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Problems associated with creep and relaxation of viscoelastic membrane structures have been documented, and range in extremes from aesthetic considerations to loss of prestress. The creep/relaxation response leading to a loss of prestress in viscoelastic membrane structures should accelerate the formation of wrinkles. Also of interest are problems of dynamic wrinkling for its effects on fatigue analysis and on snap-loading. These problems have importance in the application of polymer films and coated fabrics to air-supported and tensioned fabric structures.

Transient viscoelastic analysis is treated by the direct solution of the hereditary integro-differential constitutive equation. The constitutive equation is approximated by a finite difference equation and embedded within the finite element spatial discretization. This has the advantage that no assumption need be made as to how the stress varies over a time increment and is applicable to a wide class of problems including those with time-dependent boundary conditions. Isoparametric quadrilateral membrane elements are employed. Implicit temporal integration is used

and iterations within a time increment are performed using the modified Newton-Raphson method.

The stress-strain hereditary relation is formally derived from thermodynamic considerations. Use of strain-energy and dissipation functions facilitates the description of wrinkling during the deformation process. Certain details of the wrinkling phenomena can be ascribed to the inherent (albeit small) bending stiffness of the membrane material, a quantity left out of membrane analysis. These details can be introduced into the analysis by use of modified strain-energy and dissipation functions. The relaxation function is taken in the form of a Prony series, thus reducing the computational memory requirements to the previous time increment only, instead of over all previous time. Relaxation functions are derived from experimentally determined creep curves for a specific material.

Analytical validation of the method is by comparison to previous studies on the inflation of a plane membranes. Experimental validation is carried out on a cylindrical membrane submerged in water and excited by surface gravity waves. Other experimental methods applicable to highly flexible membrane structures are reviewed. Extensions of the method and applications are discussed.

Transient Nonlinear Deformations of
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CHAPTER I

INTRODUCTION

Membrane structures may be regarded as thin shells which, at some level of approximation, are devoid of any ability to resist bending moments. This at once provides for a dichotomy of analysis. One the one hand, the equations of shell theory are greatly simplified when one can disregard those effects due to bending. On the other, physical attributes of the structure may now have exceeded the relevant domain of the original shell theory. With their loss of bending stiffness, membranes are highly flexible structures that can suffer large deformations and unstable configurations for even quite small loads.

Leonard Euler (1707-1783) was one of the earliest researchers to consider membrane deformations. By extending the theory for the oscillation of a one-dimensional string to two dimensions, he was able to formulate an expression which gave the small deflection and vibration response of a perfectly flexible membrane. Early in this century, Ludwig Prandtl recognized the analogy between Euler's theory and the stress function describing the deformation of a bar in torsion - a connection which provided for many simplifications. Shortly thereafter, Th. v. Kármán, A. Föppl, and H. Hencky formulated expressions that would adequately describe larger deformations of plates and membranes plates.

Since that time, many researchers have participated in the development of a theory which satisfactorily accounts for large and complex deformations of membrane structures. It is within this realm that the present work has developed. The ability to accurately predict the nonlinear dynamic response of membranes, including the effects of viscoelasticity and wrinkling, is crucial for many applications - terrestrial, marine, and aerospace structures to name but a few - and it is this which the present work attempts to accomplish. To the best of the author's knowledge, no prior work has been published concerning the transient nonlinear deformations and wrinkling of viscoelastic membranes.

This thesis is the compilation of three papers previously submitted for publication, each of which comprises a separate chapter herein. Chapter II [Jenkins C., and Leonard J.W. (1991). Nonlinear dynamic response of membranes: State of the art, *Appl Mech Rev* **44**] reviews the theory and problems of nonlinear dynamic membrane analysis. The bulk of work cited dates from 1970; however, earlier research is included where necessary to establish theoretical or experimental foundations. Chapter III [Jenkins, C., and Leonard, J.W. (1991). Transient nonlinear deformation of viscoelastic membrane structures, *Struct Eng Rev* (in press)] presents the development of the theory and application for a viscoelastic constitutive equation in nonlinear finite element analysis. Indeed, most modern membrane materials will, under some conditions of loading and observation, exhibit time-dependent properties. Chapter IV [Jenkins, C., and Leonard, J.W. (1991). Dynamic wrinkling of viscoelastic membranes, *J Appl Mech* (in review)] incorporates the ability to analyze membrane wrinkling, a physical phenomenon which violates the fundamental tenet of membrane analysis.

Intrinsic to engineering is the process of putting theory into practice. In Chapters III and IV, numerous examples are given to validate and apply the formulation as presently developed. Additionally, Chapter IV describes details of an experiment performed for validation purposes. Any errors are the sole responsibility of the author; critical discussion and suggestions for improvement are greatfully solicited.

Chapter V gives conclusions and future directions so that others may expand upon that which is presented here. Finally, a composite Bibliography is provided that combines all references cited in the preceding chapters.

No work of this scope is developed *in vacuo*, but connects to the deliberations of many, many others. In that regard, this thesis is humbly offered as a small ripple in the ever widening stream of knowledge.

CHAPTER II

NONLINEAR DYNAMIC RESPONSE OF MEMBRANES: STATE OF THE ART

> "In many cases the engineer can linearize his problem by means of simplifying assumptions and a mathematical text will easily supply him with all the help he needs. However, if the engineer has a real nonlinear problem, that is, one which loses its sense by linearization, very often he has to grapple with it by himself." [von Kármán (1940)]

II.1 INTRODUCTION

The first membrane structures were biological organisms [Gordon (1988)] which probably represent the widest usage of this structure type. Examples range from dragonfly and bat wings to the bullfrog's inflatable throat [Herzog (1976)]. Research in the biomechanics of flexible membranes is a current activity. Cells may be represented as highly flexible membrane structures to aid in understanding the mechanism of cell replication and motility. Zarda et al. (1977), considered elastic deformations of red blood cells while Cheng (1987) considered the same for sea urchin eggs. Both used a finite element solution; neither considered the viscoelastic nature of biomaterials. Danielson (1973) and Dehoff and Key (1981) considered skin as an elastic membrane. [For a discussion of the nature of viscoelastic biomembranes, see Fung (1968) and Vincent (1982).] Danielson and Natarajan (1975) used tension field theory to study the stresses in sutured skin. Other works of interest include: Christie and Medland (1982), Lee and Tseng (1982), and Skalak and Tozeren (1982).

Historical use of membranes in engineering structures may be traced to the sail and the tent [Herzog (1976); Drew (1986); Tsuboi (1986)]. Kites, parachutes, balloons, and other flying structures followed. In modern times, membranes have seen increasing use in building structures such as radar domes, temporary storage, and aero-space structures [Otto (1962); Szilard (1967); Dent (1971); Hass (1971); Bushnell (1980); Bird (1986); Geiger (1986); Murphy (1986); Jackson and Christie (1987); and Kiang and Dharan (1989)].

In the marine environment, membrane structures have been considered for a variety of applications. The use of membranes as breakwaters goes back at least to 1960, and a summary of much of the work through 1970 is given in NCEL (1971). Additional membrane breakwater research has been conducted by Fredriksen (1971), who studied fluid-filled bags, both floating and not. Ijima et al. (1985, 1986) considered an array of 'sea-balloons'. Lo (1981) used a finite element solution to analyze an infinitely-long submerged cylindrical membrane with a crude model of ocean wave loading. Oyama et al. (1989) reported on the analysis of wave deformation due to a bottom-mounted, flexible mound. None of the above considered viscoelastic effects. Modi and Poon (1978) investigated inflated viscoelastic tapered cantilevers for underwater applications. Then Modi and Misra (1979) considered an inflatable offshore platform. Szyskowski and Glockner (1987a,b,c) considered membrane structures as floating storage vessels. Their analysis was limited to static loads and did not include viscoelastic effects. Leeuwrik (1987) and Bolzon et al. (1988) considered flexible membrane dams [case studies of membrane dams are also given in Koerner (1980)]. Membranes in the form of parachutes or cones attached to cables have been investigated for use as sea anchors ("drogues") by Hervey and Jordan

(1987). Inflatable boats have been used successfully for many years [Watney (1973); Freakley (1978)].

Research into the more generic membrane problem has encompassed the study of both linear and nonlinear membrane mechanics (only the latter will be discussed here), and included other aspects such as wrinkling. Specific types of membrane materials studied have included hyperelastic, viscoelastic, and fabric. Appropriate aspects of the above will be discussed.

The following conventions are used: small strain/large rotation and large strain/large rotation problems are hereafter called large deflection and large deformation, respectively; Latin indices take the values 1,2,3; Greek indices take the values 1,2; capital and lower case Latin letters refer to the undeformed and deformed state, respectively; bold type indicates vector or tensor quantities; other conventions are given where required. A list of nomenclature is given in Appendix A.

II.2 NONLINEAR MEMBRANE FIELD EQUATIONS

II.2.1 General. The equation formulation of the combined initial-boundary value problem which models the response of membrane structures has roots in classical elasticity. The extreme thinness of membrane structures generates two fundamental assumptions. One, membrane stiffness is much, much greater than bending stiffness; thus, stress couple effects may be disregarded. The subsequent simplification afforded the equilibrium equations is exploited. Two, the ratio of thickness to smallest radius of curvature is much, much less than one; thus, terms $O[(x_3)^2]$ or higher may be disregarded. This effectively decouples the strain-displacement relations from the curvature tensor.

One must be careful to distinguish between the membrane response of an idealized structure and the response of a true membrane. Thus, membrane field equations are applicable in two distinct categories [see Libai and Simmonds (1988)]. First, a plate or shell structure may be idealized to respond as a membrane away from its boundaries, in its 'interior'. That the stress couples may be neglected in this interior region is an approximation the analyst need be aware of.

Second, structure models that cannot support stress couples anywhere are by definition 'true membranes', examples of which are discussed throughout the present work. True membranes (hereafter simply called 'membranes') are inherently nonlinear, with the degree of nonlinearity incorporated into the field equations dependent on the formulation philosophy. Thus, theories may be developed ranging from linear to small strain - finite rotation (geometric nonlinearity only) to fully nonlinear both in geometry and materials (geometric and physical nonlinearity). The linear theory will not be discussed here [see, e.g., Firt (1983) and Leonard (1988) for further discussion].

The governing field equations are most easily analyzed in static loading configurations, thus it is not surprising that the bulk of work in the membrane problem has been of this type. This is not necessarily as restrictive as it may seem since many loading regimes may be considered as quasi-static, i.e., when the inherent time scale (e.g., the period of oscillation) of the loading is such that inertial effects (e.g., added mass for a surrounding fluid) may be satisfactorily ignored. The slow inflation of a plane membrane would be of this type. Furthermore, since membranes are inherently thin, light structures, their weight and hence inertial effects may be ignored even for more rapid loading schemes.

In either case, whether by virtue of shear volume of literature or by quasi-static approximation, some aspects of the static membrane problem will be discussed. The more recent investigations into the fully dynamic problem will be reported separately. Also, for a more complete review of membrane investigations prior to 1973, see Leonard (1974).

II.2.2 Föppl-von Kármán Theory. The nonlinear (small strain, 'moderate' rotation) theory for plate bending was first developed by Föppl, later extended by von Kármán, with a numerical solution to the membrane equations given by Hencky [Love (1944)]. By formally equating the plate stiffness to zero, the following field equations form the Föppl-von Kármán theory [Stoker (1968)]:

$$\nabla^4\Phi(x_1,x_2) = \tilde{E}\left(u_{3,12}^2 - u_{3,11}u_{3,22}\right) \tag{2.2.1}$$

$$\Phi_{,22}u_{3,11} + \Phi_{,11}\,u_{3,22} - 2\Phi_{,12}\,u_{3,12} + \frac{p(x_1,x_2)}{h} = 0 \tag{2.2.2}$$

where x_1, x_2, and x_3 are Cartesian coordinates with x_1 and x_2 defining the (initial) plane of the membrane, u_3 is the displacement in the x_3 direction, $\nabla^4 = \nabla^2 \bullet \nabla^2$, ∇^2 is the Laplacian operator, $\tilde{E}$ is Young's modulus, h is the membrane thickness, p is the applied force/unit area normal to the membrane, a comma followed by subscripts denotes partial differentiation with respect to the coordinates corresponding to the subscripts, and Φ is a stress function defined by

$$\Phi_{,22} = \frac{1}{h}\,N_{11} \tag{2.2.3a}$$

$$\Phi_{,11} = \frac{1}{h} N_{22} \qquad (2.2.3b)$$

$$\Phi_{,12} = -\frac{1}{h} N_{12} \qquad (2.2.3c)$$

where N_{ij} is the ij stress resultant (force/length). Due to its lineage from plate theory, the Föppl theory has generally been applied to the out of plane deflection of an initially plane membrane. Existence and uniqueness considerations have been investigated by Callegari and Reiss (1968) and Callegari et al. (1971).

Equations (2.2.1) and (2.2.2) are coupled nonlinear partial differential equations which are difficult to solve in closed form. Berger (1955) used the Föppl theory to formulate the strain energy density of a deformed plate. He then made the simplifying (but non-rational) assumption of ignoring the term containing the second invariant of strain (relative to the first invariant of strain). This leads to a set of two uncoupled partial differential equations. This approach has been applied to large deflections of membranes by Jones (1974), who used a Prandtl stress function solution, and by Schmidt and DaDeppo (1974), who used a perturbation technique. For further discussion on Berger's hypothesis and equations, see Schmidt and DaDeppo (1974), Schmidt (1974), and Mazumdar and Jones (1974).

Numerical methods are usually employed to solve the governing equations. Shaw and Perrone (1954) used a finite difference/relaxation iteration method after recasting (2.2.1) and (2.2.2) entirely in terms of displacement components, resulting in three coupled nonlinear second-order partial differential equations. Specifically, they considered a rectangular membrane and displayed stress contours. For associated numerical solutions see: Dickey (1967); Kao and Perrone (1971,1972). Recently, Allen and Al-Qarra (1987) have used a small strain/large rotation extension of the theory, solved by the finite element method, and considered both square and circular membrane problems.

Caution must be exercised, however, in the application of the Föppl theory to the large deflection problem. For the case of the axisymmetric deformation of an annular membrane, Weinitschke (1980) considered a parameter κ, where

$$\kappa = (2r_a p/\tilde{E}h)^{\frac{1}{3}} \tag{2.2.4}$$

and r_a is the radius of the outer edge of a circular membrane. Comparisons were made between the Föppl and the large rotation Reissner shell (with zero bending stiffness) theories. For values of $\kappa > 1/2$, the difference between theories exceeds 10%. The appropriateness of a Reissner-like theory for $\kappa > 1/2$ is reiterated in Grabmuller and Weinitschke (1986), and in Weinitschke (1987). Storakers (1983) notes the same effect for a circular membrane. Additional comparisons are discussed in section 2.3. [For further details on the Reissner theory, see Libai and Simmonds (1988) and Weinitschke (1989).]

II.2.3 General Membrane Shell Theory for (Quasi-) Static Response. A considerable amount of work in nonlinear membrane response follows from the theory for the large deformation problem as set forth in Green and Adkins (1970). [For some earlier work preceding this see Leonard (1974)].

Consider the Cartesian coordinates X_i of a point X on the undeformed midsurface which becomes point x with coordinates x_i on the deformed midsurface (see Figure I.1). Now

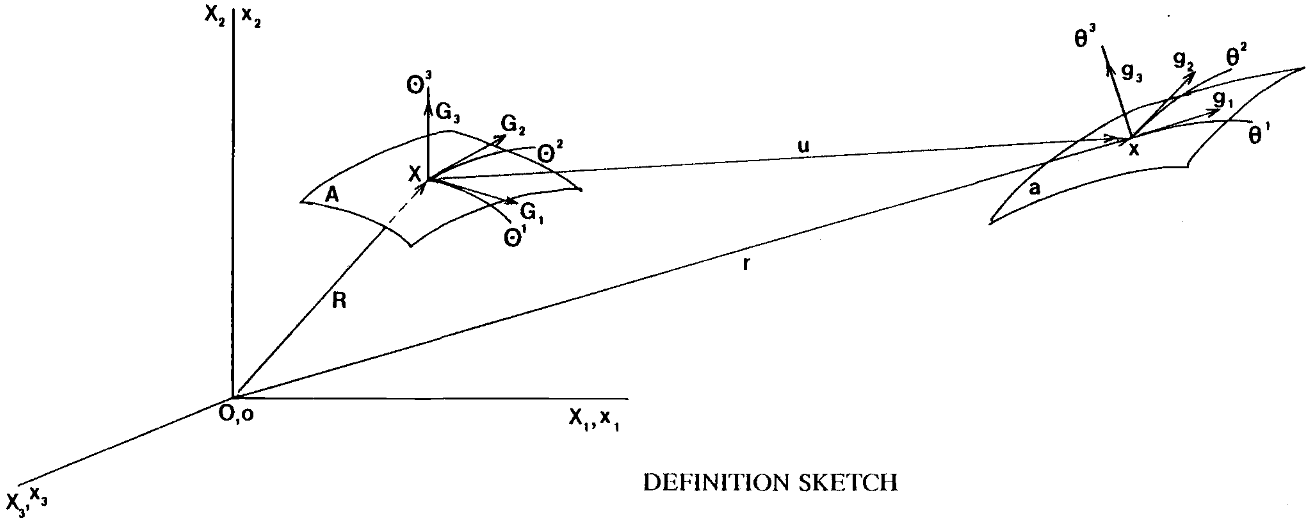

Figure II.1 Membrane deformation definition sketch also define convected (also known as intrinsic, embedded) curvilinear midsurface coordinates $\Theta^i = \theta^i$, i.e., the Θ^i coordinates of X are numerically equal to the θ^i coordinates of x. The Green deformation tensor is given by

$$\tilde{G}_{k\ell} = g_{k\ell} \frac{\partial \theta^k}{\partial \Theta^k} \frac{\partial \theta^\ell}{\partial \Theta^\ell} \tag{2.3.1}$$

where $g_{k\ell} = \mathbf{g}_k \bullet \mathbf{g}_\ell$, $\mathbf{g}_i = \partial \mathbf{r}/\partial\theta^i$, and $\mathbf{r}$ is the position vector from the origin o to x. However, by virtue of the definition of the convected coordinate θ^i,

$$\tilde{G}_{k\ell} = g_{k\ell} \tag{2.3.2}$$

The Green-Lagrange strain tensor is given by

$$E_{k\ell} = \frac{1}{2}\left(\tilde{G}_{k\ell} - G_{k\ell}\right) \tag{2.3.3}$$

where $G_{k\ell} = \mathbf{G}_k \bullet \mathbf{G}_\ell$, $\mathbf{G}_i = \partial \mathbf{R}/\partial\theta^i$, and $\mathbf{R}$ is the position vector from the origin 0 to X. Now substituting (2.3.2) into (2.3.3) gives the strain tensor for convected coordinates.

$$E_{k\ell} = \frac{1}{2}\left(g_{k\ell} - G_{k\ell}\right) \tag{2.3.4}$$

The Cauchy stress tensor s^{ij}, measured per unit area of deformed midsurface, may be related to the second Piola-Kirchoff stress tensor S^{ij}, measured per unit area of undeformed midsurface, by

$$S_{ij} = \left(\tilde{I}_3\right)^{-\frac{1}{2}} s_{ij} \tag{2.3.5}$$

where $\tilde{I}_3 = G/g$, $G = \det G_{ij}$, and $g = \det g_{ij}$.

The (hyper-) elastic stress-strain relation is given by

$$S^{ij} = \frac{1}{2(\tilde{I})^{\frac{1}{2}}} \left[\frac{\partial \tilde{W}}{\partial E_{ij}} + \frac{\partial \tilde{W}}{\partial E_{ji}} \right] \tag{2.3.6}$$

where $\tilde{W}$ is the strain energy function.

Stress resultants are defined as

$$n^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} s^{\alpha\beta} \, d\theta^3 \tag{2.3.7}$$

For pressure loads Δp normal to the membrane and in the outward $\hat{g}_3$ direction, the equilibrium equations are

$$n^{\alpha\beta}_{;\alpha} = 0 \tag{2.3.8a}$$

$$n^{\alpha\beta} \, c_{\alpha\beta} + \Delta p = 0 \tag{2.3.8b}$$

where a semicolon denotes covariant differentiation with respect to θ^α, $\hat{g}_3$ is the outward unit normal to the midsurface, and $c_{\alpha\beta}$ is the curvature tensor defined as

$$c_{\alpha\beta} = \frac{1}{2} \left[g_\alpha \bullet \frac{\partial \hat{g}_3}{\partial \theta^\beta} + g_\beta \bullet \frac{\partial \hat{g}_3}{\partial \theta^\alpha} \right] \tag{2.3.9}$$

The equations constraining the curvature tensor are the Codazzi equations

$$c_{\alpha 1;2} = c_{\alpha 2;1} \tag{2.3.10}$$

[For further details, consult Green and Adkins (1970) and Leonard (1988)].

A tractable application of the above is the case of axisymmetric deformations of an isotropic incompressible membrane of uniform undeformed thickness H. In this case, the above reduce to eight equations in eight unknowns [see Green and Adkins (1970) for further details]. Cases of in-plane loading will not be discussed except as regards membrane wrinkling (see Section 4).

Many investigators have studied the inflation of a plane circular membrane, originally outlined by Adkins and Rivlin (1952). (As above, the field equations reduce to eight in eight unknowns.) The equations may be integrated numerically after assigning values for the stretch (extension) ratios. The Mooney form of the strain energy function was used. Results were compared to experimental work of Treloar (1944).

Trostel (1962) outlined a closed form solution of the circular membrane using the so-called Kappus constitutive law. Foster (1967) gave closed form solutions for a neo-Hookean material and provided experimental verification. Others providing various numerical solutions of the plane circular membrane with nonlinear elastic constitutive relations are: Yang and Feng (1970); Tielking and Feng (1974); and Pujara and Lardner (1978). Wineman (1976), Wineman (1978), and Roberts and Green (1980) have considered a similar problem with a nonlinear viscoelastic constitutive relation. Joshi and Murphy (1990) extended the analysis to a laminated composite membrane.

A finite element solution of the inflation of a plane circular membrane was given by Oden and Sato (1967), with additional results given in Oden and Kubitza (1967) and Oden (1972). In these works, flat triangular elements and a Mooney-Rivlin material model were used. Haug and Powell (1972) used isoparametric quadrilateral elements with straight sides to analyze large membrane displacements; wrinkling analysis was included. Verma (1974) used flat superparametric quadrilateral elements for a similar problem. This was improved upon by Leonard and Verma (1976) who used doubly curved quadrilateral elements, and by Lo (1981) using curved isoparametric quadrilateral elements. Fried (1982) also gave finite element solutions for a number of membrane configurations. Warby and Whiteman (1988) presented a finite element solution for the inflation of an initially plane circular linear viscoelastic membrane up to a rigid obstacle. Douven et al. (1989) considered a finite element formulation for a viscoelastic transversely isotropic membrane.

Other axisymmetric membrane problems investigated include (i) the inflation of a spherical membrane: Green and Adkins (1970), Feng and Yang (1973), Li and Leonard (1973); (ii) annular membrane: Fulton and Simmonds (1986), Tezduyar et al. (1987); (iii) cylindrical membrane: Pipkin (1968), Wu (1970), Callegari and Keller (1974), Matsikoudi-Iliopoulou (1987); (iv) and toroidal membrane: Kydoniefs (1967), Feng (1976).

At very large deformations, asymptotic solutions are possible which greatly simplify the analysis [see: Wang and Shield (1969), Wu (1972), Fenner and Wu (1981)].

Comparisons between the aforementioned Föppl theory and the Green-Adkins theory are difficult due to the use of different constitutive theories (see Section 3). For the inflation of an initially plane circular membrane, Pujara and Lardner (1978) compare results from the Green-Adkins theory and neo-Hookean constitutive relation to work done by Dickey (1967) who used the Föppl theory and Hookean material model. Beyond values of the ratio of center deflection to initial radius of about 0.3, results begin to diverge sharply. A similar comparison has also been given by Murphy (1987).

II.2.4 Additional Formulation Philosophies for (Quasi-) Static Response. Nonlinear membrane equilibrium equations in terms of Lagrangian displacement components were formulated by Vishwanath and Glockner (1972a) and applied to a spherical Hookean membrane. Closed form series solutions were generated by perturbation methods. This formulation was also used by Vishwanath and Glockner (1972b) for a spherical membrane of either linear or nonlinear elastic material; Glockner and Vishwanath (1972) for both plane circular and spherical nonlinear elastic membranes; and Vishwanath and Glockner (1973) to analyze the effects of axisymmetric external loads on pressurized spherical membranes.

A new formulation was developed because of an interest in predicting the response of spherical inflatables to ponding collapse [Malcom and Glockner (1978)], to symmetric concentrated loads [Szyszkowski and Glockner (1984a)], and to axisymmetric hydrostatic loading [Szyszkowski and Glockner (1984b)]. The above membranes are considered to be inextensible and wrinkling analysis is included. Some additional work in this area included further analytical studies [Szyszkowski and Glockner (1987d)]; an experimental study [Szyszkowski and Glockner (1987e)]; and a review article [Glockner (1987)]. However, Dacko and Glockner (1988) include extensibility of the membrane to adjust for difficulties in some of the previous work.

II.2.5 Nonlinear Dynamics. The quasi-static approximation to the dynamic membrane problem has already been discussed (see Introduction). Dynamic relaxation is a solution technique for the quasi-static problem that borrows from computational methods of solution for structural dynamic response. In this explicit (see below), iterative method, fictitious values of mass and damping are chosen so that the static solution is achieved with the smallest number of steps [see, e.g., Barnes (1980a); Barnes and Wakefield (1988)].

A complete dynamic analysis takes into account the inertia of the membrane and its surrounding medium (added mass) as well as the damping of the membrane and its surrounding medium (radiation damping)[see Davenport (1988)]. As in the quasi-static case, various approximations will be made to afford tractability and satisfy levels of accuracy.

Nonlinear membrane equations following from the large deflection theory have been used by Chobotov and Binder (1969) and Yen and Lee (1975) to study the free and forced vibrations of a plane circular membrane using perturbation methods. Plaut and Leeuwrik (1988) derive a nonlinear equation of motion based on the inextensibility assumption and use Galerkin's method to analyze the nonlinear oscillations of a cylindrical membrane [see also Plaut (1990)].

The equations of motion, in the D'Alembert sense, for the large deformation problem can be developed from the principle of virtual work as [Leonard (1988)]

$$\iint_A \left(S^{\alpha\beta}\delta E_{\alpha\beta} - P_i \delta U_i\right)\sqrt{G}\, d\theta^1 d\theta^2 - H \iint_A M\ddot{U}_i \delta U_i \sqrt{G}\, d\theta^1 d\theta^2 = 0 \tag{2.5.1}$$

where A is the undeformed mid-surface with uniform thickness H and density M, U_i are displacements in the θ^i direction, and the over-dot symbols represent partial differentiation with respect to time. Note that no constitutive relation is specified in (2.5.1).

The equations of motion are usually solved by a spatial discretization using the finite element method and a temporal discretization using either an implicit or explicit difference method [see, e.g., Barnes (1980a)]. Since the system stiffness is usually nonconstant, iteration (e.g., Newton-Raphson) during each time step will be required [Barnes (1980a)].

Briefly, implicit methods combine the equations of motion (evaluated at time t + Δt) with finite difference formulas for velocities and accelerations (at times t and t + Δt) to obtain displacements (at time t + Δt) directly using some form of numerical integration (e.g., Newmark's method). Implicit methods require solution of simultaneous equations at each time step but are unconditionally stable. In explicit methods, the accelerations are determined from the equations of motion (at time t) and are then integrated to obtain the displacements and velocities at time t + Δt without the need for solving simultaneous equations (provided the mass matrix is diagonal). Explicit methods are conditionally stable and thus require smaller time steps.

Oden et al. (1974) considered the problem of a plane square membrane subject to a central impulsive load using constant-strain triangular elements and the central difference method (explicit). Benzley and Key (1976) used cubic quadrilateral elements and the central difference method to examine the free vibration of a square membrane. Leonard and Lo (1987) considered the same problem with an implicit Newmark's method using a variety of element types, and showed quadratic quadrilaterals to be advantageous. Lo (1982) also investigated the transient response of an infinitely long fluid-filled cylindrical membrane submerged in water, moored with cables, and loaded by gravity waves; quadratic membrane elements and Newmark's method were used for solution. Barnes (1980b) considered the transient vibrations of a pneumatic dome both numerically (explicit method) and experimentally. Hsu (1987) describes the use of a commercial general purpose nonlinear finite element code to study the dynamic response of a tension fabric structure. Notable in the latter two studies are the use of viscoelastic constitutive equations. Krakowska and Barnes (1987) reported on an explicit dynamic analysis of coated fabric membranes including on/off slackening, viscoelastic material, and pneumatic stiffening, damping, and added mass effects.

II.3 CONSTITUTIVE RELATIONS

Generalized linear elastic relations, while suitable for large deflection problems, are well documented in the literature [see for example Love (1944)] and will not be discussed here.

The more general large deformation problem uses the nonlinear isotropic constitutive relation based on the strain energy function, i.e., the so-called 'hyperelastic' materials. [For excellent discussions, see Rivlin (1948) and Ward (1983).] The strain energy function, $\tilde{W}$, is taken to be some function of the strain invarients, $\tilde{I}_i$, i.e.,

$$\tilde{W} = f\left(\tilde{I}_1 , \tilde{I}_2 , \tilde{I}_3\right) \tag{3.1}$$

where the strain invarients are functions of the extension ratios Λ_i. The more common forms for isotropic materials were first given by Mooney (1940), also known as 'Mooney-Rivlin':

$$\tilde{W} = \tilde{D}_1 \left[\left(\tilde{I}_1 - 3\right) + \frac{\tilde{D}_2}{\tilde{D}_1}\left(\tilde{I}_2 - 3\right)\right] \tag{3.2}$$

and Treloar (1944), also known as 'neo-Hookean':

$$\tilde{W} = \tilde{D}_1(\tilde{I}_1 - 3) \tag{3.3}$$

where $\tilde{D}_1$ and $\tilde{D}_2$ are constants, and $\tilde{I}_3 = 1$ for incompressible materials. Many other forms have been proposed [for a brief summary, see Libai and Simmonds (1988)].

Creep problems of membrane structures are documented and range in extremes from aesthetic considerations to loss of prestress [Ansel and Harris (1980); Happold and Dickson (1980); Liddell (1986); Srivastava (1987); Geiger (1989)]. Viscoelastic effects are usually temperature dependent as well. Coated fabrics deserve consideration as composite structures and also include the problem of fracture [Racah (1980)]. Certain membrane materials at small strain, like polymers, may be adequately modeled for some purposes by a linear viscoelastic relation. Others at small strain, and most at large strain, require a nonlinear viscoelastic formulation.

Linear viscoelastic relations may be either of two types, differential or integral. Differential equations are written for mechanical models consisting of combinations of springs and dashpots, the general form of which is, for uniaxial stress (s) and strain (e) [Christensen (1982)]:

$$\kappa_0 s(t) + \kappa_1 \dot{s}(t) + \kappa_2 \ddot{s}(t) + \ldots = \lambda_0 e(t) + \lambda_1 \dot{e}(t) + \lambda_2 \ddot{e}(t) + \ldots \tag{3.4}$$

where κ_i and λ_i are constants. For the so-called 'standard linear solid' consisting of a parallel spring and dashpot of intensity $\tilde{E}_2$ and η_2, respectively, in series with a spring of intensity $\tilde{E}_1$,

For, say, constant stress $s = s_0$, the 'creep' solution to the above is

$$s + \frac{\eta_2}{\tilde{E}_1 + \tilde{E}_2} \dot{s} = \frac{\tilde{E}_1 \tilde{E}_2}{\tilde{E}_1 + \tilde{E}_2} e + \frac{\tilde{E}_1 \eta_2}{\tilde{E}_1 + \tilde{E}_2} \dot{e} \tag{3.5}$$

$$e = s_0 \frac{\left(\tilde{E}_1 + \tilde{E}_2\right)}{\tilde{E}_1 \tilde{E}_2} \left[1 - \left[\frac{\tilde{E}_1}{\tilde{E}_1 + \tilde{E}_2}\right] \exp\left[-\frac{\tilde{E}_2}{\eta_2} t\right]\right] \tag{3.6}$$

The alternative integral form is based on the so-called 'hereditary integral' (also known as convolution integral, Stieltjes integral). For uniaxial stress and strain, this may be shown to be [Findley et al. (1976)]:

$$e(t) = s(t)\tilde{J}(0) - \int_0^t \frac{d\tilde{J}(t-\tau)}{d\tau} s(\tau)\, d\tau \tag{3.7}$$

or

$$s(t) = e(t)\tilde{Y}(0) - \int_0^t \frac{d\tilde{Y}(t-\tau)}{d\tau} e(\tau)\, d\tau \tag{3.8}$$

where $\tilde{J}$ (t) is the creep compliance and $\tilde{Y}$ (t) is the relaxation modulus. All of the above are generalizable to multiaxial stress/strain. For suitable selection of the creep and relaxation functions, the correspondence between the differential and integral forms can be shown [see, for example, Garbarski (1989)].

Warby and Whiteman (1988) have proposed a generalization of the Mooney-Rivlin relation to include viscoelastic effects. The principal stresses (s_α) for the Mooney-Rivlin material are known as

$$s_{\alpha} = \Lambda_{\beta} \frac{\partial \tilde{W}}{\partial \Lambda_{\beta}} = \left[\Lambda_{\beta} \frac{\partial \overset{@}{\tilde{W}}}{\partial \Lambda_{\beta}}\right] = \tilde{D}_1 \overset{@}{e}_{\alpha} \tag{3.9}$$

where

$$\overset{@}{\tilde{W}} = \left(\tilde{I}_1 - 3\right) + \frac{\tilde{D}_2}{\tilde{D}_1}\left(\tilde{I}_2 - 3\right) , \qquad \overset{@}{e}_{\alpha} = \Lambda_{\beta} \frac{\partial \overset{@}{\tilde{W}}}{\partial \Lambda_{\beta}}$$

The corresponding viscoelastic relation is taken as

$$s_{\alpha}(t) = \int_0^t \tilde{Y}(t-\tau) \frac{\partial e_{\alpha}(\tau)}{\partial \tau} \, d\tau$$

Let

$$e_{\alpha}(t) = \overset{@}{e}_{\alpha}(t) = \Lambda_{\beta}(t) \frac{\partial \overset{@}{\tilde{W}}(t)}{\partial \Lambda_{\beta}(t)} ,$$
$$\overset{@}{\tilde{W}}(t) = \left[\tilde{I}_1(t) - 3\right] + \frac{\tilde{D}_2}{\tilde{D}_1}\left[\tilde{I}_2(t) - 3\right]$$

Then

$$s_{\alpha}(t) = \int_0^t \tilde{Y}(t-\tau) \frac{\partial}{\partial \tau} \left[\Lambda_{\beta}(\tau) \frac{\partial \overset{@}{\tilde{W}}(\tau)}{\partial \Lambda_{\beta}(\tau)}\right] d\tau \tag{3.10}$$

Nonlinear viscoelastic constitutive relations are fundamentally of the multiple-integral type [Findley et al. (1976)]. However, for tractability, certain assumptions (e.g., superposed small loading on a large constant loading) can lead to single integral representations. Christensen (1982) gave a finite nonlinear viscoelastic constitutive relation for rubber-like materials.

The most common membrane material anisotropies are orthotropic and transversely isotropic symmetries due, for example, to fabric weave or oriented polymers, respectively. Fabrics and polymers may be combined as a coated fabric composite. An orthotropic Hookean model requires nine material constants in general, five for plane stress/strain. Other models have been developed [see, e.g., Shanahan et al. (1978); Skelton (1980); Stubbs and Fluss (1980); Dimitrov and Schock (1986); and Testa and Yu (1987)]. An anisotropic viscoelastic (linear or nonlinear) constitutive relation, while available in principle, has been little used in practice [Hsu (1987)]. The successful model must handle (nearly) incompressible materials.

II.4 WRINKLING

Thin membranes are inherently no-compression structures. Potential compressive stress and/or loss of prestress are handled via changes in membrane geometry, i.e., large out-of-plane deformations. These 'wrinkles' are a localized buckling phenomenon. Analysis of wrinkling is important to prediction of structural response.

Wagner (1929) introduced the ideas of wrinkling and 'tension field theory' in connection with flat sheet metal girders in the very thin metal webs used in airplane construction. The basic idea is that, under the action of a specific loading, one of the principal stresses goes to zero. The other remains nonnegative, and if greater than zero, defines a 'tension field.' The crests and troughs of 'wrinkle waves' align with the direction of the nonzero principal stress. Reissner (1938) generalized Wagner's results by introducing an artificial orthotropy into the membrane model whereby a separate elastic modulus is associated with each principal direction. The wrinkling condition is thus given by the diminution of one of the moduli to zero.

In the above, as well as most other wrinkling analyses, results are only in terms of average strains and displacements, while no detailed information is generated for each wrinkle. This is particularly evident in the works by Wu (1974, 1978), and by Wu and Canfield (1981) wherein a 'pseudo deformed surface' and 'pseudo stress resultants' are defined. They also defined the 'wrinkle strain' which is the difference between stretch ratios in the pseudo deformed and actual deformed surfaces. Furthermore, Wu (1978) discussed the stochastic nature of the wrinkling phenomenon in the sense that a given wrinkle distribution is difficult to replicate.

A membrane need not be wrinkled over its entire surface. Stein and Hedgepeth (1961) introduced a concept of a 'variable Poisson's ratio' to study partly wrinkled membranes. Mikulas (1964) extended the work and provided experimental details; finite element implementations are presented in Miller and Hedgepeth (1982), and Miller et al. (1985). Croll (1985) describes the wrinkling of a plane membrane with in-plane elastic constraints. Geometrically nonlinear finite element analysis was used by Contri and Schrefler (1988) to study the wrinkling of an inflated airbag, and by Fujikake et al. (1989) to analyze wrinkling in fabric tension structures.

Mansfield (1968, 1970) reformulated the theory using an energy approach: the true distribution of wrinkles maximizes the tensile strain energy. Recently, Pipkin and coworkers have extended membrane theory to include tension field theory by use of a so-called "relaxed energy density" [Pipkin (1986a); Steigmann and Pipkin (1989a,b,c)].

The relaxed energy density represents the average energy per unit initial area over a region containing many wrinkles.

Other related references of interest include: Moriya and Uemura (1971); Yokoo et al. (1971); Danielson and Natarajan (1975); Zak (1982, 1983); Magara et al. (1983); Ikemoto et al. (1986); Pipkin (1986b); Lukasiewicz and Glockner (1986); Glockner (1987); Glockner et al. (1987); Szyszkowski and Glockner (1987a,b,c); Pipkin and Rogers (1987); Honma et al. (1987); Roddeman et al. (1987); Libai and Simmonds (1988); and Libai (1990).

A creep/relaxation response leading to a loss of prestress in viscoelastic membrane structures should accelerate the formation of wrinkles. This phenomenon has been little studied, as has the relation between 'wrinkle wave length' and material properties. Also of interest is the problem of dynamic wrinkling (e.g., panel flutter [Srivastava (1987); Leonard (1988)]) and its effects on fatigue analysis (e.g., consideration of tension field effects on mean stress distribution) and on snap-loading, i.e., when a wrinkled region suddenly regains the lost principal stress.

II.5 FLUID-STRUCTURE INTERACTION

Unless existing in a vacuum, there will be some interaction between a membrane structure and its immediate environment. Since most membranes will be adjacent to a fluid (either gas or liquid), the fluid-structure interaction is of interest. Assumptions for the fluid field include considering the associated pressure field to be unaffected by structural deformations, interactive analysis whereby the deformations do affect the fluid field, and real and inviscid fluid models.

The interaction of a fluid with a rigid body or a body undergoing small deformations has been investigated for many years and is still an area of current interest [see Belytschko (1980)].

Large deformations of both fluid and structure have only recently been investigated, and then more often for the frequency-domain rather than the time-domain [Huang et al. (1985); and Minakawa (1986)]. Lee and Leonard (1988) reported on nonlinear time-domain models for floating rigid body problems. For the membrane problem, the literature is sparse. Some work has been motivated by the areas of suspension rheology and biomechanics [Zahalak et al.(1987)]. Wind-membrane structure interaction has recently been investigated by Kunieda et al. (1981) and Han and Olsen (1987).

As mentioned in Section 1, there has been an interest in the interaction of ocean waves and highly deformable structures. Here the problem is fully nonlinear in the sense that the fluid, structure, and the coupling (via boundary conditions) are all governed by nonlinear equations. This coupled problem has recently been reviewed by Broderick and Leonard (1990) when boundary element models are used.

II.6 EXPERIMENTAL METHODS

Experiments play many roles in the nonlinear membrane problem. They may provide material and loading characterizations, model verification and range of validity, determination of structural properties (e.g., modal parameters), or as part of a hybrid attack on the problem (e.g., determination of boundary conditions). Some general modeling guidelines may be found in Otto (1962) and Muller (1986).

General methods applicable to the large deflection/deformation of membranes are available. Due to the low rigidity/low elastic moduli of typical membrane materials, non-invasive, non-contact methods are desirable, such as the optical grid and Moire methods. Basic to both techniques is the use of a grid or ruled grating surface. In the grid method, direct photographic measurement of grid deformation generates displacement/strain information. In the Moire method, interference fringes produced by a second, reference grating are analyzed for such information. Further details are outlined in Durelli et al. (1967), Durelli et al. (1970), Dykes (1970), Durelli et al. (1971), Durelli and Chen (1973), Dally and Riley (1978), Sciammarella (1982), and Laermann (1988).

Simple video analysis techniques were used by Gilbert et al. (1990) to study the large deformation of tissue cultures. Results are compared with a commercial FEM model. Recently, digital imaging techniques have been applied to the large deformation problem [Peters and Ranson (1982)]. Here the method used is speckle metrology, wherein a specimen is coated with a random distribution of 'speckles' (e.g., white spray paint); then, via digital camera and computer, a correlation of the deformed and undeformed speckle intensity pattern is performed [Sutton et al. (1983)]. Applications of the method are demonstrated by Chu et al. (1985) and Lee et al. (1989).

Other methods relevant to membranes are the use of a capacitance type displacement transducer with a conductive mylar membrane [Chubotov and Binder (1964)] and the use of a combination of strain gages and high speed film camera [Konovalov (1988)]. Perry (1985) discusses the use of strain gages on low-modulus materials.

Some early material characterization experiments have been presented previously (see Section 3). Many are summarized in Green and Adkins (1970). Other results reported include: Fritzsche (1967); Hajeck and Holub (1967); Kawaguchi et al. (1971); Shimanura and Takeuchi (1971); Skelton (1971); Reinhardt (1976); Ansell and Harris (1980); Day (1986); Ishii (1986); Itoh, et al. (1986); Minami et al. (1986); Willy et al. (1986); and Testa et al. (1987).

Experimental determination of viscoelastic parameters of bulk materials is well known [see, for example, the review articles by Kolsky (1976); Hobaica and Sweet (1976); Hobaica (1979,1982); and Caseiro (1986)]. However, corresponding procedures specifically for thin membrane materials are not as well documented; of interest are articles by Darlinton and Saunders (1970,1971); Ladizesky and Ward (1971a,b); Clayton et al. (1973); and Ward (1983). It should be noted that viscoelastic properties may be different in the static and dynamic regimes (stress or strain rate sensitive) [see Cole (1979)].

Experiments may also reveal structural characteristics, e.g., vibration frequencies and mode shapes. Interesting full scale tests on a variety of inflated 'gap-crossing' devices (e.g., inflated bridges) were reported by Bulson (1967). A large scale structural model test was reported by Mainstone et al. (1980). Takeda et al. (1986) performed vibration tests on a large scale (36m x 24m) structural model. Numerous model tests were conducted by Yoshida et al. (1986) and by Ban et al. (1986). Ishizu and Minami (1986) made in-situ tests of membrane tensile stress.

Experimental data reported concerning environmental loads on membrane structures are sketchy at best. In the case of wind loads, this is partly due to the fact that membranes are often large structures that disturb the flow fields around them, thus rendering the pressure distributions configuration specific. Thus the need for wind tunnel testing of scaled aeroelastic models of tension structures is apparent. An early (ca. 1954) 'radome' wind tunnel test is discussed by Newman and Goland (1982) and a wind tunnel test on a cylindrical 'tent' (ca. 1963) is discussed by Ross (1969). Other reports of wind tunnel testing and wind loading in general may be found in Berger and Macher (1967); Rontsch and Bohme (1967); Howell (1980); Turkkan et al. (1983); Fukao et al. (1986); Ikemoto et al. (1986); Ikoma (1987); Miyamura et al. (1987); Leonard (1988);and Geiger (1989).

II.7 FUTURE STUDIES

The use of membrane structures in architecture, biomechanics, mechanics, and the ocean environment has been outlined. Certainly, their use is not quiescent.

Some aspects of the membrane problem are well investigated, especially quasi-static axisymmetric deformations as well as hyperelastic and linear viscoelastic constitutive theory.

Other areas are not so well studied, including biaxial material properties of thin membranes; non-invasive experimental techniques for large out-of-plane membrane deformations, especially for field testing and in real time; documentation of environmental loads; fluid-membrane structure interaction; analysis of hydrodynamic response of viscoelastic membranes; and advances in the wrinkling theory of nonlinear membranes.

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CHAPTER III

TRANSIENT NONLINEAR DEFORMATION OF VISCOELASTIC MEMBRANE STRUCTURES

III.1 INTRODUCTION

III.1.1 Background. The first membrane structures were biological organisms, which probably represent the widest usage of this structure type. Examples range from dragonfly and bat wings to the bullfrog's inflatable throat. Cells may be represented as highly flexible viscoelastic membrane structures. Historical use of membranes in engineering structures may be traced to the sail and the tent. Kites, parachutes, balloons, and other flying structures followed.

In modern times, membranes have seen increasing use in building structures such as radar domes, temporary storage, and aero-space structures. In the marine environment, membrane structures have been considered for a variety of applications: membranes as breakwaters (e.g., fluid-filled bags, 'sea-balloons', submerged cylindrical membrane, bottom-mounted flexible mound); inflated viscoelastic cantilevers for underwater applications; inflatable offshore platforms; floating storage vessels; flexible membrane dams; and inflatable boats. For a recent review of the above and other applications, see Jenkins and Leonard (1991).

III.1.2 Previous Research. Apparently Wineman was the first to consider the nonlinear deformation of viscoelastic membranes. He analyzed finite planar deformations of annular membranes with singular integral nonlinear viscoelastic constitutive relations

[Wineman (1972)]. The resulting integro-differential equations were solved directly by numerical integration. Wineman (1976) considered the large out-of-plane deformations of a nonlinear viscoelastic circular membrane; the system equations are again solved by numerical integration. A similar problem was addressed subsequently [Wineman (1978] but this time using a viscoelastic fluid rather than solid constitutive equation.

Roberts and Green (1980) used a method similar to Wineman's to study the finite deformation of an initially plane circular membrane loaded by its own weight. Warby and Whiteman (1988) used the finite element method and a viscoelastic generalization of the Mooney-Rivlin constitutive equation to study the circular viscoelastic membrane inflated up to a rigid obstacle. The finite planar deformation of a fibre-reinforced, transversely isotropic, linear viscoelastic square membrane with central hole was considered by Douven et al., (1989) using the finite element method. None of the above reported experimental verification.

III.1.3 Conventions. The following conventions are used: the summation convention is implied unless explicitly stated otherwise; Latin indices take the values 1,2,3 unless explicitly stated otherwise; Greek indices take the values 1,2; capital and lower case Latin letters refer to the undeformed and deformed state, respectively; bold type indicates vector or tensor quantities.

III.1.4 Kinematics. Consider the Cartesian coordinates X_i of a point X on the undeformed midsurface which becomes point x with coordinates x_i on the deformed mid-surface. Also define convected curvilinear midsurface coordinates $\Theta_i = \theta_i$, i.e., the Θ^i coordinates of X are numerically equal to the θ^i coordinates of x. The metric in the deformed state is $g_{k\ell} = \mathbf{g}_k \bullet \mathbf{g}_\ell$, $\mathbf{g}_i = \partial\mathbf{r}/\partial\theta^i$, and $\mathbf{r}$ is the position vector from the origin o to x; similarly, $G_{k\ell} = \mathbf{G}_k \bullet \mathbf{G}_\ell$, $\mathbf{G}_i = \partial\mathbf{R}/\partial\theta^i$, and $\mathbf{R}$ is the position vector from the origin 0 to X.

The convected Green-Lagrange strain tensor components, E_{kl}, are given by

$$E_{k\ell} = \frac{1}{2}\left(g_{k\ell} - G_{k\ell}\right) \tag{1.4.1}$$

The contravarient components of the Cauchy stress tensor s^{ij}, measured per unit area of deformed midsurface, may be related to the convected second Piola-Kirchoff stress tensor components S^{ij}, measured per unit area of undeformed midsurface, by

$$s^{ij} = \left(\bar{I}_3\right)^{-\frac{1}{2}} S^{ij} \tag{1.4.2}$$

where $\tilde{I}_3 = G/g$, $G = \det G_{ij}$, and $g = \det g_{ij}$.

III.2 VISCOELASTIC ANALYSIS AND CONSTITUTIVE RELATION

III.2.1 Analysis. In general, viscoelastic analysis may follow one of three procedures: quasi-elastic, integral transform, or direct. In the quasi-elastic formulation, the investigation proceeds as an elastic analysis except that the time-independent material constants are simply replaced by their corresponding time-dependent viscoelastic quantities. By the use of integral transforms, the problem can be removed from the time domain and the corresponding elastic problem analyzed; inversion of the solution back to the time domain will, except for the simplest of problems, require numerical methods. Finally, one may attack the governing equations directly by closed form solution, classical approximation methods, or modern numerical techniques.

Solution of complicated viscoelastic problems will generally require numerical techniques. The choice of either transform or direct methods may not be clear, each having its own proponents, advantages, and disadvantages. Use of transform methods is generally limited to problems which do not exhibit time dependent boundary conditions. [Christensen (1982) employs an 'extended correspondence principle' as a general approach to solving a specific class of problems for which the direct application of correspondence principle does not apply.]

Different approaches to the direct formulation are possible. Zienkiewicz and Watson (1966) presented an incremental method such that within each time increment the applied stress is assumed constant, creep strain from the previous interval is considered as 'initial strain' at the beginning of the current interval, and elastic finite elements are used to calculate the creep strain at the end of the interval. In the method proposed by White (1968) [see also Wang and Tsai (1988); and Yadagiri and Reddy (1985)], the governing integro-differential constitutive equation is approximated by a finite-difference equation and embedded within the spatial discretization thus making a viscoelastic finite element. This is the method followed in the present work.

III.2.2 Constitutive relation. In the present work, we postulate the existence of a strain-energy function, $\tilde{W}$, and a dissipation function $\tilde{V}$, for viscoelastic membranes. For isotropic material symmetry, we obtain the finite linear viscoelastic constitutive relation as $S^{\alpha\beta}=\partial\tilde{W}/\partial E_{\gamma\zeta} + \partial\tilde{V}/\partial E_{\gamma\zeta}$ or:

$$S^{\alpha\beta}(t) = \tilde{C}^{\alpha\beta\gamma\zeta}(0)\, E_{\gamma\zeta}(t) - \int_0^t \frac{d\tilde{C}^{\alpha\beta\gamma\zeta}(t-\tau)}{d\tau} E_{\gamma\zeta}(\tau)\, d\tau \quad (2.2.1)$$

where $\tilde{C}^{\alpha\beta\gamma\zeta}(t)$ is a linear viscoelastic material tensor whose dependence on the current strain has been neglected.

III.3 COMPUTATIONAL METHODS

III.3.1 Nonlinear finite element method. The starting point for computations is a combined incremental/iterative finite element method [Lo (1981)]. An implicit (Newmark's), incremental method is used to solve the equations of motion. Within each time-step, the modified Newton-Raphson iterative method is used to converge on the nonlinear solution.

We begin with a virtual work expression,

$$\iint_A \left(S^{\alpha\beta}\delta E_{\alpha\beta} - P_i \delta U_i\right)\sqrt{G}\, d\theta^1 d\theta^2 - H\iint_A M\ddot{U}_i \delta U_i \sqrt{G}\, d\theta^1 d\theta^2 = 0 \quad (3.1.1)$$

where **S** = stress resultant tensor, **P** = vector of surface tractions, **U** = vector of displacements, M = initial membrane mass, H = initial membrane thickness, and A = initial membrane area. Note that body forces have been neglected.

Following the usual finite element discretization and recasting (3.1.1) into a form suitable for the 'modified' Newton-Raphson method, the combined incremental/iterative membrane equation of motion is written as [Lo (1981)]:

$$\left[\tilde{M}_{ijIJ}\right]\left\{\ddot{U}_j^J\,(t+\Delta t;k+1)\right\} + \left[\tilde{K}_{ijIJ}\,(t+\Delta t)\right]\left\{\Delta U_j^J\,(t+\Delta t)\right\} = \left\{\tilde{P}_{iI}\,(t+\Delta t;k+1)\right\} - \left\{\tilde{F}_{iI}\,(t+\Delta t;k)\right\} \tag{3.1.2}$$

where

$[\widetilde{M}_{ijIJ}]$ = consistent mass matrix

$$= \int_{-1}^{1}\int_{-1}^{1} \psi_I\,\psi_J\,\delta_{ij}\,HM\sqrt{G}\;d\xi_1\,d\xi_2 \tag{3.1.3}$$

$[\widetilde{K}_{ijIJ}(t+\Delta t)]$ = tangent stiffness matrix

$$\begin{aligned} = \int_{-1}^{1}\int_{-1}^{1} \Big\{ &\frac{1}{2}\,(\psi_{K,\alpha}\,\psi_{I,\beta} + \psi_{I,\alpha}\,\psi_{K,\beta})\left[X_i^K + U_i^K\,(t)\right]\tilde{C}^{\alpha\beta\gamma\zeta}\,(t) \\ &\cdot(\psi_{J,\alpha}\psi_{I,\zeta} + \psi_{I,\gamma}\psi_{J,\zeta})\;[X_j^J + U_j^J(t)] \\ &+ \frac{1}{2}(\psi_{J,\alpha}\,\psi_{I,\beta} + \psi_{I,\alpha}\,\psi_{J,\beta})\,\delta_{ij}\,S^{\alpha\beta}\,(t)\Big\} H\sqrt{G}\,d\xi_1\,d\xi_2 \end{aligned} \tag{3.1.4}$$

$\{\widetilde{P}_{iI}(t+\Delta t;k+1)\}$ = external force vector

$$= \int_{-1}^{1}\int_{-1}^{1} P_i\,(t+\Delta t;k+1)\;\;\psi_I\,\sqrt{G}\;d\xi_1\,d\xi_2 \tag{3.1.5}$$

$\{\widetilde{F}_{iI}(t+\Delta t;k)\}$ = internal force vector

$$\begin{aligned} = \int_{-1}^{1}\int_{-1}^{1} &\frac{1}{2}(\psi_{J,\alpha}\psi_{I,\beta} + \psi_{I,\alpha}\,\psi_{J,\beta})\left[X_I^J + U_I^J\,(t+\Delta t;k)\right]\cdot \\ &S^{\alpha\beta}\,(t+\Delta t;k)\;H\sqrt{G}\,d\xi_1\,d\xi_2 \end{aligned} \tag{3.1.6}$$

$$\Delta S^{\alpha\beta}\,(t+\Delta t) = \tilde{C}^{\alpha\beta\gamma\zeta}\,\Delta E_{\gamma\zeta}\,(t+\Delta t) \tag{3.1.7}$$

$$\Delta S^{\alpha\beta}(t+\Delta t) = S^{\alpha\beta}(t+\Delta t;k+1) - S^{\alpha\beta}(t+\Delta t;k) \qquad (3.1.8)$$

$$\Delta E_{\gamma\zeta}(t+\Delta t) = \frac{1}{2}(\Psi_{J,\gamma}\Psi_{I,\zeta} + \Psi_{I,\gamma}\Psi_{J,\zeta})\cdot \left[X_i^J + U_i^J(t+\Delta t;k)\right]\Delta U_i^I(t+\Delta t) \qquad (3.1.9)$$

$$\Delta U_j^J(t+\Delta t) = U_j^J(t+\Delta t;k+1) - U_j^J(t+\Delta t;k) \qquad (3.1.10)$$

and Ψ_I = isoparametric shape function for node I, X_i^I = ith initial coordinate of node I, U_i^I = ith displacement of node I, ξ_i = natural coordinate of the element, I = 1,..., number of nodes (per element), and k=1,...,no. of iterations per time step.

III.3.2 Computional constitutive equation. Following (3.1.7), we write the linear viscoelastic relation (2.2.1) as

$$\Delta S^{\alpha\beta}(t) = \tilde{C}^{\alpha\beta\gamma\zeta}(0)\,\Delta E_{\gamma\zeta}(t) - \int_0^t \frac{d\tilde{C}^{\alpha\beta\gamma\zeta}(t-\tau)}{d\tau}\,\Delta E_{\gamma\zeta}(\tau)\,d\tau \quad (3.2.1)$$

where $\tilde{C}^{\alpha\beta\gamma\zeta}(t) = \tilde{C}^{\alpha\beta\gamma\zeta}\,\tilde{Y}(t)$. To reduce memory requirements, we represent the relaxation modulus, $\tilde{Y}$, by a Prony series [Zienkiewicz and Watson (1966)], viz.,

$$\tilde{Y}(t) = \tilde{A}_0 + \sum_{j=1}^{J} \tilde{A}_j \exp(-\tilde{B}_j t) \qquad (3.2.2)$$

In what follows, we use a three-term Prony series, the use of which leads to a recurrence relation requiring storage of results from the previous time-step only.

For an isotropic, compressible material, we rewrite (3.2.1) using a trapezoidal approximation. After considerable algebra, (3.2.1) follows as

$$\Delta S^{\alpha\beta}(t+\Delta t)=\left(\tilde{Y}_0-\tilde{A}_\mu\tilde{B}_\mu\Delta t\right)\tilde{C}^{\alpha\beta\gamma\zeta}\Delta E_{\gamma\zeta}(t+\Delta t)-Q^{\alpha\beta}(t+\Delta t) \quad (3.2.3)$$

where $\tilde{Y}_o = \tilde{Y}(0)$. The memory term, $Q^{\alpha B}$, is given by the recurrence relation

$$Q^{\alpha\beta}(t+\Delta t)=\exp(-\tilde{B}_\mu\Delta t)\left[\tilde{A}_\mu\tilde{B}_\mu\Delta t\,\tilde{C}^{\alpha\beta\gamma\zeta}\Delta E_{\gamma\zeta}(t)+Q^{\alpha\beta}(t)\right] \quad (3.2.4)$$

with $Q^{\alpha\beta}(0) = 0$ and (assuming constant Poisson's ratio ν)

$$\tilde{C}^{\alpha\beta\gamma\zeta}=\frac{\nu}{(1+\nu)(1-2\nu)}G^{\alpha\beta}G^{\gamma\zeta}+\frac{1}{2(1+\nu)}\left(G^{\alpha\gamma}G^{\beta\zeta}+G^{\alpha\zeta}G^{\beta\gamma}\right) \quad (3.2.5)$$

Substituting (3.2.3) into (3.1.2), we obtain

$$\begin{aligned}&\left[\tilde{M}_{iJIJ}\right]\left\{\ddot{U}_j^J(t+\Delta t;k+1)\right\}+\left[\tilde{K}_{iJIJ}(t+\Delta t)\right]\left\{\Delta U_j^J(t+\Delta t)\right\}=\\&\left\{\tilde{P}_{iI}(t+\Delta t;k+1)\right\}-\left\{\tilde{F}_{iI}(t+\Delta t;k)\right\}+\left\{\tilde{Q}_{iI}(t+\Delta t)\right\}\end{aligned} \quad (3.2.6)$$

where

$$\begin{aligned}\left[\tilde{K}_{iJIJ}(t+\Delta t)\right]=\int_{-1}^{1}\int_{-1}^{1}\Big\{&\frac{1}{4}\left(\psi_{K,\alpha}\psi_{I,\beta}+\psi_{I,\alpha}\psi_{K,\beta}\right)\left[X_i^K+U_i^K(t)\right]\cdot\\&\left(\tilde{Y}_0-\tilde{A}_\mu\tilde{B}_\mu\Delta t\right)\tilde{C}^{\alpha\beta\gamma\zeta}\left(\psi_{L,\gamma}\psi_{J,\zeta}+\psi_{L,\zeta}\psi_{J,\gamma}\right)\cdot\\&\left[X_j^L+U_j^L(t)\right]+\\&\frac{1}{2}\left(\psi_{J,\alpha}\psi_{I,\beta}+\psi_{I,\alpha}\psi_{J,\beta}\right)S^{\alpha\beta}(t)\,\delta_{ij}\Big\}\cdot\\&H\sqrt{G}\,d\zeta_1\,d\zeta_2\end{aligned} \quad (3.2.7)$$

and $\{\tilde{Q}_{iI}(t+\Delta t)\}$ = memory load vector

$$\begin{aligned}=\int_{-1}^{1}\int_{-1}^{1}\frac{1}{2}\left(\psi_{K,\alpha}\psi_{I,\beta}+\psi_{I,\alpha}\psi_{K,\beta}\right)\left[X_i^K+U_i^K(t)\right]\cdot\\Q^{\alpha\beta}(t)\,H\sqrt{G}\,d\zeta_1\,d\zeta_2\end{aligned} \quad (3.2.8)$$

and $\{\tilde{Q}_{iI}(0)\} = \{0\}$.

III.4 ANALYTICAL VALIDATION AND APPLICATIONS

III.4.1 <u>In-plane deformation</u>. Using the correspondence principle [Christensen

(1982)], a closed-form solution is readily available for the viscoelastic deformation of a biaxial tension specimen with sides B and 2B and thickness H (B = 0.61 m, H = 0.25 mm in Figure III.1). (Using symmetry, only one-quarter of the structure is modeled.) The creep and relaxation moduli are taken respectively as:

$\tilde{J}$ (t) = [7.59 - 2.41exp(-0.7t) - 0.803exp(-0.05t)] GPa^{-1};

$\tilde{Y}$ (t) = [132 + 78.5exp(-1.07t) + 17.5exp(-0.07t)] MPa.

For a uniform 689 kPa load with ν = 0.45 and Δt = 0.5 hr (see Appendix F), theoretical results and computational results (using 8 - node, isoparametric quadrilateral elements) are shown in Figure III.2. The agreement is good.

III.4.2 Out-of-plane viscoelastic deformation. For verification purposes, we employ the improved Berger's equations of Schmidt and DaDeppo (1974), and invoke the correspondence principle. For the case of the uniform inflation of an initially plane circular membrane of radius R fixed at its perimeter, the viscoelastic equation for the vertical deflection of the central node (node 1 - see Figure III.3) is

$$U_3^{(1)} = \left[\frac{(1-\nu^2)\,P_0\,R^4}{4\,H}\right]^{\frac{1}{3}} \left[1 + \frac{1-\nu}{36} + \frac{35\,(1-\nu)^2}{324} - \ldots\right] [\tilde{J}\,(t)]^{\frac{1}{3}} \quad (4.2.1)$$

The following values are used: R = 101.5 mm, ν = 0.3, H = 1.27 mm, and Δt and $\tilde{J}$ (t) are taken as in Sec. 4.1. The comparison of theoretical and numerical values for the central vertical displacement are shown in Figure III.4, for a pressure of 6.89 kPa. Good agreement is observed.

III.4.3 Out-of-plane visco-hyperelastic deformation. The method is extended to consider the example of Sec. 4.2 but for larger deformations using a hyperelastic

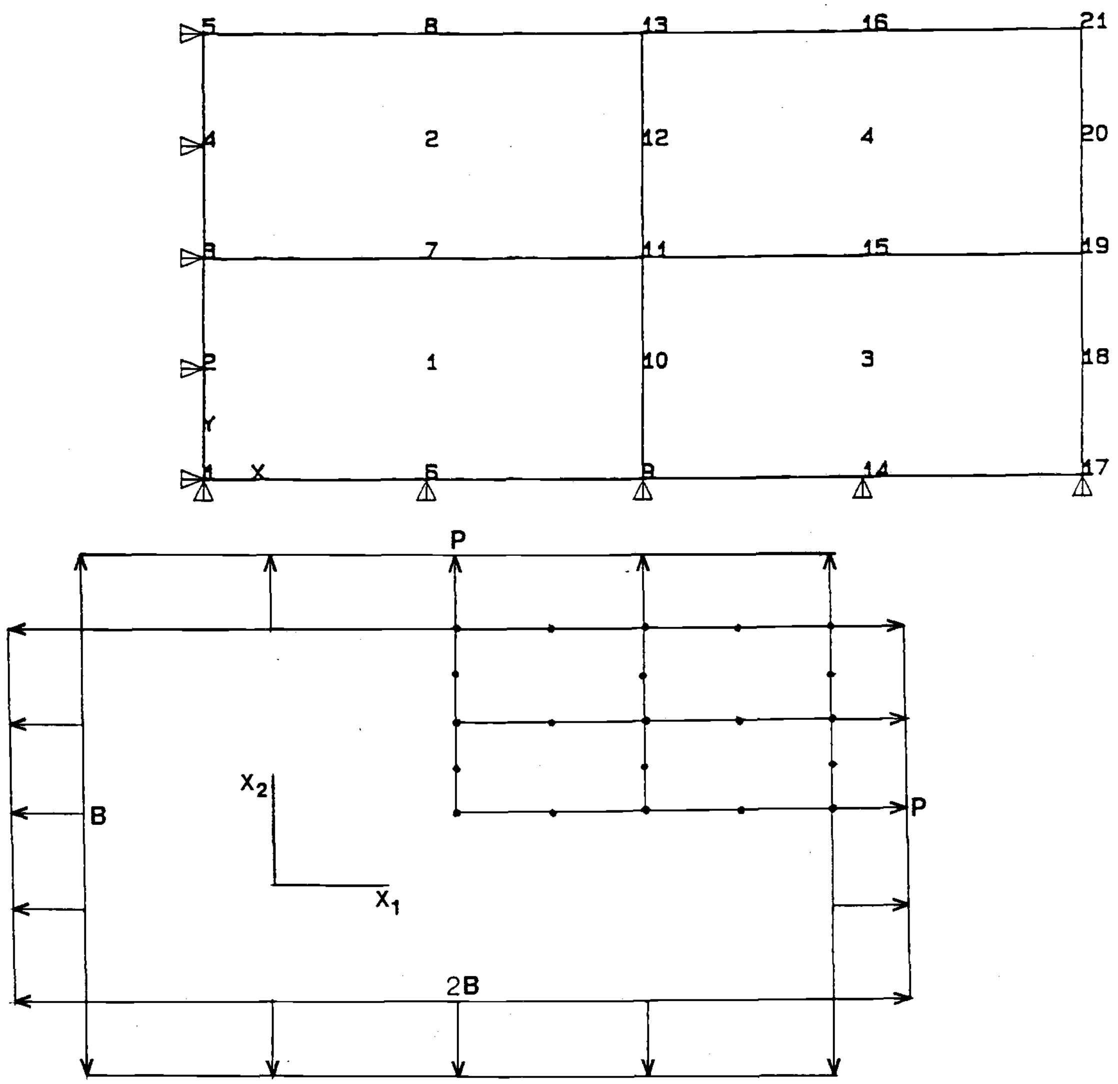

Figure III.1 Biaxial tension specimen configuration.

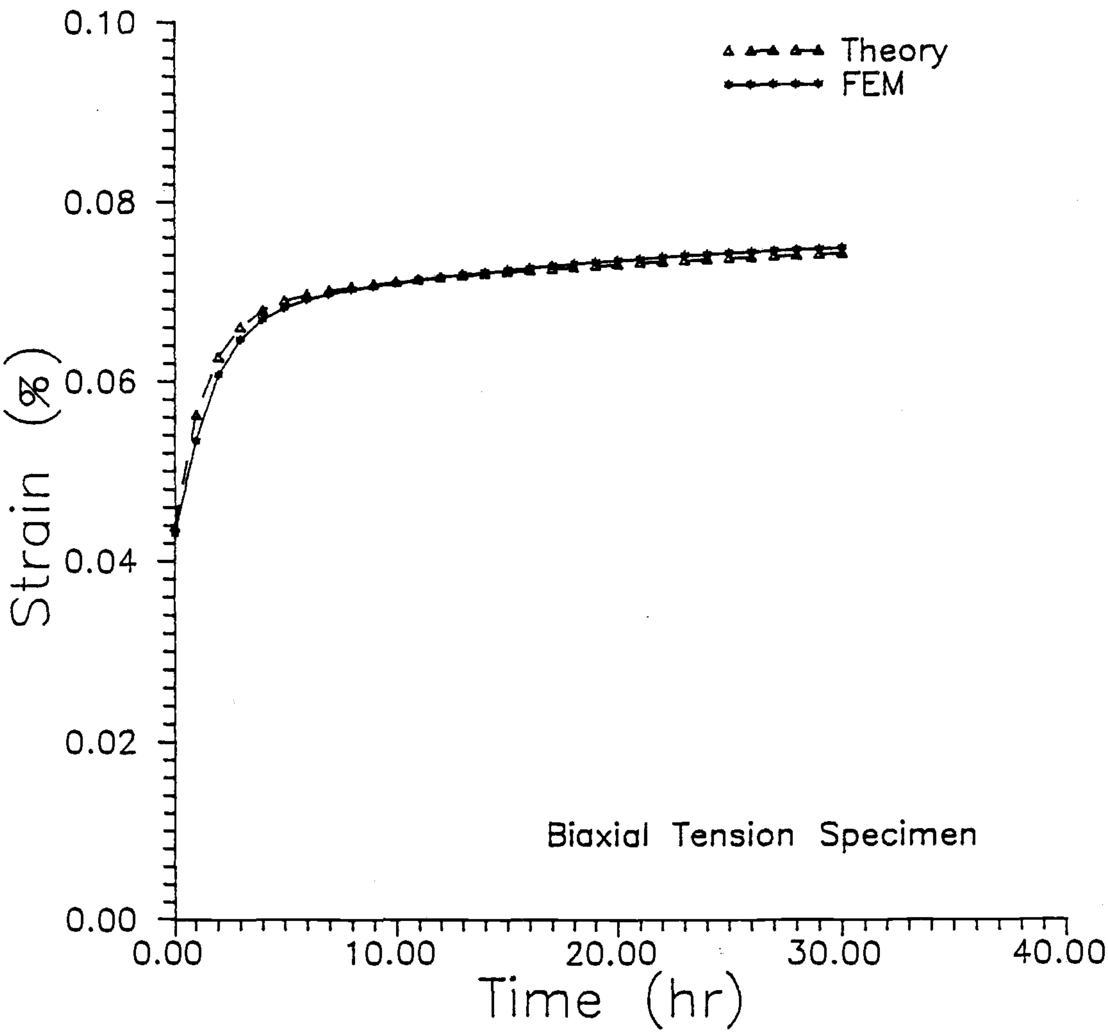

Figure III.2 Results for biaxial tension specimen

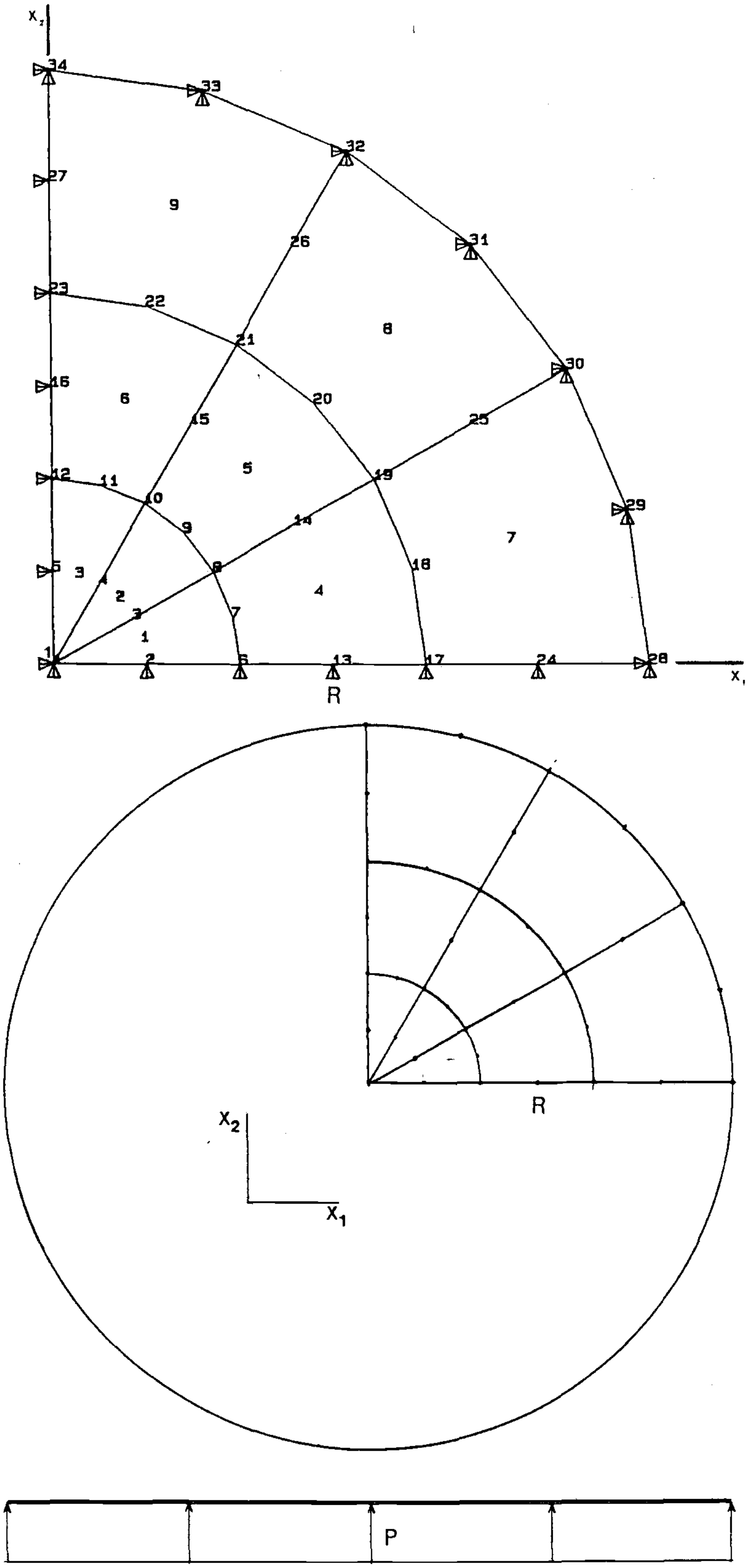

Figure III.3 Plane circular membrane configuration

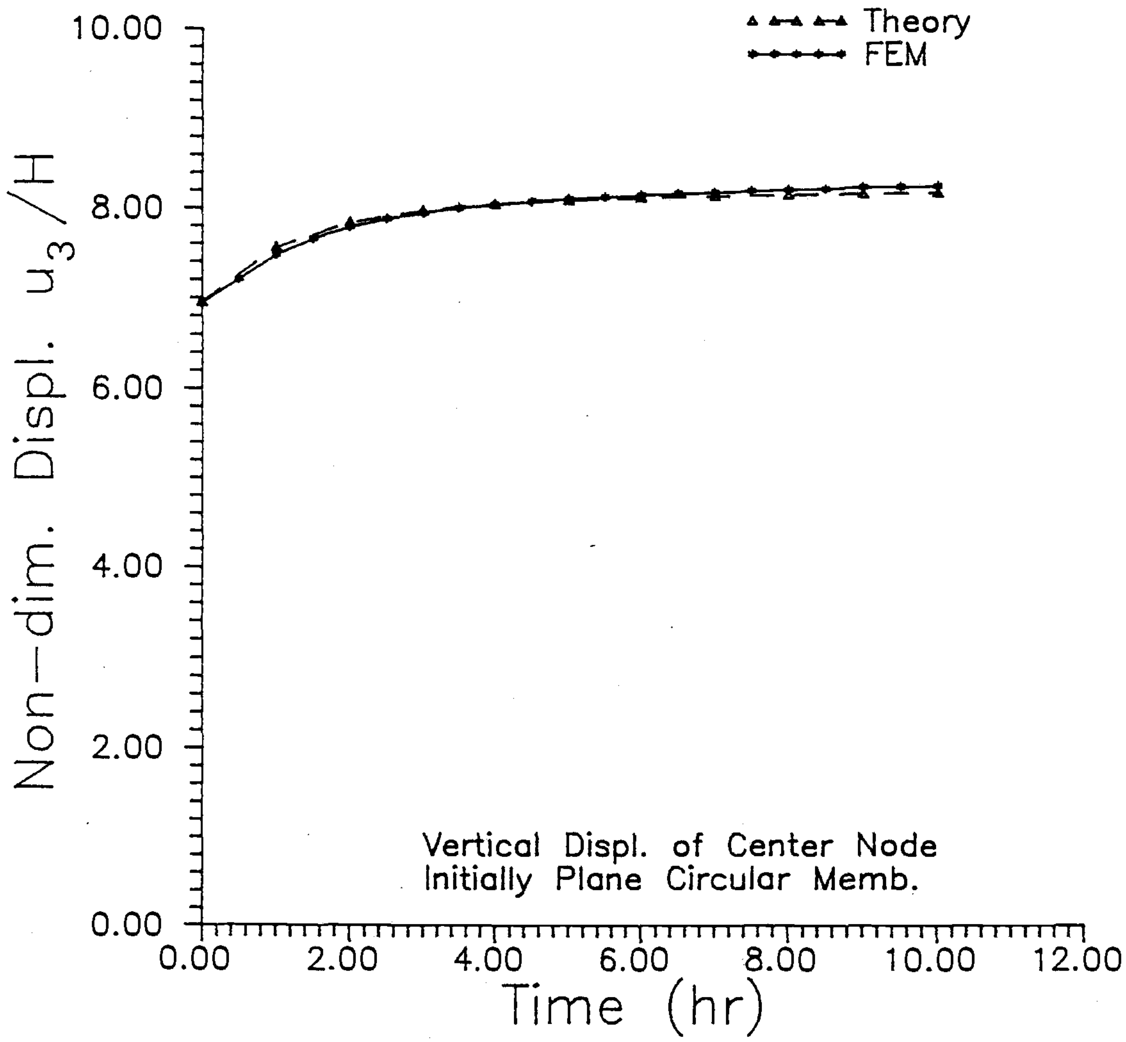

Figure III.4 Results for uniform inflation of initially plane circular viscoelastic membrane material, i.e., where $S^{\alpha\beta}=\partial \tilde{W}/\partial E_{\alpha\beta}$. For the so-called 'neo-Hookean' material [see, e.g., Jenkins and Leonard (1991)] this can be shown to be

$$S^{\alpha\beta} = 2\,\tilde{D}\,G^{\alpha\beta} + 2\,\frac{G}{g}\,\tilde{D}\,g^{\alpha\beta} \tag{4.3.1}$$

where $\tilde{D}$ is a material constant.

Recalling (3.1.7), a nonlinear relation such as (4.3.1) can be considered incrementally linear. Then using $\tilde{C}^{\alpha\beta\gamma\zeta} = \partial S^{\alpha\beta}/\partial E_{\alpha\beta}$, it can be shown [Lo (1981)] that the incremental neo-Hookean constitutive coefficients are

$$\tilde{C}^{\alpha\beta\gamma\zeta} = 8g^{\alpha\beta}g^{\gamma\zeta}\frac{G}{g}\tilde{D} - \frac{2}{g}\left(\varepsilon^{\alpha\gamma}\varepsilon^{\beta\zeta} + \varepsilon^{\alpha\zeta}\varepsilon^{\beta\gamma}\right)\frac{G}{g}\tilde{D} \tag{4.3.2}$$

Upon substitution of (4.3.2) into (3.2.7), a visco-hyperelastic material is modeled.

Figure III.5 shows the central vertical deflection for the uniform loading of an initially plane circular visco-hyperelastic (VHE) PVC membrane. Dimensions are as in Sec. 4.2 with $\tilde{Y}(t) = [3.97 + 4.24\exp(-2.6\times10^{-2}t) + 1.07\exp(-1.8\times10^{-3}t)]$ MPa, and a uniform pressure of 6.895 kPa. Also shown are results for the same problem but using a 'Hookean' viscoelastic material (VE) with the relaxation modulus equal to $6\tilde{Y}(t)$ and $\nu = 0.4995$, as well as the time-independent hyperelastic material with $\tilde{D} = \tilde{Y}(0)$. Finally, we show in Figure III.6 a profile history for an inflation pressure of 13.79 kPa. In each case, $\Delta t = 0.5$ s (see Appendix F).

III.4.4 Ocean application. The aforementioned method is applied to the case of a submerged membrane loaded by surface waves. A 3.05 m diameter circular cylindrical

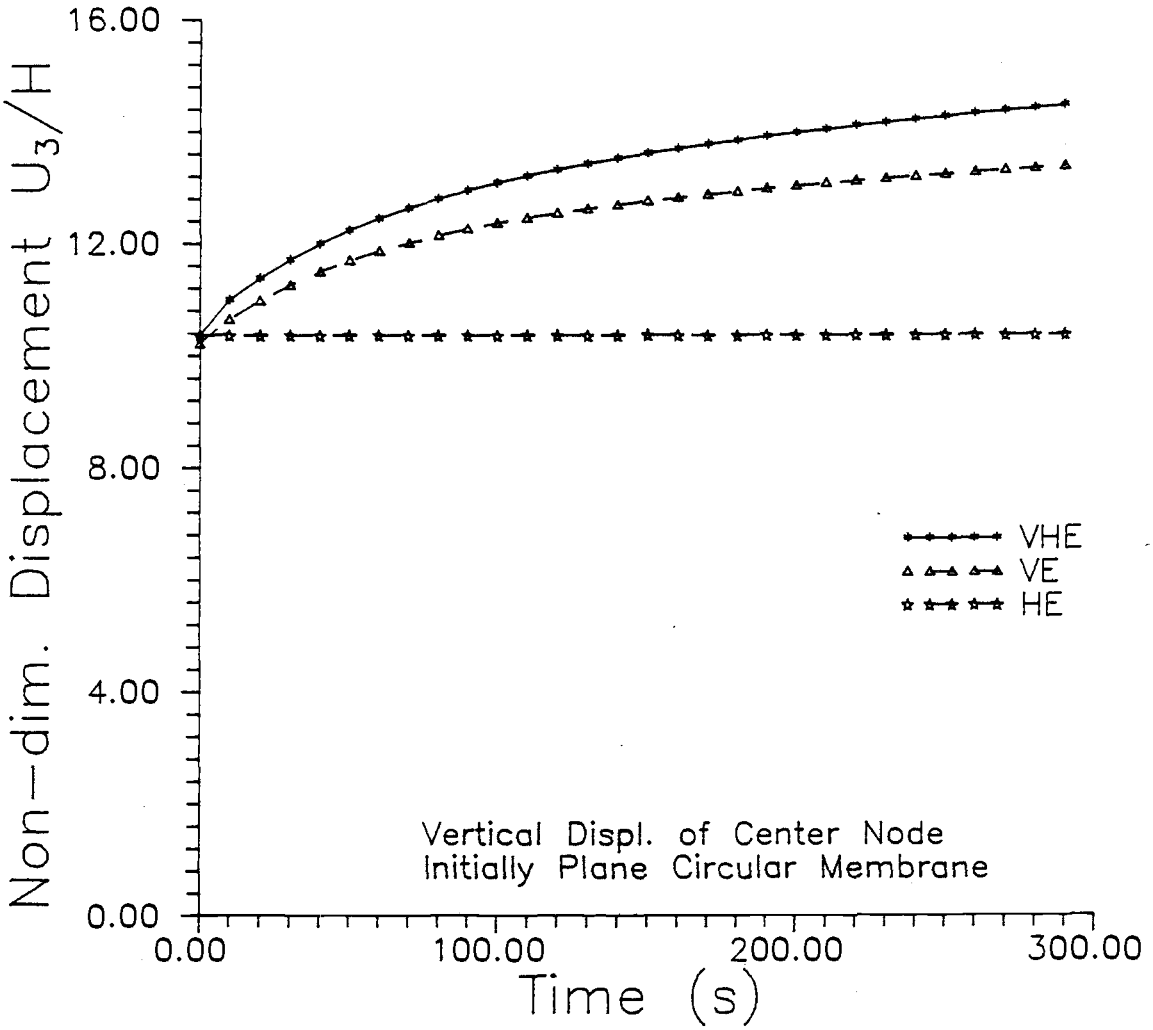

Figure III.5 Results for uniform inflation of initially plane visco-hyperelastic circular membrane

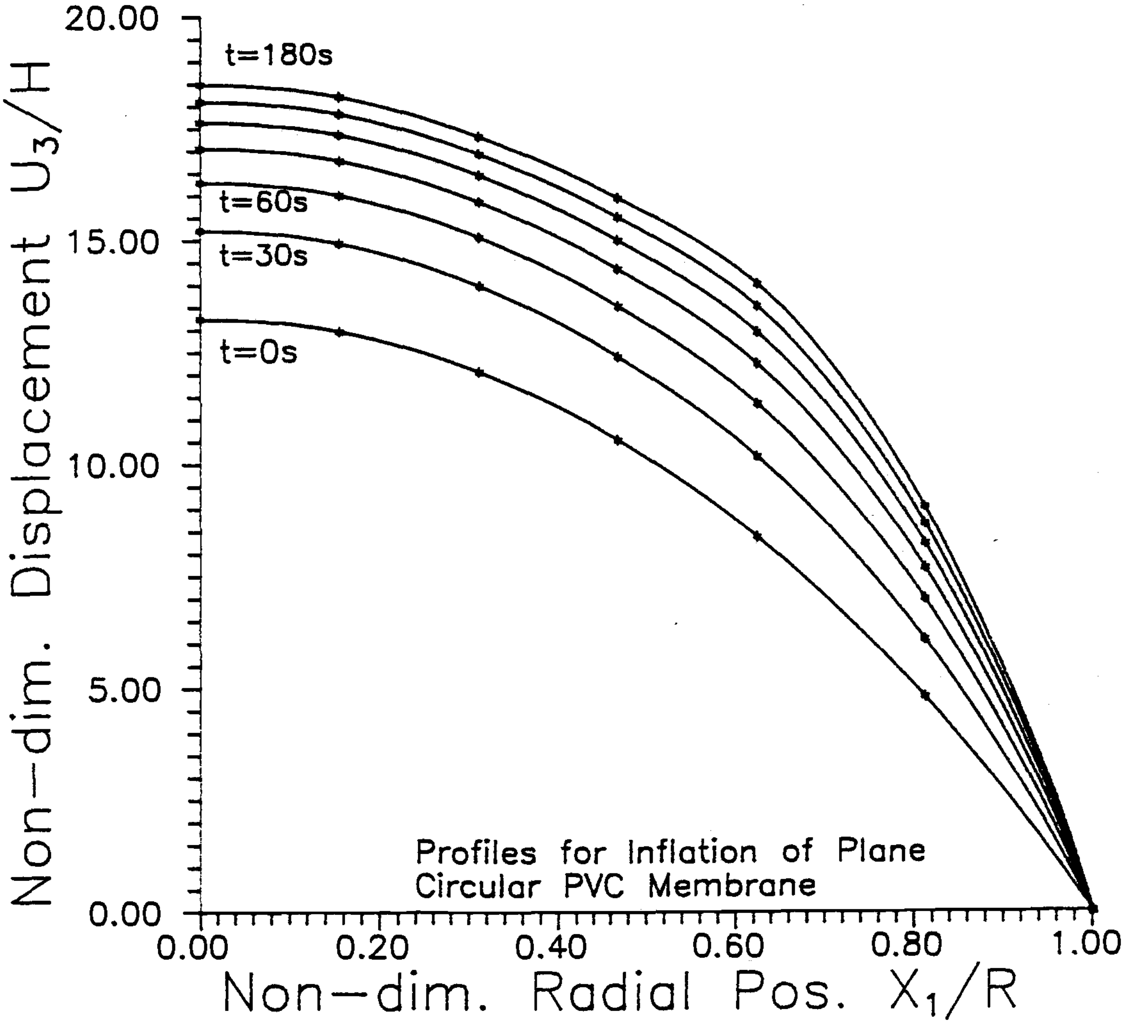

Figure III.6 Profile history for visco-hyperelastic circular membrane.

membrane of length 3.05 m is fixed horizontally at a depth of 10.7 m in 15.2 m of water depth as shown in Figure III.7; both ends are clamped. A 1.22 m amplitude wave of 4 s period (wavelength = 25.0 m) is incident on the membrane, normal to the axis of the cylinder. Figure III.8 shows a comparison of the horizontal excursion of a central node for both a linear elastic and a trial viscoelastic material with $\widetilde{E} = \widetilde{Y}_0$ and $\widetilde{Y}(t) = [5.74 + 3.42\exp(-3.0t) + 0.765\exp(-0.1t)]$ GPa, respectively. Six, 8-node quadrilateral elements model the structure, and $\Delta t = 0.1$ s.

III.5 CONCLUSION

A direct solution of the constitutive equation has been shown to be effective when applied within a nonlinear finite element formulation. Validation is achieved for both in-plane and out-of-plane examples, where transients are masked by a combination of viscous relaxation technique (IV.4.1) and time step size (II.2.5). The method is also applied to a transient membrane problem. The growth of deformations with time is clearly seen in all results.

Finite deformations can be handled by noting that a nonlinear constitutive equation can be considered incrementally linear. Results for larger deformations, using the nonlinear constitutive equation (visco-hyperelastic), diverge from those relying on the linear relation (viscoelastic).

Future directions point to the inclusion of a fully nonlinear constitutive equation, extension to anisotropic media, and the capability to analyze wrinkled membranes.

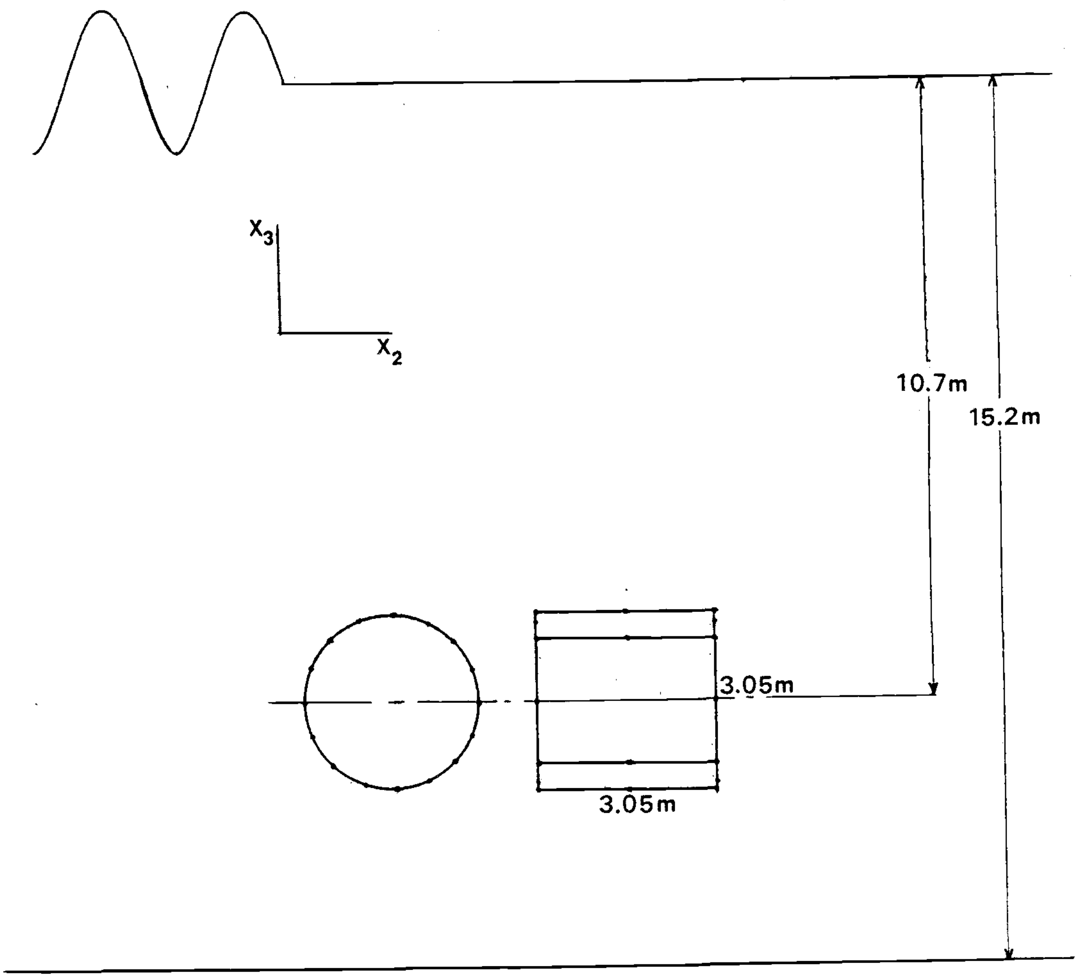

Figure III.7 Submerged membrane configuration.

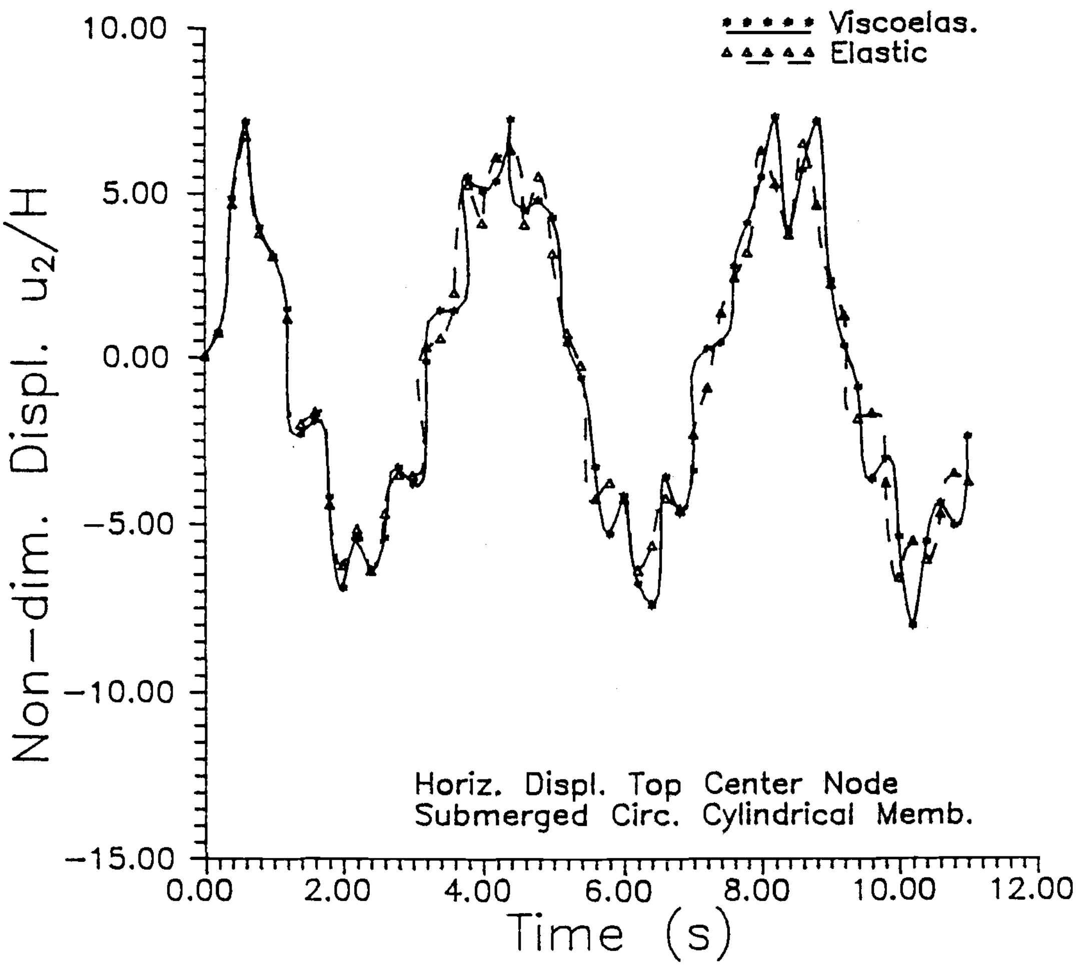

Figure III.8 Results for submerged membrane

III.6 ACKNOWLEDGEMENT

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CHAPTER IV

TRANSIENT DEFORMATIONS OF WRINKLED VISCOELASTIC MEMBRANES

IV.1 INTRODUCTION

IV.1.1 Background. Thin membranes are inherently no-compression structures. Compressive stress, of sufficient magnitude to overcome tensile prestress, will be handled via changes in membrane geometry, i.e., by an out-of-plane deformation or localized buckling called 'wrinkling'.

Analysis of wrinkling is important to the prediction of membrane structural response. In long-term loading situations, the creep/relaxation response of viscoelastic materials will tend to decrease the level of prestress, thus increasing the formation of wrinkles. Problems of dynamic wrinkling (e.g., panel flutter) are of interest for the effects on fatigue analysis (e.g., tension field effects on mean stress distribution) and on snap loading (e.g., when a wrinkled region suddenly regains the lost principal stress).

IV.1.2 Prior Research. Wagner (1929) introduced the ideas of wrinkling and 'tension field theory' in connection with flat sheet metal girders in the very thin metal webs used in airplane construction. Under the action of a specific loading, one of the principal stresses goes to zero while the other remains nonnegative. If the nonnegative principal stress remains greater than zero, a 'tension field' is defined; if it is zero, a 'slack' region results. The crests and troughs of 'wrinkle waves' align with the direction of the nonzero principal stress. Reissner (1938) generalized Wagner's results by introducing an artificial orthotropy into the membrane model whereby a separate elastic modulus is associated with each principal direction. The wrinkling condition is thus given by the diminution of one of the moduli to zero.

In typical wrinkling analysis, results are only in terms of average strains and displacements, while no detailed information is generated for each wrinkle. Furthermore, a membrane need not be wrinkled over its entire surface. Stein and Hedgepeth (1961) introduced the concept of a 'variable Poisson's ratio' to study partly wrinkled membranes. [A review of membrane wrinkling research is given by Jenkins and Leonard (1991a)].

The formation of a finite number of wrinkles during membrane deformation relies on the inherent (albeit small) bending stiffness of the material. Detailed description of the wrinkling phenomena is absent in membrane analysis since the bending stiffness is disregarded. Pipkin (1986) and Steigmann and Pipkin (1989a,b,c) discuss this further and postulate the existence of a 'relaxed strain-energy density', which represents the average energy per unit initial area over a region containing many wrinkles. The relaxed energy density is constrained such that its derivatives (stresses) are nonnegative, thus incorporating tension field theory into membrane theory automatically.

IV.1.3 Conventions. The following conventions are used: the summation convention is implied unless explicitly stated otherwise; Latin indices take the values 1,2,3 unless explicitly stated otherwise; Greek indices take the values 1,2; capital and lower case Latin letters (except indices) refer to the undeformed and deformed state, respectively; bold type indicates vector or tensor quantities; superscripts or subscripts enclosed in parentheses indicate no sum.

IV.1.4 Strain and stress measure. Consider the Cartesian coordinates X_i of a point X on the undeformed midsurface which becomes point x with coordinates x_i on the deformed midsurface. Also define convected curvilinear midsurface coordinates $\Theta^i = \theta^i$, i.e., the Θ^i coordinates of X are numerically equal to the θ^i coordinates of x. The metric tensor in the deformed state is $g_{k\ell} = \mathbf{g}_k \bullet \mathbf{g}_\ell$, $\mathbf{g}_i = \partial \mathbf{r}/\partial \theta^i$, and $\mathbf{r}$ is the position vector from o to x; similarly, in the undeformed state, $G_{k\ell} = \mathbf{G}_k \bullet \mathbf{G}_\ell$, where $\mathbf{G}_i = \partial \mathbf{R}/\partial \theta^i$, and $\mathbf{R}$ is the position vector from 0 to X.

The convected Green-Lagrange strain tensor components, E_{kl}, are given by [Green and Zerna (1968)]

$$E_{k\ell} = \frac{1}{2}\left(g_{k\ell} - G_{k\ell}\right) \qquad (1.4.1)$$

The contravariant components of the Cauchy stress tensor s^{ij}, measured per unit area of deformed midsurface, may be related to the convected Piola-Kirchoff stress tensor components S^{ij}, measured per unit area of undeformed midsurface, by [Green and Zerna (1968)]

$$s^{ij} = \left(\bar{I}_3\right)^{-\frac{1}{2}} S^{ij} \qquad (1.4.2)$$

where $\bar{I}_3 = G/g$, $G = \det G_{ij}$, and $g = \det g_{ij}$.

IV.2 VISCOELASTIC FINITE ELEMENTS

IV.2.1 Nonlinear finite element method. The starting point for computations is a combined incremental/iterative finite element method [Lo (1981)]. An implicit (Newmark's), incremental method is used to solve the equations of motion. Within each time-step, the modified Newton-Raphson iterative method is used to converge on the nonlinear solution.

We begin with a virtual work expression with plane stress assumption [Leonard (1988)],

$$\iint_A \left(HS^{\alpha\beta}\delta E_{\alpha\beta} - P_i \delta U_i\right)\sqrt{G}\, d\theta^1 d\theta^2 - H\iint_A M\ddot{U}_i \delta U_i \sqrt{G}\, d\theta^1 d\theta^2 = 0 \qquad (2.1.1)$$

where $\mathbf{P}$ = vector of surface tractions, $\mathbf{U}$ = vector of displacements, M = initial membrane mass density, H = initial membrane thickness, and A = initial membrane area. Note that, due to the relatively low mass of thin membrane structures, body forces have been neglected with respect to surface tractions.

We follow the usual finite element discretization and recast (2.1.1) into a form suitable for the 'modified' Newton-Raphson method, using an isoparametric formulation (the same shape function is used to describe both position and displacement). With the notation that the increment of time is Δt, Ψ_I = isoparametric shape function for node I, X_i^I = ith initial coordinate of node I, U_i^I = ith displacement of node I, ξ_i = natural coordinate of the element, I = 1,..., number of nodes (per element), and k=1,...,no. of iterations per time step, we write the combined incremental/iterative membrane equation of motion as [Lo (1981)]:

$$[\tilde{M}_{iJIJ}]\left\{\ddot{U}_j^J(t+\Delta t;k+1)\right\} + \left[\tilde{K}_{ijIJ}(t+\Delta t)\right]\left\{\Delta U_j^J(t+\Delta t)\right\} = \left\{\tilde{P}_{iI}(t+\Delta t;k+1)\right\} - \left\{\tilde{F}_{iI}(t+\Delta t;k)\right\} \tag{2.1.2}$$

where

$[\widetilde{M}_{ijIJ}]$ = consistent mass matrix

$$= \int_{-1}^{1}\int_{-1}^{1} \psi_I \psi_J \delta_{ij} HM\sqrt{G}\, d\xi_1\, d\xi_2 \tag{2.1.3}$$

$[\widetilde{K}_{ijIJ}(t+\Delta t)]$ = tangent stiffness matrix

$$= \int_{-1}^{1}\int_{-1}^{1} \left\{\frac{1}{2}(\psi_{K,\alpha}\psi_{I,\beta} + \psi_{I,\alpha}\psi_{K,\beta})\left[X_i^K + U_i^K(t)\right]\tilde{C}^{\alpha\beta\gamma\zeta} \cdot(\psi_{L,\gamma}\psi_{J,\zeta} + \psi_{L,\zeta}\psi_{J,\gamma})\,[X_j^L + U_j^L] + \frac{1}{2}(\psi_{J,\alpha}\psi_{I,\beta} + \psi_{I,\alpha}\psi_{J,\beta})\,\delta_{ij}\, S^{\alpha\beta}(t)\right\} H\sqrt{G}\, d\xi_1\, d\xi_2 \tag{2.1.4}$$

$\{\widetilde{P}_{iI}(t+\Delta t;k+1)\}$ = external force vector

$$= \int_{-1}^{1}\int_{-1}^{1} P_i(t+\Delta t;k+1)\;\psi_I\sqrt{G}\, d\xi_1\, d\xi_2 \tag{2.1.5}$$

$\{\widetilde{F}_{iI}(t+\Delta t;k)\}$ = internal force vector

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}(\psi_{J,\alpha}\psi_{I,\beta} + \psi_{I,\alpha}\psi_{J,\beta})\left[X_i^J + U_i^J(t+\Delta t;k)\right]\cdot S^{\alpha\beta}(t+\Delta t;k)\, H\sqrt{G}\, d\xi_1\, d\xi_2 \tag{2.1.6}$$

and

$$\Delta U_j^J(t+\Delta t) = U_j^J(t+\Delta t;k+1) - U_j^J(t+\Delta t;k) \qquad (2.1.7)$$

We note that the above formulation describes a 'mapping' from the general curvilinear convected coordinates Θ^i to the element coordinates ξ_i.

IV.2.2 Constitutive Equation - General. The method of 'local state' [see Germain (1973); Lemaitre and Chaboche (1990)] postulates that the thermodynamic state of a continuum at a specific location and time is completely defined by the values of certain variables (state variables) at that time and location, and which are a function of that location only. These state variables (also called thermodynamic or independent variables) are of two classes: observable and internal. The observable variables, temperature and deformation, define elastic (reversible) phenomena uniquely as a function of time. Internal variables (e.g., deformation rate) are required for the representation of dissipative phenomena, since the current state also depends on the state history. State laws are derived from postulated thermodynamic potentials which are functions of the state variables. In order to satisfy the Clausius-Duhem inequality (second law of thermodynamics), potentials must be nonnegative, convex functions with zero values at the origin of state variable space; a typical choice is that of a positive-definite quadratic form.

In light of the above discussion, we postulate the existence of strain energy and dissipation functions $\tilde{W} = \tilde{W}(E, \tilde{C}, t)$ and $\tilde{V} = \tilde{V}(\dot{E}, \tilde{C}, t)$, respectively, such that

$$S^{ij}(t) = \frac{\partial \tilde{W}}{\partial E_{ij}}(t) + \frac{\partial \tilde{V}}{\partial \dot{E}_{ij}}(t) \tag{2.2.1}$$

In Appendix B we show the connection between (2.2.1) and the linear hereditary constitutive equation. We now generalize as follows: for suitable choices of state variables in $\tilde{W}$ and $\tilde{V}$, and for a suitable material function approximation by a Prony series, the following finite linear viscoelastic constitutive relation is obtained from (2.2.1) (where the dependence of the material function on current strain has been neglected)

$$S^{\alpha\beta}(t) = \tilde{C}^{\alpha\beta\gamma\zeta}(0)\, E_{\gamma\zeta}(t) - \int_0^t \frac{d\tilde{C}^{\alpha\beta\gamma\zeta}(t-\tau)}{d\tau} E_{\gamma\zeta}(\tau)\, d\tau \tag{2.2.2}$$

We note that Glockner and Szyszkowski presented in a series of papers [Szyszkowski and Glockner (1987); Glockner and Szyszkowski (1989a,b); Szyszkowski and Glockner (1989); and Glockner and Szyszkowski (1990)] a similar derivation of a constitutive equation using the hereditary integral by postulating the existence of a 'complementary power potential'.

IV.2.3 Constitutive Equation - Computational Form. Solution of complicated viscoelastic problems will generally require numerical techniques. The choice of either transform or direct methods may not be clear, each having its own proponents, advantages, and disadvantages. Use of transform methods is generally limited to problems which do not exhibit time-dependent boundary conditions. In the direct method proposed by White (1968), the governing integro-differential constitutive equation is approximated by a finite-difference equation and embedded within the spatial discretization, thus making a viscoelastic finite element. This is the method followed in the present work. [For a review of computational methods in viscoelasticity, see Jenkins and Leonard (1991b).]

For a stress tensor **S** that is a continuous function of the strain tensor **E** during a displacement increment, a Taylor series approximation can be made as follows:

$$S = S(E)$$

$$\begin{aligned} S^{\alpha\beta}(t+\Delta t;k+1) &= S^{\alpha\beta}(t+\Delta t;k) \\ &\quad + \frac{\partial S^{\alpha\beta}(t+\Delta t;k)}{\partial E_{\gamma\zeta}} \Delta E_{\gamma\zeta}(t+\Delta t) + \ldots \end{aligned}$$

where

$$\begin{aligned} \Delta E_{\gamma\zeta}(t+\Delta t) &= \frac{1}{2}(\psi_{J,\gamma}\psi_{I,\zeta} + \psi_{I,\gamma}\psi_{J,\zeta}) \cdot \\ &\quad \left[X_i^J + U_i^J(t+\Delta t;k)\right] \Delta U_i^I(t+\Delta t) \end{aligned}$$

Then neglecting terms smaller than $O(\Delta E_{\gamma\zeta})$ we have

$$\begin{aligned} \Delta S^{\alpha\beta}(t+\Delta t) &= S^{\alpha\beta}(t+\Delta t;k+1) - S^{\alpha\beta}(t+\Delta t;k) \\ &= \bar{C}^{\alpha\beta\gamma\zeta} \Delta E_{\gamma\zeta}(t+\Delta t) \end{aligned} \tag{2.3.1}$$

where ΔU_i^I is given by (2.1.7), and

$$\bar{C}^{\alpha\beta\gamma\zeta} = \frac{\partial S^{\alpha\beta}(t+\Delta t;k)}{\partial E_{\gamma\zeta}}$$

is the incremental constitutive tensor. For a linear material, (2.3.1) is exact.

Following (2.3.1), we write the linear viscoelastic relation (2.2.2) as

$$\Delta S^{\alpha\beta}(t) = \tilde{C}^{\alpha\beta\gamma\zeta}(0)\,\Delta E_{\gamma\zeta}(t) - \int_0^t \frac{d\,\tilde{C}^{\alpha\beta\gamma\zeta}(t-\tau)}{d\tau}\,\Delta E_{\gamma\zeta}(\tau)\,d\tau \quad (2.3.2)$$

where $\tilde{C}^{\alpha\beta\gamma\zeta}(t) = \tilde{C}^{\alpha\beta\gamma\zeta}\,\tilde{Y}(t)$. To reduce memory requirements, we represent the relaxation modulus, $\tilde{Y}$, by a Prony series [Zienkiewicz and Watson (1966)], viz.,

$$\tilde{Y}(t) = \tilde{A}_0 + \sum_{j=1}^{J} \tilde{A}_j \exp(-\tilde{B}_j t) \quad (2.3.3)$$

In what follows, we use a three-term Prony series, which leads to a recurrence relation requiring storage of results from the previous time-step only. The adequacy of this double exponential model for solid polymers has been discussed by Garbarski (1989).

For an isotropic, compressible material, we rewrite (2.3.2) using a trapezoidal approximation [White (1968)]. After considerable algebra, (2.3.2) becomes

$$\Delta S^{\alpha\beta}(t+\Delta t) = (\tilde{Y}_0 - \tilde{A}_\mu \tilde{B}_\mu \Delta t)\,\tilde{C}^{\alpha\beta\gamma\zeta}\,\Delta E_{\gamma\zeta}(t+\Delta t) - Q^{\alpha\beta}(t+\Delta t) \quad (2.3.4)$$

where $\tilde{Y}_o = \tilde{Y}(0)$. The memory term, $Q^{\alpha\beta}$, is given by the recurrence relation

$$Q^{\alpha\beta}(t+\Delta t) = \exp(-\tilde{B}_\mu \Delta t)\left[\tilde{A}_\mu \tilde{B}_\mu \Delta t\,\tilde{C}^{\alpha\beta\gamma\zeta}\,\Delta E_{\gamma\zeta}(t) + Q^{\alpha\beta}(t)\right] \quad (2.3.5)$$

with $Q^{\alpha\beta}(0) = 0$ and (assuming constant Poisson's ratio ν)

$$\tilde{C}^{\alpha\beta\gamma\zeta} = \frac{\nu}{(1-\nu^2)}\,G^{\alpha\beta} G^{\gamma\zeta} + \frac{1}{2(1+\nu)}\left(G^{\alpha\gamma} G^{\beta\zeta} + G^{\alpha\zeta} G^{\beta\gamma}\right) \quad (2.3.6)$$

Substituting (2.3.4) into (2.1.2), we obtain

$$[\tilde{M}_{ijIJ}]\{\ddot{U}_j^J(t+\Delta t;k+1)\}+[\tilde{K}_{ijIJ}(t+\Delta t)]\{\Delta U_j^J(t+\Delta t)\}= \\ \{\tilde{P}_{iI}(t+\Delta t;k+1)\}-\{\tilde{F}_{iI}(t+\Delta t;k)\}+\{\tilde{Q}_{iI}(t+\Delta t)\} \tag{2.3.7}$$

where

$$[\tilde{K}_{ijIJ}(t+\Delta t)]=\int_{-1}^{1}\int_{-1}^{1}\Big\{\frac{1}{4}(\psi_{K,\alpha}\psi_{I,\beta}+\psi_{I,\alpha}\psi_{K,\beta})[X_i^K+U_i^K(t)]\cdot \\ (\tilde{Y}_0-\tilde{A}_\mu\tilde{B}_\mu\Delta t)\tilde{C}^{\alpha\beta\gamma\zeta}(\psi_{L,\gamma}\psi_{J,\zeta}+\psi_{L,\zeta}\psi_{J,\gamma})\cdot \\ [X_j^L+U_j^L(t)]+ \\ \frac{1}{2}(\psi_{J,\alpha}\psi_{I,\beta}+\psi_{I,\alpha}\psi_{J,\beta})S^{\alpha\beta}(t)\,\delta_{ij}\Big\}\cdot \\ H\sqrt{G}\,d\zeta_1\,d\zeta_2 \tag{2.3.8}$$

and

$\{\tilde{Q}_{iI}(t+\Delta t)\}$ = memory load vector

$$=\int_{-1}^{1}\int_{-1}^{1}\frac{1}{2}(\psi_{K,\alpha}\psi_{I,\beta}+\psi_{I,\alpha}\psi_{K,\beta})[X_i^K+U_i^K(t)]\cdot \\ Q^{\alpha\beta}(t)\,H\sqrt{G}\,d\zeta_1 d\zeta_2 \tag{2.3.9}$$

with $\{\tilde{Q}_{iI}(0)\} = \{0\}$.

IV.3 WRINKLING ANALYSIS

IV.3.1 Formulation. For wrinkling under a plane stress assumption, we define principal stresses S^β (see Appendix C):

$$S^1(t)=\frac{\partial\tilde{W}}{\partial E_1(t)}+\frac{\partial\tilde{V}}{\partial\dot{E}_1(t)}\ ,\quad S^2(t)=\frac{\partial\tilde{W}}{\partial E_2(t)}+\frac{\partial\tilde{V}}{\partial\dot{E}_2(t)} \tag{3.1.1}$$

where E_β are the principal strains. Following Steigmann and Pipkin (1989a,b,c), a 'natural width' (in simple tension), $\overset{*}{E}_2[E_1(t)]$, is defined such that when $S^2(t)\rightarrow\overset{*}{S}{}^2 = 0$, $E_2(t) = \overset{*}{E}_2[E_1(t)]$, and $S^1\rightarrow\overset{*}{S}{}^1[E_1, \overset{*}{E}_2, t]$, where starred quantities denote values at wrinkling. When $E_2(t) \leq \overset{*}{E}_2\,[E_1(t)]$ (with $E_1 > 0$), $E_2 \rightarrow \overset{*}{E}_2$, and 'relaxed' strain energy and dissipation functions are defined as

$$\tilde{W}^{*} = \tilde{W}^{*}(\tilde{\mathbf{C}}, E_1, \overset{*}{E}_2, t)\ , \quad \tilde{V}^{*} = \tilde{V}^{*}(\tilde{\mathbf{C}}, \dot{E}_1, \overset{*}{\dot{E}}_2, t) \tag{3.1.2}$$

from which the stresses during wrinkling may be formally found.

We note that the above differs from the approach of some authors [see, e.g., Contri and Schrefler (1988)] who assume S^1 remains fixed instead of E_1 above. In either case, the strain energy after wrinkling is never greater than the strain energy before wrinkling.

For the moderate deformation of compressible isotropic <u>elastic</u> membranes (dissipation function equals zero), the wrinkling condition can be shown to be (see Appendix C):

$$\overset{*}{E}_2(E_1) = -\frac{\tilde{C}^{21}}{\tilde{C}^{22}} E_1 \tag{3.1.3}$$

The wrinkling condition for isotropic finite linear <u>viscoelastic</u> membranes under plane stress is found from

$$\overset{*}{S}{}^2(t) = \tilde{C}^{2\beta}(0)\,\overset{*}{E}_\beta(t) - \int_0^t \frac{d\tilde{C}^{2\beta}(t-\tau)}{d\tau}\,\overset{*}{E}_\beta(\tau)\,d\tau = 0 \tag{3.1.4}$$

or

$$\tilde{C}^{22}(0)\overset{*}{E}_2(t)-\int_0^t\frac{d\tilde{C}^{22}(t-\tau)}{d\tau}\overset{*}{E}_2(\tau)\,d\tau=$$
$$-\tilde{C}^{21}(0)E_1(t)+\int_0^t\frac{d\tilde{C}^{21}(t-\tau)}{d\tau}E_1(\tau)\,d\tau \qquad (3.1.5)$$

Then $S^1 \rightarrow \overset{*}{S}{}^1\,[E_1, \overset{*}{E}_2, t]$ in the wrinkling region, or

$$\begin{aligned}\overset{*}{S}{}^1(t)=&\tilde{C}^{11}(0)E_1(t)+\tilde{C}^{12}(0)\overset{*}{E}_2(t)\\&-\int_0^t\frac{d}{d\tau}[\tilde{C}^{11}(t-\tau)E_1(\tau)+\tilde{C}^{12}(t-\tau)\overset{*}{E}_2(t-\tau)]\,d\tau\\=&\tilde{C}^{11}(0)E_1(t)-\int_0^t\frac{d\tilde{C}^{11}(t-\tau)}{d\tau}E_1(\tau)\,d\tau\\&+\tilde{C}^{12}(0)\overset{*}{E}_2(t)-\int_0^t\frac{d\tilde{C}^{12}(t-\tau)}{d\tau}\overset{*}{E}_2(\tau)\,d\tau\end{aligned} \qquad (3.1.6)$$

For constant Poisson's ratio, ν ,

$$\tilde{C}^{12}(t)=\nu\frac{\overset{*}{G}{}^{11}}{\overset{*}{G}{}^{22}}\tilde{C}^{22}(t)=\nu\frac{\overset{*}{G}{}^{22}}{\overset{*}{G}{}^{11}}\tilde{C}^{11}$$

Then

$$\begin{aligned}\overset{*}{S}{}^1(t)=&\tilde{C}^{11}(0)E_1(t)-\int_0^t\frac{d\tilde{C}^{11}(t-\tau)}{d\tau}E_1(\tau)\,d\tau\\&+\nu\frac{\overset{*}{G}{}^{11}}{\overset{*}{G}{}^{22}}[C^{22}(0)\overset{*}{E}_2(t)-\int_0^t\frac{d\tilde{C}^{22}(t-\tau)}{d\tau}\overset{*}{E}_2(\tau)\,d\tau]\end{aligned} \qquad (3.1.7)$$

where, finally

$$
\begin{aligned}
&= \tilde{C}^{11}(0)E_1(t) - \int_0^t \frac{d\tilde{C}^{11}(t-\tau)}{d\tau} E_1(\tau)\,d\tau \\
&+ \nu \frac{\overset{*}{G}{}^{11}}{\overset{*}{G}{}^{22}} \left[-C^{21}(0)E_1(t) - \int_0^t \frac{d\tilde{C}^{21}(t-\tau)}{d\tau} E_1(\tau)\,d\tau \right]
\end{aligned}
\qquad (3.1.8)
$$

$$
\begin{aligned}
\overset{*}{S}_1(t) &= (1-\nu^2)\,\tilde{C}^{11}(0)E_1(t) \\
&\quad - (1-\nu^2) \int_0^t \frac{d\tilde{C}^{11}(t-\tau)}{d\tau} E_1(\tau)\,d\tau
\end{aligned}
\qquad (3.1.9)
$$

IV.4 COMPUTER IMPLEMENTATION

IV.4.1 General Program Structure. The general nonlinear finite element program has previously been discussed by Lo [1981]. In what follows, use is made of quadratic isoparametric quadrilateral (8 - node) membrane elements. Three solution routines are used: initial configuration, static, and dynamic. To determine the initial equilibrium configuration of a structure to applied static loads, a viscous relaxation technique is used [Lo (1981)]. Nonlinear static and dynamic analyses are performed using a combined incremental/iterative method as discussed in Sec. 2.1 above. Newmark's method is used to solve the dynamical equations of motion. Pressure loads, an example of non-conservative loads, are accounted for by iteration, thus eliminating the need to compute non-symmetric matrices [Leonard (1988)].

IV.4.2 Viscoelastic and Wrinkling Algorithms. Subroutine MBRANE performs the calculation of the structure stiffness matrix and internal force vector. Due to the geometrical nonlinearities involved, the stiffness matrix ('stress' or 'tangent' stiffness) is a function of the stress state [see equation (2.1.4) above]. The stress state at the previous time-step is computed by subroutine MEMSTR. After computation of metric tensor components and selection of the appropriate material (linear elastic or viscoelastic, hyperelastic, and visco-hyperelastic models), MEMSTR computes the necessary strains and material coefficients, and then calls subroutine WRNKL for a check on wrinkling. WRNKL computes principal strains and directions, then checks for wrinkling against a specified wrinkling condition (which is material dependent); modified strains are computed if the wrinkling condition is met. MEMSTR then calculates element stresses, as well as memory terms for the viscoelastic and visco-hyperelastic materials. MBRANE proceeds to assemble the stiffness matrix and internal force vector as required. Appendix D contains flow charts representing the above description.

IV.4.3 Validation. For validation purposes, we consider the problem of the uniform inflation of an initially plane elastic rectangular membrane with initial sides of length $2B_1$ and $2B_2$ and thickness H. Classical analysis of this problem originates with Föppl and Hencky [see Timoshenko and Woinowsky-Krieger (1959)]. Various other authors have also discussed this problem from the point of view of both the original Föppl-Henky plate theory [see, e.g., Shaw and Perrone (1954); Kao and Perrone (1972); and Allen and Al-Qarra (1987)] and from higher order shell theory [Yang and Lu (1973); Feng and Huang (1974)]. It is interesting to note that only relatively few investigators report that compressive stress develops in the corners of the inflated rectangular membrane [Trostel (1962); Haug and Powell (1971); Kondo and Uemura (1971); and Kabrits, et al. (1983)].

The present analysis method is used for the specific case of a square membrane with sides $2B_1 = 2B_2 = 2B$; a plasticized PVC is considered for the membrane material with the following relevant properties:

Thickness = 1.3 mm (0.050 in)

Initial elastic modulus = 55.16 MPa (8000 psi)

Relaxation mod. =[23+26exp(-2.6x10^{-2} t)+6.2exp(-1.8x10^{-3})] MPa

([3.4+3.7exp(-2.6x10^{-2} t)+0.90exp(-1.8x10^{-3})]x10^{3} psi)

Poisson's ratio = 0.3 (to compare with previous elastic results);

= 0.45 (for viscoelastic results)

Mass density = 1068 N s^2/m^4 (1.0 x 10^{-4} lb s^2/in^4)

In order to determine the relaxation modulus, a number of 15-minute creep tests were first performed (see Appendix F), to which a 3-term Prony series (2.3.3) was fit (Figure IV.1). The creep compliance and relaxation modulus are connected in Laplace transform space by

$$\tilde{Y}(\chi) = \frac{1}{\chi^2 \tilde{J}(\chi)}$$

An inverse transformation (which can be done by hand for the 3-term Prony series) gives the above described relaxation modulus. The linearity of plasticized PVC for strains of less than 10% has been demonstrated by Leaderman (1962).

For the elastic case, present results are compared with the Föppl-Henky theory and the work of Allen and Al-Qarra (1987). (Advantage is taken of symmetry and only the upper right quadrant is modeled as shown in Figure IV.2) Displacement results are

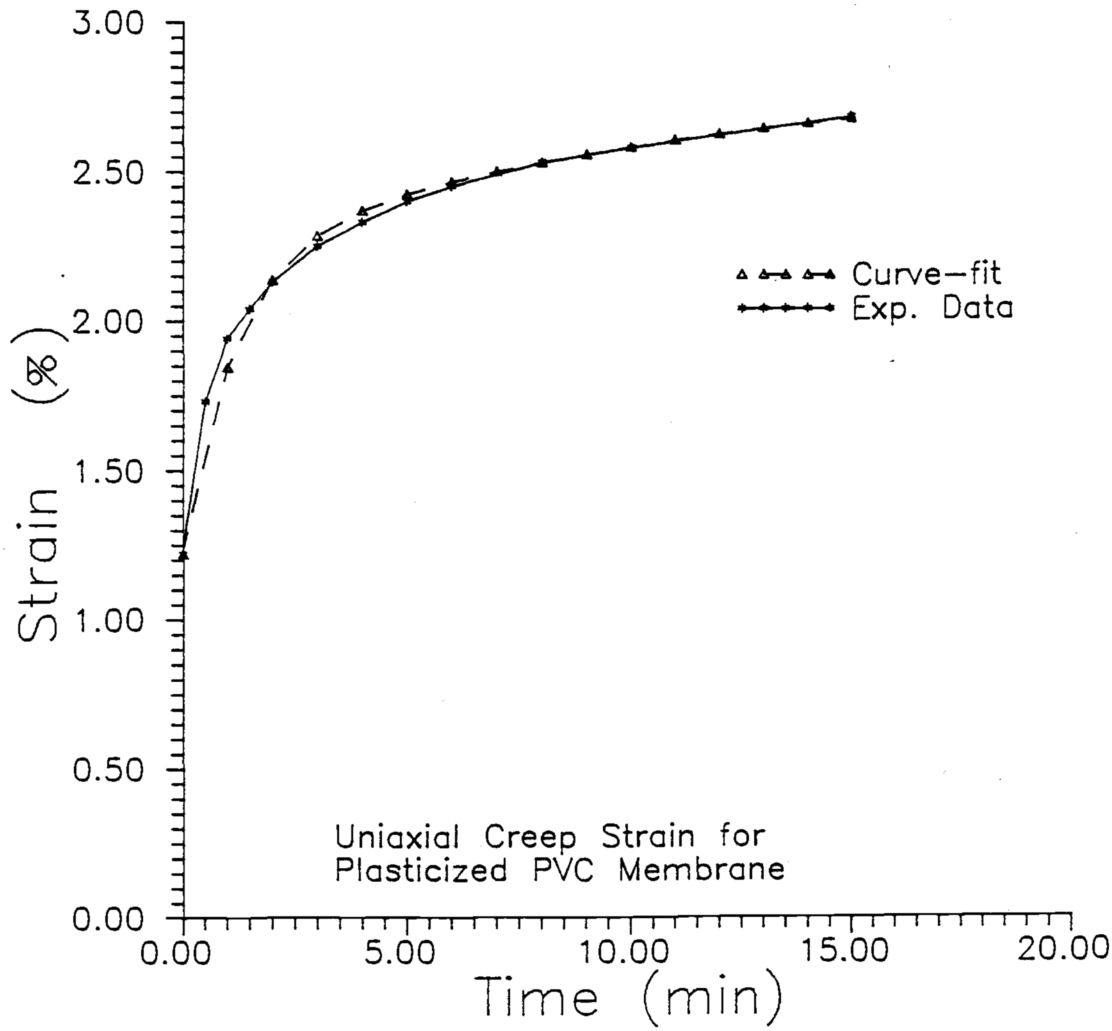

Figure IV.1 Curve-fit to 15-minute creep test.

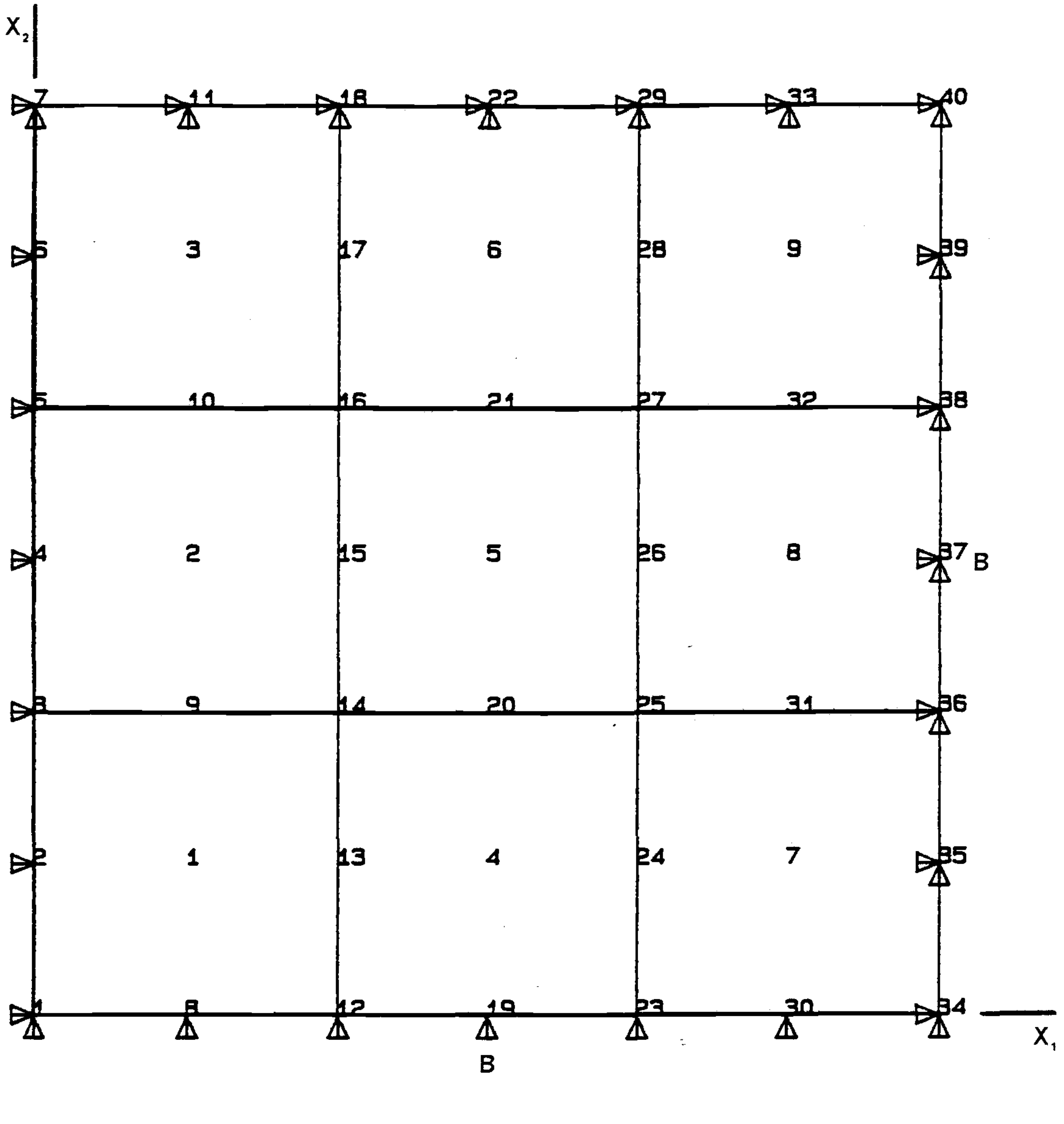

Figure IV.2 Plane square membrane configuration.

given for the centerline (X_1 - axis) and along a line connecting nodes 5 and 38 (B/3 line). For both a moderate deformation (p = 1.15 kPa (0.167 psi), maximum u_3/H = 29.5) shown in Figure IV.3 and for a larger

deformation (p = 14.94 kPa (2.167 psi), maximum u_3/H = 68.8) shown in Figure IV.4, displacement results are virtually identical with theory. Stress results for the moderate deformation case also compare well (Figure IV.5). In the above, care was taken not to exceed the limits of the Föppl-Henky theory [Weinitschke (1980)], i.e., insuring that the quantity $\sqrt[3]{(2pB/\widetilde{E}\,H)}$ does not exceed 1/2. Principal stress results are shown in Figure IV.6 for selected integration points as follows: wherever uniaxial stress ('tension field') occurs it is shown, while biaxial stress is shown only where required for clarity; magnitudes are not indicated, however, the longer line corresponds to the larger principal stress; orientation of principal stress is as shown. Wrinkling in the corner region of the membrane is clearly evident in Figure IV.6, and a sense of the 'tension lines' is given. We note that transients in the deformation are masked by a combination of viscous relaxation (Sec. IV.4.1) and time step size (Sec. II.2.5).

For the viscoelastic case, Figure IV.7 shows deformation profiles at selected times, and Figure IV.8 indicates principal stress results for selected integration points at t/T_1 = 2.0, where T_1 is the relaxation time of the first exponential term in the relaxation modulus ($\approx$ 38 s) and Δt = 1.0 s (see Appendix F). Since this is a constant stress problem, we expect the wrinkling pattern to also be constant in time;

this is observed in comparing Figure IV.6 with Figure IV.8.

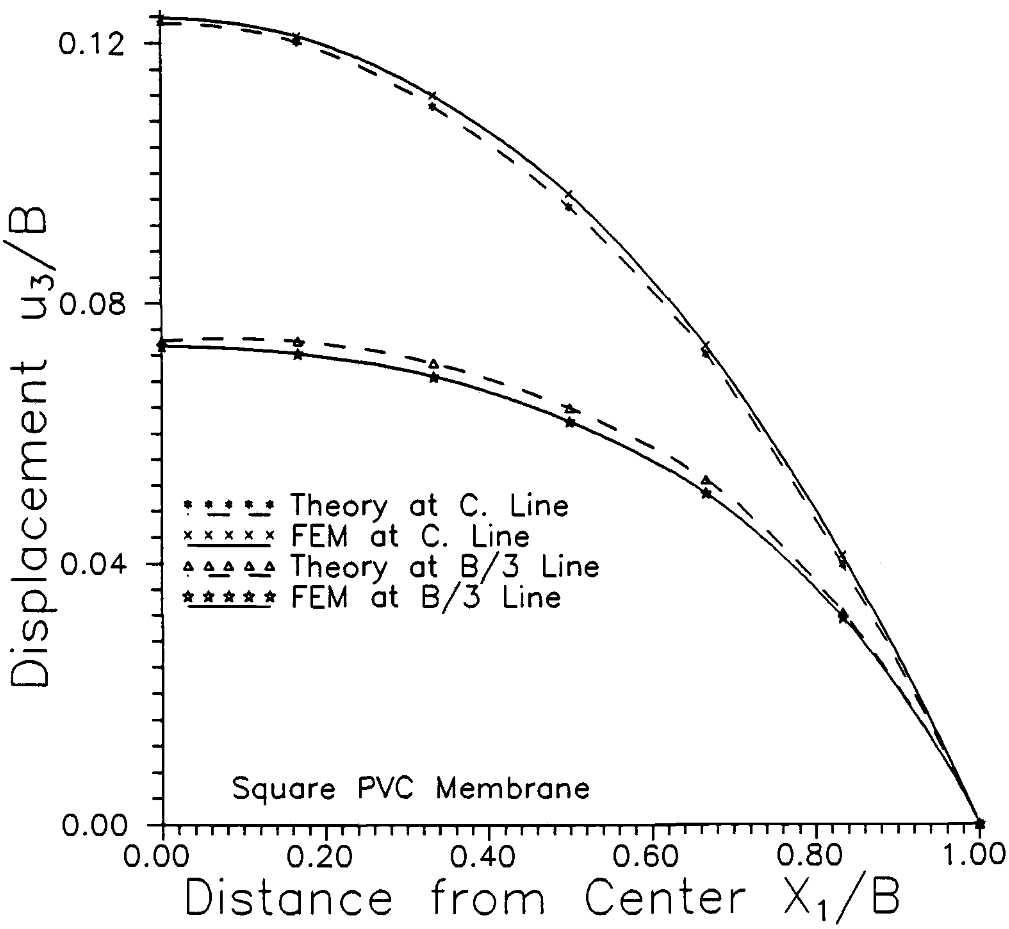

Figure IV.3 Displacement results for uniform inflation of a plane square elastic membrane P = 1.15 kPa (0.167 psi).

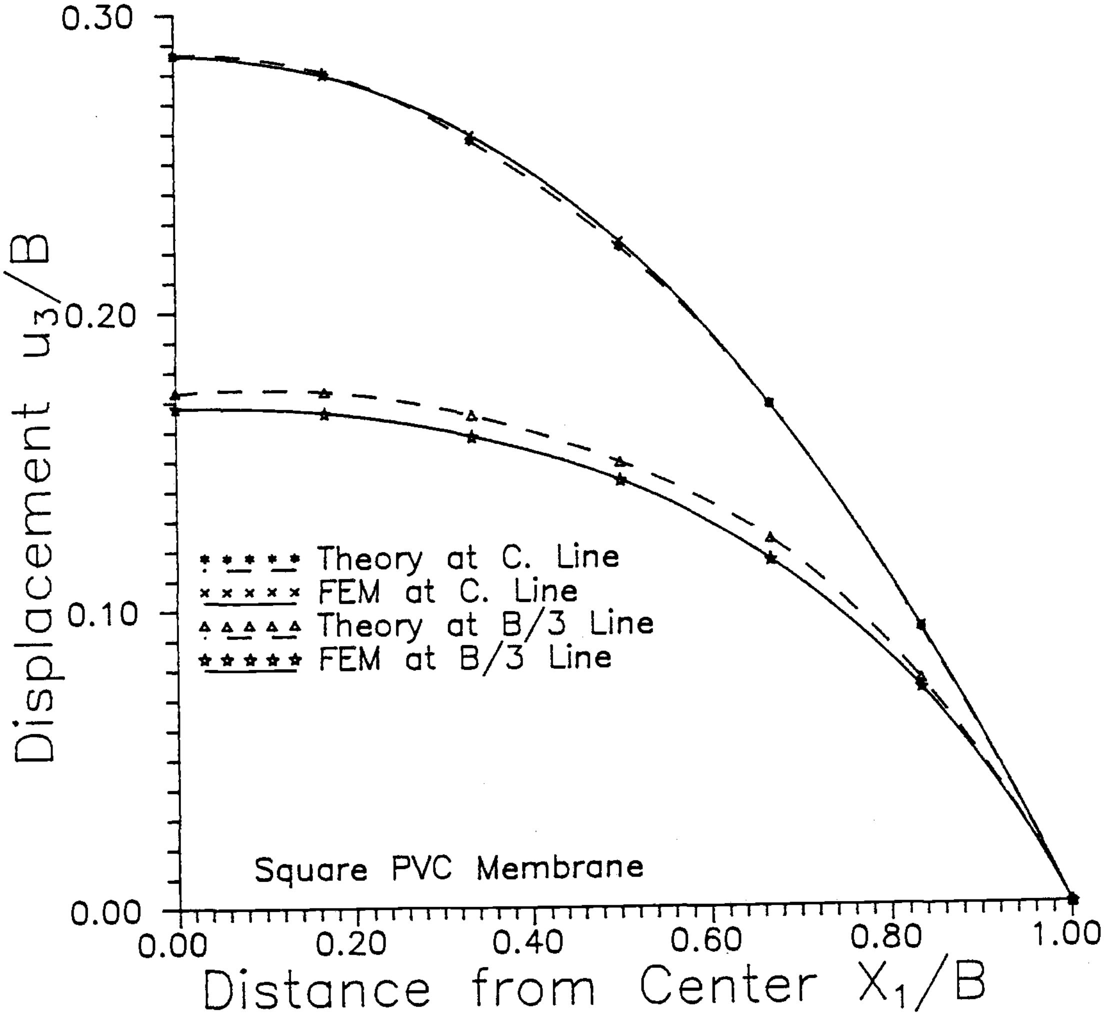

Figure IV.4 Displacement results for uniform inflation of plane square elastic membrane P = 14.94 kPa (2.167 psi).

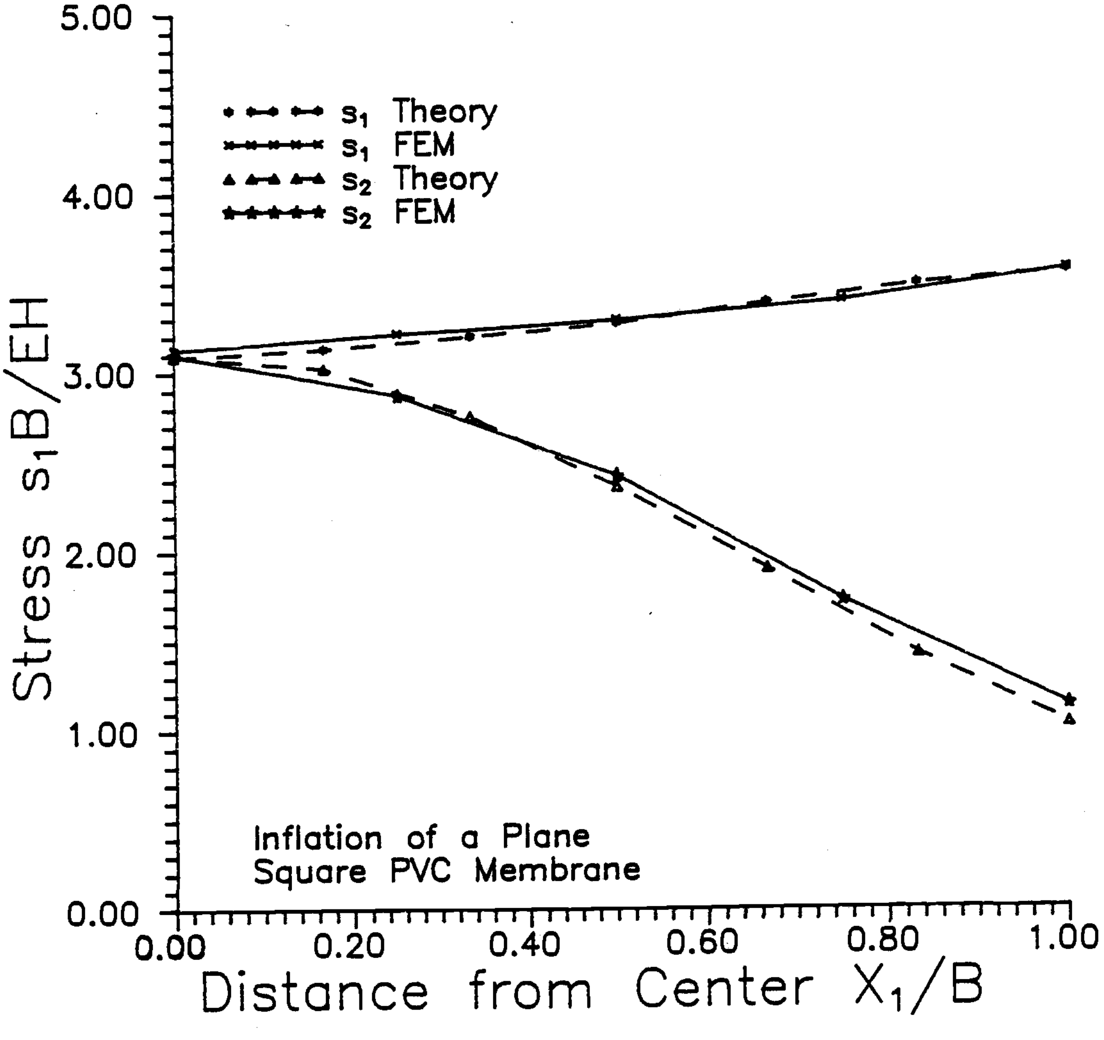

Figure IV.5 Stress results for uniform inflation of plane square elastic membrane $P = 1.15$ kPa (0.167 psi).

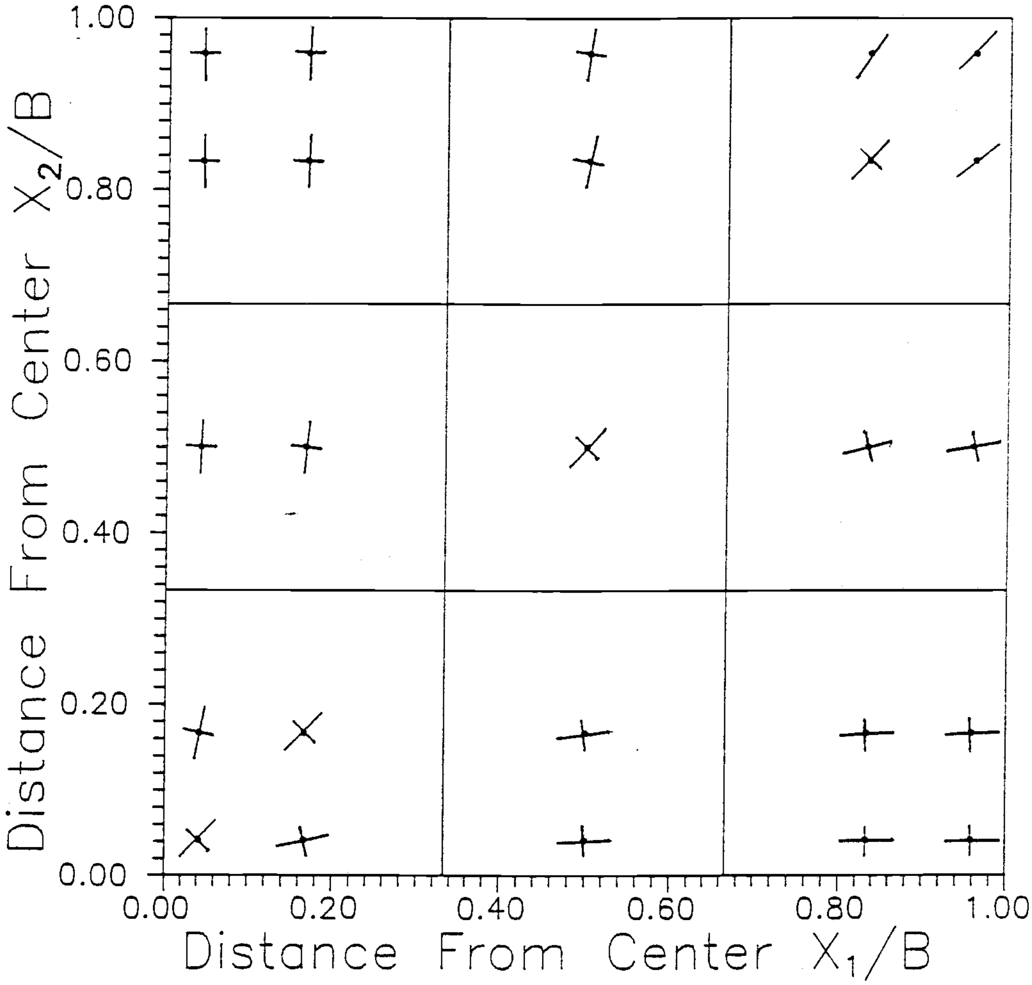

Figure IV.6 Principal stress results for uniform inflation of plane square elastic membrane.

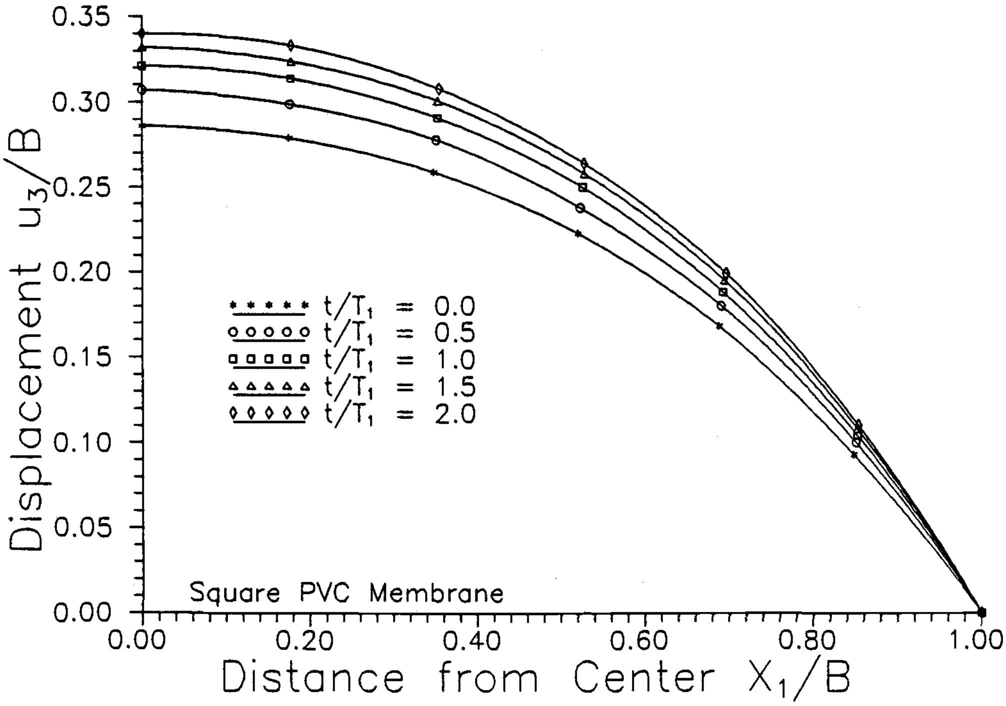

Figure IV.7 Deformation profiles for uniform inflation of plane square viscoelastic membrane P = 13.8 kPa (2.00 psi).

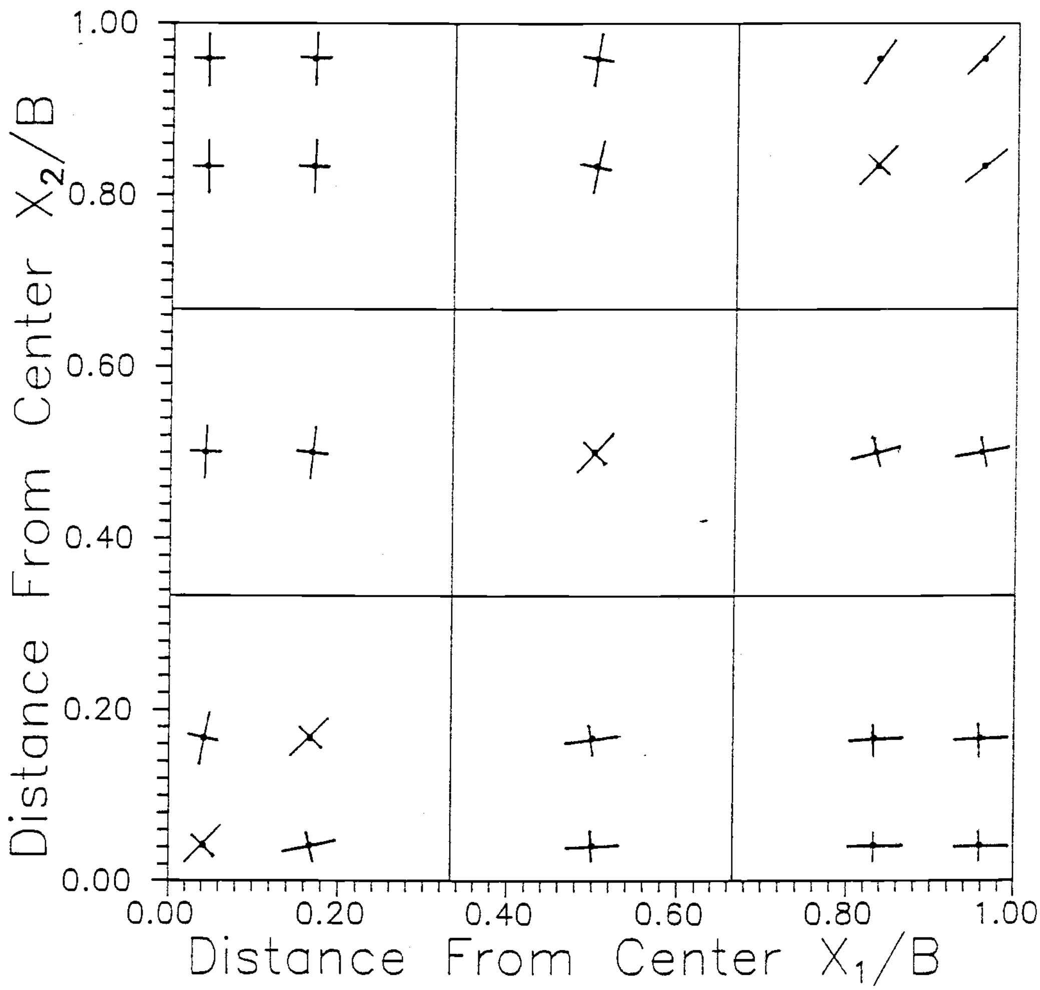

Figure IV.8 Principal stress results for uniform inflation of plane square viscoelastic membrane $t/T_1 = 2.0$.

IV.5 APPLICATIONS

IV.5.1 Deformation of an Inflated Cylindrical Membrane Cantilever. Various investigators have considered the inflated cylindrical membrane structure, with particular interest in the stability of equilibrium. An early NASA interest in inflatable reentry vehicles motivated a study by Leonard, et al. (1960). They briefly reported on a simple strength analysis of an inflated cantilever cylinder and its experimental validation. In another NASA document, Stein and Hedgepeth (1961) studied a partly wrinkled inflated cylinder undergoing pure bending. The stability of small deformations (e.g., axial compression) superimposed on finite deformations (simultaneous inflation and extension) of a hyperelastic cylinder was investigated by Corneliussen and Shield (1961). Comer and Levy (1963) analyzed the deflections and stresses between incipient buckling and final collapse for an inflated cylindrical cantilever beam. Koga (1972) extended the work of Corneliussen and Shield (1961) to consider the pre-wrinkled state of an inflated cylinder undergoing pure bending, and then applied the wrinkling analysis of Stein and Hedgepeth (1961) to the post-wrinkled state.

Leonard, et al. (1960) assumed the onset of buckling (wrinkling) would occur when the tensile axial stress due to initial pressurization was just canceled by compressive bending stress. They further assumed that collapse would occur when the pressurization moment due to the pressure force on the free end cap acting through a moment arm R was just canceled by the bending moment from the applied load. Then

$$f_{\max} B = p \pi R^2 R$$
$$f_{\max} = \frac{p \pi R^3}{B}$$

Brief experimental results are reported that corroborate these assumptions and the authors note that the collapse load is approximately twice the buckling load. Absent any evidence to the contrary, we assume the above load levels to apply in what follows.

In the present work, we first apply a sinusoidal tip load of amplitude 1.17 kN (264 lb) and period T = 1.0 s to an elastic cylindrical cantilever beam initially inflated to a pressure of 4.76 kPa (0.694 psi). (Advantage is taken of axial symmetry as shown in Figure IV.9). Cylinder dimensions are radius R = 0.46 m (1.5 ft) and length B = 2.4 m (8.0 ft). Material properties are as given in Sec. 4.3 and Δt = 0.125 s. The displacement of a free end node with time is given in Figure IV.10 as well as a deformation profile of the free end at various times. Figures IV.11 - IV.17 indicate the principal stresses at selected integration points at various times during one cycle of loading. In Figs. IV.11 - IV.17, the finite element model has been 'unrolled' into a plane surface for ease of viewing. Wrinkling results are as expected: as the load forces the free end up, compressive stresses build in the upper half of the cylinder causing wrinkling waves aligned perpendicular to the compressive stress; the same follows as the load forces the free end down and wrinkling develops in the bottom half of the cylinder.

Next, we apply a tip load of 587 N (132 lb) as a step function to the same elastic cylinder. In Figure IV.18 we record the vertical displacements of a tip node for a few cycles of vibration (Δt = 0.125). The period of oscillation, T_0, is 0.18s (5.5 Hz), a

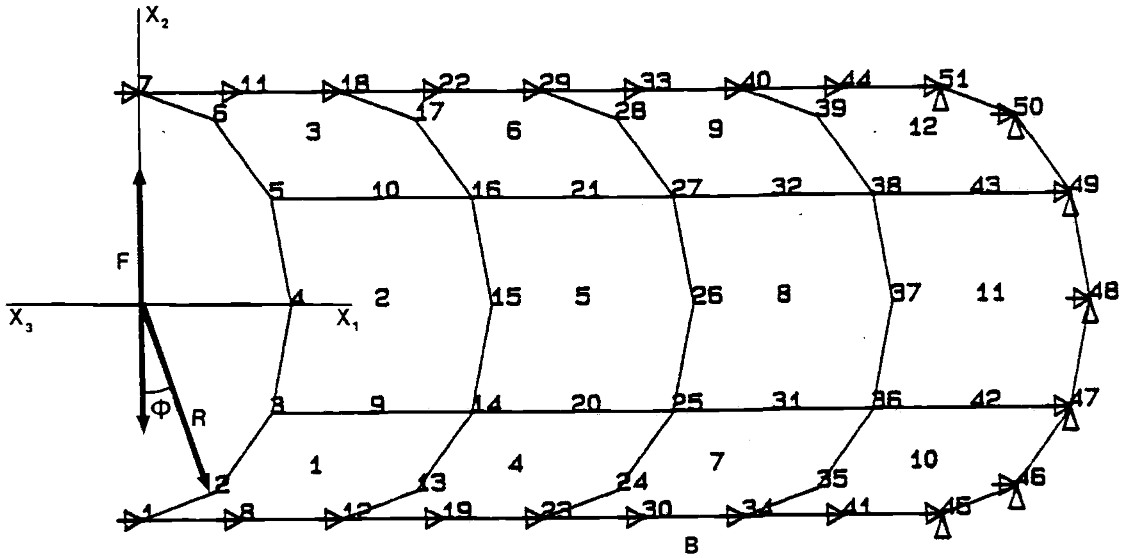

Figure IV.9 Cantilever cylinder configuration

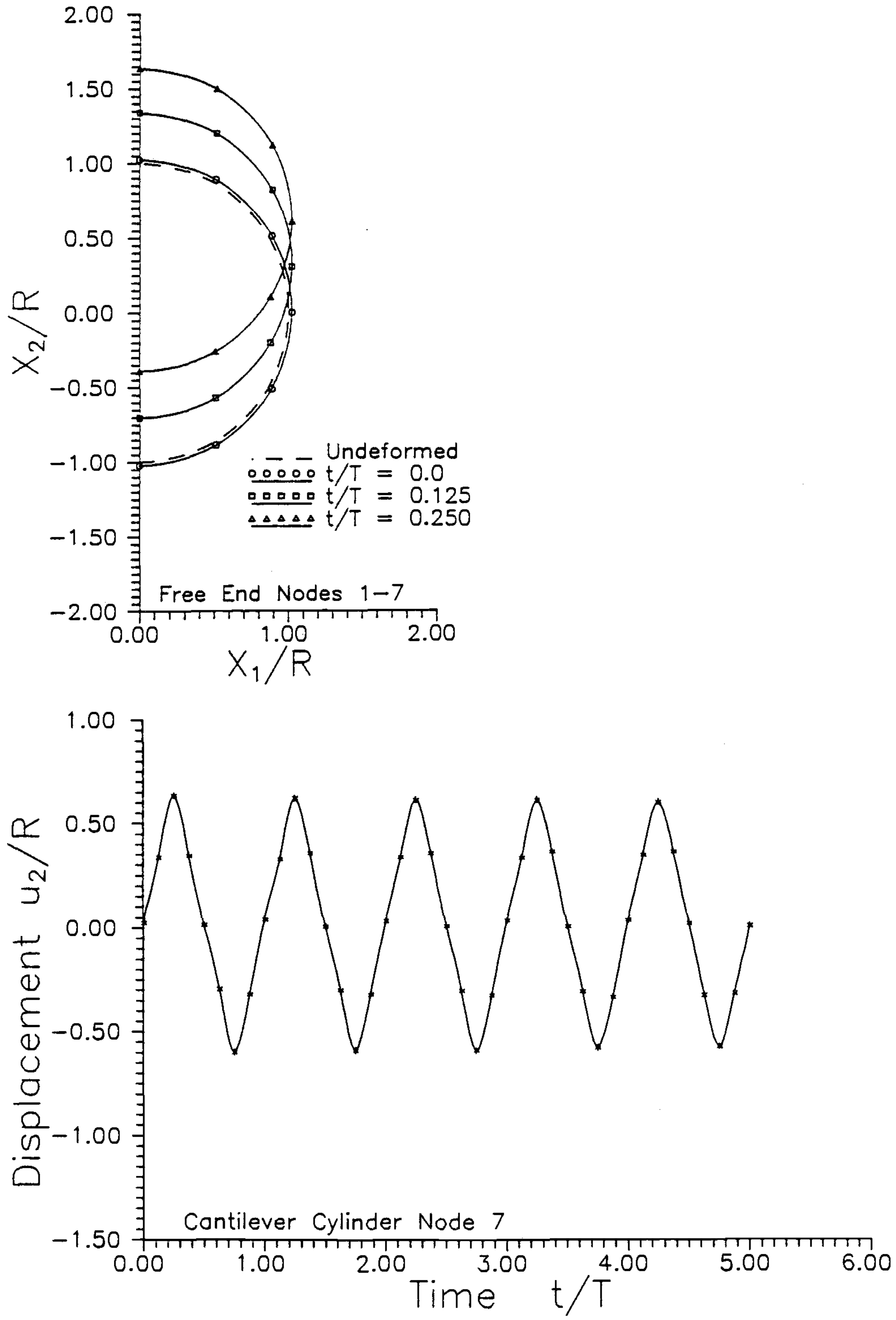

Figure IV.10 Displacement results for sinusoidal loading of elastic cantilever cylinder.

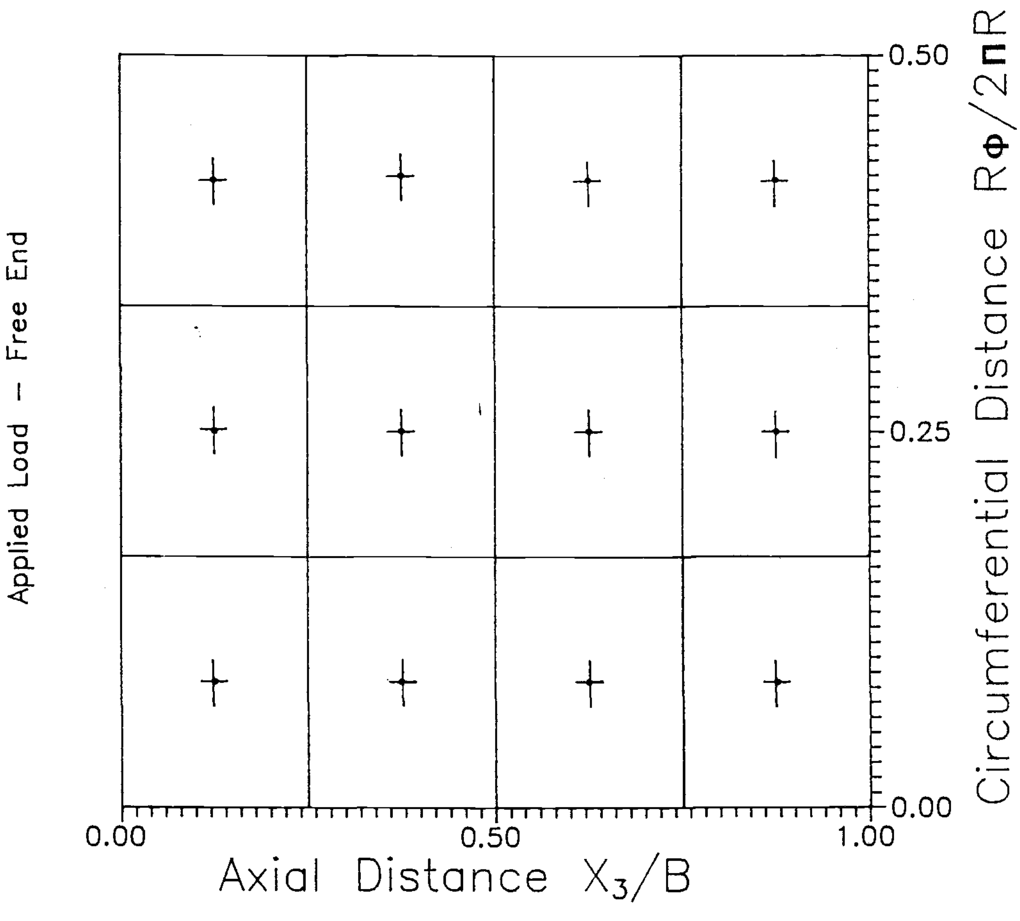

Figure IV.11 Principal stress results for sinusoidal loading of elastic cantilever cylinder, t/T = 0.0.

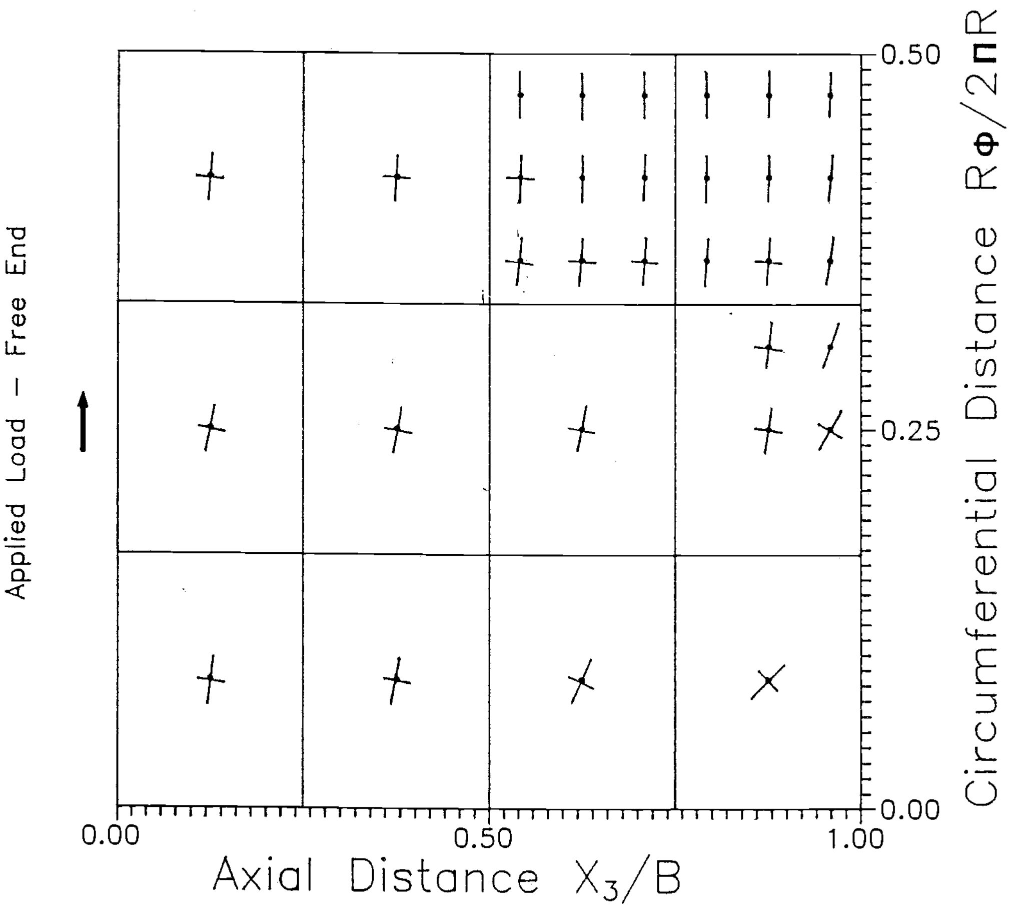

Figure IV.12 Principal stress results ... t/T = 0.125

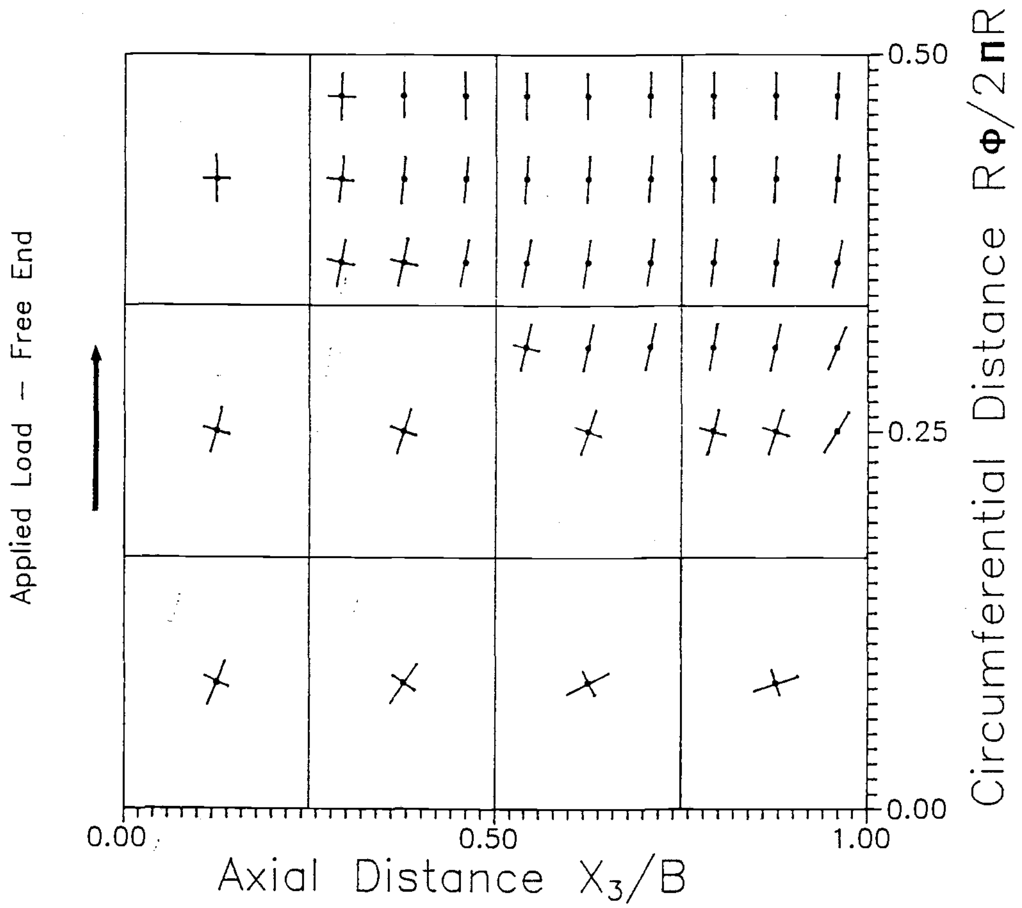

Figure IV.13 Principal stress results ... t/T = 0.25

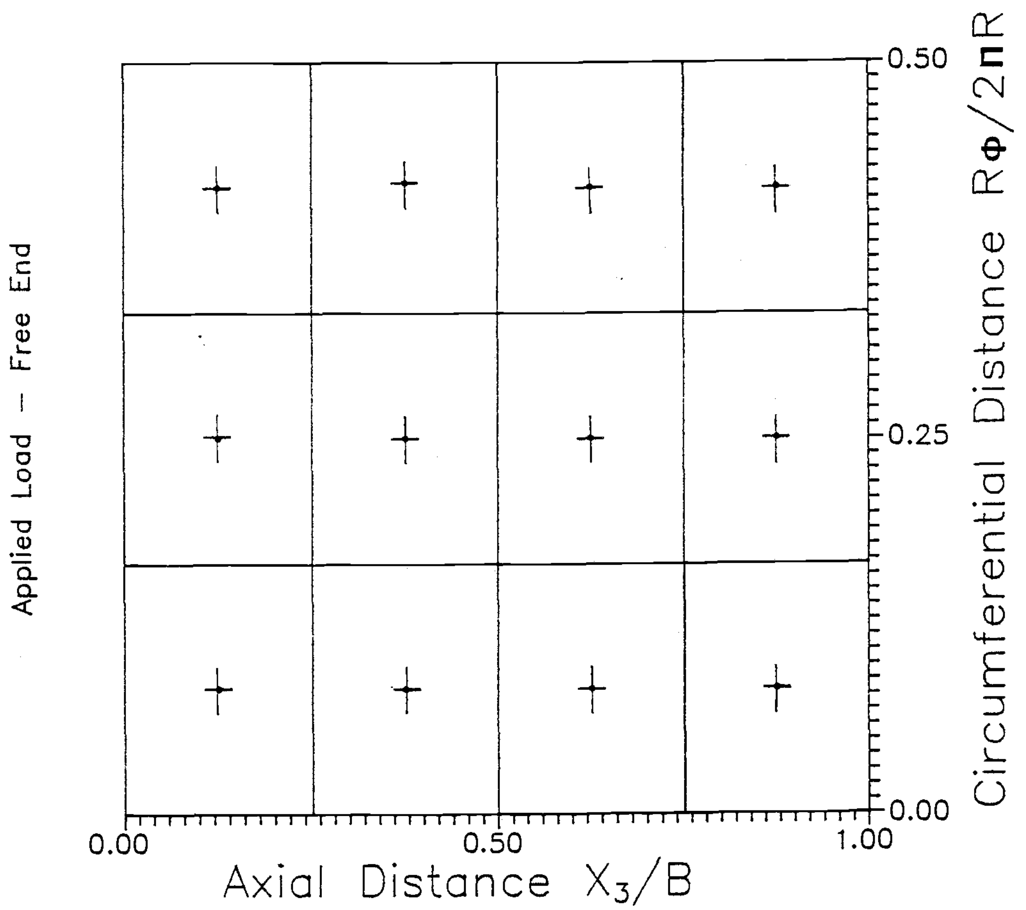

Figure IV.14 Principal stress results ... t/T = 0.5

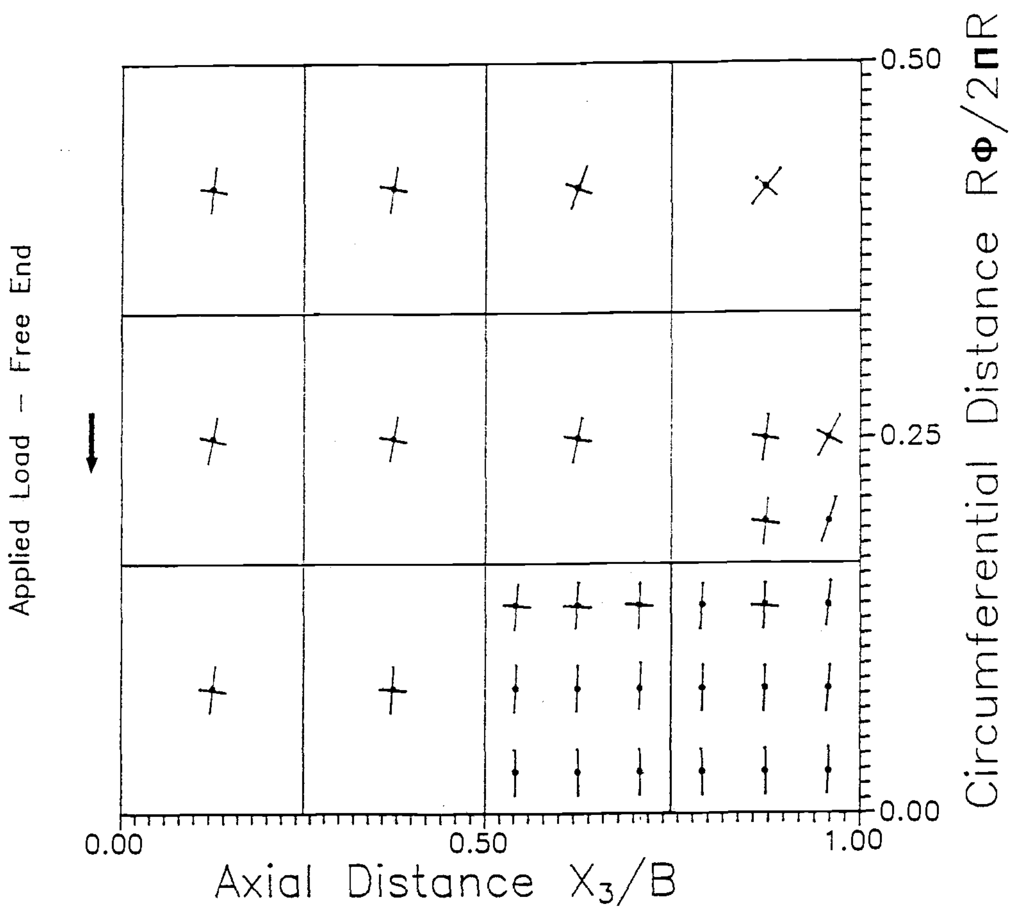

Figure IV.15 Principal stress results ... t/T = 0.625

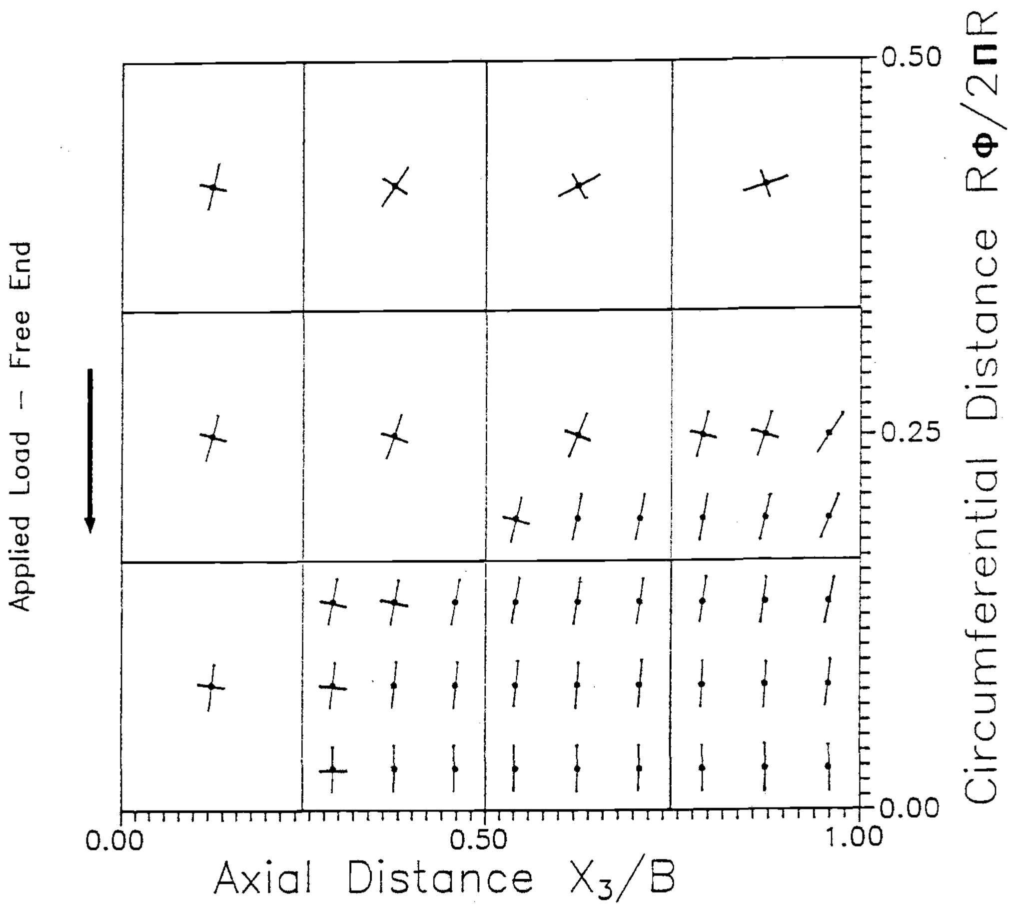

Figure IV.16 Principal stress results ... t/T = 0.75

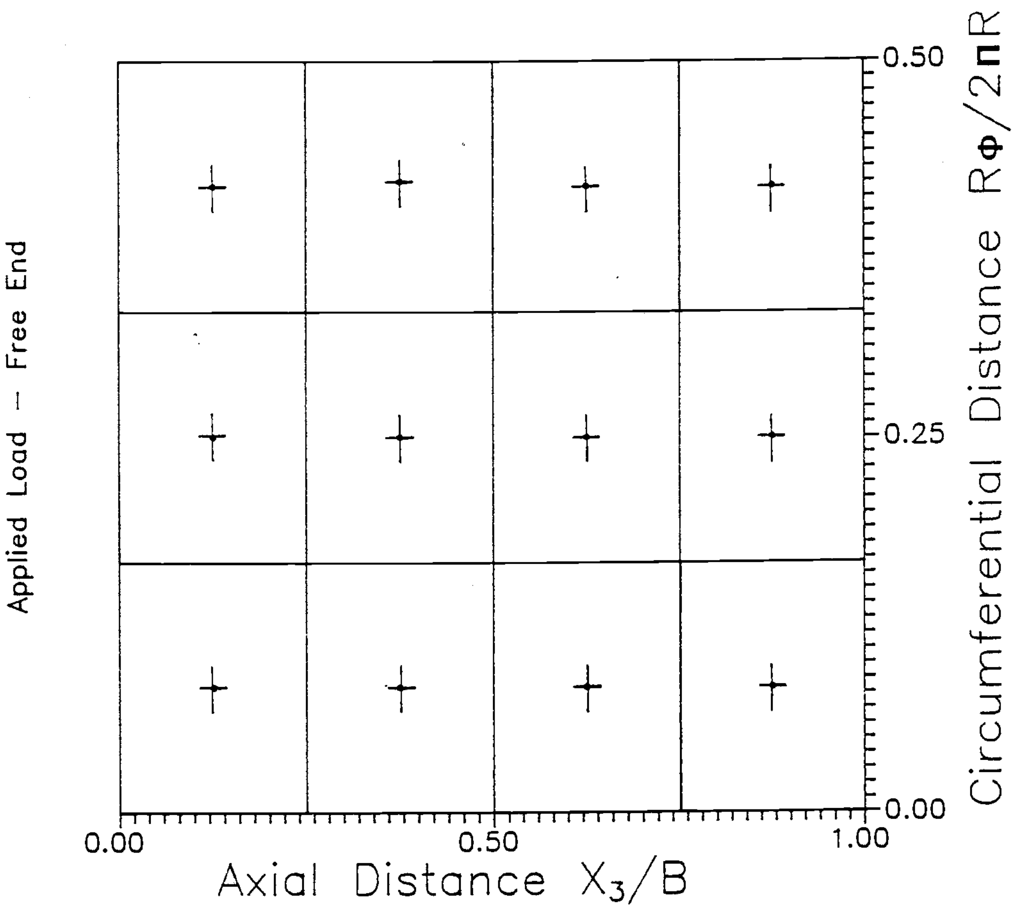

Figure IV.17 Principal stress results ... t/T = 1.0

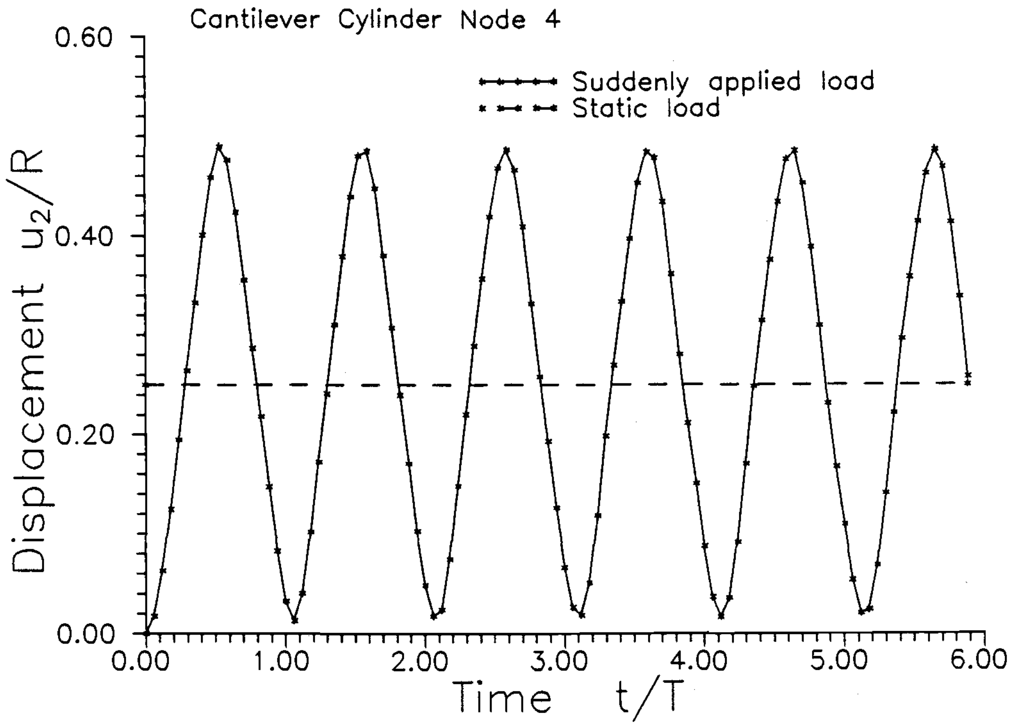

Figure IV.18 Displacement results for sudden loading of elastic cantilever cylinder.

value equal to an approximate prediction for this configuration [Blevins (1979)].

The results from Figures IV.10 and IV.18 are used to assess the system nonlinearity in the following way. First, we note the monochromatic response of Figure IV.10 verifies that no transients are excited in this case, due to zero value initial conditions (displacement and velocity). Second, when excited at a period ($T = 1.0$ s) much greater than the fundamental period ($T_0 = 0.18$ s), the system tends toward quasi-static behavior. Finally, we excite the system nearer to resonance ($T = 0.5$ s) and observe that multiples of the forcing period are present in the nonlinear dynamic response (Figure IV.19).

Finally, we apply a constant tip load (587 N) to a viscoelastic model of the above cantilever cylinder using the same relaxation modulus in Section 4.3 above. Time-dependent displacement results are given in Figure IV.20 where the time has been non-dimensionalized by the first relaxation time constant T_1 (= 1/0.0262 s^{-1} = 38.2 s) of the viscoelastic constitutive relation, and $\Delta t = 0.125$ s (see Appendix F). Corresponding static elastic results are also shown for comparison purposes. We note that since wrinkling is a function of compressive stress, and that the load (thus stress) here is constant, the membrane wrinkling does not change appreciably with time. Wrinkling results are the same as the above described elastic case for corresponding displacements.

IV.5.2 Hydrodynamic Loading of a Submerged Membrane Cylinder. Membrane structures have been considered for use in the marine environment in a variety of situations including storage containers, dwellings, and breakwaters [Jenkins and Leonard

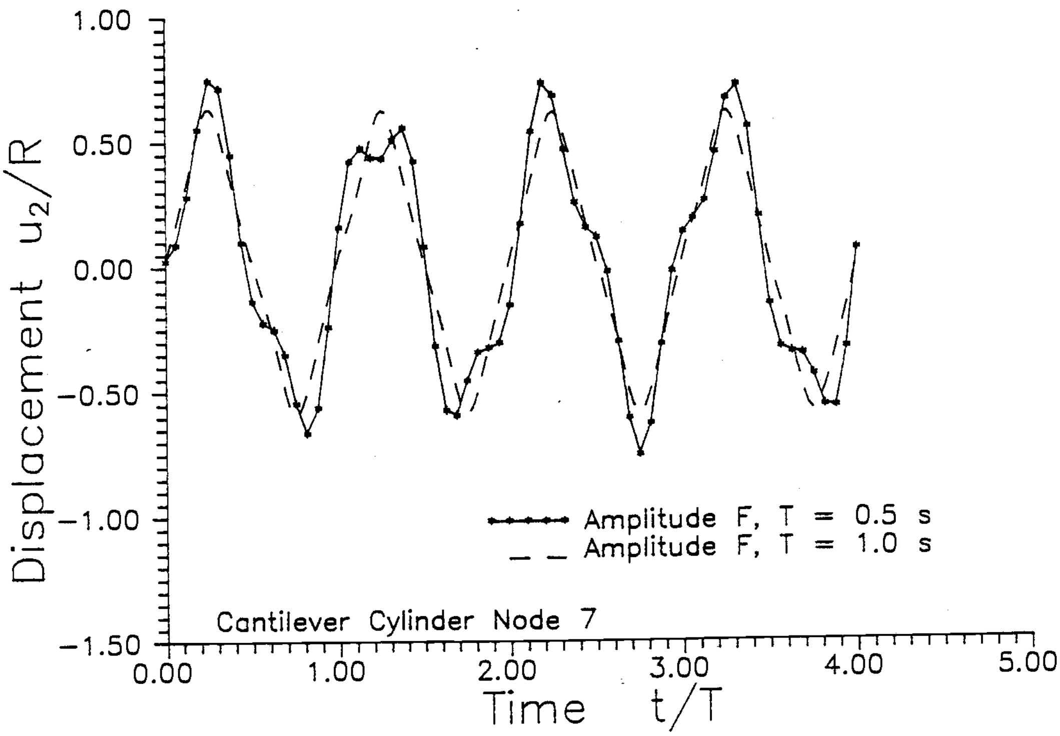

Figure IV.19 Comparison of displacement results for sinusoidal loading of elastic cantilever cylinder for two different excitation periods.

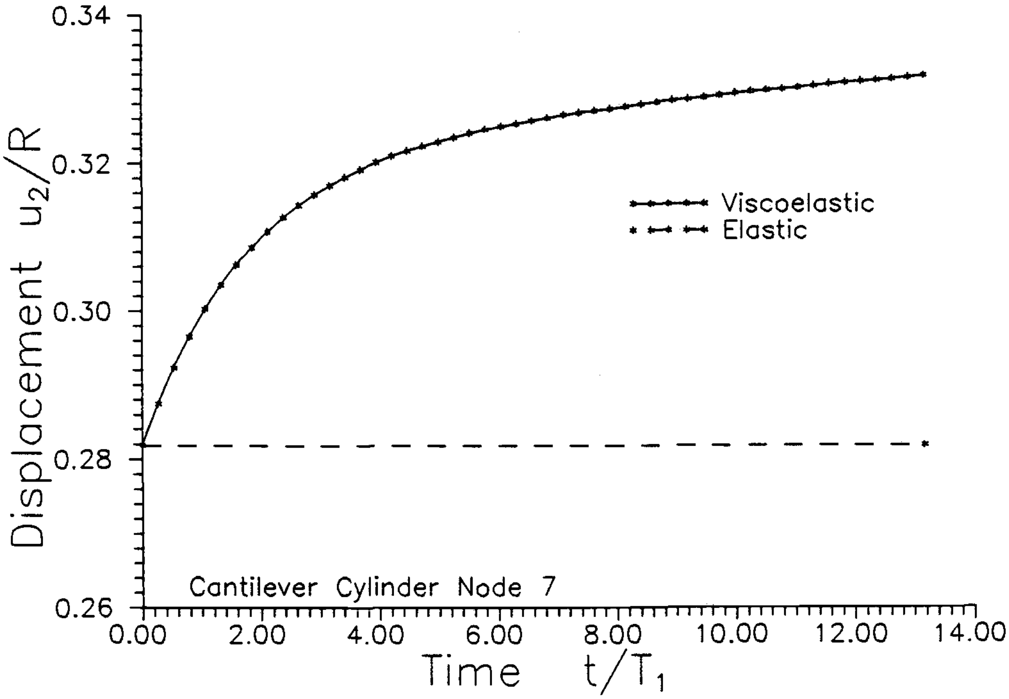

Figure IV.20 Displacement results for static loading of viscoelastic cantilever cylinder

(1991a)]. To examine the latter case, the numerically predicted response of an experimental cylindrical breakwater model was considered. (For experimental details see Appendix E.) A 0.91 m (3.0 ft) diameter right circular viscoelastic PVC cylinder of length 2B = 3.7 m (12 ft) and thickness 1.3 mm (0.050 in) is submerged in 2.7 m (9.0 ft) of water depth with the (X_3) axis of the cylinder 3 ft below the still water level and parallel to it (see Figures IV.21 and IV.22). Material properties are as in Sec. 4.3 and Δt = 0.05 s (see Appendix F). Due to symmetry of loading, only 1/2 the length of the cylinder is modeled; one end has a fixed boundary condition, the other is fixed only in the axial direction. The cylinder contains water and is subjected to an overpressure of 0.48 kPa (10 psf), a value corresponding to the experimentally observed mean value (Appendix E). Surface waves of 0.15 m (0.5 ft) height and period T = 2 s are incident on the cylinder in the positive X_1-direction.

Incident wave pressure on the cylinder is accounted for in the numerical model by a linear Froude-Krylof model, i.e., no diffraction or radiation effects are considered to modify the incident wave field. Internal pressure of the cylinder is assumed constant. The mass of the contained fluid is distributed to the cylinder as lumped masses at the corner of the elements.

Selected finite element vertical displacement results along the top membrane centerline are compared to experimental values in Figure IV.23 for approximately 7 cycles of loading (results are non-dimensionalized by 1/2 the wave height to match experimental data); the comparison is quite good, considering the lack of a more sophisticated hydrodynamic model. A detailed view of selected finite element vertical

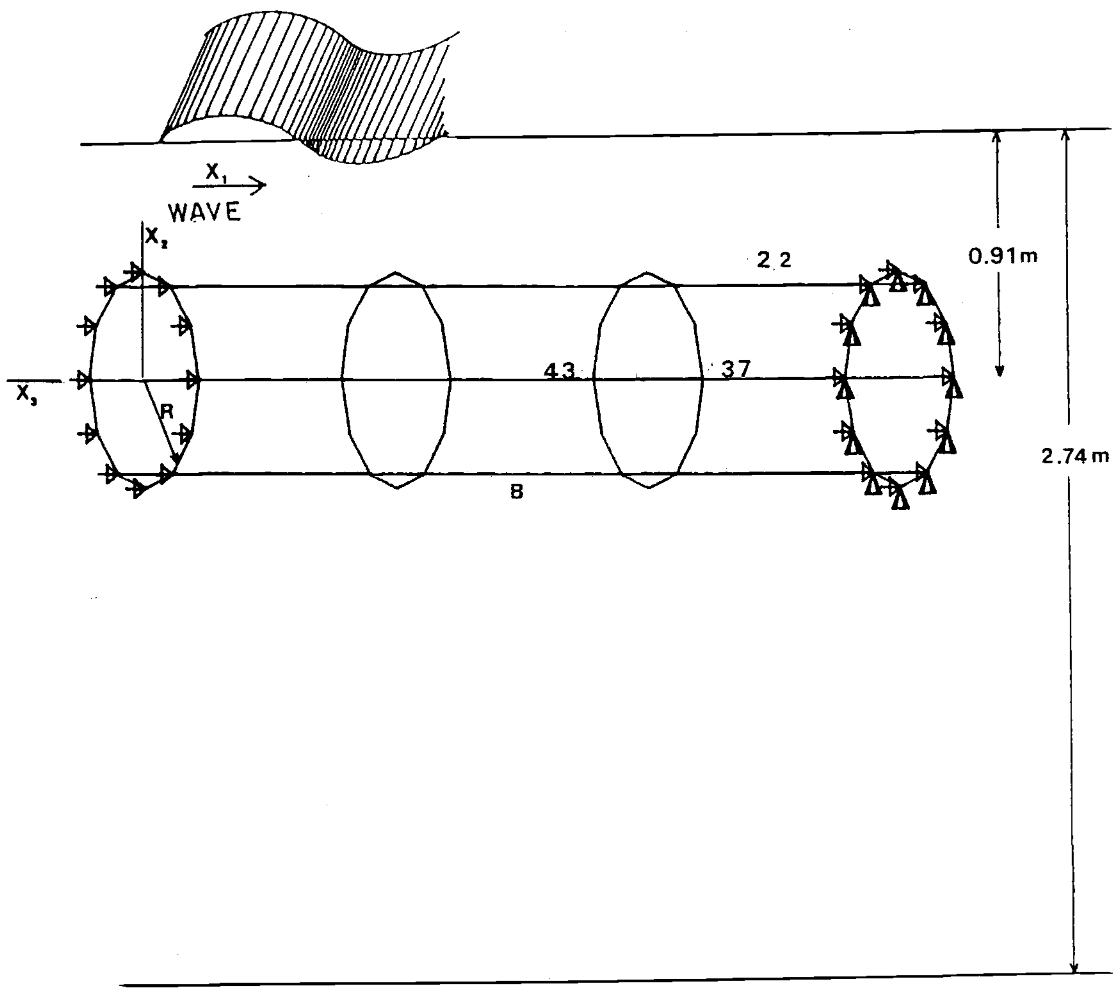

Figure IV.21 Submerged cylinder definition sketch.

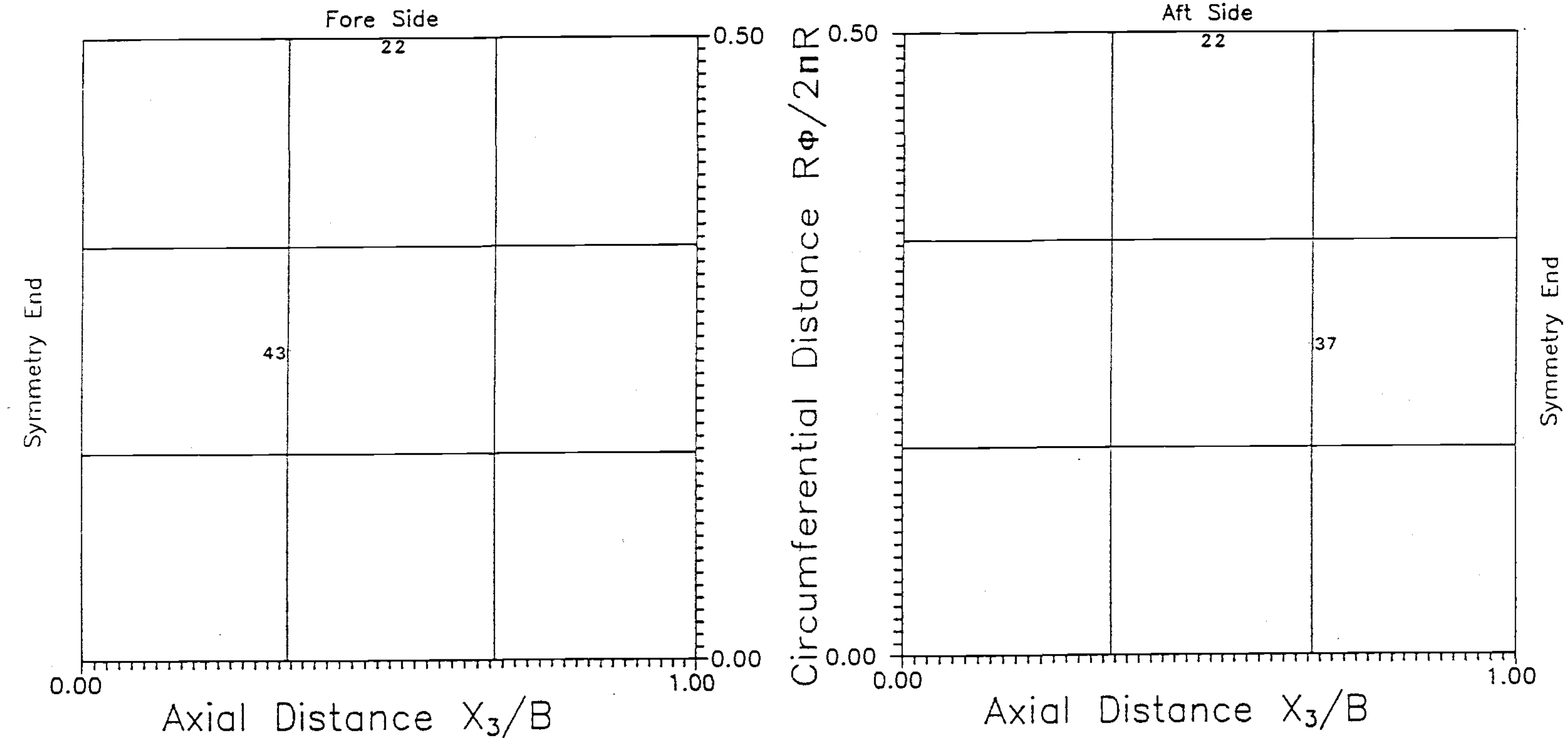

Figure IV.22 Submerged membrane configuration

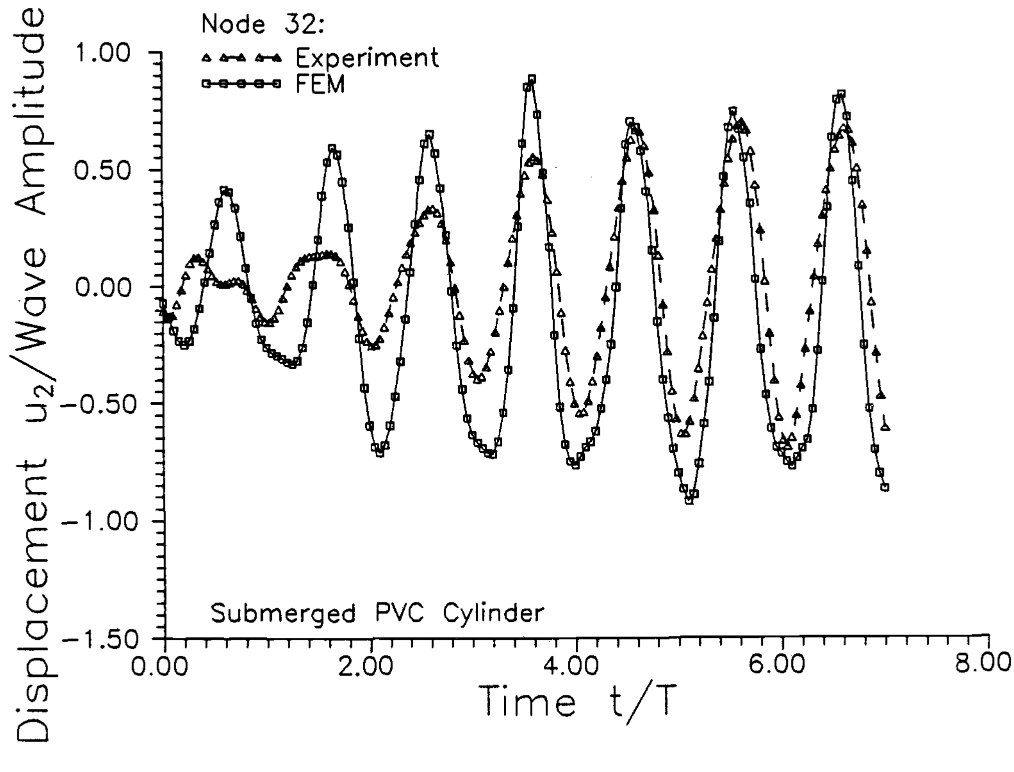

Figure IV.23 Validation comparison for vertical displacements of submerged viscoelastic cylinder: numerical vs. experiment.

and horizontal displacements along the membrane midplane centerline for approximately 2 cycles are given in Figure IV.24 with corresponding cross-section profiles shown in Figure IV.25.

Figures IV.26 - IV.31 indicate the associated principal stresses during these 2 cycles of loading. Stress interpretation is as described in Sec. 4.3 with the addition that slack regions are denoted by a circle enclosing the integration point. Both fore and aft sides (left and right sides of figures, respectively) are represented in plane 'unwrapped' view; the symmetry end corresponds to the tank centerline. A significant amount of the membrane is wrinkled, which is indicative of the complexity of the loading environment. We would expect increased internal pressure to decrease the amount of wrinkling to some extent.

Video footage of the membrane deformation was obtained during the experiment (Appendix E). Figures IV.32 and IV.33 represent two typical frames, where the lines of wrinkling have been artificially outlined for emphasis. A comparison of Figures IV.32 and IV.33 with the numerical results shows that the essential description of the wrinkled surface has been captured.

IV.6 ACKNOWLEDGEMENT

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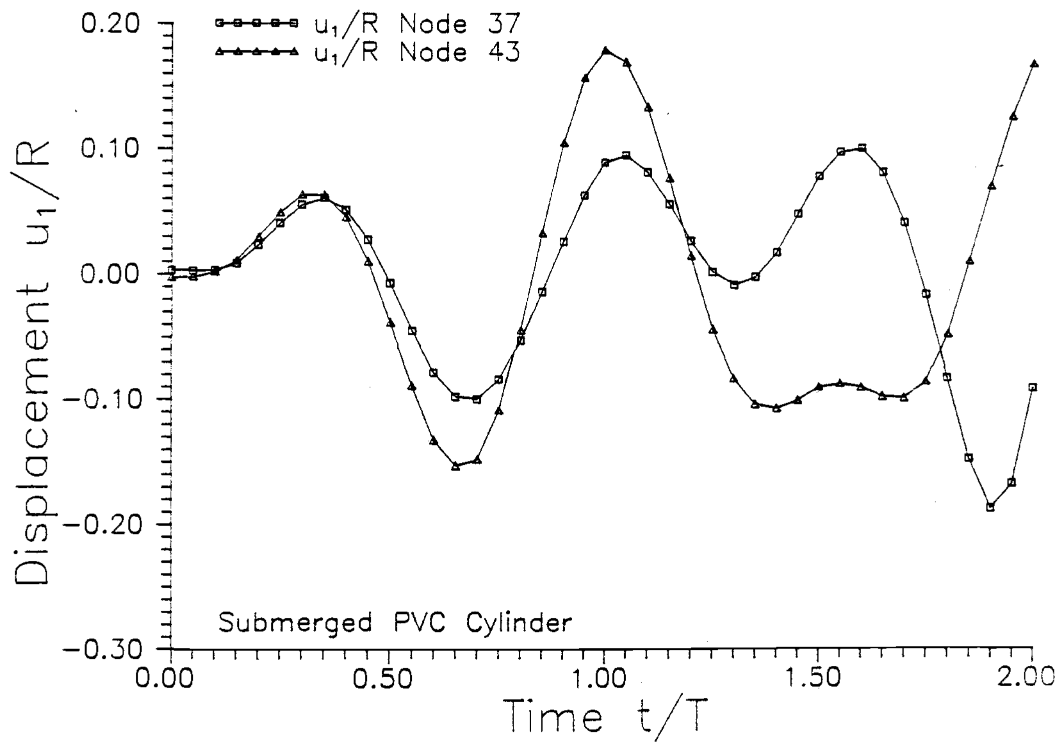

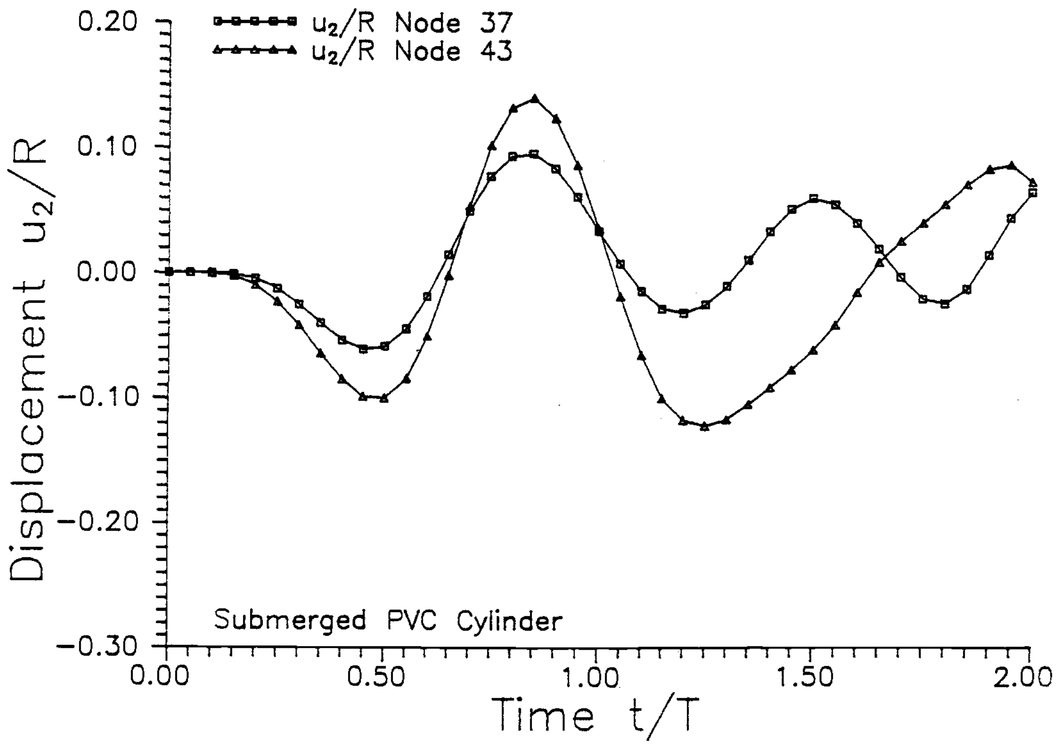

Figure IV.24 Horizontal and vertical numerical displacement results for submerged viscoelastic cylinder.

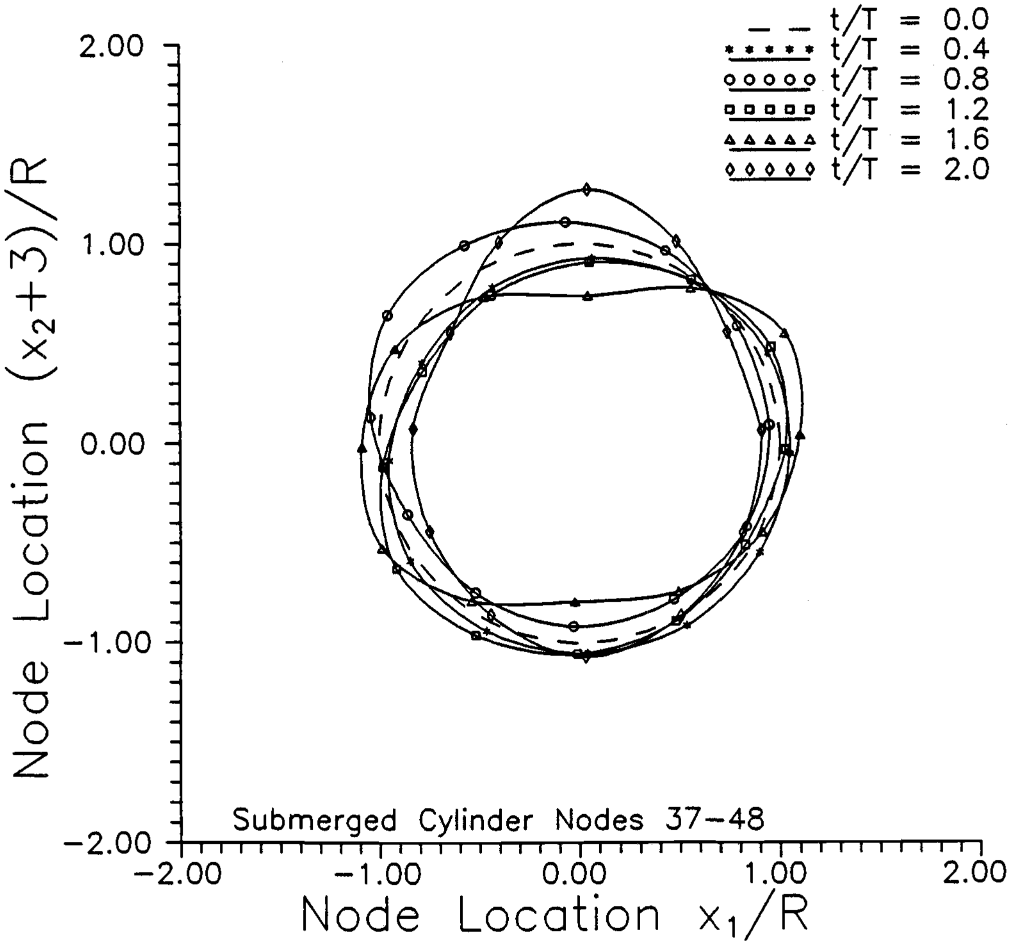

Figure IV.25 Cross-section (numerical) profile history for submerged viscoelastic cylinder.

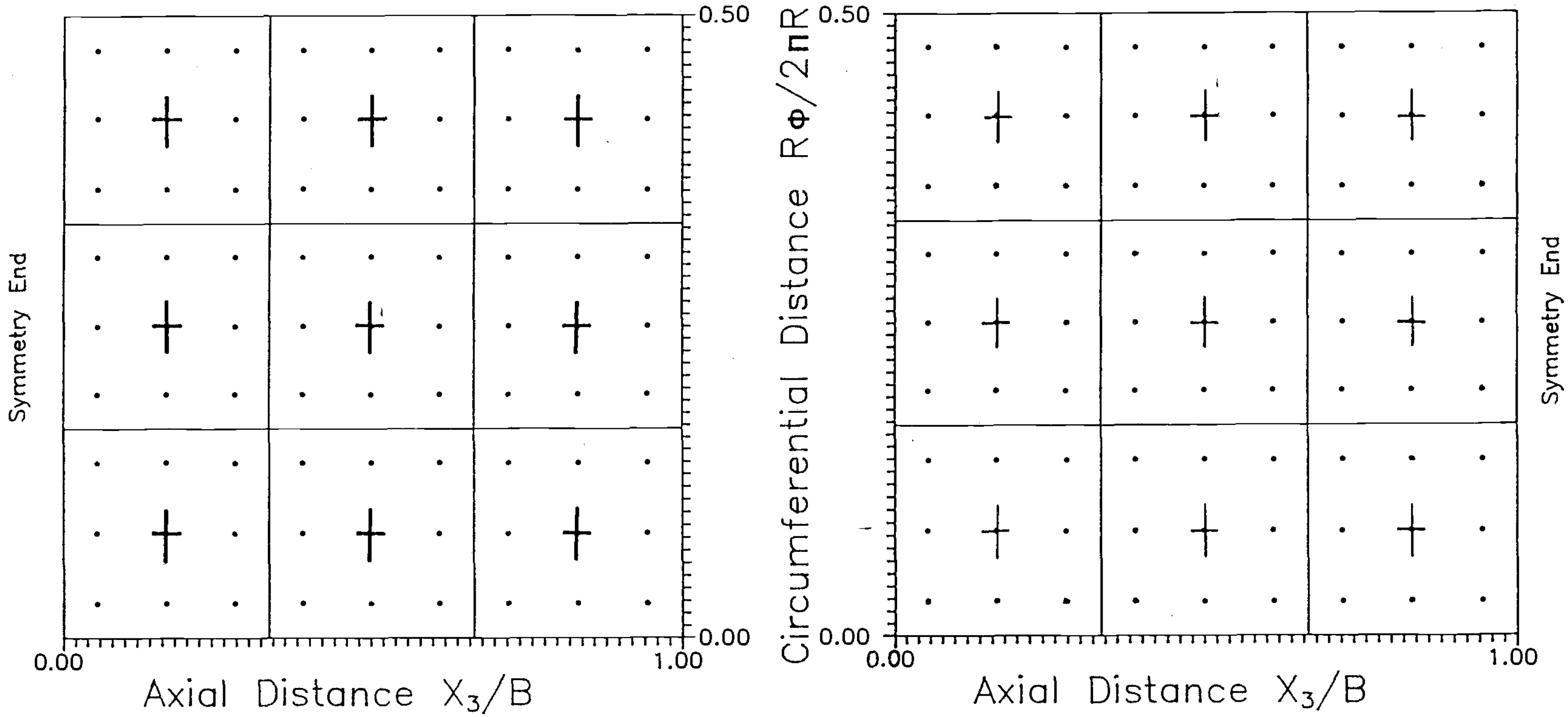

Figure IV.26 Principal stress results t/T = 0.0

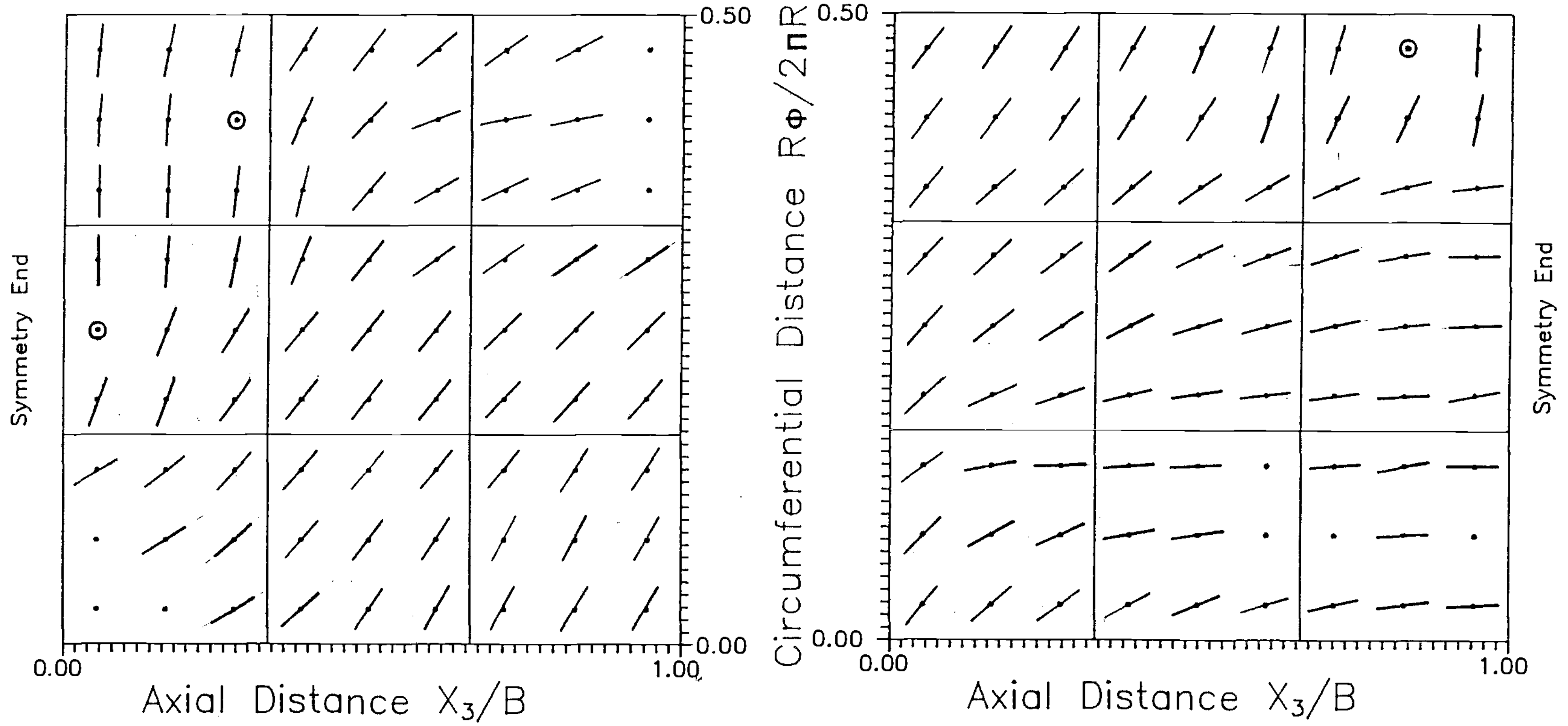

Figure IV.27 Principal stress results t/T = 0.4

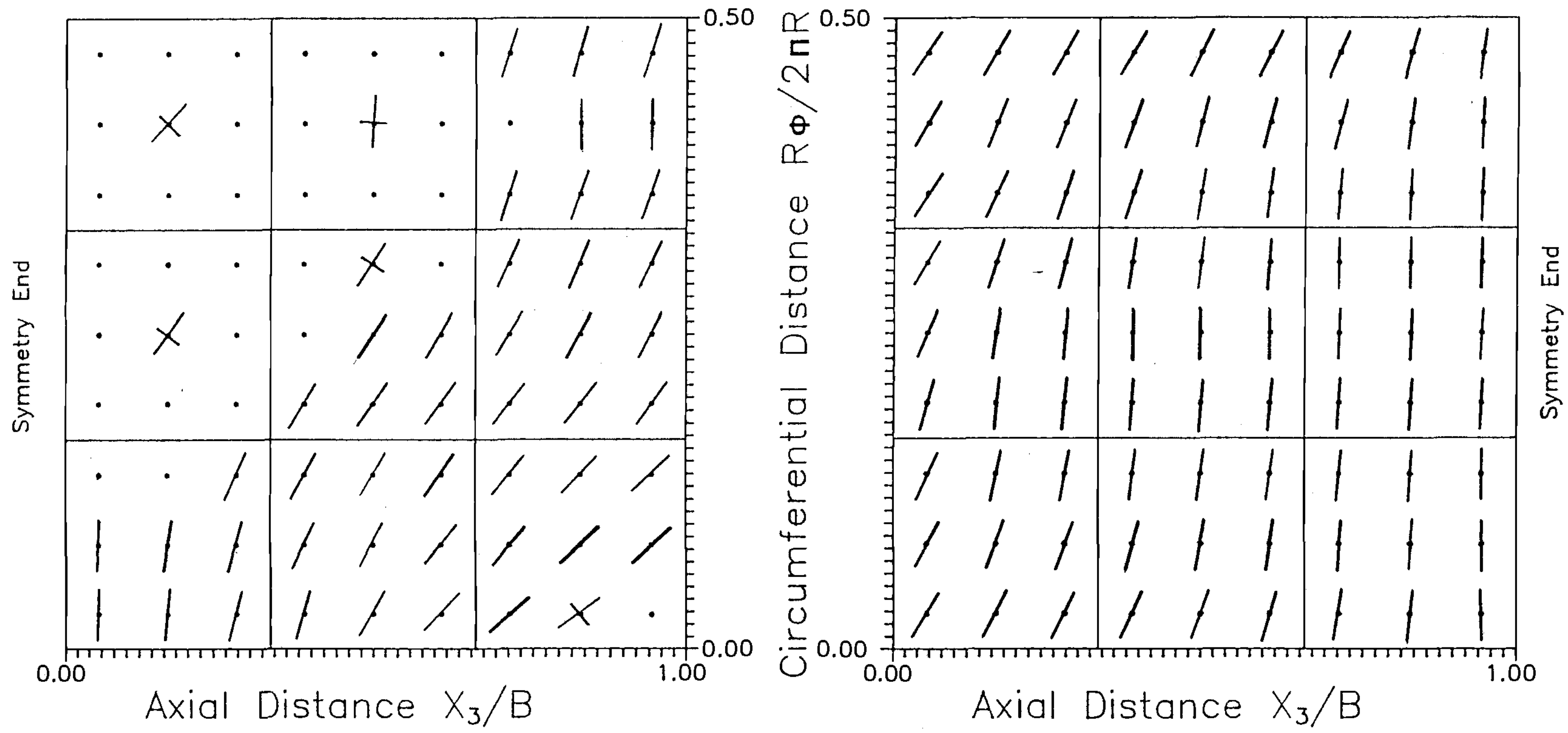

Figure IV.28 Principal stress results t/T = 0.8

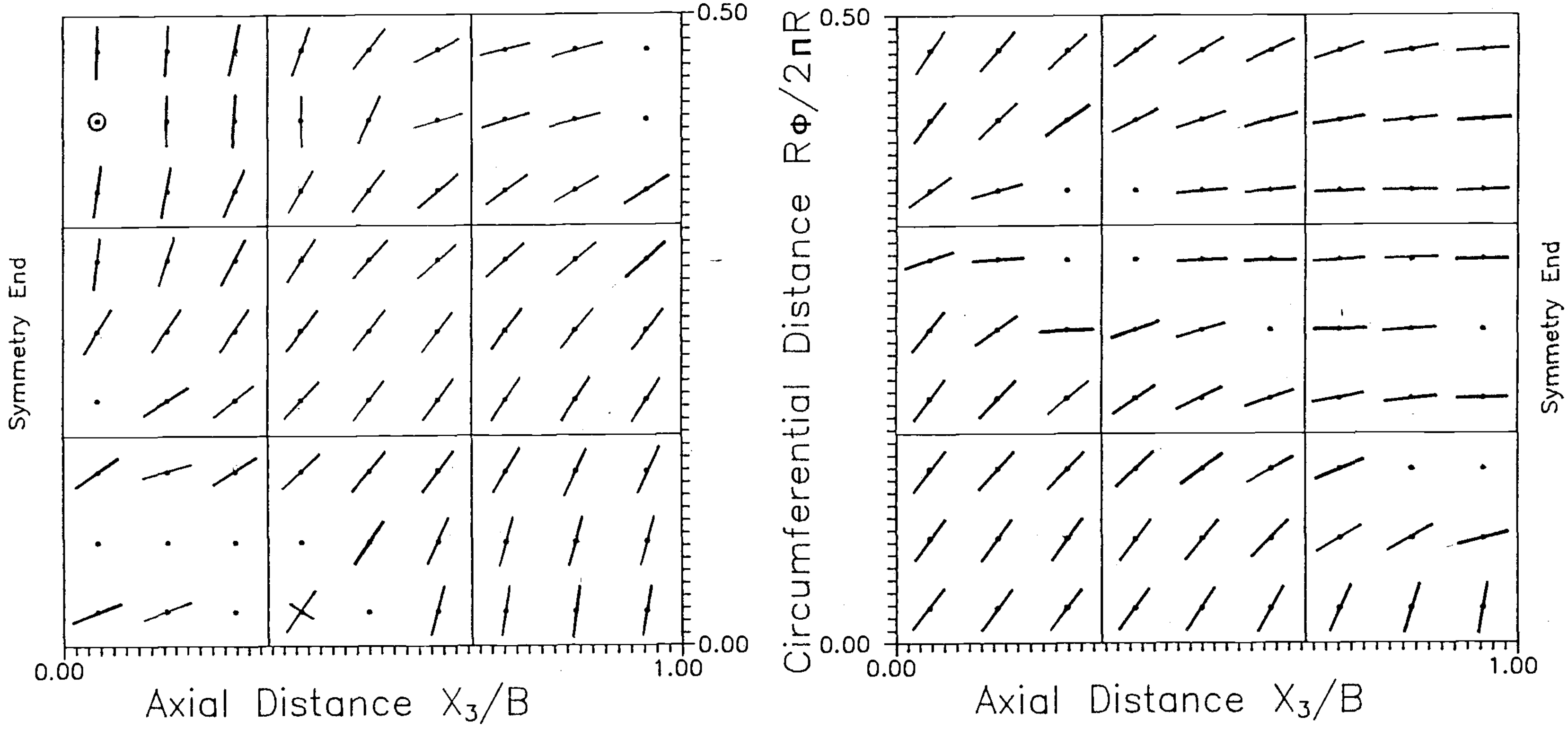

Figure IV.29 Principal stress results t/T = 1.2

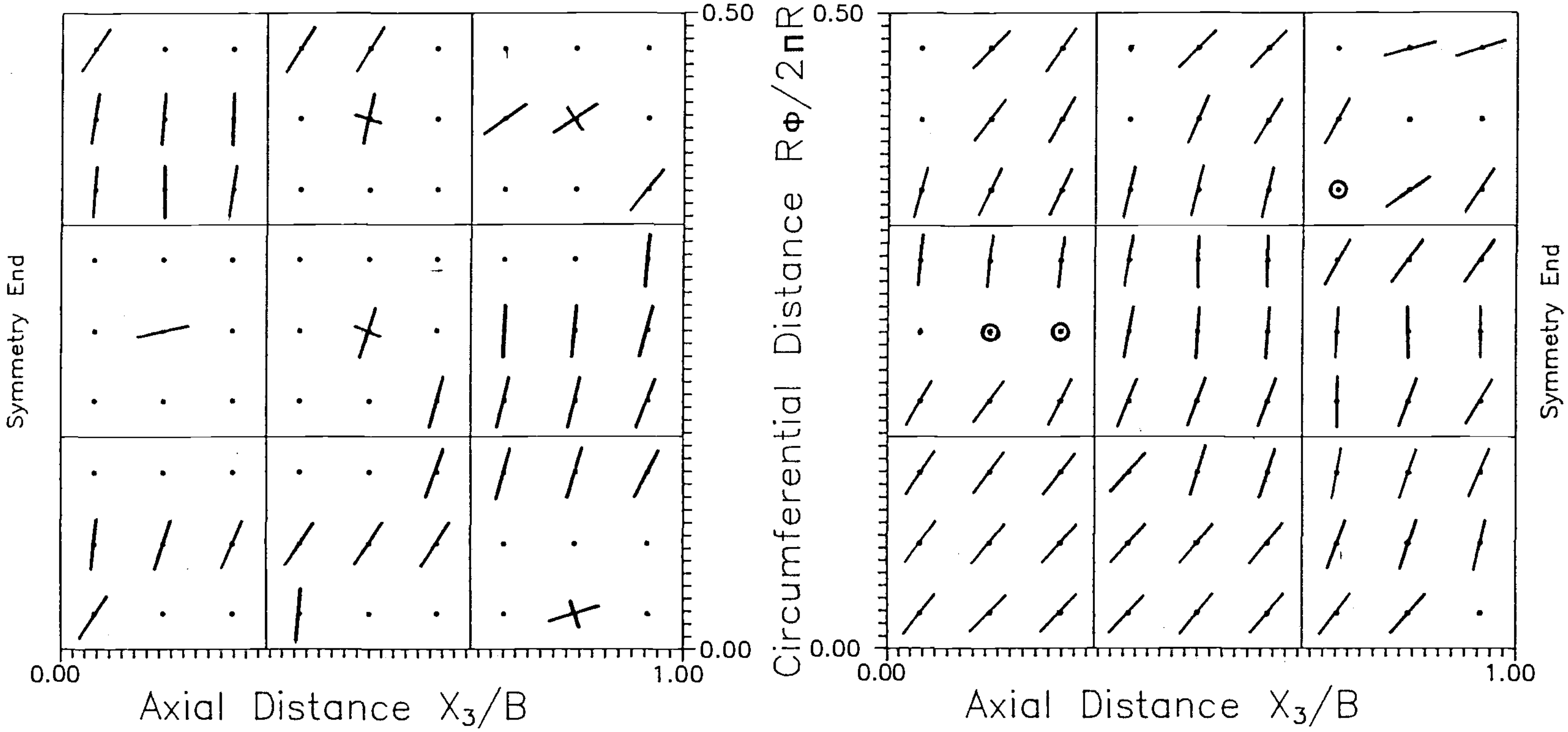

Figure IV.30 Principal stress results t/T = 1.6

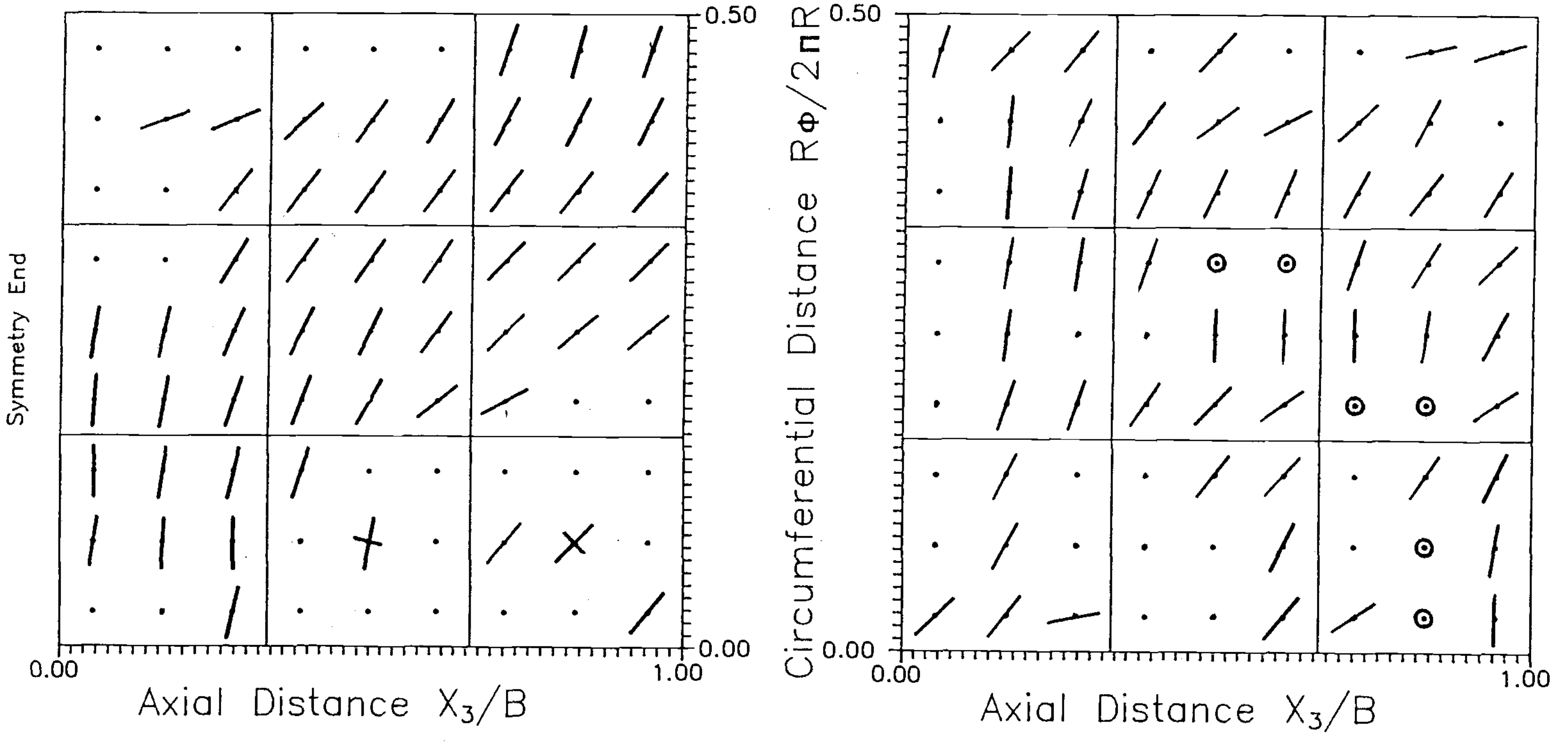

Figure IV.31 Principal stress results t/T = 2.0

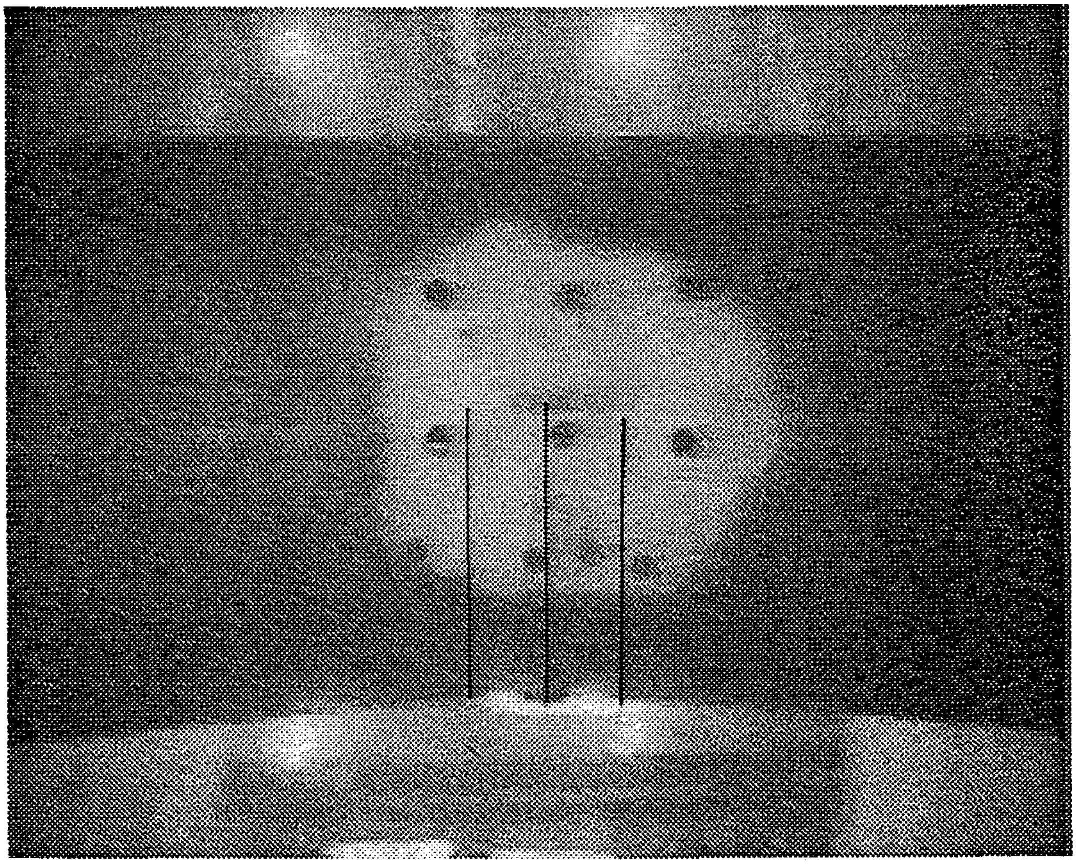

Figure IV.32 Bottom view of submerged cylinder

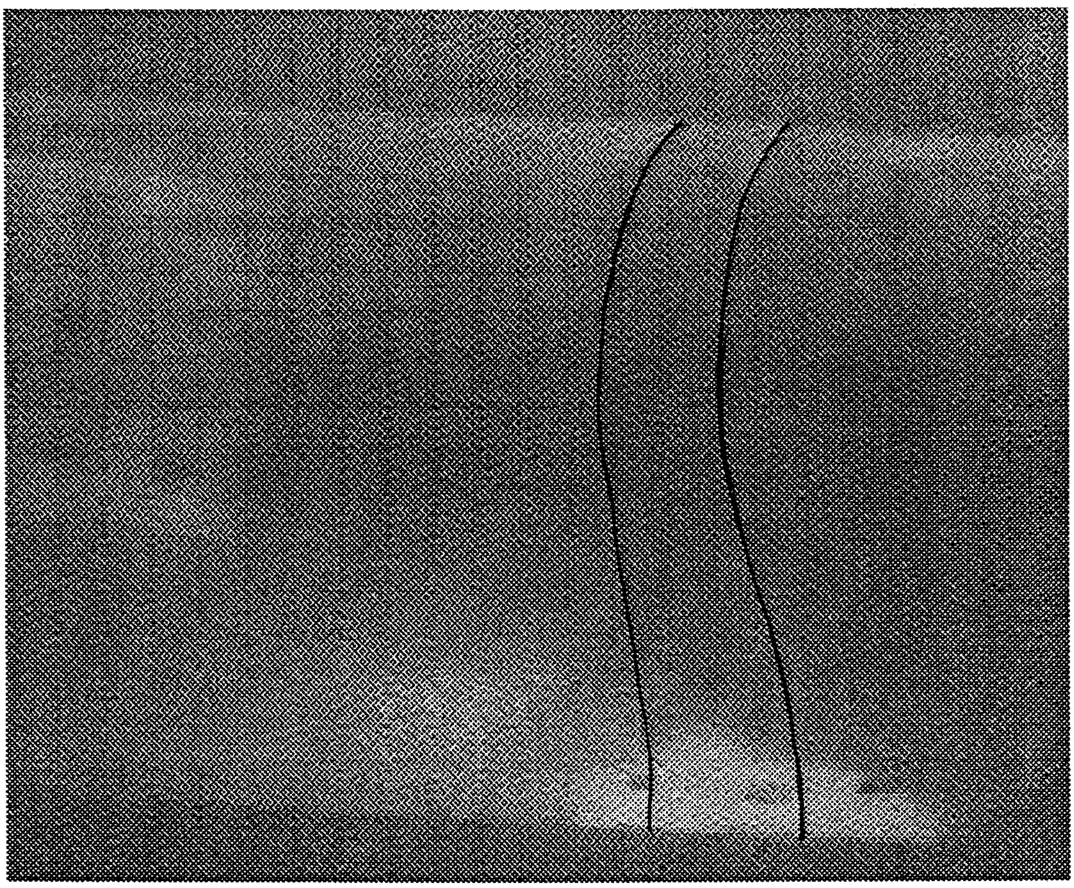

Figure IV.33 Side view of submerged cylinder

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CHAPTER V

CONCLUSION AND FUTURE STUDIES

V.1 CONCLUSION

Membrane analysis presents at once an interesting and challenging dichotomy. One the on hand, the governing field equations are greatly simplified when stress-couple effects are neglected. On the other hand, the lack of significant bending stiffness results in a highly flexible structure, portending certain challenge for the analyst. Additionally, material nonlinearities associated with many common membrane materials complicates the problem further.

The approach taken to predict nonlinear transient membrane response in this thesis may be roughly grouped into three categories: the general nonlinear finite element formulation, viscoelastic material modeling, and special consideration of membrane wrinkling.

The nonlinear finite element formulation presented above relies on a combined incremental/iteration method. Finite elements are used for spatial descretization, while an implicit (Newmark's) numerical method is used to descretize and solve the equation of motion; the modified Newton-Raphson method is used within each time increment to converge to a solution. Non-conservative pressure forces are handled in an approximate sense within the iteration procedure. The method is particularly robust for configurations where curvature is not too extreme nor loading to complex. Highly curved structures subjected to complex loadings require more careful analysis, as expected. It is possible that in these situations the assumptions used to compute the non-conservative load vector and to neglect unsymmetric stiffness contributions have been pressed to the limit; convergence problems have been discussed in the literature [see, e.g., Han (1986)]. However, the success in modeling the submerged cylinder, as evidenced by the validation comparisons, cannot be denied. Including structural damping in the analysis would also be an enhancement.

Viscoelastic materials are modeled as materials with fading memory, thus use is made of the hereditary integral formulation. This viscoelastic stress-strain relation is derived formally from strain energy and dissipation functions. For computational purposes, the resulting integro-differential equation is represented by a trapezoidal approximation and the relaxation modulus is represented by a Prony series. This method is robust wherever the viscoelastic material parameters match the loading conditions. That is to say, the experiment used to determine the viscoelastic material parameters must closely approximate loading conditions, e.g., quasi-static for quasi-static. In the submerged cylinder examples, lack of constitutive values for oscillating loads probably contributed to loss of viscoelastic detail.

With in-plane stiffness much greater than bending stiffness, membranes wrinkle under compressive load. They do not, however, wrinkle infinitely (they have non-zero bending stiffness) which violates membrane theory (assumes zero bending stiffness). Introduction of relaxed strain energy and dissipation functions incorporates wrinkling analysis into membrane analysis. As formulated herein, the method is robust in all examples considered. Increased internal pressure in the submerged cylinder example would likely lead to less wrinkling and a more stable model. The need for graphics routines to plot principal stresses is obvious.

V.2 FUTURE STUDIES

It is correct that research should raise as many questions as it answers. We list some of those questions for future studies:

<u>Nonlinear finite element analysis</u>

- need to more accurately model non-conservative loads
- need to incorporate structural damping into the analysis
- need for graphical routines to help in interpreting results

<u>Viscoelastic material model</u>

- need constitutive values for oscillatory loading
- need a fully visco-hyperelastic material model (nonlinear)

<u>Wrinkling analysis</u>

- need for graphical routines to help in interpreting results
- extend to visco-hyperelastic material

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APPENDICES

Appendix A: Nomenclature

A - undeformed membrane area

$\tilde{A}$ - relaxation modulus coefficient

B - characteristic length of undeformed body

$\tilde{B}$ - relaxation modulus coefficient

$c_{\alpha\beta}$ - curvature tensor

$\tilde{C}$ - material tensor

$\tilde{D}$, $\tilde{D}_1$, $\tilde{D}_2$ - material constants

E - convected Green - Lagrange strain tensor

$\tilde{E}$ - Young's modulus

$\tilde{F}$ - internal force vector

G_i, g_i - undeformed and deformed mid-surface basis vectors

$G_{\alpha\beta}$, $g_{\alpha\beta}$ - undeformed and deformed metric tensor components

G, g - undeformed and deformed metric tensor determinants

$\tilde{G}$ - Green deformation tensor

H - undeformed membrane thickness

i, I - indice

$\tilde{I}_i$ - strain invariant

j, J - indice

$\tilde{J}$ - creep compliance

k, K - indice

$\widetilde{K}$ - stiffness matrix

l, L - indice

M - undeformed membrane mass density

$\widetilde{M}$ - mass matrix

$n^{\alpha\beta}$ - stress resultant tensor

$\widetilde{P}$ - vector of surface tractions

P - surface traction

Q - memory term

$\widetilde{Q}$ - memory load vector

R - undeformed membrane radius

s, S - Cauchy, 2nd Piola - Kirchoff stress tensors

t - time

T - specific time: period, time constant, etc.

U_i, u_i - displacement components

X_i, x_i - undeformed and deformed membrane coordinates

$\widetilde{V}$ - dissipation energy function

$\widetilde{W}$ - strain energy function

$\widetilde{Y}$ - relaxation modulus

α, β, γ, ζ, μ - indice

δ - variation symbol

δ_{ij} - Kronecker delta

Δ - difference

$\epsilon^{\alpha\beta}$ - permutation tensor

η - viscosity coefficient

κ - constant

λ - constant

Λ_i - extension ratio

θ^i, Θ^i - convected coordinates

ν - Poisson's ratio

ξ_i - element natural coordinate

Σ - summation

τ - time (integration variable)

ψ_I - element shape function

χ - transform variable

Appendix B - Viscoelastic Constitutive Relation

Following Lemaitre and Chaboche (1990), we consider the specific plane stress case when, for small strain,

$$\tilde{W} = \frac{1}{2}\left[\frac{\nu\tilde{E}}{(1-\nu^2)} E_{\alpha\alpha}E_{\beta\beta} + 2\frac{\tilde{E}}{(1+\nu)} E_{\gamma\zeta}E_{\gamma\zeta}\right] \tag{B.1a}$$

$$\tilde{V} = \frac{1}{2}\left[\frac{\nu\tilde{E}\,T}{(1-\nu^2)} \dot{E}_{\alpha\alpha}\dot{E}_{\beta\beta} + 2\frac{\tilde{E}\,T^*}{(1+\nu)} \dot{E}_{\gamma\zeta}\dot{E}_{\gamma\zeta}\right] \tag{B.1b}$$

where T and T^* are characteristic retardation times in tension and shear, respectively, $\tilde{E}$ is the initial elastic modulus, and ν is the Poisson's ratio. The stress can now be shown to be

$$\begin{aligned} S_{\alpha\beta} &= \left[\frac{\nu\tilde{E}}{(1-\nu^2)} E_{\gamma\gamma}\delta_{\alpha\beta} + \frac{\tilde{E}}{(1+\nu)} E_{\alpha\beta}\right] \\ &\quad + \left[\frac{\nu\tilde{E}\,T}{(1+\nu)(1-\alpha\nu)} \dot{E}_{\zeta\zeta}\delta_{\alpha\beta} + \frac{\tilde{E}\,T^*}{(1+\nu)} \dot{E}_{\alpha\beta}\right] \\ &= \frac{\nu\tilde{E}}{(1-\nu^2)} (E_{\gamma\gamma} + T\dot{E}_{\zeta\zeta})\,\delta_{\alpha\beta} + \frac{\tilde{E}}{(1+\nu)} (E_{\alpha\beta} + T^*\dot{E}_{\alpha\beta}) \end{aligned} \tag{B.2}$$

The shear retardation time, T^*, can be determined during a shear test, i.e., when

$$S_{\alpha\beta} = \text{constant} = \frac{\tilde{E}}{(1+\nu)} (E_{\alpha\beta} + T^*\dot{E}_{\alpha\beta}), \quad \alpha \neq \beta \tag{B.3}$$

The solution of this differential equation is readily shown to be

$$E_{\alpha\beta}(t) = \frac{1+\nu}{\tilde{E}} S_{\alpha\beta}[1-\exp(-t/T^*)], \quad \alpha \neq \beta \tag{B.4}$$

Similarly, T is identified through a tension test, viz.,

$$S_{(\alpha)(\alpha)} = \text{constant} = \frac{\tilde{E}\nu}{(1-\nu^2)}(E_{\gamma\gamma} + T\dot{E}_{\zeta\zeta})\,\delta_{(\alpha)(\alpha)} + \frac{\tilde{E}}{(1+\nu)}[E_{(\alpha)(\alpha)} + T^*\dot{E}_{(\alpha)(\alpha)}] \tag{B.5}$$

Using the facts that $E_{(\beta)(\beta)} = -\nu E_{(\alpha)(\alpha)}$, where υ = constant, and

$1-\nu^2 = (1-\nu)(1+\nu)$, we combine terms to get

$$\begin{aligned} S_{(\alpha)(\alpha)} &= \tilde{E}E_{(\alpha)(\alpha)} + \frac{\tilde{E}}{(1+\nu)}(\nu T + T^*)\,\dot{E}_{(\alpha)(\alpha)} \\ &= \tilde{E}E_{(\alpha)(\alpha)} + \eta\dot{E}_{(\alpha)(\alpha)}, \quad \eta = \frac{\tilde{E}}{(1+\nu)}(\nu T + T^*) \end{aligned} \tag{B.6}$$

We recognize (B.6) as the governing equation for a Kelvin-Voigt mechanical-analogic model (Figure B.1) with response

$$E_{(\alpha)(\alpha)}(t) = \frac{1}{\tilde{E}}S_{(\alpha)(\alpha)}[1-\exp(-\tilde{E}t/\eta)] \tag{B.7}$$

The model can be generalized by forming assembles of Kelvin-Voigt models as shown in Figure B.2. For linear viscoelasticity, the strain responses may be summed, i.e.

$$E_{(\alpha)(\alpha)}(t) = \{\frac{1}{\tilde{E}_1} + \frac{1}{\tilde{E}_2}[1-\exp(-\tilde{E}_2 t/\eta_2)] + \frac{1}{\tilde{E}_3}[1-\exp(-\tilde{E}_3 t/\eta_3)]\}S_{(\alpha)(\alpha)} \tag{B.8}$$

This result can also be reached by use of the Prony series representation of the material function in the hereditary constitutive relation. Consider the three term Prony series

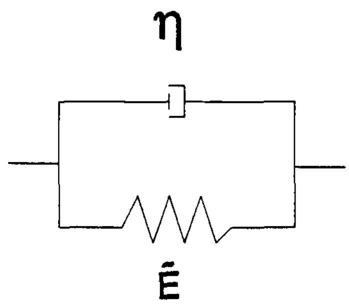

Figure B.1 Kelvin-Voigt model

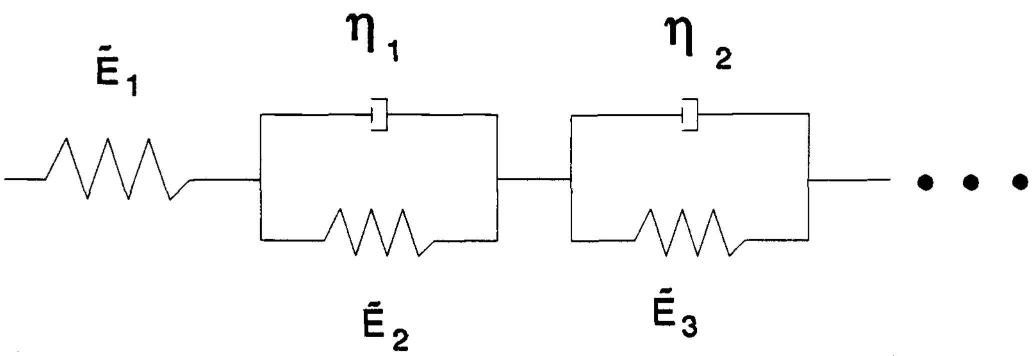

Figure B.2 Generalized Kelvin-Voigt model

$$\tilde{J}(t) = \tilde{A}_1 + \tilde{A}_2 \exp(-\tilde{B}_2 t) + \tilde{A}_3 \exp(-\tilde{B}_3 t) \tag{B.9}$$

substituted into the linear hereditary integral

$$E_{\alpha\beta}(t) = \tilde{J}(0) S_{\alpha\beta}(t) - \int_0^t \frac{d\tilde{J}(t-\tau)}{d\tau} S_{\alpha\beta}(\tau)\, d\tau \tag{B.10}$$

For constant $S_{\alpha\beta}$ in (B.10) we integrate to get equation (B.8) where

$$\frac{1}{\tilde{E}_1} = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3$$
$$\frac{1}{\tilde{E}_2} = \tilde{A}_2$$
$$\frac{1}{\tilde{E}_3} = \tilde{A}_3$$
$$\tilde{B}_2 = \frac{\tilde{E}_2}{\eta_2}$$
$$\tilde{B}_3 = \frac{\tilde{E}_3}{\eta_3}$$

Appendix C - Wrinkling Condition

Define principal stresses and strains:

$$S^{\alpha} \equiv [\, S^{(\alpha)(\alpha)} \,]_{\max,\min} \;,\; E_{\beta} \equiv [\, E_{(\beta)(\beta)} \,]_{\max,\min}$$

where $S^1 > S^2$, $E_1 > E_2$, and parentheses around superscripts or subscripts indicate no sum. Now also define

$$\tilde{C}^{\alpha\beta} \equiv \tilde{C}^{(\alpha)(\alpha)(\beta)(\beta)}$$

In general, the principal axes (defined by the basis vectors $\overset{*}{G}_{\alpha}$) will differ from the local coordinate axes (defined by the basis vectors G_{α}), and the metric tensor components $G^{\alpha\beta}$ transformed to the principal coordinates are given by

$$\overset{*}{G}{}^{\alpha\beta} = \lambda_{\gamma}^{\alpha} \, \lambda_{\xi}^{\beta} \, G^{\gamma\xi}$$

where the backward change of basis coeffecients $\lambda_{\gamma}^{\alpha}$ [Malvern (1969)] are given by the transformation

$$G_{\gamma} = \lambda_{\gamma}^{\alpha} \, \overset{*}{G}_{\alpha}$$

For an isotropic Hookean material

$$S^{\alpha} = \tilde{C}^{\alpha\beta} E_{\beta}$$

Now for wrinkling, $S^2=0$, then

$$0 = \tilde{C}^{21} E_1 + \tilde{C}^{22} E_2$$

and the wrinkling conditon is given by

$$\overset{*}{E}_2(E_1) = -\frac{\tilde{C}^{21}}{\tilde{C}^{22}} E_1$$

The principal stress under wrinkling is then given by

$$\begin{aligned} \overset{*}{S}{}^1 &= \tilde{C}^{11} E_1 + \tilde{C}^{12} \overset{*}{E}_2 \\ &= \tilde{C}^{11} E_1 - \frac{\tilde{C}^{12}\tilde{C}^{21}}{\tilde{C}^{22}} E_1 \\ &= (\tilde{C}^{11} - \frac{\tilde{C}^{12}\tilde{C}^{12}}{\tilde{C}^{22}}) E_1 \end{aligned}$$

Appendix D - Algorithm Flow Charts

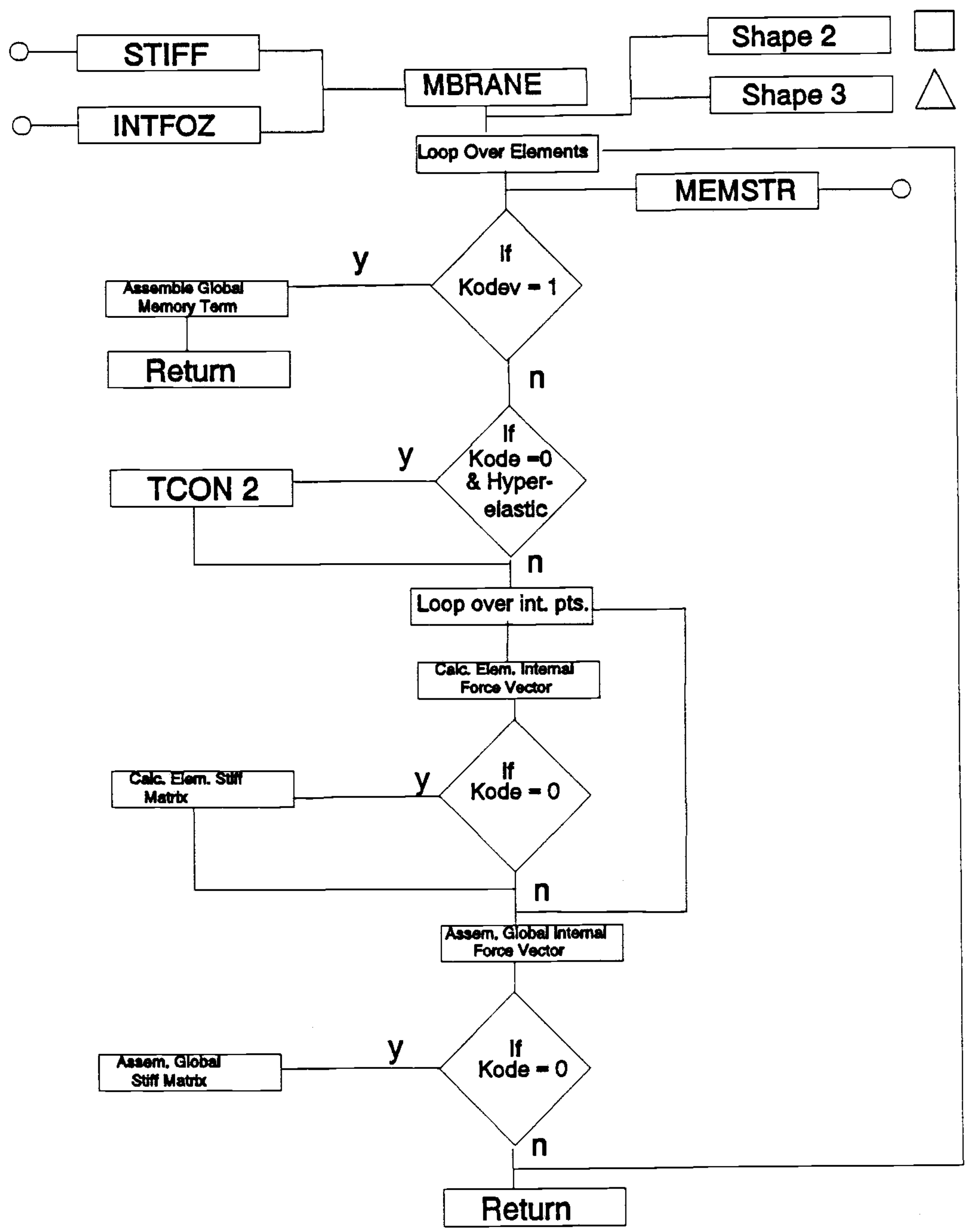

Figure D.1 Flowchart no. 1

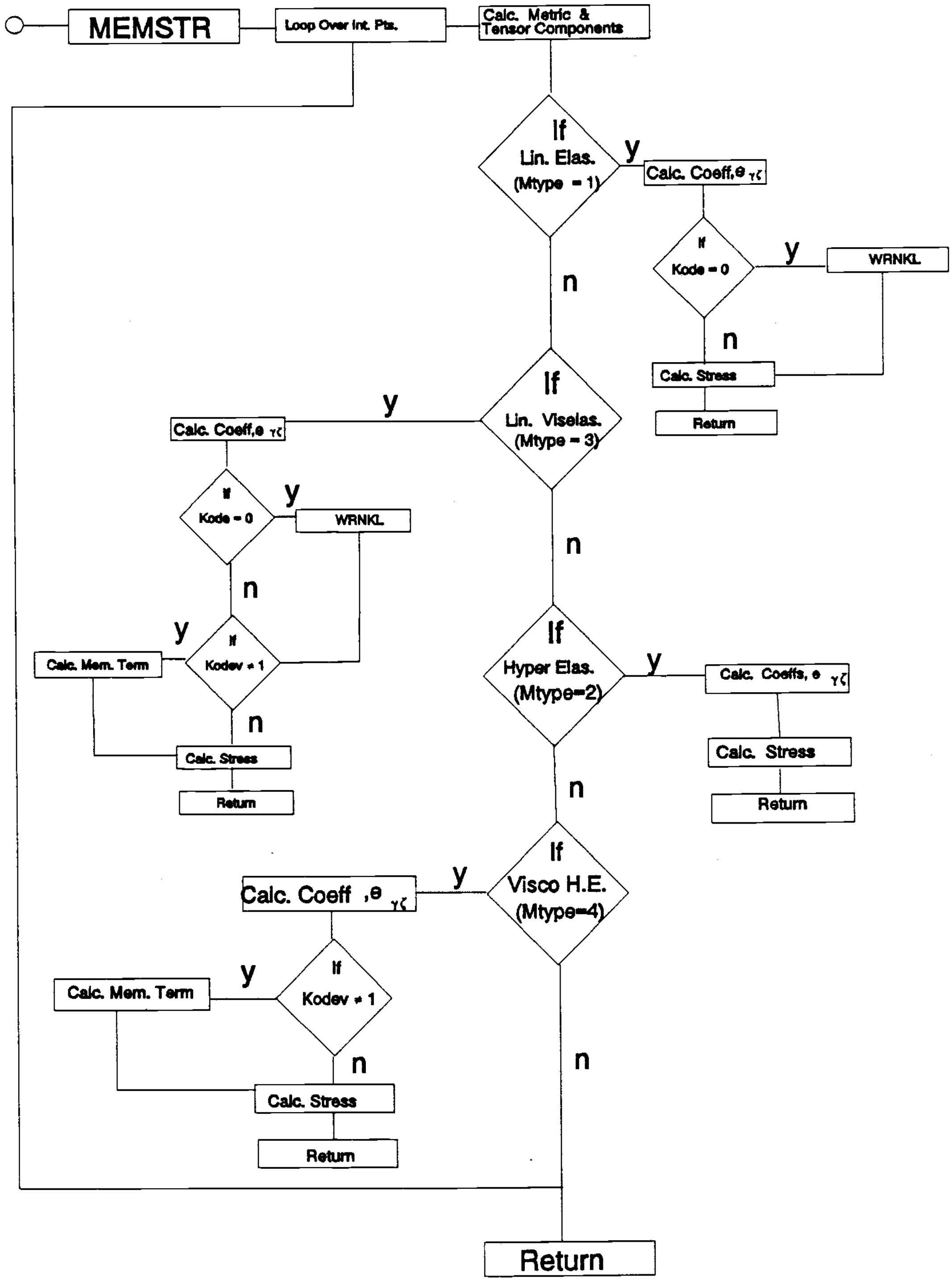

Figure D.2 Flowchart no. 2

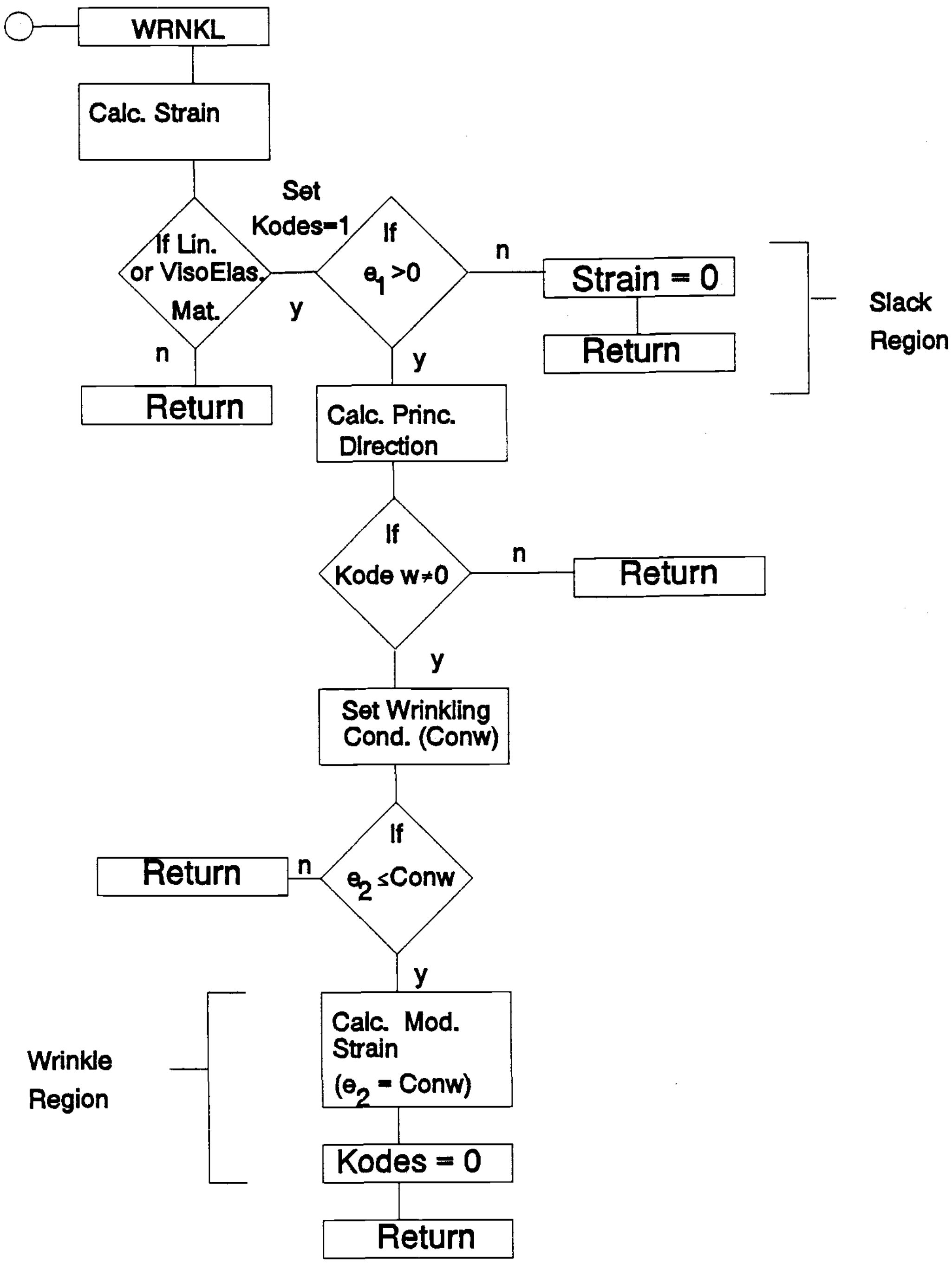

Figure D.3 Flowchart no. 3

Appendix E - Cylindrical Membrane Experiment

[Excerpted from Broderick (1991)]

To verify the finite element model of the nonlinear interaction of water waves and deformable bodies, a large-scale physical model test was conducted in the large wave channel at the O.H. Hinsdale Wave Research Laboratory at Oregon State University. A membrane cylinder was placed horizontally in the wave tank. Waves were then generated that induced motions in the cylinder, which in turn affected the waves. The deformations of the free surface were recorded above, fore, and aft of the cylinder. Displacements of the membrane were physically recorded with string pots as well as visually recorded on video tape. The internal pressure of the fluid enclosed by the membrane was also recorded. The resulting data allows for the verification of the coupled model by comparing the predicted deformations of the free surface and displacements of the membrane to those measured in the physical model.

The large wave channel in the O.H. Hinsdale Wave Research Laboratory is 342 feet long, 12 feet wide, and 15 feet deep. A hinged-flap waveboard, hydraulically driven and servo-controlled, can generate random and monochromatic waves up to five feet high. This channel provides a facility to validate theoretical wave models with large scale experiments in an environment with minimal Reynolds Number distortion.

The three-foot-diameter cylinder was placed horizontally across the tank and rigidly attached to the side walls, see Figures E.1 and E.2. The center of the cylinder was located 113 feet from the wave board and 6 feet above the floor. The membrane

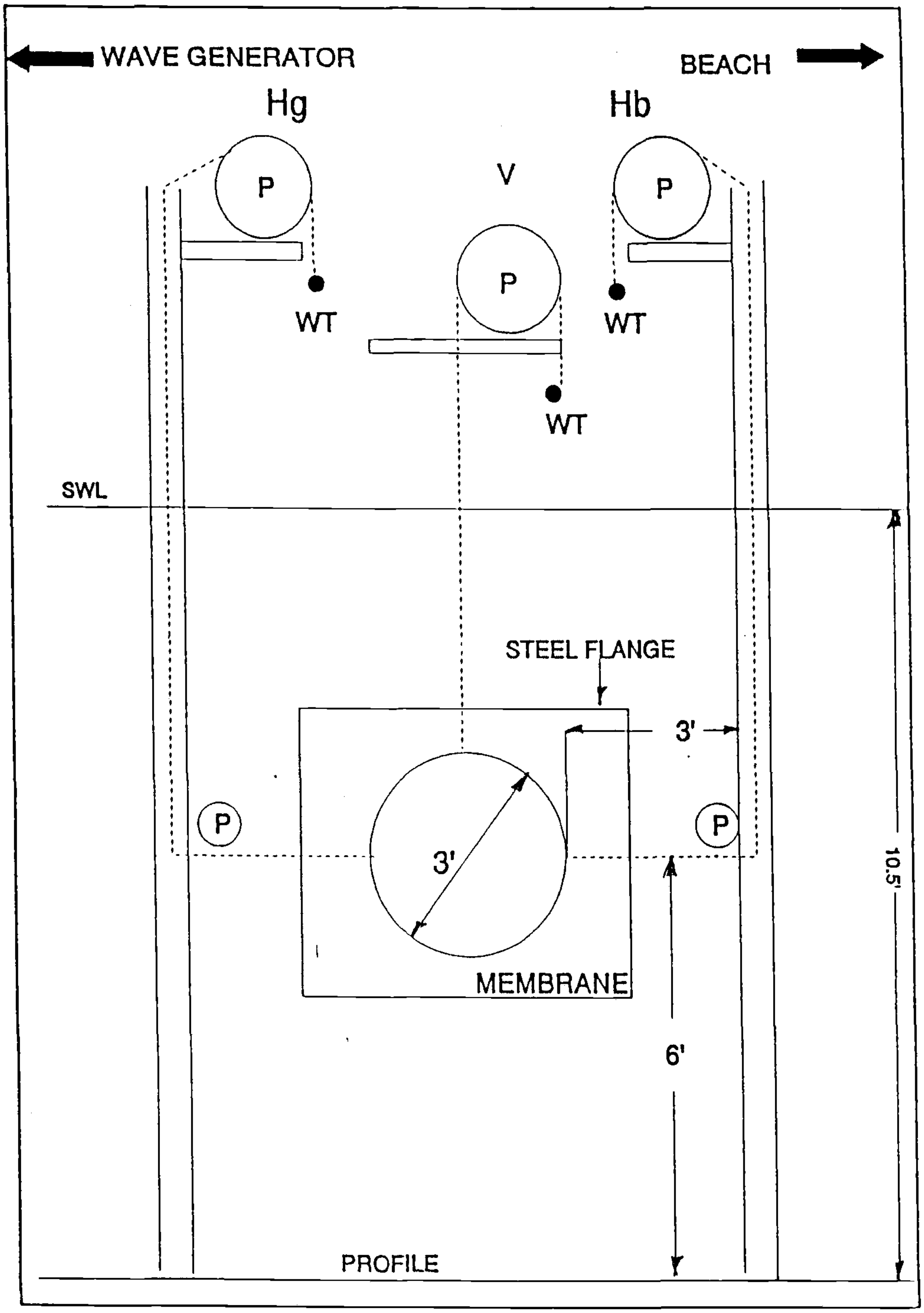

Figure E.1 Profile of flexible cylinder experiment.

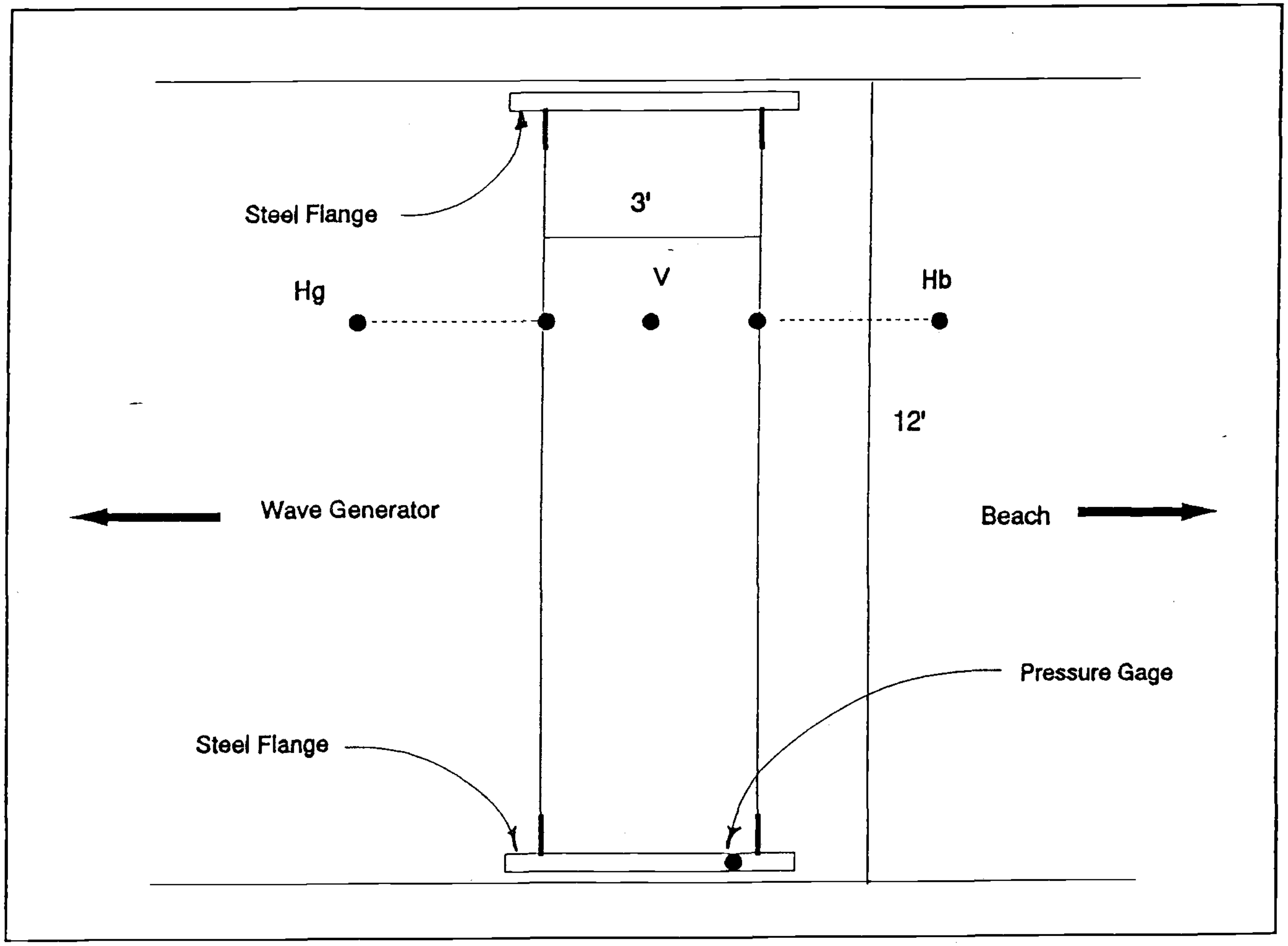

Figure E.2 Plan of flexible cylinder experiment material was plasticized polyvinyl chloride (pvc) with a thickness of 50 mil. The modulus of elasticity is approximately 8000 psi and the density of the material 0.05 lb/in^3. The material comes in rolls approximately 6 feet wide. To form cylinder, two 4.8 X 12 foot sections were attached using glue. The seams were heat-sealed and appeared to be fairly air-tight. The seams were at the top and bottom of the cylinder when placed in the wave channel.

The cylinder shape was formed by clamping the membrane over circular steel flanges that were rigidly attached to the side walls of the tank. To insure that the membrane did not slide off of the steel flanges, a one inch strip of the membrane was glued around the edge of the flange to give the clamps something to seat against. (During the trial the strips had not been glued to the flanges and the membrane did come off the flanges.)

The steel flanges were equipped with hose fittings at the top and bottom. One of the fittings on the bottom of the flange allowed the inside of the cylinder to be filled with water as the water level was raised in the tank. This same fitting allowed the internal pressure to be varied before the start of a test by adding water. During a run this fitting was sealed, allowing no change in the internal volume of fluid. A pressure transducer was placed in one of the fittings located at the top of the cylinder to record the internal pressure of the fluid enclosed in the membrane.

The center line of the cylinder was 6 feet off the tank floor, see Figure E.1. Two water levels were tested, 9 feet and 10.5 feet, which gave a submergence of 1.5 feet and 3, feet respectively. Monochromatic waves were generated with wave periods ranging from 1.5 seconds to 6 seconds. Random waves were also generated at both water depths. Wave heights varied from a couple of inches to over a foot. Sonic profilers were used to record the free surface above, fore, and aft of the cylinder. String pots and piano wire were used to measure the displacement of the membrane at three locations around the cylinder, see Figures E.1 and E.2. Various weights were tried on the end of the string pots to see if the point force affected the displacements; there was no noticeable effect. The displacement measurements were taken at the quarter point across the tank.

Seven channels of data were collected; the data was then edited and the appropriate scale factor applied. Various data analysis programs (provided by David Standley, staff member of the O.H. Hinsdale Wave Research Laboratory) were run on the data and provided a variety of information.

Originally, the plan had been to measure the displacements of the membrane by video techniques. Two video cameras, one mounted vertically under the center of the cylinder and one mounted 20 feet in front of the cylinder at an angle of 30 degrees, recorded the membrane's motion. The string pots were added because it was not clear how accurate the cameras would be in measuring the displacements. The video tapes do add qualitative information to the wave/structure interaction.

An attempt was made to strain gauge the membrane and this appeared to work during the trial. However, when the membrane was placed back into the wave tank for the experiment, the strain gauges ceased to operate, most likely because the in water-tight seals leaked.

Appendix F - Relaxation Modulus

In order to provide realistic results, real material properties are required. This Appendix outlines the method used to determine the relaxation modului for two different membrane materials: a plasticized polyvinyl chloride (PVC) geomembrane and a low-density polyethylene (LDPE) film.

Creep testing was preferred to relaxation testing since applying a step-load was more practical than applying a step-displacement. An approximately 25 mm (1.0 in) wide by 0.61 m (2.0 ft) long strip of material was placed in a dead-load type creep testing machine. A load was applied and displacement results taken over a number of different trials [for a typical result, see Figure IV.1]. For the PVC material, a 15-minute creep test used a 2.22 N (0.498 lb) load; for LDPE, a 50-hour creep test used a 0.290 N (1.29 lb) load. A three-term Prony series (see Sec. IV.2.3, equation 2.3.3) was fit to the experimental data (see Figure IV.1). Normalized to unit stress, the Prony series becomes the creep compliance for the material.

The creep compliance and relaxation modulus are related in transform space by

$$\bar{Y}(\chi) = \frac{1}{\chi^2 \bar{J}(\chi)}$$

From the above equation, the corresponding relaxation modulus was determined.

It should be noted that the creep compliance is strain-rate sensitive, that is, different values of Prony series coefficients and time constants will be recorded for different values of loading; a similar statement can be made for the relaxation modulus.

(The time constant of an exponential term, e.g., T_1 in $\lambda = \lambda_0 \exp(-t/T_1)$, is defined such that when $t = T_1$, $\lambda = 0.37\lambda_0$.) The most appropriate modulus to use is one which has been determined from a test which closely approximates the problem at hand; obviously, this is not always possible.

For a quasi-static problem (e.g., the slow inflation of an initially plane membrane), a check on appropriateness of the modulus can be made by observing if a typical value of the membrane stress can be matched to the stress level present during the (quasi-static) creep test. In the above described test, this corresponds to 679 kPa (98.5 psi) and 1.62 MPa (235 psi) for the 15-minute PVC and 50-hour LDPE creep tests, respectively.

The range of dynamic loads that can be successfully modeled as quasi-static is subject to verification. For high frequency harmonic loading, a 'complex' modulus may be desirable [see Christensen (1982)]. In this thesis, no complex modulus was used due to lack of necessary measurement apparatus.

Finally, it is noted that for the numerical results reported in this thesis, the time step used for dynamic viscoelastic analysis was generally kept at less than, or equal to, 1/10 of the time constant of the first exponential term in the relaxation modulus. This corresponds to $\Delta t \leq 3.8$ s and 0.5 hr for the 15-minute PVC and 50-hour LDPE moduli, respectively.