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Abstract Approved
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Systems of rational differential equations

$$y_i' = f_i(x, y_j), \quad i,j = 1,2,\ldots,n,$$

where $f_i(x,y_j)$ is a rational function of the dependent variables y_j and the independent variable x, are discussed with the purpose in view of setting up problems for solution on electronic computing machines.

In the first chapter is developed a general theory of rational systems. In chapter 2 it is found that a large number of the elementary functions of mathematics may arise as solutions of systems of rational differential equations. A necessary and sufficient condition for a function to be a solution of a system of rational differential equations is that it be an algebraic or algebraically transcendental function; among these functions are included the exponential, circular, hyperbolic, elliptic, hyperelliptic, and Abelian functions, and their inverses, but not the gamma function or the Riemann zeta function.

A list of basic functions and their corresponding systems of rational differential equations (if any) is given in the appendix. Systems corresponding to various combinations of these basic functions may be constructed by means of eight general rules developed for the purpose in chapter 2.

The emphasis is on differential equations, but the method is also applied to problems not involving differential equations. For example, the solution of the transcendental equation

$$\tanh y_0 + \cot y_0 = 0$$

is shown to be the solution of the system of rational differential equations

$$y_0' = -\frac{1}{2y_1^2 - 2xy_1 + x^2},$$

$$y_1' = \frac{y_1^2 - 1}{2y_1^2 - 2xy_1 + x^2},$$

when x = 0. Amongst other problems which are set up so that their

numerical solution will depend on the solution of a system of rational differential equations are the problem of obtaining an approximate real root to a polynomial equation and that of evaluation of elliptic and hyperelliptic integrals. The possibility of using this method in the computation of lengthy tables of certain of the functions of higher mathematics is discussed.

SYSTEMS OF RATIONAL DIFFERENTIAL EQUATIONS

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SYSTEMS OF RATIONAL DIFFERENTIAL EQUATIONS

INTRODUCTION

Normal systems of differential equations have been quite thoroughly studied by many authors. Analytic systems are the most general type usually discussed, algebraic systems have been exhaustively treated (from a more general standpoint) by J. F. Ritt and a few others, and linear systems have of course been the subject of a much greater volume of theory.

An approximate solution to a system of analytic differential equations may be obtained by any one of several numerical methods. However, if this is to be done by means of an electronic computing machine, the machine must be given a "memory" for any transcendental functions in the equations.

In this paper, rational normal systems of differential equations are considered. If an approximate solution were to be worked out by a computer using an ordinary calculating machine, a rational system would ordinarily have no advantage over an analytic system, and an extensive theory of systems of rational differential equations has in the past been unnecessary. However, an electronic computing machine can be set up to perform very rapidly the operations of addition, subtraction, multiplication, and division; it can be set to repeat at high speed any definite routine of these operations. Hence it might be advantageous to propose rational differential equations (which involve only the fundamental operations of arithmetic) for solution by the machine. It is true that tables of values of various irrational or transcendental functions may be stored in the machine or on a magnetic tape, but this will in general slow the operation somewhat and will make the setting up of the problem on the machine much more difficult. Thus we consider the problem of replacing various differential equations by equivalent systems of rational differential equations, which will in general be of higher order than the original equation or system of equations.

In addition, many other problems not involving differential equations may be "reduced" to the problem of obtaining the solution of a system of rational differential equations with suitable initial conditions.

I. GENERAL THEORY OF SYSTEMS OF RATIONAL DIFFERENTIAL EQUATIONS

We consider the system of first order rational differential equations

$$y_i' = f_i(x,y_j), \qquad i,j = 1,2,\ldots\ldots\ldots,n, \tag{1.1}$$

where $f_i(x,y_j)$ is a rational function of the dependent variables y_j and and the independent variable x. It will be assumed that y_2, y_3, .., y_n each actually appears in at least one of the equations (1.1). For if a certain y_p, where p is any integer from 2 to n inclusive, did not appear in any of the equations (1.1), the equation containing y_p' could be removed and the resulting system solved first. If several y_p's did not appear in any of the equations (1.1), the equations involving the derivatives of all these variables could be removed and the remaining system solved first. In the general theory of systems of rational differential equations y_1 could also be treated in this manner. However, in the use to which these equations will be put in this paper, y_1 will be the major dependent variable, and it will not be desirable to reduce the system by eliminating y_1.

It sometimes happens that the system (1.1) may be replaced by an equivalent rational system of lower order. If the first of equations (1.1) is differentiated (n-1) times, and if at each step the values of y_j are substituted from equations (1.1), we obtain the n rational equations

$$y_1^{(k)} = f_1^{(k-1)}(x,y_j), \qquad j,k = 1,2,\ldots\ldots\ldots,n. \tag{1.2}$$

One would expect that it might be possible to reduce the order of the system if the Jacobian

$$J\left(\frac{y_1', y_1'', \ldots\ldots, y_1^{(n)}}{y_1, y_2, \ldots\ldots, y_n}\right) \equiv 0,$$

although this is by no means a sufficient condition. Suppose first that the rank of J is (n-1). Then between the first n derivatives of y_1 there exists a functional relationship (perhaps containing x) which is an identity in y_1. We consider this relationship as a differential equation of nth order with the y_1 term missing. If it were possible to integrate it once, the result would be a differential equation of order (n-1). If a corresponding normal system of (n-1) differential equations were rational, we would have reduced the order of our original system by one. In case the rank of J is (n-k), where k is any integer between 2 and (n-1) inclusive, between the first n derivatives of y_1 there exist k functional relationships (perhaps containing x) which are identities in y_1. This indicates that the given system of equations is composed of certain independent subsystems. These subsystems should be treated separately.

We now consider some simple examples of reducible systems. First take the system

$$(1.3) \qquad \begin{aligned} y_1' &= 2y_2y_3, \\ y_2' &= y_3, \\ y_3' &= -y_2. \end{aligned}$$

In this case y_1 does not appear in the system. We have

$$\begin{aligned} y_1' &= 2y_2y_3, \\ y_1'' &= 2y_3^2 - 2y_2^2, \end{aligned}$$

$$y_1''' = -8y_2y_3,$$

and

(1.4) $$y_1''' + 4y_1' = 0.$$

This is equivalent to the equation

(1.5) $$y_1'' + 4y_1 + C = 0,$$

where C is an arbitrary constant; this in turn is equivalent to the system

(1.6) $$\begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 - C. \end{aligned}$$

Hence equations (1.3) have been reduced to the system (1.6). The three initial conditions of (1.3) determine uniquely the value of C and also will provide the two initial conditions for the system (1.6).

As another example consider the system

(1.7) $$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\frac{xy_2}{x^2+1}. \end{aligned}$$

Here again y_1 does not appear in either equation. We have the functional relationship

$$y_1'' + \frac{x}{x^2+1} y_1' = 0.$$

This is equivalent to the equation

(1.8) $$y_1' = C(x^2+1)^{-\frac{1}{2}},$$

but although this is of lower order than equations (1.7), it is no longer rational. The system (1.7) may not be reduced to a lower order rational system.

We now consider a simple example where y_1 actually appears in the original system of equations:

$$(1.9)\qquad \begin{aligned} y_1' &= y_1 + y_2 + 2y_3, \\ y_2' &= -y_1 - 2y_2 - 2y_3, \\ y_3' &= -5y_1 - y_2 - 3y_3. \end{aligned}$$

Then

$$(1.10)\qquad \begin{aligned} y_1' &= y_1 + y_2 + 2y_3, \\ y_1'' &= -10y_1 - 3y_2 - 6y_3, \\ y_1''' &= 23y_1 + 2y_2 + 4y_3. \end{aligned}$$

In this case

$$J\left(\frac{y_1', y_1'', y_1'''}{y_1, y_2, y_3}\right) = \begin{vmatrix} 1 & 1 & 2 \\ -10 & -3 & -6 \\ 23 & 2 & 4 \end{vmatrix} = 0,$$

and the three functions are dependent. If we let y_3 be arbitrary and solve equations (1.10) for y_1, we obtain

$$y_1 = \frac{\begin{vmatrix} (y_1' - 2y_3) & 1 \\ (y_1'' + 6y_3) & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -10 & -3 \end{vmatrix}} = \frac{-3y_1' - y_1''}{7}.$$

Then

$$(1.11)\qquad y_1'' + 3y_1' + 7y_1 = 0$$

is the differential equation equivalent to the system (1.9). A rational system of equations corresponding to (1.11) is

$$(1.12)\qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= -7y_1 - 3y_2. \end{aligned}$$

This is a reduced system equivalent to the system (1.9).

It is to be noted that if equations (1.1) make up a homogeneous linear system with constant coefficients (as in the example just discussed), the vanishing of J will mean that the functions $y_1^{(k)}$ are linearly dependent, and the system will always be reducible; the reduction may be carried out just as in the case of the system (1.9).

II. OBTAINING SYSTEMS OF RATIONAL DIFFERENTIAL EQUATIONS SATISFIED BY VARIOUS FUNCTIONS

Most of the elementary functions of mathematics may arise as solutions of systems of rational differential equations.

1. Any algebraic function may be considered as the solution of a single rational differential equation.

An algebraic function is a solution of an equation of the type

$$\sum_{i=o}^{n} P_i(x)\, y^i = 0,$$

where $P_i(x)$ is a polynomial in x. Differentiating this sum with respect to x and solving for y', we obtain the rational differential equation

$$y' = -\frac{\sum_{i=o}^{n} P_i'(x)\, y^i}{\sum_{i=1}^{n} i\, P_i(x)\, y^{i-1}} \quad . \tag{2.1}$$

2. The circular and hyperbolic functions and their inverses may be exhibited as solutions of systems of rational differential equations. The system usually has two, but sometimes only one equation. Individual cases have been worked out in the appendix.

3. Elliptic or hyperelliptic functions and their inverses may be exhibited as solutions of systems of two rational differential equations.

For an elliptic or hyperelliptic function (1, p. 252-253) satisfies a differential equation of the type

$$y_1'^2 = \frac{R_1^2(y_1)}{R_2^2(y_1)} P(y_1), \tag{2.2}$$

where R_1 and R_2 are polynomials in y_1 with no factor in common, and where $P(y_1)$ is a polynomial which is prime to its derivative and which has degree greater than or equal to three. If the degree of P is three or four, we have the case of elliptic functions; if the degree is five or greater, we have the hyperelliptic case.

Upon differentiating (2.2) with respect to x we obtain the equation

$$y_1'' = \frac{1}{2}\left[P(y_1) \frac{d}{dy_1}\left(\frac{R_1^2(y_1)}{R_2^2(y_1)}\right) + \frac{R_1^2(y_1)}{R_2^2(y_1)} \frac{dP(y_1)}{dy_1} \right].$$

If we let

$$\frac{1}{2}\frac{d}{dy_1}\left(\frac{R_1^2(y_1)}{R_2^2(y_1)}\right) = \frac{R_3(y_1)}{R_4(y_1)}$$

and

$$\frac{1}{2}\frac{dP(y_1)}{dy_1} = P_1(y_1),$$

the desired system of rational differential equations becomes

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= \frac{R_3(y_1)}{R_4(y_1)} P(y_1) + \frac{R_1^2(y_1)}{R_2^2(y_1)} P_1(y_1). \end{aligned} \tag{2.3}$$

If in equation (2.2) we interchange x and y_1 we find that an inverse elliptic or inverse hyperelliptic function satisfies a differential equation

$$y_1'^2 = \frac{R_2^2(x)}{R_1^2(x)} P(x),$$

where $P(x)$, $R_1(x)$, and $R_2(x)$ have the same meanings as in (2.2). Upon differentiating with respect to x we obtain the equation

$$2y_1'y_1'' = P(x)\,\frac{d}{dx}\left(\frac{R_2^2(x)}{R_1^2(x)}\right) + \frac{R_2^2(x)}{R_1^2(x)}\,P(x)\ .$$

The desired system of rational differential equations is

$$(2.4)\qquad \begin{aligned} y_1' &= y_2, \\ y_2' &= \frac{1}{2y_2}\left[P(x)\,\frac{d}{dx}\left(\frac{R_2^2(x)}{R_1^2(x)}\right) + \frac{R_2^2(x)}{R_1^2(x)}\,P'(x)\right] . \end{aligned}$$

4. The elliptic and hyperelliptic functions just discussed are special cases of Abelian functions.

Let $F(y,u) = 0$ be the equation of an algebraic curve, and let $R(y,u)$ be a rational function of y and u. Then

$$x = \int_{y_0}^{y} R(t,u)\ dt$$

is called an Abelian integral with respect to the curve $F(y,u) = 0$. Its inverse is an Abelian function; an Abelian function will satisfy a differential equation of the form

$$y' = [R(y,u)]^{-1} .$$

Suppose

$$F(y,u) \equiv \sum_{i=0}^{n} P_i(y)\ u^i,$$

where $P_i(y)$ is a polynomial in y. Then the system of rational differential equations satisfied by the Abelian function is

$$(2.5)\qquad y_1' = \left[R(y_1,y_2)\right]^{-1},$$

$$y_2' = - \frac{\sum_{i=o}^{n} y_2^i \frac{dP_i}{dy_1}}{R(y_1,y_2) \sum_{i=1}^{n} i P_i y_2^{i-1}} .$$

If in the definition of the Abelian integral we interchange x and y, we find that the inverse Abelian function satisfies a differential equation

$$y' = R(x,u).$$

Thus an inverse Abelian function will satisfy a system of rational differential equations of the form

$$(2.6) \qquad \begin{aligned} y_1' &= R(x,y_2), \\ y_2' &= \frac{- \sum_{i=o}^{n} P_i'(x) y_2^i}{\sum_{i=1}^{n} i P_i(x) y_2^{i-1}} . \end{aligned}$$

We now look at the problem from a more general standpoint. Consider any n meromorphic functions $f_i(x)$, i = 1,2,3,.....,n, with a common open region of existence. E. H. Moore (4, p. 50-52) defines a <u>finite realm of rationality</u>

$$R\{x\} \equiv R\left[f_1(x), f_2(x), \ldots\ldots, f_n(x)\right]$$

as the totality of rational functions of these n fundamental functions, the coefficients involved being constants. If the derivative of any function in R{x} is also in R{x}, Moore calls R{x} a <u>perfect realm of rationality</u>. J. F. Ritt uses the term <u>field</u> (5, p. 37) instead of <u>perfect realm of rationality</u>, and we will follow this usage.

By a differential form $E(y, y', y'', \ldots\ldots, y^{(m)})$ over the realm of rationality $R\{x\}$ is meant a polynomial in y and its first m derivatives with coefficients in $R\{x\}$. Setting the form equal to zero yields a differential equation.

An algebraically transcendental function with respect to the realm $R\{x\}$ is defined as a transcendental function which satisfies a differential equation $E(y, y', \ldots\ldots, y^{(m)}) = 0$, where E is a differential form over $R\{x\}$. A transcendental function which is not the solution of any such equation is called a transcendentally transcendental function with respect to $R\{x\}$.

Theorem: A necessary and sufficient condition that a function be a solution of a system of first order rational differential equations is that it be algebraic or algebraically transcendental with respect to the field $R[x]$.

1. It has already been shown that an algebraic function is a solution of a single rational differential equation.

2. An algebraically transcendental function will satisfy a differential equation of the type $E(y_1, y_1', y_1'', \ldots\ldots, y_1^{(m)}) = 0$, where E is a rational integral function of y_1 and its first m derivatives, with coefficients rational functions of x (with constant coefficients). It is assumed that $y_1^{(m)}$ actually appears in the form E.

Let

$$y_1' = y_2,$$
$$y_2' = y_3,$$
$$\ldots\ldots\ldots$$
$$y_m' = y_{m+1}.$$

Then consider $E(y_1,y_2,\ldots\ldots,y_{m+1}) = 0$ as a polynomial in y_{m+1}. We have

$$E(y_1,y_2,\ldots\ldots,y_{m+1}) \equiv \sum_{j=o}^{s} F_j(x,y_1,\ldots\ldots,y_m)\, y_{m+1}^j = 0,$$

where $F_j(x,y_1,\ldots\ldots,y_m)$ is a rational function of $x,y_1,\ldots\ldots,y_m$. Differentiating with respect to x, we obtain the equation

$$y'_{m+1} \sum_{j=1}^{s} j\, F_j\, y_{m+1}^{j-1} + \sum_{j=o}^{s} F'_j\, y_{m+1}^j = 0,$$

where

$$F'_j = \frac{\partial F_j}{\partial x} + \frac{\partial F_j}{\partial y_1} y'_1 + \ldots\ldots + \frac{\partial F_j}{\partial y_m} y'_m ,$$

which is again a rational function (of $x,y_1,\ldots\ldots,y_{m+1}$).

Then the system of rational differential equations will be

$$
\begin{aligned}
y'_1 &= y_2, \\
y'_2 &= y_3 \\
&\cdots\cdots \\
&\cdots\cdots \\
y'_m &= y_{m+1}, \\
y'_{m+1} &= - \frac{\sum_{j=o}^{s} F'_j\, y_{m+1}^j}{\sum_{j=1}^{s} j\, F_j\, y_{m+1}^{j-1}} ,
\end{aligned}
\tag{2.7}
$$

where

$$F'_j = \frac{\partial F_j}{\partial x} + \frac{\partial F_j}{\partial y_1} y_2 + \frac{\partial F_j}{\partial y_2} y_3 + \ldots\ldots + \frac{\partial F_j}{\partial y_m} y_{m+1} ,$$

and

$$j = 0,1,2,\ldots\ldots,s.$$

It is to be noted that in the special case s = 1, the last of equations (2.7) is not necessary. Then

$$
(2.8)\qquad
\begin{aligned}
y_1' &= y_2,\\
y_2' &= y_3,\\
&\cdots\cdots\\
&\cdots\cdots\\
y_{m-1}' &= y_m,\\
y_m' &= -\frac{F_o}{F_1}.
\end{aligned}
$$

3. To prove the necessity of the condition, consider any function which satisfies the equations (1.1). Since a rational function may be represented as the quotient of two polynomials, we may write equations (1.2) in the form

$$P_k(x,y_j,y_1^{(k)}) = 0, \quad j,k = 1,\ldots,n,$$

where P_k is a polynomial in x, $y_1^{(k)}$, and the y_j's. We have n equations in (n-1) unknowns $y_2,y_3,\ldots,y_n$ and may in general eliminate the latter. We consider the resultant of these n polynomials and are led to a polynomial

$$(2.9)\qquad Q(x,y_1,y_1^{(k)}) = 0, \quad k = 1,\ldots\ldots,n;$$

that is, to a differential equation, which may be considered as equivalent to the system (1.1). But a solution of (2.9) would by definition be algebraic or algebraically transcendental with respect to $R[x]$. This completes the proof of the theorem.

Moore states several theorems showing that a function which is algebraic or algebraically transcendental with respect to one field is algebraic or algebraically transcendental with respect to any field. (4, p. 54-55). However, in most investigations it is desirable to specify the field. (6, p. 19-20). As an example, consider the Mathieu function; such a function is usually exhibited as a solution of the second order differential equation

$$y'' + (4a - 16q \cos 2x)y = 0. \tag{2.10}$$

Here it would be advantageous to consider the field $R[\sin x, \cos x]$. If we wished to use the field of constants $R[1]$, the Mathieu functions would be considered as solutions of the much more complicated fourth order differential equation

$$y^2y^{(iv)} - 2yy'y''' - yy''^2 + 2y'^2y'' + 4y^2y'' + 16ay^3 = 0.$$

In order to treat equations such as (2.10) directly and in a general manner, we consider the realm of rationality $R\left[f_{11}(x), f_{21}(x), \ldots, f_{n1}(x)\right]$, where $f_{j1}(x)$ is any algebraic or algebraically transcendental function with respect to $R[x]$. We assume y_1 is exhibited as an algebraically transcendental function with respect to $R\left[f_{11}, f_{21}, \ldots, f_{n1}\right]$; the corresponding system of rational differential equations is required.

By definition y_1 satisfies a differential equation of the type

$$E(y_1, y_1', \ldots, y_1^{(m)}) = 0,$$

where E is a rational integral function of y_1 and its first m derivatives with coefficients rational functions of the $f_{j1}(x)$'s. We suppose that $y_1^{(m)}$ actually appears in the form E.

Let

$$y_1' = y_2,$$
$$y_2' = y_3,$$
$$\cdots\cdots$$
$$y_m' = y_{m+1}.$$

Consider $E(y_1, y_2, \ldots\ldots, y_{m+1}) = 0$ as a polynomial in y_{m+1}. We have

(2.11) $$E(y_1, y_2, \ldots\ldots, y_{m+1}) \equiv \sum_{k=o}^{N} R_k(f_{11}, f_{21}, \ldots\ldots, f_{n1}, y_1, y_2, \ldots\ldots, y_m) y_{m+1}^k = 0,$$

where R_k is a rational integral function of $y_1, y_2, \ldots\ldots, y_m$, with coefficients rational functions of the $f_{h1}(x)$'s.

By the preceding theorem $f_{h1}(x)$ may be considered as the solution of the system of equations

$$f'_{h1} = f_{h2},$$
$$f'_{h2} = f_{h3},$$
$$\cdots\cdots\cdots$$
$$f'_{hq} = f_{h(q+1)},$$
$$f'_{h(q+1)} = -\frac{\sum_{j=o}^{S_h} F'_{hj} f^j_{h(q+1)}}{\sum_{j=1}^{S_h} j F_{hj} f^{j-1}_{h(q+1)}},$$

where F_{hj} is a rational function of $x, f_{h1}, f_{h2}, \ldots\ldots, f_{hq}$, and

$$F'_{hj} = \frac{\partial F_{hj}}{\partial x} + \frac{\partial F_{hj}}{\partial f_{h1}} f_{h2} + \ldots\ldots + \frac{\partial F_{hj}}{\partial f_{hq}} f_{h(q+1)}.$$

Here $h = 1, 2, \ldots\ldots, n$. It is to be noted that strictly speaking the constant q should have been written q_h, since it will in general differ for different values of h. However, it should cause no confusion to omit the subscript.

If (2.11) is differentiated with respect to x and the resulting equation solved for y'_{m+1}, we have

$$y'_{m+1} = - \frac{\sum_{k=0}^{N} y_{m+1}^{k} \left[\frac{\partial R_k}{\partial f_{11}} f_{12} + \frac{\partial R_k}{\partial f_{21}} f_{22} + \cdots\cdots + \frac{\partial R_k}{\partial f_{n1}} f_{n2} \right]}{\sum_{k=1}^{N} k \, R_k \, y_{m+1}^{k-1}}$$

$$- \frac{\sum_{k=o}^{N} y_{m+1}^{k} \left[\frac{\partial R_k}{\partial y_1} y_2 + \frac{\partial R_k}{\partial y_2} y_3 + \cdots\cdots + \frac{\partial R_k}{\partial y_m} y_{m+1} \right]}{\sum_{k=o}^{N} k \, R_k \, y_{m+1}^{k-1}},$$

which is a rational function of $y_1, \ldots\ldots, y_{m+1}, f_{11}, \ldots\ldots, f_{n1}, f_{12}, \ldots\ldots, f_{n2}$.

The desired system of rational differential equations is then

$$
\begin{aligned}
& y'_1 = y_2, \\
& y'_2 = y_3, \\
(2.12) \qquad & \cdots\cdots \\
& y'_m = y_{m+1}, \\
& y'_{m+1} = - \frac{\sum_{k=o}^{N} y_{m+1}^{k} \left[\frac{\partial R_k}{\partial f_{11}} f_{12} + \cdots\cdots\cdots + \frac{\partial R_k}{\partial f_{n1}} f_{n2} \right]}{\sum_{k=1}^{N} k \, R_k \, y_{m+1}^{k-1}} \\
& \qquad - \frac{\sum_{k=o}^{N} y_{m+1}^{k} \left[\frac{\partial R_k}{\partial y_1} y_2 + \cdots\cdots\cdots + \frac{\partial R_k}{\partial y_m} y_{m+1} \right]}{\sum_{k=1}^{N} k \, R_k \, y_{m+1}^{k-1}}, \\
& f'_{h1} = f_{h2}, \\
& f'_{h2} = f_{h3}, \\
& \cdots\cdots\cdots
\end{aligned}
$$

$$f'_{h(q+1)} = - \frac{\sum_{j=o}^{S_h} F'_{hj} f^j_{h(q+1)}}{\sum_{j=1}^{S_h} j F_{hj} f^{j-1}_{h(q+1)}},$$

where

$$F'_{hj} = \frac{\partial F_{hj}}{\partial x} + \frac{\partial F_{hj}}{\partial f_{h1}} f_{h2} + \frac{\partial F_{hj}}{\partial f_{h2}} f_{h3} + \cdots\cdots + \frac{\partial F_{hj}}{\partial f_{hq}} f_{h(q+1)},$$

and $h = 1, 2, \ldots\ldots, n$.

Although most algebraic and algebraically transcendental functions will in practice arise as solutions of differential equations obtained by setting differential forms equal to zero, this is not always true. Consider the equation for the motion of the simple pendulum

(2.13) $$y'' + \frac{g}{L} \sin y = 0.$$

The expression on the left is not a differential form. However, the solution of this equation is an elliptic function and elliptic functions are algebraically transcendental with respect to the field $R[x]$. Differential forms are polynomials in the dependent variable and its first m derivatives, with coefficients rational functions of a given set of meromorphic functions. We suppose that each function of this set is algebraic or algebraically transcendental with respect to $R[x]$ and that we have constructed for each a system of rational differential equations of which it is a solution. (See list of common functions for which this has been done in the appendix). If now, instead of working with differential polynomials, we wish to allow these same functions of the dependent variable and its first m derivatives to enter into the differential equation, we may still set up an equivalent system of rational equations. For example, a system of rational equations equivalent to equation (2.13) would be

$$
\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
y_3' &= y_2 y_4, \\
y_4' &= -\, y_2 y_3 \;.
\end{aligned}
\tag{2.14}
$$

We now consider an equation made up to show in more detail the methods to be used:

$$y_1''^{\,2} + x^{1/3}\left[\sin^{-1}(y_1')^{2/5}\right](\mathrm{cn}\; y_1)^{1/2} + x^7 y_1^{-3/4} = 0. \tag{2.15}$$

Let

$$
\begin{aligned}
y_2 &= y_1', \\
y_3 &= y_1'' , \\
y_4 &= y_1^{1/4} , \\
y_5 &= x^{1/3} , \\
y_6 &= y_2^{2/5} , \\
y_7 &= \sin^{-1} y_6, \\
y_9 &= \mathrm{cn}\; y_1, \\
y_{11} &= y_9^{1/2} ,
\end{aligned}
$$

Then

$$
\begin{aligned}
y_4' &= \tfrac{1}{4}\, y_1^{-3/4}\, y_1' = \tfrac{1}{4}\, y_4^{-3}\, y_2, \\
y_5' &= \tfrac{1}{3}\, x^{-2/3} = \tfrac{1}{3}\, y_5^{-2} , \\
y_6' &= \tfrac{2}{5}\, y_2^{-3/5}\, y_2' = \tfrac{2}{5}\, y_2^{-1}\, y_3 y_6, \\
y_7' &= y_8 y_6' = \tfrac{2}{5}\, y_2^{-1}\, y_3 y_6 y_8,
\end{aligned}
$$

which defines the function y_8,

$$y_8' = y_6 y_8^3 y_6' = \frac{2}{5} y_2^{-1} y_3 y_6^2 y_8^3 ,$$

$$y_9' = - y_{10} y_1' = - y_2 y_{10}, \text{ which defines } y_{10},$$

$$y_{10}' = y_9(2k^2 y_9^2 + 1 - 2k^2)\, y_1' = y_2 y_9 (2k^2 y_9^2 + 1 - 2k^2),$$

$$y_{11}' = \frac{1}{2} y_9^{-1/2} y_9' = - \frac{1}{2} y_2 y_{10} y_{11}^{-1}.$$

Equation (2.15) becomes

$$y_3^2 + y_5 y_7 y_{11} + x^7 y_4^{-3} = 0.$$

Differentiating this with respect to x and solving for y_3, we have

$$y_3' = - \frac{1}{2} y_3^{-1} (y_5 y_7 y_{11}' + y_5 y_7' y_{11} + y_5' y_7 y_{11} + 7\, x^6 y_4^{-3} - 3\, x^7 y_4^{-4} y_4').$$

Thus equation (2.15) is replaced by a system of eleven rational differential equations in eleven dependent variables:

$$\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
y_3' &= \frac{1}{4} y_2 y_3^{-1} y_5 y_7 y_{10} y_{11}^{-1} - \frac{1}{5} y_2^{-1} y_5 y_6 y_8 y_{11} \\
&\quad - \frac{1}{6} y_3^{-1} y_5^{-2} y_7 y_{11} - \frac{7}{2} x^6 y_3^{-1} y_4^{-3} + \frac{3}{8} x^7 y_2 y_3^{-1} y_4^{-7}, \\
y_4' &= \frac{1}{4} y_4^{-3} y_2, \\
y_5' &= \frac{1}{3} y_5^{-2}, \qquad (2.16) \\
y_6' &= \frac{2}{5} y_2^{-1} y_3 y_6, \\
y_7' &= \frac{2}{5} y_2^{-1} y_3 y_6 y_8, \\
y_8' &= \frac{2}{5} y_2^{-1} y_3 y_6^2 y_8^3, \\
y_9' &= - y_2 y_{10}, \\
y_{10}' &= y_2 y_9 (2k^2 y_9^2 + 1 - 2k^2), \\
y_{11}' &= - \frac{1}{2} y_2 y_{10} y_{11}^{-1}.
\end{aligned}$$

In the appendix are listed systems of rational equations which are satisfied by various elementary functions. A set of rules will now be considered by which these systems may be used to build up systems corresponding to more complicated functions. In each case a simple example will first be given, after which a general formula will be laid down. Initial conditions for the new system will be given except in cases 1, 5, and 6, where the initial value of the new function is merely a constant multiple or a constant power of the initial value of the original function. By "function" will always be meant "function which satisfies a rational system".

1. <u>A constant multiplied by a function of x</u>. ($y_o = C\, y_1$).

Example; We find the system of rational differential equations satisfied by the function $y_o = 5 \sec x$ by setting $y_o = 5\, y_1$, where $y_1 = \sec x$ satisfies the rational system

$$y_1' = y_1 y_2,$$
$$y_2' = y_1^2.$$

Then

$$y_o' = 5\, y_1' = 5\, y_1 y_2 = y_o y_2,$$
$$y_2' = y_1^2 = \frac{1}{25}\, y_o^2.$$

In general, suppose $y_o = C\, y_1$, where y_1 is a solution of the rational system

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_n' = f_n(x, y_1, \ldots\ldots, y_n).$$

Then y_o is a solution of the system

$$y_o' = C\, f_1(x, C^{-1}y_o, y_2, \ldots\ldots, y_n),$$

$$y_2' = f_2(x, C^{-1}y_o, y_2, \ldots\ldots, y_n)$$

(2.17) $$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_n' = f_n(x, C^{-1}y_o, y_2, \ldots\ldots, y_n).$$

2. <u>The sum of n functions of x.</u>

Example: We find the system of rational differential equations satisfied by the function $y_o = \sinh^{-1}x + \cos x$ by setting $y_o = y_1 + y_3$, where $y_1 = \sinh^{-1}x$ satisfied the rational system

$$y_1' = y_2,$$
$$y_2' = -\,xy_2^{\;3},$$

with initial values

$$x_o = 0,$$
$$(y_1)_o = 0,$$
$$(y_2)_o = 1,$$

and where $y_3 = \cos x$ satisfies the rational system

$$y_3' = y_4,$$
$$y_4' = -\,y_3,$$

with initial values

$$x_o = 0,$$
$$(y_3)_o = 1,$$
$$(y_4)_o = 0.$$

Then

$$y_o' = y_1' + y_3' = y_2 + y_4,$$
$$y_2' = -\,x\,y_2^{\,3},$$

$$y_3' = y_4$$

$$y_4' = -y_3.$$

Initial values are

$$x_o = 0,$$

$$(y_o)_o = (y_1)_o + (y_3)_o = 1,$$

$$(y_2)_o = 1,$$

$$(y_3)_o = 1,$$

$$(y_4)_o = 0.$$

In general, suppose

$$y_{m_o} = \sum_{k=o}^{n-1} y_{m_o + m_1 + \cdots + m_k + 1},$$

where $y_{m_o + m_1 + \cdots + m_k + 1}$ is a solution of the rational system of $m_{(k+1)}$ equations

$$y'_{m_o + \cdots + m_k + i} = f_{m_o + \cdots + m_k + i}(x, y_{m_o + \cdots + m_k + j}),$$

with $i, j = 1, 2, \ldots, m_{(k+1)}$; suppose initial values are x_o and $(y_{m_o + \cdots + m_k + i})_o$. The rational system determining y_{m_o} will be

$$y'_{m_o} = f_{m_o+1}\left(x, (y_{m_o} - \sum_{k=1}^{n-1} y_{m_o + \cdots + m_k + 1}), y_{m_o+p}\right)$$

$$+ \sum_{k=1}^{n-1} f_{m_o + \cdots + m_k + 1}(x, y_{m_o + \cdots + m_k + j}),$$

where $j = 1, 2, \ldots, m_{(k+1)}$, and $p = 2, 3, \ldots, m_1$;

$$y'_{m_o + q} = f_{m_o + q}\left(x, (y_{m_o} - \sum_{k=1}^{n-1} y_{m_o + \cdots + m_k + 1}), y_{m_o+p}\right)$$

(2.18) for $p, q = 2, 3, \ldots, m_1$;

$$y'_{m_o+\cdots\cdots+m_k+i} = f_{m_o+\cdots\cdots+m_k+i}(x, y_{m_o+\cdots\cdots+m_k+j})$$

for $i,j = 1,2,\ldots\ldots,m_{(k+1)}$, and $k = 1,2,\ldots\ldots,(n-1)$.

The initial conditions are the same as for the original systems except that the value of $(y_{m_o+1})_o$ is superfluous, and a value of $(y_{m_o})_o$ is now needed.

$$(y_{m_o})_o = \sum_{k=o}^{n-1} (y_{m_o+\cdots\cdots+m_k+1})_o.$$

3. <u>The product of two functions of x</u>. $(y_o = y_1 y_{n+1})$.

Example: We find the system of rational differential equations satisfied by the function $y_o = (\cosh x)(\cos x)$, where $y_1 = \cosh x$ satisfies the rational system

$$y'_1 = y_2,$$

$$y'_2 = y_1,$$

with initial values

$$x_o = 0,$$

$$(y_1)_o = 1,$$

$$(y_2)_o = 0,$$

and where $y_3 = \cos x$ satisfies the system

$$y'_3 = y_4,$$

$$y'_4 = -y_3,$$

with initial values

$$x_o = 0,$$

$$(y_3)_o = 1,$$

$$(y_4)_o = 0.$$

Then $$y_o = y_1 y_3,$$

and the system of rational equations satisfied by y_o is

$$y_o' = y_1 y_3' + y_1' y_3 = y_o y_3^{-1} y_4 + y_2 y_3,$$

$$y_2' = y_1 = y_o y_3^{-1},$$

$$y_3' = y_4,$$

$$y_4' = - y_3,$$

where

$$x_o = 0,$$

$$(y_o)_o = (y_1)_o (y_3)_o = 1,$$

$$(y_2)_o = 0,$$

$$(y_3)_o = 1,$$

$$(y_4)_o = 0.$$

This particular system may be simplified by setting

$$Y_1 = y_o,$$

$$Y_2 = y_o y_3^{-1} y_4 + y_2 y_3,$$

$$Y_3 = 2\, y_2 y_4$$

$$Y_4 = 2\, y_o y_3^{-1} y_4 - 2\, y_2 y_3.$$

Then

$$Y_1' = Y_2,$$

$$Y_2' = Y_3,$$

$$Y_3' = Y_4,$$

$$Y_4' = - 4\, Y_1,$$

and

$$x_o = (Y_2)_o = (Y_3)_o = (Y_4)_o = 0,$$
$$(Y_1)_o = 1.$$

In general, suppose $y_o = y_1 y_{n+1}$, where y_1 and y_{n+1} are respectively solutions of the systems

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_n' = f_n(x, y_1, \ldots\ldots, y_n),$$

with initial values x_o, $(y_1)_o, \ldots\ldots, (y_n)_o$, and

$$y_{n+1}' = f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_{n+m}' = f_{n+m}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

with initial values x_o, $(y_{n+1})_o, \ldots\ldots, (y_{n+m})_o$. Then the system determining y_o will be

$$y_o' = y_o y_{n+1}^{-1} f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m})$$
$$+ y_{n+1} f_1(x, y_o y_{n+1}^{-1}, y_2, \ldots\ldots, y_n),$$
$$y_2' = f_2(x, y_o y_{n+1}^{-1}, y_2, \ldots\ldots, y_n),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(2.19) \qquad \cdots\cdots\cdots\cdots\cdots$$
$$y_n' = f_n(x, y_o y_{n+1}^{-1}, y_2, \ldots\ldots, y_n),$$
$$y_{n+1}' = f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$

$$y'_{n+m} = f_{n+m}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

with initial values x_o, $(y_o)_o = (y_1)_o (y_{n+1})_o$, $(y_2)_o, \ldots\ldots, (y_{n+m})_o$.

4. <u>The quotient of two functions of x</u>. $(y_o = \dfrac{y_1}{y_{n+1}})$.

Example: We find the system of rational differential equations satisfied by the function $y_o = \dfrac{\cosh x}{\cos x}$, where $y_1 = \cosh x$ satisfies the rational system

$$y'_1 = y_2,$$
$$y'_2 = y_1,$$

with initial values

$$x_o = 0,$$
$$(y_1)_o = 1,$$
$$(y_2)_o = 0,$$

and where $y_3 = \cos x$ satisfies the system

$$y'_3 = y_4,$$
$$y'_4 = - y_3,$$

with initial values

$$x_o = 0,$$
$$(y_3)_o = 1,$$
$$(y_4)_o = 0.$$

Then

$$y_o = \frac{y_1}{y_3},$$

and the system of rational equations satisfied by y_o is

$$y'_o = \frac{y_3 y'_1 - y_1 y'_3}{y_3^2} = \frac{y_2 y_3 - y_1 y_4}{y_3^2} = \frac{y_2 - y_o y_4}{y_3},$$

$$y_2' = y_1 = y_o y_3,$$

$$y_3' = y_4,$$

$$y_4' = -y_3,$$

where

$$x_o = 0,$$

$$(y_o)_o = \frac{(y_1)_o}{(y_3)_o} = 1,$$

$$(y_2)_o = 0,$$

$$(y_3)_o = 1,$$

$$(y_4)_o = 0.$$

In general, suppose $y_o = \dfrac{y_1}{y_{n+1}}$, where y_1 and y_{n+1} are respectively solutions of the systems

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$y_n' = f_n(x, y_1, \ldots\ldots, y_n),$$

with initial values x_o, $(y_1)_o, \ldots\ldots, (y_n)_o$, and

$$y_{n+1}' = f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_{n+m}' = f_{n+m}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

with initial values x_o, $(y_{n+1})_o, \ldots\ldots, (y_{n+m})_o$. Then the system determining y_o will be

$$y_o' = \frac{f_1(x, y_o y_{n+1}, y_2, \ldots\ldots, y_n) - y_o f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m})}{y_{n+1}},$$

$$y_2' = f_2(x, y_0 y_{n+1}, y_2, \ldots\ldots, y_n),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(2.20) \qquad y_n' = f_n(x, y_0 y_{n+1}, y_2, \ldots\ldots, y_n),$$

$$y_{n+1}' = f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_{n+m}' = f_{n+m}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

with initial values $x_o, (y_o)_o = \dfrac{(y_1)_o}{(y_{n+1})_o}, (y_2)_o, \ldots\ldots, (y_{n+m})_o$.

5. _A function of x raised to a power which is the reciprocal of an integer._ ($y_o = y_1^{1/q}$).

Example: We find the system of rational differential equations satisfied by the function $y_o = (\cos x)^{1/5}$, where $y_1 = \cos x$ satisfies the system of equations

$$y_1' = y_2,$$

$$y_2' = -y_1.$$

Then

$$y_o = y_1^{1/5},$$

and the system of rational equations satisfied by y_o is

$$y_o' = \frac{1}{5} y_1^{-4/5} y_1' = \frac{1}{5} y_o^{-4} y_2,$$

$$y_2' = -y_1 = -y_o^5.$$

In general, suppose $y_o = y_1^{1/q}$, where q is an integer and where y_o is a solution of the rational system

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$y_n' = f_n(x, y_1, \ldots\ldots, y_n).$$

Then y_o is a solution of the system

$$y_o' = q^{-1} y_o^{1-q} f_1(x, y_o^q, y_2, \ldots\ldots, y_n),$$

$$y_2' = f_2(x, y_o^q, y_2, \ldots\ldots, y_n),$$

(2.21) $$\cdots\cdots\cdots\cdots\cdots$$

$$y_n' = f_n(x, y_o^q, y_2, \ldots\ldots, y_n).$$

6. <u>A function of x raised to a constant power</u>. ($y_o = y_1^m$, where m is a constant not the reciprocal of an integer).

Example: We find the system of rational differential equations satisfied by the function $y_o = (\cos^{-1} x)^{\sqrt{2}}$, where $y_1 = \cos^{-1} x$ satisfies the system of equations

$$y_1' = y_2,$$

$$y_2' = x\, y_2^3.$$

Then

$$y_o = y_1^{\sqrt{2}},$$

and the system of rational equations satisfied by y_o is

$$y_o' = \sqrt{2}\, y_1^{\sqrt{2}-1} y_1' = \sqrt{2}\; y_o y_1^{-1} y_2,$$

$$y_1' = y_2,$$

$$y_2' = x\, y_2^3.$$

In general, suppose $y_o = y_1^m$, where m is a constant not the reciprocal of an integer and where y_1 is a solution of the rational system

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$y_n' = f_n(x, y_1, \ldots\ldots, y_n).$$

Then y_o is a solution of the system

$$
\begin{aligned}
y_o' &= m\, y_o y_1^{-1} f_1(x, y_1, \ldots\ldots, y_n), \\
y_1' &= f_1(x, y_1, \ldots\ldots, y_n), \\
&\; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\
y_n' &= f_n(x, y_1, \ldots\ldots, y_n).
\end{aligned}
\tag{2.22}
$$

7. <u>A function of x which is raised to a power which is a function of x.</u> $(y_o = y_1^{y_{n+1}})$.

Example: We find the system of rational differential equations satisfied by the function $y_o = (\cos mx)^{\sinh^{-1} nx}$, where $y_1 = \cos mx$ satisfies the rational system

$$y_1' = m\, y_2,$$

$$y_2' = -\, m\, y_1,$$

with initial values

$$x_o = 0,$$

$$(y_1)_o = 1,$$

$$(y_2)_o = 0,$$

and where $y_3 = \sinh^{-1} nx$ satisfies the system

$$y_3' = n\, y_4,$$

$$y_4' = -\, n^2\, x y_4^3\,,$$

with initial values

$$x_o = 0,$$

$$(y_3)_o = 0,$$

$$(y_4)_o = 1.$$

Then

$$y_o = y_1^{\,y_3},$$

and

$$\ln y_o = y_3 \ln y_1.$$

We let $z = \ln y_1$. Then

$$\ln y_o = y_3 z,$$

and y_o is a solution of the system of equations

$$y_o' = y_o(y_3 z' + y_3' z) = y_o(y_1^{-1} y_3 y_1' + n y_4 z) = m y_o y_1^{-1} y_2 y_3 + n y_o y_4 z,$$

$$z' = y_1^{-1} y_1' = m y_1^{-1} y_2,$$

$$y_1' = m y_2,$$

$$y_2' = - m y_1,$$

$$y_3' = n y_4,$$

$$y_4' = - n^2 x y_4^3 ,$$

where

$$x_o = 0,$$

$$(y_o)_o = (y_1)_o^{(y_3)_o} = 1$$

$$(y_1)_o = 1$$

$$(y_2)_o = (y_3)_o = 0,$$

$$(y_4)_o = 1$$

$$z_o = \ln (y_1)_o = \ln 1 = 0.$$

In general suppose $y_o = y_1^{\,y_{n+1}}$, where y_1 and y_{n+1} are respectively solutions of the rational systems

$$y_1' = f_1(x,y_1,\ldots\ldots,y_n),$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_n' = f_n(x,y_1,\ldots\ldots,y_n),$$

with initial values x_o, $(y_1)_o,\ldots\ldots,(y_n)_o$, and

$$y_{n+1}' = f_{n+1}(x,y_{n+1},\ldots\ldots,y_{n+m}),$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_{n+m}' = f_{n+m}(x,y_{n+1},\ldots\ldots,y_{n+m}),$$

with initial values x_o, $(y_{n+1})_o$, $\ldots\ldots,(y_{n+m})_o$. Then y_o is a solution of the system

$$y_o' = y_o y_1^{-1}(y_{n+1}f_1(x,y_1,\ldots\ldots,y_n) + y_1 z\, f_{n+1}(x,y_{n+1},\ldots,y_{n+m})),$$

$$z' = y_1^{-1} f_1(x,y_1,\ldots\ldots,y_n),$$

$$y_1' = f_1(x,y_1,\ldots\ldots,y_n),$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_n' = f_n(x,y_1,\ldots\ldots,y_n),$$

$$y_{n+1}' = f_{n+1}(x,y_{n+1},\ldots\ldots,y_{n+m}),$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_{n+m}' = f_{n+m}(x,y_{n+1},\ldots\ldots,y_{n+m}),$$

with initial values x_o, $(y_o)_o = (y_1)_o^{(y_{n+1})_o}$, $z_o = \ln (y_1)_o$, $(y_1)_o,\ldots$ $\ldots,(y_{n+m})_o$.

8. <u>A function of a function of x</u>. $(y_1 = y_1[y_{n+1}(x)])$.

Example: We find the system of rational differential equations satisfied by the function $y_1 = \cosh(\sec^{-1} x)$, where $y_3 = \sec^{-1} x$ satisfies the system

$$y_3' = x^{-1} y_4^{-1},$$

$$y_4' = x\, y_4^{-1},$$

with initial values

$$x_0 = \sqrt{2},$$

$$(y_3)_0 = \pi/4,$$

$$(y_4)_0 = 1,$$

and where $y_1 = \cosh y_3$ satisfies the system

$$\frac{dy_1}{dy_3} = y_2,$$

$$\frac{dy_2}{dy_3} = y_1,$$

with initial values

$$(y_3)_0 = \pi/4,$$

$$(y_1)_0 = \cosh \pi/4,$$

$$(y_2)_0 = \sinh \pi/4.$$

Then y_1 is a solution of the single rational system

$$y_1' = \frac{dy_1}{dy_3} y_3' = x^{-1} y_2 y_4^{-1},$$

$$y_2' = \frac{dy_2}{dy_3} y_3' = x^{-1} y_1 y_4^{-1},$$

$$y_3' = x^{-1} y_4^{-1},$$

$$y_4' = xy_4^{-1},$$

with initial values

$$x_o = \sqrt{2},$$

$$(y_1)_o = \cosh \pi/4,$$

$$(y_2)_o = \sinh \pi/4,$$

$$(y_3)_o = \pi/4,$$

$$(y_4)_o = 1.$$

However, it is to be noted that y_3 does not appear in the system of equations, so that in this particular example the equation for y_3' is superfluous. The simplest system of rational equations satisfied by y_1 is

$$y_1' = x^{-1}y_2y_4^{-1},$$

$$y_2' = x^{-1}y_1y_4^{-1},$$

$$y_4' = xy_4^{-1}.$$

In general, suppose $y_1 = y_1\left[y_{n+1}(x)\right]$, where y_{n+1} is a solution of the rational system

$$y_{n+1}' = f_{n+1}(x,y_{n+1},\ldots\ldots,y_{n+m}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_{n+m}' = f_{n+m}(x,y_{n+1},\ldots\ldots,y_{n+m}),$$

with initial values x_o, $(y_{n+1})_o,\ldots\ldots,(y_{n+m})_o$, and where y_1 is a solution of the rational system

$$\frac{dy_1}{dy_{n+1}} = f_1(y_{n+1},y_1,y_2,\ldots\ldots,y_n),$$

. . . .

. . . .

$$\frac{dy_n}{dy_{n+1}} = f_n(y_{n+1}, y_1, y_2, \ldots\ldots, y_n),$$

with initial values $(y_{n+1})_o$, $(y_1)_o, \ldots\ldots, (y_n)_o$. Then y_1 is a solution of the single rational system

$$y_1' = f_1(y_{n+1}, y_1, \ldots\ldots, y_n)\ f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

. . . .

. . . .

$$y_n' = f_n(y_{n+1}, y_1, \ldots\ldots, y_n)\ f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

(2.24) $$y_{n+1}' = f_{n+1}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

. . . .

. . . .

$$y_{n+m}' = f_{n+m}(x, y_{n+1}, \ldots\ldots, y_{n+m}),$$

with initial values x_o, $(y_1)_o, \ldots\ldots, (y_{n+m})_o$.

It is to be particularly noted that in rules 2, 3, 4, and 7, the equations of the functions to be combined must have the same initial values x_o, and that in rule 8 the initial value $(y_{n+1})_o$ of the principal dependent variable in the first system must be the same as the initial value of the independent variable in the second system.

As far as the previous investigations indicate, it might be supposed that when working with systems of rational differential equations it would always be advantageous to work with a system of least possible order. However, if an electronic computing machine were used this would not necessarily be the case. For example, the elliptic function dn (u,k) satisfies the system

$$(2.25)\qquad \begin{aligned} y_1' &= -y_2, \\ y_2' &= y_1(k^2 - 2 + 2y_1^2). \end{aligned}$$

It also satisfies the third order system

$$\begin{aligned} y_1' &= -k^2 y_2 y_3, \\ y_2' &= y_1 y_3, \\ y_3' &= -y_1 y_2. \end{aligned}$$

This may be further simplified by setting $Y_2 = ky_2$ and $Y_3 = ky_3$. Then

$$\begin{aligned} y_1' &= -Y_2 Y_3, \\ Y_2' &= y_1 Y_3, \\ Y_3' &= -y_1 Y_2. \end{aligned}$$

A set of initial conditions is

$$\begin{aligned} u_o &= 0, \\ (y_1)_o &= 1, \\ (Y_2)_o &= 0, \\ (Y_3)_o &= k. \end{aligned}$$

This system could quite conceivably be easier to handle in an electronic computing machine than system (2.25).

As a further example, consider the system

$$(2.27)\qquad \begin{aligned} y_1' &= y_1 y_2(y_3^2 - 2y_3y_4 + y_4^2 + k), \\ y_2' &= y_2^2(y_3^2 - 2y_3y_4 + y_4^2 + k), \\ y_3' &= y_1 y_4, \end{aligned}$$

$$y_4' = - y_1 y_3.$$

We let

$$y_5 = y_3^2 - 2\, y_3 y_4 + y_4^2 + k.$$

Then

$$\begin{aligned} y_5' &= 2\, y_3 y_3' - 2\, y_3' y_4 - 2\, y_3 y_4' + 2\, y_4 y_4' \\ &= 2\, y_3 y_1 y_4 - 2\, y_1 y_4^2 + 2\, y_1 y_3^2 - 2\, y_1 y_3 y_4 \\ &= 2\, y_1 (y_3^2 - y_4^2). \end{aligned}$$

Let

$$y_6 = \frac{y_3^2 - y_4^2}{4}.$$

Then

$$y_6' = \frac{1}{4} (2\, y_1 y_3 y_4 + 2\, y_1 y_3 y_4) = y_1 y_3 y_4.$$

Thus we have the sixth order system of equations

$$\begin{aligned} y_1' &= y_1 y_2 y_5, \\ y_2' &= y_2^2 y_5, \\ y_3' &= y_1 y_4, \\ y_4' &= - y_1 y_3, \\ y_5' &= 8\, y_1 y_6, \\ y_6' &= y_1 y_3 y_4. \end{aligned} \tag{2.28}$$

It might be easier to set up an electronic computing machine to solve these equations than to set it up to solve equations (2.27), because in equations (2.28) each derivative is obtained by multiplying either two or three quantities together; the operations in equations (2.27) are much more varied.

However, it is not always possible to simplify a system of equations by increasing the order as was done in the preceding examples. The derivatives of rational functions are likely in general to be more complicated than the functions themselves, and setting up a new dependent variable as a rational function of the others may complicate rather than simplify the system.

III. APPLICATIONS TO PROBLEMS NOT INVOLVING DIFFERENTIAL EQUATIONS

It has been shown that differential equations of quite wide generality may be replaced by equivalent rational systems, which in turn might be solved numerically more easily than the original equations. However, the idea might also be applied to the evaluation of any formula $y = f(x, a_1, a_2, \ldots\ldots, a_n)$, where f, considered as a function of x, may be expressed as the solution of a system of rational differential equations.

As a simple example consider the formula for the area of the segment of a circle cut off by a chord x units from the center:

$$y_o = \tfrac{1}{2}\pi r^2 - x(r^2 - x^2)^{\frac{1}{2}} - r^2 \sin^{-1}\left(\frac{x}{r}\right). \tag{3.1}$$

We let $Y_2 = \sin^{-1}(\frac{x}{r})$, and by formula 7 in the appendix,

$$Y_2' = r^{-1} Y_1,$$
$$Y_1' = x r^{-2} Y_1^3.$$

Then

$$y_o = \tfrac{1}{2}\pi r^2 - r x Y_1^{-1} - r^2 Y_2,$$

and

$$y_o' = -\frac{rY_1 - r^{-1} Y_1^3 x^2}{Y_1^2} - rY_1 = -\frac{r^2 + (-x^2 + r^2)\, Y_1^2}{rY_1} = -\,2\, rY_1^{-1}.$$

The desired system of equations is then

$$\begin{aligned} y_o' &= -\,2\, rY_1^{-1}, \\ Y_1' &= r^{-2} x Y_1^3. \end{aligned} \tag{3.2}$$

This system may be further simplified by setting $y_1 = -\, rY_1^{-1}$. Then

$$y_o' = 2\, y_1, \tag{3.3}$$

$$y_1' = - xy_1^{-1},$$

and a set of initial conditions would be

$$x_o = 0,$$

$$(y_o)_o = \frac{1}{2}\pi r^2,$$

$$(y_1)_o = - r.$$

The actual numerical integration will now be carried out for the case $r = 10$ by Milne's method. (3, p. 455-460). The formulas used in this method,

$$(3.4) \quad y_n = y_{n-4} + \frac{4h}{3}(2\, y_{n-1}' - y_{n-2}' + 2\, y_{n-3}'),$$

$$(3.5) \quad y_n = y_{n-2} + \frac{h}{3}\,(y_n' + 4\, y_{n-1}' + y_{n-2}'),$$

$$(3.6) \quad y_1 = y_o + \frac{h}{24}(-\, y_2' + 13\, y_1' + 13\, y_o' - y_{-1}'),$$

$$(3.7) \quad y_n = y_{n-3} + \frac{h}{24}(8\, y_n' + 31\, y_{n-1}' + 21 y_{n-2}' + 13\, y_{n-3}' - y_{n-4}'),$$

are derived from Newton's interpolation formula

$$P(x) = u_o + \Delta u_o \frac{x}{h} + \Delta^2 u_o \frac{x(x-h)}{2!h^2} + \Delta^3 u_o \frac{x(x-h)(x-2h)}{3!h^3} + \Delta^4 u_o \frac{x(x-h)(x-2h)(x-3h)}{4!h^4},$$

which is a polynomial of fourth degree fitted to the set of five points $(0,u_o)$, (h,u_1), $(2h,u_2)$, $(3h,u_3)$, $(4h,u_4)$.

To simplify the notation, equations (3.3) are transformed by letting $v = y_o$ and $w = y_1$. Suppose the value of v for $x = 1$ is desired correct to six significant figures. The differential equations and initial conditions are

$$v' = 2w,$$

$$w' = -xw^{-1},$$

$$x_0 = 0, \tag{3.8}$$

$$w_0 = -10,$$

$$v_0 = 50\pi. \quad (v_0 = 157.0796, \text{ correct to 4 decimal places}).$$

We take $\Delta x = h = 0.1$ and apply formulas (3.5) and (3.6) to correct the values of w_{-1}, w_1, w_2, v_{-1}, v_1, and v_2 given by the approximate formulas

$$w_{-1} = w_0 - hw'_0,$$

$$w_1 = w_0 + hw'_0,$$

$$w_2 = w_0 + 2\,hw'_0,$$

$$v_{-1} = v_0 - hv'_0,$$

$$v_1 = v_0 + hv'_0,$$

$$v_2 = v_0 + 2\,hv'_0.$$

The results, placed in tabular form, are as follows:

	First Approximation				Second Approximation		
x	w	w'	v	v'	w	w'	v
-0.1	-10	-0.01	159.0796	-19.999	-9.99950	-0.01000	159.0796
0.0	-10	0.00	157.0796	-20.000	-10.00000	0.00000	157.0796
0.1	-10	0.01	155.0796	-19.999	-9.99950	0.01000	155.0796
0.2	-10	0.02	153.0796	-19.996	-9.99800	0.02000	153.0799

Formula (3.4) is now used for integrating ahead, and at each step formula (3.5) is used as a check. Every third value is checked by formula (3.7). The results, placed in tabular form, are as follows:

x	w	w'	w(checked 3.5)	w(checked 3.7)
0.3	-9.99550	0.03001	-9.99550	
0.4	-9.99200	0.04003	-9.99200	-9.99200
0.5	-9.98749	0.05006	-9.98749	
0.6	-9.98199	0.06011	-9.98199	
0.7	-9.97547	0.07017	-9.97547	-9.97547
0.8	-9.96795	0.08026	-9.96795	
0.9	-9.95941	0.09037	-9.95942	
1.0	-9.94988	0.10050	-9.94988	-9.94988

x	v	v'	v(checked 3.5)	v(checked 3.7)
0.3	151.0805	-19.99100	151.0805	
0.4	149.0817	-19.98400	149.0818	149.0817
0.5	147.0837	-19.97498	147.0838	
0.6	145.0868	-19.96398	145.0869	
0.7	143.0910	-19.95094	143.0911	143.0911
0.8	141.0968	-19.93590	141.0968	
0.9	139.1040	-19.91884	139.1040	
1.0	137.1131	-19.89976	137.1131	137.1131

Thus at $x = 1$, $v = 137.113$, correct to six significant figures.

Another simple formula, which however leads to a more complicated system of differential equations, is the formula for an angle y_o of a triangle when the opposite side x and the adjacent sides b and c are given:

$$y_o = 2\cos^{-1}\frac{1}{2}\sqrt{\frac{(b+c)^2 - x^2}{bc}} \tag{3.9}$$

Let

$$y_1 = \frac{1}{2}\sqrt{\frac{(b+c)^2 - x^2}{bc}}\,.$$

Then

$$y_1' = \frac{-x}{4bcy_1}\,.$$

Then y_o is a solution of the system of equations

$$y_o' = -\,2\,y_2 y_1',$$

$$y_2' = y_1 y_2^3 y_1'.$$

Thus the desired system of equations is

$$y_o' = \frac{x}{2bc}\,\frac{y_2}{y_1},$$

(3.10)
$$y_1' = \frac{-x}{4bc}\,\frac{1}{y_1},$$

$$y_2' = \frac{-x}{4bc}\,y_2^3,$$

and a set of initial values is

$$x_o = (b^2 + c^2)^{\frac{1}{2}},$$

$$(y_o)_o = \pi/2,$$

$$(y_1)_o = \frac{1}{2}\sqrt{2},$$

$$(y_2)_o = \sqrt{2}\,.$$

In a similar manner, considering the trigonometric formula

(3.11)
$$y_1^2 = a^2 + b^2 - 2\,ab\cos x,$$

we let

$$y_2 = -\,ab\cos x,$$

and exhibit y_1 as a solution of the system of rational equations

$$y_1' = y_1^{-1} y_3,$$

(3.12)
$$y_2' = y_3,$$

$$y_3' = - y_2.$$

A set of initial values is

$$x_o = \pi/2,$$

$$(y_1)_o = (a^2 + b^2)^{\frac{1}{2}},$$

$$(y_2)_o = 0,$$

$$(y_3)_o = ab.$$

As a further example, consider the problem of obtaining an approximate real root to the polynomial equation

(3.13) $$\sum_{i=o}^{n} a_i y^i = 0.$$

Consider the function

(3.14) $$x = \sum_{i=o}^{n} a_i y^i .$$

Then

(3.15) $$y' = \left[\sum_{i=1}^{n} i a_i y^{i-1} \right]^{-1} .$$

An approximation to the root will be obtained by solving this differential equation for y when x = 0. Suitable initial values may be obtained from equation (3.14).

Various algebraically transcendental equations may also be solved numerically by the same method. For example, consider the equation

(3.16) $$\tanh y + \cot y = 0.$$

Let

(3.17)
$$x = \tanh y_o + \cot y_o,$$
$$y_1 = \tanh y_o.$$

Then

$$\frac{dx}{dy_o} = 1 - \tanh^2 y_o - \cot^2 y_o - 1 = - y_1^{\;2} - (x - y_1)^2.$$

The system of rational differential equations determining y_o is then

(3.18)
$$y_o' = \frac{-1}{2y_1^{\;2} - 2xy_1 + x^2},$$
$$y_1' = \frac{y_1^{\;2} - 1}{2y_1^{\;2} - 2xy_1 + x^2}.$$

Suitable initial conditions are obtained from equations (3.17).

Many definite integrals may be evaluated numerically by use of systems of rational differential equations. For example, consider the elliptic integral

(3.19)
$$y = \int_0^{0.3} \frac{dS}{\sqrt{(S^2 - 1)(3S^2 - 4)}} .$$

Let

$$y_o = \int_0^{x} \frac{dS}{\sqrt{(S^2 - 1)(3S^2 - 4)}} ,$$

and let

$$y_1 = y_o' = \left[(x^2 - 1)(3x^2 - 4) \right]^{-\frac{1}{2}} .$$

The system of rational differential equations will be

(3.20)
$$y_o' = y_1,$$
$$y_1' = (7x - 6x^3)y_1^{\;3},$$

with initial values

$$x_o = 0,$$
$$(y_o)_o = 0,$$
$$(y_1)_o = \frac{1}{2} .$$

The definite integral (3.19) will be obtained by solving this system numerically for y_o when $x = 0.3$.

It will be no more difficult to obtain an approximation to a definite elliptic integral such as

$$(3.21) \qquad y_o = \int_0^{0.75} \frac{dS}{\sqrt{S^4 - 6S^2 + 4}},$$

where the polynomial under the radical in the integrand has no rational root. The solution will be the value of y_o when $x = 0.75$ obtained from the system of equations

$$(3.22) \qquad \begin{aligned} y_o' &= y_1, \\ y_1' &= (6x - 2x^3)y_1{}^3, \end{aligned}$$

with initial values

$$\begin{aligned} x_o &= 0, \\ (y_o)_o &= 0, \\ (y_1)_o &= \frac{1}{2} . \end{aligned}$$

Hyperelliptic integrals offer no new difficulty. The hyperelliptic integral

$$(3.23) \qquad y_o = \int_0^{0.5} \frac{dS}{\sqrt{S^5 + S^4 - 6S^3 - 6S^2 + 4S + 4}}$$

will be obtained as the solution when $x = 0.5$ of the system of equations

$$(3.24) \qquad \begin{aligned} y_o' &= y_1, \\ y_1' &= (-\frac{5}{2}x^4 - 2x^3 + 9x^2 + 6x - 2)y_1{}^3, \end{aligned}$$

with initial values

$$x_o = 0,$$

$$(y_0)_0 = 0,$$

$$(y_1)_0 = \frac{1}{2}.$$

As a final application we consider the problem of constructing tables of prolate spheroidal functions which arise as solutions within a prolate spheroid of the equation

$$\Delta^2 W + k^2 W = 0$$

in certain physical problems. Stratton, Morse, Chu, and Hutner (7, p. 61-63) define the prolate spheroidal functions $S_{mL}^{(1)}(c, \cos\theta)$, which are solutions of the differential equation

$$v'' + (\cot\theta)v' - (A + c^2\cos^2\theta + m^2\csc^2\theta)v = 0. \tag{3.25}$$

In this equation $c = \frac{1}{2}dk$, where d is the distance between the foci of the generating ellipse of the spheroid, and A is a separation constant determined by the conditions of finiteness and periodicity over the surface of the spheroid. These authors have shown how to actually evaluate some of these prolate spheroidal functions in terms of the associated Legendre functions by use of tables which they have constructed. They stated that some time in the future they hoped to construct tables of the functions themselves, but that it had taken many years to complete the tables of coefficients which they were publishing, and they intimated that the complete tables would probably not be forthcoming for many years.

As a special example we consider the particular function $S_{12}^{(1)}(2, \cos\theta)$ and show how tables of this function might be constructed by means of an electronic computing machine designed to solve systems of rational differential equations by numerical methods.

$S_{12}^{(1)}(2, \cos\theta)$ is a solution of the differential equation

$$(3.26) \qquad v'' + (\cot\theta)v' - (-13.88150 + 4\cos^2\theta + \csc^2\theta)v = 0.$$

We let $\sin\theta = s$, $\cos\theta = t$, and $v' = w$. Then the system of rational differential equations equivalent to the equation (3.26) is

$$(3.27) \qquad \begin{aligned} v' &= w, \\ w' &= (-13.88150 + 4t^2 + s^{-2})v - s^{-1}tw, \\ s' &= t, \\ t' &= -s, \end{aligned}$$

and a set of initial values is

$$\begin{aligned} \theta_o &= \pi/2, \\ v_o &= -1.5, \\ w_o &= 0, \\ s_o &= 1, \\ t_o &= 0. \end{aligned}$$

We take $\Delta\theta = -1^\circ = -0.0174533$ radians and solve equations (3.27) numerically by Milne's method. The following table contains the results of these calculations:

θ	Variable	1st Approx.	2nd Approx.	3rd Approx.
91°	s_{-1}	1.000000	0.999848	0.999848
90°	s_o	1.000000	1.000000	1.000000
89°	s_1	1.000000	0.999848	0.999848
88°	s_2	1.000000	0.999391	0.999391
87°	s_3	0.998630	0.998630	0.998630
86°	s_4	0.997564	0.997564	
85°	s_5	0.996195	0.996195	

θ	Variable	1st Approx.	2nd Approx.	3rd Approx.
84°	s_6	0.994522	0.994522	0.994522
91°	t'_{-1}	-1.000000	-1.000000	-0.999848
90°	t'_o	-1.000000	-1.000000	-1.000000
89°	t'_1	-1.000000	-0.999848	-0.999848
88°	t'_2	-1.000000	-0.999391	-0.999391
87°	t'_3	-0.998630	-0.998630	
86°	t'_4	-0.997564	-0.997564	
85°	t'_5	-0.996195	-0.996195	
84°	t'_6	-0.994522	-0.994522	
91°	t_{-1}	-0.017453	-0.017452	-0.017452
90°	t_o	0.000000	0.000000	0.000000
89°	t_1	0.017453	0.017452	0.017452
88°	t_2	0.034907	0.034899	0.034899
87°	t_3	0.052336	0.052336	0.052336
86°	t_4	0.069756	0.069756	
85°	t_5	0.087155	0.087156	
84°	t_6	0.104528	0.104528	0.104528
91°	s'_{-1}	-0.017453	-0.017452	-0.017452
90°	s'_o	0.000000	0.000000	0.000000
89°	s'_1	0.017453	0.017452	0.017452
88°	s'_2	0.034907	0.034899	0.034899
87°	s'_3	0.052336	0.052336	
86°	s'_4	0.069756	0.069756	
85°	s'_5	0.087155	0.087156	
84°	s'_6	0.104528	0.104528	

θ	Variable	1st Approx.	2nd Approx.	3rd Approx.	4th Approx
91°	w_{-1}	0.337176	0.337163	0.337037	0.337037
90°	w_o	0.000000	0.000000	0.000000	0.000000
89°	w_1	-0.337176	-0.337353	-0.337036	-0.337037
88°	w_2	-0.674352	-0.674642	-0.672873	-0.672878
87°	w_3	-1.006331	-1.006332	-1.006332	
86°	w_4	-1.336218	-1.336219		
85°	w_5	-1.661373	-1.661374		
84°	w_6	-1.980655	-1.980656	-1.980656	
91°	w'_{-1}	19.325850	19.287773	19.287961	19.287961
90°	w'_o	19.322250	19.322250	19.322250	19.322250
89°	w'_1	19.325850	19.287790	19.287960	19.287960
88°	w'_2	19.336660	19.185053	19.185223	19.185223
87°	w'_3	19.014524	19.014524		
86°	w'_4	18.776612	18.776612		
85°	w'_5	18.472571	18.472571		
84°	w'_6	18.103761	18.103761		
91°	v_{-1}	-1.500000	-1.497055	-1.497058	-1.497058
90°	v_o	-1.500000	-1.500000	-1.500000	-1.500000
89°	v_1	-1.500000	-1.497056	-1.497058	-1.497058
88°	v_2	-1.500000	-1.488224	-1.488242	-1.488242
87°	v_3	-1.473584	-1.473584	-1.473584	
86°	v_4	-1.453135	-1.453135		
85°	v_5	-1.426969	-1.426969		
84°	v_6	-1.395177	-1.395176	-1.395176	
91°	v'_{-1}	0.337163	0.337037	0.337037	0.337037

θ	Variable	1st Approx.	2nd Approx.	3rd Approx.	4th Approx
90°	v'_0	0.000000	0.000000	0.0000000	0.000000
89°	v'_1	-0.337353	-0.337036	-0.337037	-0.337037
88°	v'_2	-0.674642	-0.672873	-0.672878	-0.672878
87°	v'_3	-1.006331	-1.006332		
86°	v'_4	-1.336218	-1.336219		
85°	v'_5	-1.661373	-1.661374		
84°	v'_6	-1.980655	-1.980656		

The values of $S_{12}^{(1)}(2,\cos\theta)$ taken from the above computations are as follows:

θ	$S_{12}^{(1)}(2,\cos\theta)$
90°	-1.50000
89°	-1.49706
88°	-1.48824
87°	-1.47358
86°	-1.45313
85°	-1.42697
84°	-1.39518

In the tables of Stratton, Morse, Chu, and Hutner, (7, p. 99), we find that

$$(3.28)\quad S_{12}^{(1)}(2,\cos\theta) = 0.12740\,P_1^1(\cos\theta) + 1.02938\,P_3^1(\cos\theta) - 0.043462\,P_5^1(\cos\theta) + 0.00082742\,P_7^1(\cos\theta) - 0.00000931\,P_9^1(\cos\theta) + 0.00000007\,P_{11}^1(\cos\theta).$$

Upon substituting particular values of θ, we obtain the following table:

θ	$S_{12}^{(1)}(2,\cos\theta)$
90°	-1.50000
89°	-1.49705
88°	-1.48824
87°	-1.47359
86°	-1.45313
85°	-1.42697
84°	-1.39518

Only two of these values are different from those found by the other method, and this difference in each case is only one unit in the fifth decimal place. The accuracy of the entries in the last table is limited by the number of significant figures given in the tables of coefficients used to obtain equation (3.28).

It is to be noted that by the method of numerical integration we automatically obtain values of $\frac{d}{d\theta} S_{12}^{(1)}(2,\cos\theta)$, so that tables of the derivative function could be constructed at the same time as tables of the function itself. The values for $\frac{d}{d\theta} S_{12}^{(1)}(2,\cos\theta)$ obtained in this manner and also the values obtained from the equation

$$(3.29)\quad \frac{d}{d\theta} S_{12}^{(1)}(2,\cos\theta) = 0.12740 \frac{d}{d\theta} P_1^1(\cos\theta) + 1.02938 \frac{d}{d\theta} P_3^1(\cos\theta) - 0.043462 \frac{d}{d\theta} P_5^1(\cos\theta) + 0.00082742 \frac{d}{d\theta} P_7^1(\cos\theta) - 0.00000931 \frac{d}{d\theta} P_9^1(\cos\theta) + 0.00000007 \frac{d}{d\theta} P_{11}^1(\cos\theta)$$

are shown in the following table:

θ	$\frac{d}{d\theta} S_{12}^{(1)}(2, \cos\theta)$ (Numerical Integration)	$\frac{d}{d\theta} S_{12}^{(1)}(2, \cos\theta)$ (by (3.29))
90°	0.00000	0.00000
89°	-0.33704	-0.33703
88°	-0.67288	-0.67288
87°	-1.00633	-1.00633
86°	-1.33622	-1.33621
85°	-1.66137	-1.66137
84°	-1.98066	-1.98065

Here again the difference is at most one unit in the fifth decimal place.

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APPENDIX

SYSTEMS OF RATIONAL DIFFERENTIAL EQUATIONS SATISFIED BY VARIOUS ELEMENTARY FUNCTIONS AND FUNCTIONS SATISFYING NO SUCH SYSTEM

1. $y = x^{n}$

$y' = nx^{-1}y$ $\qquad x_o = 1$

$y_o = 1$

2. $y = \log_a x$

$y' = (x \ln a)^{-1}$ $\qquad x_o = a$

$y_o = 1$

3. $y = a^{mx}$

$y' = my(\ln a)$ $\qquad x_o = 0$

$y_o = 1$

4. $y_1 = \begin{matrix} \sin mx \\ \cos mx \end{matrix}$

	sin mx	cos mx
$y_1' = my_2$	$x_o = 0$	$x_o = 0$
$y_2' = -\,my_1$	$(y_1)_o = 0$	$(y_1)_o = 1$
	$(y_2)_o = 1$	$(y_2)_o = 0$

5. $y = \begin{matrix} \tan mx \\ \operatorname{ctn} mx \end{matrix}$

	tan mx	ctn mx
$y' = \pm\, m(1 + y^2)$	(+ sign)	(- sign)
	$x_o = 0$	$x_o = \pi/2m$
	$y_o = 0$	$y_o = 0$

6. $y_1 = \begin{matrix} \sec mx \\ \csc mx \end{matrix}$

	sec mx	csc mx
$y_1' = my_1y_2$	$x_o = 0$	$x_o = \pi/2m$
$y_2' = my_1{}^2$	$(y_1)_o = 1$	$(y_1)_o = 1$

	$(y_2)_o = 0$	$(y_2)_o = 0$
7. $y_1 = \begin{matrix}\sin^{-1} mx \\ \cos^{-1} mx\end{matrix}$	$\sin^{-1} mx$	$\cos^{-1} mx$
$y_1' = my_2$	$x_o = 0$	$x_o = 0$
$y_2' = m^2 x y_2^3$	$(y_1)_o = 0$	$(y_1)_o = \pi/2$
	$(y_2)_o = 1$	$(y_2)_o = -1$

And an alternate system is

$y_1' = \frac{m}{1 - m^2x^2} y_2 = \frac{m}{y_2}$	$x_o = 0$	$x_o = 0$
$y_2' = \frac{-m^2 x}{1 - m^2x^2} y_2 = -\frac{m^2 x}{y_2}$	$(y_1)_o = 0$	$(y_1)_o = \pi/2$
	$(y_2)_o = 1$	$(y_2)_o = -1$
8. $y_1 = \begin{matrix}\tan^{-1} mx \\ \text{ctn}^{-1} mx\end{matrix}$	$\tan^{-1} mx$	$\text{ctn}^{-1} mx$
$y' = \pm \frac{m}{1 + m^2x^2}$	(+ sign)	(- sign)
	$x_o = 0$	$x_o = 0$
	$y_o = 0$	$y_o = \pi/2$
9. $y_1 = \begin{matrix}\sec^{-1} mx \\ \csc^{-1} mx\end{matrix}$	$\sec^{-1} mx$	$\csc^{-1} mx$
$y_1' = (mx)^{-1} y_2$	$x_o = \frac{\sqrt{2}}{m}$	$x_o = \frac{\sqrt{2}}{m}$
$y_2' = -xy_2^3$	$(y_1)_o = \pi/4$	$(y_1)_o = \pi/4$
	$(y_2)_o = m$	$(y_2)_o = -m$

An alternate system is

$y_1' = \frac{m}{x(m^2x^2 - 1)} y_2 = \frac{1}{mxy_2}$	$x_o = \frac{\sqrt{2}}{m}$	$x_o = \frac{\sqrt{2}}{m}$
$y_2' = \frac{m^2 x}{m^2x^2 - 1} y_2 = \frac{x}{y_2}$	$(y_1)_o = \pi/4$	$(y_1)_o = \pi/4$
	$(y_2)_o = \frac{1}{m}$	$(y_2)_o = -\frac{1}{m}$

10. $y_1 = \begin{matrix}\sinh mx \\ \cosh mx\end{matrix}$

$y_1' = my_2$

$y_2' = my_1$

	sinh mx	cosh mx
	$x_o = 0$	$x_o = 0$
	$(y_1)_o = 0$	$(y_1)_o = 1$
	$(y_2)_o = 1$	$(y_2)_o = 0$

11. $y = \begin{matrix}\tanh mx \\ \coth mx\end{matrix}$

$y' = m(1 - y^2)$

	tanh mx	coth mx
	$x_o = 0$	$x_o = 1$
	$y_o = 0$	$y_o = \coth m$

12. $y_1 = \begin{matrix}\text{sech } mx \\ \text{csch } mx\end{matrix}$

$y_1' = \mp my_1y_2$

$y_2' = my_1^2$

	sech mx	csch mx
	(- sign)	(+ sign)
	$x_o = 0$	$x_o = 1$
	$(y_1)_o = 1$	$(y_1)_o = \text{csch } m$
	$(y_2)_o = 0$	$(y_2)_o = -\coth m$

13. $y_1 = \begin{matrix}\sinh^{-1} mx \\ \cosh^{-1} mx\end{matrix}$

$y_1' = my_2$

$y_2' = -m^2xy_2^3$

	$\sinh^{-1} mx$	$\cosh^{-1} mx$
	$x_o = 0$	$x_o = \frac{2}{m}$
	$(y_1)_o = 0$	$(y_1)_o = \cosh^{-1} 2$
	$(y_2)_o = 1$	$(y_2)_o = \frac{\sqrt{3}}{3}$

An alternate system is

$y_1' = \frac{m}{m^2x^2 \pm 1} \quad y_2 = \frac{m}{y_2}$

$y_2' = \frac{m^2x}{m^2x^2 \pm 1} \quad y_2 = \frac{m^2x}{y_2}$

	(+ sign)	(- sign)
	$x_o = 0$	$x_o = \frac{2}{m}$
	$(y_1)_o = 0$	$(y_1)_o = \cosh^{-1} 2$
	$(y_2)_o = 1$	$(y_2)_o = \sqrt{3}$

14. $y = \begin{matrix}\tanh^{-1} mx \\ \coth^{-1} mx\end{matrix}$

$y' = \dfrac{m}{1 - m^2x^2}$

$\tanh^{-1} mx$	$\coth^{-1} mx$
$x_o = 0$	$x_o = 2/m$
$y_o = 0$	$y_o = \coth^{-1} 2$

15. $y_1 = \begin{matrix}\operatorname{sech}^{-1} mx \\ \operatorname{csch}^{-1} mx\end{matrix}$

$y_1' = x^{-1} y_2$

$y_2' = \pm\, m^2 x y_2^{\ 3}$

$\operatorname{sech}^{-1} mx$	$\operatorname{csch}^{-1} mx$
(Assuming $0<m<1$)	(- sign)
(+ sign)	$x_o = 1/m$
$x_o = 1$	$(y_1)_o = \sinh^{-1} 1$
$(y_1)_o = \operatorname{sech}^{-1} m$	$(y_2)_o = -\sqrt{2}/2$
$(y_2)_o = (1-m^2)^{-\frac{1}{2}}$	$(mx > 0)$

An alternate system is

$$y_1' = \frac{y_2}{x(1 \mp m^2x^2)} = \frac{1}{xy_2}$$

$$y_2' = \frac{m^2 x y_2}{(1 \mp m^2x^2)} = \mp \frac{m^2 x}{y_2}$$

(- sign)	(+ sign)
$x_o = 1$	$x_o = 1/m$
$(y_1)_o = \operatorname{sech}^{-1} m$	$(y_1)_o = \sinh^{-1} 1$
$(y_2)_o = (1-m^2)^{\frac{1}{2}}$	$(y_2)_o = -\sqrt{2}$
	$(mx > 0)$

16. $y_1 = \operatorname{gd} mx$

$y_1' = my_2$	$x_o = 0$
$y_2' = -\, my_2y_3$	$(y_1)_o = 0$
$y_3' = my_2^{\ 2}$	$(y_2)_o = 1$
	$(y_3)_o = 0$

17. Hypergeometric functions $y_1 = F(a,b,c,x)$.

These functions satisfy the differential equation

$$x(1 - x)y_1'' + \left[c - (a + b + 1)x\right]y_1' - aby_1 = 0,$$

and the corresponding rational system is

$$y_1' = y_2,$$
$$y_2' = \frac{aby_1 + y_2\left[(a + b + 1)x - c\right]}{x(1 - x)}.$$

18. Confluent hypergeometric functions $y_1 = M(\alpha, \gamma, x)$.

These functions satisfy the differential equation

$$xy_1'' + (\gamma - x)y_1' - \alpha y_1 = 0,$$

and the corresponding rational system is

$$y_1' = y_2,$$
$$y_2' = \frac{\alpha y_1 + y_2(x - \gamma)}{x}.$$

19. Legendre's functions $y_1 = P_n(x)$.

These functions satisfy the differential equation

$$(x^2 - 1)y_1'' + 2xy_1' - n(n + 1)y_1 = 0,$$

and the corresponding rational system is

$$y_1' = y_2,$$
$$y_2' = \frac{n(n + 1)y_1 - 2xy_2}{x^2 - 1}.$$

20. Bessel's functions $y_1 = J_n(x)$.

These functions satisfy the differential equation

$$y_1'' + x^{-1}y_1' + (1 - n^2x^{-2})y_1 = 0,$$

and the corresponding rational system is

$$y_1' = y_2,$$

$$y_2' = (n^2x^{-2} - 1)y_1 - x^{-1}y_2.$$

21. Weierstrassian elliptic functions $y_1 = \wp(x)$.

These functions satisfy the differential equation

$$y_1'^2 = 4y_1^3 - g_2y_1 - g_3,$$

where g_2 and g_3 are called the invariants of the function. The corresponding rational system is

$$y_1' = y_2,$$

$$y_2' = 6y_1^2 - \tfrac{1}{2}g_2.$$

22. Jacobian elliptic functions $y_1 = \begin{matrix} \text{sn }(x,k) \\ \text{cn }(x,k) \\ \text{dn }(x,k) \end{matrix}$.

	sn (x,k)	cn (x,k)	dn (x,k)
$y_1' = -y_2$			
$y_2' = y_1(a + by_1^2)$	$a = 1 + k^2$	$a = 1 - 2k^2$	$a = k^2 - 2$
	$b = -2k^2$	$b = 2k^2$	$b = 2$
	$x_o = 0$	$x_o = 0$	$x_o = 0$
	$(y_1)_o = 0$	$(y_1)_o = 1$	$(y_1)_o = 1$
	$(y_2)_o = -1$	$(y_2)_o = 0$	$(y_2)_o = 0$

A third order system, which might be more satisfactory in some cases, is as follows:

$$y_1' = y_2y_3, \qquad x_o = 0,$$

$$y_2' = -y_1y_3, \qquad (y_1)_o = 0,$$

$y_3' = - k^2 y_1 y_2.$ $\qquad (y_2)_o = 1,$

$(y_3)_o = 1.$

To obtain sn (x,k), solve for y_1; to obtain cn (x,k), solve for y_2; to obtain dn (x,k), solve for y_3.

23. Mathieu functions $y_1 = \begin{matrix} ce_m(x,q) \\ se_m(x,q). \end{matrix}$

These functions satisfy the differential equation

$$y_1'' + (4a - 16q \cos 2x)y_1 = 0,$$

and if we set $y_3 = 16q \cos 2x$, the desired system of rational differential equations becomes

$$y_1' = y_2,$$
$$y_2' = (y_3 - 4a)y_1,$$
$$y_3' = 2y_4,$$
$$y_4' = - 2y_3.$$

The initial conditions for y_1 and y_2 will depend on the particular function chosen. If $x_o = 0$, $(y_3)_o = 16q$, and $(y_4)_o = 0$.

24. Lamé functions $y_1 = E_n{}^m(x)$.

These functions satisfy the differential equation

$$y_1'' = \left[n(n + 1)k^2 sn^2 x + A\right] y_1,$$

and if we set $y_3 = n^{\frac{1}{2}}(n + 1)^{\frac{1}{2}} k \text{ sn } x$, the desired system of rational differential equations becomes

$$y_1' = y_2,$$
$$y_2' = (y_3{}^2 + A)y_1,$$
$$y_3' = - y_4,$$

$$y_4' = y_3(1 + k^2 - \frac{2y_3^2}{n^2 + n}).$$

25. Gamma function $y = \Gamma(x)$.

This function may not be exhibited as a solution of a system of rational differential equations. It is the most widely used function which is transcendentally transcendental with respect to the field $R[x]$. (2, p 1-13).

26. Riemann Zeta function $y_1 = \zeta(x)$.

This function is, like the gamma function, transcendentally transcendental with respect to the field $R[x]$. This fact is not mentioned in most treatises on the subject, but it may be proved with the help of Riemann's formula linking $\zeta(x)$ and $\zeta(1-x)$. (8, p 269). This formula is

$$2^{1-x}\Gamma(x)\,\zeta(x)\,\cos(\tfrac{1}{2}\pi x) = \pi^x\,\zeta(1-x).$$

In this formula, let

$$y_0 = \Gamma(x),$$
$$y_1 = \zeta(x),$$
$$y_{n+1} = \zeta(1-x),$$
$$y_{2n+1} = \pi^x,$$
$$y_{2n+2} = 2^{x-1},$$
$$y_{2n+3} = \cos\tfrac{1}{2}\pi x.$$

We suppose that $y_1 = \zeta(x)$ satisfies a system of rational differential equations

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$
$$\cdots\cdots\cdots\cdots\cdots$$

. . . .

$$y_n' = f_n(x, y_1, \ldots\ldots, y_n),$$

and we wish to show that this leads to a contradiction. Solving Riemann's formula for $y_0 = \Gamma(x)$, we have

$$y_0 = \frac{y_{n+1} y_{2n+1} y_{2n+2}}{y_1 y_{2n+3}}.$$

We differentiate this expression with respect to x and obtain the result

$$y_0' = y_1^{-2} y_{2n+3}^{-2} \Big[y_1 y_{2n+3} (y_{n+1} y_{2n+1} y_{2n+2}' + y_{n+1} y_{2n+1}' y_{2n+2} + y_{n+1}' y_{2n+1} y_{2n+2}) - y_{n+1} y_{2n+1} y_{2n+2} (y_1 y_{2n+3}' + y_1' y_{2n+3}) \Big].$$

Then $y_0 = \Gamma(x)$ would satisfy the system of rational equations

$$y_0' = \frac{y_{n+1} y_{2n+1} y_{2n+2}}{y_1 y_{2n+3}} \left[\ln 2 + \ln \pi - \frac{f_1(1-x, y_{n+1}, \ldots\ldots, y_{2n})}{y_{n+1}} - \frac{f_1(x, y_1, \ldots\ldots, y_n)}{y_1} - \frac{\pi}{2} \frac{y_{2n+4}}{y_{2n+3}} \right],$$

$$y_1' = f_1(x, y_1, \ldots\ldots, y_n),$$

. . . .

. . . .

$$y_n' = f_n(x, y_1, \ldots\ldots, y_n),$$

$$y_{n+1}' = - f_1(1-x, y_{n+1}, \ldots\ldots, y_{2n}),$$

. . . .

. . . .

$$y_{2n}' = - f_n(1-x, y_{n+1}, \ldots\ldots, y_{2n}),$$

$$y_{2n+1}' = y_{2n+1} \ln \pi,$$

$$y'_{2n+2} = y_{2n+2} \ln 2,$$

$$y'_{2n+3} = \frac{\pi}{2} y_{2n+4},$$

$$y'_{2n+4} = -\frac{\pi}{2} y_{2n+3},$$

which would contradict the fact that the gamma function is a transcendentally transcendental function.