1) + Ax + By + Cz &= 0 \\ y + z &= 0 \end{aligned} \right\}$$

$$\ell_2: \left\{ \begin{aligned} D(x^2+y^2+z^2+1) + Ax + By + Cz &= 0 \\ x^2 + y^2 + z^2 + 1 + 2y + z &= 0 \end{aligned} \right\}$$

Since no point with $y = -z$ is on ℓ_2 the lines are distinct. From theorem 3a there exist two distinct points on ℓ_1. Let $P_1: (x_1, y_1, -y_1)$ and $P_2: (x_2, y_2, -y_2)$ be two points with either $x_1 \neq x_2$ or $y_1 \neq y_2$ on ℓ_1. Let $P_3: (x_3, y_3, z_3)$ be a point on ℓ_2. P_1, P_2 and P_3 are all points of the plane since they must satisfy the equation of the plane.

Theorem 3c. There exists a plane.

Proof. $x + y + z = 0$ is a plane.

Definition 3. If every point of a set of points is a point of one and only one line, the points are called collinear, the set a collinear set; if every point (line) of a set of points (lines) is a point (line) of one and the same plane, they are called coplanar, and the set a coplanar set.

Lemma 3. The rank of the matrix M in Lemma 2 is two iff the three points are collinear.

Proof. The rank of M can only be two or three. If the rank of M is two then there are two rows that are not proportional (assume they are the first two), and the third row is a linear combination of the first two. That is, the four equations

$$x_3 = t_1x_1 + t_2x_2,$$

$$y_3 = t_1y_1 + t_2y_2,$$

$$z_3 = t_1z_1 + t_2z_2,$$

$$x_3^2+y_3^2+z_3^2+1 = t_1(x_1^2+y_1^2+z_1^2+1) + t_2(x_2^2+y_2^2+z_2^2+1)$$

are simultaneously satisfied.

Consider the line ℓ determined by P_1 and P_2. Since P_3 is a linear combination of P_1 and P_2 it is easily seen that P_3 is also a point of ℓ, and hence the three points are collinear.

To show that if the three points are collinear then the rank of

M is two, we shall show that if the rank of M is three then the points are not collinear.

Let the rank of M be three. Let a line ℓ be defined as follows:

$$\ell: \begin{cases} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 \\ D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 \end{cases}$$

If the three points all lie on ℓ then, since they satisfy both equations defining ℓ, it follows that the rank of the matrix

$$\begin{vmatrix} D & A & B & C \\ D' & A' & B' & C' \end{vmatrix}$$

is not two. But this implies a contradiction. Hence the three points are not collinear.

In the proof of Lemma 3 we have also deduced the fact that the rank of M is three iff the three points are not collinear, in which case they are called non-collinear.

Metadef. 5. A plane $D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$ is <u>incident upon three distinct, non-collinear points</u> $P_1: (x_1, y_1, z_1)$, $P_2: (x_2, y_2, z_2)$ and $P_3: (x_3, y_3, z_3)$ iff

$$D(x_1^2+y_1^2+z_1^2+1) + Ax_1 + By_1 + Cz_1 = 0$$

$$D(x_2^2+y_2^2+z_2^2+1) + Ax_2 + By_2 + Cz_2 = 0$$

$$D(x_3^2+y_3^2+z_3^2+1) + Ax_3 + By_3 + Cz_3 = 0 .$$

Theorem 4. There exists one and only one plane incident upon three non-collinear points, $P_1: (x_1, y_1, z_1)$, $P_2: (x_2, y_2, z_2)$ and $P_3: (x_3, y_3, z_3)$.

Proof. Consider the system of equations

$$D(x_1^2+y_1^2+z_1^2+1) + Ax_1 + By_1 + Cz_1 = 0$$

$$D(x_2^2+y_2^2+z_2^2+1) + Ax_2 + By_2 + Cz_2 = 0$$

$$D(x_3^2+y_3^2+z_3^2+1) + Ax_3 + By_3 + Cz_3 = 0 .$$

Since the rank of the coefficient matrix of the unknowns D, A, B and C is three, the system has the following solution:

$$D = k \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \quad A = k \begin{vmatrix} x_1^2+y_1^2+z_1^2+1 & z_1 & y_1 \\ x_2^2+y_2^2+z_2^2+1 & z_2 & y_2 \\ x_3^2+y_3^2+z_3^2+1 & z_3 & y_3 \end{vmatrix},$$

$$B = k \begin{vmatrix} x_1^2+y_1^2+z_1^2+1 & x_1 & z_1 \\ x_2^2+y_2^2+z_2^2+1 & x_2 & z_2 \\ x_3^2+y_3^2+z_3^2+1 & x_3 & z_3 \end{vmatrix},$$

$$C = k \begin{vmatrix} x_1^2+y_1^2+z_1^2+1 & y_1 & x_1 \\ x_2^2+y_2^2+z_2^2+1 & y_2 & x_2 \\ x_3^2+y_3^2+z_3^2+1 & y_3 & x_3 \end{vmatrix},$$

where k is any real number, and at least one of D, A, B and C is non-zero. For any value of $k \neq 0$, the constants are uniquely determined and satisfy Metadef. 2.

Theorem 5. Any three, distinct, non-collinear points of a plane $\Pi : D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$, have Π incident upon them.

Proof. The three points $P_1: (x_1, y_1, z_1)$, $P_2: (x_2, y_2, z_2)$ and $P_3: (x_3, y_3, z_3)$ satisfy

$$D(x_1^2+y_1^2+z_1^2+1) + Ax_1 + By_1 + Cz_1 = 0$$

$$D(x_2^2+y_2^2+z_2^2+1) + Ax_2 + By_2 + Cz_2 = 0$$

$$D(x_3^2+y_3^2+z_3^2+1) + Ax_3 + By_3 + Cz_3 = 0$$

where Π is the plane in question.

Theorem 6. If two distinct points P_1 and P_2 of a line are points of a plane then every point of the line is a point of the plane.

Proof. Let ℓ be a line incident upon P_1 and P_2. By theorem 2 this line is unique, hence one of the equations in the pair of equations defining ℓ is the equation of the plane. Since all points of ℓ satisfy both equations defining ℓ they satisfy the equation of the plane and hence are on the plane.

Theorem 7. If a point P is a point of each of two distinct planes then there exists another point of these same two planes.

Proof. Let Π: $D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$ and Π': $D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0$ be two distinct planes with

P on each. Let ℓ be the line represented by the equations

$$\ell: \quad \begin{cases} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 \\ D'(x^2+y^2+z^2+1)+A'x+B'y+C'z = 0 \end{cases} .$$

By theorem 3a, there exist at least two distinct points of ℓ, hence there exists another point on ℓ distinct from P. This point satisfies both equations of ℓ and hence is a point of Π as well as of Π'.

Theorem 8. There exist four points that are not all on the same plane.

Proof. The points $P_1: (0,0,0)$, $P_2: (\frac{1}{2}, 0, 0)$, $P_3: (0, \frac{1}{2}, 0)$ and $P_4: (0, 0, \frac{1}{2})$ are not all in the same plane. For if there exists a plane $\Pi: D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$ incident upon the four points, then P_1 on Π implies $D = 0$, P_2 on Π implies $2A = -5D = 0$, P_3 on Π implies $2B = -5D = 0$, and P_4 on Π implies $2C = -5D = 0$. But then $A^2 + B^2 + C^2 = 4D^2$ which implies a contradiction.

II. AXIOMS OF ORDER

Lemma 4. There exist real, monotone functions $f(t)$, $g(t)$ and $h(t)$ such that the set of points on a line ℓ,

$$\ell: \quad \left\{ \begin{array}{l} D(x^2+y^2+z^2+1) + Ax + By + Cz = 0 \\ D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0 \end{array} \right\}$$

is the same as the set of points (x, y, z) such that

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

where t is an element of an open subset of the real numbers.

Proof. We shall consider two cases.

Case 1. At least one of D or D' is non-zero. Assume $D \neq 0$. The equations defining ℓ are equivalent to

$$\ell_1: \quad \left\{ \begin{array}{r} x^2+y^2+z^2+1 + 2ax + 2by + 2cz = 0 \\ a'x + b'y + c'z = 0 \end{array} \right\}$$

where $2a = \frac{A}{D}$, $2b = \frac{B}{D}$, $2c = \frac{C}{D}$ and if $D' = 0$, then $A' = a'$, $B' = b'$ and $C' = c'$. If $D' \neq 0$, then $a' = \frac{A}{D} - \frac{A'}{D'}$,

$b' = \frac{B}{D} - \frac{B'}{D'}$ and $c' = \frac{C}{D} - \frac{C'}{D'}$. We shall call the second plane in the representation of ℓ_1, the r-plane of ℓ_1, and note that all points of ℓ_1 are on the r-plane. We now translate our xyz coordinate system to the Euclidean point (x_c, y_c, z_c), called the center of ℓ_1 where

$$x_c = -a + a'u_0$$

$$y_c = -b + b'u_0$$

$$z_c = -c + c'u_0$$

with $u_0 = \frac{aa' + bb' + cc'}{a'^2 + b'^2 + c'^2}$. We note further that the center of ℓ_1 is not a point in the model for $x_c^2+y_c^2+z_c^2 > 1$. The equations of our translation are

$$(1) \quad \begin{aligned} x' &= x + a - a'u_0 \\ y' &= y + b - b'u_0 \\ z' &= z + c - c'u_0 \end{aligned} \qquad (2) \quad \begin{aligned} x &= x' - a + a'u_0 \\ y &= y' - b + b'u_0 \\ z &= z' - c + c'u_0 \end{aligned}$$

System (1) transforms the coordinates of an Euclidean point and hence a point in our geometry, in the xyz system into the coordinates of an Euclidean point in the x'y'z' system, while system (2) transforms the coordinates in the reverse direction. The equations defining our original line in the x'y'z' system is the equivalence class

$$x'^2+y'^2+z'^2+2a'u_0x'+2b'u_0y'+2c'u_0z' = a^2+b^2+c^2-1-u_0(aa'+bb'+cc'),$$

$$a'x' + b'y' + c'z' = 0 \ .$$

If either a' or b' are non-zero then we rotate our $x'y'z'$ system by an Euclidean rotation so that the z'-axis goes through the Euclidean point $(-a'u_0, \ -b'u_0, \ -c'u_0)$. The equations in matrix form of the Euclidean rotation are

$$\begin{pmatrix} \frac{-a'^2c' + b'R}{R(c'^2-R^2)} & \frac{a'b'}{R(-c'+R)} & \frac{-a'}{R} \\ \frac{a'b'}{R(-c'+R)} & \frac{a'^2R-b'^2c'}{R(c'^2-R^2)} & \frac{-b'}{R} \\ \frac{-a'}{R} & \frac{-b'}{R} & \frac{-c'}{R} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$

with $R = (a'^2+b'^2+c'^2)^{\frac{1}{2}}$.

If $a' = b' = 0$ then there is no need to rotate the $x'y'z'$ system of coordinates. In any case the equations defining our line in the $x''y''z''$ system are equivalent to

$$x''^2+y''^2+z''^2+2u_0Rz'' = a^2+b^2+c^2-1-u_0(aa'+bb'+cc'),$$

$$z'' = 0.$$

We now let

$$(3) \quad \begin{aligned} x'' &= r\cos t \\ y'' &= r\sin t \\ z'' &= 0 \end{aligned}$$

where $r = [-1+a^2+b^2+c^2 - \frac{(-aa'-bb'-cc')^2}{a'^2+b'^2+c'^2}]^{\frac{1}{2}}$ and t is a real number such that $t_m < t < t_n$ if $t_m < t_n$ or $t_n < t < t_m$ if $t_n < t_m$, with t_m and t_n determined as follows.

There are two distinct Euclidean points $P_n:(x_n, y_n, z_n)$ $P_m:(x_m, y_m, z_m)$ whose coordinates satisfy the equation $x^2+y^2+z^2 = 1$ and the line ℓ. Therefore they satisfy the following system of independent equations

$$a'x + b'y + c'z = 0$$

$$ax + by + cz = -1.$$

This implies that at least one of $b'c-bc'$, $ac'-a'c$ and $a'b-ab'$ is non-zero. Assume $ac'-a'c \neq 0$. We have on solving the system for the coordinates of P_m and P_n that

$$x_m = \frac{-c'}{\lambda} + \phi v_m \qquad x_n = \frac{-c'}{\lambda} + \phi v_n$$

$$y_m = \lambda v_m \qquad y_n = \lambda v_n$$

$$z_m = \frac{a'}{\lambda} - \theta v_m \qquad z_n = \frac{a'}{\lambda} - \theta v_n$$

where $\lambda = c'a-ca'$, $\phi = c'b-cb'$, $\theta = a'b-ab'$ and

$$v_m = \frac{(c'\phi+a'\theta)+[(c'\phi+a'\theta)^2-(\phi^2+\lambda^2+\theta^2)(c'^2+a'^2-\lambda^2)]^{\frac{1}{2}}}{\lambda(\phi^2+\lambda^2+\theta^2)}$$

$$v_n = \frac{(c'\phi+a'\theta)-[(c'\phi+a'\theta)^2-(\phi^2+\lambda^2+\theta^2)(c'^2+a'^2-\lambda^2)]^{\frac{1}{2}}}{\lambda(\phi^2+\lambda^2+\theta^2)} .$$

Similar equations are obtained with the assumption that ϕ or θ are non-zero.

If the points P'_m, P'_n, P''_m and P''_n are the points corresponding to P_m and P_n in the $x'y'z'$ and $x''y''z''$ systems respectively, then there corresponds to P''_m and P''_n a unique real number t_m and t_n respectively by the system (3) of equations above.

Since our steps can be reversed, the lemma is proved in both directions for case 1.

Case 2. $D = D' = 0$. The equations defining ℓ are equivalent to

$$\ell_2: \begin{cases} Ax + By + Cz = 0 \\ A'x + B'y + C'z = 0 \end{cases}.$$

The points on ℓ_2 are equivalent to the set of points (x, y, z) where

$$x = \begin{vmatrix} B & C \\ B' & C' \end{vmatrix} t$$

$$y = \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} t$$

$$z = \begin{vmatrix} A & B \\ A' & B' \end{vmatrix} t$$

and t is a real number with $t_m < t < t_n$ where

$$t_n = -t_m = \frac{1}{[(BC'-B'C)^2 + (CA'-AC')^2 + (AB'-BA')^2]^{\frac{1}{2}}} .$$

Again since our steps can be reversed the lemma is proved in both directions for case 2 and hence the lemma is proved.

Remark: The system of equations

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t),$$

which are equivalent to

$$D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$$

$$D'(x^2+y^2+z^2+1) + A'x + B'y + C'z = 0$$

are referred to as the parametric equations of the line.

Metadef. 6. Let

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

be the parametric equations of a line. We say that three distinct points P_1, P_2 and P_3 on the line, are such that P_2 is <u>between</u> P_1 and P_3 iff one of the following is true,

$$t_{P_1} < t_{P_2} < t_{P_3} \quad \text{or} \quad t_{P_3} < t_{P_2} < t_{P_1} ,$$

where t_{P_1}, t_{P_2} and t_{P_3} are the values of the parameter t corresponding to P_1, P_2 and P_3 respectively in the above equations.

Theorem 9. If P, Q and R are points, then if Q is between P and R, it is between R and P.

Proof. By symmetry.

Theorem 10. If P, Q and R are points, then at most one of them is between the other two.

Proof. At most one of the parameters t_P, t_Q and t_R

corresponding to the points P, Q and R respectively is between the other two in the ordering of the real numbers.

Theorem 11. If P and Q are distinct points of a line, then there exists a point R of the line such that Q is between P and R.

Proof. Let t_P and t_Q be the values of the parameters corresponding to P and Q respectively. Then since they are elements of an open interval of the real numbers, $t_P < t_Q$ implies there exists t_R such that $t_P < t_Q < t_R$ and $t_Q < t_P$ implies there exists t_R such that $t_R < t_Q < t_P$.

Definition 4. A <u>line segment</u> P_1P_2 is the set of points between P_1 and P_2, where P_1 and P_2 are distinct points of the line.

Definition 5. A <u>closed line segment</u> $\overline{P_1P_2}$ is the union of P_1P_2 and the two points P_1 and P_2.

Definition 6. If P_1, P_2 and P_3 are three points of the same plane and are non-collinear, then the <u>triangle</u> $P_1P_2P_3$ is the union of the sets, $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_3P_1}$. P_1, P_2 and P_3 are called the vertices of the triangle, and the sets P_1P_2, P_2P_3 and P_3P_1 are called the <u>sides</u> of the triangle.

There is a theorem in Euclidean geometry that states: if a closed Jordan Curve on a surface has a point of the boundary of a region of the surface and a point of the region, then it has another point of the boundary on it, (for definitions, see [5]).

We shall make use of this theorem in proving the very important Pasch's Axiom.

Theorem 12. If P_1, P_2 and P_3 are vertices of a triangle and ℓ is a line of the plane incident upon P_1, P_2 and P_3 such that ℓ has on it a point of one side of the triangle then ℓ has another point of the triangle on it.

Proof. It is evident that a triangle is a region with its sides, the boundary, and the plane determined by the vertices of the triangle is a surface, while the curve defining a line is a closed Jordan Curve. Therefore the theorem from Euclidean geometry above applies and thus, this theorem is proved.

III. AXIOM OF PARALLELS

Theorem 13. Let ℓ be a line and P a point not on ℓ. Then in the plane determined by P and ℓ there is more than one line passing through P that does not intersect ℓ.

Proof. Let ℓ be defined as

$$\ell: \quad \begin{Bmatrix} z = 0 \\ \\ y = 0 \end{Bmatrix}$$

and let the point P be $(0, \frac{1}{2}, 0)$. Clearly P is not on ℓ. The plane determined by P and ℓ is

$$\Pi: \quad \{ z = 0 \}.$$

Now the lines

$$\ell': \quad \begin{Bmatrix} x^2+y^2+z^2+1 - \frac{5}{2}y = 0 \\ \\ z = 0 \end{Bmatrix}$$

and

$$\ell'': \quad \begin{Bmatrix} x^2+y^2+z^2+1+x-\frac{5}{2}y = 0 \\ \\ z = 0 \end{Bmatrix}$$

do not intersect ℓ since no point with $z = 0$ and $y = 0$ is a point of

ℓ' or ℓ''. ℓ' and ℓ'' are distinct lines since the point $\left(\frac{5^{\frac{1}{2}}}{4}, \frac{3}{4}, 0\right)$ is on ℓ' and not on ℓ''.

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