AN ABSTRACT OF THE THESIS OF

DONALD JAY CRESSWELL for the Doctor of Philosophy
(Name) (Degree)

in Mathematics presented on May 12, 1967
(Major) (Date)

Title THE RELATIONS AMONG THE MODES OF CONVERGENCE OF SEQUENCES OF MEASURABLE FUNCTIONS

Abstract approved Redacted for Privacy
(Edward L. Kaplan)

This study undertakes to determine the existence or non-existence of an implication in either direction between any two out of nine different modes of convergence, with the use of any subset of a set of ten auxiliary hypotheses. The functions are real finite-valued measurable functions defined on an arbitrary abstract measure space. A collection of 25 counterexamples is used to establish the invalidity of the unproved implications. When the desired convergence cannot be concluded, the existence of a convergent subsequence is investigated. An appendix investigates the possible non-uniqueness of the limit function when the definitions of two of the modes are slightly relaxed.

The Relations Among the Modes of Convergence of Sequences of Measurable Functions

by

Donald Jay Cresswell

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1967

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented May 12, 1967

Typed by Carol Baker for Donald Jay Cresswell

ACKNOWLEDGEMENT

I wish to express my appreciation to Dr. E. L. Kaplan for guiding the development and aiding in the writing of this thesis.

TABLE OF CONTENTS

		Page
1.	INTRODUCTION	1
2.	THE MODES OF CONVERGENCE AND BASIC THEOREMS	14
3.	IMPLICATIONS AMONG THE AUXILIARY HYPOTHESES	18
4.	LIST OF COUNTEREXAMPLES	28
5.	CONVERGENCE IN MEAN - CONVERGENCE IN MEASURE	44
6.	ALMOST UNIFORM CONVERGENCE - CONVERGENCE IN MEASURE	51
7.	CONVERGENCE IN MEAN - UNIFORM CONVERGENCE	53
8.	CONVERGENCE IN MEAN - ALMOST UNIFORM CONVERGENCE	57
9.	ALMOST UNIFORM CONVERGENCE - CONVERGENCE a. e.	60
10.	CONVERGENCE IN MEASURE - CONVERGENCE a. e.	66
11.	UNIFORM CONVERGENCE - CONVERGENCE a. e.	72
12.	UNIFORM CONVERGENCE - ALMOST UNIFORM CONVERGENCE	73
13.	UNIFORM CONVERGENCE - CONVERGENCE IN MEASURE	74
14.	CONVERGENCE IN MEAN - CONVERGENCE a. e.	75
15.	CONVERGENCE IN MEAN - L_p CONVERGENCE	79
16.	L_p CONVERGENCE - CONVERGENCE IN MEASURE	86
17.	L_p CONVERGENCE - UNIFORM CONVERGENCE	89
18.	L_p CONVERGENCE - ALMOST UNIFORM CONVERGENCE	93
19.	L_p CONVERGENCE - CONVERGENCE a. e.	96

TABLE OF CONTENTS (Continued)

Page

20. WEAK L_1 CONVERGENCE - CONVERGENCE a.e. 99

21. WEAK L_1 CONVERGENCE - CONVERGENCE IN MEASURE 104

22. WEAK L_1 CONVERGENCE - CONVERGENCE IN MEAN 108

23. WEAK L_1 CONVERGENCE - ALMOST UNIFORM CONVERGENCE 110

24. WEAK L_1 CONVERGENCE - UNIFORM CONVERGENCE 114

25. WEAK L_1 CONVERGENCE - L_p CONVERGENCE 117

26. WEAK L_p CONVERGENCE - CONVERGENCE a.e. 120

27. WEAK L_p CONVERGENCE - CONVERGENCE IN MEASURE 125

28. WEAK L_p CONVERGENCE - ALMOST UNIFORM CONVERGENCE 130

29. WEAK L_p CONVERGENCE - UNIFORM CONVERVERGENCE 133

30. WEAK L_p CONVERGENCE - CONVERGENCE IN MEAN 136

31. WEAK L_p CONVERGENCE - L_p CONVERGENCE 142

32. WEAK L_1 CONVERGENCE - WEAK L_p CONVERGENCE 145

33. WEAK * L_∞ CONVERGENCE - CONVERGENCE a.e. 156

34. WEAK * L_∞ CONVERGENCE - CONVERGENCE IN MEASURE 165

TABLE OF CONTENTS (Continued)

Page

35. WEAK * L_∞ CONVERGENCE - ALMOST UNIFORM CONVERGENCE 171

36. WEAK * L_∞ CONVERGENCE - UNIFORM CONVERGENCE 175

37. WEAK * L_∞ CONVERGENCE - CONVERGENCE IN MEAN 178

38. WEAK * L_∞ CONVERGENCE - L_p CONVERGENCE 183

39. WEAK * L_∞ CONVERGENCE - WEAK L_1 CONVERGENCE 189

40. WEAK * L_∞ CONVERGENCE - WEAK L_p CONVERGENCE 198

BILBIOGRAPHY 207

APPENDIX 208

THE RELATIONS AMONG THE MODES OF CONVERGENCE OF SEQUENCES OF MEASURABLE FUNCTIONS

1. INTRODUCTION

This thesis studies the logical relations among nine definitions of convergence (Section 2) of sequences of real almost-everywhere finite-valued measurable functions $\{f_n\}$ defined on an abstract measure space X provided with a non-negative measure μ defined on a σ-algebra S of measurable subsets of X. All possible subsets of a set of ten auxiliary hypotheses (Section 3) are considered. The non-existence of the unproved theorems is established by means of a collection of 25 counterexamples (Section 4).

Each of the next 36 sections (5-40) is devoted to one pair of the nine modes of convergence.

The Appendix gives the results of some work done with two slightly different modes of convergence.

The nine modes of convergence previously indicated with abbreviations that are used in the diagrams are the following:

a. un. - almost uniform convergence
a. e. - convergence almost everywhere
meas. - convergence in measure
unif. - uniform (L_∞) convergence (except on a null set)
L_p - convergence in L_p norm
L_1 - convergence in mean (L_1 norm)
wk * L_∞ - weak * L_∞ convergence
wk L_p - weak L_p convergence
wk L_1 - weak L_1 convergence

The definitions given in Section 2 for these nine modes of convergence are such that the limit function f is always unique except on a set of measure zero. This is not true for the two modes of convergence defined in the appendix.

The ten auxiliary hypotheses will now be listed together with the numbers by which they will hereafter be identified. It is taken for granted that the conditions may be violated by a finite number of the functions of the sequence $\{f_n\}$ or by any of the functions on a null subset of X.

1. Each member of the sequence of functions $\{f_n\}$ is integrable.

2. The function f toward which the sequence of functions converges in the specified manner is integrable.

3. The measure space is totally finite: $\mu(X) < \infty$.

4. The indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are uniformly absolutely continuous; that is given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_E |f_n| d\mu < \varepsilon$ $n = 1, 2, \cdots$ for measurable E for which $\mu(E) < \delta$.

5. The indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equi-continuous from above at 0; that is for every decreasing sequence of sets $\{E_n\}$ for which $\lim_n E_n = \phi$, and for every $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$ $\int_{E_n} |f_m| d\mu < \varepsilon$ $m = 1, 2, \cdots$.

6. The sequence of functions $\{f_n\}$ is uniformly essentially bounded.

7. There is an integrable function g such that $|f_n| \leq g$ a.e., $n = 1, 2, \cdots$.

8. $\lim_n \int f_n d\mu = \int f d\mu$. (It is assumed that $\lim_n \int f_n d\mu$ exists in the sense that it is finite, equal to ∞, or equal to $-\infty$.)

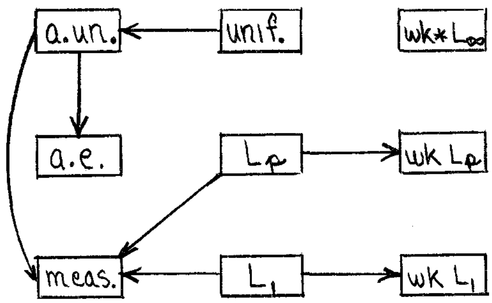
9. The sequence of functions $\{f_n\}$ is monotonic a.e.

10. The support of the functions $E = \bigcup_{n=1}^{\infty} \{x : f_n(x) \neq 0\}$ has σ-finite measure.

The consideration of possible convergence of subsequences in the event that we have no implication with a given set of hypotheses gives a simple result. It turns out that in six instances (meas. $\Rightarrow$ a.un., meas. $\Rightarrow$ a.e., $L_1 \Rightarrow$ a.un., $L_1 \Rightarrow$ a.e., $L_p \Rightarrow$ a.un., $L_p \Rightarrow$ a.e.) always have a convergent subsequence and in all other instances we never have one.

In referring to previous results, a number such as 5.2 will specify the second theorem in Section 5. If we are referring to a theorem in the same section, the section number will be omitted.

When no auxiliary hypotheses are assumed the following well-known implications are valid:

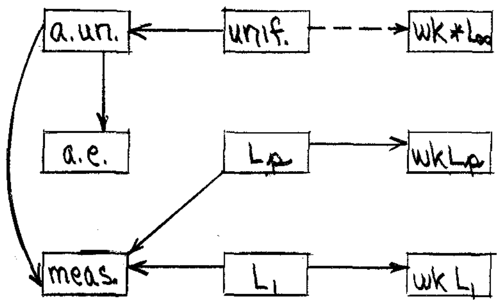


We now give 37 additional diagrams which summarize the results of the thesis exclusive of the appendix. Each of the diagrams 29 through 37 shows how one mode of convergence interacts with the other 8. The arrows indicate implications and an arrow which passes through a box containing sets of hypotheses indicates that each set of hypotheses in the box completes the implication when included in the antecedent.

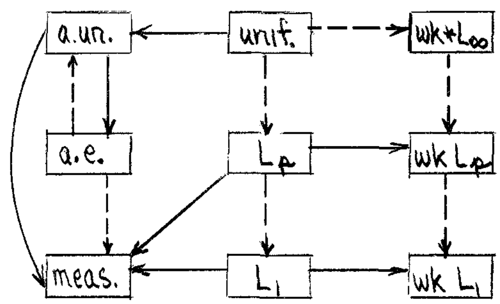
Diagrams 1 through 28 each refer to 1, 2, or 3 sets of auxiliary hypotheses as indicated in the following table. In each case the additional implications so obtained are superimposed upon the basic diagram given above. When more than one set of hypotheses is listed after a diagram number, the same diagram is valid for both. The hypotheses given in the list below are the only non-redundant sets of hypotheses which alter the basic diagram.

Diagram number	Set of hypotheses satisfied	Diagram number	Set of hypotheses satisfied
1	(1) (2) (10)	15	(5,9)
2	(3)	16	(6,7)
3	(5)	17	(6,9)
4	(6)	18	(7,9)
5	(7)	19	(1,2,9)
6	(9)	20	(1,5,6)
7	(1,5) (5,10)	21	(1,5,9) (5,9,10)
8	(1,6) (2,6) (6,10)	22	(2,5,8)
9	(1,9) (9,10)	23	(2,8,9)
10	(3,4) (3,5)	24	(5,6,10)
11	(3,6)	25	(5,8,9)
12	(3,9)	26	(1,2,6,9)
13	(5,6)	27	(2,5,6,8)
14	(5,8)	28	(2,6,8,9)

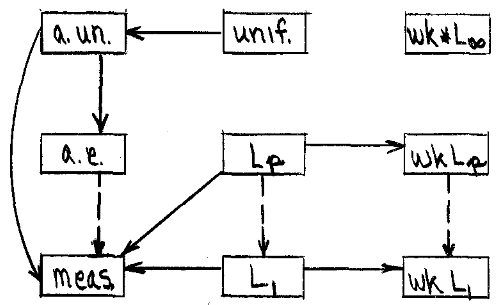
<u>Diagram 1</u>, (1) (2) (10) satisfied



<u>Diagram 2</u>, (3) satisfied



<u>Diagram 3</u>, (5) satisfied



<u>Diagram 4</u>, (6) satisfied

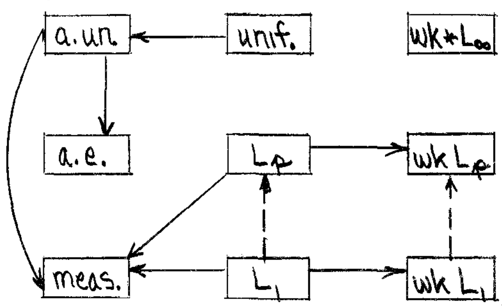


Diagram 5, (7) satisfied

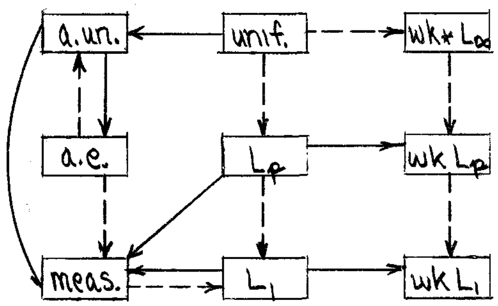


Diagram 6, (9) satisfied

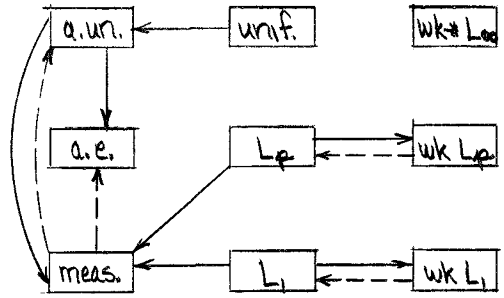


Diagram 7, (1,5) (5,10) satisfied

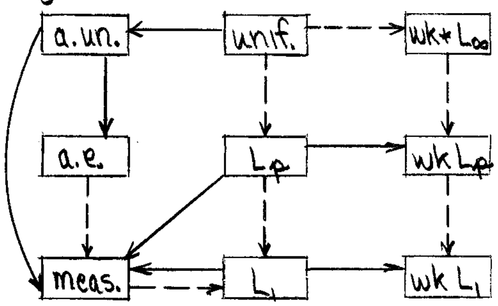


Diagram 8, (1,6) (2,6) (6,10) satisfied

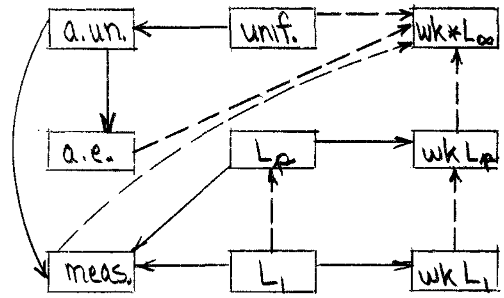


Diagram 9, (1,9) (9,10) satisfied

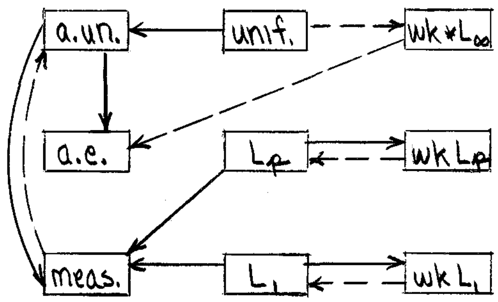


Diagram 10, (3,4) (3,5) satisfied

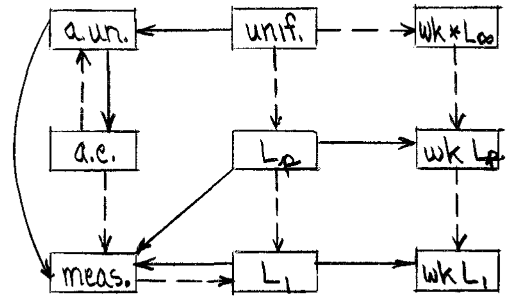


Diagram 11, (3,6) satisfied

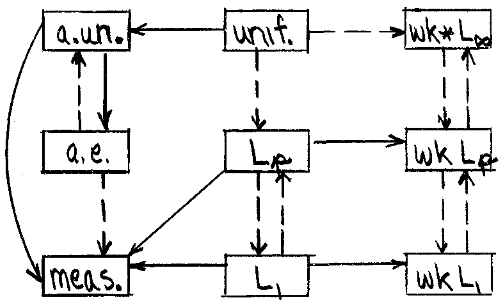


Diagram 12, (3,9) satisfied

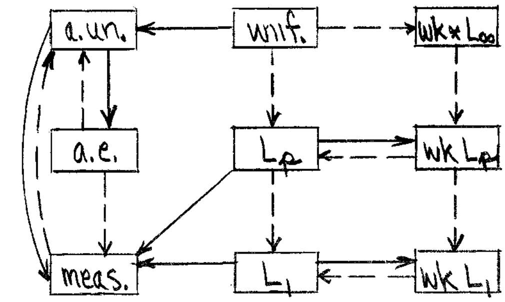


Diagram 13, (5,6) satisfied

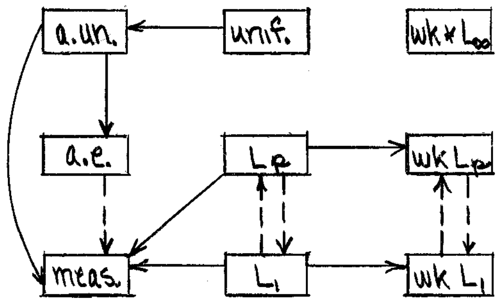


Diagram 14, (5,8) satisfied

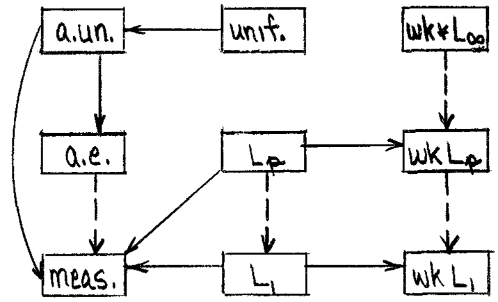


Diagram 15, (5,9) satisfied

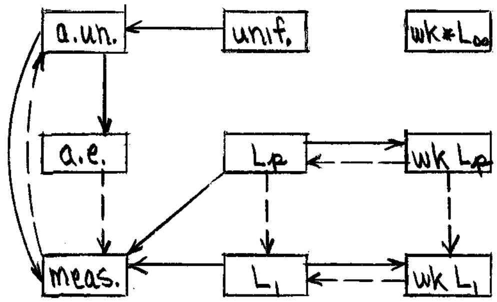


Diagram 16, (6,7) satisfied

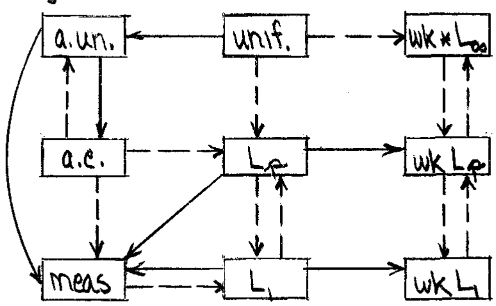


Diagram 17, (6,9) satisfied

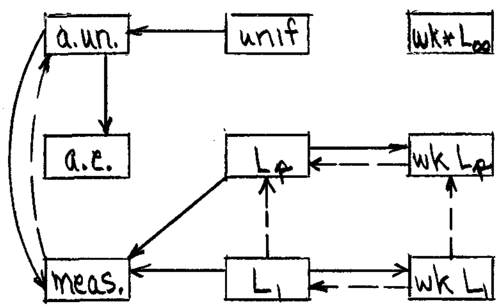


Diagram 18, (7,9) satisfied

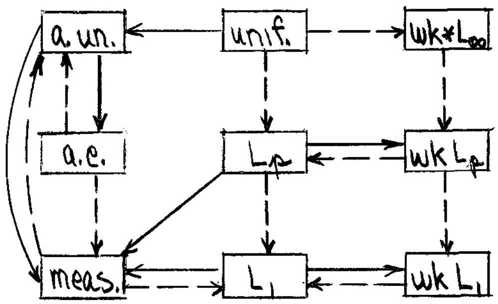


Diagram 19, (1,2,9) satisfied.

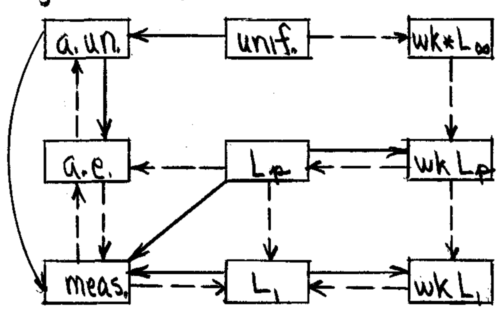


Diagram 20, (1,5,6) satisfied

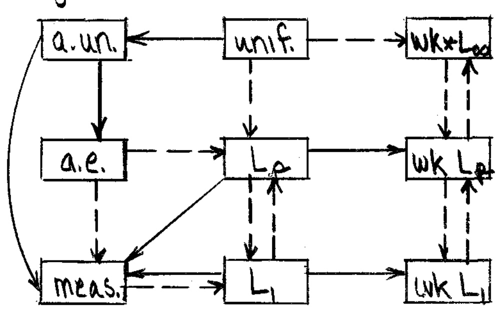


Diagram 21, (1,5,9) (5,9,10) satisfied

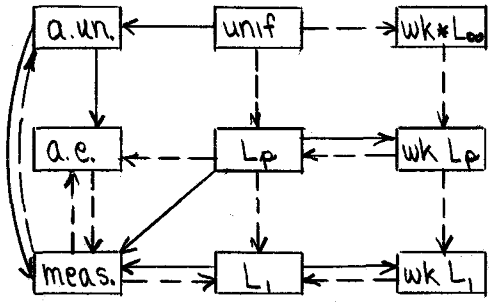


Diagram 22, (2,5,8) satisfied

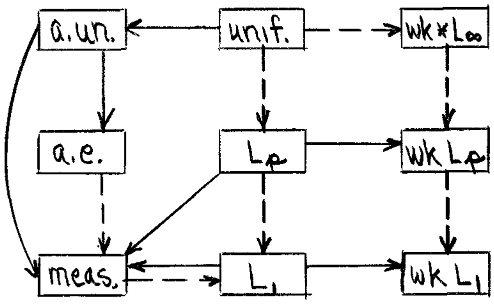


Diagram 23, (2,8,9) satisfied

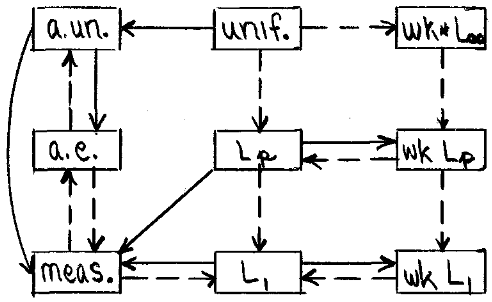


Diagram 24, (5,6,10) satisfied

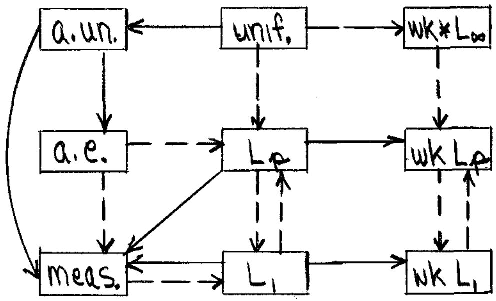


Diagram 25, (5,8,9) satisfied

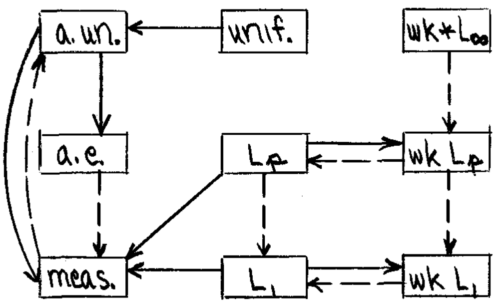


Diagram 26, (1,2,6,9) satisfied

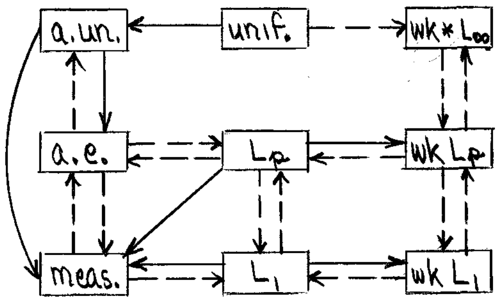


Diagram 27, (2,5,6,8) satisfied

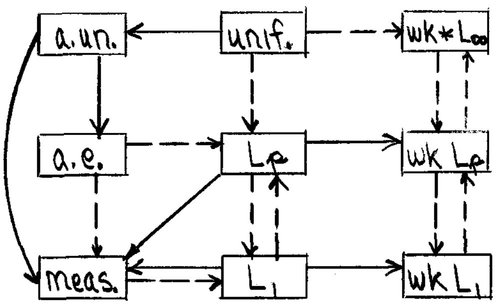


Diagram 28, (2,6,8,9) satisfied

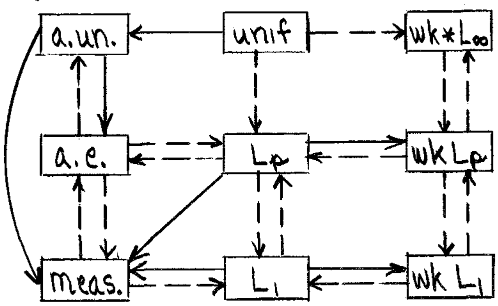


Diagram 29

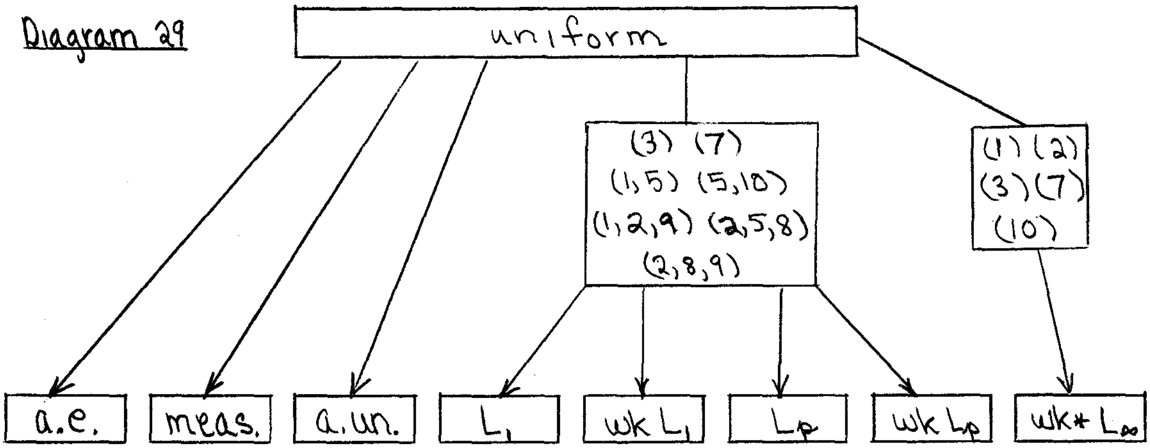
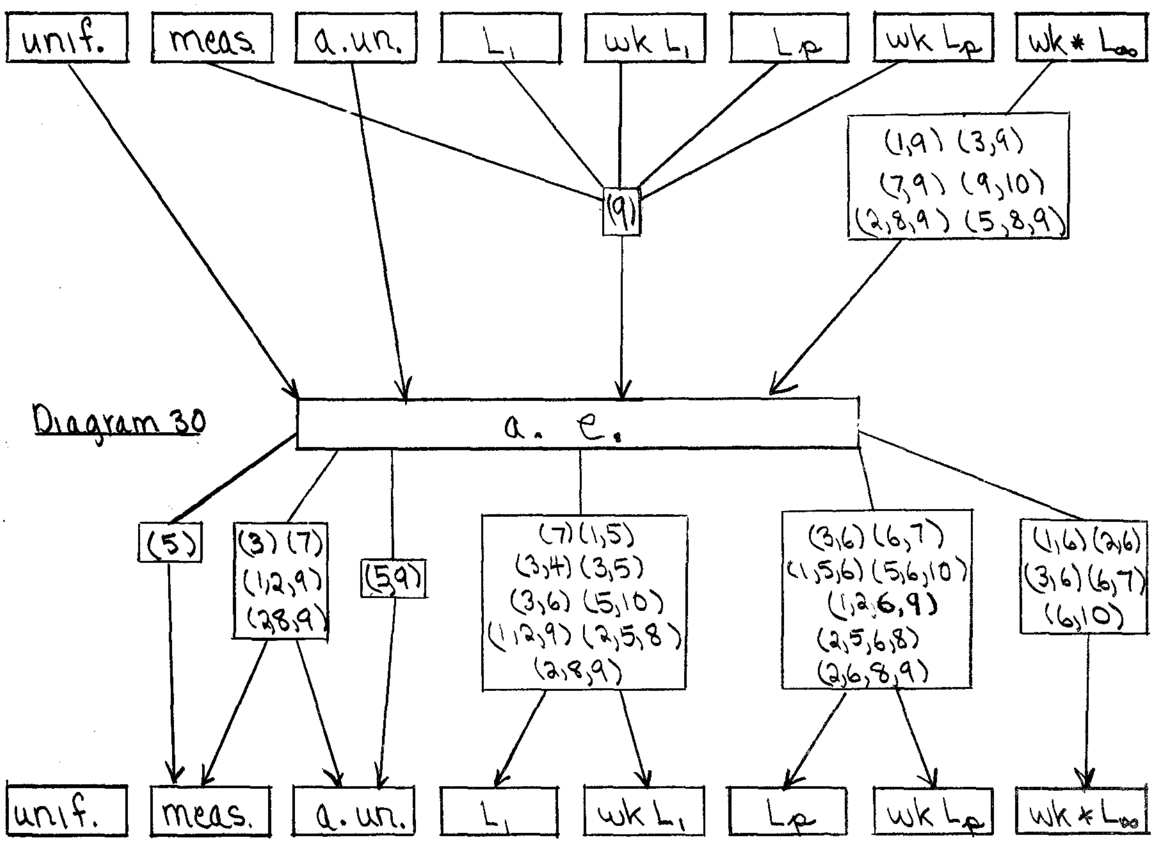


Diagram 30



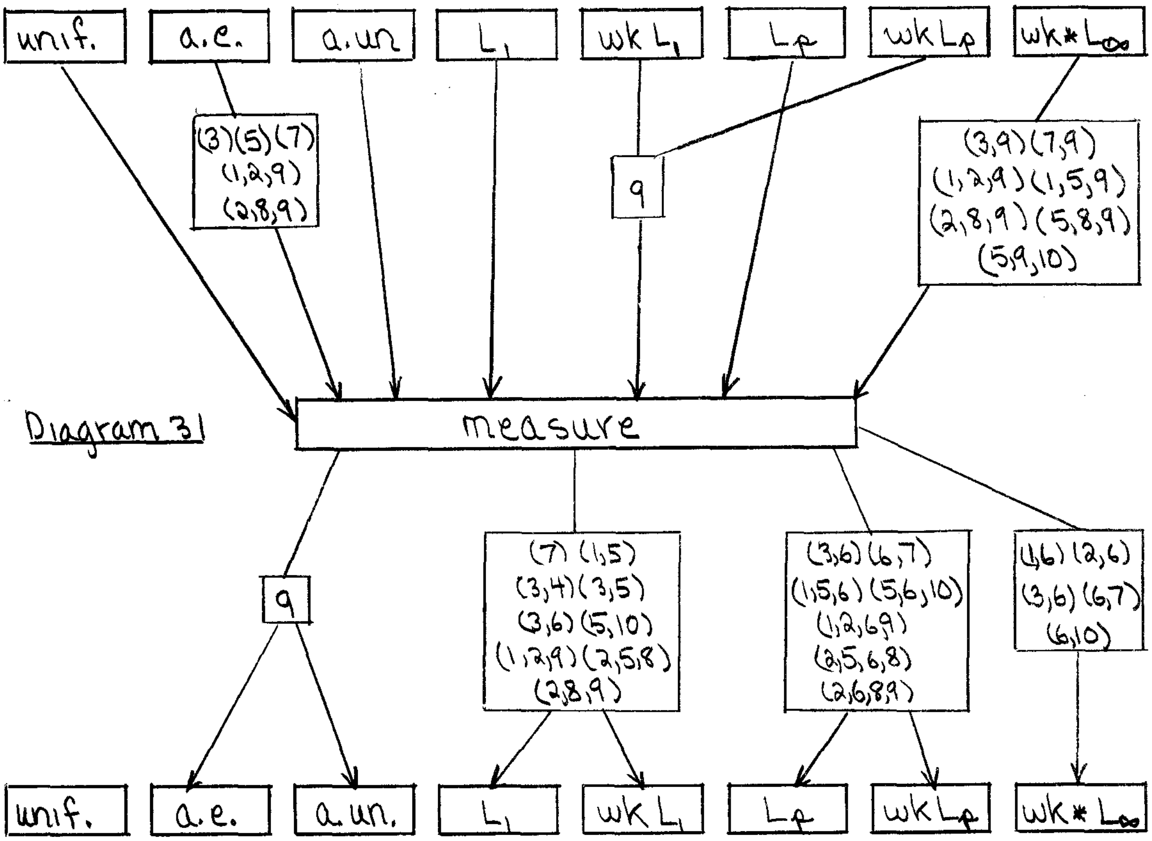


Diagram 31

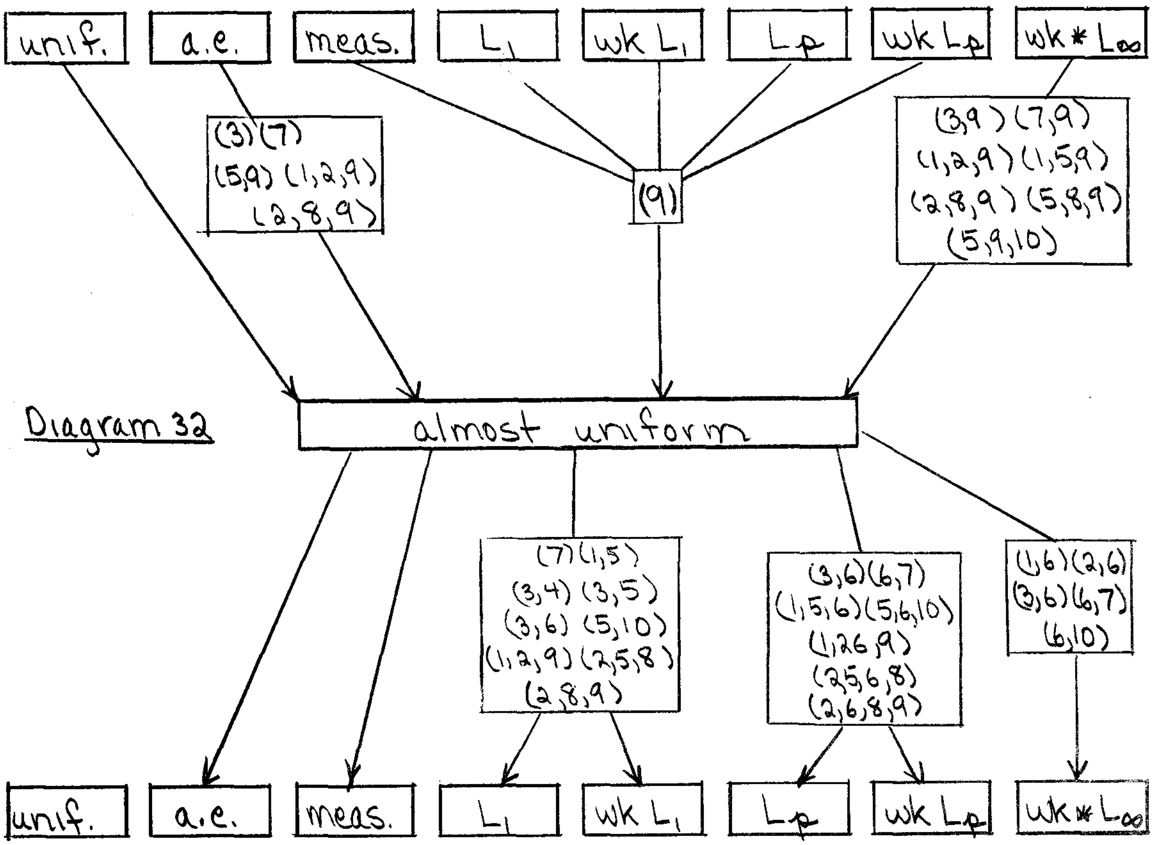


Diagram 32

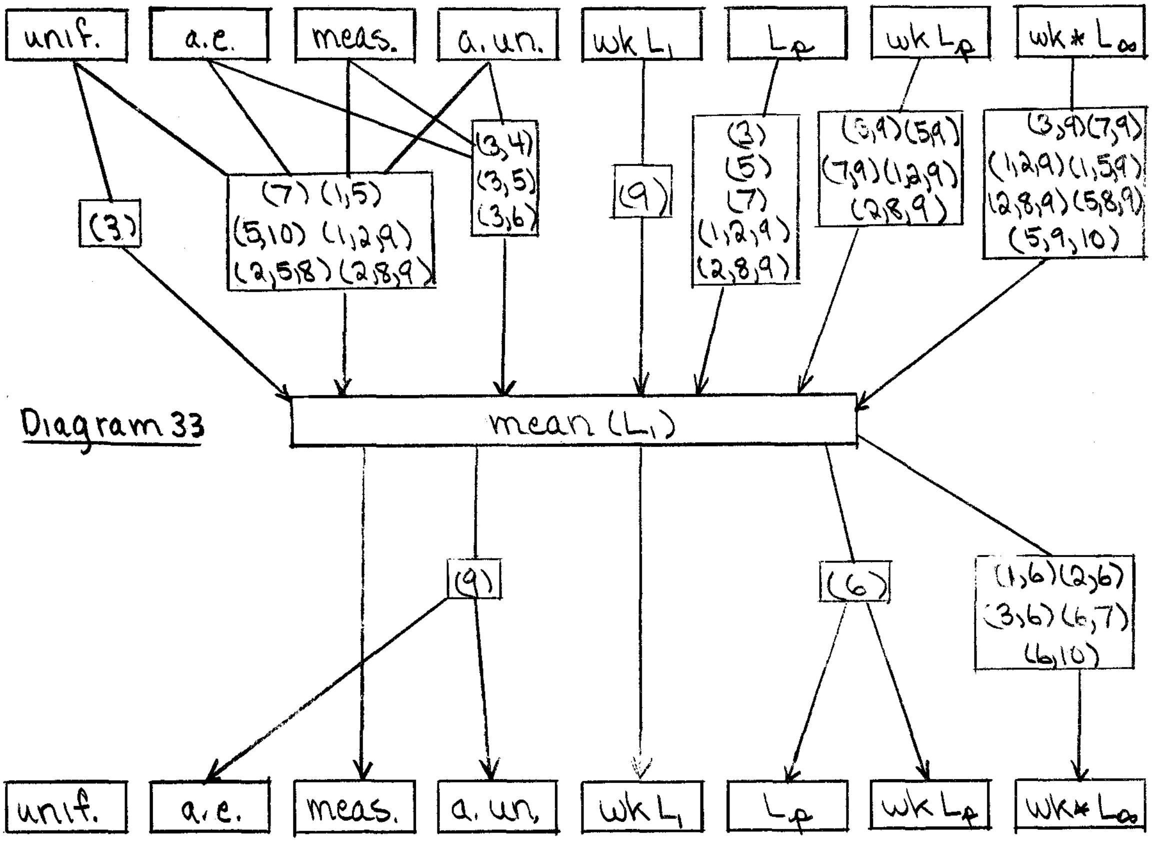


Diagram 33

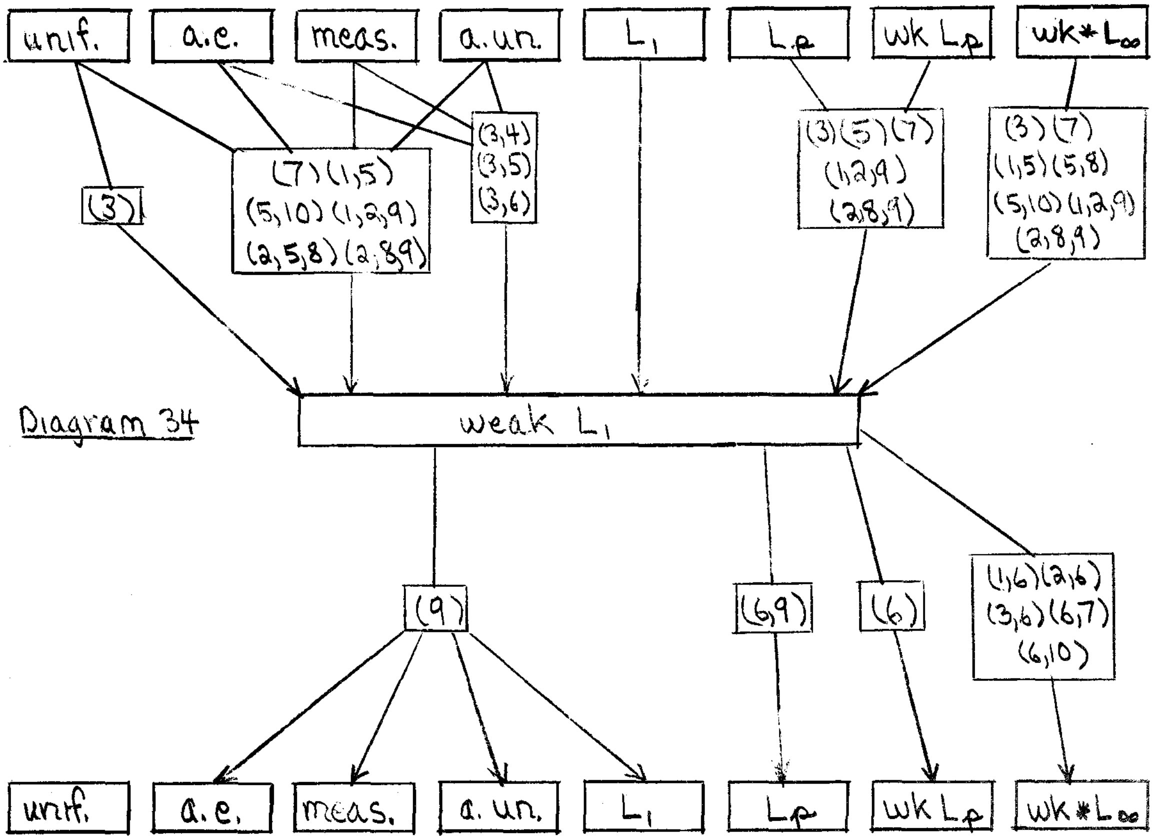


Diagram 34

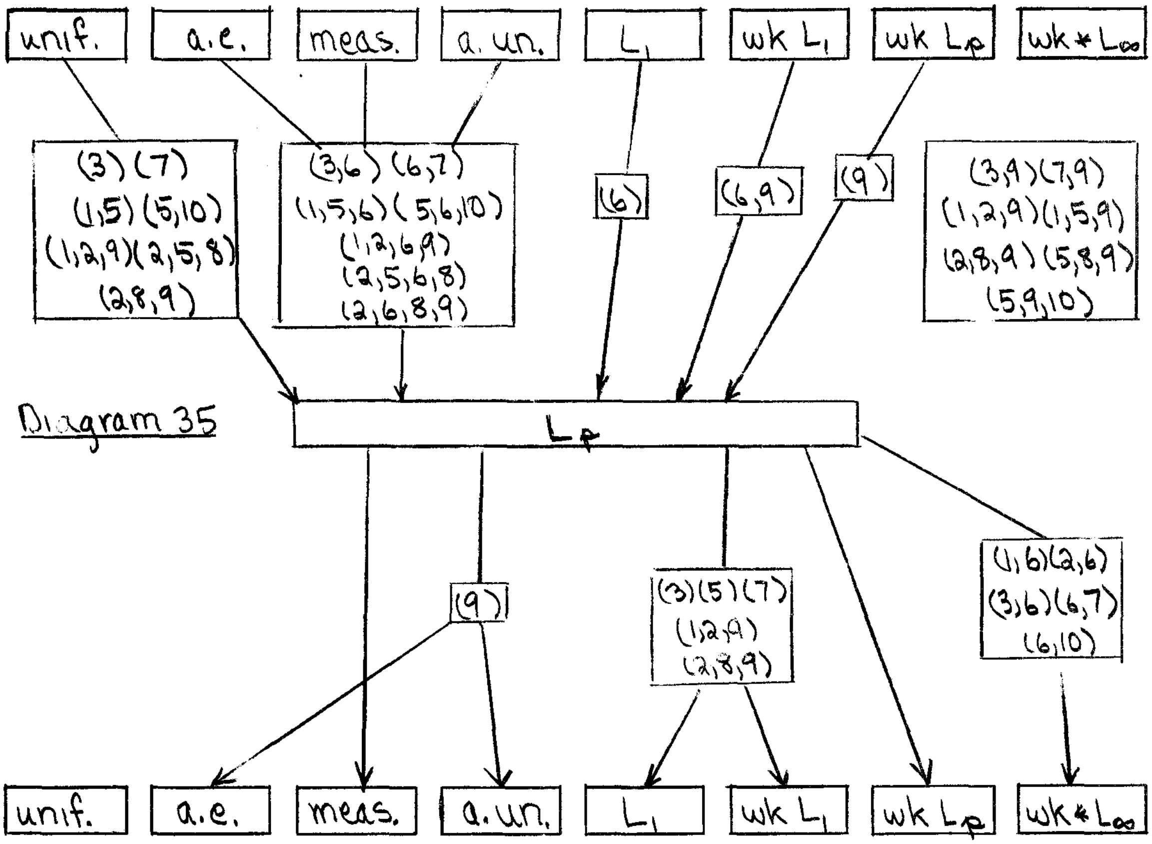


Diagram 35

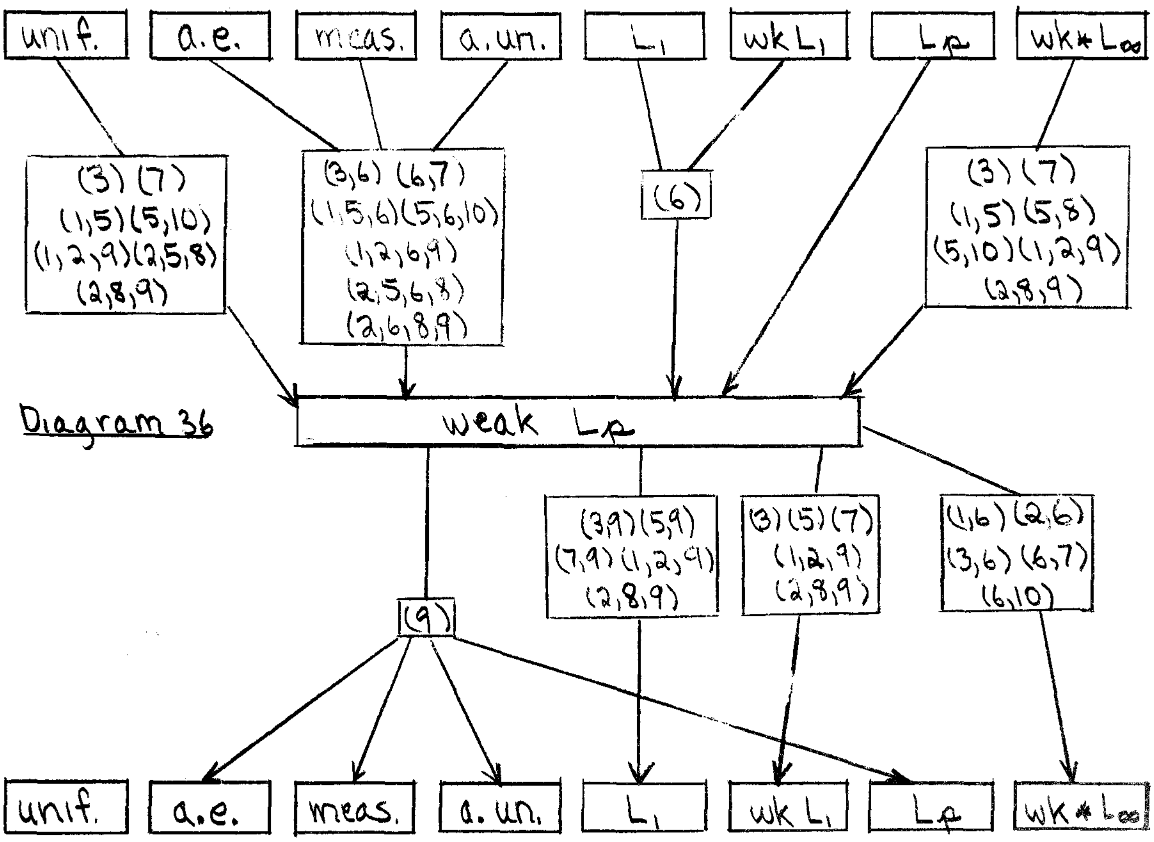


Diagram 36

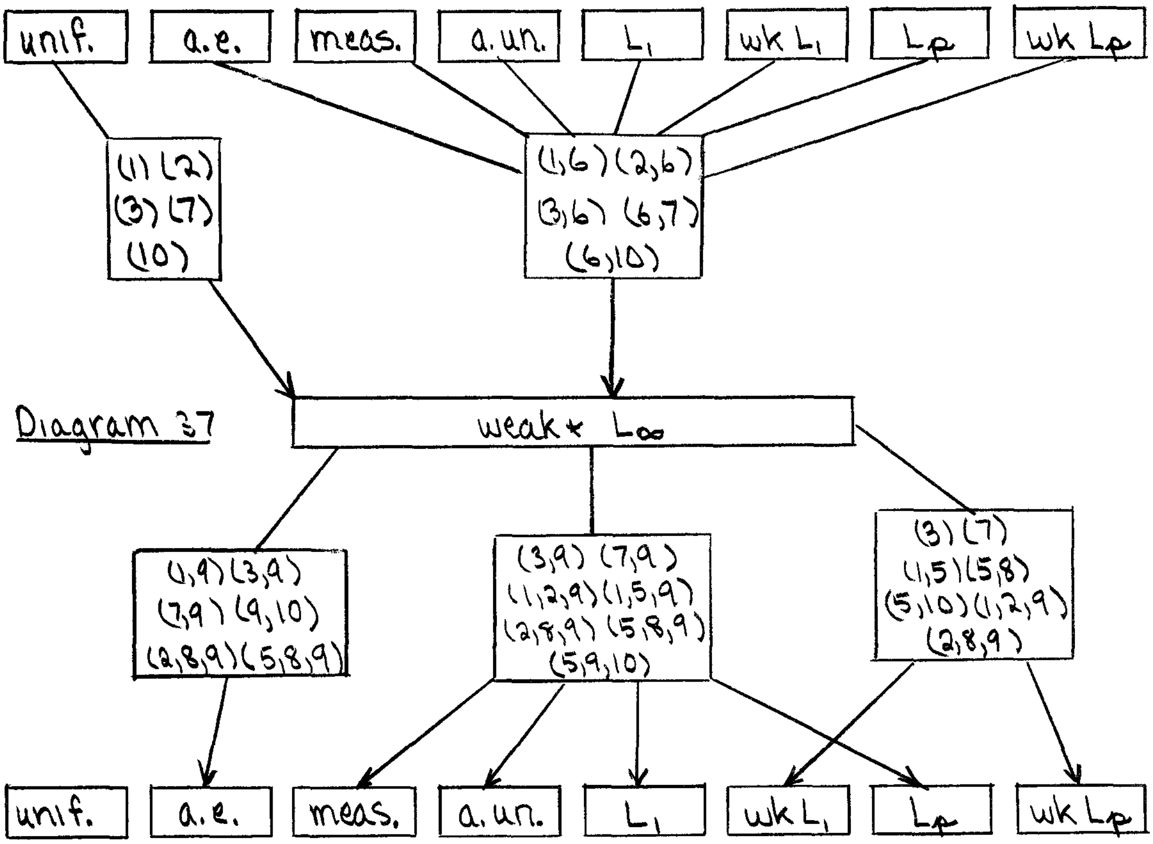

unif.
a.e.
meas.
a.un.
L1
wk L1
Lp
wk Lp
(1) (2) (3) (7) (10)
(1,6) (2,6) (3,6) (6,7) (6,10)
weak* L∞
(1,9) (3,9) (7,9) (9,10) (2,8,9) (5,8,9)
(3,9) (7,9) (1,2,9) (1,5,9) (2,8,9) (5,8,9) (5,9,10)
(3) (7) (1,5) (5,8) (5,10) (1,2,9) (2,8,9)
unif.
a.e.
meas.
a.un.
L1
wk L1
Lp
wk Lp

Diagram 37

2. THE MODES OF CONVERGENCE AND BASIC THEOREMS

Below are given a group of definitions and theorems many of which can be found in the book Measure Theory by Paul R. Halmos. This list is not complete as the author assumes many concepts are well known. Any not given here may be found in Halmos. The functions f_n and f are assumed to be measurable and a. e. finite-valued.

1. Definition: A sequence of functions $\{f_n\}$ converges a. e. to the function f if and only if given $\varepsilon > 0$, there exists $n_0(x, \varepsilon)$ such that for $n \geq n_0(x, \varepsilon)$, $|f_n(x)-f(x)| < \varepsilon$.

2. Definition: A sequence of functions $\{f_n\}$ converges in measure to the function f if and only if $\mu\{x : |f_n(x)-f(x)| > \varepsilon\} \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

3. Definition: A sequence of functions $\{f_n\}$ converges uniformly to the function f if and only if given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$, $|f_n(x)-f(x)| < \varepsilon$ for all x except possibly for a set of measure zero.

4. Definition: A sequence of functions converges almost uniformly to the function f if and only if given $\varepsilon > 0$, there exists a measurable set E such that $\mu(E) < \varepsilon$ and the sequence $\{f_n\}$ converges to f uniformly on E^c.

5. Definition: A sequence of functions $\{f_n\}$ converges to f in mean (L_1) if and only if $\lim_n \int |f_n - f| d\mu = 0$.

6. Definition: A sequence of functions $\{f_n\}$ L_p converges to f if and only if $\lim_n \int |f_n - f|^p d\mu = 0$ where $1 < p < \infty$.

7. Definition: A sequence of functions $\{f_n\}$ weak L_1 converges to f if and only if $\lim_n \int_E (f_n - f) d\mu = 0$ for measurable E

8. Definition: A sequence of functions $\{f_n\}$ weak L_p converges to the function f if and only if

a) $1 < p < \infty$ and

b) there exists N such that $f_n - f \epsilon L_p$ for $n \geq N$ and

c) $\lim_n \int (f_n - f) g d\mu = 0$ for all $g \epsilon L_q$ $(q = \frac{p}{p-1})$.

9. Definition: A sequence of functions $\{f_n\}$ weak * L_∞ converges to f if and only if

a) f has σ-finite support $\{x: f(x) \neq 0\}$ and

b) there exists N such that $f_n - f \epsilon L_\infty$ for $n \geq N$ and

c) $\lim_n \int (f_n - f) g d\mu = 0$ for all $g \epsilon L_1$.

10. Theorem: (Theorem H, p. 97 of Halmos) The indefinite integral of an integrable function is absolutely continuous.

11. Theorem: (Theorem I, p. 98 of Halmos) The indefinite integral of an integrable function is countably additive.

12. Corollary: The indefinite integral of a non-negative integrable function is a finite measure.

13. Theorem: (Theorem F, p. 105 of Halmos) If f is an integrable function, then the set $N(f) = \{x : f(x) \neq 0\}$ has σ-finite measure.

14. Corollary: The union of the supports of a countable number of integrable functions has σ-finite measure.

15. Theorem: (Lebesgue's bounded convergence theorem) If $\{f_n\}$ is a sequence of measurable functions which converges in measure to f [or else converges to f a.e.], and if g is an integrable function such that $|f_n| \leq g$ a.e., $n = 1, 2, \cdots$, then f is integrable and the sequence $\{f_n\}$ converges to f in mean.

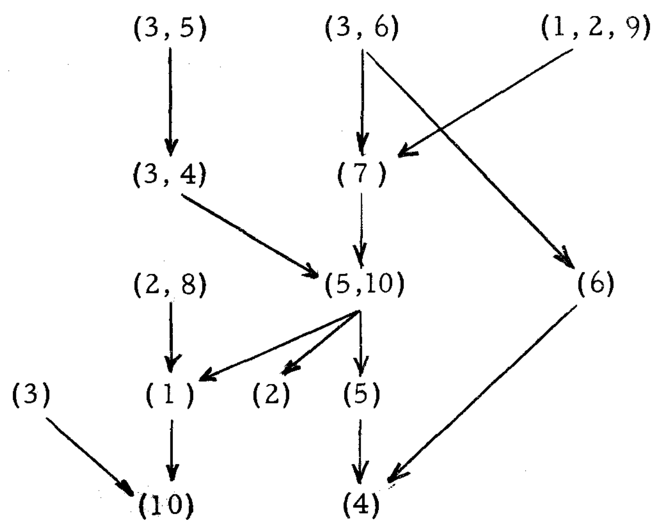
16. Theorem: (Theorem D, p. 38 of Halmos) If μ is a measure on a ring R and if $\{E_n\}$ is an increasing sequence of sets in R for which $\lim_n E_n \in R$, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.

17. Theorem: (Theorem E, p. 38 of Halmos) If μ is a measure on a ring R and if $\{E_n\}$ is a decreasing sequence of sets in R

of which at least one has finite measure and for which $\lim_n E_n \in R$, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.

3. IMPLICATIONS AMONG THE AUXILIARY HYPOTHESES

In order to shorten the length of many proofs we have proved some implications among the ten hypotheses listed in the introduction. Below the results are diagrammed where the numbers enclosed in parentheses are hypotheses and the arrows indicate implications. Sets of hypotheses involving hypotheses 2 and 8 were not originally considered in this investigation since they make assertions about the limit function which depends on the mode of convergence for its definition. Three implications involving hypotheses 2 and 8 are given in the diagram. These three are included in the diagram since they were observed to be true regardless of the mode of convergence assumed. Following the diagram, the results are stated and proved.



Theorem 1: If a sequence of functions $\{f_n\}$ is such that the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equicontinuous from above at 0, then the indefinite integrals of $|f_n|$, $n = 1, 2, \cdots$, are uniformly absolutely continuous.

Proof:

Without loss of generality assume each member of the sequence is finite valued everywhere.

Let $E_n^m = \{x : |f_n(x)| > m\}$ and let $E^m = \bigcup_{n=1}^{\infty} E_n^m$.

$\{E^m\}$ is a decreasing sequence of sets such that $\lim_m E^m = \phi$.

By hypothesis, given $\frac{\varepsilon}{2} > 0$, there exists an $m_0(\varepsilon)$ such that for $m \geq m_0(\varepsilon)$ $\int_{E^m} |f_n| d\mu < \frac{\varepsilon}{2}$ $n = 1, 2, \cdots$.

But for $x \in (E^{m_0})^c$, $|f_n| < m_0$ for $n = 1, 2, \cdots$.

To have uniform absolute continuity we must exhibit for $\varepsilon > 0$ a $\delta > 0$ such that for any E for which $\mu(E) < \delta$, $\int_E |f_n| d\mu < \varepsilon$, $n = 1, 2, \cdots$.

Thus we have only to choose $\delta = \frac{\varepsilon}{2m_0(\varepsilon)}$ and we have uniform absolute continuity.

Theorem 2: If a sequence of functions $\{f_n\}$ is uniformly essentially bounded, then the indefinite integrals of $|f_n|$,

$n = 1, 2, \cdots,$ are uniformly absolutely continuous.

Proof:

Suppose the uniform essential bound is K.

$$\int_E |f_n| d\mu \leq \int_E K d\mu \leq K\mu(E) .$$

If $\mu(E) < \frac{\varepsilon}{K}$, then $\int |f_n| d\mu < \varepsilon$ for $n = 1, 2, \cdots$.

To have uniform absolute continuity we must exhibit for $\varepsilon > 0$,

a $\delta > 0$ such that for any E for which $\mu(E) < \delta$,

$\int_E |f_n| d\mu < \varepsilon$ for $n = 1, 2, \cdots$.

Thus choose $\delta = \frac{\varepsilon}{K}$ and we have shown uniform absolute continuity.

<u>Theorem 3:</u> If there exists integrable g such that $|f_n| \leq g$ a.e. for $n = 1, 2, \cdots,$ then each member of the sequence of functions $\{f_n\}$ is integrable.

Proof:

Immediate result of Theorem A, p. 112 and Theorem C, p. 113 of Halmos.

<u>Theorem 4:</u> If there exists integrable g such that $|f_n| \leq g$ a.e. for $n = 1, 2, \cdots,$ then the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots,$ are uniformly absolutely continuous.

Proof:

Theorem 2.10 tells us g is absolutely continuous.

$$\int_E |f_n| d\mu \leq \int_E g d\mu \quad n = 1, 2, \cdots .$$

Theorem 5: If there exists integrable g such that $|f_n| \leq g$ a.e. for $n = 1, 2, \cdots$, then the indefinite integrals of $|f_n|$, $n = 1, 2, \cdots$, are equicontinuous from above at 0.

Proof:

By Corollary 2.11, $\nu(E) = \int_E g d\mu$ is a finite measure.

Let $\{E_n\}$ be a decreasing sequence of sets such that

$$\lim_n E_n = \phi .$$

By Theorem 2.17, $\lim_n \nu(E_n) = \nu(\lim_n E_n) = \nu(\phi) = 0$

Given $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$

$$\int_{E^m} g d\mu < \varepsilon .$$

But $\int_{E_n} |f_n| d\mu \leq \int_{E_n} g d\mu$ for all n.

Thus the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equicontinuous from above at 0.

Theorem 6: If the sequence of functions $\{f_n\}$ is uniformly essentially bounded and the measure μ is totally finite, then each member of the sequence $\{f_n\}$ is integrable.

Proof:

Let K be the uniform essential bound.

On a totally finite measure space the constant function K is integrable.

$$\int |f_n| d\mu \leq \int K d\mu .$$

A measurable function is integrable if and only if its absolute value if integrable.

Thus each f_n is integrable.

Theorem 7: If the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are uniformly absolutely continuous and the measure μ is totally finite, then the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equicontinuous from above at 0.

Proof:

Let $\{E_n\}$ be a decreasing sequence of sets such that

$\lim_n E_n = \phi$.

By Theorem 2.17 $\lim_n \mu(E_n) = \mu(\lim_n E_n) = \mu(\phi) = 0$.

Given $\varepsilon > 0$ we can, by hypothesis, find a $\delta > 0$ such that for any E for which $\mu(E) < \delta$, $\int_E |f_n| d\mu < \varepsilon$ $n = 1, 2, \cdots$.

But since $\lim_n \mu(E_n) = 0$ we can find an n_0 such that for

$n \geq n_0 \quad \mu(E_n) < \delta$.

Thus we have equicontinuity.

<u>Theorem 8:</u> If the sequence of functions $\{f_n\}$ is uniformly essentially bounded and the measure μ is totally finite, then the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equicontinuous from above at 0.

Proof:

It is a result of Theorems 2 and 7.

<u>Theorem 9:</u> If the sequence of functions $\{f_n\}$ is uniformly essentially bounded and the measure μ is totally finite, then there exists integrable g such that $|f_n| \leq g$ a.e., $n = 1, 2, \cdots$.

Proof:

Let K be the uniform essential bound.

$|f_n| \leq K$ a.e., $n = 1, 2, \cdots$.

$\int K d\mu < \infty$ since a constant function is integrable on a totally finite measure space.

<u>Theorem 10:</u> If the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are uniformly absolutely continuous and the measure μ is totally finite, then each member of the sequence of functions $\{f_n\}$ is integrable.

Proof:

Let f_{n_0} be an arbitrary but fixed member of the sequence.

Without loss of generality assume that f_{n_0} is finite valued everywhere.

Let $E_m = \{x : |f_{n_0}| > m\}$.

Clearly $\{E_m\}$ is a decreasing sequence of sets and $\lim_m E_m = \phi$.

By Theorem 2.17, $\lim_m \mu(E_m) = \mu(\lim_m E_m) = \mu(\phi) = 0$.

By hypothesis, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any E for which $\mu(E) < \delta$ $\int_E |f_n| d\mu < \varepsilon$ $n = 1, 2, \cdots$.

Since $\lim_m \mu(E_m) = 0$, there exists an m_0 such that for $m \geq m_0$, $\mu(E_m) < \delta$.

$$\int_X |f_{n_0}| d\mu = \int_{E_{m_0}} |f_{n_0}| d\mu + \int_{X-E_{m_0}} |f_{n_0}| d\mu \leq \varepsilon + m_0\mu(X) < \infty .$$

Thus f_{n_0} is integrable.

<u>Theorem 11:</u> If the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equicontinuous from above at 0 and the measure μ is totally finite, then each member of the sequence of functions $\{f_n\}$ is integrable.

Proof:

It is a result of Theorems 1 and 10.

Theorem 12: If each member of the sequence of functions $\{f_n\}$ is integrable, then the support of the sequence $\{f_n\}$ has σ-finite measure.

Proof:

Follows from 2.14.

Theorem 13: If there exists integrable g such that $|f_n| \leq g$ a.e. $n = 1, 2, \cdots$, then the support of the sequence of functions $\{f_n\}$ has σ-finite measure.

Proof:

Follows from Theorems 3 and 12.

Theorem 14: If the measure μ is totally finite, then the support of the sequence of functions $\{f_n\}$ has σ-finite measure.

Proof:

A set which has finite measure has σ-finite measure.

Theorem 15: If the support of the sequence of functions $\{f_n\}$ has σ-finite measure and the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are equicontinuous from above at 0, then each member of the sequence is integrable.

Proof:

Without loss of generality assume the measure is totally σ-finite and each member of the sequence $\{f_n\}$ is finite valued everywhere.

Then there exists a sequence $\{E_n\}$ of sets such that

$$X = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad \mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $F_n = X - \bigcup_{i=1}^{n} E_i$ and notice that $\{F_n\}$ is a decreasing sequence of sets such that $\lim_n F_n = \phi$.

By hypothesis, given $\varepsilon > 0$ there exists an n_0 such that

$$\text{for} \quad n \geq n_0 \quad \int_{F_n} |f_k| d\mu < \varepsilon \quad k = 1, 2, \cdots .$$

Let f_{k_0} be an arbitrary but fixed member of the sequence $\{f_n\}$.

Let $K_m = \{x : |f_{k_0}| > m\}$ and notice that $\{K_m\}$ is a decreasing sequence of sets such that $\lim_m K_m = \phi$.

By hypothesis, given $\varepsilon > 0$ there exists an m_0 such that

$$\text{for} \quad m \geq m_0 \quad \int_{K_m} |f_{k_0}| d\mu < \varepsilon .$$

$$\int_X |f_{k_0}|\,d\mu = \int_{F^c_{n_0}} |f_{k_0}|\,d\mu + \int_{F_{n_0}} |f_{k_0}|\,d\mu < \int_{F^c_{n_0}} |f_{k_0}|\,d\mu + \varepsilon$$

$$< \int_{F^c_{n_0} \cap K^c_{m_0}} |f_{k_0}|\,d\mu + \int_{F^c_{n_0} \cap K_{m_0}} |f_{k_0}|\,d\mu + \varepsilon \le \int_{F^c_{n_0} \cap K^c_{m_0}} |f_{k_0}|\,d\mu + 2\varepsilon .$$

For $x \in K^c_{m_0}$, $|f_{k_0}| \le m_0$.

Thus $\int_X |f_{k_0}|\,d\mu < m_0 \mu[F^c_{n_0} \cap K^c_{m_0}] + 2\varepsilon \le m_0 \mu(F^c_{n_0}) + 2\varepsilon$.

But $\mu(F^c_{n_0}) < \infty$.

Thus $\int_X |f_{k_0}|\,d\mu < \infty$.

Since f_{k_0} was an arbitrary member of the sequence, we conclude that each member of the sequence is integrable.

4. LIST OF COUNTEREXAMPLES

We now list a number of counterexamples. Each counterexample will have a sequence of functions $\{f_n\}$ and a limit function f defined on a measure space (X, S, μ). In each counterexample we will indicate the measure space by specifying X, S, and μ. These counterexamples are used throughout the paper and will be referred to by number.

1. X - any non-empty set.

   S - $\{X, \phi\}$.

   μ - defined by $\mu(X) = \infty$, $\mu(\phi) = 0$.

   $f_n(x) = \frac{1}{n}$ for all x.

   $f(x) = 0$ for all x.

   $\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in weak * L_∞ sense.

   $\{f_n\}$ does not converge to f in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

   These hypotheses are satisfied: 2, 4, 5, 6, 9.

2. X - the interval $[-1, 1]$ on the real line.

   S - the Lebesgue measurable sets on $[-1, 1]$.

   μ - Lebesgue measure on $[-1, 1]$.

$$f_n(x) = \begin{cases} n & \text{if} \quad 0 < x \leq \frac{1}{n} \\ -n & \text{if} \quad -\frac{1}{n} \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a. e., in measure, almost uniformly.

$\{f_n\}$ does not converge to f uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak $*$ L_∞ sense.

These hypotheses are satisfied: 1, 2, 3, 8, 10.

3. X - the real line.

S - the Lebesgue measurable sets on the real line.

μ - Lebesgue measure

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if} \quad 0 < x \leq n \\ -\frac{1}{n} & \text{if} \quad -n \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a. e., in measure, uniformly, almost uniformly, in L_p sense, in weak L_p sense, in weak $*$ L_∞ sense.

$\{f_n\}$ does not converge to f in mean, in weak L_1 sense .

These hypotheses are satisfied: 1, 2, 4, 6, 8, 10.

4. X - the real line.

S - the Lebesgue measurable sets on the real line.

μ = Lebesgue measure

$$f(x) = \begin{cases} \dfrac{1}{[x]+1} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad [x] \text{ is the greatest integer } \leq x.$$

$f_n(x) = f(x) \cdot \chi_{[0,n]}$.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in L_p sense, in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in mean, in weak L_1 sense.

These hypotheses are satisfied: 1, 4, 6, 8, 9, 10.

5. X - any non-empty set.

S - $\{X, \phi\}$.

μ - defined by $\mu(X) = \infty$, $\mu(\phi) = 0$.

$f_n(x) = 1 - \frac{1}{n}$ for all x.

$f(x) = 1$ for all x.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly.

$\{f_n\}$ does not converge to f in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

These hypotheses are satisfied: 4, 5, 6, 8, 9.

6. X - the real line.

S - the Lebesgue measurable sets on the real line.

μ - Lebesgue measure.

$f_n(x) = \frac{1}{n}$ for all x.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 2, 4, 6, 9, 10.

7. X - the interval $[0,1]$ on the real line.

S - the Lebesgue measurable sets on $[0,1]$.

μ - Lebesgue measure.

For $n = 1, 2, \cdots$, let $E_n^i = [\frac{i-1}{n}, \frac{i}{n}]$ $i = 1, 2, \cdots, n$.

Let χ_n^i be the characteristic function for E_n^i and consider the following sequence

$\{\chi_1^1, \chi_2^1, \chi_2^2, \chi_3^1, \chi_3^2, \chi_3^3, \cdots,\}$. Let $\{f_n\}$ be this sequence.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f in measure, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f a. e., uniformly, almost uniformly.

These hypotheses are satisfied: 1, 2, 3, 4, 5, 6, 7, 8, 10.

8. X - the interval $[0,1]$ on the real line.

S - the Lebesgue measurable sets on $[0,1]$.

μ - Lebesgue measure on $[0,1]$.

$f_n(x) = x^n$ for all x.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a. e., in measure, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f uniformly.

These hypotheses are satisfied: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

9. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure

$$f_n(x) = \begin{cases} 1 & \text{if } n - \frac{1}{n} \le x \le n \\ 0 & \text{otherwise.} \end{cases}$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in measure, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f uniformly, almost uniformly.

These hypotheses are satisfied: 1, 2, 4, 5, 6, 8, 10.

10. X - the set of positive integers.

S - all subsets of the positive integers.

μ - defined by $\mu(x) = 2^{-x}$, $\mu(E) = \sum_{x \in E} \mu(x)$.

$$f_n(x) = \begin{cases} 2^x & \text{for } x \geq n \\ 0 & \text{otherwise} \end{cases}.$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in measure, almost uniformly.

$\{f_n\}$ does not converge to f uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

These hypotheses are satisfied: 2, 3, 9, 10.

11. X - the set of positive integers.

S - all subsets of the positive integers.

μ - defined by $\mu(x) = 2^{-x}$ $\mu(E) = \sum_{x \in E} \mu(x)$.

$$f_n(x) = \begin{cases} 2^x & \text{for } x \leq n \\ 0 & \text{otherwise}. \end{cases}$$

$f(x) = 2^x$ for all x.

$\{f_n\}$ converges to f a.e., in measure, almost uniformly.

$\{f_n\}$ does not converge to f uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

These hypotheses are satisfied: 1, 3, 8, 9, 10.

12. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

$f_n(x) = \chi_{[n,\infty)}$.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 2, 4, 6, 9, 10.

13. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

$f_n(x) = \chi_{[0,n]}$.

$f(x) = \chi_{[0,\infty)}$.

$\{f_n\}$ converges to f a.e., in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 1, 4, 6, 8, 9, 10.

14. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

$$f_n(x) = \begin{cases} 1 & \text{if} \quad n \leq x \leq n+1 \\ -1 & \text{if} \quad -n-1 \leq x \leq -n \\ 0 & \text{otherwise} . \end{cases}$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense.

These hypotheses are satisfied: 1, 2, 4, 6, 8, 10.

15. X - the interval $[0,1]$ on the real line.

S - the Lebesgue measurable sets on $[0,1]$.

μ - Lebesgue measure.

$f_n(x) = \dfrac{1}{nx^{\frac{1}{r}}}$ for all x $(1 < r < \infty)$.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in measure, almost uniformly, in mean, in weak L_1 sense.

$\{f_n\}$ does not converge to f uniformly, in L_p sense, in weak L_p sense, in weak * L_∞ sense.

These hypotheses are satisfied: 1, 2, 3, 4, 5, 7, 8, 9, 10.

16. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } 0 < x \le n^r \\ -\frac{1}{n} & \text{if } -n^r \le x \le 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (r \text{ is some positive constant} > p).$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 1, 2, 4, 6, 8, 10.

17. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure

$$f(x)=\begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } 0\le x\le 1 \\ \frac{1}{2} & \text{if } 1\le x<1+2^r \\ \frac{1}{3} & \text{if } 1+2^r\le x<1+2^r+3^r \\ \vdots & \end{cases}$$ (r is a real number > p-1)

$f_n(x) = f(x) \cdot \chi_{[0,n]}$.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 1, 4, 6, 8, 9, 10.

18. X - the real line

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

$$f_n(x) = \begin{cases} \dfrac{1}{[x]+1} & \text{for } x \ge n \\ 0 & \text{otherwise} \end{cases}$$ ([x] is the greatest integer $\le$ x).

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in L_p sense, in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f in mean, in weak L_1 sense.

These hypotheses are satisfied: 2, 4, 6, 9, 10.

19. X - the interval $[0, 1]$ on the real line.

S - the Lebesgue measurable sets on $[0, 1]$.

μ - Lebesgue measure on $[0, 1]$.

$f_n(x) = \sin(nx)$ for all x.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f in weak L_1 sense, in weak L_p sense, in weak * L_∞ sense.

$\{f_n\}$ does not converge to f a.e., in measure, uniformly, almost uniformly, in mean, in L_p sense.

These hypotheses are satisfied: 1, 2, 3, 4, 5, 6, 7, 8, 10.

Show $\{f_n\}$ weak L_p converges to f:

Suppose $g \in L_q$.

Show $g \in L_1$; this is shown as follows:

Let $E_1 = \{x: |g| \geq 1\}$, $E_2 = \{x: |g| < 1\}$.

$$\left|\int g\,dx\right| = \left|\int_{E_1} g\,dx\right| + \left|\int_{E_2} g\,dx\right| \le \int |g|^q dx + 1\mu(E_2)$$

$$\le \int |g|^q dx + 1 .$$

By the theorem on p. 33 of Riesz-Nagy, given $\varepsilon > 0$, there exists a stepfunction s such that

$$\int |g-s|\,dx < \varepsilon .$$

$$\left|\int (f_n - f) g\,dx - \int (f_n - f) s\,dx\right| = \left|\int \sin (nx)(g-s)\,dx\right|$$

$$\le \int |\sin nx|\,|g-s|\,dx$$

$$\le \int |g-s|\,dx < \varepsilon .$$

But clearly $\lim\limits_n \int \sin nx\, s\,dx = 0$.

Thus $\lim\limits_n \int (f_n - f) g\,dx = \lim\limits_n \int_0^1 \sin nx\, g\,dx = 0$.

20. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

Let $g(x) = \begin{cases} \frac{1}{x} & \text{for } 1 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$ and choose the sequence of numbers $m_1, m_2, \cdots$ such that $\int_{m_i}^{m_{i+1}} g\,d\mu = 1$.

$f_n(x) = \chi_{[-m_{n+1}, -m_n]} - \chi_{[m_n, m_{n+1}]}$.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in weak $* \ L_\infty$ sense.

$\{f_n\}$ does not converge to f in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 1, 2, 4, 6, 8, 10.

21. X - the real line.

S - the Lebesgue measurable sets.

μ - Lebesgue measure.

Choose a sequence of natural numbers $k_1, k_2, \cdots$ such that

$$\sum_{n=k_r}^{k_{r+1}} \frac{1}{n} > 1 \quad \text{and} \quad k_{r+1} - k_r \quad \text{is even} \quad r = 1, 2, \cdots .$$

Let $h(x) = \begin{cases} (-1)^{[x]+1} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ ($[x]$ is the greatest integer $\leq x$).

$f_n(x) = \chi_{[k_n, k_{n+1}]} \cdot h(x)$.

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a.e., in weak $* \ L_\infty$ sense.

$\{f_n\}$ does not converge to f in measure, uniformly, in weak L_1 sense, almost uniformly, in mean, in L_p sense, in weak L_p sense.

These hypotheses are satisfied: 1, 2, 4, 6, 8, 10.

22. X - $\{1,2\}$.

S - $\{1,2\}$, $\{1\}$, $\{2\}$, $\{\phi\}$.

μ - defined by $\mu\{1,2\}, = \mu\{1\} = \infty$, $\mu\{2\} = 1$, $\mu(\phi) = 0$.

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ \frac{1}{n} & \text{if } x = 2 \end{cases} .$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f in weak * L_∞ sense.

$\{f_n\}$ does not converge to f a.e., in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 2, 4, 5, 6, 9.

23. X - the set of positive integers.

S - all subsets of the positive integers.

μ - defined by $\mu(1) = \infty$, $\mu(x) = 1$ for $x \geq 2$.

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ 1 + \frac{1}{n} & \text{if } x \geq 2 \end{cases} .$$

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x \geq 2 \end{cases} .$$

$\{f_n\}$ converges to f in weak * L_∞ sense.

$\{f_n\}$ does not converge to f a.e., in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

These hypotheses are satisfied: 4, 6, 8, 9 .

24. X - any non-empty set.

S - X, ϕ .

μ - $\mu(X) = \infty$, $\mu(\phi) = 0$.

$f_n(x) = 1$ for all x.

$f(x) = 1$ for all x.

$\{f_n\}$ converges to f a.e., in measure, uniformly, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

$\{f_n\}$ does not converge to f in weak * L_∞ sense.

These hypotheses are satisfied: 4, 5, 6, 8, 9.

25. X - the set of positive integers.

S - all subsets of the positive integers.

μ - defined by $\mu(x) = 2^{-x^2}$, $\mu(E) = \sum_{x \in E} \mu(x)$.

$$f_n(x) = \begin{cases} 2^x & \text{if } x \geq n \\ 0 & \text{otherwise.} \end{cases}$$

$f(x) = 0$ for all x.

$\{f_n\}$ converges to f a. e., in measure, almost uniformly, in mean, in L_p sense, in weak L_1 sense, in weak L_p sense.

$\{f_n\}$ does not converge to f in weak * L_∞ sense, uniformly.

These hypotheses are satisfied: 1, 2, 3, 4, 5, 7, 8, 9, 10.

5. CONVERGENCE IN MEAN - CONVERGENCE IN MEASURE

First investigate the conditions under which convergence in mean implies convergence in measure. The following theorem finishes this investigation.

Theorem 1: If a sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

This is theorem A, p. 103 of Halmos.

Now investiagte the conditions under which convergence in measure implies convergence in mean. Each set of hypotheses listed below, together with convergence in measure, implies convergence in mean. Following the list, the results are stated and proved. Counterexamples 1, 2, 3, 4, 5, 6, 10, 11 show that these are the only implications with a non-redundant set of hypotheses.

(7)	(3, 4)	(3, 6)	(1, 2, 9)	(2, 8, 9)
(1, 5)	(3, 5)	(5, 10)	(2, 5, 8)	

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

This is Theorem 2.15.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

By 3.1 we may assume that we have hypothesis 4.

The conclusion is then that of Theorem C, p. 108 of Halmos.

Theorem 4: Suppose hypotheses 3 and 4 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

3.10 tells us that we may assume hypothesis 1.

Choose $\varepsilon > 0$.

Let $E_{mn} = \{x : |f_n - f_m| \geq \frac{\varepsilon}{3\mu(X)}\}$ and note that $\mu(E_{mn}) \to 0$

as $m, n \to \infty$.

Since we assume hypothesis 4, there exists m_0 such that

$$\text{for} \quad n, m \geq m_0 \int_{E_{mn}} |f_k| \, d\mu < \frac{\varepsilon}{3} \quad k = 1, 2, \cdots .$$

For $m, n \geq m_0$ $\int_X |f_n - f_m| d\mu = \int_{X-E_{mn}} |f_n - f_m| d\mu + \int_{E_{mn}} |f_n - f_m| d\mu$

$$< \frac{\varepsilon}{3\mu(X)} \cdot \mu(X-E_{mn}) + \int_{E_{mn}} |f_n| d\mu + \int_{E_{mn}} |f_m| d\mu \leq \varepsilon .$$

Since ε is arbitrary, $\lim\limits_{m,n} \int_X |f_n - f_m| d\mu = 0$.

Thus $\{f_n\}$ is a mean fundamental sequence of integrable functions and according to Theorem B, p. 107 of Halmos there is an integrable function g such that

$$\lim \int |f_n - g| d\mu = 0.$$

But then $\{f_n\}$ converges in measure to g.

$$\{x : |f-g| \geq \varepsilon\} \subseteq \{x : |f_n - f| \geq \tfrac{\varepsilon}{2}\} \cup \{x : |f_n - g| \geq \tfrac{\varepsilon}{2}\} .$$

Thus $f = g$ a.e.

Thus $\{f_n\}$ converges in mean to f.

<u>Theorem 5:</u> Suppose hypotheses 3 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

By 3.1 we may assume that we have hypothesis 4.

The conclusion then follows from Theorem 4.

Theorem 6: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Let C be the uniform essential bound for the sequence of functions $\{f_n\}$.

On a totally finite measure space a constant function is integrable.

The conclusion is a result of 2.15.

Theorem 7: Suppose hypotheses 5 and 10 are satisfied. If a sequence of functions $\{f_n\}$ converges in measure to f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 3.15 and Theorem 3.

Theorem 8: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$|f_n| \leq \max(|f|, |f_1|)$ which is integrable.

The conclusion is then a result of 2.15.

Theorem 9: Suppose hypotheses 2, 5 and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

By hypothesis, $\lim_n \int f_n d\mu = \int f d\mu$ and f is integrable.

Thus there exists n_0 such that for $n \geq n_0$, f_n is integrable.

Thus without loss of generality we may assume hypothesis 1.

Let $E = \bigcup_{n=1}^{\infty} \{x : f_n(x) \neq 0\}$.

By 3.12, E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and notice that $\{E^m\}$ is a decreasing sequence of sets such that $\lim_m E^m = \phi$.

Choose $\varepsilon > 0$.

Since we assume hypothesis 5, there exists m_0 such that

$$\text{for} \quad m \geq m_0 \quad \int_{E^m} |f_n| \, d\mu < \frac{\varepsilon}{6} \qquad n = 1, 2, \cdots .$$

By 3.1 we may assume we have hypothesis 4.

Let $E_{mn} = \{x: |f_n - f_m| \geq \frac{\varepsilon}{3\mu(E-E^{m_0})}\}$ and notice that

$$\mu(E_{mn}) \to 0 \quad \text{as} \quad m,n \to \infty.$$

Since we have hypothesis 4, there exists N such that for

$$m,n \geq N, \quad \int_{E_{mn}} |f_k|\, d\mu < \frac{\varepsilon}{6} \quad k = 1, 2, \cdots .$$

For $m,n \geq N$

$$\int_X |f_n - f_m|\, d\mu = \int_E |f_n - f_m|\, d\mu = \int_{E-E^{m_0}} |f_n - f_m|\, d\mu + \int_{E\cap E^{m_0}} |f_n - f_m|\, d\mu$$

$$< \int_{E-E^{m_0}} |f_n - f_m|\, d\mu + \frac{\varepsilon}{3} < \int_{(E-E^{m_0})-E_{mn}} |f_n - f_m|\, d\mu$$

$$+ \int_{(E-E^{m_0})\cap E_{mn}} |f_n - f_m|\, d\mu + \frac{\varepsilon}{3}$$

$$\leq \frac{\varepsilon}{3\mu(E-E^{m_0})} \cdot \mu(E-E^{m_0}) + \int_{(E-E^{m_0})\cap E_{mn}} |f_n|\, d\mu$$

$$+ \int_{(E-E^{m_0})\cap E_{mn}} |f_m|\, d\mu + \frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon .$$

Since ε is arbitrary $\lim_{n,m} \int |f_n - f_m| d\mu = 0$.

With the same closing argument as that in Theorem 3 we conclude $\{f_n\}$ converges in mean to f.

<u>Theorem 10:</u> Suppose hypotheses 2, 8 and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

By hypothesis, $\lim_{n} \int f_n d\mu = \int f d\mu$ and f is integrable.

Thus there exists n_0 such that for $n \geq n_0$ f_n is integrable.

Thus without loss of generality assume we have hypothesis 1.

$|f_n| \leq \max(|f|, |f_1|)$ $n = 1, 2, \cdots$.

The conclusion is then the result of 2.15.

6. ALMOST UNIFORM CONVERGENCE-CONVERGENCE IN MEASURE

First investigate the conditions under which almost uniform convergence implies convergence in measure. The following theorem finishes this investigation.

Theorem 1: If a sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

This is Theorem B, p. 92 of Halmos.

Now investigate the conditions under which convergence in measure implies almost uniform convergence. In counterexample 7 $\{f_n\}$ converges in measure to $f = 0$, hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge almost uniformly to $f = 0$. The one missing hypothesis gives a reault which is now stated and proved. An interesting point is that convergence in measure of $\{f_n\}$ to f implies the existence of a subsequence that converges almost uniformly to f.

Theorem 2: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Let $E_n^k = \{x : |f_n(x) - f(x)| > \frac{1}{2^k}\}$.

Notice that for each k, E_n^k decreases with n.

Choose $\varepsilon > 0$.

There exists n_1 such that for $n \geq n_1$, $\mu(E_n^1) < \frac{\varepsilon}{2}$.

For $k = 2, 3, \cdots$, there exists $n_k > n_{k-1}$ such that for

$$n \geq n_k, \quad \mu(E_n^k) < \frac{\varepsilon}{2^k}.$$

Let $E = \bigcup_{k=1}^{\infty} E_{n_k}^k$.

$$\mu(E) = \mu\left[\bigcup_{k=1}^{\infty} E_{n_k}^k\right] \leq \sum_{k=1}^{\infty} \mu(E_{n_k}^k) < \varepsilon.$$

$$E^c = \left[\bigcup_{k=1}^{\infty} E_{n_k}^k\right]^c = \bigcap_{k=1}^{\infty} \left[E_{n_k}^k\right]^c.$$

Thus for all $x \in E^c$ and $n \geq n_k$, $|f_n(x) - f(x)| < \frac{1}{2^k}$.

Since this is true for every k, we have uniform convergence on E^c.

Since ε is arbitrary, we have almost uniform convergence.

7. CONVERGENCE IN MEAN - UNIFORM CONVERGENCE

First investigate the conditions under which convergence in mean implies uniform convergence. In counterexample 8 $\{f_n\}$ converges in mean to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no results with convergence in mean implying uniform convergence.

Now investigate the conditions under which uniform convergence implies convergence in mean. Each set of hypotheses listed below, together with uniform convergence, implies convergence in mean. Following the list, the results are stated and proved. Counterexamples 1, 3, 4, 5, 6 show that these are the only implications with a non-redundant set of hypotheses.

(3) (1,5) (1, 2, 9) (2, 8, 9)

(7) (5, 10) (2, 5, 8)

Theorem 1: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$,

$$|f_n(x)-f(x)| < \varepsilon .$$

On a totally finite measure space a constant function is

integrable and hence $f_n - f \in L_1$ for $n \geq n_0(\varepsilon)$.

For $n \geq n_0(\varepsilon)$, $\int |f_n - f| \, d\mu < \varepsilon \mu(X)$.

Since ε is arbitrary $\lim_n \int |f_n - f| \, d\mu = 0$.

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Uniform convergence implies convergence in measure and so

$\{f_n\}$ converges in measure to f.

Now the conclusion is a result of 5. 2.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 5. 3 and the fact that uniform convergence implies

convergence in measure.

Theorem 4: Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 5.7 and the fact that uniform convergence implies convergence in measure.

<u>Theorem 5:</u> Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 5.8 and the fact that uniform convergence implies convergence in measure.

<u>Theorem 6:</u> Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 5.9 and the fact that uniform convergence implies convergence in measure.

<u>Theorem 7:</u> Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 5.10 and the fact that uniform convergence implies convergence in measure.

8. CONVERGENCE IN MEAN - ALMOST UNIFORM CONVERGENCE

First investigate the conditions under which convergence in mean implies almost uniform convergence. In counterexample 7 $\{f_n\}$ converges in mean to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge almost uniformly to $f = 0$. The one remaining hypothesis gives an implication which is stated and proved below. An interesting fact is that convergence in mean of a sequence of functions $\{f_n\}$ to the function f, implies the existence of a subsequence which converges almost uniformly to f.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 5.1 and 6.2.

Now investigate the conditions under which almost uniform convergence implies convergence in mean. Each set of hypothesis listed below, together with almost uniform convergence, implies convergence in mean. Following the list, the results are stated because we need them numbered for future use; the proofs are all consequences of the same numbered theorems in Section 5 and the fact that almost uniform convergence implies convergence in measure. Counterexamples 1, 2, 3, 4, 5, 6, 10, 11 show that there are no other implications with a non-redundant set of hypotheses.

(7)	(3, 5)	(1, 2, 9)
(1, 5)	(3, 6)	(2, 5, 8)
(3, 4)	(5, 10)	(2, 8, 9)

<u>Theorem 2:</u> Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 3:</u> Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 4:</u> Suppose hypotheses 3 and 4 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 5:</u> Suppose hypotheses 3 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 6:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 7:</u> Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 8:</u> Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 9:</u> Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

<u>Theorem 10:</u> Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges in mean to f.

9. ALMOST UNIFORM CONVERGENCE - CONVERGENCE a. e.

First investigate the conditions under which almost uniform convergence implies convergence a. e. The following theorem finishes this investigation.

Theorem 1: If a sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ converges a. e. to f.

Proof:

This is Theorem B, p. 89 of Halmos.

Now investigate the conditions under which convergence a. e. implies almost uniform convergence. Each set of hypotheses listed below, together with convergence a. e., implies almost uniform convergence. Following the list, the results are stated and proved. Counterexamples 9, 12, 13 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(5, 9)	(2, 8, 9)
(7)	(1, 2, 9)	

Theorem 2: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converges a. e. to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Given on p. 88 of Halmos.

Theorem 3: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

By hypothesis, there exists integrable g such that

$$|f_n| \leq g \quad \text{a.e.} \quad n = 1, 2, \cdots .$$

Clearly $|f| \leq g$ a.e.

Since we are trying to prove almost uniform convergence, we can without loss of generality assume $\{f_n\}$ converges pointwise to f everywhere.

Let $E_n^m = \{x : |f_n - f| > \frac{1}{m}\}$ and let $A_i^m = \bigcup_{n=i}^{\infty} E_n^m$.

For each i, A_i^m has finite measure since

$$\frac{1}{m}\mu(A_i^m) = \int_{A_i^m} \frac{1}{m} d\mu \leq 2 \int g d\mu < \infty.$$

Since we assume convergence everywhere

$$\lim_i A_i^m = \bigcap_i \bigcup_{n=i}^{\infty} \{x : |f_n - f| > \frac{1}{m}\} = \phi \quad \text{for all } m .$$

$$\lim_i \mu(A_i^m) = \mu[\lim_i A_i^m] = \mu(\phi) = 0 .$$

Choose $\varepsilon > 0$.

For each m, there exists an $i_0(m)$ such that for $i \geq i_0(m)$,

$$\mu(A_i^m) < \frac{\varepsilon}{2^m}.$$

Let $F = \bigcup_{m=1}^{\infty} A_{i_0(m)}^m$.

$$\mu(F) = \mu\left[\bigcup_{m=1}^{\infty} A_{i_0(m)}^m\right] < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon \ .$$

$$F^c = \left[\bigcup_{m=1}^{\infty} A_{i_0(m)}^m\right]^c = \bigcap_{m=1}^{\infty} \left[A_{i_0(m)}^m\right]^c .$$

Thus for $x \in F^c$ and $n \geq i_0(m)$, $|f_n - f| < \frac{1}{m}$.

Since this must hold for each m, we have shown uniform convergence on F^c.

Since ε is arbitrary, we have almost uniform convergence.

<u>Theorem 4:</u> Suppose hypotheses 5 and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a. e. to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Without loss of generality assume $\{f_n\}$ converges to f everywhere.

Let k be an arbitrary fixed positive integer.

Let $E_k^n = \{x : |f_n - f| > \frac{1}{k}\}$ and notice that the sequence of sets $\{E_k^n\}$ decreases with n and $\lim_n E_k^n = \phi$.

Because of hypothesis 5, there exists $n_0(k)$ such that

$$\int_{E_k^{n_0(k)}} |f_j| \, d\mu < 1 \quad j = 1, 2, \cdots .$$

On the set $E_k^{n_0(k)}$, hypotheses 1, 5, and 9 are satisfied and of course $\{f_n\}$ converges pointwise to f.

Theorem 10.3 tells us $\{f_n\}$ converges in measure to f on $E_k^{n_0(k)}$.

Theorem 5.3 tells us $\{f_n\}$ converges in mean to f on $E_k^{n_0(k)}$, that is $\lim_i \int_{E_k^{n_0(k)}} |f_i - f| \, d\mu = 0$.

Thus there exists $i_0 > n_0$ such that $\int_{E_k^{n_0(k)}} |f_{i_0} - f| \, d\mu < \infty$.

Since $E_k^{i_0} \subseteq E_k^{n_0(k)}$, $\int_{E_k^{i_0}} |f_{i_0} - f| \, d\mu < \infty$.

$E_k^{i_0}$ has finite measure since

$$\frac{1}{k} \mu(E_k^{i_0}) = \int_{E_k^{i_0}} \frac{1}{k} \, d\mu \leq \int_{E_k^{i_0}} |f_{i_0} - f| \, d\mu < \infty .$$

Thus $\lim_n \mu(E_k^n) = \mu[\lim_n E_k^n] = \mu(\phi) = 0$ for all k.

Choose $\varepsilon > 0$.

There exists n_1 such that for $n \geq n_1$, $\mu(E_1^n) < \frac{\varepsilon}{2}$.

For $k = 2, 3, \cdots$, there exists $n_k > n_{k-1}$ such that for

$n \geq n_k$, $\mu(E_k^n) < \frac{\varepsilon}{2^k}$.

Let $F = \bigcup_{k=1}^{\infty} E_k^{n_k}$.

$$\mu(F) = \mu\left[\bigcup_{k=1}^{\infty} E_k^{n_k}\right] \leq \sum_{k=1}^{\infty} \mu(E_k^{n_k}) < \varepsilon .$$

$$F^c = \left[\bigcup_{k=1}^{\infty} E_k^{n_k}\right]^c = \bigcap_{k=1}^{\infty} [E_k^{n_k}]^c .$$

Thus for $x \in F^c$ and $n \geq n_k$, $|f_n(x) - f(x)| < \frac{1}{k}$.

Since the above statement is true for every k we have shown

almost uniform convergence on F^c.

Since ε was arbitrary, we have shown almost uniform

convergence.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

We may assume we have hypothesis 7 since $\max[|f_1|, |f|]$

is integrable and $|f_n| \leq \max[|f_1|, |f|]$ a.e.

$n = 1, 2, 3, \cdots$.

The conclusion is then a result of Theorem 3.

Theorem 6: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Since we have hypothesis 8, $\lim_n \int f_n d\mu = \int f d\mu$.

There exists an n_0 such that for $n \geq n_0$, f_n is integrable. Thus without loss of generality we may assume we have hypothesis 1.

The conclusion is a result of Theorem 5.

10. CONVERGENCE IN MEASURE - CONVERGENCE a. e.

First investigate the conditions under which convergence in measure implies convergence a. e. In counterexample 7 $\{f_n\}$ converges in measure to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge a. e. to $f = 0$. The one remaining hypothesis gives an implication which is stated and proved below. Of interest here is the fact that a sequence of functions which converges in measure to a particular function necessarily has a subsequence which converges a. e. to that function.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ converges a. e. to f.

Proof:

Follows from 6. 2 and 9. 1.

Now investigate the conditions under which convergence a. e. implies convergence in measure. Each set of hypotheses listed below, together with convergence a. e. implies convergence in measure. Following the list, the results are stated and proved. Counterexamples 12, 13, 14 show that these are the only implications with a non-redundant set of hypotheses.

(3) (7) (2, 8, 9)

(5) (1, 2, 9)

Theorem 2: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converge a. e. to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 9. 2 and 6. 1.

Theorem 3: Suppose hypothesis 5 is satisfied. If the sequence of functions $\{f_n\}$ converges a. e. to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Let $E_n^m = \{x : |f_m - f_n| > \varepsilon\}$.

Let $E^m = \bigcup_{n=m}^{\infty} E_n^m$.

Let $A_i = \bigcup_{m=i}^{\infty} E^m$.

Let $A = \lim_i A_i$.

Clearly $\{A_i\}$ is a decreasing sequence of sets and since we assume convergence a. e., $\mu(A) = 0$.

$\lim_i (A_i - A) = \phi$.

Since we have hypothesis 5, there exists an i_0 such that

$$\int_{A_{i_0}-A} |f_n| \, d\mu < 1 \qquad n = 1, 2, \cdots .$$

Thus on $A_{i_0}-A$ each member of the sequence $\{f_n\}$ is integrable.

Let $F = \bigcup_{n=1}^{\infty} \{x : f_n(x) \neq 0, \quad x \in A_{i_0}-A\}$.

E has σ-finite measure, that is $F = \bigcup_{n=1}^{\infty} F_n$ where $\mu(F_n) < \infty, \quad n = 1, 2, \cdots$.

Let $F^m = F - \bigcup_{n=1}^{m} F_n$ and note that $\{F^m\}$ is a decreasing sequence of sets such that $\lim_m F^m = \phi$.

Since we have hypothesis 5, given $\varepsilon_1 > 0$, there exists $n_0(\varepsilon_1)$ such that for $m \geq m_0$, $\int_{F^m} |f_n| \, d\mu < \frac{\varepsilon_1}{5}$ $n = 1, 2, \cdots$.

$$\int_{A_{i_0}-A} |f_m - f_n| \, d\mu = \int_F |f_m - f_n| \, d\mu = \int_{F-F^{m_0}} |f_m - f_n| \, d\mu + \int_{F \cap F^{m_0}} |f_m - f_n| \, d\mu$$

$$< \int_{F-F^{m_0}} |f_m - f_n| \, d\mu + \frac{2\varepsilon_1}{5} .$$

Since $\mu(F-F^{m_0}) < \infty$, Theorem 2 tells us $\{f_n\}$ converges in measure to f on $F-F^{m_0}$, and thus is fundamental in measure on $F - F^{m_0}$.

Thus $\lim_{m,n} \mu[G_n^m] = 0$ where

$$G_n^m = \{x : |f_m - f_n| > \frac{\varepsilon_1}{5\mu(F - F^{m_0})}, \quad x \in F - F^{m_0}\}.$$

Theorem 3.1 allows us to conclude that the indefinite integrals of $|f_n|$ $n = 1, 2, \cdots$ are uniformly absolutely continuous.

Thus there exists N such that for $m, n \geq N$

$$\int_{G_n^m} |f_j| \, d\mu < \frac{\varepsilon_1}{5} \qquad j = 1, 2, \cdots .$$

Thus for $m, n \geq N$

$$\int_{A_{i_0} - A} |f_m - f_n| \, d\mu < \int_{F - F^{m_0}} |f_m - f_n| \, d\mu + \frac{2\varepsilon_1}{5}$$

$$\leq \int_{(F - F^{m_0}) - G_n^m} |f_m - f_n| \, d\mu + \int_{(F - F^{m_0}) \cap G_n^m} |f_m - f_n| \, d\mu + \frac{2\varepsilon_1}{5}$$

$$< \frac{\varepsilon_1}{5\mu(F - F^{m_0})} \cdot \mu(F - F^{m_0}) + \frac{4\varepsilon_1}{5} = \varepsilon_1 .$$

Since ε_1 is arbitrary, $\lim_{m,n} \int_{A_{i_0} - A} |f_m - f_n| \, d\mu = 0$.

Note that for $m,n \geq i_0$, $E_n^m \subseteq A_{i_0} - A$ except for a set of measure zero, and so $0 = \lim_{m,n} \int_{A_{i_0}-A} |f_m - f_n| d\mu \geq \int_{E_n^m} |f_m - f_n| d\mu$

$$\geq \varepsilon \mu(E_n^m) .$$

Thus $\mu(E_n^m) \to 0$ as $m,n \to \infty$.

By Theorem E, p. 93 of Halmos, there exists a function g such that $\{f_n\}$ converges in measure to g.

$$\{x: |f_n - g| > \varepsilon\} \subseteq \{x: |f_n - g| > \tfrac{\varepsilon}{2}\} \cup \{x: |f_m - g| > \tfrac{\varepsilon}{2}\}.$$

By Theorem D, p. 93 of Halmos and 9.1 we know there is a subsequence of $\{f_n\}$ which converges a. e. to g.

Thus $f = g$ a. e. and thus $\{f_n\}$ converges in measure to f.

Theorem 4: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges a. e. to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 9.3 and 6.1.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a. e. to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 9.5 and 6.1.

<u>Theorem 6:</u> Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 9.6 and 6.1.

11. UNIFORM CONVERGENCE - CONVERGENCE a.e.

First investigate the conditions under which uniform convergence implies convergence a.e. The following theorem finishes this investigation.

Theorem 1: If a sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

Obvious.

Now investigate the conditions under which convergence a.e. implies uniform convergence. In counterexample 8 $\{f_n\}$ converges a.e. to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no implications with convergence a.e. implying uniform convergence.

12. UNIFORM CONVERGENCE - ALMOST UNIFORM CONVERGENCE

First investigate the conditions under which uniform convergence implies almost uniform convergence. The following theorem finishes this investigation.

Theorem 1: If a sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges almost uniformly to f.

Now investigate the conditions under which almost uniform convergence implies uniform convergence. In counterexample 8 $\{f_n\}$ converges almost uniformly to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we have no results with almost uniform convergence implying uniform convergence.

13. UNIFORM CONVERGENCE - CONVERGENCE IN MEASURE

First investigate the conditions under which uniform convergence implies convergence in measure. The following theorem finishes this investigation.

Theorem 1: If a sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Obvious.

Now investigate the conditions under which convergence in measure implies uniform convergence. In counterexample 8, $\{f_n\}$ converges in measure to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no implications with convergence in measure implying uniform convergence.

14. CONVERGENCE IN MEAN - CONVERGENCE a.e.

First investigate the conditions under which convergence in mean implies convergence a.e. In counterexample 7 $\{f_n\}$ converges in mean to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge a.e. to $f = 0$. The one remaining hypothesis gives an implication which is stated and proved below. Of interest here is the fact that a sequence of functions which converges in mean to a particular function necessarily has a subsequence which converges a.e. to that function.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

Follows from 9.1 and 10.1 .

Now investigate the conditions under which convergence a.e. implies convergence in mean. Each set of hypotheses listed below, together with convergence a.e., implies convergence in mean. Following the list, the results are stated and proved. Counterexamples 1, 2, 3, 5, 10, 11, 12, 13 show that these are the only implications with a non-redundant set of hypotheses.

(7)	(3, 5)	(1, 2, 9)
(1, 5)	(3, 6)	(2, 5, 8)
(3, 4)	(5, 10)	(2, 8, 9)

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 10.4 and 5.2.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 10.3 and 5.3.

Theorem 4: Suppose hypotheses 3 and 4 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 3.7, 3.10 and Theorem 3.

<u>Theorem 5:</u> Suppose hypotheses 3 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 3.11 and Theorem 3.

<u>Theorem 6:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 3.9 and Theorem 2.

<u>Theorem 7:</u> Suppose hypotheses 10 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 3.15 and Theorem 3.

<u>Theorem 8:</u> Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$|f_n| \leq \max[|f_1|, |f|] \quad \text{for} \quad n = 1, 2, \cdots.$$

The conclusion is then a result of 2.15.

Theorem 9: Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int f_n d\mu = \int f\, d\mu .$$

There is an n_0 such that for $n \geq n_0$, f_n is integrable. Thus without loss of generality we may assume we have hypothesis 1.

The conclusion is then a result of Theorem 3.

Theorem 10: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int f_n d\mu = \int f d\mu .$$

There exists n_0 such that for $n \geq n_0$, f_n is integrable.

Thus without loss of generality assume we have hypothesis 1.

$|f_n| \leq \max[|f_1|, |f|]$ for $n = 1, 2, \cdots$.

The conclusion is then a result of Theorem 2.

15. CONVERGENCE IN MEAN - L_p CONVERGENCE

First investigate the conditions under which convergence in mean implies L_p convergence. In counterexample 15, $\{f_n\}$ converges in mean to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 7, 8, 9, 10 are satisfied, but $\{f_n\}$ does not L_p converge to $f = 0$. The one missing hypothesis gives an implication which is now stated and proved.

Theorem 1: Suppose hypothesis 6 is satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

$$\lim_n \int |f_n - f| \, d\mu = 0.$$

Since we have hypothesis 6, there exists a constant c such

that $|f_n| \leq c$ a. e. $n = 1, 2, \cdots$.

Let $E = \{x : |f| \geq c + 1\}$.

$$0 = \lim_n \int |f_n - f| \, d\mu \geq \lim_n \int_E |f_n - f| \, d\mu \geq 1\mu(E) .$$

Thus $\mu(E) = 0$ and we conclude that $|f| \leq c + 1$ a. e.

$$\int |f_n - f|^p d\mu \leq [2(c+1)]^{p-1} \int |f_n - f| \, d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $\{f_n\}$ L_p converges to f.

Now investigate the conditions under which L_p convergence implies convergence in mean. Each set of hypotheses listed below, together with L_p convergence, implies convergence in mean. Following the list, the results are stated and proved. Counter-examples 3, 4, 18 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(7)	(2, 8, 9)
(5)	(1, 2, 9)	

<u>Theorem 2:</u> Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Since we assume L_p convergence, there exists an n_0 such that for $n \geq n_0$, $|f_n - f|^p$ is integrable.

Define q so that $\frac{1}{p} + \frac{1}{q} = 1$.

Since the measure is totally finite, any constant is integrable.

For $n \geq n_0$, Holder's inequality (p. 175 of Halmos) asserts that $|f_n - f|$ is integrable and that

$$\int |f_n - f| \, d\mu \leq \left[\int d\mu \right]^{\frac{1}{q}} \left[\int |f_n - f]^p \, d\mu \right]^{\frac{1}{p}} .$$

Clearly the term on the right goes to 0 as $n \to \infty$.

Lemma: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Let E be the support of the functions f_n $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and note that $\{E^m\}$ is a decreasing sequence of sets and $\lim_m E^m = \phi$.

Since we have hypothesis 5, given $\varepsilon > 0$ there exist m_0 such that for $m \geq m_0$ $\int_{E^m} |f_n| d\mu < \frac{\varepsilon}{5}$ $n = 1, 2, \cdots$.

By 16.1 we know that $\{f_n\}$ converges in measure to f.

Let $F_n^m = \{x : |f_n - f_m| > \dfrac{\varepsilon}{5\mu(E - E^{m_0})}\}$ and note that

$$\mu(F_n^m) \to 0 \quad \text{as} \quad n, m \to \infty.$$

By 3.1 we may assume we have hypothesis 4.

Thus there exists n_0 such that for $m, n \geq n_0$

$$\int_{F_n^m} |f_j| d\mu < \frac{\varepsilon}{5} \qquad j = 1, 2, \cdots .$$

For $n, m \geq n_0$

$$\int |f_n - f_m| d\mu = \int_E |f_n - f_m| d\mu = \int_{E - E^{m_0}} |f_n - f_m| d\mu + \int_{E \cap E^{m_0}} |f_n - f_m| d\mu$$

$$\leq \int_{(E-E^{m_0}) - F_n^m} |f_n - f_m| d\mu + \int_{(E-E^{m_0}) \cap F_n^m} |f_n - f_m| d\mu + \frac{2\varepsilon}{5}$$

$$\leq \frac{\varepsilon}{5\mu(E - E^{m_0})} \cdot \mu(E - E^{m_0}) + \frac{4\varepsilon}{5} = \varepsilon .$$

Since ε is arbitrary $\lim_{n,m} \int |f_n - f_m| d\mu = 0$.

By Theorem B, p. 107 of Halmos, there exists integrable g such that $\{f_n\}$ converges in mean to g.

5.1 tells us that $\{f_n\}$ converges in measure to g.

But $\mu\{x: |f-g| > \varepsilon\} \leq \mu\{x: |f_n - f| > \frac{\varepsilon}{2}\} + \mu\{x: |f_n - g| > \frac{\varepsilon}{2}\}$.

Thus $\mu\{x: |f-g| > \varepsilon\} = 0$.

Thus $\{f_n\}$ converges in mean to f.

<u>Theorem 3:</u> Suppose hypothesis 5 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int |f_n - f|^p d\mu = 0.$$

Without loss of generality we may assume that $f_n - f \in L_p$ for all n.

Let E be the support of the functions $f_n - f \quad n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and note that $\{E^m\}$ is a decreasing sequence of sets and $\lim_m E^m = \phi$.

Since we assume hypothesis 5, there exists m_0 such that

$$\int_{E^{m_0}} |f_n| d\mu < 1 \quad n = 1, 2, \cdots .$$

Since $\mu(E - E^{m_0}) < \infty$, Theorem 2 tells us that $\{f_n\}$ converges in mean to f on $E - E^{m_0}$ and the preceding lemma tells us the same on E^{m_0}.

Since the limit exists, the limit of the sum is the sum of the limits.

Thus $\{f_n\}$ converges in mean to f.

Theorem 4: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

By 16.1 we may assume $\{f_n\}$ converges in measure to f.

Follows from 5.2.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Let F be the support of the functions $f_n - f$ $n = 1, 2, \cdots$. Since $f_n - f$ is integrable for all n, F has σ-finite measure, that is $F = \bigcup_{n=1}^{\infty} F_n$ where $\mu(F_n) < \infty$ $n = 1, 2, \cdots$.

Let $F^m = F - \bigcup_{n=1}^{m} F_n$ and note that $\{F^m\}$ is a decreasing sequence of sets and $\lim_m F^m = \phi$.

Because of hypothesis 9, $\int |f_n - f| \, d\mu \leq \int |f_1 - f| \, d\mu$ for all n.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$

$$\int_{F^m} |f_n - f| \, d\mu < \frac{\varepsilon}{3} \quad n = 1, 2, \cdots .$$

By 16.1 we know $\{f_n\}$ converges in measure to f.

Let $E_n = \{x : |f_n - f| > \dfrac{\varepsilon}{3\mu(F - F^{m_0})}\}$ and note that $\mu(E_n) \to 0$ as $n \to \infty$.

There exists n_0 such that for $n \geq n_0$ $\int_{E_n} |f_1 - f| \, d\mu < \frac{\varepsilon}{3}$.

For $n \geq n_0$

$$\int |f_n - f|\,d\mu = \int_F |f_n - f|\,d\mu = \int_{F - F^{m_0}} |f_n - f|\,d\mu + \int_{F \cap F^{m_0}} |f_n - f|\,d\mu$$

$$\leq \int_{(F - F^{m_0}) - E_n} |f_n - f|\,d\mu + \int_{(F - F^{m_0}) \cap E_n} |f_n - f|\,d\mu + \frac{\varepsilon}{3}$$

$$\leq \frac{\varepsilon}{3\mu(F - F^{m_0})} \cdot \mu(F - F^{m_0}) + \frac{2\varepsilon}{3} = \varepsilon .$$

Since ε is arbitrary, $\lim_n \int |f_n - f|\,d\mu = 0.$

<u>Theorem 6:</u> Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int f_n\,d\mu = \int f\,d\mu .$$

Since f is integrable, there exists n_0 such that for

$n \geq n_0$, f_n is integrable.

Thus without loss of generality we may assume hypothesis 1.

Follows from Theorem 5.

16. L_p CONVERGENCE - CONVERGENCE IN MEASURE

First investigate the conditions under which L_p convergence implies convergence in measure. The following theorem finishes this investigation.

<u>Theorem 1.</u> If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Let $E_n = \{x : |f_n - f| > \varepsilon\}$.

$$\int_X |f_n - f|^p d\mu \geq \int_{E_n} |f_n - f|^p d\mu \geq \varepsilon^p \mu(E_n) \quad .$$

Since ε and p are fixed, $\mu(E_n) \to 0$ as $n \to \infty$.

Now investigate the conditions under which convergence in measure implies L_p convergence. Each set of hypotheses listed below, together with convergence in measure, implies L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 15, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3, 6)	(5, 6, 10)	(2, 6, 8, 9)
(6, 7)	(1, 2, 6, 9)	
(1, 5, 6)	(2, 5, 6, 8)	

Theorem 2: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.6 and 15.1.

Theorem 3: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.2 and 15.1.

Theorem 4: Suppose hypotheses 1, 5, and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.3 and 15.1.

Theorem 5: Suppose hypotheses 5, 6, and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.7 and 15.1.

Theorem 6: Suppose hypotheses 1, 2, 6, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.8 and 15.1.

Theorem 7: Suppose hypotheses 2, 5, 6, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.9 and 15.1.

Theorem 8: Suppose hypotheses 2, 6, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 5.10 and 15.1.

17. L_p CONVERGENCE - UNIFORM CONVERGENCE

First investigate the conditions under which L_p convergence implies uniform convergence. In counterexample 8, $\{f_n\}$ L_p converges to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no implications with L_p convergence implying uniform convergence.

Now investigate the conditions under which uniform convergence implies L_p convergence. Each set of hypotheses listed below, together with uniform convergence, implies L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(1, 5)	(1, 2, 9)	(2, 8, 9)
(7)	(5, 10)	(2, 5, 8)	

<u>Theorem 1:</u> Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Given $0 < \varepsilon < 1$, there exists $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$

$$|f_n - f| < \varepsilon .$$

On a totally finite measure space a constant function is

integrable; thus $|f_n - f|$ is integrable for $n \geq n_0$.

For $n \geq n_0$, $\int |f_n - f| \, d\mu \leq \varepsilon \, \mu(X)$.

Since ε is arbitrary, $\lim_n \int |f_n - f| \, d\mu = 0$.

For $n \geq n_0$, $\int |f_n - f|^p d\mu \leq \int |f_n - f| \, d\mu$.

Thus $\lim_n \int |f_n - f|^p \, d\mu = 0$ and we have L_p convergence.

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

By 13.1 we may assume $\{f_n\}$ converges in measure to f.

By 2.15 we may assume $\{f_n\}$ converges in mean to f.

Given $0 < \varepsilon < 1$, there exists $n_0(\varepsilon)$ such that for $n \geq n_0(\varepsilon)$

$|f_n - f| < \varepsilon$.

For $n \geq n_0$, $\int |f_n - f|^p \, d\mu \leq \int |f_n - f| \, d\mu$.

Thus $\lim_n \int |f_n - f|^p \, d\mu = 0$ and we have L_p convergence.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

By 13.1 we may assume $\{f_n\}$ converges in measure to f.

By 5.3 we may assume $\{f_n\}$ converges in mean to f.

The remainder of the argument is the same as that of the theorem above.

Theorem 4: Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 3.15 and Theorem 3.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

By 13.1 we may assume $\{f_n\}$ converges in measure to f.

By 5.8 we may assume $\{f_n\}$ converges in mean to f.

The argument concludes exactly the same as that of Theorem 2.

Theorem 6: Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

By 13.1 we may assume $\{f_n\}$ converges in measure to f.

By 5.9 we may assume $\{f_n\}$ converges in mean to f.

The argument concludes exactly the same as that of Theorem 2.

Theorem 7: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

$$\lim_n \int f_n d\mu = \int f d\mu .$$

Since f is integrable, there exists n_0 such that for $n \geq n_0$, f_n is integrable.

Thus without loss of generality we may assume we have hypothesis 1.

Follows from Theorem 5.

18. L_p CONVERGENCE - ALMOST UNIFORM CONVERGENCE

First investigate the conditions under which L_p convergence implies almost uniform convergence. In counterexample 7, $\{f_n\}$ L_p converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge almost uniformly to $f = 0$. The one missing hypothesis gives a result which is stated and proved below. Of interest here is the fact that a sequence of functions L_p converging to a particular function, necessarily has a subsequence which converges almost uniformly to that function.

<u>Theorem 1:</u> Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 16. 1 and 6. 2.

Now investigate the conditions under which almost uniform convergence implies L_p convergence. Each set of hypotheses listed below, together with almost uniform convergence, implies L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 15, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3, 6)	(1, 5, 6)	(1, 2, 6, 9)	(2, 6, 8, 9)
(6, 7)	(5, 6, 10)	(2, 5, 6, 8)	

Theorem 2: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.2.

Theorem 3: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.3.

Theorem 4: Suppose hypotheses 1, 5, and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.4.

Theorem 5: Suppose hypotheses 5, 6, and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.5.

Theorem 6: Suppose hypotheses 1, 2, 6, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.6.

Theorem 7: Suppose hypotheses 2, 5, 6, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.7.

Theorem 8: Suppose hypotheses 2, 6, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 6.1 and 16.8.

19. L_p CONVERGENCE - CONVERGENCE a.e.

First investigate the conditions under which L_p convergence implies convergence a.e. In counterexample 7, $\{f_n\}$ L_p converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge a.e. to $f = 0$. The one missing hypothesis gives a result which is stated and proved below. Of interest here is the fact that a sequence of functions L_p converging to a particular function necessarily has a subsequence which converges a.e. to that function.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

Follows from 16.1 and 10.1.

Now investigate the conditions under which convergence a.e. implies L_p convergence. Each set of hypotheses listed below, together with convergence a.e., implies L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 15, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3, 6)	(1, 5, 6)	(1, 2, 6, 9)	(2, 6, 8, 9)
(6, 7)	(5, 6, 10)	(2, 5, 6, 8)	

Theorem 2: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.2 and 16.2.

Theorem 3: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.4 and 16.3.

Theorem 4: Suppose hypotheses 1, 5, and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.4 and 16.4.

Theorem 5: Suppose hypotheses 5, 6, and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.3 and 16.5.

<u>Theorem 6:</u> Suppose hypotheses 1, 2, 6, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.5 and 16.6.

<u>Theorem 7:</u> Suppose hypotheses 2, 5, 6, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.3 and 16.7.

<u>Theorem 8:</u> Suppose hypotheses 2, 6, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 10.6 and 16.8.

20. WEAK L_1 CONVERGENCE - CONVERGENCE a.e.

First investigate the conditions under which weak L_1 convergence implies convergence a.e. In counterexample 19, $\{f_n\}$ weak L_1 converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge a.e. to $f = 0$. The one missing hypothesis gives a result which is now stated and proved.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

$$\lim_n \left| \int (f_n - f)d\mu \right| = 0.$$

But $\left| \int (f_n - f)d\mu \right| = \int |f_n - f| d\mu$ for each n.

Thus $\{f_n\}$ converges in mean to f.

The conclusion then follows from 14.1.

Now investigate the conditions under which convergence a.e. implies weak L_1 convergence. Each set of hypotheses listed below, together with convergence a.e., implies weak L_1 convergence. Counterexamples 1, 2, 3, 4, 5, 6, 10, 11 show that these are the only implications with a non-redundant set of hypotheses.

(7)	(3, 5)	(1, 2, 9)
(1, 5)	(3, 6)	(2, 5, 8)
(3, 4)	(5, 10)	(2, 8, 9)

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.2 we know $\{f_n\}$ converges in mean to f.

But $\left|\int_E (f_n - f)d\mu\right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.3 we know $\{f_n\}$ converges in mean to f.

But $\left|\int_E (f_n - f)d\mu\right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

<u>Theorem 4:</u> Suppose hypotheses 3 and 4 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.4 we know $\{f_n\}$ converges in mean to f.

But $\left| \int_E (f_n - f) d\mu \right| \leq \int_E |f_n - f| \, d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f) d\mu = 0$ for measurable E.

<u>Theorem 5:</u> Suppose hypotheses 3 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.5 we know $\{f_n\}$ converges in mean to f.

But $\left| \int_E (f_n - f) d\mu \right| \leq \int_E |f_n - f| \, d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f) \, d\mu = 0$ for measurable E.

<u>Theorem 6:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.6 we know $\{f_n\}$ converges in mean to f.

But $\left|\int_E (f_n - f)d\mu\right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

<u>Theorem 7:</u> Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.7 we know $\{f_n\}$ converges in mean to f.

But $\left|\int_E (f_n - f)d\mu\right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

<u>Theorem 8:</u> Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.8 we know $\{f_n\}$ converges in mean to f.

But $\left|\int_E (f_n - f)d\mu\right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

Theorem 9: Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.9 we know $\{f_n\}$ converges in mean to f.

But $\left| \int_E (f_n - f)d\mu \right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

Theorem 10: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 14.10 we know $\{f_n\}$ converges in mean to f.

But $\left| \int_E (f_n - f)d\mu \right| \leq \int_E |f_n - f| d\mu$ for measurable E.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

21. WEAK L_1 CONVERGENCE - CONVERGENCE IN MEASURE

First investigate the conditions under which weak L_1 convergence implies convergence in measure. In counterexample 19, $\{f_n\}$ weak L_1 converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge in measure to $f = 0$. The one remaining hypothesis gives a result which is now stated and proved.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

$$\lim_n \left| \int_X (f_n - f)\, d\mu \right| = 0 .$$

$$\text{But} \quad \left| \int_X (f_n - f)\, d\mu \right| = \int_X |f_n - f|\, d\mu .$$

Thus $\{f_n\}$ converges in mean to f.

The conclusion then follows from 5.1.

Now investigate the conditions under which convergence in measure implies weak L_1 convergence. Each set of hypotheses listed below, together with convergence in measure, implies weak L_1 convergence. Following the list, the results are stated and proved.

Counterexamples 1, 2, 3, 4, 5, 6, 10, 11 show that these are the only implications with a non-redundant set of hypotheses.

(7)	(3, 5)	(1, 2, 9)
(1, 5)	(3, 6)	(2, 5, 8)
(3, 4)	(5, 10)	(2, 8, 9)

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.2 we know $\{f_n\}$ converges in mean to f.

The conclusion is then obvious.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.3 we know $\{f_n\}$ converges in mean to f.

The conclusion is then obvious.

Theorem 4: Suppose hypotheses 3 and 4 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.4 we know $\{f_n\}$ converges in mean to f.

The conclusion is then obvious.

<u>Theorem 5:</u> Suppose hypotheses 3 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.5 we know $\{f_n\}$ converges in mean to f.

The conclusion is then obvious.

<u>Theorem 6:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.6 we know $\{f_n\}$ converges in mean to f.

The conclusion is then obvious.

<u>Theorem 7:</u> Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.7 we know that $\{f_n\}$ converges in mean to f.

The conclusion is then obvious.

Theorem 8: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.8 we know that $\{f_n\}$ converges in mean to f. The conclusion is then obvious.

Theorem 9: Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.9 we know that $\{f_n\}$ converges in mean to f. The conclusion is then obvious.

Theorem 10: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

By 5.10 we know that $\{f_n\}$ converges in mean to f. The conclusion is then obvious.

22. WEAK L_1 CONVERGENCE - CONVERGENCE IN MEAN

First investigate the conditions under which weak L_1 convergence implies convergence in mean. In counterexample 19, $\{f_n\}$ weak L_1 converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge in mean to $f = 0$. The one missing hypothesis gives a result which is now stated and proved.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \left| \int (f_n - f) d\mu \right| = 0 .$$

$$\text{But} \quad \left| \int (f_n - f) d\mu \right| = \int |f_n - f| d\mu .$$

Now investigate the conditions under which convergence in mean implies weak L_1 convergence. The following theorem finishes this investigation.

Theorem 2: If the sequence of function $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\left| \int (f_n - f)\,d\mu \right| \le \int |f_n - f|\,d\mu \ .$$

23. WEAK L_1 CONVERGENCE - ALMOST UNIFORM CONVERGENCE

First investigate the conditions under which weak L_1 convergence implies almost uniform convergence. In counterexample 19, $\{f_n\}$ weak L_1 converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge almost uniformly to $f = 0$. The one remaining hypothesis gives a result which is now stated and proved.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 22.1 and 8.1.

Now investigate the conditions under which almost uniform convergence implies weak L_1 convergence. Each set of hypotheses listed below, together with almost uniform convergence, implies weak L_1 convergence. Following the list the results are stated and proved. Counterexamples 1, 2, 3, 4, 5, 6, 10, 11 show that these are the only implications with a non-redundant set of hypotheses.

(7)	(3, 5)	(1, 2, 9)
(1, 5)	(3, 6)	(2, 5, 8)
(3, 4)	(5, 10)	(2, 8, 9)

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.2 and 22.2.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.3 and 22.2.

Theorem 4: Suppose hypotheses 3 and 4 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.4 and 22.2

Theorem 5: Suppose hypotheses 3 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.5 and 22.2.

<u>Theorem 6:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.6 and 22.2.

<u>Theorem 7:</u> Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.7 and 22.2.

<u>Theorem 8:</u> Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.8 and 22.2.

Theorem 9: Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.9 and 22.2.

Theorem 10: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 8.10 and 22.2.

24. WEAK L_1 CONVERGENCE - UNIFORM CONVERGENCE

First investigate the conditions under which weak L_1 convergence implies uniform convergence. In counterexample 8, $\{f_n\}$ weak L_1 converges to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no implications with weak L_1 convergence implying uniform convergence.

Now investigate the conditions under which uniform convergence implies weak L_1 convergence. Each set of hypotheses listed below, together with uniform convergence, implies weak L_1 convergence. Following the list, the results are stated and proved. Counterexamples 1, 3, 4, 5, 6 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(1, 5)	(1, 2, 9)	(2, 8, 9)
(7)	(5, 10)	(2, 5, 8)	

<u>Theorem 1:</u> Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7. 1 and 22. 2.

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7.2 and 22.2.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7.3 and 22.2.

Theorem 4: Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7.4 and 22.2.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7.5 and 22.2.

Theorem 6: Suppose hypotheses 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converge uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7.6 and 22.2.

Theorem 7: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 7.7 and 22.2.

25. WEAK L_1 CONVERGENCE - L_p CONVERGENCE

First investigate the conditions under which weak L_1 convergence implies L_p convergence. There is only one result which is stated and proved below. Counterexamples 15 and 19 show that this is the only implication with a non-redundant set of hypotheses.

Theorem 1: Suppose hypotheses 6 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 22.1 and 15.1.

Now investigate the conditions under which L_p convergence implies weak L_1 convergence. Each set of hypotheses listed below, together with L_p convergence, implies weak L_1 convergence. Following the list, the results are stated and proved. Counterexamples 3, 4, 18 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(7)	(2, 8, 9)
(5)	(1, 2, 9)	

Theorem 2: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 15.2 and 22.2.

Theorem 3: Suppose hypothesis 5 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 15.3 and 22.2.

Theorem 4: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 15.4 and 22.2.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 15.5 and 22.2.

Theorem 6: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 15.6 and 22.2.

26. WEAK L_p CONVERGENCE - CONVERGENCE a.e.

First investigate the conditions under which weak L_p convergence implies convergence a.e. In counterexample 19, $\{f_n\}$ weak L_p converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge a.e. to $f = 0$. The one remaining hypothesis gives an implication which is now stated and proved.

<u>Theorem 1:</u> Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_q .$$

Let E be the support of the functions $f_n - f$ $n = 1, 2, \cdots$.

Without loss of generality we may assume $f_n - f \in L_p$ for all n.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

It suffices to show pointwise convergence on E_{n_0} for n_0 arbitrary but fixed.

χ_E clearly belongs to L_q for $E \subseteq E_{n_0}$.

Thus $\lim_n \int_E (f_n - f)d\mu = 0$ for $E \subseteq E_{n_0}$.

Since $f_n - f$ has the same sign for each n

$$\left|\int_E (f_n - f)d\mu\right| = \int_E |f_n - f|\, d\mu \quad \text{for} \quad E \subseteq E_{n_0}.$$

The result then follows from 14.1.

Now investigate the conditions under which weak L_p convergence implies convergence a.e. Each set of hypotheses listed below, together with convergence a.e., implies weak L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 15, 17, 31 show that there are no other implications with a non-redundant set of hypotheses.

(3, 6)	(1, 5, 6)	(1, 2, 6, 9)	(2, 6, 8, 9)
(6, 7)	(5, 6, 10)	(2, 5, 6, 8)	

Theorem 2: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

There exists a constant c such that $|f_n| \leq c$ a.e. for all n.

From 2.15 we know $\{f_n\}$ converges in mean to f.

From 15.1 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \quad .$$

Theorem 3: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

From 2.15 we know $\{f_n\}$ converges in mean to f.

From 15.1 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \quad .$$

Theorem 4: Suppose hypotheses 1, 5, and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

From 19.4 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \quad .$$

Theorem 5: Suppose hypotheses 5, 6, and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

From 19.5 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \, .$$

<u>Theorem 6:</u> Suppose hypotheses 1, 2, 6, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

From 19.6 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \, .$$

<u>Theorem 7:</u> Suppose hypotheses 2, 5, 6, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

From 19.7 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \, .$$

<u>Theorem 8:</u> Suppose hypotheses 2, 6, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

From 19.8 we know $\{f_n\}$ L_p converges to f.

$| \int (f_n - f) g d\mu | \leq \| f_n - f \|_p \, \| g \|_q$.

27. WEAK L_p CONVERGENCE - CONVERGENCE IN MEASURE

First investigate the conditions under which weak L_p convergence implies convergence in measure. In counterexample 19, $\{f_n\}$ weak L_p converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge in measure to $f = 0$. The one reamining hypothesis gives a result which is now stated and proved.

<u>Theorem 1:</u> Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Let E be the support of the functions $f_n - f$, $n = 1, 2, \cdots$.

Since $\{f_n - f\}$ eventually belongs to L_p, we may without loss of generality assume that $f_n - f \in L_p$ for all n.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$, $n = 1, 2, \cdots$.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and note that $\{E^m\}$ is a decreasing sequence of sets and $\lim_n E^m = \phi$.

$|f_1 - f| \geq |f_n - f|$ for all n and so $\int |f_1 - f|^p d\mu \geq \int |f_n - f|^p d\mu$

for all n.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$,

$$\int_{E^m} |f_1 - f|^p d\mu < \frac{\varepsilon}{3} .$$

Let $F_n = \{x : |f_n - f| > (\frac{\varepsilon}{3\mu(E-E^{m_0})})^{\frac{1}{p}}\}$ and note that $\{F_n\}$

is a decreasing sequence and $\lim_n F_n = \phi$ (or a set of measure zero).

There exists n_0 such that for $n \geq n_0$ $\int_{F_n} |f_1 - f|^p d\mu < \frac{\varepsilon}{3}$.

For $n \geq n_0$,

$$\int |f_n - f|^p d\mu \leq \int_{E-E^{m_0}} |f_n - f|^p d\mu + \int_{E^{m_0}} |f_n - f|^p d\mu \leq \int_{E-E^{m_0}} |f_n - f|^p d\mu + \frac{\varepsilon}{3}$$

$$\leq \int_{(E-E^{m_0})-F_n} |f_n - f|^p d\mu + \int_{(E-E_{m_0})\cap F_n} |f_n - f|^p d\mu + \frac{\varepsilon}{3}$$

$$\leq \frac{\varepsilon}{3\mu(E-E^{m_0})} \cdot \mu(E-E^{m_0}) + \frac{2\varepsilon}{3} = \varepsilon .$$

Since ε is arbitrary, $\lim_n \int |f_n - f|^p d\mu = 0$.

The result then follows from 16.1.

Now investigate the conditions under which convergence in measure implies weak L_p convergence. Each set of hypotheses listed below, together with convergence in measure, implies weak L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 15, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3, 6)	(1, 5, 6)	(1, 2, 6, 9)	(2, 6, 8, 9)
(6, 7)	(5, 6, 10)	(2, 5, 6, 8)	

<u>Theorem 2:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

There exists a constant c such that $|f_n| \leq c$ a.e. $n = 1, 2, \cdots$.

By 2.15 we know $\{f_n\}$ converges in mean to f.

By 15.1 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q \, .$$

<u>Theorem 3:</u> Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

There exists integrable h such that $|f_n| \leq h$ a.e. $n=1,2,\cdots$.

By 2.15 we know $\{f_n\}$ converges in mean to f.

By 15.1 we know $\{f_n\}$ L_p converges to f.

$$\left|\int (f_n - f) g \, d\mu\right| \leq \|f_n - f\|_p \, \|g\|_q .$$

Theorem 4: Suppose hypotheses 1, 5, and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 16.4 we know $\{f_n\}$ L_p converges to f.

$$\left|\int (f_n - f) g \, d\mu\right| \leq \|f_n - f\|_p \, \|g\|_q .$$

Theorem 5: Suppose hypotheses 5, 6, and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 16.5 we know $\{f_n\}$ L_p converges to f.

$$\left|\int (f_n - f) g \, d\mu\right| \leq \|f_n - f\|_p \, \|g\|_q .$$

Theorem 6: Suppose hypotheses 1, 2, 6, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 16. 6 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q .$$

Theorem 7: Suppose hypotheses 2, 5, 6, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 16. 7 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q .$$

Theorem 8: Suppose hypotheses 2, 6, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 16. 8 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q .$$

28. WEAK L_p CONVERGENCE - ALMOST UNIFORM CONVERGENCE

First investigate the conditions under which weak L_p convergence implies almost uniform convergence. In counterexample 19, $\{f_n\}$ weak L_p converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not converge almost uniformly to $f = 0$. The one remaining hypothesis gives a result which is now stated and proved.

Theorem 1: Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 27.1 and 6.2.

Now investigate the conditions under which almost uniform convergence implies weak L_p convergence. Each set of hypotheses listed below, together with almost uniform convergence, implies weak L_p convergence. Counterexamples 1, 5, 6, 15, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3, 6)	(1, 5, 6)	(1, 2, 6, 9)	(2, 6, 8, 9)
(6, 7)	(5, 6, 10)	(2, 5, 6, 8)	

Theorem 2: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.2.

Theorem 3: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.3.

Theorem 4: Suppose hypotheses 1, 5, and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.4.

Theorem 5: Suppose hypotheses 5, 6, and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.5.

Theorem 6: Suppose hypotheses 1, 2, 6, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.6.

Theorem 7: Suppose hypotheses 2, 5, 6, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.7.

Theorem 8: Suppose hypotheses 2, 6, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 6.1 and 27.8.

29. WEAK L_p CONVERGENCE - UNIFORM CONVERGENCE

First investigate the conditions under which weak L_p convergence implies uniform convergence. In counterexample 8, $\{f_n\}$ weak L_p converges to $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no results where weak L_p convergence implies uniform convergence.

Now investigate the conditions under which uniform convergence implies weak L_p convergence. Each set of hypotheses listed below, together with uniform convergence, implies weak L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 5, 6, 16, 17 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(1, 5)	(1, 2, 9)	(2, 8, 9)
(7)	(5, 10)	(2, 5, 8)	

<u>Theorem 1:</u> Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.1 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \quad .$$

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.2 we know that $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \quad .$$

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.3 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \, \| g \|_q \quad .$$

Theorem 4: Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.4 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q \, .$$

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.5 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q \, .$$

Theorem 6: Suppose hypothes 2, 5, and 8 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.6 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f\_g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q \, .$$

Theorem 7: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 17.7 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \, \|g\|_q \, .$$

30. WEAK L_p CONVERGENCE - CONVERGENCE IN MEAN

First investigate the conditions under which weak L_p convergence implies convergence in mean. Each set of hypotheses listed below, together with weak L_p convergence, implies convergence in mean. Following the list, the results are stated and proved. Counterexamples 4, 18, 19 show that these are the only implications with a non-redundant set of hypotheses.

(3, 9) (7, 9) (2, 8, 9)

(5, 9) (1, 2, 9)

Theorem 1: Suppose hypotheses 3 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \left| \int (f_n - f) g \, d\mu \right| = 0 \quad \text{for} \quad g \in L_q .$$

Since we have hypothesis 3, $g = 1$ belongs to L_q.

Thus $0 = \lim_n \left| \int (f_n - f) d\mu \right| = \lim_n \int |f_n - f| \, d\mu$.

Theorem 2: Suppose hypotheses 5 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function

f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_q .$$

Since $f_n - f$ eventually belongs to L_p, we may assume without loss of generality that $f_n - f \in L_p$ for all n.

Let E be the support of the functions $f_n - f$ $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

Let F be the support of the functions $f_n - f_m$ $n, m = 1, 2, \cdots$.

$F \subseteq E$. This is proved as follows:

Suppose $x_0 \notin E$.

Then $f_n(x_0) - f(x_0) = 0$ for all n.

Thus $f_n(x_0) = f_m(x_0)$ for all m, n.

Thus $x_0 \notin F$.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and note that $\{E^m\}$ is a decreasing sequence of sets and $\lim_n E^m = \phi$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$

$$\int_{E^m} |f_n| d\mu < \frac{\varepsilon}{5} \qquad n = 1, 2, \cdots .$$

By 26.1 we know $\{f_n\}$ converges a.e. to f.

Let $G^n = \bigcup_{m=n}^{\infty} \{x : |f_n - f_m| > \frac{\varepsilon}{5\mu(E - E^{m_0})}\}$ and note that

$\{G_n\}$ is a decreasing sequence of sets and $\lim_n G^n = \phi$ (or a set of measure zero).

There exists n_0 such that for $n \geq n_0$, $\int_{G^n} |f_m| d\mu < \frac{\varepsilon}{5}$ $m = 1, 2, \cdots .$

For $n, m \geq n_0$

$$\int |f_n - f_m| d\mu = \int_E |f_n - f_m| d\mu = \int_{E - E^{m_0}} |f_n - f_m| d\mu$$

$$+ \int_{E \cap E^{m_0}} |f_n - f_m| d\mu$$

$$\leq \int_{(E - E^{m_0}) - G^n} |f_n - f_m| d\mu + \int_{(E - E^{m_0}) \cap G^n} |f_n - f_m| d\mu + \frac{2\varepsilon}{5}$$

$$\leq \frac{\varepsilon}{5\mu(E - E^{m_0})} \cdot \mu(E - E^{m_0}) + \frac{4\varepsilon}{5} = \varepsilon .$$

Since ε is arbitrary $\lim_{n,m} \int |f_n - f_m| d\mu = 0.$

From 3.15 we know each f_n is integrable on E.

By Theorem B, p. 107 of Halmos we know there exists integrable h such that $\{f_n\}$ converges in mean to h.

By 27.1 we know that $\{f_n\}$ converges in measure to f.

By 5.1 we know that $\{f_n\}$ converges in measure to h.

$\mu\{x : |h-f| > \varepsilon\} \leq \mu\{x : |f_n - h| > \frac{\varepsilon}{2}\} + \mu\{x : |f_n - f| > \frac{\varepsilon}{2}\}$.

Thus $\mu\{x : |h-f| > \varepsilon\} = 0$.

Since ε is arbitrary $f = h$ a.e.

Thus $\{f_n\}$ converges in mean to f.

Theorem 3: Suppose hypotheses 7 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 3.5 and Theorem 2.

Theorem 4: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Let $h(x) = \max[\,|f(x)|, |f_1(x)|\,]$.

$h(x)$ is integrable and $|f_n(x)| \leq h(x)$ $n = 1, 2, \cdots$.

Follows then from Theorem 3.

Theorem 5: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int f_n d\mu = \int f d\mu .$$

Since f is integrable, there exists n_0 such that for

$n \geq n_0$, f_n is integrable.

Thus without loss of generality we may assume we have

hypothesis 1.

The result then follows from Theorem 4.

Now investigate the conditions under which convergence in mean implies weak L_p convergence. In counterexample 15, $\{f_n\}$ converges in mean to $f = 0$, hypotheses 1, 2, 3, 4, 5, 7, 8, 9, 10 are satisfied, but $\{f_n\}$ does not weak L_p converge to $f = 0$. The one remaining hypothesis gives us a result which is now stated and proved.

Theorem 6: Suppose hypothesis 6 is satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 15.1 we know $\{f_n\}$ L_p converges to f.

$$\left| \int (f_n - f) g \, d\mu \right| \leq \|f_n - f\|_p \; \|g\|_q \; .$$

31. WEAK L_p CONVERGENCE - L_p CONVERGENCE

First investigate the conditions under which weak L_p convergence implies L_p convergence. In counterexample 19, $\{f_n\}$ weak L^p converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 6, 7, 8, 10 are satisfied, but $\{f_n\}$ does not L_p converge to $f = 0$. The one remaining hypothesis gives us an implication which is now stated and proved.

<u>Theorem 1:</u> Suppose hypothesis 9 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_q .$$

Without loss of generality we may assume $f_n - f \in L_p$ for all n.

Let E be the support of the functions $f_n - f$ $n = 1, 2, \cdots$.

E has σ-finite measure ; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and note that $\{E^m\}$ is a decreasing sequence of sets and $\lim_n E^m = \phi$.

Since we have hypothesis 9, $\int |f_n - f|^p d\mu \le \int |f_1 - f|^p d\mu$

$n = 1, 2, \cdots$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \ge m_0$,

$$\int_{E^m} |f_n - f|^p d\mu < \frac{\varepsilon}{3} \quad n = 1, 2, \cdots .$$

Let $F_n = \{x : |f_n - f| > \dfrac{\varepsilon}{3\mu(E - E^{m_0})}\}$.

From 27.1 we know $\{F_n\}$ is a decreasing sequence of sets

and $\lim_n F_n = \phi$ (or a set of measure zero).

There exists n_0 such that for $n \ge n_0$, $\int_{F_n} |f_1 - f|^p d\mu < \frac{\varepsilon}{3}$.

For $n \ge n_0$,

$$\int |f_n - f|^p d\mu = \int_E |f_n - f|^p d\mu = \int_{E - E^{m_0}} |f_n - f|^p d\mu + \int_{E^{m_0}} |f_n - f|^p d\mu$$

$$\le \int_{(E - E^{m_0}) - F_n} |f_n - f|^p d\mu + \int_{(E - E^{m_0}) \cap F_n} |f_n - f|^p d\mu + \frac{\varepsilon}{3}$$

$$\le \frac{\varepsilon}{3\mu(E - E^{m_0})} \cdot \mu(E - E^{m_0}) + \frac{2\varepsilon}{3} = \varepsilon .$$

Since ε is arbitrary, $\lim_n \int |f_n - f|^p d\mu = 0$.

Now investigate the conditions under which L_p convergence implies weak L_p convergence. The following theorem finishes this investigation.

Theorem 2: If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

$$\left| \int (f_n - f) g \, d\mu \right| \leq \| f_n - f \|_p \ \| g \|_q \, .$$

32. WEAK L_1 CONVERGENCE - WEAK L_p CONVERGENCE

First consider the conditions under which weak L_1 convergence implies weak L_p convergence. In counterexample 15 $\{f_n\}$ weak L_1 converges to $f = 0$ and hypotheses 1, 2, 3, 4, 5, 7, 8, 9, 10 are satisfied, but $\{f_n\}$ does not weak L_p converge to $f = 0$. The one remaining hypothesis gives a result which is stated and proved below. First we state and prove a lemma which will be used in the proof of the theorem.

Lemma: Suppose the space X is totally σ-finite, $f_n - f$ is integrable for all n and there exists K such that $|f_n - f| \leq K$ a.e. for all n. If $\lim_n \int_E (f_n - f) d\mu = 0$ for measurable E, then there exists C such that $\int |f_n - f| d\mu < C \quad n = 1, 2, \cdots$.

Proof:

The proof will be accomplished by using the contrapositive; that is assume that $\int |f_n - f| d\mu$ is not bounded and show that there exists a measurable set E such that $\lim_n \int_E (f_n - f) d\mu \neq 0$.

$X = \bigcup_{n=1}^{\infty} F_n$ where $\mu(F_n) < \infty \quad n = 1, 2, \cdots$.

Let $F^m = \bigcup_{n=1}^{m} F_n$.

There exists n_1 such that $\int |f_{n_1} - f| d\mu > 4$.

Since $f_{n_1} - f$ is integrable, either $\int (f_{n_1} - f)^+ d\mu > 2$ or

$\int (f_n - f)^- d+ > 2$; for the sake of argument assume

that the first inequality is true.

There exists m_1 such that $\int_{F^{m_1}} (f_{n_1} - f)^+ d\mu > 1$ and

$\int_{X - F^{m_1}} |f_{n_1} - f| d\mu < \frac{1}{4}$.

Let $G_{n_1} = \{x: (f_{n_1} - f)^+ \neq 0\} \cap F^{m_1}$.

There exists $n_2 > n_1$ such that

$$\int_{(F^{m_1})^c} |f_{n_2} - f| d\mu > 2K\mu(F^{m_1}) + 4 .$$

Either $\int_{(F^{m_1})^c} (f_{n_2} - f)^+ d\mu > K\mu(F^{m_1}) + 2$ or

$\int_{(F^{m_1})^c} (f_{n_2} - f)^- d\mu > K\mu(F^{m_1}) + 2$; for the sake of argument

assume that the first inequality is true.

There exists m_2 such that $\int_{F^{m_2}} (f_{n_2}-f)^+ d\mu > K\mu(F^{m_1}) + 1$

and $\int_{X-F^{m_2}} |f_{n_2}-f| d\mu < \frac{1}{4}$.

Let $G_{n_2} = \{x : (f_{n_2}-f)^+ \neq 0\} \cap (F^{m_2}-F^{m_1})$.

There exists $n_3 > n_2$ such that $\int_{(F^{m_2})^c} (f_{n_3}-f)\, d\mu > 2K\mu(F^{m_2}) + 4$.

Either $\int_{(F^{m_2})} (f_{n_3}-f)^+ d\mu > K\mu(F^{m_2}) + 2$ or

$\int_{(F^{m_2})^c} (f_{n_3}-f)^- d\mu > K\mu(F^{m_2}) + 2$; for the sake of argument assume the first inequality is true.

There exists m_3 such that $\int_{F^{m_3}} (f_{n_3}-f)^+ d\mu > K\mu(F^{m_2}) + 1$

and $\int_{X-F^{m_3}} |f_{n_3}-f| d\mu < \frac{1}{4}$.

Let $G_{n_3} = \{ x : (f_{n_3}-f)^+ \neq 0 \} \cap (F^{m_3}-F^{m_2})$.

Continuing in this manner we get a subsequence $\{f_{n_k}-f\}$ and a sequence of sets $\{G_{n_k}\}$ such that for all k, $\left|\int_H (f_{n_k}-f) d\mu\right| > \frac{3}{4}$ where $H = \bigcup_{n=1}^{\infty} G_{n_k}$.

Thus $\lim_n \int_H (f_n - f)d\mu \neq 0$.

<u>Theorem 1:</u> Suppose hypothesis 6 is satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to f, then $\{f_n\}$ weak L_p converges to f.

Proof:

$\lim_n \int_E (f_n - f)d\mu = 0$ for measurable E.

Let K be the uniform essential bound for the f_n $n = 1,2,\cdots$.

$K+1$ is an essential bound for f.

Without loss of generality we may assume $f_n - f$ is integrable for all n.

$f_n - f \in L_p$ for all n since $\int |f_n - f|^p d\mu \leq 2(K+1)^{p-1} \int |f_n - f| d\mu$.

Let F be the support of the functions $f_n - f$ $n = 1, 2, \cdots$.

$F = \bigcup_{n=1}^{\infty} F_n$ where $\mu(F_n) < \infty$ $n = 1, 2, \cdots$.

Let $F^m = F - \bigcup_{n=1}^{m} F_n$.

Let $g \in L_q$.

By the above lemma we know that there exists a constant C such that $\int |f_n - f| d\mu \leq C$ for all n.

Given $\varepsilon > 0$, there exists n_0 such that for $m \geq m_0$

$$\int_{F^m} |g|^q d\mu < \left(\frac{\varepsilon}{[2(K+1)]^{p-1} C} \right)^{\frac{p}{q}}$$

$$\left| \int (f_n - f) g d\mu \right| \leq \left| \int_{F^{m_0}} (f_n - f) g d\mu \right| + \left| \int_{F - F^{m_0}} (f_n - f) g d\mu \right|$$

$$\leq \|f_n - f\|_p \int_{F^{m_0}} |g|^q d\mu + \left| \int_{F - F^{m_0}} (f_n - f) g d\mu \right|$$

$$< \varepsilon + \left| \int (f_n - f) g \cdot \chi_{F - F^{m_0}} d\mu \right| .$$

$g \cdot \chi_{F - F^{m_0}} \in L_1$.

Thus there exists an integrable simple function s such that

$$\int \left| g \cdot \chi_{F - F^{m_0}} - s \right| d\mu < \frac{\varepsilon}{2(K+1)} .$$

$$\left| \int (f_n - f) g \cdot \chi_{F - F^{m_0}} d\mu - \int (f_n - f) s d\mu \right| \leq 2(K+1) \int \left| g \cdot \chi_{F - F^{m_0}} - s \right| d\mu < \varepsilon .$$

But clearly $\lim_n \int (f_n - f) s d\mu = 0$.

Thus $\lim_n \int (f_n - f) g \cdot \chi_{F - F^{m_0}} d\mu = 0$.

Thus $\limsup_n \left| \int (f_n - f) g d\mu \right| \leq \varepsilon$.

Since ε is arbitrary, $\lim_n \int (f_n - f) g d\mu = 0$.

Now consider the conditions under which weak L_p convergence implies weak L_1 convergence. Each set of hypotheses listed below, together with weak L_p convergence, implies weak L_1 convergence. Following the list, the results are stated and proved. Counterexamples 3, 4, 18 show that these are the only implications with a non-redundant set of hypotheses.

(3) (7) (2, 8, 9)

(5) (1, 2, 9)

Theorem 2: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_q .$$

But $\chi_E \in L_q$ for measurable E.

Thus $\lim_n \int_E (f_n - f) \, d\mu = 0$ for measurable E.

Theorem 3: Suppose hypothesis 5 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_q .$$

Without loss of generality we may assume $f_n - f \in L_p$ for all n.

Let E be the support of the functions $f_n - f$ $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$ and note that $\{E^m\}$ is a decreasing sequence of sets and $\lim_n E^m = \phi$.

Let $F_n = \{x: |f| \leq n\}$ and note that $\{F_n^c\}$ is a decreasing sequence of sets and $\lim_n F_n^c = \phi$.

Given $\varepsilon > 0$, there exists m_0 and n_0 such that for

$$m \geq m_0, \quad n \geq n_0, \quad \int_{E^m} |f_k|\, d\mu < \frac{\varepsilon}{2} \quad \text{and}$$

$$\int_{F_n^c} |f_k|\, d\mu < \frac{\varepsilon}{2} \qquad k = 1, 2, \cdots .$$

Note $\chi_{(E-E^{m_0}) \cap F_{n_0}} \in L_q$.

f is integrable on $(E-E^{m_0}) \cap F_{n_0}$ and $\lim_n \int_{(E-E^{m_0}) \cap F_{n_0}} (f_n - f)\, d\mu = 0$.

Thus eventually all of the f_n are integrable on $(E-E^{m_0}) \cap F_{n_0}$.

For **n** sufficiently large

$$\left|\int_E f_n d\mu\right| \leq \left|\int_{E^{m_0}} f_n d\mu\right| + \left|\int_{E-E^{m_0}} f_n d\mu\right|$$

$$\leq \frac{\varepsilon}{2} + \left|\int_{(E-E^{m_0})\cap F_{n_0}} f_n d\mu\right| + \left|\int_{(E-E^{m_0})\cap F^c_{n_0}} f_n d\mu\right|$$

$$\leq \varepsilon + \left|\int_{(E-E^{m_0})\cap F_{n_0}} f_n d\mu\right| < \infty.$$

Without loss of generality we may assume f_n is integrable on E for all n.

$\lim_{n,m} \int (f_n - f_m) g d\mu = 0$ for $g \in L_q$ since

$$\left|\int (f_n - f_m) g d\mu\right| \leq \left|\int (f_n - f) g d\mu\right| + \left|\int (f - f_m) g d\mu\right| .$$

For measurable $F \subseteq E$

$$\left|\int_F f_n d\mu - \int_F f_m d\mu\right| \leq \left|\int_{F\cap(E-E^{m_0})} (f_n - f_m) d\mu\right| + \left|\int_{F\cap E^{m_0}} (f_n - f_m) d\mu\right|$$

$$\leq \varepsilon + \left|\int (f_n - f_m)\chi_{F\cap(E-E^{m_0})} d\mu\right| .$$

Since $\chi_{F\cap(E-E^{m_0})} \in L_q$ the last term on the right approaches 0 as $n, m \to \infty$.

Thus $\limsup_{n,m} \left| \int_F f_n d\mu - \int_F f_m d\mu \right| \leq \varepsilon$.

Since ε is arbitrary $\lim_n \int_F f_n d\mu$ exists and is finite

for measurable $F \subseteq E$.

Let $H_1 = \{x : f > 0, \quad x \in E\}$ $H_2 = \{x : f \leq 0, \quad x \in E\}$.

Clearly $\lim_m \int_{H_1 \cap (E-E^m)} f \, d\mu = \int_{H_1} f d\mu$.

But for each m f is integrable on $H_1 \cap (E-E^m)$ and

$$\lim_n \int_{H_1 \cap (E-E^m)} f_n d\mu = \int_{H_1 \cap (E-E^m)} f \, d\mu \quad .$$

Thus $\lim_n \int_{H_1} f_n d\mu = \int_{H_1} f d\mu$.

Since $\lim_n \int_{H_1} f_n d\mu$ exists and is finite, f is integrable on

H_1 .

Similarly we conclude f is integrable on H_2 .

Thus f is integrable on E.

There exists m_1 such that for $m \geq m_1$

$$\int_{E^m} |f| d\mu < \frac{\varepsilon}{2} \quad \text{and} \quad \int_{E^m} |f_n| d\mu < \frac{\varepsilon}{2} \quad n = 1, 2, \cdots .$$

For measurable F,

$$\left|\int_F (f_n - f)d\mu\right| \le \left|\int_{F\cap E^{m_1}} (f_n - f)d\mu\right| + \left|\int_{F\cap(E-E^{m_1})} (f_n - f)d\mu\right|$$

$$\le \quad \varepsilon + \left|\int (f_n - f)\chi_{F\cap(E-E^{m_1})} d\mu\right| .$$

$$\lim_n \int (f_n - f)\chi_{F\cap(E-E^{m_1})} d\mu = 0 \quad \text{since} \quad \chi_{F\cap(E-E^{m_1})} \in L_q .$$

Thus $\limsup_n \left|\int_F (f_n - f)d\mu\right| \le \varepsilon$ for measurable F.

Since ε is arbitrary $\lim_n \int_F (f_n - f)d\mu = 0$ for measurable F.

Theorem 4: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 3.5 and Theorem 3.

Theorem 5: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 30.4 and 22.2.

<u>Theorem 6:</u> Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 30.5 and 22.2.

33. WEAK * L_∞ CONVERGENCE - CONVERGENCE a.e.

First investigate the conditions under which weak * L_∞ convergence implies convergence a.e. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies convergence a.e. Following the list, the results are stated and proved. Counter-examples 19, 22, 23 show that these are the only implications with a non-redundant set of hypotheses.

(9, 10)	(3, 9)	(2, 8, 9)
(1, 9)	(7, 9)	(5, 8, 9)

Theorem 1: Suppose hypotheses 9 and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

$$\lim_n \int (f_n - f) g d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

Recall that the support of f has σ-finite measure.

Let E be the support of the functions f and f_n $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

It suffices to show pointwise convergence on E_{n_0} for n_0

arbitrary but fixed.

$\chi_{E_{n_0}} \in L_1$ and hence $\lim_n \int_{E_{n_0}} (f_n - f)d\mu = 0$.

But $\left| \int_{E_{n_0}} (f_n - f)d\mu \right| = \int_{E_{n_o}} |f_n - f| \, d\mu$.

Thus $\{f_n\}$ converges in mean to f on E_{n_o}

The conclusion then follows from 14.1.

Theorem 2: Suppose hypotheses 1 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

Follows from 3.12 and Theorem 1.

Theorem 3: Suppose hypotheses 3 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

Follows from 3.3. and Theorem 2.

Theorem 4: Suppose hypotheses 7 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges a.e. to f.

Proof:

Follows from 3. 3 and Theorem 2.

Theorem 5: Suppose hypotheses 2, 8, and 9 are satisifed. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges a. e. to f.

Proof:

$$\lim_n \int f_n d\mu = \int f d\mu \quad \text{and} \quad f \quad \text{is integrable.}$$

Thus, without loss of generality, we may assume that we have hypothesis 1.

The conclusion then follows from Theorem 2.

Theorem 6: Suppose hypotheses 5, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges a. e. to f.

Proof:

$$\lim_n \int (f_n - f) g d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

Let E be the support of the function f.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$,

$$\int_{E^m} |f_n| \, d\mu < \frac{\varepsilon}{2} \quad n = 1, 2, \cdots .$$

Let $F_n = \{x : |f| \leq n, \quad x \in E - E^{m_0}\}$.

There exists n_0 such that for $n \geq n_0$, $\displaystyle\int_{(E-E^{m_0}) \cap F_n^c} |f_m| \, d\mu < \frac{\varepsilon}{2}$

$m = 1, 2, \cdots .$

$\chi_{F_{n_0}} \in L_1$ and hence $\displaystyle\lim_n \int_{F_{n_0}} (f_n - f) \, d\mu = 0$.

But on F_{n_0} f is integrable and hence $\displaystyle\lim_k \int_{F_{n_0}} f_k \, d\mu = \int_{F_{n_0}} f \, d\mu .$

There exist k_0 such that for $k \geq k_0$, f_k is integrable on F_{n_0} .

Thus for $k \geq k_0$, f_n is integrable on E.

$$0 = \lim_n \left| \int_{E-E^{m_0}} (f_n - f) \, d\mu \right| = \lim \int_{E-E^{m_0}} |f_n - f| \, d\mu .$$

Thus $\displaystyle\lim_{n,m} \int_{E-E^{m_0}} |f_n - f_m| \, d\mu = 0$.

For $n, m \geq k_0$

$$\int_E |f_n - f_m| \, d\mu \leq \int_{E^{m_0}} |f_n - f_m| \, d\mu + \int_{E-E^{m_0}} |f_n - f_m| \, d\mu$$

$$\leq \varepsilon + \int_{E-E^{m_0}} |f_n - f_m| \, d\mu .$$

Since the last term on the right approaches 0 as $n, m \to \infty$

$$\limsup_{n,m} \int_E |f_n - f_m| d\mu \leq \varepsilon .$$

Since ε is arbitrary, $\lim_{n,m} \int_E |f_n - f_m| d\mu = 0$.

Thus $\lim_n \int_E f_n d\mu$ exists and is finite.

By Theorem B, p. 107 of Halmos we know that there exists an

integrable function h such that $\lim_n \int_E |f_n - h| d\mu = 0$.

It follows from 8.1 and Theorem B, p. 89 of Halmos that

there is a subsequence of $\{f_n\}$ which converges a.e.

to h on E.

Theorem 1 tells us that $\{f_n\}$ converges a.e. to f on E.

Thus $\{f_n\}$ converges in mean to f on E and hence

$$\lim_n \int_E f_n d\mu = \int f d\mu .$$

Thus f is integrable.

The conclusion then follows from Theorem 5.

Now investigate the conditions under which convergence a.e. implies weak * L_∞. Each set of hypotheses listed below, together with convergence a.e., implies weak * L_∞ convergence. Following the list, the results are stated and proved. Counterexamples 5 and 15 show that these are the only implications with a non-redundant set of hypotheses.

(6, 10) (2, 6) (6, 7)

(1, 6) (3, 6)

<u>Theorem 7:</u> Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

$\lim_n f_n(x) = f(x)$.

Let K be the uniform essential bound for the f_n $n = 1, 2, \cdots$.

K is an essential bound for f.

Clearly $f_n - f \in L_\infty$ for $n = 1, 2, \cdots$.

Hypothesis 10, together with pointwise convergence of $\{f_n\}$ to f tells us that f has σ-finite support.

Let E be the support of f.

Since E has σ-finite measure, $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

Let $g \in L_1$.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$

$$\int_{E^m} |g| \, d\mu < \frac{\varepsilon}{6K}.$$

Let $F_n = \{x : |f_n - f| > \frac{\varepsilon}{3\int |g| d\mu}, \quad x \in E - E^{m_0}\}$.

Since $\mu(E - E^{m_0}) < \infty$, 10.2 tells us that $\mu(F_n) \to 0$ as $n \to \infty$.

There exists n_0 such that for $n \geq n_0$, $\int_{F_n} |g| d\mu < \frac{\varepsilon}{6K}$.

For $n \geq n_0$

$$|\int (f_n - f) g d\mu| = |\int_E (f_n - f) g d\mu| \leq |\int_{E^{m_0}} (f_n - f) g d\mu| + |\int_{E-E^{m_0}} (f_n - f) g d\mu|$$

$$\leq 2K \int_{E^{m_0}} |g| d\mu + |\int_{(E-E^{m_0}) \cap F_n} (f_n - f) g d\mu| + |\int_{(E-E^{m_0}) - F_n} (f_n - f) g d\mu|$$

$$< \frac{\varepsilon}{3} + 2K \int_{(E-E^{m_0}) \cap F_n} |g| d\mu + \frac{\varepsilon}{3\int |g| d\mu} \int |g| d\mu$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Since ε is arbitrary $\lim_n \int (f_n - f) g d\mu = 0$.

<u>Theorem 8:</u> Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a. e. to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 3.12 and Theorem 7.

<u>Theorem 9:</u> Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Let $g \epsilon L_1$.

Let $F_n = \{x : |f_n - f| > \frac{\varepsilon}{2\int |g| d\mu}\}$ and let $F^m = \bigcup_{n=m}^{\infty} F_n$.

Note that $\{F^m\}$ is a decreasing sequence of sets and

$\lim_m F^m = \phi$ (or a set of measure zero).

There exists m_0 such that for $m \geq m_0$, $\int_{F^m} |g| d\mu < \frac{\varepsilon}{4K}$ where K is the uniform essential bound for the f_n $n = 1, 2, \cdots$.

For $m \geq m_0$ $\left|\int (f_n - f) g d\mu\right| \leq \left|\int_{F^m} (f_n - f) g d\mu\right| + \left|\int_{[F^m]^c} (f_n - f) g d\mu\right|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\int |g| d\mu} \cdot \int |g| d\mu = \varepsilon \ .$$

Since ε is arbitrary, $\lim_n \int (f_n - f) g d\mu = 0$.

It is obvious that f has σ-finite support and clearly $f_n - f \epsilon L_\infty$

for all n.

<u>Theorem 10:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 3.14 and Theorem 7.

<u>Theorem 11:</u> Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges a.e. to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 3.13 and Theorem 7.

34. WEAK * L_∞ CONVERGENCE - CONVERGENCE IN MEASURE

First investigate the conditions under which weak * L_∞ convergence implies convergence in measure. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies convergence in measure. Following the list, the results are stated and proved. Counterexamples 12, 13, 19, 22 show that these are the only implications with a non-redundant set of hypotheses.

(3, 9)	(1, 2, 9)	(2, 8, 9)	(5, 9, 10)
(7, 9)	(1, 5, 9)	(5, 8, 9)	

Theorem 1: Suppose hypotheses 3 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.3 and 10.2.

Theorem 2: Suppose hypotheses 7 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.4 and 10.4.

Theorem 3: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.2 and 10.5.

Theorem 4: Suppose hypotheses 1, 5, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.2 and 10.3.

Theorem 5: Suppose hypotheses 2, 8, and 9 are satisifed. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.5 and 10.6.

Theorem 6: Suppose hypotheses 5, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.6 and 10.3.

Theorem 7: Suppose hypotheses 5, 9, and 10 are satisifed. If the sequence of functions $\{f_n\}$ weak $*$ L_∞ converges to the function f, then $\{f_n\}$ converges in measure to f.

Proof:

Follows from 33.1 and 10.3.

Now investigate the conditions under which convergence in measure implies weak $*$ L_∞ convergence. Each set of hypotheses listed below, together with convergence in measure implies weak $*$ L_∞ convergence. Following the list, the results are stated and proved. Counterexamples 5 and 15 show that these are the only implications with a non-redundant set of hypotheses.

(6, 10)	(2, 6)	(6, 7)
(1, 6)	(3, 6)	

Theorem 8: Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Let K be the uniform essential bound for the f_n $n = 1, 2, \cdots$.

K+1 is an essential bound for f.

Let E be the support of the functions f_n $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

$\mu\{x : |f_n - f| > \varepsilon\} \to 0$ as $n \to \infty$ and hence

$$\mu\{x : |f| > \varepsilon, \quad x \in E^c\} = 0 .$$

Since ε is arbitrary f has σ-finite support.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

Let $g \in L_1$.

There exists m_0 such that for $m \geq m_0$, $\int_{E^m} |g| d\mu < \frac{\varepsilon}{6(K+1)}$.

Let $F_n = \{x : |f_n - f| > \frac{\varepsilon}{3 \int |g| d\mu}\}$.

There exists n_0 such that for $n \geq n_0$, $\int_{F_n} |g| d\mu < \frac{\varepsilon}{6(K+1)}$.

For $n \geq n_0$

$$\left|\int (f_n - f) g d\mu\right| = \left|\int_E (f_n - f) g d\mu\right| \leq \left|\int_{E^{m_0}} (f_n - f) g d\mu\right| + \left|\int_{E - E^{m_0}} (f_n - f) g d\mu\right|$$

$$< \frac{\varepsilon}{3} + \left|\int_{(E - E^{m_0}) \cap F_n} (f_n - f) g d\mu\right| + \left|\int_{(E - E^{m_0}) - F_n} (f_n - f) g d\mu\right|$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3 \int |g| d\mu} \cdot \int |g| d\mu = \varepsilon .$$

Since ε is arbitrary, $\lim_n \int (f_n - f) g d\mu = 0$.

Clearly $f_n - f \in L_\infty$ for all n.

Theorem 9: Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Follows from 3.12 and Theorem 9.

Theorem 10: Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Since f is integrable, the support of f has σ-finite measure.

Let K be the uniform essential bound for the f_n $n = 1, 2, \cdots$.

Clearly $K+1$ is an essential bound for f.

Let $g \in L_1$.

Let $F_n = \{x : |f_n - f| > \dfrac{\varepsilon}{2\int |g|\,d\mu}\}$.

There exists n_0 such that for $n \geq n_0$, $\int_{F_n} |g|\,d\mu < \dfrac{\varepsilon}{4(K+1)}$.

For $n \geq n_0$

$$\left|\int (f_n - f) g\,d\mu\right| \leq \left|\int_{F_n} (f_n - f) g\,d\mu\right| + \left|\int_{F_n^c} (f_n - f) g\,d\mu\right| \leq 2(K+1)\int_{F_n} |g|\,d\mu$$

$$+ \frac{\varepsilon}{2\int |g|\,d\mu} \cdot \int |g|\,d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Since ε is arbitrary, $\lim_n \int (f_n - f) g d\mu = 0$.

Clearly $f_n - f \in L_\infty$ for all n.

<u>Theorem 11:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Follows from 3.14 and Theorem 9.

<u>Theorem 12:</u> Suppose hypotheses 6 and 7 are satisifed. If the sequence of functions $\{f_n\}$ converges in measure to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Follows from 3.13 and Theorem 9.

35. WEAK * L_∞ CONVERGENCE-ALMOST UNIFORM CONVERGENCE

First investigate the conditions under which weak * L_∞ convergence implies almost uniform convergence. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies almost uniform convergence. Following the list, the results are stated and proved. Counterexamples 12, 13, 19, 22 show that these are the only implications with a non-redundant set of hypotheses.

(3, 9)	(1, 2, 9)	(2, 8, 9)	(5, 9, 10)
(7, 9)	(1, 5, 9)	(5, 8, 9)	

Theorem 1: Suppose hypotheses 3 and 9 are satisifed. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34. 1 and 6. 2.

Theorem 2: Suppose hypotheses 7 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34. 2 and 6. 2.

Theorem 3: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34. 3 and 6. 2.

Theorem 4: Suppose hypotheses 1, 5, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34. 4 and 6. 2.

Theorem 5: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functiions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34. 5 and 6. 2.

Theorem 6: Suppose hypotheses 5, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34. 6 and 6. 2.

Theorem 7: Suppose hypotheses 5, 9, and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges almost uniformly to f.

Proof:

Follows from 34.7 and 6.2.

Now investigate the conditions under which almost uniform convergence implies weak * L_∞ convergence. Each set of hypotheses listed below, together with almost uniform convergence, implies weak * L_∞ convergence. Following the list, the results are stated and proved. Counterexamples 5 and 15 show that these are the only implications with a non-redundant set of hypotheses.

(1, 6)	(3, 6)	(6, 10)
(2, 6)	(6, 10)	

Theorem 8: Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 6.1 and 34.9.

Theorem 9: Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 6.1 and 34.10.

<u>Theorem 10:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 6.1 and 34.11.

<u>Theorem 11:</u> Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 6.1 and 34.12.

<u>Theorem 12:</u> Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges almost uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 6.1 and 34.8.

36. WEAK $*$ L_∞ CONVERGENCE - UNIFORM CONVERGENCE

First investigate the conditions under which weak $*$ L_∞ convergence implies uniform convergence. In counterexample 8, $\{f_n\}$ weak $*$ L_∞ converges to the function $f = 0$ and all ten hypotheses are satisfied, but $\{f_n\}$ does not converge uniformly to $f = 0$. Thus we get no results with weak $*$ L_∞ convergence implying uniform convergence.

Now investigate the conditions under which uniform convergence implies weak $*$ L_∞ convergence. Each set of hypotheses listed below, together with uniform convergence, implies weak $*$ L_∞ convergence. Following the list, the results are stated and proved. Counterexample 5 shows that these are the only implications with a non-redundant set of hypotheses.

(1)	(3)	(10)
(2)	(7)	

Theorem 1: Suppose hypothesis 1 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Clearly f has σ-finite support.

Let $g \in L_1$.

Given $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{\int |g| d\mu} \quad \text{for all } x.$$

For $n \geq n_0$, $\left| \int (f_n - f) g d\mu \right| < \frac{\varepsilon}{\int |g| d\mu} \cdot \int |g| d\mu = \varepsilon$.

Since ε is arbitrary, $\lim_n \int (f_n - f) g d\mu = 0$.

Clearly $f_n - f$ eventually belongs to L_∞ .

Theorem 2: Suppose hypothesis 2 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Since f is integrable, f has σ-finite support.

The proof is now the same as that of Theorem 1.

Theorem 3: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Clearly f has σ-finite support.

The proof is now the same as that of Theorem 1.

<u>Theorem 4:</u> Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 3.3 and Theorem 1.

<u>Theorem 5:</u> Suppose hypothesis 10 is satisfied. If the sequence of functions $\{f_n\}$ converges uniformly to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Clearly the support of f has σ-finite measure.

The proof is now the same as that of Theorem 1.

37. WEAK * L_∞ CONVERGENCE - CONVERGENCE IN MEAN

First investigate the conditions under which weak * L_∞ convergence implies convergence in mean. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies convergence in mean. Following the list, the results are stated and proved. Counterexamples 1, 4, 6, 19 show that these are the only implications with a non-redundant set of hypotheses.

(3, 9)	(1, 2, 9)	(2, 8, 9)	(5, 9, 10)
(7, 9)	(1, 5, 9)	(5, 8, 9)	

<u>Theorem 1:</u> Suppose hypotheses 3 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

But $g = 1$ belongs to L_1 and so $\lim\limits_n \int (f_n - f) d\mu = 0$.

Since we have hypotheses 9, $\left| \int (f_n - f) d\mu \right| = \int |f_n - f| \, d\mu$.

Thus $\lim\limits_n \int (f_n - f) d\mu = 0$.

Theorem 2: Suppose hypotheses 7 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 34.2 and 5.2.

Theorem 3: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 34.3 and 5.8.

Theorem 4: Suppose hypotheses 1, 5, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 34.4 and 5.3.

Theorem 5: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 34.5 and 5.10.

Theorem 6: Suppose hypotheses 5, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

The same argument as that in 33.6 shows that f is integrable. The conclusion then follows from Theorem 5.

Theorem 7: Suppose hypotheses 5, 9, and 10 are satisifed. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ converges in mean to f.

Proof:

Follows from 34.7 and 5.7.

Now investigate the conditions under which convergence in mean implies weak * L_∞ convergence. Each set of hypotheses listed below, together with convergence in mean implies weak * L_∞ convergence. Counterexamples 15 and 24 show that these are the only implications with a non-redundant set of hypotheses.

(1, 6)	(3, 6)	(6, 10)
(2, 6)	(6, 7)	

Theorem 8: Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f,

then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 5.1 and 34.9.

Theorem 9: Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 5.1 and 34.10.

Theorem 10: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 5.1 and 34.11.

Theorem 11: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 5.1 and 34.12.

<u>Theorem 12:</u> Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ converges in mean to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 5.1 and 34.8.

38. WEAK * L_∞ CONVERGENCE - L_p CONVERGENCE

First investigate the conditions under which weak * L_∞ convergence implies L_p convergence. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 6, 13, 19 show that these are the only implications with a non-redundant set of hypotheses.

(3, 9)	(1, 2, 9)	(2, 8, 9)	(5, 9, 10)
(7, 9)	(1, 5, 9)	(5, 8, 9)	

Theorem 1: Suppose hypotheses 3 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

But $g = 1$ belongs to L_1 and hence $\lim_n \int (f_n - f) d\mu = 0$.

Since we have hypotheses 9, $\left| \int (f_n - f) d\mu \right| = \int |f_n - f| d\mu$ and hence $\{f_n\}$ converges in mean to f.

$f_1 - f \in L_\infty$; call the essential bound K.

$|f_n - f| \le |f_1 - f|$ for all n. (See 14.1)

$$\int |f_n - f|^p d\mu \leq K^{p-1} \int |f_n - f| d\mu \to 0 \quad \text{as} \quad n \to \infty .$$

Theorem 2: Suppose hypotheses 7 and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

$$\lim_n \int (f_n - f) g d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

By 37.2 we know that $\{f_n\}$ converges in mean to f.

Without loss of generality we may assume that $f_n - f \in L_\infty$ for all n.

Let K be the essential bound for $f_1 - f$.

$|f_n - f| \leq |f_1 - f|$ for all n and hence K is a uniform essential bound for $f_n - f$ $n = 1, 2, \cdots$. (See 14.1)

$$\int |f_n - f|^p d\mu \leq K^{p-1} \int |f_n - f| d\mu \to 0 \quad \text{as} \quad n \to \infty .$$

Theorem 3: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Without loss of generality we may assume $f_n - f \in L_\infty$ for all n.

Let K be the essential bound for $f_1 - f$.

By 37.3 we know that $\{f_n\}$ converges in mean to f.

$|f_n - f| \leq |f_1 - f|$ for all n. (See 14.1)

$$\int |f_n - f|^p d\mu \leq K^{p-1} \int |f_n - f| d\mu \to 0 \quad \text{as} \quad n \to \infty .$$

<u>Theorem 4:</u> Suppose hypotheses 1, 5, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Without loss of generality we may assume $f_n - f \in L_\infty$ for all n.

Let K be the essential bound for $f_1 - f$.

By 37.4 we know that $\{f_n\}$ converges in mean to f.

$|f_n - f| \leq |f_1 - f|$ for all n. (See 14.1)

$$\int |f_n - f|^p d\mu \leq K^{p-1} \int |f_n - f| d\mu \to 0 \quad \text{as} \quad n \to \infty .$$

<u>Theorem 5:</u> Suppose hypotheses 2, 8, and 9 are satisifed. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

$\lim_n \int f_n d\mu = \int f d\mu$ and f is integrable.

Thus without loss of generality we may assume hypothesis 1.

The conclusion then follows from Theorem 3.

Theorem 6: Suppose hypotheses 5, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Without loss of generality we may assume that $f_n - f \in L_\infty$ for all n.

Let K be the essential bound for $f_1 - f$.

By 37.6 we know that $\{f_n\}$ converges in mean to f.

$|f_n - f| \geq |f_1 - f|$ for all n. (See 14.1)

$$\int |f_n - f|^p d\mu \leq K^{p-1} \int |f_n - f| d\mu \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Theorem 7: Suppose hypotheses 5, 9, and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ L_p converges to f.

Proof:

Follows from 3.15 and Theorem 4.

Now investigate the conditions under which L_p convergence implies weak * L_∞ convergence. Each set of hypotheses listed below, together with L_p convergence, implies weak * L_∞ convergence. Following the list, the results are stated and proved. Counterexamples 24 and 25 show that these are the only implications with a non-redundant set of hypotheses.

(1, 6) (3, 6) (6, 10)

(2, 6) (6, 7)

Theorem 8: Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 16.1 and 34.9.

Theorem 9: Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 16.1 and 34.10.

Theorem 10: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 16.1 and 34.11.

Theorem 11: Suppose hypotheses 6 and 7 are satisifed. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 16.1 and 34.12.

Theorem 12: Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 16.1 and 34.8.

39. WEAK * L_∞ CONVERGENCE - WEAK L_1 CONVERGENCE

First investigate the conditions under which weak * L_∞ convergence implies weak L_1 convergence. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies weak L_1 convergence. Following the list, the results are stated and proved. Counterexamples 1, 3, 4, 6 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(1, 5)	(5, 10)	(2, 8, 9)
(7)	(5, 8)	(1, 2, 9)	

Theorem 1: Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

But $\chi_E \in L_1$ for measurable E and so $\lim_n \int_E (f_n - f) d\mu = 0$ for measurable E.

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int (f_n - f) g\, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

$$\left| \int (f_n - f_m) g\, d\mu \right| \le \left| \int (f_n - f) g\, d\mu \right| + \left| \int (f - f_m) g\, d\mu \right| \quad \text{and so}$$

$$\lim_{n,m} \int (f_n - f_m) g\, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

Let E be the support of the functions f and f_n $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \ge m_0$

$$\int_{E^m} |f_n|\, d\mu < \frac{\varepsilon}{2} \quad n = 1, 2, \cdots .$$

$$\left| \int f_n\, d\mu - \int f_m\, d\mu \right| \le \left| \int_{E^{m_0}} (f_n - f_m)\, d\mu \right| + \left| \int_{E - E^{m_0}} (f_n - f_m)\, d\mu \right|$$

$$\le \varepsilon + \left| \int (f_n - f_m)\, \chi_{E - E^{m_0}}\, d\mu \right| .$$

The last term on the right approaches 0 as $n, m \to \infty$

since $\chi_{E - E^{m_0}} \in L_1$.

Since ε is arbitrary, $\lim_n \int f_n\, d\mu$ exists and is finite.

Let $F_1 = \{x : f \ge 0\}$ and $F_2 = \{x : f < 0\}$.

Clearly $\lim_m \int_{(E-E^m)\cap F_1} f\,d\mu = \int_{F_1} f\,d\mu$.

But $\mu[(E-E^m)\cap F_1] < \infty$ for each m and hence by Theorem 1 we have $\lim_n \int_{(E-E^m)\cap F_1} f_n\,d\mu = \int_{(E-E^m)\cap F_1} f\,d\mu$ for each m.

Thus $\lim_n \int_{F_1} f_n\,d\mu = \int_{F_1} f\,d\mu$ and hence f is integrable on F_1 .

Similarly we can show f is integrable on F_2 .

Thus f is integrable.

There exists m_1 such that for $m \geq m_1$, $\int_{E^m} |f|\,d\mu < \frac{\varepsilon}{2}$ and $\int_{E^m} |f_n|\,d\mu < \frac{\varepsilon}{2}$, $n = 1, 2, \cdots$.

For measurable G, $\left|\int_G (f_n - f)\,d\mu\right| \leq \left|\int_{G\cap E^{m_1}} (f_n - f)\,d\mu\right| + \left|\int_{G\cap(E-E^{m_1})} (f_n - f)\,d\mu\right|$

$$\leq \varepsilon + \left|\int (f_n - f)\,\chi_{G\cap(E-E^{m_1})}\,d\mu\right| .$$

From Theorem 1 we know that the last term on the right approaches 0 as $n \to \infty$.

Since ε is arbitrary, $\lim \int_G (f_n - f)\,d\mu = 0$ for measurable G.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ weak $*$ L_∞ converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

Let E be the support of the functions f and f_n $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$,

$$\int_{E^m} |f_n| \, d\mu < \frac{\varepsilon}{2} \quad n = 1, 2, \cdots .$$

$$\left| \int (f_n - f_m) g \, d\mu \right| \leq \left| \int (f_n - f) g \, d\mu \right| + \left| \int (f - f_m) g \, d\mu \right|$$ and hence

$$\lim_{n,m} \int (f_n - f_m) g \, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

$$\left| \int f_n \, d\mu - \int f_m \, d\mu \right| \leq \left| \int_{E^{m_0}} (f_n - f_m) \, d\mu \right| + \left| \int_{E - E^{m_0}} (f_n - f_m) \, d\mu \right|$$

$$\leq \varepsilon + \left| \int (f_n - f_m) \, \chi_{E - E^{m_0}} \, d\mu \right| .$$

The last term on the right approaches 0 as $n \to \infty$ since

$$\chi_{E-E^{m_0}} \in L_1 .$$

Since ε is arbitrary, $\lim_n \int f_n d\mu$ exists and is finite.

The proof is completed in the same way as that of Theorem 2.

<u>Theorem 4:</u> Suppose hypotheses 5 and 8 are satisfied. If the sequence of functions $\{f_n\}$ weak $*$ L_∞ converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int (f_n - f) g d\mu = 0 \quad \text{for all} \quad g \in L_1 .$$

Let E be the support of the function f.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$$\mu(E_n) < \infty \quad n = 1, 2, \cdots .$$

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

There exists m_0 such that for $m \geq m_0$, $\int_{E^m} |f_n| d\mu < 1$

$$n = 1, 2, \cdots .$$

Let $F_n = \{x : |f| \leq n, \ x \in E - E^{m_0}\}$.

There exists n_0 such that for $n \geq n_0$, $\int_{F_n^c} |f_m| d\mu < 1$

$$m = 1, 2, \cdots .$$

$\chi_{F_{n_0}} \in L_1$ and so $\lim_n \int_{F_{n_0}} (f_n - f) d\mu = 0$.

Since f is integrable on F_{n_0}, $\lim_n \int_{F_{n_0}} f_n d\mu = \int_{F_{n_0}} f d\mu$.

Thus there exists n_1 such that for $n \geq n_1$, f_n is integrable on F_{n_0}.

Thus f_n is integrable for $n \geq n_1$.

Without loss of generality we may assume that f_n is integrable for all n.

The conclusion then follows from Theorem 3.

<u>Theorem 5:</u> Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak $*$ L_∞ converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 3.15 and Theorem 3.

<u>Theorem 6:</u> Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak $*$ L_∞ converges to the function f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

Follows from 37.3 and 22.2.

<u>Theorem 7:</u> Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak $*$ L_∞ converges to the function

f, then $\{f_n\}$ weak L_1 converges to f.

Proof:

$$\lim_n \int f_n d\mu = \int f d\mu \quad \text{and} \quad f \text{ is integrable.}$$

Thus, without loss of generality, we may assume that we have hypothesis 1.

The conclusion then follows from Theorem 6.

Now investigate the conditions under which weak L_1 convergence implies weak * L_∞ convergence. Each set of hypotheses listed below, together with weak L_1 convergence, implies weak * L_∞ convergence. Following the list, the results are stated and proved. Counterexamples 15 and 24 show that these are the only implications with a non-redundant set of hypotheses.

(1, 6)	(3, 6)	(6, 10)
(2, 6)	(6, 7)	

<u>Theorem 8:</u> Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

$$\lim_n \int_E (f_n - f) d\mu = 0 \quad \text{for measurable} \quad E.$$

Clearly f has σ-finite support.

Let K be the uniform essential bound for the f_n $n = 1,2,\cdots$.

Clearly $K+1$ is an essential bound for f.

Let $g \epsilon L_1$.

Given $\varepsilon > 0$, there exists an integrable simple function s

such that $\int |g-s| d\mu < \frac{\varepsilon}{2(K+1)}$.

$$\left| \int (f_n - f) g d\mu - \int (f_n - f) s d\mu \right| \leq 2(K+1) \int |g-s| d\mu < \varepsilon .$$

But clearly $\lim\limits_n \int (f_n - f) s d\mu = 0$.

Since ε is arbitrary, $\lim\limits_n \int (f_n - f) g d\mu = 0$ for $g \epsilon L_1$.

<u>Theorem 9:</u> Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Since f is integrable, f has σ-finite support.

The proof is now completed in the same manner as that of Theorem 8.

<u>Theorem 10:</u> Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Obviously f has σ-finite support.

The proof is now completed in the same way as that of Theorem 8.

<u>Theorem 11:</u> Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 3.3 and Theorem 8.

<u>Theorem 12:</u> Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak L_1 converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Let E be the support of the functions f_n $n = 1, 2, \cdots$.

$\lim_n \int_F (f_n - f)d\mu = 0$ for measurable F.

Thus $\int_F f d\mu = 0$ for measurable $F \subseteq E^c$.

Thus $f = 0$ a.e. on E^c.

Since E has σ-finite measure, the support of f has σ-finite measure.

The proof is then completed in the same way as that of Theorem 8.

40. WEAK * L_∞ CONVERGENCE - WEAK L_p CONVERGENCE

First investigate the conditions under which weak * L_∞ convergence implies weak L_p convergence. Each set of hypotheses listed below, together with weak * L_∞ convergence, implies weak L_p convergence. Following the list, the results are stated and proved. Counterexamples 1, 6, 13, 20 show that these are the only implications with a non-redundant set of hypotheses.

(3)	(1, 5)	(5, 10)	(2, 8, 9)
(7)	(5, 8)	(1, 2, 9)	

<u>Theorem 1:</u> Suppose hypothesis 3 is satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

$$\lim_n \int (f_n - f) g \, d\mu = 0 \quad \text{for} \quad g \in L_1 .$$

Without loss of generality we may assume $f_n - f \in L_\infty$ for all n. Since we have hypothesis 3, $f_n - f \in L_p$ for all n. Let $h \in L_q$ and let $F_1 = \{x: |h| \le 1\}$, $F_2 = \{x: |h| > 1\}$.

$$\int |h| \, d\mu \le \int_{F_1} |h| \, d\mu + \int_{F_2} |h| \, d\mu \le \mu(F_1) + \int_{F_2} |h|^q d\mu .$$

Thus $h \in L_1$ and hence $\lim_n \int (f_n - f) h \, d\mu = 0$ for $h \in L_q$.

Theorem 2: Suppose hypothesis 7 is satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 39.2 we know that $\{f_n\}$ weak L_1 converges to f.

Since we have hypothesis 7, we know that $\lim_n \int_F f_n d\mu = \int_F f d\mu$ for measurable F.

Thus f is integrable.

Let K_n be the essential bound for $f_n - f$.

$\int |f_n - f|^p d\mu \leq (K_n)^{p-1} \int |f_n - f| d\mu < \infty$.

Thus each $f_n - f$ belongs to L_p.

Let E be the support of the functions f and f_n $n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

Let $E^m = E - \bigcup_{n=1}^{m} E_n$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$,

$$\int_{E^m} |f| d\mu < \frac{\varepsilon}{2} \quad \text{and} \quad \int_{E^m} |f_n| d\mu < \frac{\varepsilon}{2} \quad n = 1, 2, \cdots .$$

Let $h \in L_q$ and let $G_1 = \{x : |h| \leq 1\}$, $G_2 = \{x : |h| > 1\}$.

$$\left|\int (f_n - f)h\,d\mu\right| \le \left|\int_{G_1} (f_n - f)h\,d\mu\right| + \left|\int_{G_2} (f_n - f)h\,d\mu\right|$$

$$\le \left|\int_{G_1 \cap E^{m_0}} (f_n - f)h\,d\mu\right| + \left|\int_{G_1 \cap (E - E^{m_0})} (f_n - f)h\,d\mu\right| + \left|\int (f_n - f)h \cdot \chi_{G_2}\,d\mu\right|$$

$$< \varepsilon + \left|\int_{G_1 \cap (E - E^{m_0})} (f_n - f)h\,d\mu\right| + \left|\int (f_n - f)h \cdot \chi_{G_2}\,d\mu\right| .$$

Since $\mu[G_1 \cap (E - E^{m_0})] < \infty$, Theorem 1 tells us that

$$\lim_n \left|\int_{G_1 \cap (E - E^{m_0})} (f_n - f)h\,d\mu\right| = 0 .$$

Since $h \cdot \chi_{G_2} \epsilon L_1$, the last term on the right approaches 0 as $n \to \infty$.

Thus $\limsup_n \left|\int (f_n - f)h\,d\mu\right| \le \varepsilon$.

Since ε is arbitrary, $\lim_n \int (f_n - f)h\,d\mu = 0$ for $h \epsilon L_q$.

Theorem 3: Suppose hypotheses 1 and 5 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

By 39. 3 we know that $\{f_n\}$ weak L_1 converges to f.

Thus there exists n_0 such that for $n \ge n_0$, $f_n - f$ is integrable.

The proof is now completed in the same way as that of Theorem 2.

Theorem 4: Suppose hypotheses 5 and 8 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Let E be the support of the function f.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ $n = 1, 2, \cdots$.

Let $E^m = E - \bigcup_{n=1}^{\infty} E_n$.

There exists m_0 such that for $m \geq m_0$, $\int_{E^m} |f_n| \, d\mu < 1$ $n = 1, 2, \cdots$.

Let $F_n = \{x : |f| \leq n, \ x \in E - E^{m_0}\}$.

There exists n_0 such that for $n \geq n_0$, $\int_{F_n^c} |f_m| \, d\mu < 1$ $n = 1, 2, \cdots$.

By 39.4 we know that $\{f_n\}$ weak L_1 converges to f.

But on F_{n_0} f is integrable and hence $\lim \int_{F_{n_0}} f_n \, d\mu = \int_{F_{n_0}} f \, d\mu$.

Thus there exists n_1 such that for $n \geq n_1$, f is integrable.

Without loss of generality we may assume that we have hypothesis 1.

The conclusion then follows from Theorem 3.

Theorem 5: Suppose hypotheses 5 and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 3.15 and Theorem 3.

Theorem 6: Suppose hypotheses 1, 2, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 38.3 and 31.2.

Theorem 7: Suppose hypotheses 2, 8, and 9 are satisfied. If the sequence of functions $\{f_n\}$ weak * L_∞ converges to the function f, then $\{f_n\}$ weak L_p converges to f.

Proof:

Follows from 38.5 and 31.2.

Now investigate the conditions under which weak L_p convergence implies weak * L_∞ convergence. Each set of hypotheses listed below, together with weak L_p convergence implies weak * L_∞ convergence. Following the list, the results are stated and proved. Counterexamples 24 and 25 show that these are the only implications with a non-redundant set of hypotheses.

(6, 10) (2, 6) (6, 7)

(1, 6) (3, 6)

Theorem 8: Suppose hypotheses 6 and 10 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Without loss of generality we may assume $f_n - f \in L_p$ for all n.

Let E be the support of the functions f_n and $f_n - f$,

$n = 1, 2, \cdots$.

E has σ-finite measure; that is $E = \bigcup_{n=1}^{\infty} E_n$ where

$\mu(E_n) < \infty$, $n = 1, 2, \cdots$.

Clearly f has σ-finite support.

Let K be the uniform essential bound for the f_n $n=1, 2, \cdots$.

Let $F_1 = \{x: f \geq K+1\}$, $F_2 = \{x: f \leq -K-1\}$.

Show $\mu(F_1) = \mu(F_2) = 0$ as follows:

Assume this is not so; that is $\mu(F_1) > 0$.

There exists n_0 such that $\mu(F_1 \cap E_{n_0}) > 0$.

Let $H_m = \{x: |f| \leq m, \ x \in F_1 \cap E_{n_0}\}$ and note that

$\{H_m\}$ is an increasing sequence of sets and

$F_1 \cap E_{n_0} = \bigcup_{m=1}^{\infty} H_m$ except for perhaps a set of measure zero.

There exists m_0 such that $\mu[F_1 \cap E_{n_0} \cap H_{m_0}] > 0$.

By 32.2 we know that $\{f_n\}$ weak L_1 converges to

f on $F_1 \cap E_{n_0} \cap H_{m_0}$.

Since f is integrable on $F_1 \cap E_{n_0} \cap H_{m_0}$,

$$\lim \int_{F_1 \cap E_{n_0} \cap H_{m_0}} f_n \, d\mu = \int_{F_1 \cap E_{n_0} \cap H_{m_0}} f \, d\mu .$$

But $\left| \int_{F_1 \cap E_{n_0} \cap H_{m_0}} f_n \, d\mu \right| \leq K\mu[F_1 \cap E_{n_0} \cap H_{m_0}]$ and

$$\left| \int_{F_1 \cap E_{n_0} \cap H_{m_0}} f \, d\mu \right| \geq (K+1)\mu[F_1 \cap E_{n_0} \cap H_{m_0}] .$$

This is a contradiction and so $\mu(F_1) = 0$.

In a similar way we can show that $\mu(F_2) = 0$.

Thus $K+1$ is an essential bound for f.

Clearly each $f_n - f \in L_\infty$.

Let $g \in L_1$.

Given $\varepsilon > 0$, there exists m_0 such that for $m \geq m_0$,

$$\int_{E^m} |g| \, d\mu < \frac{\varepsilon}{2(K+1)} .$$

$$\left|\int (f_n - f) g \, d\mu\right| \leq \left|\int_{E^{m_0}} (f_n - f) g \, d\mu\right| + \left|\int_{E-E^{m_0}} (f_n - f) g \, d\mu\right| < \varepsilon + \left|\int (f_n - f) g \cdot \chi_{E-E^{m_0}} \, d\mu\right| .$$

Show $\lim_n \left| \int_{E-E^{m_0}} (f_n - f) g \cdot \chi \, d\mu \right| = 0$ as follows:

By 32.2 we know that $\{f_n\}$ weak L_1 converges to f on $E-E^{m_0}$.

Given $\varepsilon_1 > 0$, there exists an integrable simple function s such that $\int |g \cdot \chi_{E-E^{m_0}} - s| d\mu < \frac{\varepsilon}{2(K+1)}$.

$$\left| \int (f_n - f) g \cdot \chi_{E-E^{m_0}} d\mu - \int (f_n - f) s \, d\mu \right| \leq 2(K+1) \int |g \cdot \chi_{E-E^{m_0}} - s| \, d\mu < \varepsilon.$$

But clearly $\lim_n \int (f_n - f) s \, d\mu = 0$.

Since ε_1 is arbitrary, $\lim_n \int (f_n - f) g \cdot \chi_{E-E^{m_0}} d\mu = 0$.

Thus $\limsup_n \left| \int (f_n - f) g \, d\mu \right| \leq \varepsilon$.

Since ε is arbitrary, $\lim_n \int (f_n - f) g \, d\mu = 0$.

Theorem 9: Suppose hypotheses 1 and 6 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak * L_∞ converges to f.

Proof:

Follows from 3.12 and Theorem 8.

Theorem 10: Suppose hypotheses 2 and 6 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Clearly f has σ-finite support.

Without loss of generality we may assume that $f_n - f \in L_p$ for all n.

Thus we may assume that we have hypothesis 10.

The conclusion then follows from Theorem 8.

Theorem 11: Suppose hypotheses 3 and 6 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Follows from 3.14 and Theorem 8.

Theorem 12: Suppose hypotheses 6 and 7 are satisfied. If the sequence of functions $\{f_n\}$ weak L_p converges to the function f, then $\{f_n\}$ weak $*$ L_∞ converges to f.

Proof:

Follows from 3.13 and Theorem 8.

BIBLIOGRAPHY

1. Dunford, N. and J. T. Schwartz. Linear operators part I: general theory. New York, Interscience, 1964. 858 p.

2. Halmos, P. R. Measure theory. Princeton, New Jersey, Van Nostrand, 1961. 304 p.

3. Munroe, M. E. Introduction to measure and integration. New York, Addison-Wesley, 1953. 304 p.

4. Riesz, F. and B. Sz-Nagy. Functional analysis. New York, Frederick Unger, 1955. 467 p.

5. Royden, H. L. Real analysis. New York, The Macmillan Co., 1963. 284 p.

APPENDIX

APPENDIX

When the work on convergence relations was first being investigated, the definitions of weak L_1 and weak * L_∞ convergences were replaced respectively by these two kinds of convergence:

Definition: A sequence of functions $\{f_n\}$ weak L_1' converges to the function f if $\lim_n \int_E f_n d\mu = \int_E f d\mu$ for every measurable set E.

Definition: A sequence of functions $\{f_n\}$ weak * L_∞' converges to the function f if $\lim_n \int (f_n - f) g d\mu = 0$ for $g \in L_1$.

The disadvantage of these two definitions is the fact when $\{f_n\}$ converges in either of the two ways to f, f is not necessarily unique. The non-uniqueness of these limit functions occurs when we have a measure space with atoms of infinite measure. Non-uniqueness of the limit functions seemed undesirable and so the definitions were altered so as to make the limit functions unique. A question that is of interest when one of the above convergences are involved is described as follows:

Suppose we have selected two types of convergence, call them type I and type II. If $\{f_n\}$ converges in the type I sense to f and there is a subsequence $\{f_{n_k}\}$ which converges in the type II sense to u, is $f = u$ a. e. ?

The answers to this question, along with the questions previously considered are given in the tables and diagrams below. About half of the work with weak L_1' convergence relied heavily on one theorem which is now stated and proved.

<u>Theorem:</u> Suppose hypothesis 10 is satisfied. If the sequence of functions $\{f_n\}$ weak L_1' converges to the function f and converges a. e. to the function u, then $f = u$ a. e.

Proof:

$$\lim_n \int_E f_n d\mu = \int_E f d\mu \quad \text{for measurable } E.$$

Let F be the support of the functions f_n $n = 1, 2, \cdots$.

$$0 = \lim_n \int_E f_n d\mu = \int_E f d\mu \quad \text{for measurable } E \subseteq F^c .$$

Upon applying Theorem E, p. 105 of Halmos we see $f = 0$ a. e. on F^c.

Thus $f = u$ a. e. on F^c.

By hypothesis, F has σ-finite measure; that is $F = \bigcup_{n=1}^{\infty} F_n$ where $\mu(F_n) < \infty$ $n = 1, 2, \cdots$.

It suffices to show $f = u$ a. e. on each F_n.

Thus without loss of generality we may assume that the measure space is totally finite.

Let $h_n(x) = \sup\{f_i(x) : i = n, n+1, \cdots\}$ and

$g_n(x) = \inf\{f_i(x) \colon i = n, n+1, \cdots\}$; by Theorem A, p. 84 of Halmos these functions are measurable.

Clearly $\{g_n\}$ and $\{h_n\}$ converge a. e. to u.

Let $E_n = \{x : |u(x)| \leq n\}$ and note that $X = \bigcup_{n=1}^{\infty} E_n$ except for a set of measure zero.

It suffices to show $f = u$ a. e. on E_{n_0} for n_0 arbitrary but fixed.

10. 2 tells us that given $\varepsilon > 0$, $\mu\{x : |g_n - u| > \varepsilon,\ x \in E_{n_0}\} \to 0$ as $n \to \infty$.

Let $Z_i = \{x : |g_i - u| < \varepsilon,\ x \in E_{n_0}\}$ and note that $\{Z_i\}$ is an increasing sequence of sets and $E_{n_0} = \bigcup_{i=1}^{\infty} Z_i$.

It suffices to show $f = u$ a. e. on Z_{i_0} for i_0 arbitrary but fixed.

Since $|g_n| \leq n_0 + \varepsilon$ for $x \in Z_{i_0}$ and $n \geq i_0$, 2. 15 tells us $\{g_n\}$ converges in mean to u in Z_{i_0}.

$\int_C f d\mu = \lim_n \int_C f_n d\mu \geq \lim_n \int_C g_n d\mu = \int_C u d\mu$ for measurable $C \subseteq Z_{i_0}$.

Thus $f = u$ a.e. on Z_{i_0} which is proved as follows:

If not, $f < u$ on $A \subseteq Z_{i_0}$ such that $\mu(A) > 0$.

$u-f$ is non-negative on A.

Either $u-f$ is integrable or $u-f$ is not integrable on A.

Part 1: Assume $u-f$ is integrable on A.

$$\int (u-f)d\mu \geq 0 .$$

By Theorem B, p. 104 of Halmos $\int_A (u-f)d\mu = 0$

if and only if $u-f = 0$ a.e. on A.

Since $f < u$, $\int_A fd\mu < \int_A ud\mu$.

This is a contradiction since $\int_A fd\mu \geq \int_A ud\mu$.

Part 2: Assume $u-f$ is not integrable on A.

Let $B^m = \{x : u-f \leq m, x \in A\}$ and note that

$\{B^m\}$ is an increasing sequence of

sets and $A = \bigcup_{m=1}^{\infty} B^m$.

$$\lim_m \mu(B^m) = \mu[\lim_m B^m] > 0 .$$

Thus there exists m_0 such that $\mu(B^{m_0}) > 0$.

$$\int_{B^{m_0}} (u-f)d\mu \geq 0 \ .$$

$$\int_{B^{m_0}} (u-f)d\mu = 0 \quad \text{if and only if} \quad u-f = 0 \ \text{a.e.}$$

Since $f < u$ $\int_{B^{m_0}} fd\mu < \int_{B^{m_0}} ud\mu$.

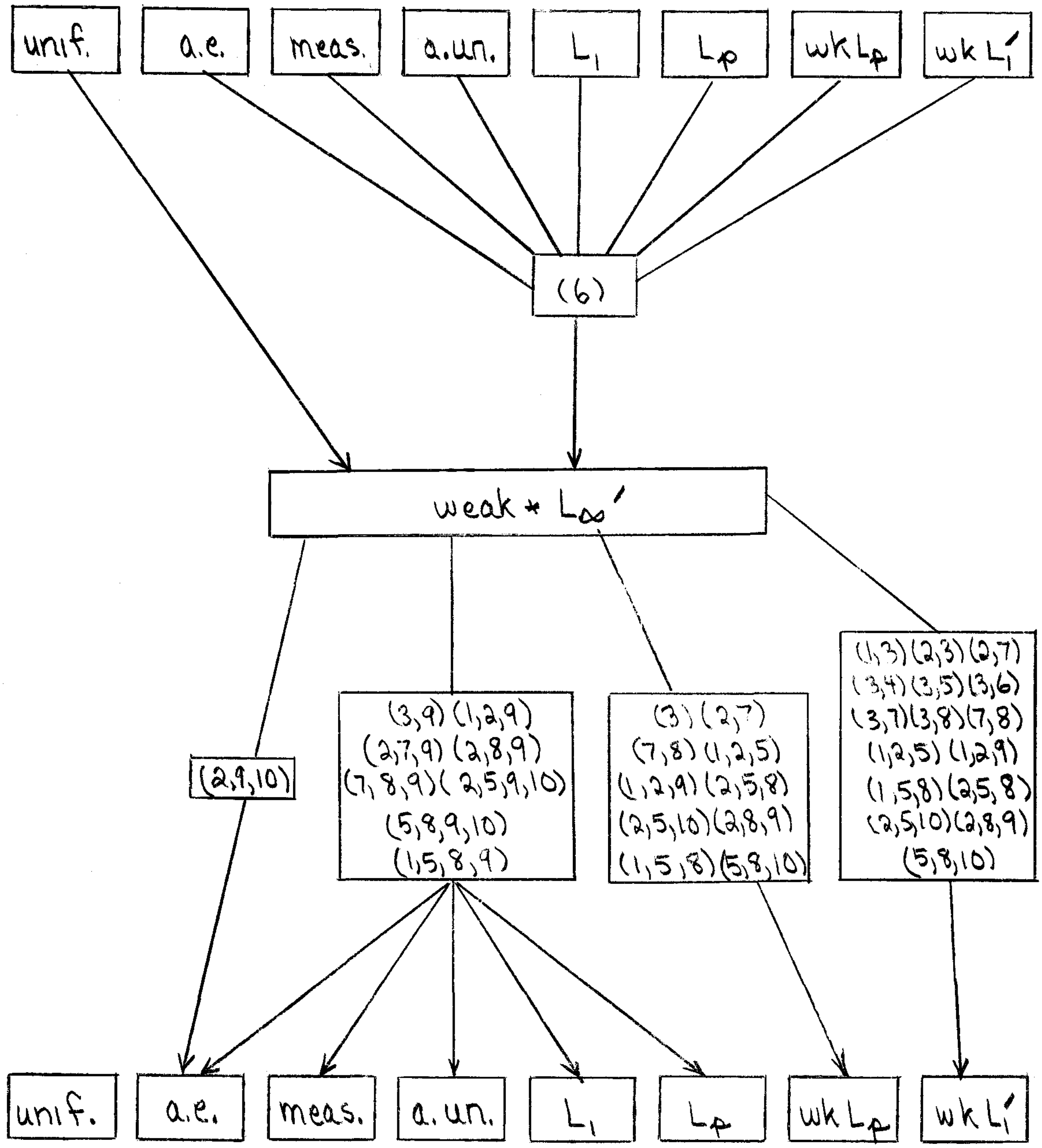
This is a contradiction since $\int_{B^{m_0}} fd\mu \geq \int_{B^{m_0}} ud\mu$.

With the above argument modified by replacing g_n by h_n, $u-f$ by $f-u$ and reversing some of the inequalities we can conclude that $f \leq u$ a.e. on Z_{i_0} .

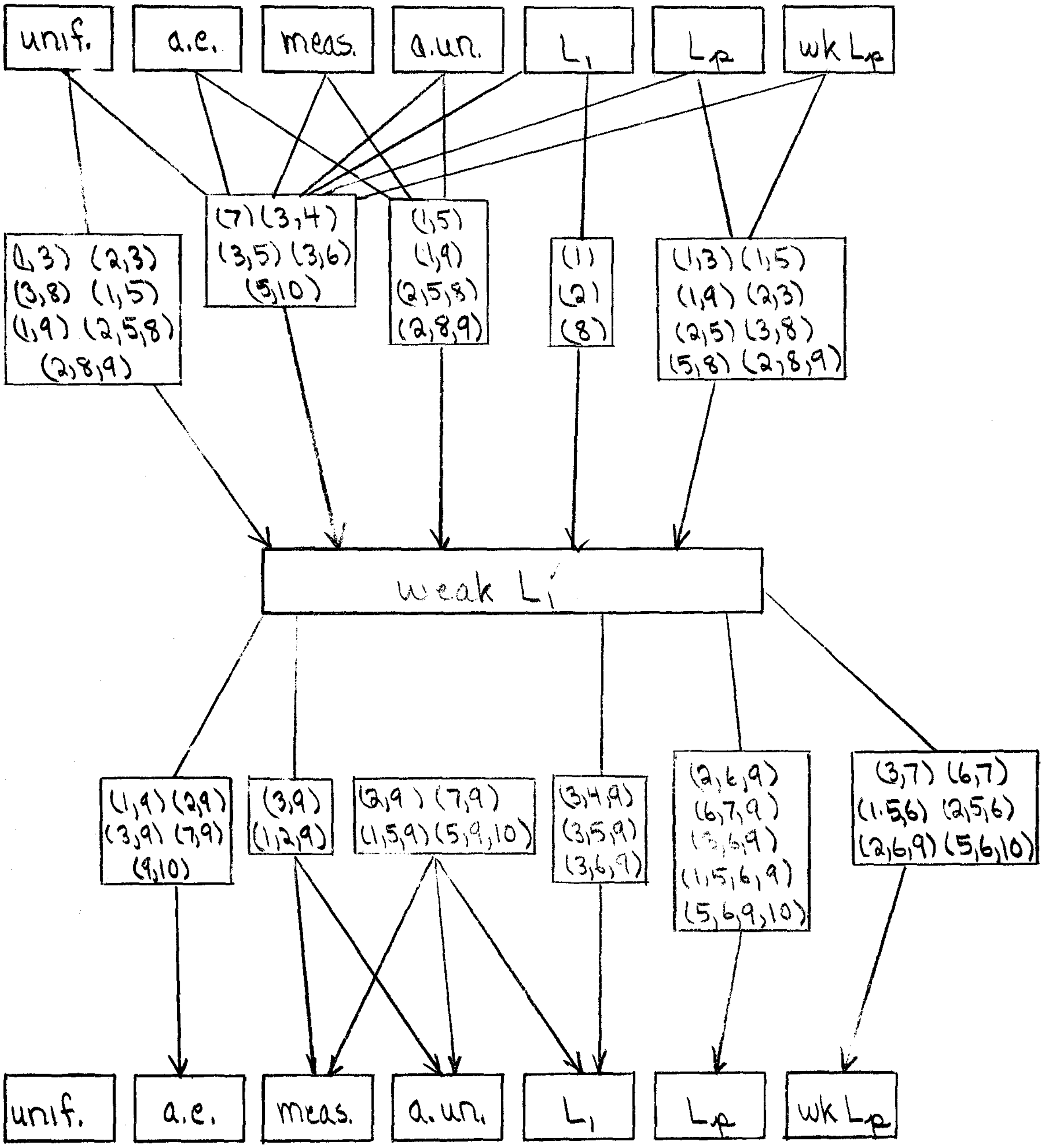
Thus $f = u$ a.e. on z_{i_0}, hence on E_{n_0} and hence on E.

We now give two digarams which treat the questions considered throughout the thesis. Tables are then given which answer the question posed in this appendix.

The diagram given below indicates how weak $*$ L'_∞ interacts with the other eight modes of convergence. The arrows indicate cases where we have an implication. The numbers in parentheses are sets of hypotheses. If an arrow goes to a box containing sets of hypotheses, this indicates that at least one set of hypotheses is needed for the implication. As an illustration: Weak * L'_∞ convergence implies weak L_p convergence if hypothesis 3 is satisfied. Notice that any of the other nine sets of hypotheses in the box also give this implication.



The diagram given below is read in the same way as the one above except that it shows how weak L_1' interacts with seven modes of convergence. (The interaction with weak $*$ L_∞' was shown in the last diagram.)



The table below answers the question posed earlier in this appendix when weak $*$ L'_∞ convergence is one of the two types of convergence under consideration. We now illustrate how the table is read and what information it yields.

Choose any one of the eight modes of convergence listed along the top of the table; suppose mean convergence is chosen. Now pick one of the sets of hypotheses listed along the left side of the table; suppose $(7, 8^f)$ is selected. At the intersection of the column containing mean convergence and the row containing $(7, 8^f)$ we see there is an X This indicates that the following theorem is true.

Theorem: Suppose hypothesis 2 is satisfied and hypothesis 8 is satisfied with f. If $\{f_n\}$ weak $*$ L'_∞ converges to f and converges in mean to u, then $f = u$ a.e.

If there had been a O at the intersection we would have gotten no theorem.

Notice that this theorem answers the question posed earlier in this appendix regardless of the roles taken by weak $*$ L'_∞ and mean convergence.

	a.e.	meas.	a.un.	unif.	(L_1) mean	L_p	wk L_p	wk L_1'
(3)	X	X	X	X	X	X	X	X
$(1, 2^f)$	X	X	X	X	X	X	X	X
$(1, 8^f)$	O	O	O	O	X	O	O	O
$(2^u, 2^f)$	X	X	X	X	X	X	X	X
(2, 7)	X	X	X	X	X	X	X	X
$(2^f, 8^f)$	X	X	X	X	X	X	X	X
$(2^u, 8^f)$	O	O	O	O	X	O	O	X
$(2^f, 10)$	X	X	X	X	X	X	X	X
$(7, 8^f)$	X	O	O	O	X	X	X	X
$(1, 5, 8^f)$	X	X	X	X	O	X	X	X
$(1, 2^u, 9)$	X	O	O	O	O	O	O	O
$(2^u, 5, 8^f)$	O	O	O	O	O	O	X	O
$(2^u, 8^u, 8^f)$	X	X	X	X	O	X	X	O
$(5, 10, 8^f)$	X	X	X	X	X	X	X	X
$(1, 2^u, 8^f, 9)$	X	X	X	X	O	O	O	O

The following table is read in the same was as the one above except that weak L_1' convergence is treated instead of weak $* L_\infty'$.

	a.e.	meas.	a.un.	unif	(L_1) mean	L_p	wk L_p
(1)	X	X	X	X	X	X	X
(2^f)	X	X	X	X	X	X	X
(2^u)	O	O	O	O	X	X	X
(3)	X	X	X	X	X	X	X
(7)	X	X	X	X	X	X	X
(10)	X	X	X	X	X	X	X
$(2^u, 8^u)$	X	X	X	X	X	X	X