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DESIGN CURVES FOR THE BUCKLING OF SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE

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UNITED STATES DEPARTMENT OF AGRICULTURE
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DESIGN CURVES FOR THE BUCKLING OF SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE[1]

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Abstract

This report contains curves and formulas for the calculation of the critical external pressure of sandwich cylinders of finite length. The facings of the sandwich are equal and isotropic and their individual stiffness is not taken into account. The core is isotropic or orthotropic having natural axes in the axial, tangential, and radial directions of the cylinder. Curves are given for isotropic cores and for orthotropic cores having certain relative elastic properties. If the cores are very rigid, the method yields results that are substantially those of von Mises.

Introduction

This report presents design curves for the critical external pressure on sandwich cylinders, calculated according to the formulas developed in Forest

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[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Products Laboratory report 1844-B (9).[3] The sandwich cylinders have isotropic facings and orthotropic or isotropic cores. The natural axes of the orthotropic cores are axial, tangential, and radial. These formulas reduce substantially to those developed by von Mises (7, 10, 14) when the core is very rigid so that the stiffness of the spaced facings of the sandwich (no reduction of the stiffness due to shear strains in the core) can be used as suggested in 3.1.5 of ANC-23 (11).

A great deal of investigative work has been done on isotropic cylindrical shells subjected to external pressure since report 1844-B giving theoretical analysis of sandwich cylinders was written. It has been found that experiment sometimes yields critical loads that are less than those predicted by von Mises' theory (13). This has been attributed to two causes. First, the experimental cylinders contained imperfections that lowered the critical load (1, 2, 3, 8, 12). Second, lower energy levels are associated with post-buckling configurations of the cylinder than with those just at buckling, the former being reached without the necessity of passing through (snap-through) the latter or the energy necessary for passing through the latter being supplied by vibration or shocks (4, 5, 6, 8).

The curves published in this report do not consider snap-through buckling or cylinders with imperfections. Sandwich cylinders, however, are much more perfect than their solid counterparts because they are thicker and the effect of an imperfection is in proportion to the ratio of its amplitude to the thickness of the cylindrical shell. Also, the curves neglect the stiffnesses of the individual facings. These stiffnesses add a considerable amount to the critical loads when the cylinders are short, and it is for short cylinders that "snap-through" is likely to occur (6). Thus it seems that these curves are useful in the design of sandwich cylinders.

Development of Formula from which Design Curves Were Calculated

The critical external pressure is found by placing the determinant on page 23 of report No. 1844-B (9) equal to zero and solving for α. This determinant can be simplified if the transverse modulus of elasticity of the core (E_c) is assumed to be infinite. For most core materials except possibly for low density foams E_c is sufficiently large so that this assumption yields values of the critical pressure that are only very slightly too great.

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[3] Underlined numbers in parentheses refer to the references.

Before E_c is allowed to approach infinity, the first and fourth columns of the determinant are multiplied by $\frac{G_{Rz}}{E_c}$ and the third column is divided by E_c. Then when E_c approaches infinity, the expressions in rows 3, 4, 5, and 6 in column 3 approach zero.

The expressions in each row, excepting the first, are replaced by new expressions, as indicated by the following formulas in which R represents the expression in the row designated by its subscript and in some column. The primed values are the new ones to be substituted for the old. These substitutions can be made without changing the value of α because of the well-known properties of determinants and because the determinant is equated to zero.

$$R_2' = R_2 \frac{a^2}{b^2} + R_1$$

$$R_3' = R_3 + R_2'$$

$$R_4' = R_4 + R_2'$$

$$R_5' = 2R_5 + (n^2 + 3\lambda^2) R_2'$$

$$R_6' = 2R_6 + (n^2 + 3 \frac{b^2}{a^2} \lambda^2) R_2'$$

These substitutions cause the expressions in column 3 and rows 2, 3, 4, 5, and 6 and those in column 6 and rows 3, 4, 5, and 6 to become zero, and the determinant is readily reduced (by minors) from a 6-by-6 to a 4-by-4 determinant. This determinant is simplified slightly by replacing the second row of expressions by the second row minus the first row.

After the determinant was written in this form, a change in parameters was made using the following nomenclature:

R -- mean radius of the sandwich cylinder

h -- thickness of the sandwich

c -- thickness of the core of the sandwich

n -- number of half waves in the circumference of the cylinder

$r = \frac{G_{Rz}}{G_{R\theta}}$

$$V = \frac{E}{(1 - \mu^2)\, G_{R\theta}} \frac{t}{h}$$

$$K = \alpha \frac{h}{a} = \frac{q(1 - \mu^2)}{E} \frac{h}{t}$$

where G_{Rz} and $G_{R\theta}$ are the moduli of rigidity of the core associated with the radial and axial directions and with the radial and tangential directions; E, μ, and t are the modulus of elasticity, Poisson's ratio, and thickness of the facings; and q is the external critical pressure on the cylinder.

The radii a and b were eliminated by the following equations obtained from the geometry of the cylinder:

$$a = R + \frac{1}{4}(h + c)$$

$$b = R - \frac{1}{4}(h + c)$$

and the following substitutions made:

$$\Phi = \frac{4R}{h}$$

$$\phi = \frac{h + c}{h}$$

The expressions in the final determinant are:

Row 1, column 1

$$\frac{n^2 - 1}{n^2} \frac{\Phi + \phi}{\Phi - \phi} - 1 + \frac{2V}{\Phi + \phi} \left[1 - \frac{\pi^2}{3n^2} \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 \right]$$

Row 2, column 1

$$\frac{1}{n^2} \frac{2\Phi}{\Phi - \phi} + \frac{8\Phi\phi}{\Phi^2 - \phi^2} V$$

Row 3, column 1

$$\frac{n^2 - 1}{n^2} \frac{2\phi}{\Phi - \phi} \left[n^2 + 3\pi^2 \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 \right] + 4\pi^2 V \frac{\Phi + \phi}{\Phi^2} \left(\frac{R}{\ell}\right)^2 \left[\frac{1}{3} - \frac{\pi^2}{n^2} \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 - \frac{\Phi^2 - \phi^2}{8\Phi} K \right]$$

Row 4, column 1

$$\frac{n^2 - 1}{n^2} \frac{2\phi}{\Phi - \phi}\left[n^2 + 3\pi^2 \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi - \phi}{\Phi}\right)^2\right] + 4\pi^2 V \frac{\Phi + \phi}{\Phi^2}\left(\frac{R}{\ell}\right)^2 \left[\frac{1}{3} - \frac{\pi^2}{n^2}\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi - \phi}{\Phi}\right)^2 - \frac{(\Phi + \phi)^2}{8\Phi} K\right]$$

Row 1, column 2

$$(n^2 - 1) \frac{(\Phi + \phi)^2}{4\Phi} K \frac{\Phi^2 + \phi^2}{\Phi^2 - \phi^2} - n^2 + 1 + \frac{\pi^2}{3}\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2$$

Row 2, column 2

$$- \frac{2\phi}{\Phi + \phi}\left[(n^2 - 1)\frac{\Phi + \phi}{\Phi - \phi} + \frac{\pi^2}{3}\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2\right]$$

Row 3, column 2

$$2\pi^2\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2\left[- \frac{n^2 - 1}{3} + \pi^2\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 + (n^2 - 1)\frac{\Phi^2 - \phi^2}{8\Phi} K\right] + (n^2 - 1)\frac{(\Phi + \phi)^2}{4\Phi}\frac{\Phi^2 + \phi^2}{\Phi^2 - \phi^2} K \left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2\right]$$

Row 4, column 2

$$2\pi^2\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2\left[- \frac{n^2 - 1}{3}\frac{\Phi - \phi}{\Phi + \phi} + \pi^2\left(\frac{R}{\ell}\right)^2 \frac{(\Phi - \phi)^3}{\Phi(\Phi + \phi)} + (n^2 - 1)\frac{\Phi^2 - \phi^2}{8\Phi} K\right] + (n^2 - 1)\frac{(\Phi + \phi)^2}{4\Phi}\frac{\Phi^2 + \phi^2}{\Phi^2 - \phi^2} K \left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi - \phi}{\Phi}\right)^2\right]$$

Row 1, column 3

$$\frac{2\phi}{\Phi + \phi} + \frac{4}{3r}\frac{V}{\Phi + \phi}\log\frac{\Phi - \phi}{\Phi + \phi}$$

Row 2, column 3

$$- \frac{8}{3r}\frac{V}{\Phi + \phi}\log\frac{\Phi - \phi}{\Phi + \phi}$$

Row 3, column 3

$$- 2 + \frac{2\phi}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2\right] + \frac{4}{3r}\,\frac{V}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2\right]\log\frac{\Phi - \phi}{\Phi + \phi}$$

Row 4, column 3

$$2\,\frac{\Phi - \phi}{\Phi + \phi} + \frac{2\phi}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2\right] - \frac{4}{3r}\,\frac{V}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2\right]\log\frac{\Phi - \phi}{\Phi + \phi}$$

Row 1, column 4

$$n^2 - \frac{\pi^2}{3}\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2$$

Row 2, column 4

$$- \frac{4\pi^2}{3}\left(\frac{R}{\ell}\right)^2\frac{\phi}{\Phi}$$

Row 3, column 4

$$4\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2 n^2\left[\frac{1}{3} - \frac{\Phi^2 - \phi^2}{16\Phi}K\right] - \frac{2\pi^2}{3}\left(\frac{R}{\ell}\right)^2\frac{\Phi^2 + \phi^2}{\Phi^2}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2\right]$$

Row 4, column 4

$$4\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2 n^2\left[\frac{1}{3} - \frac{(\Phi + \phi)^2}{16\Phi}K\right] - \frac{2\pi^2}{3}\left(\frac{R}{\ell}\right)^2\frac{\Phi^2 + \phi^2}{\Phi^2}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2\right]$$

This determinant was placed equal to zero and values of K determined by an electronic computer for various values of $\frac{\ell}{R}$, n, V, $\frac{R}{h}$, and $\frac{c}{h}$.

The limits of these curves for very long and very short cylinders may be found from Forest Products Laboratory Reports 1844-A and 1844-B (9). For infinitely long cylinders having membrane facings and for which the modulus of elasticity of the core in the radial direction is infinite, equation (72) of Report 1844-A becomes:

$$K = \frac{3\,(\Phi - \phi)\,\phi}{2\,(\Phi^2 + \phi^2)\left[\dfrac{\Phi^2 - \phi^2}{16\phi} + \dfrac{Et}{G_{R\theta}\,(1 - \mu^2)\,h}\right]}$$

For very short cylinders having membrane facings and for which the modulus of elasticity of the core in the radial direction is infinite, the equation on page 28 of report 1844-B becomes:

$$K = \frac{2\Phi}{\Phi + \phi}\;\frac{G_{R\theta}\,(1 - \mu^2)\,h}{Et}$$

From this equation the critical hoop compression per unit length of cylinder is found to be:

$$N_\theta = \frac{1}{2}\,(h + c)\,G_{R\theta}$$

which is the usual limit imposed on the edge compression of sandwich constructions with membrane facings by the shear instability of the core.

Description of Design Curves

The design curves (figs. 1 to 6) apply to sandwich cylinders having equal isotropic facings and isotropic cores or orthotropic cores having thin natural axes parallel to the axial, tangential, and radial directions of the cylinders. They are plots of K against $\frac{\ell}{R}$ where the critical pressure is given by

$$q = \frac{Et}{(1 - \mu^2)\,h}\,K$$

and E, μ, and t are the modulus of elasticity, Poisson's ratio, and thickness of the facings; h the thickness of the sandwich; ℓ and R the length and mean radius of the cylinder. Six curve sheets are presented, two for each of three values (10, 50, and 100) of $\frac{R}{h}$. One of each pair of curve sheets applies to sandwich having very thin facings for which the ratio of the thickness of the core to the thickness of the sandwich (c/h) is substantially unity. The other applies to sandwich for which this ratio is 0.7. Each curve applies to sandwich having a particular value of:

$$V = \frac{E}{(1 - \mu^2)\, G_{R\theta}} \frac{t}{h}$$

where $G_{R\theta}$ is the modulus of rigidity of the core associated with the radial and tangential directions of the cylinder. It was found that the modulus of rigidity of the core associated with the radial and axial directions (G_{Rz}) has very little influence on the critical pressure. It does not enter the formulas for the critical pressure of very long or very short cylinders. Calculations were made for cylinders having three values of

$$r = \frac{G_{Rz}}{R_{R\theta}}$$

These values were 2.5, 1.0, and 0.4 to agree with the values appropriate for some honeycomb and isotropic core materials. Many of the curves are not affected by the use of these values. Some of them are affected slightly. This is shown in the figures by the use of three adjacent curves; the greatest, intermediate, and least values of K are associated with the greatest, intermediate, and least values of r.

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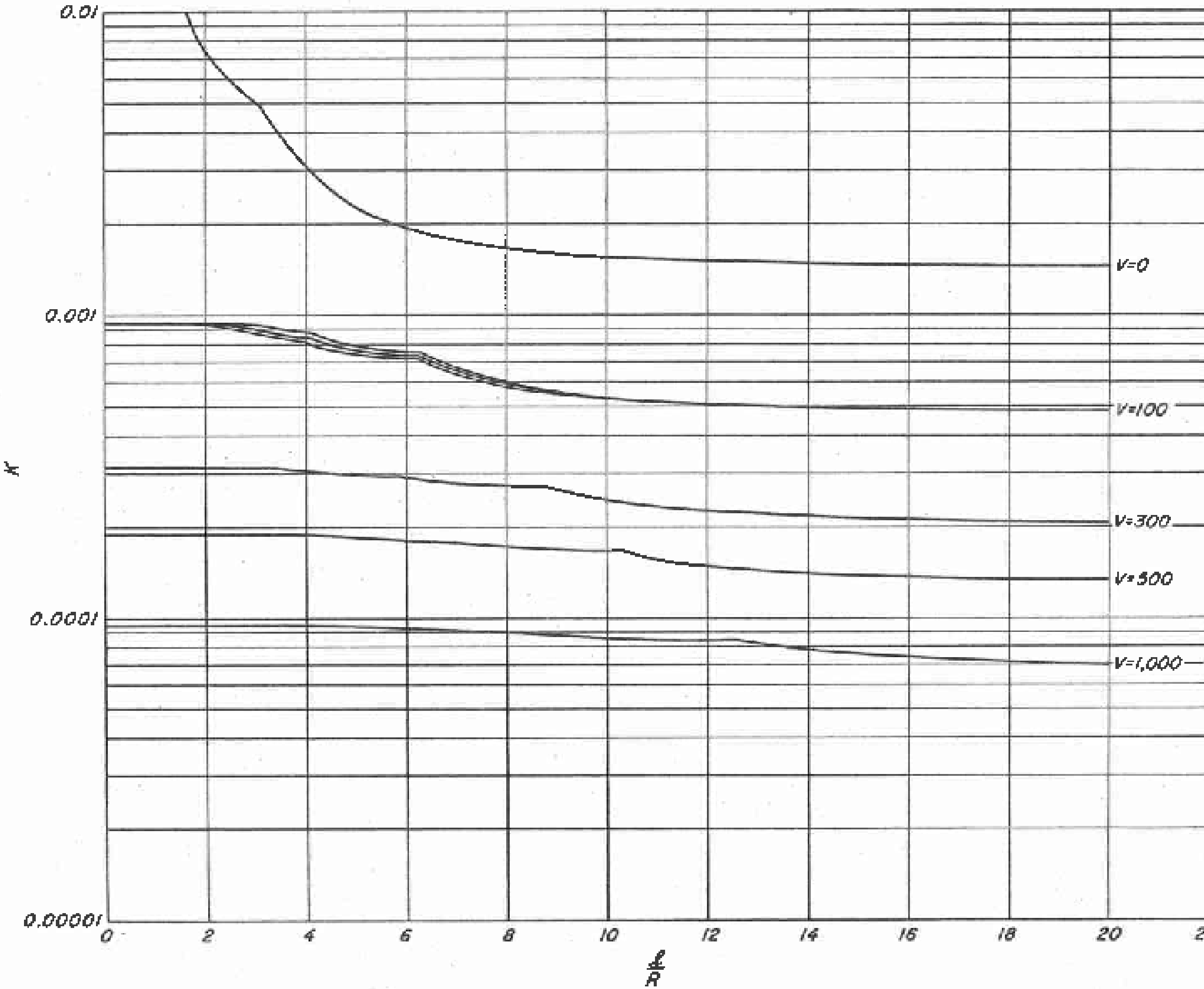

Figure 1.--Values of K for $\frac{R}{h} = 10$ and $\frac{c}{h} = 1$ for various values of V and $\frac{\ell}{R}$.

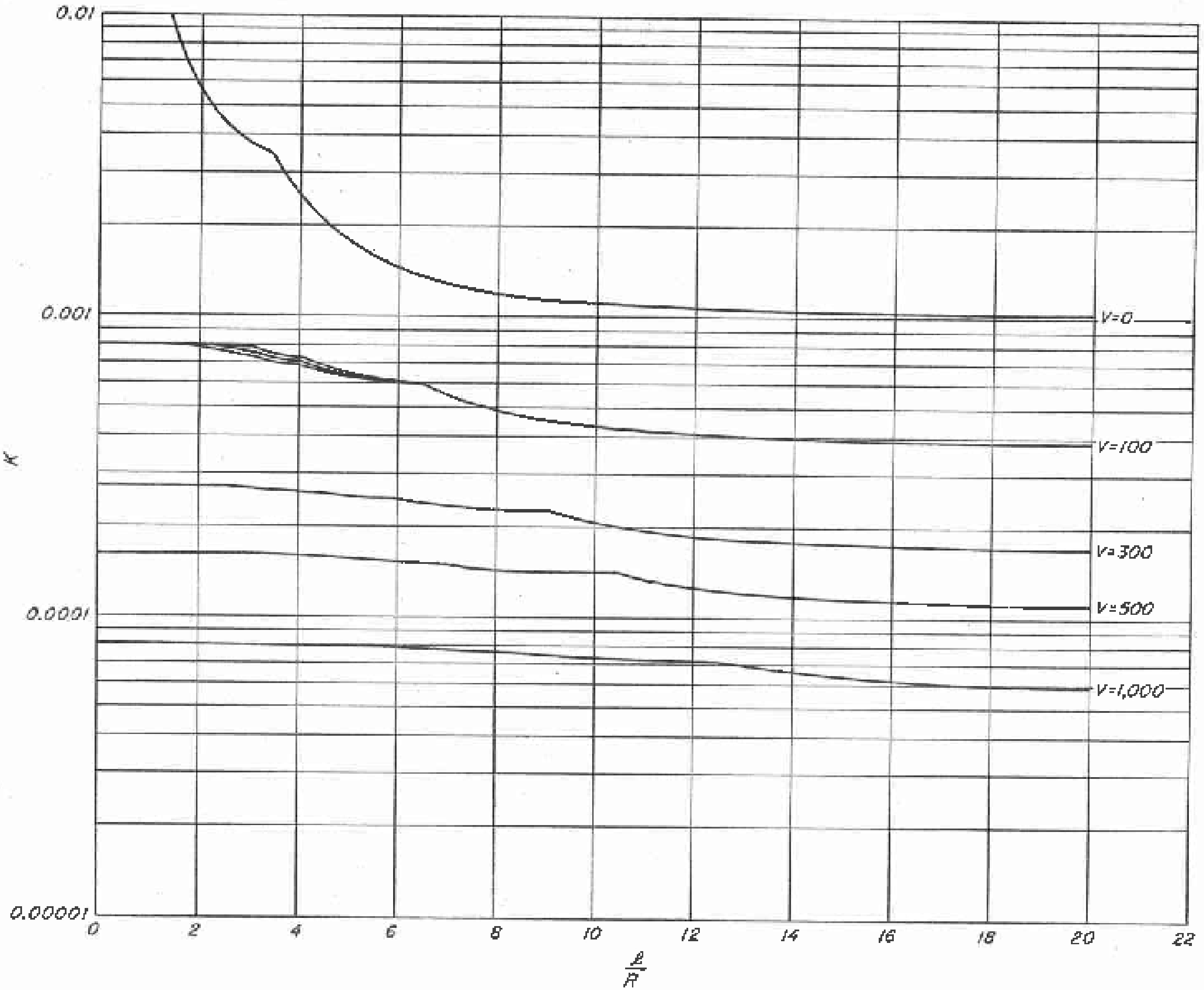

Figure 2. --Values of K for $\frac{R}{h} = 10$ and $\frac{c}{h} = 0.7$ for various values of V and $\frac{\ell}{R}$.

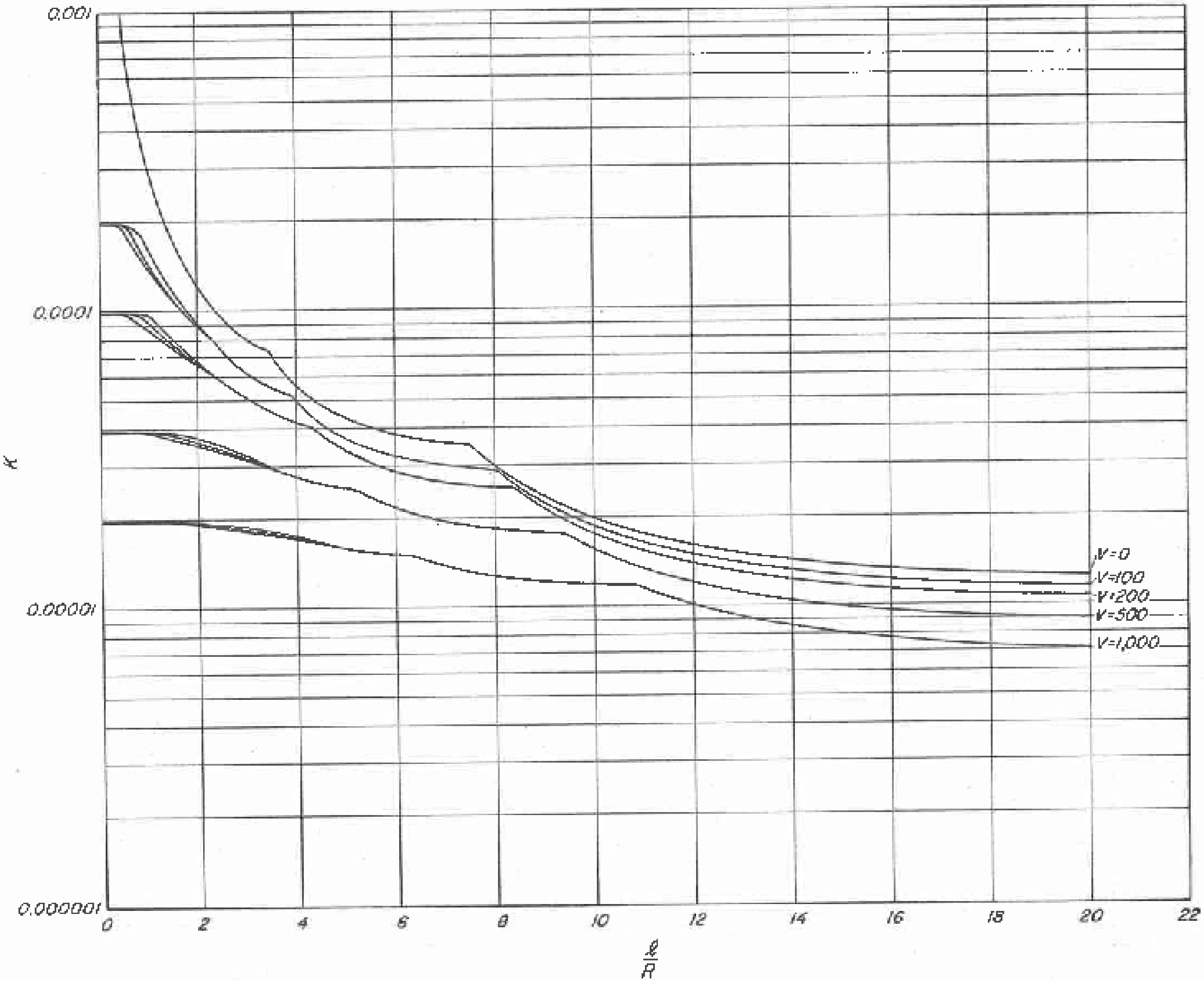

Figure 3. --Values of K for $\frac{R}{b}$ = 50 and $\frac{c}{h}$ = 1 for various values of V and $\frac{l}{R}$.

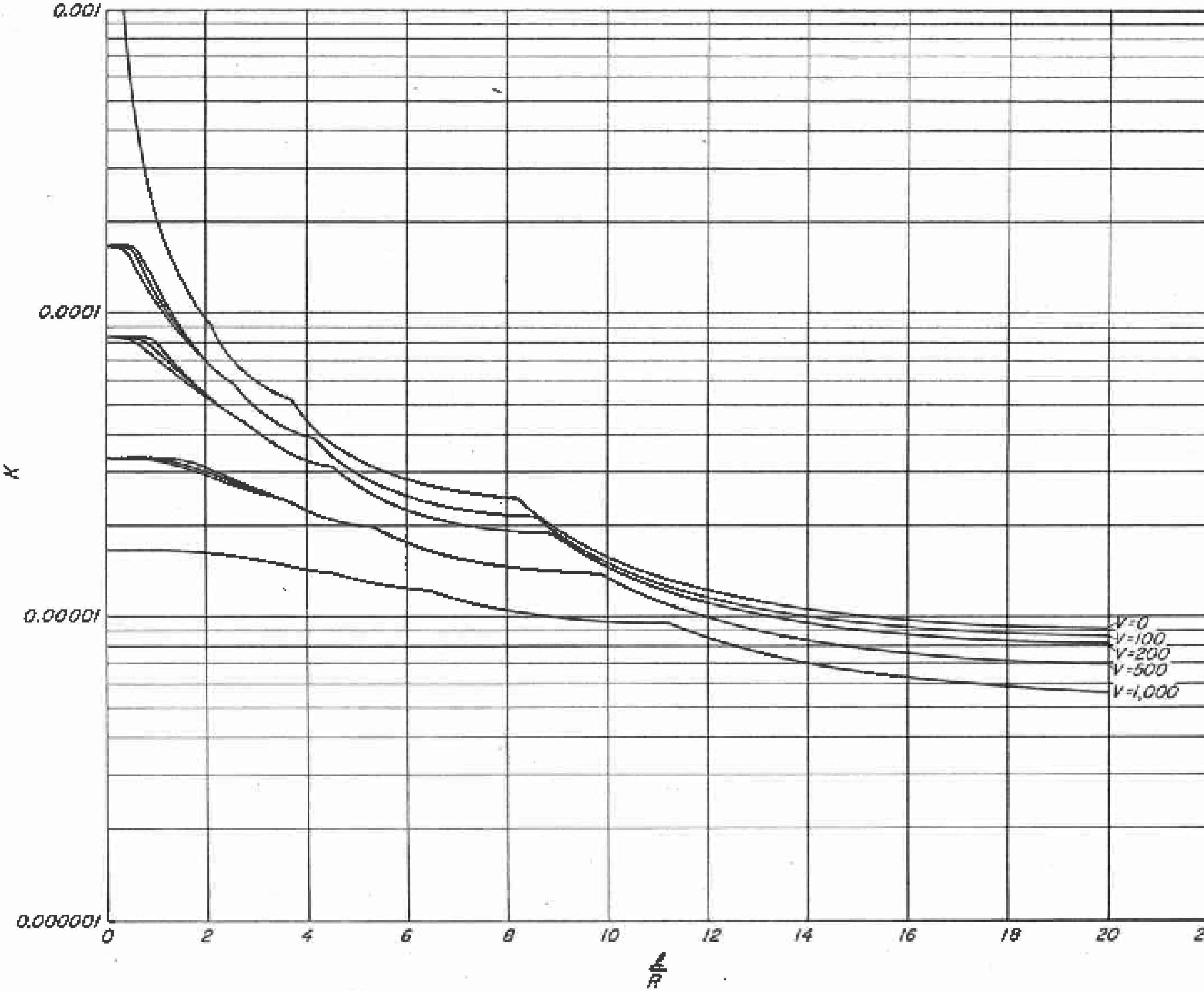

Figure 4. --Values of K for $\frac{R}{h} = 50$ and $\frac{c}{h} = 0.7$ for various values of V and $\frac{\ell}{R}$.

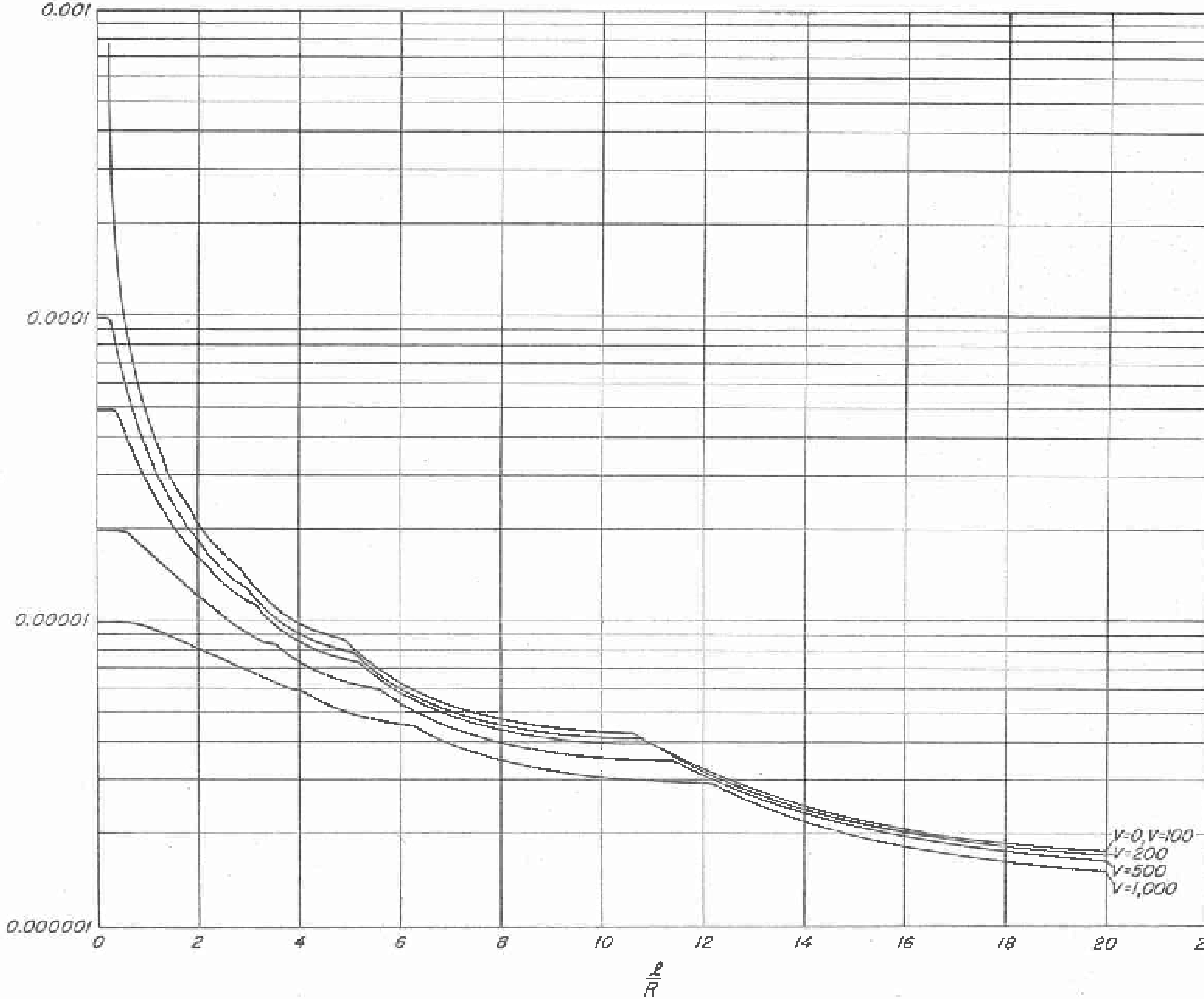

Figure 5. --Values of K for $\frac{R}{h} = 100$ and $\frac{c}{h} = 1$ for various values of V and $\frac{\ell}{R}$.

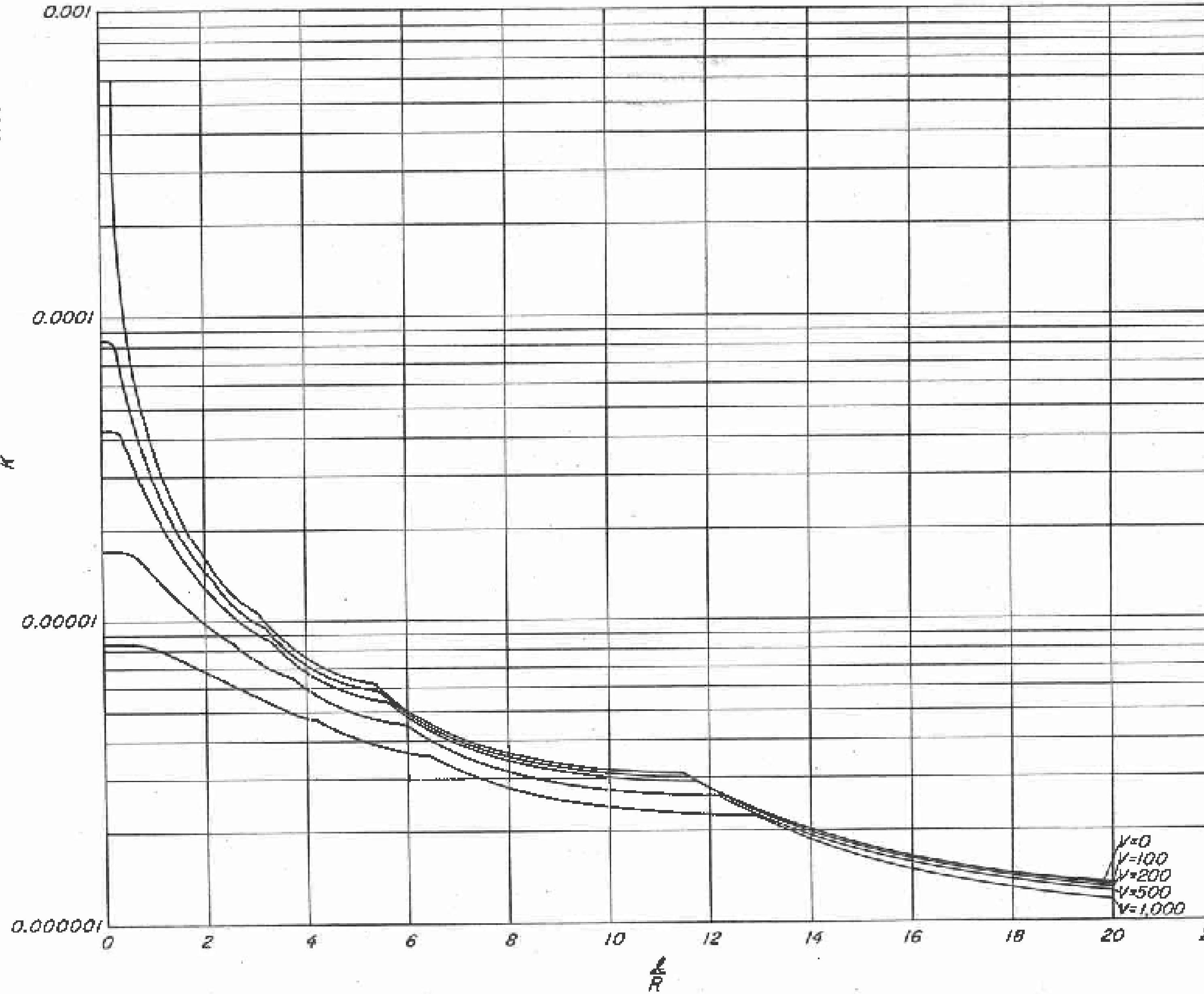

Figure 6. --Values of K for $\frac{R}{h} = 100$ and $\frac{c}{h} = 0.7$ for various values of V and $\frac{\ell}{R}$.