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Abstract approved

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The paper concerns itself with the problem of heat transport in a homogeneous, incompressible, isotropic, semi-infinite solid whose boundary is nonstationary. The methods of solution in the one dimensional case are discussed. Numerical examples and a method for arriving at the solutions published by Bailey and Lakin in their paper concerning underground combustion with convection effects are given. The convergence of the approximate solutions obtained by the method of successive approximations as well as the maximum and minimum principles and the uniqueness theorem is discussed.
ON THE THEORY OF HEAT TRANSPORT IN A SEMI-INFINITE SOLID

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# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>I.  Formulation of the Problem</td>
<td>4</td>
</tr>
<tr>
<td>II. Methods of Solution</td>
<td>8</td>
</tr>
<tr>
<td>Constant v</td>
<td>8</td>
</tr>
<tr>
<td>Formulation of an Integral Equation</td>
<td>10</td>
</tr>
<tr>
<td>Perturbation Method</td>
<td>12</td>
</tr>
<tr>
<td>Heat Transport Equations in Cylindrical and Spherical Coordinates</td>
<td>14</td>
</tr>
<tr>
<td>III. Some Numerical Examples</td>
<td>17</td>
</tr>
<tr>
<td>IV. Convergence of the Approximate Solutions</td>
<td>29</td>
</tr>
<tr>
<td>Bibliography</td>
<td>38</td>
</tr>
<tr>
<td>Appendices</td>
<td>39</td>
</tr>
</tbody>
</table>
ON THE THEORY OF HEAT TRANSPORT IN A SEMI-INFINITE SOLID

INTRODUCTION

The problem of heat conduction was first studied by Fourier. This field has received a great amount of attention and has been of central interest in mathematical physics (see [5, p. 449]).

Heat transport in a porous solid by conduction coupled with convection effects due to the movement of fluids arises in many instances in nature. For instance, percolation of rain water into the ground is an example provided by nature. Another example is the heat transport by migrating underground water, gas, or oil which may be upward, lateral depending on the geophysical conditions existing. Also, the movement of magma provides another natural phenomenon of heat transport by a fluid. In all these cases heat energy is transferred through the solid by the mass flow of fluid.

Under modest restrictions a mathematical equation derived by Bödvarsson [3] governing the heat transport phenomenon in a solid moving at a given velocity takes the form

\[
\frac{\partial T}{\partial t} + (\mathbf{u} + \frac{s}{\rho_c} \mathbf{q}) \cdot \nabla T = \text{div} (a(T) \nabla T) + A,
\]

where \( T \) denotes temperature; \( a(T) \) denotes thermal diffusivity; \( t \) denotes time; \( A \) denotes the temperature production in the wet
solid; $s$ and $c$ are specific heats of the fluid and the wet solid, respectively; $\rho$ is the density of the wet solid; $\mathbf{u}$ denotes the velocity vector of the solid; and, $\mathbf{q}$ denotes the mass flow vector of the fluid.

The equation is obtained under the assumption that the specific heats are constant, the thermal contact between the solid and the fluid is perfect, and both phases are incompressible. Therefore, the equation in the above form is not strictly applicable to the problem involving mass flow of gases, since gases are compressible media.

The problem of heat transport by the mass flow of gas arises in connection with the thermal methods of oil production. A partial differential equation governing the process of underground combustion has been developed by Bailey and Lakin [1]. It is interesting to find that the equation developed by them has a similar form as the one-dimensional heat transport equation previously cited.

So far we have assumed that the boundaries of the solids involved are stationary. In nature the heat transport problem may be further complicated by the presence of a time dependent boundary position due to erosion or sedimentation. For instance, the surface of the earth or of a glacier is subjected to accumulation and ablation. Examples illustrating this type of problem may be cited from the literature. Dewey [6] has studied the problem of heat
distribution in a slab due to structural damage on the surfaces of a missile by ablation. Other examples could be cited since a considerable amount of research has been done on the topic; however, most if not all, of which was concerned with cases in which the positions of the boundaries were unknown. For this reason, this paper will attempt to solve the case in which the rate of the movement of the boundary is known.
FORMULATION OF THE PROBLEM

The present investigation will concern itself with the one-dimensional heat distribution in a semi-infinite solid whose boundary position is time dependent. For definiteness and simplicity, the following conditions are assumed:

1. The body $D$ involved is homogeneous, incompressible and isotropic.

2. The body is porous; the porosity is assumed to be small and the infinitesimally small pores are uniformly distributed throughout the body.

3. An incompressible fluid is moving through $D$ at the rate $q$, where $q$ represents the mass flow of fluid per unit area of the solid and is measured relative to the solid.

4. The solid itself is moving relative to a fixed frame of reference at the velocity $v$.

5. Heat may be generated in the solid at a rate $S$ per unit mass and time.

6. The thermal contact between both phases is perfect;
that is, the temperature of the solid is equal to the

temperature of the fluid at any point.

7. The range of variation in temperature is relatively small

in the sense that the thermal properties of the media

involved are independent of the temperature.

8. The initial and the boundary conditions are prescribed.

Then under these eight assumptions the one-dimensional heat

transport equation takes the form

\[
\frac{\partial T}{\partial t} + (v + \frac{sq}{\rho c}) \frac{\partial T}{\partial x} = a \frac{\partial^2 T}{\partial x^2} + A, \quad 0 < x, \quad 0 < t,
\]

\[T(x, 0) = g(x),\]

\[T(0, t) = f(t),\]

where \( A = \frac{S}{c}, \) temperature production.

The initial-boundary value problem (1) holds for a fixed

boundary at \( x = 0.\)

We will now assume that the boundary moves at the velocity

\( u(t) \) and that the temperature there is \( f(t). \) Then the new bound-

ary condition is

\[
T \left( \int_0^t u(t') dt', t \right) = f(t), \quad T(x, 0) = g(x).
\]
The system may be simplified by the following transformation

\[ z(t) = x - \int_0^t u(t') dt', \]

under which the transformed equation becomes

\[ \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2} + (u - v - \frac{sA}{pc}) \frac{\partial T}{\partial z} + A, \quad 0 < z, \quad 0 < t, \]

\[ T(z, 0) = g(z), \]

\[ T(0, t) = f(t), \]

where \( A \) is defined only for \( z > 0 \); this follows from the transformation we used and assumption (5) above.

The initial-boundary value problem (1') shows that in the one-dimensional case mass transport of heat and nonstationary boundary transport lead to the same type of equation. Consequently, these two phenomena may therefore be studied by the same methods. For this reason, without loss of generality, the investigation will concentrate on the problem
\[
(2) \quad \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2} + v \frac{\partial T}{\partial z} + A, \quad 0 < z, \quad 0 < t,
\]

\[
T(z, 0) = g(z),
\]

\[
T(0, t) = f(t),
\]

throughout the remainder of the paper; that is, with a suitable interpretation on the function \( v \) in equation (2), the system describes different physical phenomena.
METHODS OF SOLUTION

It is well known that the differential equations arising in research are, in general, difficult to solve. Sometimes an exact solution for a given equation may not be feasible analytically. Then, one has to rely on some approximation methods to obtain an approximate solution for the equation involved. Consequently, the finding of an approximate solution may very well become a major task encountered by the researcher.

In this section our prime interest is to solve equation (2) by various methods. Approximation methods such as the perturbation technique as well as the method of successive approximations are employed. However, the closed-form solution for equation (2) is feasible and can readily be obtained if \( v \) is constant in time. (Here we have used the phrase "the solution"; in Part IV we shall show that equation (2) has one and only one continuous solution in the domain of definition.)

**Constant \( v \)**

For the case in which the function \( v \) is a constant, say \( k \), equation (2) can be solved by means of a suitable transformation which reduces the equation to a "well-known" form. The transformation we are concerned with is of the form (Appendix 1)
under which equation (2) with constant rate $k$ becomes

\[
\frac{\partial U}{\partial t} = a \frac{\partial^2 U}{\partial z^2} + A e^{\frac{k}{2a} \frac{(z + kt)}{2}}, \quad 0 < z, \quad 0 < t,
\]

\[
U(z, 0) = g(z) e^{\frac{kz}{2a}},
\]

\[
U(0, t) = f(t) e^{\frac{kt}{4a}}.
\]

Now equation (4) has the same form as that of equation (1) of Appendix 2; consequently, the solution of (4), by formula (2) of Appendix 2, is given by

\[
U(z, t) = \int_0^\infty g(z') e^{\frac{kz'}{2a}} G(z, t; z', 0) dz'
\]

\[
+ \frac{z}{2\sqrt{\pi a}} \int_0^t f(t') e^{\frac{1}{4a} \left( \frac{k^2 t'^2}{(t-t'^2)} - \frac{z^2}{(t-t'^2)} \right)} \frac{\left( \frac{k}{2a} \frac{(z' + kt')}{2} \right)}{(t-t'^2)^{3/2}} dt'
\]

\[
+ \int_0^t \int_0^\infty A e^{\frac{k}{2a} \frac{(z' + kt')}{2}} G(z, t; z', t') dz' dt',
\]

where

\[
G(z, t; z', t') = \frac{1}{2\sqrt{\pi a(t-t')}} \left\{ e^{\frac{-(z-z')^2}{4a(t-t')}} - e^{\frac{-(z+z')^2}{4a(t-t')}} \right\},
\]
and it is called the Green's function for the semi-infinite solid.

It follows that equations (3) and (5) give us the desired solution.

**Formulation of an Integral Equation**

To begin with it is sufficient to find a transformation under which the coefficient of the flux in equation (2) vanishes. It can be shown [Appendix 4] that the transformation we seek is of the form

\[ T(z, t) = U(z, t)e^{\frac{v(t)z}{2a}}. \]

Equation (2) now becomes

\[
\frac{\partial U}{\partial t} = a \frac{\partial^2 U}{\partial z^2} + HU + Ae^{\frac{vz}{2a}}, \quad 0 < z, \quad 0 < t,
\]

\[ U(z, 0) = g(z)e^{\frac{v(0)z}{2a}}, \]

\[ U(0, t) = f(t), \]

where

\[
H(z, t) = \left\{ \frac{v'(t)z}{2a} - \left[ \frac{v(t)}{4a} \right]^2 \right\},
\]
and \( v \) is a differentiable function.

By formula (2) of Appendix 2, the integral equation corresponding to equation (6) is, see Appendix 3 also,

\[
U(z, t) = \int_{0}^{\infty} g(z') e^{\frac{v(0)z'}{2a}} G(z, t; z', 0) dz' + \frac{z}{2\sqrt{\pi a}} \int_{0}^{t} f(t') e^{-\frac{z^2}{4a(t-t')}} \frac{dt'}{(t-t')^{3/2}}
\]

\[
+ \int_{0}^{t} \int_{0}^{\infty} Ae^{\frac{v(t')z'}{2a}} G(z, t; z', t') dz' dt' + \int_{0}^{t} \int_{0}^{\infty} H(z', t') G(z, t; z', t') U(z', t') dz' dt'.
\]

Equation (8) can be solved by the method of successive approximations. The sequence of approximate functions converges uniformly in the domain of definition (see part IV). We start with an initial guess value \( U_0 \). Then the \( n \)th approximation of the solution of (8) is
\( U_n(z, t) = \int_0^\infty g(z')e^{\frac{v(0)z'}{2a}} G(z, t; z', 0)dz' \
\)

\[ + \frac{z}{2\sqrt{\pi a}} \int_0^t f(t')e^{-\frac{z^2}{4a(t-t')}} \frac{1}{(t-t')^{3/2}} \, dt' \]

\[ + \int_0^t \int_0^\infty A \cdot e^{\frac{v(t')z'}{2a}} G(z, t; z', t')dz' \, dt' \]

\[ + \int_0^t \int_0^\infty H(z', t')G(z, t; z', t')U_{n-1}(z', t')dz' \, dt' \]

Hence, the \( n \)th approximate solution of (2) is given by

\[ T_n(z, t) = U_n(z, t)e^{\frac{-v(t)z}{2a}}, \]

where \( U_n(z, t) \) is defined by (9).

Perturbation Method

Suppose that the rate of movement of the boundary is small.
Then the rate function in equation (2) may be written as \( \varepsilon v \) where \( \varepsilon \) is small so that \( \varepsilon v(t) \) remains small for all time under consideration. Then the perturbation technique may be used to obtain an approximate solution of (2),

\[
T(z, t) = T_0(z, t) + \varepsilon T_1(z, t),
\]

where \( T_0 \) is the solution of

\[
\frac{\partial T_0}{\partial t} + a \frac{\partial^2 T_0}{\partial z^2} = A, \quad 0 < z, \quad 0 < t,
\]

\( T_0(z, 0) = g(z), \)

\( T_0(0, t) = f(t), \)

and

\[
\frac{\partial T_1}{\partial t} + a \frac{\partial^2 T_1}{\partial z^2} + v \frac{\partial T_0}{\partial z} = 0, \quad 0 < z, \quad 0 < t,
\]

\( T_1(z, 0) = 0, \)

\( T_1(0, t) = 0. \)
So $T_0$ and $T_1$ can be obtained by applying formula (5) to the unperturbed system (10) and the first perturbed system (11), respectively.

**Heat Transport Equations in Cylindrical and Spherical Coordinates**

In the cylindrical and spherical cases the total heat transports by an incompressible fluid through the porous medium are given by

$$H = -2\pi rk \frac{\partial T}{\partial r} + sqT,$$

$$H = -4\pi r^2 k \frac{\partial T}{\partial r} + sqT,$$

respectively, where the first terms on the right-hand side of both equations are heat transport due to conduction and the second terms due to mass flow of fluid. Here $r$ denotes the radial variable; $q$ is the mass flow of fluid per unit time per unit length of the cylinder, whereas in the spherical case it denotes mass per unit time; and the others are defined as above.
On the basis of the law of conservation of energy, the heat transport equations become

\[ \frac{\partial T}{\partial t} = a \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) - \frac{sq}{2\pi\rho c} \frac{\partial T}{\partial r} \]

(12) Cylindrical:

\[ \frac{\partial T}{\partial t} = a \left( \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) - \frac{sq}{4\pi\rho c r^2} \frac{\partial T}{\partial r} . \]

(13) Spherical:

Here we have used the fact that \( \frac{\partial q(t)}{\partial r} = 0 \).

We conclude this section with a discussion concerning the possibility of finding a transformation under which the heat transport equations in the cylindrical and spherical coordinates may be transformed into some well-known forms. It is interesting to find that there exists no transformation which reduces the equations to either the form of equation (4) or the form in which the Hankel transformation is applicable even under the assumption that \( q \) is a constant function different from zero (Of course, if \( q(t) \equiv 0 \), the equations become classical, and their solutions are well established in the literature; see, for example, Carslaw and Jaeger [5, p. 230, 260]). The nonexistence of such a transformation is due to the fact that
the function $g$ in Appendix 1 is not only a function of time but also a function of spatial variable.
SOME NUMERICAL EXAMPLES

In this section a few solutions to problems of practical interest are obtained based on the formulae developed in the last section. However, the author wishes to point out that on carrying out some of the integrations the calculations become, sometimes, quite involved. For this reason a zero source of heat generated in the medium considered is assumed throughout the chapter unless the contrary is explicitly stated. Also, verifications to some of the results published by Bailey and Lakin [1] are given by employing the Laplace transformation as well as the formulae established in this paper.

1. Suppose that a semi-infinite solid erodes at a constant rate \( k \) when \( t > 0 \), and that the initial temperature distribution is a linear function given by \( T(z, 0) = gz + T_i \) for \( z > 0 \). Also, the boundary of the solid is kept at constant temperature \( V \) when \( t > 0 \).

By formulae (3) and (5) and the integral formulae (1) and (4) of Appendix 5, the temperature distribution in the solid after erosion has started is

\[
T(z, t) = U(z, t)e^{-\frac{k}{2a}(z + \frac{kt}{2})}, \quad 0 < z, \quad 0 < t,
\]
where

\[ U(z,t) = \frac{1}{2} \left\{ \left[ g(z+kt)+T_1 \right] e^{\frac{-2zk+k^2t}{4a}} \left[ 1 + \text{erf}\left(\frac{z+kt}{2\sqrt{at}}\right)\right] \right. \]

\[ + \left[ g(z-kt)-T_1 \right] e^{\frac{-2zk+k^2t}{4a}} \left[ 1 - \text{erf}\left(\frac{z-kt}{2\sqrt{at}}\right)\right] \]

\[ + Ve^{\frac{k^2t}{4a}} \left[ 2 \cosh \frac{kz}{2a} + e^{\frac{-kz}{2a}} \text{erf}\left(\frac{k^2t}{2} - \frac{z}{2\sqrt{at}}\right)\right] \]

\[ - e^{\frac{kz}{2a}} \text{erf}\left(\frac{k^2t}{a} + \frac{z}{2\sqrt{at}}\right) \} . \]

We can see easily that the function \( T(z,t) \) satisfies the initial and the boundary conditions of the problem; however, it is not immediately apparent that \( T(z,t) \) satisfies the equation corresponding to our example.

II. Suppose that the surface of a glacier erodes at a constant rate \( B \) when \( 0 < t < t_1 \), and that the erosion stops at \( t = t_1 \) and sedimentation (ice forming) occurs when \( t > t_1 \) at a constant rate \( C \); also, assume that the initial temperature \( T(z,0) = gz \), and the temperature vanishes at the boundary.

Then the temperature is given by
\[ T(z,t) = \frac{1}{2} \left\{ g(z+Bt)[ 1 + \text{erf} \left( \frac{z+Bt}{2\sqrt{at}} \right) ] \right\} \\
+ e^{-\frac{Bz}{2a}} g(z-Bt)[ 1-\text{erf} \left( \frac{z-Bt}{2\sqrt{at}} \right) ] \right\}, \quad 0 < z, \ 0 < t < t_1 \\
T(z,t) = U(z,t) e^{-\frac{C(t)}{2}}, \quad 0 < z, \ t_1 < t, \\
\]

where

\[ U(z,t) = e^{-\frac{Ct}{4a}} \left\{ \int_0^\infty g(z'+B_t)e^{2a} G(z,t-t_1;z',0)dz' \right\} \\
+ \int_0^\infty g(z'-B_t)e^{2a} \text{erf} \left( \frac{z'-B_t}{2\sqrt{at_1}} \right) G(z,t-t_1;z',0)dz' \\
+ \int_0^\infty g(z'+B_t)e^{2a} \text{erf} \left( \frac{z'+B_t}{2\sqrt{at_1}} \right) G(z,t-t_1;z',0)dz' \\
+ \int_0^\infty g(z-B_t)e^{2a} \text{erf} \left( \frac{z-B_t}{2\sqrt{at_1}} \right) G(z,t-t_1;z',0)dz'. \\
\]

The first two integrals on the right hand side of the last formula may be evaluated with the help of integral formula (4) of Appendix 5, and the same formula may also be used to evaluate the last two integrals if \( t_1 \) is small. Otherwise, it seems numerical integration is necessary.

The following two examples were taken from Bailey and Lakin's
paper concerning underground combustion with convection effects.
The notations used in the examples were well defined in the literature.
For convenience the reader is referred to the end of this paper.

III. $T$ constant at $x = 0$

Equation:

$$a \frac{\partial^2 T}{\partial x^2} - \beta \frac{\partial T}{\partial x} - \frac{\partial T}{\partial t} + \frac{T}{M_f} \delta(x-x_f) = 0.$$ 

Boundary conditions:

$$T = T_i \text{ at } t = 0, \quad T = T_L \text{ at } x = 0,$$

$$T \to T_i \text{ as } x \to \infty.$$ 

Applying the Laplace transformation, we obtain

$$a \frac{d^2 \bar{T}}{dx^2} - \beta \frac{d \bar{T}}{dx} - p \bar{T} = - (T_i + \frac{T}{M_f} e^{-\frac{px}{v_f}}),$$

$$\frac{T_L}{p} \text{ at } x = 0, \quad \frac{T_i}{p} \text{ as } x \to \infty.$$ 

Then the solution of the transformed equation is

$$\bar{T}(x, p) = \left\{ \frac{T_L}{p} - \frac{T_i}{p} + \frac{T}{M_f} \right\} \exp\left[ \frac{\beta - \sqrt{\beta^2 + 4ap}}{2a} \right] \cdot \frac{T}{M_f} \exp\left( -\frac{px}{v_f} \right) + \frac{T_i}{p}.$$
where

\[ K = \left( \frac{ap^2}{v_f^2} + \frac{\beta}{v_f} - p \right). \]

By Theorem 1 and the Inverse Laplace transform formulae (1) and (2) of Appendix 5, the solution is

\[ T(x, t) = T_i + \frac{T}{2} \left\{ \text{erfc}\left[ \frac{x}{2\sqrt{at}} - \frac{\beta}{2} \sqrt{\frac{t}{a}} \right] + \exp\left( \frac{\beta x}{a} \right) \text{erfc}\left[ \frac{x}{2\sqrt{at}} + \frac{\beta}{2} \sqrt{\frac{t}{a}} \right] \right\} \]

\[ + \frac{v_f T M}{2(\beta - v_f)} \left\{ \text{erfc}\left[ \frac{x}{2\sqrt{at}} - \frac{\beta}{2} \sqrt{\frac{t}{a}} \right] + \exp\left( \frac{\beta x}{a} \right) \text{erfc}\left[ \frac{x}{2\sqrt{at}} + \frac{\beta}{2} \sqrt{\frac{t}{a}} \right] \right\} \]

\[ + \frac{v_f T M}{2(v_f - \beta)} \left\{ \exp\left[ \frac{v_f t(\beta - v_f) + v_f x}{a} \right] \text{erfc}\left[ \frac{x}{2\sqrt{at}} - (\frac{\beta}{2} - \frac{v_f}{a}) \sqrt{\frac{t}{a}} \right] \right\} \]

\[ + \exp\left[ \frac{(\beta - v_f)(x - v_f t)}{a} \right] \text{erfc}\left[ \frac{x}{2\sqrt{at}} + (\frac{\beta}{2} - \frac{v_f}{a}) \sqrt{\frac{t}{a}} \right] \right\} \]

\[ + \frac{T M v_f}{(v_f - \beta)} \left\{ 1 - \exp\left[ \frac{1}{a} (v_f t - x)(v_f - \beta) \right] \right\}, \]

where the last term is replaced by zero when \( t \leq \frac{x}{v_f} \).

The solution obtained is equivalent to formula (17) on Bailey and Lakin's report.
IV. Constant Temperature Heater Turned Off After $t_1$ Hours

Equation:

$$a \frac{\partial^2 T}{\partial x^2} - \beta \frac{\partial T}{\partial x} - \frac{\partial T}{\partial t} = 0.$$ 

Boundary conditions:

$$T = T_i \text{ at } t = 0, \quad T = \begin{cases} T_H & t \leq t_1 \\ T_i & t > t_1 \end{cases} \text{ at } x = 0,$$

$$T \to T_i \text{ as } x \to \infty.$$ 

The corresponding transformed system is

$$a \frac{d^2 \bar{T}}{dx^2} - \beta \frac{d \bar{T}}{dx} - \bar{T} + T_i = 0,$$

$$\bar{T}_{x=0} = \frac{T_H}{p} + \left(\frac{T_i - T_H}{p}\right)e^{-pt_1}$$

where the second term of the last equation is replaced by zero when $t \leq t_1$

$$\frac{T}{T} \to \frac{T_i}{p} \text{ as } x \to \infty.$$
It follows that the solution of the transformed system is

$$
\bar{T} = \frac{T_i}{p} + \left[ \frac{T_H - T_i}{p} + (\frac{T_i - T_H}{p})e^{-pt} \right] \exp\left[ \frac{px}{2a} - \left( \frac{\beta^2}{4a^2} + \frac{p}{a} \right) \frac{1}{2} x \right].
$$

With the help of Theorem 1 and inverse formula 1 of Appendix 5, we again obtain the solution, which is the same as that of Bailey and Lakin,

$$
T(x,t) = T_i + \frac{T_H - T_i}{2} \left\{ \text{erfc}\left[ \frac{x}{2\sqrt{at}} - \frac{\beta}{2} \sqrt{\frac{t}{a}} \right] + \exp\left( \frac{x\beta}{a} \right) \text{erfc}\left[ \frac{x}{2\sqrt{at}} + \frac{\beta}{2} \sqrt{\frac{t}{a}} \right]
\right.

- \text{erfc}\left[ \frac{x}{2\sqrt{a(t-t_1)}} - \frac{\beta}{2} \sqrt{\frac{t-t_1}{a}} \right] - \exp\left( \frac{x\beta}{a} \right) \text{erfc}\left[ \frac{x}{2\sqrt{a(t-t_1)}} + \frac{\beta}{2} \sqrt{\frac{t-t_1}{a}} \right]

\left. + \frac{\beta}{2} \sqrt{\frac{t-t_1}{a}} \right\},
$$

where the last two terms are replaced by zeros when $t \leq t_1$.

In view of the fact that $a, \beta$ appearing in the last two examples are constant, analytical solutions are feasible by our formulae (3) and (5). We shall, therefore, solve the examples again by using our formulae. The calculations may reveal the validity of our formulae.

V. Solve example III by formulae (3) and (5) above.

By formula (3), we have
\[ T(x, t) = U(x, t) e^{-\frac{\beta}{4\alpha} (2x - \beta t)} \]

where

\[ U(x, t) = \int_0^\infty \frac{e^{-\beta x'}}{2a} \frac{\exp\left[ \frac{\beta^2 t'}{4a} - \frac{x^2}{4a(t-t')} \right]}{(t-t')^{3/2}} dt' \]

\[ + \frac{x}{2\sqrt{\pi a}} \int_0^t T_L \exp\left[ \frac{\beta^2 t'}{4a} - \frac{x^2}{4a(t-t')} \right] (t-t')^{3/2} dt' \]

\[ + T_M \frac{v}{f} \int_0^t \int_0^\infty \exp\left[ \frac{-\beta x'}{2a} + \frac{\beta^2 t'}{4a} \right] \delta(x-vt') G(x, t; x', t') dx' dt'. \]

The first two integrals can be evaluated easily by the integral formulae (4) and (1) of Appendix 5, and the last integral after carrying out the integration with respect to \( x' \) becomes

\[ I = T_M \frac{v}{f} \left\{ \int_0^t \frac{\exp\left[ (\frac{\beta t'}{4a} + \frac{\beta}{4\alpha}) \right] \exp\left[ \frac{-(x-vt')^2}{4a(t-t')} \right]}{2\sqrt{\pi a(t-t')}} dt' \right\} \]

\[ - \int_0^t \frac{\exp\left[ (\frac{\beta t'}{4a} + \frac{\beta}{4\alpha}) \right] \exp\left[ \frac{-(x+vt')^2}{4a(t-t')} \right]}{2\sqrt{\pi a(t-t')}} dt' \}

\[ = T_M \frac{v}{f} (I_1 + I_2). \]

Applying the transformation \( y = t-t' \) to \( I_1 \), we have
\[ I_1 = \exp\left[\frac{-2v_f(x-v_f t)}{4a} - \frac{(2v_f - \beta)\beta t}{4a} \right] \int_0^t e^{-b^2 y^{-a^2/y}} dy, \]

where

\[ b^2 = \frac{\left(v_f^2 + \beta^2 - 2\beta v_f\right)}{4a} \text{ and } a^2 = \frac{x-v_f t}{4a}. \]

It follows that \( a = \frac{|x-v_f t|}{2Nra}, \ b = \frac{|v_f - \beta|}{2Nra} = \frac{(v_f - \beta)}{2Nra}, \) since the frontal velocity \( v_f \) is greater than \( \beta \). Now by the integral formula (2) of Appendix 5, the value of \( I_1 \) is given by

\[ I_1 = -\frac{\exp\left[\left(-2v_f(x-v_f t) - 2\beta v_f t + \beta^2 t\right)/4a\right]}{(v_f - \beta)} \{ \sinh \frac{|x-v_f t| (v_f - \beta)}{2a} \}

\[ -\frac{1}{2} \exp\left[-|x-v_f t| (v_f - \beta)\sqrt{a} \right] \text{erf}\left[\left((v_f - \beta)\sqrt{t} - |x-v_f t|/\sqrt{t}\right)/2Nra\right], \]

\[ -\frac{1}{2} \exp\left[|x-v_f t| (v_f - \beta)\sqrt{a} \right] \text{erf}\left[\left((v_f - \beta)\sqrt{t} + |x-v_f t|/\sqrt{t}\right)/2Nra\right]. \]

Similarly,

\[ I_2 = \frac{\exp\left[2v_f(x+v_f t) - 2\beta v_f t + \beta^2 t\right]/4a]}{(v_f - \beta)} \{ \sinh \frac{|x+v_f t| (v_f - \beta)}{2a} \}

\[ -\frac{1}{2} \exp\left[-|x+v_f t| (v_f - \beta)/2a \right] \text{erf}\left[\left((v_f - \beta)\sqrt{t} - |x+v_f t|/\sqrt{t}\right)/2Nra\right], \]

\[ -\frac{1}{2} \exp\left[|x+v_f t| (v_f - \beta)/2a \right] \text{erf}\left[\left((v_f - \beta)\sqrt{t} + |x+v_f t|/\sqrt{t}\right)/2Nra\right]. \]
The absolute value signs appearing in $I_1$ may be eliminated by considering the cases in which $x \geq v_f t$ and $x < v_f t$. In $I_2$ we have $|x + v_f t| = x + v_f t$ since $x, v_f, t$ are positive quantities. Then the solution of the system may be expressed as follows:

$$T(x, t) = \frac{T_i}{2} \left\{ 1 + \text{erf} \left( \frac{x - \beta t}{2 \sqrt{a t}} \right) - e^{\alpha} \left[ 1 - \text{erf} \left( \frac{x + \beta t}{2 \sqrt{a t}} \right) \right] \right\}$$

$$+ \frac{T_L}{2} \left\{ 2e^{\frac{\beta x}{2a}} \text{cosh} \frac{\beta x}{2a} + \text{erf} \left( \frac{(\beta t - x)}{2 \sqrt{a t}} \right) - e^{\frac{\alpha}{2}} \text{erf} \left( \frac{(\beta t + x)}{2 \sqrt{a t}} \right) \right\}$$

$$+ \frac{T_M v_f}{(v_f - \beta)} \exp \left[ \frac{(v_f (x + v_f t - \beta t) + \beta x)}{2a} \right] \left\{ \sinh \frac{(x + v_f t)(v_f - \beta)}{2a} \right\}$$

$$- \frac{1}{2} \exp \left[ -(x + v_f t)(v_f - \beta)/2a \right] \text{erf} \left[ \frac{(v_f - \beta)\sqrt{t} - (x + v_f t)/\sqrt{t}}{2\sqrt{a}} \right]$$

$$- \frac{1}{2} \exp \left[ (x + v_f t)(v_f - \beta)/2a \right] \text{erf} \left[ \frac{(v_f - \beta)\sqrt{t} + (x + v_f t)/\sqrt{t}}{2\sqrt{a}} \right]$$

$$\begin{align*}
\begin{cases}
 f_1, & \text{if } x \geq v_f t, \\
 f_2, & \text{if } x < v_f t,
\end{cases}
\end{align*}$$

where
\[ f_1 = \frac{T_{Mf} v_f}{(v_f - \beta)} \exp\left[ -v_f (x-v_f t+\beta t)+\beta x\right]/2a \] \{\sinh \frac{(x-v_f t)(v_f - \beta)}{2a} \} - \frac{1}{2} \exp\left[ -(x-v_f t)(v_f - \beta)/2a \right] \text{erf}\left[ ((v_f - \beta)\sqrt{t}-(x-v_f t)/\sqrt{t})/2\sqrt{a} \right]\}

\[ f_2 = \frac{T_{Mf} v_f}{(v_f - \beta)} \exp\left[ -v_f (x-v_f t-\beta t)+\beta x\right]/2a \] \{\sinh \frac{(v_f t-x)(v_f - \beta)}{2a} \} - \frac{1}{2} \exp\left[ (x-v_f t)(v_f - \beta)/2a \right] \text{erf}\left[ ((v_f - \beta)\sqrt{t}-(x-v_f t)/\sqrt{t})/2\sqrt{a} \right]\}

Although the solution obtained seems different from that of example III above, it can be shown by the properties of the hyperbolic functions and of the error function that the two solutions are equal.

VI. Solve example IV by formulae (3) and (5).

\[ \frac{\beta x}{2a} \left( \frac{\beta^2 t}{4a} \right) \]

We have \[ T(x,t) = U(x,t) e^{\frac{\beta x}{2a} \left( \frac{\beta^2 t}{4a} \right)} \],

where
\[ U(x,t) = T_i \int_{0}^{\infty} e^{-\frac{\beta x^2}{2a}} G(x,t; x',0) dx' \]

\[ + \frac{x}{2\sqrt{\pi a}} \int_{0}^{t} \frac{T_H \exp\left[ (\beta^2 t' - x^2 / (t-t'))/4a \right] dt'}{(t-t')^{3/2}} \]

\[ + \left\{ \begin{array}{ll}
\frac{x}{2\sqrt{\pi a}} \int_{t_1}^{t} \frac{T_i - T_H \exp\left[ (\beta^2 t' - x^2 / (t-t'))/4a \right] dt'}{(t-t')^{3/2}} , & t > t_1 \\
0 , & t \leq t_1
\end{array} \right. \]

Applying formulae (4) and (1) of Appendix 5, we have

\[ T(x,t) = \frac{T_i}{2} \left\{ 1 + \text{erf}\left( (x - \beta t \sqrt{2at}) - e^{\frac{\beta x}{2a}} \left[ 1 - \text{erf}(x + \beta t \sqrt{2at}) \right] \right) \right\} \]

\[ + \frac{T_H}{2} \left\{ 2e^{\frac{\beta x}{2a}} \cosh \frac{\beta x}{2a} + \text{erf}\left( (\beta t - x \sqrt{2at}) - e^{\frac{\beta x}{2a}} \text{erf}\left( (\beta t + x \sqrt{2at}) \right) \right) \right\} \]

\[ + \frac{T_i - T_H}{2} \left\{ 2e^{\frac{\beta x}{2a}} \cosh \frac{\beta x}{2a} + \text{erf}\left( (\beta \sqrt{\frac{t-t_1}{t-t_1}} - x \sqrt{\frac{t-t_1}{t-t_1}}) / \sqrt{2} \right) \right\} \]

\[ - e^{\frac{\beta x}{a}} \text{erf}\left( (\beta \sqrt{\frac{t-t_1}{t-t_1}} + x \sqrt{\frac{t-t_1}{t-t_1}}) / \sqrt{2} \right) \right\}, \]

where the last bracket is replaced by zero when \( t \leq t_1 \).

Again the solution obtained above is equivalent to equation (19) in Bailey and Lakin's paper.
CONVERGENCE OF THE SEQUENCE OF APPROXIMATE SOLUTIONS

In this chapter we shall establish the convergence of the sequence of the approximate solutions of the initial-boundary value problem defined by (2).

By using formula (2) of Appendix 2, the initial-boundary value problem may be transformed into an integro-differential equation

\[
T(z,t) = \int_0^\infty g(z')G(z,t; z', 0)dz' + \frac{z}{2\sqrt{\pi a}} \int_0^t \frac{4a(t-t')}{(t-t')^{3/2}} dt'
\]

\[
+ \int_0^t \int_0^\infty v(t') \frac{\partial T}{\partial z'} (z', t')G(z,t; z', t')dz' dt'
\]

\[
+ \int_0^t \int_0^\infty A G(z,t; z', t')dz' dt'.
\]

Under certain conditions the above equation can be solved by the method of successive approximations as follows: For each \( n \), \( n = 1, 2, 3, \ldots \),
Here we shall show that the sequence \( \{T_n\} \) defined by formula (14) converges uniformly on a closed and bounded region \( R \), \( R = \{(z, t) | 0 < z < Z, \ 0 < t < T' \} \); consequently, we now prove the following assertion.

**Theorem 1.** Let \( f, g, \) and \( v \) be bounded functions and \( f \) has bounded first derivative on the domain of definition such that \( f(0) = g(0) = 0, \) and \( N \) be the least upper bound of \( v \). Then the sequence \( \{T_n\} \) converges uniformly on \( R \) for \( 2N(\frac{T'}{\pi})^{1/2} < 1 \).

**Proof:** In order to prove the assertion, we shall first show that the sequence \( \frac{\partial T_n}{\partial z} \) is uniformly convergent.

Since the exponential factors secure the uniform convergence, we have
\[
\frac{\partial T_n}{\partial z}(z, t) = \int_0^\infty g(z') \frac{\partial}{\partial z} G(z,t;z',0)dz' + \frac{\partial}{\partial z} \left\{ -\frac{z}{2\sqrt{\pi a}} \int_0^t f(t')e^{\frac{-z^2}{4a(t-t')}} dt' \right\} \\
+ \int_0^t \int_0^\infty v(t') \frac{\partial T_{n-1}}{\partial z'}(z', t') \frac{\partial}{\partial z} G(z,t;z',t')dz' dt' \\
+ \int_0^t \int_0^\infty A \frac{\partial}{\partial z} G(z,t;z',t')dz' dt' .
\]

From Appendix 6, \( \frac{\partial T_1}{\partial z} \) is bounded in \( \Omega \) if \( \frac{\partial T_0}{\partial z} \) is bounded there; so let

\[
K = \max \left| \frac{\partial T_1}{\partial z} - \frac{\partial T_0}{\partial z} \right| ,
\]

and define

\[
D_n = \frac{\partial T_n}{\partial z} - \frac{\partial T_{n-1}}{\partial z} .
\]

For \( n = 1, 2, 3, \cdots, k, \cdots \) (see Appendix 6 also)
\[
|D_2(z, t)| \leq \int_0^t \int_0^\infty |v(t')| \left| \frac{\partial T}{\partial z'}(z', t') - \frac{\partial T}{\partial z'}(z', t') \right| \left| \frac{\partial}{\partial z} G(z, t; z', t') \right| dz' dt'
\]

\[
\leq KN \int_0^t \frac{e^{\frac{-z^2}{4a(t-t')}}}{\sqrt{\pi a(t-t')}} dt' \leq KS N t, \text{ where } S = \frac{2N}{\sqrt{\pi a}}.
\]

\[
|D_3(z, t)| \leq \int_0^t \int_0^\infty N(KS N t') \left| \frac{\partial}{\partial z} G(z, t; z', t') \right| dz' dt' \leq K(SN t)^2
\]

\[
|D_k(z, t)| \leq K(SN t)^{k-1}
\]

Thus, for \( n > m \)

\[
\left| \frac{\partial T}{\partial z}^{n-1}(z, t) - \frac{\partial T}{\partial z}^{m-1}(z, t) \right| \leq K \sum_{\ell=m}^{n-1} (SN t)^{\ell-1}
\]

So the sequence \( \left\{ \frac{\partial T^n}{\partial z} \right\} \) converges uniformly on \( 0 \leq SN t \leq p < 1 \).

Therefore,
\[
|T_n(z, t) - T_m(z, t)| \leq \int_0^t \int_0^{\infty} |v(t')| \left| \frac{\partial T_{n-1}}{\partial z'} - \frac{\partial T_{m-1}}{\partial z'} \right| |G(z, t; z', t')|dz'dt'
\]

\[
\leq KN \int_0^t \sum_{\ell = m}^{n-1} (S\sqrt{t'})^{\ell-1} dt' = \frac{KN\pi a}{S} \sum_{\ell = m}^{n-1} \frac{(S\sqrt{t'})^{\ell+1}}{\ell+1}
\]

\[
\leq \frac{KN\pi a}{S} \sum_{\ell = m}^{n-1} \frac{(S\sqrt{T'})^{\ell+1}}{\ell+1}
\]

since the series \( \sum_{\ell=0}^{\infty} (S\sqrt{T'})^{\ell} \) converges for \( (S\sqrt{T'}) < 1 \). Thus for sufficiently large \( m, n \), the right-hand side of the last inequality can be made less than any preassigned quantity. Hence, the sequence \( \{ T_n \} \) converges uniformly on \( R \), by Cauchy's convergence theorem.

**Maximum and Minimum Principles**

It is well known that a function satisfying equation (2) attains its maximum and minimum values on the boundary; the assertion follows from Picone's theorem [Appendix 5]. Nevertheless, we shall prove here the principles of maxima and minima for the case in which the domain of definition is a bounded region. The principle may be stated as follows:
Theorem 2. Let $T(z,t)$ be a solution of equation (2). Then $T(z,t)$ assumes its maximum and minimum values on the boundary.

Proof: Without loss of generality, let's consider the homogeneous equation of (2), since equation (2) may be reduced to homogeneous character by applying the transformation $T = T^* + At$ (to equation (2)).

Let $R$ and $R_i$ be the regions such that

$$R = \{(z,t) | 0 \leq z \leq Z, \ 0 \leq t \leq T'\}$$

$$R_i = \{(z,t) | 0 < z < Z, \ 0 < t \leq T'\}.$$ 

Since $T(z,t)$ is continuous on $R$, set

$$\text{Max}_{R} T(z,t) = M, \quad \text{Max}_{R_i} T(z,t) = m,$$

where $R \sim R_i$ denotes the set of all points in $R$ but not in $R_i$. In the case of maximum value we wish to show $m \geq M$.

Assume that $M > m$. Then there exists a point $(z_0, t_0) \in R_i$.
such that \( T(z_0, t_0) = M \). On \( \mathbb{R} \) we define the function

\[
\theta(z, t) = T(z, t) + \frac{(M-m)(t_0-t)}{2T'}.
\]

On \( \mathbb{R} \sim \mathbb{R}_i \) it follows that

\[
\theta(z, t) \leq m + \frac{(M-m)}{2} < M.
\]

Also

\[
\theta(z_0, t_0) = M.
\]

Therefore, the function \( \theta(z, t) \) also attains its maximum value \( L \) on \( \mathbb{R}_i \). Then let \( (z_1, t_1) \) be a point in \( \mathbb{R}_i \) at which \( \theta(z_1, t_1) = L \). It follows that at \( (z_1, t_1) \)

\[
\frac{\partial \theta}{\partial t} - a\frac{\partial^2 \theta}{\partial z^2} - v \frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial t} - a\frac{\partial^2 \theta}{\partial z^2} > 0,
\]

since \( a > 0 \). On the other hand,
\[
\frac{\partial \theta}{\partial t} - \frac{a \partial^2 \theta}{\partial z^2} - \nu(t) \frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial t} - \frac{(M-m)}{2T'} - a \frac{\partial^2 T}{\partial z^2} - \nu(t) \frac{\partial T}{\partial t}
\]

\[\quad = - \frac{(M-m)}{2T'} < 0.\]

The two inequalities are inconsistent; therefore, the first assertion follows.

Similarly the second part of the theorem may be proved.

However, this does not require a separate proof because the solution \( T^o(z,t) = -T(z,t) \) has a maximum value when \( T(z,t) \) has a minimum.

Let us now prove the uniqueness theorem which is a consequence of the maximum and minimum principles.

Theorem 3. Suppose \( T(z,t) \) is a solution of equation (2) in \( \mathbb{R} \). Then \( T(z,t) \) is unique.

Proof: In order to prove this theorem, we examine the difference of two distinct solutions \( T_1 \) and \( T_2 \). Then

\[ W(z,t) = T_1(z,t) - T_2(z,t) \]

is also a solution of equation (2). Therefore, by Picone's theorem, \( W(z,t) \) attains extreme values on the boundary. However,
\[ W(z,0) = T_1(z,0) - T_2(z,0) = 0, \]
\[ W(0,t) = T_1(0,t) - T_2(0,t) = 0. \]

Therefore, \( W(z,t) \) must be identically zero throughout the domain. Hence, \( T_1(z,t) = T_2(z,t) \).

The solutions of equation (2) are well-posed in the sense that they depend continuously on the boundary conditions. The assertion is another consequence of the maximum value principle which may be stated as follows:

**Theorem 4.** If two solutions of \( T_1(z,t) \) and \( T_2(z,t) \) of equation (2) satisfy the inequality

\[ |T_1(z,t) - T_2(z,t)| \leq \varepsilon \]

on the boundary of the region, then

\[ |T_1(z,t) - T_2(z,t)| \leq \varepsilon \]

holds for all \( z,t \) in the domain.

**Proof:** The conclusion follows immediately from the principle of the maxima. The difference of two solutions in the domain can not exceed their difference on the boundary.
BIBLIOGRAPHY


APPENDIX 1

The purpose of this section is to find a transformation which reduces equation (2) to a well-known form — equation (1) of Appendix 2.

Consider the transformation of the form

(1) \[ T(z, t) = U(z, t) f(z)g(t) \]

such that \( f(z)g(t) \neq 0 \). Differentiating (1) and substituting into the heat transport equation, we obtain

\[
\frac{\partial U}{\partial t} f g = a f g \frac{\partial^2 U}{\partial z^2} + (2a \frac{df}{dz} g + v f g) \frac{\partial U}{\partial z}
\]
\[
+ (a \frac{d^2 f}{dz^2} g + v \frac{df}{dz} g - f \frac{dg}{dt}) U + A.
\]

Set

(2) \[ 2a \frac{df}{dz} g + v f g = 0, \]

and

(3) \[ a \frac{d^2 f}{dz^2} g + v \frac{df}{dz} g - f \frac{dg}{dt} = 0. \]
Since \( g \neq 0 \), from (2) we have

\[
2a \frac{df}{dz} + vf = 0, \tag{4}
\]

and it implies

\[
2a \frac{d^2 f}{dz^2} + v \frac{df}{dz} = 0. \tag{5}
\]

Solving (4) and (5) for the first and second derivatives in terms of \( f \) and substituting into (3), we have

\[
\frac{dg}{g} = -\frac{v^2}{4a} \, dt.
\]

\[-\int \frac{(v(t))^2}{4a} \, dt = -\frac{v(t)}{2a}z
\]

It follows that \( g(t) = Ce^{-\frac{v(t)}{2a}z} \), and from (4) \( f(z) = De^{2az} \).

If \( v(t) \) were not constant, the last equation would contradict our original assumption on \( f \). On the other hand, if \( v(t) \) is a constant function, say \( k \), then the transformation sought has the form

\[
T(z, t) = U(z, t)e^{-\left(\frac{kz}{2a} + \frac{k^2 t}{4a}\right)}
\]

for which we choose the constant of integration to be unity.
APPENDIX 2

Consider the equation

\[
\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2} + F, \quad 0 < z, \quad 0 < t,
\]

\[T(z, 0) = h(z),\]

\[T(0, t) = f(t).\]

We can split the equation into two distinct problems such that the combined solution gives us the desired solution of the system above.

The solution is given by Tikhonov [11, p. 299] and Carslaw [5, p. 357],

\[
T(z, t) = \int_0^\infty h(z')G(z, t; z', 0)dz' + \frac{z^2}{2\sqrt{\pi a}} \int_0^t f(t')e^{-\frac{4a(t-t')}{(t-t')^{3/2}}} dt'
\]

\[
+ \int_0^t \int_0^\infty F(z', t')G(z, t; z', t')dz' dt',
\]

where

\[
G(z, t; z', t') = e^{-\frac{(z-z')^2}{4a(t-t')}} - e^{-\frac{(z+z')^2}{4a(t-t')}},
\]

and it is called the Green's function for the semi-infinite solid.
APPENDIX 3

In the following discussion we shall assume $f, g$ and $v$ are continuous and bounded functions on the domain of definition. Then the first, third and fourth integrals on the right-hand side of (8) are continuous functions because the exponential factors appearing on the integrands secure the uniform convergence. Consequently, the integrals tend to zero as $z$ tends to zero. So it remains to be shown that

$$\lim_{z \to 0} \frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{a}}}^{\infty} f\left(t - \frac{z^2}{2}\right) e^{-u^2} \frac{du}{2\sqrt{a}t} = f(t).$$

Choose $\delta < 1$ so that $\delta \leq \frac{z}{2\sqrt{a}t}$. Then consider

$$\left| \frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{a}}}^{\infty} f\left(t - \frac{z^2}{2}\right) e^{-u^2} \frac{du}{2\sqrt{a}t} - f(t) \right| \leq \frac{2}{\sqrt{\pi}} \left\{ \int_{\delta}^{\infty} |f\left(t - \frac{z^2}{2}\right) - f(t)| e^{-u^2} \frac{du}{2\sqrt{a}t} + \int_{\delta}^{\infty} |f(t)| e^{-u^2} \frac{du}{2\sqrt{a}t} \right\}.$$

By hypotheses, there is a positive number $M$ such that $|f(t)| \leq M$ for all $t$'s, and given $\epsilon > 0$, $\delta_0 > 0$, there is $\eta_0(\epsilon, \delta_0) > 0$ such that

$$|f\left(t - \frac{z^2}{2}\right) - f(t)| \leq \frac{\epsilon}{2}, \text{ provided } 0 < z < \eta_0.$$
also, there is $\delta_1(\varepsilon)$ such that for all $\delta < \delta_1(\varepsilon)$

$$\int_0^\delta e^{-u^2} du < \frac{\sqrt{\pi}}{4M} \varepsilon.$$  

Thus

$$\left| \frac{2}{\sqrt{\pi}} \int_{z_0}^\infty f(t - \frac{z^2}{4at}) e^{-\frac{z^2}{4at}} du - f(t) \right| < \varepsilon,$$  

provided $\eta = \min \{ \eta_0, \delta_1 \}$.

Similarly, as $t$ tends to zero, the last three integrals on the right-hand side of (8) tend to zero because the exponential factors again secure the uniform convergence. So we want to show that

$$\lim_{t \to 0} \int_0^\infty \frac{v(0)z'}{2a} G(z; t; z', 0) dz' = g(z)e^{\frac{v(0)z}{2a}}.$$  

Let $h(z) = g(z)e^{\frac{v(0)z}{2a}}$, $\eta = \frac{z}{2\sqrt{at}}$ and $z' - z = 2\sqrt{at}u$.

Then consider
\[ J = | \int_0^\infty h(z') g(z, t; z', 0) dz' - h(z) | \]

\[ = \frac{1}{\sqrt{\pi}} | \int_{-\eta}^\infty h(z+2\sqrt{u} t) e^{-u^2} du + \int_{\eta}^\infty h(-z+2\sqrt{u} t) e^{-u^2} du - \int_{-\infty}^\infty h(z) e^{-u^2} du | \]

\[ \leq \frac{1}{\sqrt{\pi}} \left\{ \int_{-\eta}^\eta |h(z+2\sqrt{u} t) - h(z)| e^{-u^2} du + \int_{\eta}^\infty |h(z+2\sqrt{u} t)| e^{-u^2} du \right. \]

\[ + \left. \int_{\eta}^\infty |h(-z+2\sqrt{u} t)| e^{-u^2} du + 2 \int_{\eta}^\infty |h(z)| e^{-u^2} du \right\}. \]

By hypotheses, for a given \( z \), let

\[ M = \max_u \{ |g(z+2\sqrt{u} t)| e^{2a}, |g(-z+2\sqrt{u} t)| e^{2a} \}, \]

\[ N = \max |h(z)|, \]

and given \( \varepsilon > 0 \), there is \( \delta_0(\varepsilon) > 0 \) such that, for a given \( t \), for all \( \delta < \delta_0(\varepsilon) = 2\sqrt{u_0} \), we have

\[ |h(z+\delta) - h(z)| < \frac{\varepsilon}{5}. \]

So
\[ J \leq \frac{1}{\sqrt{\pi}} \left\{ \int_{-\infty}^{\infty} |h(z+2\sqrt{at}u)-h(z)| e^{-u^2} \, du + 2M \int_{\eta}^{\infty} e^{au} e^{-u^2} \, du + 2N \int_{\eta}^{\infty} e^{-u^2} \, du \right\}, \]

where \( a = \frac{|v(0)| z}{2a} \).

Hence

\[ J \leq \frac{1}{\sqrt{\pi}} \left\{ \int_{-\infty}^{-u_0} |h(z+2\sqrt{at}u)-h(z)| e^{-u^2} \, du + \int_{u_0}^{u_0} e^{-u^2} \, du \right\} \]

\[ + \int_{u_0}^{\infty} |h(z+2\sqrt{at}u)-h(z)| e^{-u^2} \, du + 2M \int_{\eta}^{\infty} e^{au} e^{-u^2} \, du + 2N \int_{\eta}^{\infty} e^{-u^2} \, du \right\}. \]

For sufficiently small \( t, \ t < t_0 \), the first and last three integrals can be made less than \( \frac{\sqrt{\pi} \varepsilon}{5} \); hence,

\[ \left| \int_{0}^{\infty} \frac{v(0)z'}{2a} G(z, t; z', 0)dz' - g(z)e^{\frac{v(0)z}{2a}} \right| < \varepsilon. \]
Here, we are seeking for a transformation under which the flux term in equation (2) vanishes.

Consider the transformation

\[ T(z,t) = U(z,t)F(z,t) \]

such that \( F(z,t) \neq 0 \) for all \( 0 \leq z, \ 0 \leq t \). It is easy to see that under the transformation equation (2) becomes

\[
\frac{\partial U}{\partial t} = a \frac{\partial^2 U}{\partial z^2} + (v + 2a \frac{\partial F}{\partial z}) \frac{\partial U}{\partial z} + \frac{1}{F} (v \frac{\partial F}{\partial z} + a \frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial t})U + \frac{A}{F}.
\]

Set \( v + \frac{2a \frac{\partial F}{\partial z}}{F} = 0 \).

It follows that

\[
F(z,t) = C(t)e^{\frac{-v(t)z}{2a}} = e^{\frac{-v(t)z}{2a}}.
\]

Since we sought for a transformation, set

\[ C(t) = 1. \]
APPENDIX 5

INTEGRAL FORMULAE

The formulae have been taken from the following sources:

[8, 9].

(1) \[ I(t) = \int_0^t x^{3/2} \exp\left(-\frac{a^2}{x}b^2\right)dx \quad a \neq 0 \]

\[ = \frac{\sqrt{\pi}}{a} \cosh 2ab + \frac{\sqrt{\pi}}{2a} \left[ e^{-2ab} \text{erf}(b\sqrt{t} - \frac{a}{\sqrt{t}}) - e^{2ab} \text{erf}(b\sqrt{t} + \frac{a}{\sqrt{t}}) \right] \]

(2) \[ J(t) = \int_0^t x^{1/2} \exp\left(-\frac{a^2}{x}b^2\right)dx = -\frac{1}{2b} \frac{dI(t)}{db} \quad a \neq 0 \quad b \neq 0 \]

\[ = -\frac{1}{2b} \left\{ 2\sqrt{\pi} \sinh 2ab + \frac{\sqrt{\pi}}{a} \left[ e^{-2ab} \frac{\sqrt{t} e^{-(b\sqrt{t}-a/\sqrt{t})^2}}{\sqrt{\pi}} \right. \right. \]

\[ - ae^{-2ab} \text{erf}(b\sqrt{t}-a/\sqrt{t}) - ae^{2ab} \text{erf}(b\sqrt{t}+a/\sqrt{t}) \]

\[ \left. - \frac{\sqrt{t}}{\sqrt{\pi}} e^{2ab} e^{-(b\sqrt{t} + 2/\sqrt{t})^2} \right\} \]

(3) \[ \int_0^\infty e^{-(ax^2+2bx+c)}dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2-ac}{2a}} \quad \left[ 1 - \text{erf}\left(\frac{b}{\sqrt{a}}\right) \right], \quad a > 0 \]
\[ \int_0^\infty \frac{2lz'}{(mz'+n)e^{4at}} G(z,t;z',t')dz' \]

\[
= \frac{1}{2} \left\{ [m(z+\ell)+n] e^{4at} \left[ 1+\text{erf}\left(\frac{z+\ell}{2\sqrt{at}}\right)\right] \\
+ [m(z-\ell)-n] e^{4at} \left[ 1-\text{erf}\left(\frac{z-\ell}{2\sqrt{at}}\right)\right] \right\}
\]

**Inverse Laplace Transforms**

\[ L^{-1}\left\{ e^{-q^x/p-a} \right\} = \frac{1}{2} e^{at} \left\{ e^{-x\sqrt{\alpha}/K} \text{erfc}\left[ x/2\sqrt{Kt} - \sqrt{at}\right] \\
+ e^{x\sqrt{\alpha}/K} \text{erfc}\left[ x/2\sqrt{Kt} + \sqrt{at}\right] \right\} \]

\[ L^{-1}\left\{ \frac{be^{-ap}}{p(p+b)} \right\} = \begin{cases} 0, & 0 < t < a, \\ 1-e^{-b(t-a)}, & t > a, \end{cases} \]

where \( \overline{\nu}(p) = \int_0^\infty e^{-pt} \nu(t)dt, \quad q = \sqrt{\frac{P}{K}}, \quad K > 0, \quad x > 0 \) and \( a \) is a real number.

**Theorems**

**Theorem 1.** If \( a \) is any constant and \( L\{\nu\} = \overline{\nu}(p) \), then \( L\{e^{-at}\nu\} = \overline{\nu}(p+a) \).

*Sources: [2, 4, 5, 7, 10].*
Theorem 2. (Picone's theorem) Let $D$ be a domain with $B$ as its boundary in the $(n+1)$-dimensional space of points $P$ with coordinates $x_1, x_2, x_3, \ldots, x_n, t$, and let $B_{-t}$ be that part of $B$ which includes all points $P$ such that (a) the interior normal to the boundary exists and is directed in the negative $t$ direction, and (b) each point $P$ is an interior point of $B_{-t}$.

Let $U(P)$ be a solution of the parabolic differential equation

$$
\sum_{i,j} a_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial U}{\partial x_i} + cU - \frac{\partial U}{\partial t} = 0,
$$

where $a_{ij}(P) = a_{ji}(P)$, $b_i(P)$ and $c(P)$ are real finite functions of $P$, and where the quadratic form $\sum_{i,j} a_{ij} x_i x_j$ is positive-definite. Then:

(a) if $U \geq 0$ on $B \sim B_{-t'}$, then $U \geq 0$ throughout $D$, and if $U \leq 0$ on $B \sim B_{-t'}$, then $U \leq 0$ throughout $D$;

(b) if $U \equiv 0$ on $B \sim B_{-t'}$, then $U \equiv 0$ throughout $D$;

(c) if $U$ is prescribed throughout $B \sim B_{-t}$, then $U$ is uniquely determined throughout $D$;

(d) the maximum value of $|U|$ occurs on $B \sim B_{-t}$.
if $C \equiv 0$, both the maximum and the minimum values of $U$ occur on $B \sim B_t$. 
APPENDIX 6

Here we shall establish the fact that \( \frac{\partial T_1}{\partial z} \) is bounded on \( Q \) if \( \frac{\partial T_0}{\partial z} \) is bounded there, where \( T_0 \) is the zeroth approximation of the solution and \( Q = \{ (z, t) \mid 0 < z < \infty, \ 0 < t < T' \} \).

From formula (15), we have

\[
\left| \frac{\partial T_1}{\partial z}(z, t) \right| \leq \left| \int_0^\infty g(z') \frac{\partial}{\partial z} G(z, t; z', 0) dz' \right|
\]

\[
+ \left| \int_0^\infty \frac{\partial}{\partial z} \frac{z^2}{2 \sqrt{at}} \right| \int_0^\infty \frac{\partial}{\partial z} f(t - \frac{z^2}{2}) e^{-\frac{z^2}{2}} du \frac{f(0)}{\sqrt{\pi at}} e^{\frac{-z^2}{4at}} \right|
\]

\[
+ \left| \int_0^t \int_0^\infty v(t') \frac{\partial T_0}{\partial z'} (z', t') \frac{\partial}{\partial z} G(z, t; z', t') dz' dt' \right|
\]

\[
+ \left| \int_0^t \int_0^\infty A \frac{\partial}{\partial z} G(z, t; z', t') dz' dt' \right|
\]

Let

\[
L = \max \left| f_z \left( t - \frac{\frac{z^2}{2}}{4au^2} \right) \right|
\]

\[
M = \max |g(z)|
\]

\[
N = \max |v(t)|
\]

and use the fact that
Then

\[
I_1 = \left| \int_{\varepsilon}^{\infty} g(z') \frac{\partial}{\partial z} G(z, t; z', 0) dz' \right| \leq \frac{M}{2\sqrt{\pi} \sigma t} \frac{-(z-z')^2}{4at} \left. \right|_{\varepsilon}^{\infty} \frac{-(z+z')^2}{4at} \left. \right|_{\varepsilon}
\]

\[
\leq \frac{M}{2\sqrt{\pi} \sigma t} e^{-\frac{z^2}{4at}} \left\{ e^{-\frac{2\varepsilon z}{4at}} + e^{-\frac{2\varepsilon z}{4at}} \right\} \rightarrow \frac{M}{\sqrt{\pi} \sigma t} e^{-\frac{z^2}{4at}}, \text{ as } \varepsilon \rightarrow 0,
\]

\[
z > 0.
\]

\[
I_2 = \left| \int_{\frac{2}{\sqrt{\pi}}}^{\infty} f_z(t, -\frac{z}{2}) \left(-\frac{2z}{4au} \right) e^{-\frac{u^2}{4au}} du \right| \leq \frac{2L}{a} \left[ \sqrt{at} e^{-\frac{z^2}{4at}} \frac{1}{2} \text{erfc} \left( \frac{z}{2\sqrt{at}} \right) \right]
\]

\[
\leq \frac{2L}{a} \sqrt{at} e^{-\frac{z^2}{4at}}
\]

\[
I_3 = \left| \int_{0}^{t} \int_{0}^{\infty} v(t') \frac{\partial T}{\partial z'} (z', t') \frac{\partial}{\partial z} G(z, t; z', t') dz' dt' \right|
\]

\[
\leq \frac{2NB}{\sqrt{\pi} \sigma t} \frac{t^{1/2}}{2}, \text{ where } B = \max \left| \frac{\partial T}{\partial z} \right|.
\]

\[
I_4 = \left| \int_{0}^{t} \int_{0}^{\infty} A \frac{\partial}{\partial z} G(z, t; z', t') dz' dt' \right| \leq \frac{2A}{\sqrt{\pi} \sigma t} \frac{t^{1/2}}{2}
\]
It follows that

\[ \frac{\partial T_1(z, t)}{\partial z} \leq I_1 + I_2 + I_3 + I_4 < \infty \text{ on } \Omega. \]
NOMENCLATURE

\( T = \) Temperature

\( a = \) thermal diffusivity

\( \beta = \frac{\rho_g C_g v_g}{\rho_m C_m} \)

\( \rho_g = \) density of gas

\( C_g = \) heat capacity of gas

\( v_g = \) flux or velocity of gas

\( \rho_m = \) density of the mass

\( C_m = \) heat capacity of the mass

\( x = \) space variable

\( T_i = \) constant, ambient temperature

\( T_L = \) constant

\( T_M = \) max. temperature rise neglecting convection effects

\( v_f = \) frontal velocity

\( t = \) time

\( x_f = v_f t \)