Undergraduate Generalization of Lists and Tables to Solve Introductory Combinatorial Problems

by
Michael Aimonetto

A THESIS

submitted to

Oregon State University

Honors College

in partial fulfillment of
the requirements for the
degree of

Honors Baccalaureate of Science in Mathematics
(Honors Scholar)

Presented May 29, 2019
Commencement June 2019

AN ABSTRACT OF THE THESIS OF

Michael Aimonetto for the degree of Honors Baccalaureate of Science in Mathematics presented on May 29, 2019. Title: Undergraduate Generalization of Lists and Tables to Solve Introductory Combinatorial Problems.

Abstract approved:____

Elise Lockwood

In this study I viewed video and corresponding transcripts of two students solving introductory combinatorial problems. Using an adapted version of Harel's (2008) concepts of Result Pattern Generalization and Process Pattern Generalization, I analyzed the work done by the two students. Both students primarily worked through the problem *How many ways are there to distribute 3 identical hats to 5 friends?* Student 4 generated tally style lists that marked friends who did have hats and left blank spaces for friends who did not receive hats. This visual structure allowed Student 4 to make generalizations about symmetry and support those generalizations contextually. Meanwhile, Student 11 made lists of sets of numbers, where numbers included in the set represented friends who received hats. This numerical structure caused Student 11 to focus more intently on the order they presented the outcomes on their list. Student 11 also was not as conscious of the students who did not receive hats, so their generalizations of symmetry were less contextualized. Overall, this observational study showed the generalizations students make in combinatorial problems is at least partially tied to the visual structure of the lists they use to count outcomes.

Key Words: Mathematics Education, Generalization, Combinatorics

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Honors Baccalaureate of Science in Mathematics project of Michael Aimonetto presented on May 29, 2019.

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I understand that my project will become part of the permanent collection of Oregon State University, Honors College. My signature below authorizes release of my project to any reader upon request.

Michael Aimonetto, Author

Introduction

Understanding the ways in which students approach problems is fundamental to teaching and learning, and in order to improve mathematics education, it is important for education research to explore ways in which students approach mathematics problems. It is useful to study how students reason about particular content areas, and also how they engage in mathematical practices. Specifically, it is worth investigating the ways students engage in *generalization*, an important aspect of students' mathematical thinking and activity. Mathematics education research has shown that students can learn effectively when they take into account what they have learned in the past (e.g., Ellis, 2007; Harel, 2008). Generalization involves looking at patterns in previous work to aid with future problems, so if researchers (and teachers) can understand how students generalize, then they could use such generalizations to help students reason about new topics and extend their work to new contexts.

In this paper, I look at the ways two different undergraduate students generalize aspects of combinatorial problems. Introductory combinatorics is a useful topic to study generalization because it is simultaneously approachable and complex (e.g., Kapur, 1970; Lockwood, 2013). Combinatorial problem solving involves problems that are typically about objects or situations with which students typically have some familiarity or experience. For example, a combinatorial problem may include counting the number of ways to give gifts to friends, or the number of PINs or passwords. Further such problems tend not to require much background mathematical knowledge – as opposed to a calculus, linear algebra, or real analysis problem that might require understanding of (what may seem to students to be) complex topics. At the same time, seemingly innocent combinatorics problems can require a significant amount of mathematical understanding to solve. Often times students do not have a framework to begin solving combinatorics problems, and small differences in the text of a question can completely change the approach required to find a solution (e.g., Annin & Lai, 2010; Lockwood, 2013). Therefore, I have chosen to research the ways students generalize combinatorial problems by adapting Harel's (2008) conceptions of Result Pattern Generalization and Process Pattern Generalization. After slightly adapting Harel's definitions to fit the problems being solved, I looked to answer the following research question: *What is the interaction between a student's listing procedure and their RPG and PPG in introductory combinatorial problem solving?*

Literature Review

Combinatorics

Combinatorics is a useful field of mathematics, and it is an interesting topic to study in educational settings, because it is simultaneously approachable and challenging (Kapur, 1970). Combinatorics and combinatorial analysis, as opposed to more commonly taught mathematical fields like calculus, requires little to no previous mathematical understanding. The language and notation of combinatorics is almost entirely self-contained, meaning a student is not required to have a working knowledge of concepts like functions, relations, or continuity in order to understand combinatorics. That being said, combinatorics is still very challenging. Some of the most difficult problems in all of mathematics come from the field of combinatorics (Kapur, 1970). Even with introductory problems, combinatorics can be very difficult to learn and difficult to teach (Annin & Lai, 2010; Lockwood, 2014). In summary, anyone is capable of studying combinatorics because it requires very little mathematical experience, but the task of successfully learning and teaching combinatorics can be a rigorous one. This ideal blend of accessibility and mathematical rigor makes combinatorics a fascinating field of study in its own right, and it is an excellent area for math education research.

More specifically, within combinatorics, it can be especially enlightening to study how participants list sets of outcomes to combinatorial questions (Lockwood 2013; 2014; Lockwood & Gibson, 2016). Lists and tables provide visual structures for how students represent combinatorial outcomes. A lot of difficulty in solving combinatorial problems comes from students not understanding what a combination, or more generally an outcome, looks like. For example, if a student is counting how many ways to give 2 hats to their 5 friends, a possibility would be giving a hat to their first and second friend but not to their third, fourth, or fifth friend.

There are several ways a student could imagine this outcome; they could picture all of your friends with some wearing hats and some not wearing hats, or they could picture just the subset of their friends who have obtained hats, or, they could number their friends and imagine a list of numbers. There are many structures that equivalently represent outcomes that need to be counted. For this reason, researches pay special attention to lists as a means by which to examine students' combinatorial reasoning (e.g., Lockwood, 2014; Lockwood & Gibson, 2016). Further, by examining lists researchers can gain insight into how students engage in other mathematical practices like justifying and generalizing.

Generalization

Generalization is an important concept to study within math education and learning in general (Harel 2008, Ellis 2007). Broadly, generalization in the context of learning refers to someone using their existing knowledge and experience to better understand and reason about novel problems or familiar problems in novel contexts. Generalization has been studied in depth with young children who are learning algebraic concepts for the first time. This research mainly discussed the generalizing actions and reflection generalizations that the researchers could observe (Ellis 2007). In contrast, the study conducted in this paper seeks to understand generalization made by undergraduate college students. Thus, although some research has been done on this demographic (e.g., Lockwood 2011; Lockwood & Reed, 2018), very little research has been done specifically about undergraduate generalization related to combinatorial lists and tables.

To expand on previous research, I plan on adapting Harel's constructs of a result pattern generalization and process pattern generalization. Harel describes two distinct "way(s) of thinking": result pattern generalization (RPG) and process pattern generalization (PPG). RPG uses repeated observations about numerical values to prove mathematical facts. For example, suppose you saw a set of numbers as follows: 10, 20, 30, 40,… One example of a RPG would be that all of these numbers have zeros at the end, so the next number that would fit would be 50. An example of a PPG, on the other hand, would be to notice that in order to get the next number of the sequence, you always add 10 to the previous number, which would make the next number 50. These ways of thinking work dynamically with one another to help students empirically derive, and justify, mathematical patterns.

In the context of collegiate level combinatorics, these concepts of PPG and RPG can be applied, but since the emphasis for combinatorial problems is more about finding solutions then proving conjectures, I am suggesting a slight modification of Harel's definitions for my goals in this paper. For the purposes of solving combinatorial problems, PPG is the approach of using similar problem-solving methods (e.g. listing, visualizing, acting, modeling) to solve problems that a student views as similar. For example, suppose a student was given two problems asking how many different combinations of numbers there are to make a 3-digit passcode, and the student listed all possible passcodes. If, when asked how many different combinations of numbers there are to make a 4-digit passcode, the student used a similar listing procedure, the student would be engaging in PPG.

In combinatorial problem solving, RPG involves finding a solution by making an observation (algebraic or otherwise) about results of previous problems. If, from a listing procedure, a student noticed that a single digit passcode had 10 options and a two-digit passcode had 100 options, the student could make an algebraic observation and claim a 3-digit passcode would have 1000 options and an n digit passcode would have 10^n options. This observation may be tied to an algebraic pattern in the results but not to a process that generated the results.

This observation of algebraic (and sometimes numerical) trends in solutions sets (particularly when not justified by some process for how the results were created) is what defines the combinatorial RPG.

It is important to note that when using the concepts of PPG and RPG in analyzing combinatorial problem solving, the person doing the problem solving need not have a complete understanding of the problem. This is because PPG and RPG are empirical strategies. They use evidence to draw conclusions instead of a chain of logical deduction. In the context of proof schemas, this result is a complete theorized understanding of mathematical principle. In the context of combinatorial problem solving, evidence is used to find solutions to a problem. This emphasis on solutions rather than theory means that PPG and RPG in combinatorial problem solving do not require the student to fully understand the mathematical reasoning behind the solutions they procure. To use the passcode example, a student could generate the correct 10^n formula through a RPG without understanding why this is the case. The student could potentially be unaware of the fact that adding another digit adds 10 more possibilities to each previous passcode, implying an exponential growth in the number of passcodes. PPG is when a student uses the same method (e.g. listing, visualizing, acting, modeling) to solve similar problems, where RPG is when a student selects a solution to a problem because of its relation (algebraic or otherwise) to previously derived results. It is important to keep in mind that logical deduction and complete mathematical understanding are not examples of RPG or PPG because RPG and PPG are empirical reasoning schemes. This means that the student may not have complete certainty about their reasoning, but instead are making choices in their approach to the problem based on the evidence that has been provided to them.

In this paper, I will use my adapted version of Harel's PPG and RPG concepts to make observations about how undergraduate students generalize when making lists and tables. In the following sections I will explain in detail the procedure with which I collected those observations and then I will relay those observations in two different case studies.

Methods

In this study, Dr. Lockwood recorded 19 videos, each approximately an hour long, of undergraduate students working through a specific set of related combinatorial problems. In these videos, the interviewees (undergraduate students) sat in an office opposite an interviewer and a witness who ran the video camera. The interviewer asked each of the student to write down their work, and they used a Livescribe pen that recorded their work digitally and captured their real-time writing and audio.

Students were asked their year in school, their area of study, and which math classes they had taken thus far in their schooling. Students were then asked to complete a series of combinatorial tasks, which increased in generality and difficulty. Occasionally, the experimenter would probe the student to explain their reasoning verbally, or she would ask about certain approaches. The intention was not for the students to necessarily obtain the correct answer, but to explain their thought processes. After an hour had passed, the interviewer would ask the subject to stop at however far along they had gotten in the collection of problems, and the video would conclude. The questions asked to the participants and their answers are provided below.

Question	Solution
There are 5 people, and I want to give 3 of them red hats. Can you find all of the ways that I can do this? (often times in addition students were asked the same problem but instead giving 2 hats).	There are $\binom{5}{3}$ solutions to this problem which evaluates to 10. The two hat case also evaluates to 10 because deciding which 3 friends get hats is the same as deciding which 2 friends do not get hats.
Suppose I want to give red hats to 4 of the 5 people. How many ways are there to do this?	There are $\binom{5}{4}$ solutions to this problem which evaluates to 5.

<table>
<tr><td>If I vary how many of the 5 people are given red hats, what are the possible cases, and how many ways are there in each case? Please make a table to record what you find.</td><td>The table of values can be found below:
<table><tr><th>Number of Hats</th><th>Ways to Distribute</th></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>5</td></tr><tr><td>2</td><td>10</td></tr><tr><td>3</td><td>10</td></tr><tr><td>4</td><td>5</td></tr><tr><td>5</td><td>1</td></tr></table></td></tr>
<tr><td>In total, if there are 5 people, how many ways are there to give red hats to any number of them?</td><td>The total ways to distribute hats to 5 people is the sum of the right side of the table shown above. This evaluates to 1+5+10+10+5+1= 32, which can also be stated as 2^5.</td></tr>
<tr><td>Now could you do the same thing for 6 people? That is, how many ways are there to give red hats to any number of the 6 people?</td><td>The table of values can be found below:
<table><tr><th>Number of Hats</th><th>Ways to Distribute</th></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>6</td></tr><tr><td>2</td><td>15</td></tr><tr><td>3</td><td>20</td></tr><tr><td>4</td><td>15</td></tr><tr><td>5</td><td>6</td></tr><tr><td>6</td><td>1</td></tr></table></td></tr>
</table>

Prior to my work analyzing the data for my thesis, these videos were transcribed for analytical purposes. I excluded several of the videos due to technical issues (Students 8, 9, 10, 13, and 18) and because some students focused on more advanced combinatorial topics than we wanted to examine in this part of the study (Students 5 and 16). Thus, I began with 13 videos of students. I watched each of the videos taking unstructured notes of my observations related to PPG and RPG. I noticed made several observations about the relationship between a student's previous mathematical background (i.e. how many math classes the student had taken and how comfortable the student felt with combinatorics) and the student's preference for PPG or RPG. I did not find any strong correlation, and I also realized for a significant amount of students it is not immediately clear whether they are performing a PPG or RPG. This was also an inefficient kind of observation to make for this study design because there is not a large enough data set to make statistically significant correlational claims.

Eventually, I decided to focus on the visual structures of lists and tables as well as the RPGs and PPGs that followed from those visual structures. This research focus was more apparent to see in the videos and there were very clear examples of relationships between RPG, PPG, listing, and tabling. In my analysis of the videos, based on my research questions, I decided to focus on two students: Student 4 and Student 11. I choose to look at these video interviews because the students had distinct and semi-consistent listing procedures. I wanted to investigate whether invariants in listing procedures indicated a RPG with regards to the visual image of a list or a PPG in the procedure of generating the list. I recorded instances of generalization involving each student's lists and in the tables they created. I then used the transcripts from the videos to provide evidentiary support for the qualitative observations.

Lastly, it is important to note that the methods for data collection used in this research is not designed for the extrapolation of larger trends. I am only citing two students' work, so I cannot make any definitive claims to the trends of generalization among students. Although I can gain rich insight into how these particular students are reasoning, I make no claims about all students. The observations presented were specific to the students being studied, and further studies must be conducted to ensure that the observed phenomena are applicable for use in teaching. That being said, the data collected did suggest possible trends or hypothesis about generalization, and those trends will be discussed further in both the results section and the conclusion.

Results

In this section, I present cases of two students and describe their generalizing activity during interviews in which they solved combinatorial problems. I present each students' work, then compare and contrast the cases and synthesize the results in the Discussion and Conclusion session.

Student 4 listing procedure case study

Generalization of the listing process. The interview began with Student 4 being asked, "So there are 5 people, and I want to give 3 of them red hats. Can you find all the ways that I can do this?" Student 4 initiated a listing procedure at the beginning of the session as a way to visualize the possible options for ways hats can be distributed to people. They started by writing out the outcomes as in Figure 1, with five lines in the top row representing the five people, and, as they noted in the excerpt below, r's to represent which people were getting hats.

S4: All right. Just go 1, 2, 3, 4, 5 represent each person.
Int: Okay.
S4: And then do an r to represent each hat. So if you did it that way you could do there and that's 1 too many hats. Just kind of do it visually.

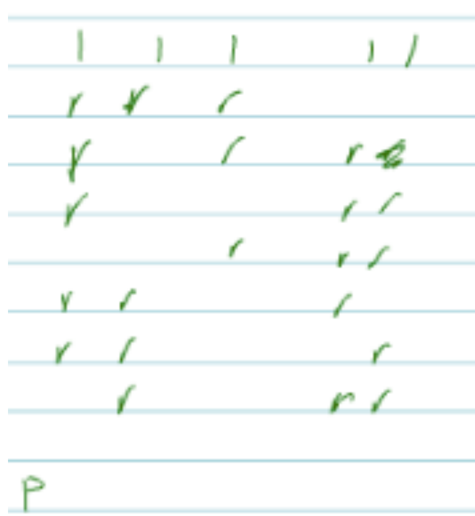


Figure 1 – Student 4's initial way of representing the list of 3 red hats to 5 people

The interviewer then asked them to explain the list, and they had the following exchange.

> Int: Okay. And can you - can you justify why you have them all?
> S4: Well, just looking at it trying different solutions in my head, I kind of went through it just systematically having all 3 and then separating them by 1 and 3 – 1 and 2. And then 1 and 2 again, then try doing the vice - versa on the other side and just so on, doing the whole thing.

During this exchange, Student 4 gestured from left to right on the rows of their list. This suggests that in order to generate all of the solutions to the problem, the student moved the position of the *r* marker back and forth in a semi-systematic way, being careful to not repeat any previously listed results. I say semi-systematic because while the student held that pattern for the initial rows, it was not clear that they continued that pattern for the latter rows. This pattern suggests a visual understanding of the problem, but the student has not made enough assertions about symmetries of visual cues within their list to suggest any RPG or PPG.

The interviewer then asked the student to solve a similar problem, "What if we wanted to give [5 people] 2 hats?". In response, the student generalized the pattern of keeping an element invariant, which we can see in Figure 2 visually through vertical lines. Note in Student 4's explanation below, they state that they viewed their listing activity on this problem as "pretty much the same thing" as their listing procedure for the previous problem, further suggesting some sort of generalization.

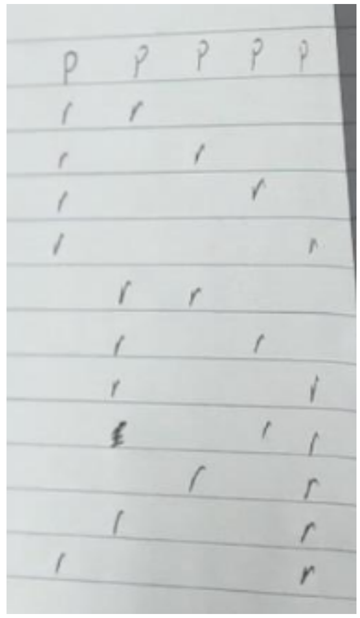


Figure 2 – Student 4's way of representing the list of 2 red hats to 5 people

Int: Okay. And can you explain what you did?
S4: Pretty much the same thing I did with the 3 hats. Just kind of lined them up systematically from one side and then the other side and threw in a couple that just came to mind as I was going through.

I interpret their work on this 2 hats problem as very clearly a generalization of their previous listing effort because they explicitly referenced back to their previous list. There is not enough evidence to definitely state if this generalization is an RPG or PPG, but it provides an interesting case to consider what might be going on. In fact, their statement in the previous excerpt is particularly ambiguous with regard to what they were actually attending to in their generalization. If the student chose to make a vertical line because they saw a line visually in the last list, then I would interpret their generalization as an RPG because they were generalizing based on the resulting structure of the physical list itself. Student 4's language of "just kind of lined them up" suggests perhaps an attention to the visual organization of the list. However, the back and forth gesturing may suggest Student 4 was replicating a *process* of holding invariants. In that case, they would be performing a PPG. The interviewer did not push for further explanation, so I do not have additional insight into their thinking about this list. I find this to be an interesting example, though, because it demonstrates that even in generalizing listing activity there could be a variety of details to which a student is attending.

I did gain some additional insight in another task, where the student was asked to solve how many ways to give 2 hats to 6 people and they created the table shown in Figure 3. Afterward the student and the interviewer had the following exchange:

S4: I'm counting 13 ways, but –
Int: – okay.
S4: – giving 2 hats to 6 people I don't think would generate an odd number.

While creating a list for $\binom{6}{3}$, Student 4 appeared to doubt their own work because the result seemed incorrect. It is uncertain whether this was an RPG based on the previous results or a spontaneous assertion based on the context of the problem. The student then made the following claim when asked about their listing procedure:

Int: Okay. Okay. Cool. And in terms of actually like which (inaudible) did where, do you feel like you were trying to go systematically, or were you –
S4: A little bit of both. I did – I started systematically with just putting 2 in each consecutive – 2 consecutive columns.
Int: Sure.
S4: Towards the middle here I jumped around a little bit –

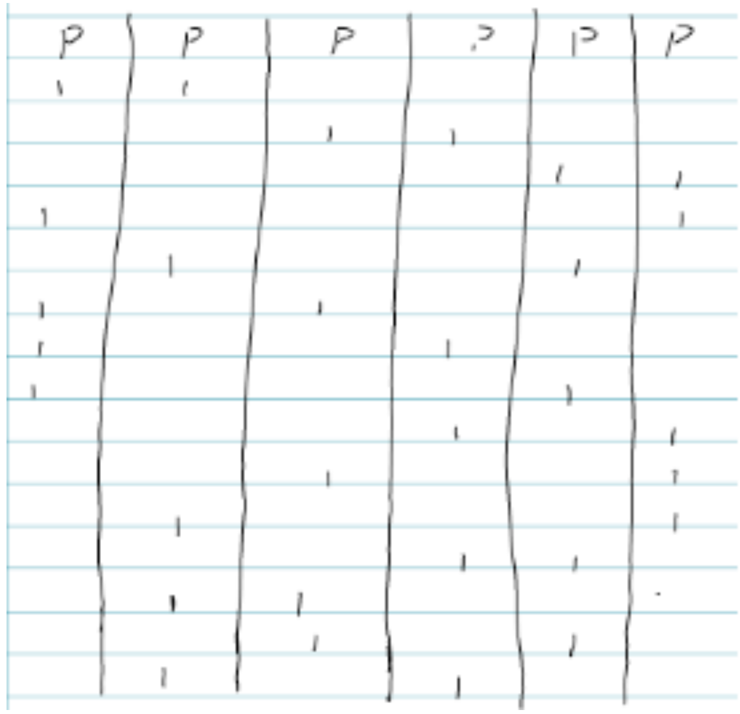


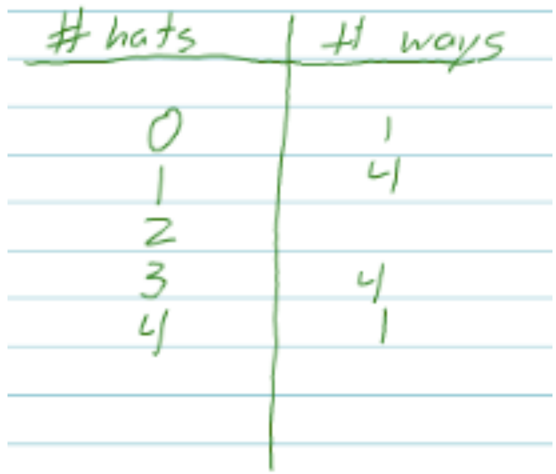
Figure 3 – Student 4's way of representing the list of 2 red hats to 6 people

In creating this list, Student 4 did, again, use the same representation as before, having made columns and marks below people who have hats. The student was using visual structure, and they explained that the visual structures come from them visualizing how to distribute hats within the problem. This list is fascinating because it shares some visual similarities from previous lists, but is different enough to require that Student 4 use a different process to create the list. The listing procedure was not maintained, but visual cues from previous lists (e.g. vertical lines and grouping) suggest a generalization still occurred from list to list. For this reason, I assert that Student 4 was engaged in an RPG with the result being the visual presentation of their list. Student 4's work provides a good example of how a procedure such as listing to count a problem can still be intricately connect to a RPG.

Generalization of making tables. A little bit later in the interview, Student 4 was asked to make tables for how many ways there are to give people numbers of hats. For example, the students' table for giving 5 people 0, 1, 2, 3, 4, or 5 hats is seen in Figure 4 (note, the 7 is incorrect in this figure).

Figure 4 – Student 4's table of the number of ways to distribute hats to 5 people based on the number of hats being distributed

I highlight an episode from Student 4's work on making a table for giving hats to 4 people. In particular, the interviewer asked, "Could you make a similar table for giving 4 people 0, 1, 2, 3



or 4 hats?" The student thought to themself for a while, and they first filled in everything except the 6 as seen below in figure 5.

Figure 5 – Student 4's initial table for the number of ways to distribute hats to 4 people based on the number of hats being distributed

We see in their discussion below that the reason they filled out the table in this manner was because they had noticed a pattern from the previous table.

> S4: Yeah. Just like with the 5 people and – which is 5, with the 5 people, you know, that there's only going to be 1 way to give out 0 hats and 1 way to give out 4 hats. And just there's going to be 4 ways to give out 1 hat and 4 ways to give out 3 hats, just because either 1 person's going to get that hat or 3 people are going to get the hat.

When Student 4 was asked to make a table of the different ways to give a number of hats to 4 people, they immediately were able to provide answers for 0,1,3, and 4 hats. The speed with which they answered and the similarity to the previous answer suggests some form of generalization. Also, the student was immediately able to explain why their new answer made sense within the context of the problem. I am inclined to believe this is a RPG because they seemed to be copying the numerical values from the 5-person table in Figure 4, but their justification may suggest something deeper than a surface level RPG. The student noticed patterns in the results that they got and made sense of those patterns using the problem as a motivation. Then when they received a new problem that had the same internal motivations in the problem, they could insure that their results generalized to a new context. Thus the student used a process (in the context of the problem) to justify the solutions at which they arrived, but they did not use this process to initially generate those solutions. Therefore, this was not an example of a PPG because they did not use a process to arrive that their answer. The result was obtained as a logical consequence of the problem and those patterns in the result (namely a 1x x1 pattern, by which I mean a symmetric pattern in the table) were generalized to the new problem. The student explained this in the following quote:

> S4: A bit, yeah. I did notice that there was a pattern between 0 and 5, 1 and 4. So I – it was faster to think of that, to reflect back to that and to think, okay, well, I already know if you have 0 hats there's only 1 way to give that out. And same with 1 hat, there's only 4 ways to give 1 hat to 4 people.

This quote confirms that the student was performing an RPG for their partial generation of a table of ways to give a number of hats to 4 people.

In trying to figure out the final entry in that table for 4 people as seen in Figure 5, the student tried to list out the ways to give 2 hats to 4 people. They created the list using the same representation of putting marks below certain people, and in their list seen in figure 6 below, they found six ways. Although there is less explanation for this portion, the lack of a consistent visual pattern in this result suggests that Student 4 was not performing a PPG in this listing procedure, but it is unclear if the student performed an RPG.

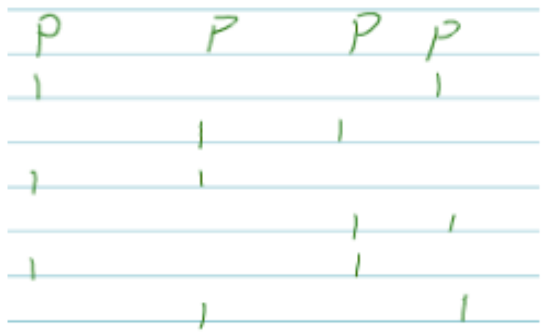


Figure 6 – Student 4's way of representing the list of 2 red hats to 4 people

To summarize Student 4's work, Student 4 showed some inconsistent examples of RPG both in their generation of lists and tables. While generating lists, the student used visual cues of vertical lines and grouping to help make lists for similar cases. While generating tables, the student generalized a 1x x1 structure. This generalization is an example of an RPG as evident by the speed of generation and the language of the student, but it is important to note that the student was able to place their table generation within the context of the problem.

Student 11 listing procedure case study

In this section I describe Student 11's listing activity. I discuss ways in which they generalized their work and connect this listing process to RPG and PPG. At the beginning of the interview process, Student 11 was asked how many ways there are to give 3 hats to 5 people.

Generalization of the listing process. Originally, Student 11 was unable to generate any efficient process for listing all of the possibilities. Evidence for this was seen in the student attempting to use mathematical notation that did not appear to be relevant for the problem. For example, Student 11 wrote an algebraic expression 5x+3y=0, where x represents people and y represents hats. After the student spent some time without making progress, the interviewer prompted Student 11 to label the people receiving hats into a numbered list, as seen in the transcript below.

> Int: Okay. So what if –so what if we called people like 1 through 5 or something – like there was a person 1, person 2, person 3, person 4, person 5. What are like some of the different possibilities for who could get hats?

The question prompted Student 11 to begin a rudimentary listing procedure that involved rewriting the numbers 1 through 5 to represent the people, then placing marks next to the numbers to represent that person getting a hat. Before Student 11 could write more than one possibility, they realized how long it would take to list the whole set of outcomes and defaulted to an unsubstantiated algebraic claim, as seen below.

> S11: –and we –okay, like so many like –I made this little chart that branched out like that, writing it all by hand, so I suppose I could just write down Person 1 with 1 hat, 2 with 2 hats, so I mean, bring it down –Person 1 is –okay, 1 –Person 1 with 1 hat, Person 2 with 1 hat, Person 3 with 1 hat, and then 4 and 5 don't have anything. Then we just (inaudible) –I mean, keep writing that out. Person 2 with 1 hat, and so I would guess somewhere in the range of 15 possibilities just because 5 times 3 is 15.

After some probing into this algebraic claim, I argue that the student's 5 times 3 pattern was not based on any contextual understanding, but instead involved a somewhat automatic use of multiplication as a familiar operation to find a singular numerical value. This assumption is based on Student 11's use of the word "guess" and "just" to indicate that there is no connection to the problem other than the numbers 5 and 3.

The next use of a listing process came when Student 11 was asked to find all possibilities of distributing 2 hats to 5 people. Student 11 continued the use of numbers as placeholders for people, but they changed the listing process to be faster. They did this by just listing the numbers of the people who were receiving hats. By doing this, each entry in the $\binom{5}{2}$ list is just a sequence of 2 numbers between 1 and 5 (note, when I say the $\binom{5}{2}$ case I mean the case of giving 2 hats to 5 people).

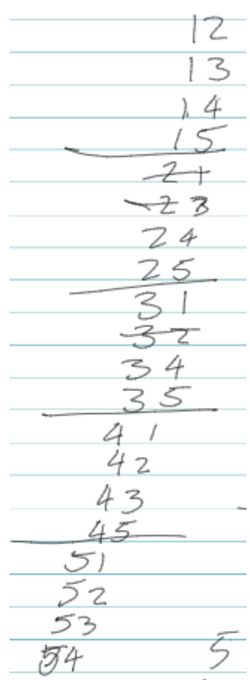


Figure 7 - Student 11's way of representing the list of 2 red hats to 5 people

This listing procedure represents a different process than Student 4, who had used a more visual listing process of dashes written in columns. Student 11 also segmented this list based on the first number listed. In its first instance, I argue that segmenting actually hindered Student 11 by making it much harder to see when numbers are repeated (this is in contrast to Student 4's more visual list). However, since Student 11 had established a consistent counting procedure, one could analyze to what extent the process and results were generalized in further cases. One could possibly make the claim that Student 11 used an RPG in creating this listing procedure. The argument would be that using numbers (as opposed to a visual representation) in the first attempt at a procedure, and then using the same numbering system in the modified listing procedure is an example of generalizing the result (where result being generalized are the lists themselves). That being said, I do not believe there is enough evidence to say with certainty that Student 11 was generalizing in this way. Perhaps a more likely explanation would be to say that the student was primed to think all of their work must include numbers. Just because a student has similarities across their work on various problems does not necessarily mean that the student generalized those elements.

Later while listing the $\binom{5}{3}$ case in a similar looking listing procedure as shown below, the student ran into some problems. The pattern of holding one number constant and filtering through the remaining numbers becomes a much more arduous task when there are three numbers that need to be chosen. It would require either two numbers to be fixed at a time or some system for counting all of the options for the two non-fixed numbers.

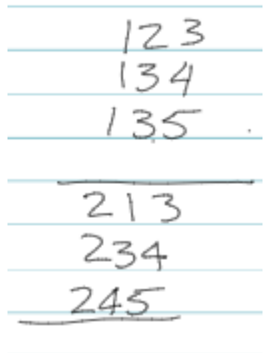


Figure 8 - Student 11's way of representing the partial list of 3 red hats to 5 people

This is not surprising, as it is more difficult to establish a lexicographic list with 3 items than it is with 2 items, and this provides further evidence that PPGs are much harder to implement for more complex cases. In addition, I interpret that lexicographic listing is something that makes more sense for a list of numbers than it does for Student 4's more visual listing procedure. That is, while a strategy of holding certain entries constant could be applied to the visual list of Student 4, it might require additional reasoning and translation to see the lexicographic strategy, which is more apparent when listing numbers or words. I suggest that this might explain why Student 4 payed very little, if any, attention to the order of the distribution.

When attempting to list $\binom{5}{3}$ in their segmented listing procedure, Student 11 made an incorrect assumption about the amount of possibilities that begin with a 1 in the $\binom{5}{2}$ case versus the $\binom{5}{3}$ case. In particular, they assumed that the amount decreases, which makes sense given the partial lists they generated for $\binom{5}{2}$ and $\binom{5}{3}$ as seen in Figure 9.

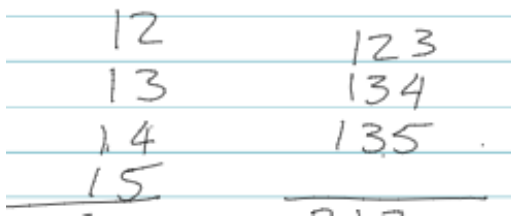


Figure 9 - Student 11's list of 2 red hats to 5 people and 3 red hats to 5 people in which person 1 receives a hat

In actuality, the decreasing pattern is not the case. The complete $\binom{5}{2}$ lexicographic list contains four elements that begin with 1 (12, 13, 14, and 15), while the $\binom{5}{3}$ lexicographic list contains six elements that begin with 1 (123, 124, 125, 134, 135, and 145). Student 11's inefficient RPG arose from an incomplete $\binom{5}{3}$ list. They stated their generalization in the following excerpt.

S11: All right. So with that there's 4 –you can give a hat 4 different ways, so 3 hats probably 2 different ways maybe. I don't know if I write that out –this is probably the really long way to do it, but if I give 3 hats –2, 1, 2, 3 –1, 3 and 4 –1, 3 and 5 –there's no 6. You can't give hats to the same person, so we've got 3 different ways now that I can think of, 1, 2, 3 –and then 2, 1 and 3, 2, 3, 4, 2, 4, 5. So there's a pattern there.

In this quote, I infer that Student 11 noticed the decreasing possibilities in segmented regions and attempted to attribute it after the fact, to an increase in the number of spaces they needed to fill with numbers. I interpret this as a perfect example of an inefficient RPG. Student 11 noticed a numerical trend in their data and applied it to generate solutions without listing all possibilities. The student then used their RPG to make conclusions about the nature of combinatorics within the context of the problem. Their false conclusions drawn from their inefficient generalizations led to a misconception about the nature of the problem itself.

A little later on, the student noticed some mistakes they had made in the $\binom{5}{2}$ list regarding repeats. The idea of permuted repetitions naturally occurred to Student 11 while they were attempting to generate the $\binom{5}{3}$ list, and they used their new observation to return to their $\binom{5}{2}$ list and make corrections.

> S11: So 2 and 1 is the same as 1 and 2. 2 and 3 is the same as 3 and 2. That changes everything [student laughing].

Using this understanding that they had not considered that 12 and 21 (and 23 and 32) should be the same thing (which I understand to be the difference between permutations and combinations, though the student did not use such language), Student 11 was able to finish an accurate list of $\binom{5}{2}$, but was not yet able to correctly write the $\binom{5}{3}$ list. In fact, during a later episode, the student was constructing a table for how many hats could be given to 5 people as seen below.

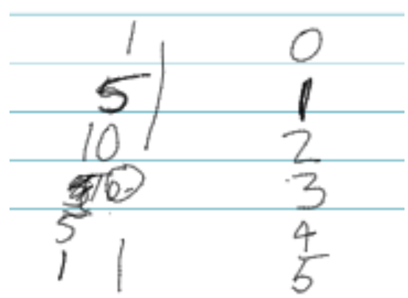


Figure 10 - Student 11's table of the number of ways to distribute hats to 5 people based on the number of hats being distributed

The student was asked to fill out the number of possibilities when giving 3 hats to 5 people, but the student hadn't completed a satisfactory list, so instead of using their list to count the possibilities, I interpreted that they used an RPG. Instead of going back and fixing their list, they attempted to generalize what they had found for $\binom{5}{1}$ and $\binom{5}{2}$. In particular, their language in the excerpt below suggests that since they saw that there were 5 hats for 1 and 10 hats for 2, then following that pattern there should be 15 hats for 3. I interpret this is an RPG because they seemed to be making a generalization based on the numerical results in their table. Furthermore, this is a regression back to Student 11's previous inefficient mathematical observation for $\binom{5}{3}$, where they claimed there were 5x3=15 possibilities.

A: So 5 different –so we've got 1 hat to give to 1, 2, 3, 4, and 5. And 2 hats, 10, so 3 –let's say, 3 hats, 15 –
Q: Okay.
A: –that –if we were pinched on time I'd say, okay, according to those first 2, 1 hat's 1, 2, 3, 4, 5. 2 hats, you know, 10, and then I'd say 3 hats, 15. I mean –but if we had, I don't know, if you want me to write that out, but if I wrote it out I'm pretty sure I would get 15.

This passage further illustrates Student 11's eagerness to use simplistic numerical RPGs instead of brute force listing methods when solving combinatorial problems. The student even commented on how much more time consuming brute force listing methods can be, rather than making conclusions based on previous results. We see that trend throughout Student 11's listing process. At several moments in the interview, they would get frustrated or tired while developing listing procedures and instead attempt to generate an answer using simpler algebraic patterns. It is worth noting that this could partially be due to Student 11's motivation in completing the tasks. In this interview process, it was made clear to the student that the interviewer did not care about the correctness of the answer. Therefore, the student might have felt free to generate answers without needing to be too careful about accuracy, and so this motivation may perhaps have lent itself to RPG.

Later on in the video, Student 11 was asked to condense their data into tables that compare the number of hats distributed to 5 people with the number of possibilities for distribution. Completing this table was initially difficult for Student 11 because they did not include the case of distributing 0 hats to 5 people, but they did include the case of distributing 5 hats to 5 people. This hid a visual cue of symmetry, as seen in their original table entry in Figure 11.

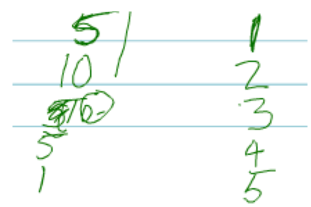


Figure 11 - Student 11's incomplete table of the number of ways to distribute hats to 5 people based on the number of hats being distributed

When Student 11 was prompted that there was 1 way to distribute 0 hats, they added that entry to their table, and they had the following reaction.

> S11: Yeah. So if we had 0, we had 1, 1, 5 and 10. And then, I mean, if –1, 5, 10, 10 –I don't know. That really doesn't make any mathematical sense, but as long as there's a pattern.

Student 11 was immediately able to notice the symmetry pattern when they were presented with a more clearly symmetrical relationship. I interpreted that they instantly performed an RPG based on the symmetry of their solution, which demonstrates how RPGs can be highly dependent on the visual representation the student uses to solve to problem.

When asked further about the context behind the pattern they recognized in their table, Student 11 was not able to make conceptual sense of why $\binom{5}{1} = \binom{5}{4}$ (this is opposed to Student 4, who was able to explain that relationship combinatorially). This is shown in the excerpt below.

> Int: All right. Question. First of all, does it –can you think of a reason why it would make sense that there's the same number of ways to give 1 hat or to give 4 hats? Like why would those both be 5?
>
> S11: I think that would be, you know, reasonable is because of I only have 1 hat to 5 people. Yeah. And then we have 4 hats to give to 5 people, and we can –I mean,

reasonably –probably in my mind that doesn't seem not possible, because, you know, we've got 5 different people and 4 hats, there's a bunch of different combinations I could work. So it seems like it's not like in my face, no, that's not possible.

Int: Sure.

S11: That doesn't ring –no alarms going off in there.

Student 11, even when prompted earlier to look at people who didn't receive hats, was unable to reach any contextual understanding in regards to their symmetry RPG. Again, this stands in stark contrast to Student 4, who was able to notice that context almost simultaneously to their generation of an RPG. Although both students developed the same generalization, the differences in their listing procedure allowed for a different level of greater understanding. I posit that this could possibly be because the people without hats might have been easier to see in Student 4's listing procedure than in Student 11's list (specifically, Student 4's recording of outcomes had gaps that represented who did not get hats, whereas Students 11's outcomes did not have a visual representation of people who did not get hats). This is further evidence that structural differences in listing procedure can make certain observations more easily visible to the list maker. As discussed earlier, Student 11's list brings attention to the order of elements, while Student 4's list brings attention to which elements are included and which elements are excluded. I address this phenomenon further in the Discussion and Conclusion section.

Later, Student 11 was asked to recreate their list for giving 3 hats to 5 people (their $\binom{5}{3}$ list). This time they were attempting to make their list match the value they obtained from their RPG. Originally they were incapable of doing it, and they provide a very fascinating list shown in Figure 12.

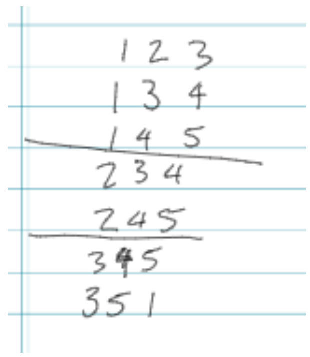


Figure 12 - Student 11's second attempt at representing the partial list of 3 red hats to 5 people

Looking closer, I infer that Student 11's process was to fix the first number and then begin an incomplete cyclic list of the remaining numbers. Although there is not enough information to make a definitive assertion, it appears that Student 11 attempted to simplify the $\binom{5}{3}$ problem by fixing the first number and performing a $\binom{4}{2}$ problem with the remaining variables. There were several inefficiencies in the listing procedure that make it difficult to say for certain, but this could be a potential PPG of a cyclic listing pattern used in $\binom{5}{2}$.

After some prompting, Student 11 decided to rewrite their entire list. This time they went more slowly and more carefully, with special attention paid to matching their results found in their table RPG of symmetry. First, they fixed the first variable, not unlike with their previous attempt. Then, they used a lexicographic listing procedure while only allowing entries with increasing values as cited in the transcript below.

Int: Well, I'm curious. Can you try to write all the ones that start with 1?

S11: So I think like that has a 1 in it, but could you organize them according to like – 1's –so 1's –let me start over. 1, 2, 3 is probably the simplest one. And then, I think I'll just start going up by 1's over here, so 1, 2,4 –

Int: Nice.

S11: –1, 2, 5. And then 1, 3, 2 they're already a thing. 1, 3, 4. 1, 3, 5. And 1, 4, 2's already a thing. 1, 4, 5. 1, 5, 2's already a thing. 1, 5, 3's already a thing. 1, 4, 5's already a thing, so that's it.

By listing in increasing order, Student 11 avoided accidentally repeating entries. It is difficult to tell whether this process was a conscious choice or just a trend they happened to follow. Since this was a new process for them, we cannot say it was a PPG or RPG, but it is important to note that without the RPG of symmetry in the tables, Student 11 would never have been numerically prompted to create a more efficient listing procedure. The RPG in the table informed Student 11 as to the result they should obtain from their $\binom{5}{3}$ list. This is specifically important in the context of the interview because the interviewer purposely refrained from commenting on the student's accuracy, meaning the student did not know if their lists were complete, but the RPG suggested whether the lists were complete or not. Now that the student had a way to check their answer, they were able to construct lists with more accuracy.

When Student 11 was later asked to make a table for 4 people instead of 5, they did seem to develop an RPG across tables. The generalization can be seen in the following dialogue.

Int: Awesome. Okay. Perfect. So I love that you made this table. I was going to ask you to make a table like that. So I'm wondering can you make this similar table for giving hats to 4 people?

S11: Hats to 4 people. I would –I suppose I would just start out with the same, you know, 1 and 0, that's 1 way to get 0 hats. 4 ways to give 1 hat. And based on the –the same thing we figured out, I mean, I suppose there would be 4 ways to give 4 hats. Oops. 1, 2, 3, 4 ways. I just lost my train of thought there. 1 for 4 people. Oh, that's ways, not people. Okay. So 1 way, 2, 3,4. 4 hats to 4 people is 1 way. So 4. That would not be the same pattern, because that's an even number. In between 1, 2, 3, 4 is but it looks nice, 1, 5, 10. 10, 5, 1. We only have 5 now so we have to pick some weird number in the middle like 1, 4 weird number 4, 1.

Here Student 11 adopted a 1x x1 structure similar to Student 4. I do not feel that there is enough evidence that Student 11 could justify why this pattern occurred – in fact, based on what they said in the earlier quote about $\binom{5}{2}$ being equal to $\binom{5}{3}$, I infer that they likely did not understand the pattern conceptually. If this is the case, it shows again that use of the 1x x1 structure and table symmetry do not necessitate that the student understands the context of the problem. Even without that extra insight, Student 11 was able to complete all but 1 entry in their table without creating any list (so, based just on numerical regularity and generalizations). For $\binom{4}{2}$, Student 11 did have to create a list, and they successfully used a lexicographic procedure. We can somewhat confirm that the motivation for the list was lexicographic in the following student quote, as the discussion of 12, 13, 14, etc. suggests holding 1 constant and going through the other people in a systematic order.

> Int: Yeah, like what was your organizational strategy?
> S11: –so, so I wrote, you know, the people –
> Int: Uh-huh. A–each getting a hat, and so in my mind I give 1 a hat and then 2 a hat. And then to stay with, you know, simplicity, I gave 1 a hat, 3 a hat, 1 a hat, 4a hat. And then I jumped to –now since I already did all the 1's, I get from 2 –2 gets a hat, 4 gets a hat. And then I don't really need to jump around. I mean, especially just giving 2 hats, and I can go 2, jump 3, 4 and go back to 1.
> Int: Sure.
> S11: So, I mean, 2 and 1 already got a hat, so there's really nothing more I can do. And then just 3 and 4.

Now, this consistent use of lexicographic listing may appear to be a PPG, but considering how simplistic lexicographic listing is for 2 variables, I cannot definitively say that Student 11 was extending the process they used to develop their $\binom{5}{3}$ list. To see if Student 11 had truly generalized the lexicographic procedure, I argue that they would need to attempt an $\binom{n}{3}$ list with n greater than 5. Thus, I can hypothesize what they were doing, but I cannot say for sure because they did not do larger cases.

As the student attempted to make a table for 6 people, they began with the 1x x1 structure, and made a numerical observation that $\binom{n}{2}$ is a triangular number. Once they had correctly listed that next entry of 15, they showed again their RPG of symmetry by writing $\binom{6}{4}$ as 15 (without doing any listing – they simply wrote the number). They were unable to obtain the final entry of $\binom{6}{3}$. If they were given the time to work on that final problem, it would solidify for us exactly the extent of PPG utilized by Student 11. It is ambiguous, given the data that was collected, and to know more we would need to see additional evidence of their listing activity on larger cases.

Overall, Student 11's lexicographic listing structure provided unique affordances and drawbacks. By distinguishing cases numerically instead of spatially, Student 11 gained more insight into the order of the possibilities they listed, but by only focusing on the people receiving hats and in no way visually cueing which students would not receive hats, Student 11 was unable to conceptualize their RPG of symmetry. Student 11's lexicographic list also prompted a recursive structure for which to make more complex lists, but because of the nature of the interview, both the time allotted and the bias toward RPG over PPG, they did not get a chance to formally generalize their recursive structure into a PPG.

Discussion and Conclusion

During the course of studying Student 11 and Student 4, I made several observations regarding their generalizations of lists and tables. Both students used lists, and then later tables, to track results to the combinatorial problems that they were presented. Although they reached some common conclusions, the lists made by the students were structured in drastically different ways. Student 4's tally-style lists provided visual cues that prompted generalizations about symmetry, while Student 11's lexicographic lists of sets of numbers led to a stronger focus on the order of solutions. In the following section I will compare and contrast the ways in which the two students generalized and report overarching observations about the nature of generalization. Because I only studied the work of two students, we cannot take these observations as being universally true or necessarily generalizable to all students. However, we still can gain insight into how some students reason combinatorially and engage in generalizing, and these insights should be used to prompt further research. If these findings can be successfully replicated in a more rigorous experimental design, only then should they be used to help instructors and students better teach and learn.

Visual Structures and Reasoning

The most notable difference between the work of Student 11 and Student 4 is the style (or representation) with which they chose to list possibilities. Student 11 listed sequences of numbers that corresponded to the people who had hats, while excluding the number of those who did not have hats. This listing procedure made it very easy for Student 11 to fix certain numbers as they listed. For example, they would write out every possibility that included person 1 by writing out every possibility that contained the number 1. This listing procedure also seemed naturally to lead to lexicographic listing in which the student focused on sequences that were strictly increasing. Lexicographic listing can be very helpful in listing a large number of outcomes because it helps prevent repetition of possibilities through permutation. Student 11 was likely not consciously thinking about this fact, but even still, the structure of Student 11's list did help them find all options for the $\binom{5}{2}$ problem and the $\binom{5}{3}$ problem. One drawback of Student 11's list is that it is difficult to produce. For the $\binom{5}{2}$ case, a lexicographic list is almost natural, especially if a student already has a process of fixing the first number, but the $\binom{5}{3}$ case becomes significantly more difficult and involves a much higher level of understanding. While listing the $\binom{5}{3}$ case in this way, a student must fix two numbers at a time and filter through options in a recursive style that can be challenging to do without making mistakes, especially for students who do not have much experience listing outcomes.

In contrast, Student 4 used a tally-style listing procedure that involved constructing a chart where each row represented a possibility and each column represented a person. A tally in the chart signified that in that possibility (row) that student (column) was given a hat. This kind of list visually cues the student to which people have hats, but it can also signify to the student which people do not have hats. Being able to see the gaps between possibilities allowed Student 4 to reason about the symmetry of the problem. Later on in the interview, both Student 4 and Student 11 were able to generalize that $\binom{n}{k}$ is equivalent to $\binom{n}{n-k}$ by looking at symmetries in their table, but only Student 4 was able to contextualize that finding. They noticed that if they were to swap the tallied boxes with the untallied boxes they would arrive at the $\binom{n}{n-k}$ list. I suggest that this reasoning that would have been much harder to obtain in a lexicographic list like the one generated by Student 11, which could explain why Student 11 was not able to reason why symmetries emerged within the context of this combinatorial problem. There may be other ways to observe the symmetry in a lexicographic list, but Student 11 did not recognize the symmetry.

With just two students it is evident that visual structures of lists provide certain affordances and drawbacks, so it would be worth investigating more into what list structures are the most efficient in prompting generalization among other students. With further study in this aspect of combinatorial problem solving, teachers could use this information to present combinatorial lists in certain ways depending on what the teacher wants their students to be focusing on. If teachers take care when choosing how to visually represent combinatorial lists, they could potentially have an impact on the kinds of lists students might generate and use. I now offer a couple more points of discussion.

RPG and contextual reasoning

To look deeper at the symmetry generalization, it is important to note that Student 4 developed an RPG about symmetry and almost simultaneously provided a rationale in the context of the problem. In contrast, Student 11 was able to make the same RPG, but was unable to provide any contextual reasoning. This means that, although both students made the same RPG, their level of understanding was different. RPG involves regularity in the results, and Student 11 is evidence that it is possible to observe regularity in the result enough to generalize without understanding the contextual mechanisms that produce the regularity. Essentially, it is possible to know how to arrive at an answer for a combinatorial problem without understanding why this answer is a sensical result to the problem being posed. The difference between these two cases could potentially inform language in math education that distinguishes between complete contextual understanding and understanding as a consequence of RPG.

Complex PPG

When discussing Student 11's listing procedure, I mentioned that it is difficult to generate a lexicographic list for $\binom{5}{3}$. This idea could easily be applied to other combinatorial problems, where working with a small number of elements makes the problem a lot easier to understand and list. It makes sense that the more elements a student has to manage in a problem the more difficult and demanding a problem becomes. In the case of Student 11, there were several instances in which they told they interviewer that they did not want to generate the more complex list. It is also interesting that the $\binom{5}{3}$ list has the same number of possibilities as the $\binom{5}{2}$, but the $\binom{5}{3}$ list was still more difficult for Student 11 to produce because of the amount of variables that needed to be managed. It would be worth studying to what extent complexities within PPGs deter students and how that could affect their understanding of combinatorial problems.

Pedagogy

Lastly, it must be said that I infer there was a bias in the interview process that favors the use of RPG over the use of PPG. In particular, by asking for the students to create lists and tables, and especially by prompting students to compare values across tables, I argue that the interviewer was focusing students on the results of their counting and listing. Thus, I do not feel that I can claim from these interviews whether students are more or less likely to use PPG or RPG when solving combinatorial problems in another context. This bias also arrives from the indifference on the part of the interviewer in regards to the accuracy of the student. It is made very clear at the beginning of the interview process that the purpose of the study is student learning and not the abilities of the students to perform problems. Therefore, students are relieved of the pressure of having their answers be correct. Students may naturally avoid doing tasks that are difficult or draining, and if their objective in the interview process is not to obtain the correct answer, but instead present an answer to the interviewer, they may be more inclined to use whatever RPG they can find to produce answers without having to go through the rigorous task of writing extensive lists or developing algorithmic listing procedures. I recognize this as a potential limitation of the research. Future studies would have to be conducted that presented the accuracy of the solutions as a higher priority, or required students to list possibilities along with their numerical solutions.

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