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Regular null curves in L^3 always have a time component parametrization. This fact puts the Frenet equations for regular null curves in L^3 in a simple form. A fundamental existence and uniqueness theorem for regular null curves in L^3 is proved, and applications to geometry are given.

A Frenet Theorem for Regular Null Curves in L^3

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I. INTRODUCTION

The purpose of this paper is to show that with the proper choice of a parameter and a frame field the Frenet equations of all regular null curves in L^3 take on a simple form. In addition an existence and uniqueness theorem is proved for null curves in L^3.

II. FRENET EQUATIONS

Definition 2.1: Let L^3 denote the space of triplets of real numbers, (a_1, a_2, a_3) with the bi-linear form defined by:

$$<a_1, a_2, a_3, b_1, b_2, b_3> = -a_1b_1 + a_2b_2 + a_3b_3$$

We will take as our basis for L^3 the following three vectors:

$$e_0 = (1, 0, 0) \quad e_1 = (0, 1, 0) \quad e_3 = (0, 0, 1)$$

A vector $A \epsilon L^3$ is called timelike, spacelike or null depending on whether (A, A) is negative, positive, or zero (1).

We will define the cross product of two vectors A, B in L^3 by:

$$A \times B = \begin{vmatrix} -e_0 & e_1 & e_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Definition 2.2: Let $a(t) : I \rightarrow L^3$ be a c^2 curve, then $a(t)$ is a null curve iff $(a'(t), a'(t)) = 0$ for every $t \epsilon I$.

Definition 2.3: $a(t)$ a null curve in L^3 is regular iff $a'(t) \neq 0$ for every $t \epsilon I$.

Theorem 2.1: Given $a(t)$ a regular null curve in L^3, there exists a re-parametrization $b(s)$ such that $s = a_1(t)$.

Proof: Since $a(t)$ is regular and null, $a_1'(t) \neq 0$ without loss of generality we can assume $a_1'(t) > 0$. This will orient towards the future. Since $a_1'(t) > 0$, we know $a_1(t)$ has a global inverse. Set $s = a_1(t)$ then $t = a_1^{-1}(s)$. Define:

$$b(s) = (a_1(a_1^{-1}(s)), a_2(a_1^{-1}(s)), a_3(a_1^{-1}(s)))$$

For convenience set:

$$a_2(a_1^{-1}(s)) = g(s)$$

$$a_3(a_1^{-1}(s)) = h(s)$$

then

$$b(s) = (s, g(s), h(s))$$

Next we want to attach a frame field to the curve given above and then look at the Frenet equations.

Definition 2.4: Let A, B, C be three linearly independent vectors in L^3. A, B, C constitute a null frame iff

(1) A, B are null

(2) $\langle A, B \rangle = -1$

(3) C is unit space-like

(4) $\det(A, B, C) = \pm 1$ (2).

We will now take our null curve $b(s) = (s, g(s), h(s)$ and define a frame field on it as follows:

$$T(s) = (1, g'(s), h'(s))$$

$$N(s) = (1/2, -g'(s)/2, -h'(s)/2)$$

$$B(s) = T(s) \times N(s) = (0, h'(s), -g'(s))$$

Calculation shows that T, N, B is a null frame.

Definition 2.5: A null frame is properly oriented iff $\langle A, B \times C\rangle = 1$. T, N, B is properly oriented. We are now in a position to express the derivatives T', N', B' in terms of T, N, B by the following scheme. Set

$$T' = w_{11}T + w_{12}N + w_{13}B$$

$$N' = w_{21}T + w_{22}N + w_{23}B$$

$$B' = w_{31}T + w_{32}N + w_{33}B \quad (3).$$

The w_{ij}'s have the following values

$$w_{11} = w_{12} = w_{21} = w_{22} = w_{33} = 0$$

and

$$w_{13} = g''h' - g'h''$$

$$w_{23} = \frac{-g''h'+g'h''}{2}$$

$$w_{31} = \frac{h''g'-g''h'}{2}$$

$$w_{32} = h'g'' - g'h''$$

Define:

$$w_{13} = k_1(s) \quad \text{(curvature)}$$

$$w_{23} = k_2(s) \quad \text{(torsion)}$$

We can see that

$$w_{13} = w_{32} \quad \text{and} \quad w_{23} = w_{31}$$

also

$$k_2(s) = -\frac{1}{2}k_1(s)$$

With this information in mind the Frenet equations take the following form.

$$T' = k_1 B \qquad\qquad b'(s) = T$$

$$N' = k_2 B$$

$$B' = k_2 T + k_1 N$$

A null curve whose Frenet equations have this form is called a cartan frame curve (4). Clearly every regular null curve in L^3 can be put in this form.

III. EXISTENCE AND UNIQUENESS

In the definition of L^3 the triple of numbers (a_1, a_2, a_3), a_1 is the time coordinate and a_2, a_3 are spacial coordinates. If we look at all the events of the form $(0, a_2, a_3)$, we get a plane in L^3 which we will call the spacelike plane. If we restrict our quadratic form from Chapter I to this plane it is positive definite. The result of this is that we have a two dimensional subspace of L^3 that is Euclidean (5). This fact gives rise to the following two lemmas:

Lemma 3.1: Given $b(s) : I \rightarrow L^3$ a regular null curve parametrized with respect to its time component, then the canonical projection of $b(s)$, call it $\bar{b}(s)$, into the spacelike plane is a regular curve parametrized with respect to its arc length.

Proof: Let π represent canonical projection, thus

$$\bar{b}(s) = \pi b(s)$$

From the hypothesis:

$$b(s) = (s, g(s), h(s)), \quad b'(s) \neq 0$$

and

$$\langle b'(s), b'(s) \rangle = 0$$

Then

$$\bar{b}(s) = \pi(s, g(s), h(s)) = (g(s), h(s))$$

Since $b(s)$ is regular we know $g'(s), h'(s)$ cannot both be zero. Therefore $\bar{b}(s)$ is regular. Also:

$$\|\bar{b}(s)\| = g'(s)^2 + h'(s)^2 = 1$$

From Lemma 3.2 we know $\bar{b}(s)$ is a regular curve parametrized in terms of its arc length. We next want to fit a frame field to $\bar{b}(s)$ and define its curvature.

<u>Definition 3.1</u>: Let $\bar{T}(s) = \bar{b}'(s) = (g'(s), h'(s))$ and $\bar{N}(s) = (-h'(s), g'(s))$.

<u>Definition 3.2</u>: Let $\bar{k}_1$ be the curvature of $\bar{b}(s)$ defined by the following equation:

$$\bar{T}'(s) = \bar{k}_1(s)\bar{N}(s) \qquad (6).$$

<u>Lemma 3.2:</u> Given $b(s)$ a regular null curve parametrized with respect to its time component. Let $\bar{b}(s)$ be the canonical projection of $b(s)$ into the space like plane. Let $k_1(s)$ be the (curvature) of $b(s)$ as defined in Chapter I, and $\bar{k}_1(s)$ be the curvature of $\bar{b}(s)$ as in Definition 2.2, then:

$$k_1(s) = -\bar{k}_1(s)$$

Proof: By Definition 2.2:

$$\bar{k}_1(s) = <\bar{T}'(s), \bar{N}(s)> = -g''(s)h'(s) + h''(s)g'(s)$$

From Chapter I

$$k_1(s) = g''(s)h'(s) - h''(s)g'(s)$$

which gives

$$k_1(s) = -\bar{k}_1(s)$$

Lemmas 2.1 and 2.2 put us in a position to prove an existence and uniqueness theorem for regular null curves in L^3.

Theorem 3.1, Existence: Given $\bar{k}_1(s)$ a continuous function defined on some interval, I there exists a regular null curve $b(s) : I \to L^3$ parametrized with respect to its time component that has $-\bar{k}_1(s)$ as its curvature.

Proof: By Frenet's Theorem in the plane $\bar{k}_1(s)$ will give rise to a regular curve in the spacelike plane parametrized with respect to its arc length. This is possible because we can identify the spacelike plane with E^2. Call this curve $\bar{b}(s)$ in component form $\bar{b}(s) = (g(s), h(s))$. Next define a mapping $F : \bar{b}(s) \to L^3$ by: $F(\bar{b}(s)) = (s, g(s), h(s))$. Then set:

$$b(s) = F(\bar{b}(s))$$

<u>Theorem 3.2, Uniqueness:</u> Given $b(s)$ and $c(\bar{s})$ two regular null curves in L^3 parametrized with respect to their time components with $k_1(s)$ and $j_1(\bar{s})$ their respective curvatures such that $k_1(s) = j_1(\bar{s})$, then $b(s) \cong c(\bar{s})$. i.e. $b(s)$ is the image of $c(\bar{s})$ under a proper Lorentz transformation.

<u>Proof:</u> Assume $b(0) = c(0)$ and let $\bar{b}(s)$ and $\bar{c}(\bar{s})$ be the projected curves in the spacelike plane for $b(s)$ and $c(\bar{s})$. Let $\bar{k}_1(s)$ and $\bar{j}_1(\bar{s})$ be the respective curvatures of $\bar{b}(s)$ and $c(\bar{s})$. By hypothesis $k_1(s) = j_1(s)$. This gives by Lemma 2.2

$$\bar{k}_1(s) = \bar{j}_1(\bar{s})$$

also

$$\bar{b}(0) = \bar{c}(0)$$

So we have

$$\bar{b}(s) = G \circ \bar{c}(\bar{s})$$

where G is a proper motion of E^2, since congruent curves have equal arc lengths $s = \bar{s}$ (7). From this information we are going to construct a proper Lorentz transformation that carries $c(s)$ into $b(s)$. Let

$$b(s) = (s, g(s), h(s)) \qquad c(s) = (s, \bar{g}(s), \bar{h}(s))$$

$$\bar{b}(s) = (g(s), h(s)) \qquad \bar{c}(s) = (\bar{g}(s), \bar{h}(s))$$

Next let us represent G by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\bar{b}(s) = G \circ \bar{c}(s) = (s, a\bar{g}(s)+b\bar{h}(s), c\bar{g}(s)+d\bar{h}(s))$$

Now define a new matrix H by

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

We want to show that,

(1) H is a proper Lorentz transformation of L^3

(2) $b(s) = H \circ c(s)$.

Definition 3.3: A 3×3 matrix X is a member of the Lorentz group of proper motions called $SO_+(1, 2)$ iff

(1) $\det X = 1$

(2) $\langle Xe_0, e_0 \rangle \leq -1$ (8).

We claim that H is a member of $SO_+(1, 2)$

(1) $\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} = 1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

$$(2)\quad He_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_0$$

which gives

$$\langle He_0, e_0 \rangle = \langle e_0, e_0 \rangle = -1 \leq -1$$

We now have to show that $b(s) \approx c(s)$

$$H \circ c(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} s \\ \bar{g}(s) \\ \bar{h}(s) \end{pmatrix}$$

$$= (s, a\bar{g}(s)+b\bar{h}(s), c\bar{g}(s)+d\bar{h}(s))$$

but

$$g(s) = a\bar{g}(s) + b\bar{h}(s)$$

$$h(s) = c\bar{g}(s) + d\bar{h}(s)$$

therefore

$$b(s) \approx c(s)$$

IV. GEOMETRIC APPLICATIONS

From the previous sections two theorems regarding null curves in L^3 can be proven.

Theorem 4.1: $a(s) : I \rightarrow L^3$ a regular null curve parameterized with respect to its time component is a straight line iff

$$k_1(s) = 0 \quad \text{for every} \quad s \in I$$

(a) $a(s)$ is a straight line implies $k_1(s) = 0$

$$a(s) = (s, b_1 s + c_1, b_2 s + c_2)$$

$$g(s) = b_1 s + c_1$$

$$h(s) = b_2 s + c_2$$

Recalling

$$k_1(s) = g''(s)h'(s) - g'(s)h''(s)$$

clearly $k_1(s) = 0$ since $g''(s) = h''(s) = 0$.

(b) $k_1(s) = 0$ implies $a(s)$ is a straight line

$$g''(s)h'(s) - g'(s)h''(s) = 0$$

and

$$g''(s)h'(s) + h''(s)h'(s) = 0$$

Thus

$$\begin{pmatrix} h'(s) & -g'(s) \\ g'(s) & h'(s) \end{pmatrix} \begin{pmatrix} g''(s) \\ h''(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Consider

$$\det \begin{pmatrix} h'(s) & -g'(s) \\ g'(s) & h'(s) \end{pmatrix} = g'(s)^2 + h'(s)^2 = 1$$

because of this $g''(s) = 0 \quad h''(s) = 0$.

<u>Theorem 4.2:</u> Let $a(s) : I \to L^3$ be a regular null curve parametrized with respect to its time component. Consider the cylinder $X(r, s) = (r, g(s), h(s))$ where $g(s)$ and $h(s)$ are the spacial components of $a(s)$. Then $a(s)$ is a null-geodesic of $X(r, s)$.

<u>Proof:</u> First set $r = s$ then $X(s, s) = (s, g(s), h(s))$ so $a(s)$ lies in $X(r, s)$. We now have to show that $a(s)$'s acceleration vector is orthogonal to $X(r, s)$ at every pt. (9). At each point of $X(r, s)$ the tangent space is spanned by $\frac{\partial X(r,s)}{\partial r}$ and $\frac{\partial X(r,s)}{\partial s}$. This is clear because $\frac{\partial X(r,s)}{\partial r} = (1, 00)$ and $\frac{\partial X(r,s)}{\partial s} = (0, g'(s), h'(s))$. A typical vector in $T_p \; X(r, s)$ say v can be written

$$v = A(1, 00) + B(0, g'(s), h'(s))$$

or

$$v = (A, Bg'(s), Bh'(s))$$

Next we have to show that

$$<a''(s), v> = 0$$

$$a''(s) = (0, g''(s), h''(s))$$

Thus

$$<a''(s), v> = <(0, g''(s)h''(s)), (A, Bg'(s), bh'(s))>$$

$$= B(g''(s)g'(s)+h''(s)h'(s))$$

$$= 0$$

V. SUMMARY

In Chapter II, it was shown that regular null curves in L^3 have a preferred parametrization. Using this fact, and a proper choice of the frame field, the Frenet equations for regular null curves in L^3 take on a simple form.

In Chapter III, an existence and uniqueness theorem was proved. That is, a continuous function defined on some interval I, is enough to determine a regular null curve in L^3 up to a proper motion in the spacelike plane.

In Chapter IV, a class of cylinders was constructed and null curves parametrized with respect to their time components were shown to be geodesics of these cylinders.

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