This study investigated a solution technique to shorten the calculation time required for determining power losses of the different resonant modes of oscillation in the Fabry-Perot optical cavity.

The first comprehensive analytic study of the Fabry-Perot optical resonator was made by A. G. Fox and Tingye Li of Bell Laboratories. Using a scalar formulation of Huygen's principle, they were able to derive a homogeneous Fredholm equation of the second kind to describe the resonant modes. A numerical iteration method with a converging solution was used to calculate the solutions of this eigenvalue problem. This method was accurate but required a considerable amount of computer time. This made the iteration solution unattractive in terms of computer time cost.

A modified solution scheme, based on iteration, was studied in this thesis to reduce the calculation time for determining the
power loss. A reduction of about 70% in the computer time was achieved without loss of accuracy by using a variational principle in addition to the iteration process.

I would like to express my gratitude to Dr. J. Fred Holmes for the guidance he has given me in this thesis study.
Numerical Calculation of Optical Resonator Losses

by

Richard Shuji Yamamoto

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Professor of Electrical and Electronic Engineering in charge of major

Redacted for privacy

Head of Department of Electrical and Electronic Engineering

Redacted for privacy

Dean of Graduate School

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NUMERICAL CALCULATION OF
OPTICAL RESONATOR LOSSES

INTRODUCTION

The Fabry-Perot etalon or cavity is an optical resonator that consists of two partially transmitting reflectors oriented in space with a common axis of symmetry. This etalon was initially used in the Fabry-Perot interferometer to measure the spectral content of light waves. The interferometer has a multiple beam output that produces interference rings or fringes at the far field. The width of these fringes which determines the spectral resolution of the input excitation, is a function of the reflectivity, flatness and geometric configuration of the reflectors which make up the resonant cavity.

Since the development of laser theory, the application of the Fabry-Perot cavity has extended into the field of amplification at optical frequencies. The cavity serves as the resonant chamber where the optical frequencies are pumped or amplified. The geometric configurations of the reflectors and their distance of separation determine the resonant modes of the etalon. Different modes of oscillations are denoted by the indices m, n, and q. The indices m and n indicate the number of phase reversals of the fields across each of the two coordinate dimensions of the reflectors. The index
q specifies the number of half wavelengths separating the reflectors.

A Fabry-Perot etalon utilizing flat rectangular reflectors is shown below:

![Figure 1. Rectangular reflector Fabry-Perot etalon.](image)

Analytical studies of this cavity have provided data for the different modes of oscillation that are excited within different reflector geometries. This information includes the characteristic field patterns and losses of the modes, and is valuable in the design of lasers which use the Fabry-Perot etalon as a resonant cavity and in optical spectrometers.

The optical maser can be considered as a source of randomly fluctuating dipole moments which drives every mode in the cavity. The output frequency of such a maser then is determined by the relative losses of the various modes in the resonant cavity (20). These losses are primarily determined by the transverse mode orders and are only very slightly affected by the longitudinal orders.
Thus the Fabry-Perot cavity is highly mode selective with respect to transverse modes (8).

It is sometimes desirable to build a single mode laser to use as a coherent monochromatic light source. An example of an application of this type of laser would be a light modulated information carrier system. In such a system the noise introduced by the carrier source would be minimized by using only a single mode of oscillation. Sustained oscillation within the Fabry-Perot resonator occurs for the modes with the highest Q, a quantity which is inversely proportional to the sum of the power losses due to reflection and diffraction off the mirror surface. All the other modes only contribute to the background noise. If analytical studies could be used to enhance a desired mode of oscillation and suppress or attenuate all other modes, a better signal to noise ratio could be obtained for the light source.

Analytic studies of the etalon are also useful in the design of spectroscopic instruments. One of the factors that determine the width of its spectral lines or fringes is the frequency content of the output radiation. If the output is composed of a dominant frequency $f_o$ with small traces of frequencies close to $f_o$, then the output fringes will not be distinct lines but instead lines that are broadened with a Gaussian shaped intensity distribution. This line broadening can be regarded as the "noise" introduced by the spectroscopic system. The results of the analytic studies can be used in designing the
resonant cavity to improve the monochromaticity of the interferometer output and increase the spectral resolution.

There have been numerous schemes to minimize or isolate modes of transverse resonances (3, 7, 8, 11, 12, 14, 17, 19). There is probably no optimum technique for achieving a single mode source, instead each scheme has its advantage in certain laser configurations.

The design of resonant cavities for monochromatic operation begins with the analytic investigation of the modes within the different cavity geometries. An integral equation has been formulated to describe the different modes of oscillations (see page 7). This integral equation is an eigenvalue problem which has an infinite number of solutions. Each solution, which represents a different resonant mode, consists of the eigenfunction or transverse field distribution and the corresponding eigenvalue or relative loss of that particular mode.

Mode selection is based upon the relative (power) losses which can be derived from the eigenvalues of the different modes, and also on the feasibility of filtering different field distributions. In the laser, the lowest order mode of resonance which has the smallest loss is most often selected as the desired output. In order to restrict the output oscillations to the lowest order mode, external energy supplied to the cavity is limited to a threshold level necessary for sustained oscillations for the lowest order mode. Since all the other
modes require more than the threshold level provided for the lowest mode, they are attenuated. The enhancement of higher modes of resonance require that the losses of the lower modes be made larger than the desired mode by altering the fields using techniques such as filtering by apertures and films. Another method that has been tried is the insertion of an additional tilted etalon to change the relative diffractional losses of the modes. These results have been verified by analytic results (3). Mode selection as described above may also be used to increase the spectral resolution of the Fabry-Perot interferometer.

Because the analytic results are so important in the study and design of lasers and interferometers, there have been many studies to compute the solutions of the Fabry-Perot cavity problem. The complexity of this problem and the absence of a "straight-forward" solution resulted in a variety of solution schemes. Most of the schemes, however are still complicated and often very lengthy. The purpose of this study was to develop a scheme to simplify the solution technique and also to reduce the time required for the calculation. This scheme is based on an accurate but time consuming technique of successive iteration. The large number of calculations required in the iteration method was very undesirable. The results of this study have shown that a modified scheme tested reduced the calculation time considerably without sacrificing accuracy. By incorporating
a variational technique within the iterational solution, computer time required for a convergent solution was reduced to about $1/3$ of the time required for the iteration method alone.

The results of this study were presented at the USNC/USRI - IEEE G-AP Symposium, Numerical Methods and Antennas session at the UCLA campus on September 23, 1971.
INTEGRAL EQUATION FOR THE FABRY-PEROT CAVITY

The resonant modes of oscillation for the Fabry-Perot Cavity are analytically represented by a homogeneous Fredholm equation of the second kind. This closed form describing the resonant behavior of the cavity was derived by Fox and Li (5) using a scalar formulation of Huygen's principle. In this approach of solving the fields at one mirror in terms of an integral of the fields of the other, the following assumptions had to be made. The dimension of the mirrors were very large compared to one wavelength, and also the fields were very nearly transverse electromagnetic. These assumptions were shown to be valid by their results.

A brief derivation as done by Fox and Li is given below:

\[ U_{A'}(R) = \int \frac{U_a(A) \cdot n}{\sin \theta} dA \]

The field \( U_{A'} \) at the surface \( A' \) due to the fields \( U_a \) radiated from the aperture \( A \) is given by the surface integral (16),
\[ U_p = \frac{jk}{4\pi} \int_A U_a \frac{e^{-jkR}}{R} (1 + \cos \Theta) \, dS \]

where: \( k \) is the propagation constant \((2\pi/\lambda)\) of the medium between A and A'.

\( \bar{R} \) is the distance from the source point to the observation point.

\( \hat{n} \) is the unit normal at the source point.

\( \Theta \) is the angle between \( \bar{R} \) and \( \hat{n} \).

When the surfaces A and A' are considered to be the mirrors of the Fabry-Perot cavity, the source fields \( U_a \) are the reflected fields from one mirror back to the other. The surface integral can then be written:

\[ U_{i+1} = \frac{jk}{4\pi} \int_A U_i \frac{e^{-jkR}}{R} (1 + \cos \Theta) \, dS \]

Where \( U_i \) represent the fields for the \( i \)th reflection. If the fields settle to a steady-state distribution, \( U_{i+1} \) will differ from \( U_i \) only by a complex constant multiplier \( \gamma \). Then the equation can be represented in the form of a homogeneous Fredholm equation of the second kind.

\[ \gamma v = \int_A K v \, dS \]

where: \( v \), the eigenfunction, represents the field distribution at the surface of the mirror.

\( \gamma \), the eigenvalue, gives the attenuation and phase
shift of the fields as it is reflected back from the mirror.

The kernel \( K = \frac{j \kappa}{4\pi R} \cdot (1 + \cos \Theta) e^{-jkR} \).

The integral equation is an eigenvalue equation which has been shown by Fox and Li and others to have a number of discrete solutions \( \nu_m \) with corresponding eigenvalues \( \gamma_m \). These different solutions represent the modes of the cavity. The eigenvalues \( \gamma_m \) which differ for each mode are complex and can be expressed as \( |\gamma| e^{-j\beta} \).

\( |\gamma| \) specifies the relative attenuation of the field amplitude, and it can be used to calculate the relative power loss as \( (1 - |\gamma|^2) \).

The phase shift of the wavefront as it is reflected off the mirror surface is specified by the imaginary part of \( \gamma \) which is \( \beta \).

The derivation of the integral equation applies to a very general case of two reflectors oriented in space. This equation can then be applied to the specific geometries of the different types of Fabry-Perot interferometer cavities. The different types of cavities are constructed by using various combinations and spacing of flat and curved mirrors. In each case, however, the axes of symmetry of both mirrors coincide with each other. The different geometries of the reflectors and the separation distance determine the field distribution and relative losses of the modes.

The Fabry-Perot interferometer is a two dimensional cross-sectional system and is represented by the two dimensional problem
in Cartesian coordinates:

\[ \gamma_{xy} \psi_2(x_2, y_2) = \int_{\text{surface}} K(x_1, x_2; y_1, y_2) \psi_1(x_1, y_1) \, dx_1 \, dy_1 \]

The subscripts 1 and 2 designate the coordinates on reflectors one and two respectively.

This problem was simplified by assuming normal modes which made the eigenfunctions separable on the surface of integration (5).

Then

\[ \psi(x, y) = \psi(x) \, \psi(y) \]

and

\[ \gamma_{xy} = \gamma_x \gamma_y \]

The equation for \( \psi(x, y) \) can now be solved using two separate one dimensional equations for \( \psi(x) \) and \( \psi(y) \).

\[ \gamma \, \psi(x_2) = \int K(x_1, x_2) \, \psi(x_1) \, dx_1 \]

There are three types of orthogonality for the eigenfunctions of the homogeneous Fredholm equation given above (2). The type of orthogonality for a given integral equation depends upon the nature of the kernel which may be real or complex. These kernels represent the scattering matrices of the end reflectors (6).

If the kernel is Hermitian, i.e. symmetric and equal to its complex conjugate, \( K(x_1, x_2) = \overline{K(x_2, x_1)} \), then the eigenvalues are all
real and the eigenfunctions are orthogonal in the Hermitian sense. The upper bar signifies the conjugated term. These eigenfunctions form a complete set of orthogonal functions and satisfy the condition:

$$\int_{X_1}^{X_2} \overline{\psi_m(x)} \psi_n(x) \, dx = 0 \quad m \neq n$$

The kernel that only possesses symmetry but not conjugate equality, results in non-Hermitian orthogonality for the eigenfunctions. The eigenvalues in this case are complex unless the kernel itself is real. These eigenfunctions do not form a complete orthogonal set, and orthogonality is defined as:

$$\int_{X_1}^{X_2} \psi_m(x) \psi_n(x) \, dx = 0$$

If the kernel is neither symmetric nor Hermitian, the orthogonality relationship involves the Hermitian adjoint problem. The original equation and the adjoint problem are given by equation 2 and 3.

$$\gamma \psi(x_2) = \int_{X_1}^{X_2} K(x_1, x_2) \psi(x_1) \, dx_1 \quad (2)$$
The orthogonality relationship is expressed as

$$\gamma \phi(x_2) = \int \bar{K}(x_1, x_2) \phi(x_1) \, dx_1$$

The orthogonality relationship is expressed as

$$\bar{x}_2 \int \phi_m(x) \phi_n(x) = 0 \quad m \neq n$$

The eigenvalues for the equation set are complex conjugates of one another.

This study investigated a solution technique applied to the integral equation for the flat rectangular mirror cavity. The mirrors were squares of side dimension 2a and separated by a distance b. The kernel for this equation was obtained by making the valid assumptions $a^2/b << (b/a)^2$ at optical frequencies (5)

$$K(x_2, x_1) = \frac{e^{j(\pi/4)}}{\sqrt{\lambda b}} e^{-jk(x_1-x_2)^2/2b}$$

This kernel is symmetric but not equal to its complex conjugate so it is a non-Hermitian kernel. The solutions obtained are orthogonal in the non-Hermitian sense.
The integral equation which was selected for this study describes the resonant behavior of a Fabry-Perot cavity with an identical pair of square flat reflectors. The losses for this cavity were shown by Fox and Li's results to be a function of the Fresnel number $N$, given by

$$N = \frac{a^2}{b\lambda}$$

The Fresnel number $N = 6.25$ was intentionally chosen so that a direct comparison of results could be made with those of Fox and Li who originally used this number in their computations. The wavelength of the Helium-Neon laser was used for this study ($\lambda = 6.328\,\text{Å}$) with $a = 25\,\lambda$ and $b = 100\,\lambda$.

The mirrors were assumed to be perfectly reflecting, although
there are very small transmission losses in practical cavities that permit electromagnetic energy to be coupled into or out of the cavity. Other losses due to absorption at the reflectors and surface irregularities were not included in the loss calculations because they were constant for all the modes and were not factors in determining separability of the different modes.
The homogeneous Fredholm equation of the second kind is difficult to solve. Since the discovery of lasers this problem has attracted the attention of many investigators who tried different schemes to solve the equation. These schemes differed considerably, ranging from iteration (5) to the method of moments (1). The first published method was that of iteration done by A. G. Fox, and Tingye Li in 1960. Unlike many other methods that were complex and difficult to follow, the iteration method was simple and gave good results. The method illustrated in Figure 2 shows the iterated solution method for the lowest order mode.

Figure 2. Iteration solution.
A uniform plane wave was used for the initial trial function which was integrated with the kernel for the flat rectangular mirror. This generated a new function which was compared to the previous one to check for convergence. If the difference between the two functions was large, the newly generated function was again integrated and tested for convergence. In order to obtain solutions for the higher order modes of resonance, the initial trial function must be modified.

The solution resulting from this process yields an accurate value for the field and losses. However, the undesirable feature of this method is its slow convergence to the steady-state value which requires considerable computer time.

After Fox and Li published their results in 1961, many other investigators have searched for a faster solution, sometimes using more complicated techniques. In many cases the results differed due to simplifying assumptions used or other computational errors in the different methods. In most cases, there seemed to be general agreement that the results of the iteration method was probably the most accurate.
COMPUTER PROGRAM FOR THE ITERATED SOLUTION

The calculation of successively iterated eigenfunctions required the integration of the trial eigenfunction with the appropriate kernel. This integral cannot be solved by direct analytical methods, so a numerical solution was used.

The lengthy nature of the iteration process demanded a simple numerical integration method that consumed a minimal amount of computer time. Two basic methods were tried. These were the rectangular summation of areas and the Simpson's method of integration. The optimum number of increments was 50, based on the computation times and accuracy of results.

The results of the integration methods tested are shown in Figure 3. The Simpson integration proved to be more accurate than the rectangular summation method for a given number of iteration. In the Simpson method, increasing the increments from 50 to 100 showed only a small change in the computed function and did not show any justification for using it, considering the increase in computation time. The final results are compared to the graphs of Fox and Li in Figure 7 and Figure 8. The functions generated by the trial methods did not coincide exactly with Fox and Li's result. The difference between the functions were probably attributed to the following reasons: a) The accuracies of the two computers used, b) Precision
Figure 3. Comparison of numerical integration methods first iteration.
of constants used in programming, c) Type and accuracy of integration scheme used.

There was a noticeable difference between the functions at the edge of the reflector. The 100 increment Simpson generated function had a smooth, monotonically decreasing curvature similar to the Fox and Li curve; but the 50 increment Simpson method exhibited an oscillatory decrease in the function. This reflected the effects of the abrupt discontinuity at the edge of the reflector, and the correctional process of iteration. Fortunately this oscillatory behavior was diminished as the number of iterations was increased. The first few iterations underwent considerable change from the initial function which was chosen to be a real constant. The error at the edges were less severe as the function approached the steady-state value.

Convergence rates of the eigenfunction values across the entire dimension of the reflector were not the same. Because of the high degree of discontinuity at the outer edges of the reflector, convergence of the eigenfunction was not complete until the outermost point of the reflector reached a steady-state value. At this time the amount of attenuation between successively generated eigenfunctions had reached a constant value over the entire face of the reflector. The amount of phase difference between successive eigenfunctions had also reached pointwise convergence. However, the phase values
were not uniform across the mirror. The discontinuity at the edge caused the eigenfunction to suffer a greater amount of phase shift in this region.

Values of attenuation and phase shift constituting the eigenvalue of the equation were computed by comparing successive eigenfunctions generated by iteration. The eigenvalues in this study were complex. The real part represented the attenuation and the imaginary part represented the phase change of the fields for one pass (back and forth) through the resonator.
Select initial trial eigenfunction
\[ \psi_1(x) = \text{const.} \ (1+j0) \]

Generate new eigenfunction
\[ \psi_2(x) = \int K(x_1, x_2) \psi_1(x) \]

Compare successive eigenfunctions to determine \( \gamma = \frac{\psi_2}{\psi_1} \) convergence, this process generates eigenvalues

Calculateatten of field at point x=0.4 after every iteration (see Table 2 for 290 to 300 iteration results)

Select eigenfunctions of iterations 1 through 10 and every tenth one thereafter for inspection

Normalize eigenfunction to the value at center to simplify plotting

Print out results (see Table 1 for 300th iteration)

Check convergence of every 10th iteration results

Figure 4. Block diagram of working program.
RESULTS OF ITERATION SOLUTION

The iteration process yielded the correct eigenfunction after approximately 300 iterations. The converging process of iteration is shown in Figure 5.

In order to continuously examine the nature of the convergence for the eigenfunction, the iterated values at a single point on the mirror was printed out after every iteration. The selection of only this single point for continuous monitoring was necessary to minimize the computer time. Data for observing the convergence of the entire eigenfunction was available only after sets of 10 iterations were completed.

The point chosen for continuous examination for convergence was half way from the center of the reflector to the edge. The results are shown in Figure 6, with sample data given in Table 2 of the Appendix. The reason for selecting this point was because of the "average" change in the eigenfunction in this region throughout the iteration process. The eigenfunction values at the center of the reflector, $d = 0$, exhibited the least amount of change and converged the fastest. The eigenfunction values at the edge of the reflector underwent drastic changes during the early iterations and required the longest time for convergence.

The results of every tenth iteration were printed out for
Figure 5. Converging process of iteration.
Figure 6. Change in magnitude of eigenfunction $\psi(x=0, t)$. 

Ratio of Successively Iterated Eigenfunction Magnitude

Number of Iterations

0 100 200 300

0.85 0.90 0.95 1.00 1.05 1.10
inspection. The results of the 300th iteration are given in Table 1 of the Appendix, and also graphically in Figures 7 and 8. The eigenfunction was expressed in normalized form in both complex and polar notation. The normalizing factor, the value at x = 0, is given at the bottom. When the eigenfunction had fully converged, there were no further changes in either its relative amplitude or phase from iteration to iteration. The attenuation or real part of the complex eigenvalue γ was determined by calculating the ratio of the eigenfunction magnitude for the n+1 and nth iteration. This ratio represented the change of attenuation of the field magnitude as it was reflected off the mirror. These values are labeled ATTEN on the normalized solutions for the different iterations. The power loss per transit was then obtained by using the expression 1-(ATTEN)².

The convergence of a selected point in the eigenfunction, (x=0.4), is shown graphically in Figure 6. The eigenvalues from these eigenfunctions showed the most drastic changes for the first 10 iterations. Once the general shape of the eigenfunction had reached some resemblance to the actual curve, the corrections became progressively smaller. From about 40 to 50 iterations the convergence of the eigenvalue or the attenuation seemed to follow a damped sine wave pattern.

The phase shift factor or imaginary part of the eigenvalue was determined by calculating the phase change between successively
Figure 7. Amplitude of lowest order eigenfunction.
Figure 8. Phase distribution of lowest order eigenfunction.
iterated eigenfunctions. The phase shift per transit for the lowest order mode computed by this method was $1.53^\circ$. This constant phase change across the entire face of the reflector enabled the phase front to be reconstructed exactly from reflection to reflection. The final phase distribution of the wave front for the lowest order mode is shown in Figure 8.

The main purpose in obtaining the eigenvalue was to compare the difference in attenuation between eigenvalues of different mode orders. This information helps to determine the threshold power requirements for sustained oscillation for the different modes under laser operation of the cavity and can be used to enhance or separate modes of oscillation.
VARIATIONAL METHOD OF CALCULATING THE EIGENVALUE

The variational principle can be used very advantageously in an iterative process. Iteration of a trial function results in a convergent solution. When convergence is complete, successively iterated eigenfunctions differ only by a complex constant multiplier. This complex constant factor represents the eigenvalue of the integral equation.

The variational principle applied to the homogeneous Fredholm equation of the second kind results in equation 4. A detailed study of this variational principle is given in the text by Morse and Feshbach (10).

\[
\gamma \psi(x_2) = \int K(x_1, x_2) \psi(x_1) \, dx_1
\]

\[
\gamma = \frac{\iint \psi(x_2) K(x_1, x_2) \psi(x_1) \, dx_2 \, dx_1}{\int \psi^2(x_2) \, dx_2}
\] (4)

This variational expression, known as the "Rayleigh quotient," represents the eigenvalue of the resonant mode. The advantage of using this variational ratio for \( \gamma \) is its error minimizing property. By using this equation, it is possible to use an approximate expression for the eigenfunction with a small error \( \epsilon \) and calculate the value of \( \gamma \) which differs from the correct value by a complex quantity of the order \( \epsilon^2 \) (9). The disadvantage of using this
method is that one must have an estimate of error in the approximate eigenfunction to determine the accuracy of the results. Often, such an estimate is not available for the trial function. Another disadvantage of using this method is the absence of an extremal property for the values of \( \gamma \) that are calculated when the kernel is non-Hermitian (4).
HYBRID METHOD

The calculus of variations has been applied to the integral equation for the Fabry-Perot cavity (2, 9, 18). The idea to use the variational principle in this study was obtained from the work of Samuel P. Morgan (9).

Morgan used the variational process to obtain a good first approximation for the eigenvalue. The trial eigenfunction employed, however, had to be chosen using a priori knowledge of the nature of the true eigenfunction. In Morgan's work, he chose as his trial approximations, trigonometric functions that seemed to fit the boundary conditions and field pattern variation of chosen transverse modes.

Because of the close resemblance of the trigonometric function and the actual eigenfunction, the results were good considering it was based on a "first approximation trial function". The disadvantage of using these trigonometric functions or other "guessed" trial function is that there is no measure of the accuracy of the results. Also in using the trigonometric function, the phase component was neglected since the sines and cosines are real functions. This means that the phase was considered to be constant or zero across the dimension of the reflector, when in fact it was shown to be varying by the iterative process. The omission of the phase term introduced some error in the calculated eigenvalues.
The results of both the iteration solution and Morgan's variational technique revealed the possibility for incorporating the desirable features of each method for solving the eigenvalue problem. In both cases, the desirable feature in one method was lacking or not as prominent in the other. The iterated solution was very accurate but required a considerable amount of calculation and computer time. The variational method, on the other hand, required very little computer time, but the accuracy of the results were difficult to determine.

To have some measure of accuracy of the eigenvalue, there must be first some information about the accuracy of the trial function used. Instead of using a "guess", based on some priori information, for the trial function, the eigenfunctions generated in the intermediate stages of the iteration process were used. The results of the iteration solution showed that the eigenfunction did converge at a regular rate after about the first 20 to 30 iterations. This number, however, probably depends on the choice of the function used as a starting point for iteration. The "guess" in the original variational approach was now replaced by an approximate function that compared to some degree of accuracy to the actual eigenfunction. When used in this relationship, the error in the calculated eigenvalue per iteration was minimized to a greater extent than by either method alone.
The Hybrid program using iteration along with the application of the variational principle is shown in the block diagram in Figure 9. This program was basically the same as the iteration program except that a subroutine was added at the end of the iteration step. This subroutine used the newly generated eigenfunction in the variational equation to calculate the eigenvalue. When the eigenvalues calculated after each iteration did not change, it had converged to the correct value.

![Block diagram of the Hybrid method](image)

Figure 9. Hybrid method (iteration and variation).

The application of this Hybrid method resulted in a considerable reduction in time to compute the correct eigenvalue. As shown in Figure 10, the eigenvalue represented by the power loss per transit was computed in about 1/3 the time required by the straight iteration process.

These results have shown that the eigenvalue computed by using an approximate eigenfunction in the variational expression had a
Figure 10. Convergence comparison of iteration, hybrid, and combination methods for the lowest order mode TEM$_{00}$.
smaller error than the eigenfunction itself. A rough measure of this error minimizing property was obtained by comparing the error in the eigenfunction and the error in the corresponding eigenvalue by variation. Figure 11 gives error values for the power loss calculated from the absolute error between corresponding points of the actual and approximate function. This graph was based on the 300th iteration results which were considered to be the true eigenfunction and eigenvalue.

Since this graph showed the power loss error and not of the eigenvalue itself, the $\epsilon^2$ error reduction was not clearly evident. However, Tables 3 and 4 in the Appendix do give the error comparison for the attenuation and phase shift component of the eigenvalue. There was no error for the attenuation and an error of 5.37% for the phase shift using the iterated eigenfunctions in the variational solution.

To test the effectiveness of the Hybrid method further, it was applied to the higher order modes of resonance, that had either or both indices equal to one. The one indicates that there is a single phase reversal across the dimension of the reflector. These modes are called odd-symmetric modes because of the $180^\circ$ phase difference between points about the axis of symmetry of the cross-sectional area.

The method of generating this odd-symmetric eigenfunction was given by Fox and Li. The original program was slightly
Figure 11. Comparison of error between power loss and eigenfunction.
modified by including a phase difference of $180^\circ$ in the trial eigenfunction for half of the reflector dimension. The results are given in Figures 12 and 13. It was noted that convergence for this higher order mode was faster than the lowest order mode and required only about one-half the number of iterations for full convergence of the eigenfunction. When the hybrid method was again applied to the iterated results, convergence of the eigenfunctions showed a considerable improvement compared to the results produced directly from iteration. The comparison of these two methods is shown in Figure 12. There was a reduction of about 75% in the computation time to obtain a fairly accurate value for the loss using the variational method.
Figure 12. Iterated eigenfunction for first odd-symmetric mode.
Figure 13. Convergence comparison of iteration and hybrid methods first odd-symmetric mode.
According to the text by Morse and Feshbach (10), the iterated variational method should produce a monotonically convergent eigenvalue. When applied to describing the Fabry-Perot etalon, the iterated variational solution did not show this type of convergence. After a rather erratic behavior for the first few iterations, the eigenvalue did tend towards convergence but not in a monotonic manner. The convergence was somewhat oscillatory.

There are two reasons for this difference. One is that the iterated function was complex, and secondly the kernel was not Hermitian. The effect of the non-Hermitian nature of the kernel has not been fully investigated mathematically as it applies to the variational method used here. The results of the problem involving a non-Hermitian kernel lacks an extremal property for the eigenvalue (4).

The cause of the oscillatory nature of the convergence was found to be the iteration of the phase component of the eigenfunction along with its magnitude. When the phase was excluded, iterating only the magnitude of the eigenfunction, the eigenfunction and also eigenvalue showed a monotonic and also faster convergence to a steady-state value. However, this steady-state eigenfunction was not the correct value because of the phase omission in the iteration. The variational method, when applied to the iterated results, also showed
a monotonic convergence of the eigenvalue. This eigenvalue, however, was about 11% less than the value computed correctly by including the phase, see Figure 15. Unlike the phase iterated variational results which fluctuated considerably with no regular pattern for the first 30 iterations, the phase omitted iterations or "real iterations" had a monotonic convergence from the initial iteration. This is the reason for a faster convergence steady-state value obtained by the real iterations.

After noting the faster convergence of the real iterations, a scheme was tested to utilize this property of the real iteration process. It was hoped that by using the faster monotonic convergence of the real iteration during the first 10 to 50 iterations and then switching to complex iterations, the correct final steady-state eigenfunction would be generated using a fewer number of iterations. This method was called the combination method. The program is shown in block diagram in Figure 14.

The computer program for the combination method was basically the same as the hybrid method program. A slight modification was made to remove the phase factor from the eigenfunction before it was used to generate a new one. This was done for the first N iterations only. Thereafter the phase factor was not removed.

Three programs were used to check for the effectiveness of this combination scheme, and also to try to determine the optimum
value of N, see Figure 14. The programs tested used N values of 10, 20 and 30, which were the number of real iterations used before switching to the complex iteration. The results are given in Figure 10. In each case, the error in the calculated power loss was smaller than that for both the iteration and hybrid methods for a given number of iterations. A definite conclusion about the optimum number for N could not be made from these short and incomplete iteration runs.

Figure 14. Combination method.
Figure 15. Convergence of power loss from variational method.
One of the advantages of using iteration for generating the solution of the integral equation was that both the eigenfunction and the eigenvalues of the equation became known. However, the steady-state solution was obtained only after many iterations were made. The preliminary results from the three test programs for the combination method showed indications that the generation of the correct eigenfunction can be achieved faster than the previous complex iteration solution requiring 300 iterations for the lowest order mode.

A comparison of the iterated eigenfunction for the combination method and the straight complex iteration method was made in Figure 16. It is evident from the graphs of the magnitude and phase of the eigenfunction that at 60 iterations there was considerably less error for the combination method as compared with the straight complex iteration method. There was about a 50% reduction in error for the phase and an even greater error reduction in the magnitude. The combination results were compared using Fox and Li's generated eigenfunction as the correct value. The iterations in the combination method were not continued until complete convergence was achieved due to a lack of sufficient computer time.

The purpose of using the real iteration in this case was to make faster gross corrections on the rough guess used for the initial trial function. This was done because it was observed that uniform rate of convergence for the complex iterated function occurred only after
Figure 16. Comparison of complex and combination iterations.
the function had been corrected to approximately resemble the true eigenfunction. The real iterations proved to be very effective for this purpose as shown by the preliminary results.
SUMMARY AND CONCLUSION

The homogeneous Fredholm equation of the second kind, which does not have a simple solution method, attracted the attention of many investigators studying laser characteristics. Compounding the difficulty in calculating the correct answer for the equation of the Fabry-Perot resonator was the fact that there were no mathematical works on which to base the validity of the different methods. This was the result of the non-Hermitian kernel in the integral equation. To this date there seems to be no comprehensive studies of the homogeneous Fredholm equation with this type of kernel.

The variational scheme, which was adopted and incorporated into the iterated solution have been shown to possess an external property for eigenfunctions associated with real or Hermitian kernels. It has been shown by this study that this extremal property is absent for a non-Hermitian kernel. This rules out the use of the Rayleigh-Ritz procedure for the solution of non-Hermitian kernel because there would be no guarantee that the eigenfunction was converging to a correct solution.

One of the investigators using the variational method (18) has based his results on the premise that an extremal principle did exist for the integral equation involved. In that study the trigonometric functions were used to represent the eigenfunctions. The results were
questioned by Fox and Li (4) who gave substantial evidence that there were no grounds for the assumption of the extremal property in this case.

Even without this extremal property, however, the variational method proved to be very useful when used with a self-correcting iterative scheme. The convergence rate of the eigenvalue was greatly increased as shown by the results.

Although this study concentrated on the solution concerning power losses within an optical cavity, the solution method employed may be applicable to the general homogeneous integral equation.
BIBLIOGRAPHY


Table 1. Results of 300th iteration.

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NORMALIZING FACTOR: -7.969405E-02, 5.204274E-01J
Table 2. Results of 290 to 300 iterations and variational method.

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ATTENUATION CALCULATED BY VARIATIONAL METHOD

| EIGENVALUE | 9.9667E-01  | 2.8051E-02 |
| AMPLITUDE  | 9.9706E-01  |            |
| PHASE      | 2.8138E-02  |            |
| ATTENUATION| 5.8691E-03  |            |
Table 1. Eigenvalue amplitude calculated by variational method.

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Comparison based on eigenvalue from iteration

Amplitude attenuation per transit --- .9971
Table 4. Eigenvalue phase shift calculated by variational method.

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Comparison based on eigenvalue from iteration

Phase shift per transit --- 1.53°
I. ITERATIVE METHOD: (FOX & LI)

\[ \gamma \psi(x_2, y_2) = \iint K(x_2, y_2; x_1, y_1) \psi(x_1, y_1) \, dx_1 \, dy_1 \]

WHERE FOR RECTANGULAR PLANE MIRROR:

\[ K(x_2, y_2; x_1, y_1) = (j/\lambda b) \exp(-jk((x_1-x_2)^2+(y_1-y_2)^2)/2b) \]

WITH, \( b \rightarrow \) REFLECTOR SEPARATION
\( k \rightarrow \) PROPAGATION CONSTANT (2\pi/\lambda)

\[ \text{LOSS} = 1 - \left( |\psi_{n+1}|^2 / |\psi_n|^2 \right) \]

II. VARIATIONAL METHOD: (S. P. MORGAN)

\[ \gamma = \frac{\iint \int \psi(x_2, y_2) K(x_2, y_2; x_1, y_1) \psi(x_1, y_1) \, dx_1 \, dy_1 \, dx_2 \, dy_2}{\iint \psi^2(x_2, y_2) \, dx_2 \, dy_2} \]

\[ \text{LOSS} = 1 - |\gamma|^2 \]

Equations used in iteration and variational methods.