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In previous papers by Awrejcewicz in 1986 and Narayanan and Jayaraman in 1991, it was claimed that the nonlinear oscillator with dry friction exhibited chaos for several forcing frequencies. The chaos determination was achieved using the characteristic exponent of Lyapunov which requires the right-hand side of the differential equation to be differentiable. With the addition of the dry friction term, the right-hand side of the equation of motion is not continuous and therefore not differentiable. Thus this approach cannot be used. The Filippov definition must be employed to handle the discontinuity in the spatial variable. The behavior of the nonlinear oscillator with dry friction is studied using a numerical solver which produces the Filippov solution. The results show that the system is not chaotic; rather it has a stable periodic limit cycle for at least one forcing frequency. Other forcing frequencies produce results that do not clearly indicate the presence of chaotic motion.

Determination of the Filippov Solutions of the Nonlinear Oscillator with Dry Friction

by

Heather L. Moreland

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Heather L. Moreland, Author

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To Mom and Dad:
their love & support made this possible

DETERMINATION OF THE FILIPPOV SOLUTIONS OF THE NONLINEAR OSCILLATOR WITH DRY FRICTION

1 INTRODUCTION

Linear mass-spring systems have many applications. The principles that govern the behavior of the mass are used in the shock absorbers of automobiles as well as other machinery. The study of this type of system has also benefitted the military industry. Large guns can drastically recoil from the force of the explosion. Engineers have learned how to minimize this recoil by studying mass-spring systems. With the inclusion of linear viscous friction and a forcing term, the system can be solved explicitly and used to predict the behavior of the mass. However, with the addition of dry friction (also known as a stick-slip condition), the system cannot always be solved explicitly. The behavior of such a system can be studied to determine what kind of motion it will exhibit.

Dry (or Coulomb) friction is responsible for many of the oscillations we experience in every day life. The squealing of a car's brakes or of a railway car's wheels are just two examples of this phenomenon. As the alternate name stick-slip hints, there are actually two different phenomena at work here. In order to differentiate between them, let us once again focus on the mass-spring system. In a realistic system, an inherent amount of friction must be overcome to get the mass in motion. This inertial friction is called the stick condition, the tendency of the mass to remain at rest. If the mass is already in motion, for example sliding or rolling, there is a resistance to this motion and this is the slip condition.

In his 1986 paper "Chaos in Simple Mechanical Systems with Friction", Professor J. Awrejcewicz states that the one-dimensional non-linear mass-spring system which incorporates dry and viscous friction, stiffness, and a harmonic excitation term exhibits chaotic motion for specific values of the parameters.[2] Awrejcewicz's determination of chaotic behavior is based on the use of Lyapunov exponents. Lyapunov originally developed this idea to study the sensitivity of a system on its initial conditions. Professors S. Narayanan and K. Jayaraman continued this analysis in their 1991 paper "Chaotic Vibration in a Non-Linear Oscillator with Coulomb Damping".[6] Here, a number of excitation frequencies are studied and the corresponding Lyapunov exponents are calculated, concluding once again that some of these systems behave chaotically. The analysis of such systems using the characteristic exponent of Lyapunov is troublesome since this theory requires the differentiability of the right-hand side of the differential equation. In these mass-spring systems, the right-hand side of the equation is not continuous and therefore not differentiable.

In the first section of this paper, we present a review of well-known mass spring systems and their behavior. The theory of differential equations with discontinuous right-hand side, i.e. Filippov theory, is developed in the second section. The search for an appropriate numerical solver is presented in section three using the software Matlab. Various combinations of the elements of the mass-spring system are then considered. Finally, the harmonically excited non-linear oscillator with dry and viscous friction is considered using the appropriate numerical solver. We will observe that these results vary from those of Awrejcewicz, Narayanan, and Jayaraman in that the system does not exhibit chaotic behavior. In fact, for certain combinations of parameters, the system approaches a periodic orbit.

2 BACKGROUND INFORMATION

2.1 SIMPLE LINEAR OSCILLATORS

One of the first real problems students solve in a differential equations course is the mass on a linear spring. Here we have a mass m attached to an elastic spring.

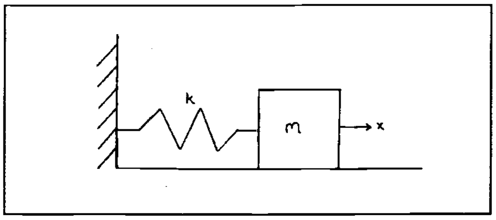

Figure 2.1: Mass-spring system.

If the mass is displaced some distance x from its equilibrium position, the spring will exert a restoring force of magnitude $k\,|x|$. The constant $k > 0$ is a measure of the stiffness of the spring and is called the spring constant. Thus, the restoring force is denoted by $F = -kx$ where the minus sign indicates the force is in the direction opposite the displacement. This relationship is known as Hooke's Law. Notice that this formulation does not include any types of friction or external forcing. By recalling that the force is equal to the product of the mass and acceleration, Newton's Second Law of Motion can be expressed as

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{2.1}$$

This is a second order, linear differential equation with constant coefficients and has a solution of the form

$$x(t) = a\cos\omega_o t + b\sin\omega_o t \tag{2.2}$$

where $\omega_o^2 = k/m$ is called the natural angular frequency of the system. This is a periodic solution which means that our mass is oscillating and will continue in this fashion since there is neither damping nor external forcing present. Of course, this is an unrealistic phenomenon since friction is always present in some form.

If the friction between the mass and the surface is included, a damping force arises in (2.1). The linear damping force opposes the direction of motion and is directly proportional to the velocity of the mass, dx/dt. If the velocity is positive, i.e. the mass is in motion, the damping force will be $-c(dx/dt)$, where $c > 0$. The forces on the mass are now $F = -kx - c(dx/dt)$ and the equation of motion becomes

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0. \tag{2.3}$$

The roots of the characteristic equation are then:

$$s_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} \quad \text{and} \quad s_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}. \tag{2.4}$$

Three cases must be considered. If $c^2 - 4km > 0$, then both s_1 and s_2 in (2.4) are negative and the solution to (2.3) is of the form

$$x(t) = ae^{s_1 t} + be^{s_2 t}. \tag{2.5}$$

This is the overdamped case and the mass will creep back to its equilibrium position in infinite time. If $c^2 - 4km = 0$, then the solution to (2.3) is

$$x(t) = (a + bt)e^{-ct/2m}. \tag{2.6}$$

We say the system is critically damped and the mass will return to its equilibrium position. The case of $c^2 - 4km < 0$ is the most interesting of the three cases. Here, the solution to (2.3) is

$$x(t) = e^{-ct/2m}[a \cos \mu t + b \sin \mu t] \tag{2.7}$$

where $\mu = \dfrac{\sqrt{4km - c^2}}{2m}$. The system is said to be underdamped and if the solution is rewritten in the form

$$x(t) = A\ e^{-ct/2m} \cos(\mu t - \delta), \tag{2.8}$$

where $A = \sqrt{a^2 + b^2}$ and $\delta = \arctan \dfrac{b}{a}$, we see that the displacement of the mass will oscillate between the curves $x(t) = \pm A\ e^{-ct/2m}$. Thus it represents a cosine curve with a decreasing amplitude. Physically, this means that any initial disturbance to the system will be dissipated by the damped system.

Now we will look at what happens if we remove the damping from our system and replace it with a forcing term. For simplicity, we require that this forcing term be periodic. Usually, forcing terms of this type are taken to be $F(t) = F_o \cos \omega t$. Equation (2.1) then becomes

$$\frac{d^2x}{dt^2} + \omega_o^2 x = \frac{F_o}{m} \cos \omega t. \tag{2.9}$$

where $\omega_o^2 = \dfrac{k}{m}$. The case where $\omega \neq \omega_o$ results in oscillatory behavior. The solution to (2.9) in this case is the sum of two periodic functions with different periods;

$$x(t) = c_1 \cos \omega_o t + c_2 \sin \omega_o t + \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos \omega t. \tag{2.10}$$

In contrast, when the frequency of the applied force equals the natural frequency of the system, i.e. $\omega = \omega_o$, a phenomenon known as resonance occurs. We will need a particular solution $\psi(t)$ of (2.10) and combining this with the solution to the homogeneous problem, we have that the solution to (2.9) is

$$x\,(t) = c_1 \cos \omega_o t + c_2 \sin \omega_o t + \frac{F_o t}{2m\omega_o} \sin \omega_o t. \tag{2.11}$$

The first two terms are periodic functions of time. The last term, however, is an oscillation with an amplitude that increases with time. Thus the resonance condition causes unbounded oscillations in the system. Resonance phenomena like this have been responsible for many mechanical catastrophes, the most well known being the collapse of the Tacoma Narrows Bridge in 1940. However, resonance can also have beneficial applications. In the linear realm, the fact that resonance amplifies the incoming signal is utilized in radios. The act of tuning a radio is really bringing the resonant frequency of the electrical circuit into agreement with an incoming signal.

Now we look at the case where both the damping term and the forcing term are included in our system. This yields the following differential equation governing the motion of the mass

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_o \cos \omega t \tag{2.12}$$

where $c > 0$ and $k > 0$. The solution, $\varphi(t)$, to the homogeneous equation is the same as that for the damped system. Recall here we have three cases to consider, but in each of the cases, the solution decays to zero as time approaches infinity. The particular solution to this equation is found by the same methods as before, the calculations are just more complicated. We see that (2.12) has a particular solution, $\psi(t)$, of the form

$$\begin{aligned}\psi(t) &= \frac{F_o}{(k-m\omega^2)^2+c^2\omega^2}\left[\left(k-m\omega^2\right)\cos\omega t + c\omega\sin\omega t\right] \\ &= \frac{F_o}{(k-m\omega^2)^2+c^2\omega^2}\left[\left(k-m\omega^2\right)^2+c^2\omega^2\right]^{1/2}\cos(\omega t-\delta), \qquad (2.13) \\ &= \frac{F_o\cos(\omega t-\delta)}{\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}}\end{aligned}$$

where $\tan\delta = \dfrac{c\omega}{(k-m\omega^2)}$. Thus every solution to (2.12) must be of the form

$$x\,(t) = \varphi(t) + \psi(t) = \varphi(t) + \frac{F_o\cos(\omega t-\delta)}{\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}}. \qquad (2.14)$$

We noted previously that every solution of the form $x(t) = \varphi(t)$ approaches zero as t goes to infinity. Therefore, for large t, $x\,(t) = \psi(t)$ will describe the position of the mass quite accurately. We often call $\psi(t)$ the steady state solution of (2.14) and $\varphi(t)$ is called the transient solution of (2.14). Since the steady state solution is now periodic in time, the unbounded oscillations that were seen in the previous case will not occur.

We have restricted ourselves to studying systems in which the equation of motion was a linear differential equation. As a result, we were still quite limited in the types of problems we could consider. If we want to broaden our repertoire, we must begin to consider oscillators whose behavior is governed by nonlinear differential equations. To this end, we will consider the oscillators of Van der Pol and Duffing. Before we study these two examples, some background on the theory of nonlinear differential equations must be introduced.

2.2 PLANAR AUTONOMOUS SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

Let $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and consider autonomous systems of ordinary differential equations of the form

$$\dot{x} = f(x) \tag{2.15}$$

where $x = (x_1, \ldots, x_n)$, $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$, and $f = (f_1, \ldots f_n)$. If we could convert second order differential equations into a system of first order equations, then the equations from the previous sections could be expressed in the form of (2.15). If f is a differentiable function, the derivative of f, denoted by Df, is given by the Jacobian matrix

$$Df = \left[\frac{\partial f_i}{\partial x_j}\right]. \tag{2.16}$$

We will now define the notion of the flow of the nonlinear system of differential equations (2.15) with the initial condition $x(0) = x_o$.

Definition 1 *Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$, that is, the first derivative of f is continuous on the set E. For $x_o \in E$, let $\phi(t, x_o)$ be the solution of the initial value problem defined on its maximal interval of existence, say $I(x_o)$. Then for $t \in I(x_o)$, the set of mappings $\phi_t = \phi(t, x_o)$ is called the flow of the differential equation (2.15). ϕ_t is also referred to as the flow of the vector field $f(x)$.*

Another concept that must be distinguished from that of the flow of a differential equation is that of a trajectory of the system. The formal definition is as follows.

Definition 2 *Let the initial point x_o be fixed and let $I = I(x_o)$ as above. Then the mapping $\phi(\cdot, x_o) : I \rightarrow E$ defines a solution curve or trajectory of the system (2.15) through the point $x_o \in E$.*

An example can help to clarify the subtle differences between flows and trajectories. Suppose that the differential equation (2.15) describes the motion of a fluid over time. A trajectory of (2.15) illustrates the motion of an individual particle in the fluid over time, while the flow of the differential equation (2.15) characterizes the motion of the entire fluid during that time interval.

The behavior of nonlinear systems of differential equations can become quite complex. Often these systems cannot be solved explicitly as in the previous section. Instead, the qualitative behavior of the system must be investigated. We will illustrate some of the techniques by considering the case of the planar pendulum. We let g be the gravitational constant and l be the length of the pendulum.

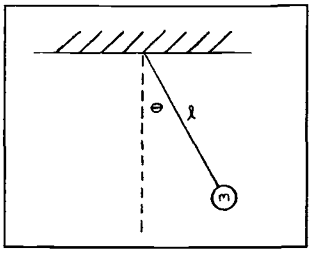

Figure 2.2: Planar pendulum of length l.

By ignoring friction, we see that the displacement angle θ from the vertical rest position of the pendulum satisfies the second order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{2.17}$$

This equation can be converted into a system of first order differential equations by introducing the change of variables

$$\begin{aligned} x_1(t) &= \theta(t) \\ x_2(t) &= \frac{d\theta}{dt}. \end{aligned} \tag{2.18}$$

Thus, x_1 is the position of the mass at time t and x_2 is the velocity of the mass at time t. Then (2.17) becomes the system of equations

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1.\end{aligned} \tag{2.19}$$

For our purposes here, we will only consider small oscillations of the pendulum so that we can assume $\sin\theta \approx \theta$. We will also assume that $g = 1$ and $l = 1$ to allow for further simplification. The system of equations (2.19) now becomes

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1\end{aligned} \tag{2.20}$$

which is known as the linear harmonic oscillator. We can plot a trajectory of this system in the three dimensional (t, x_1, x_2) space.

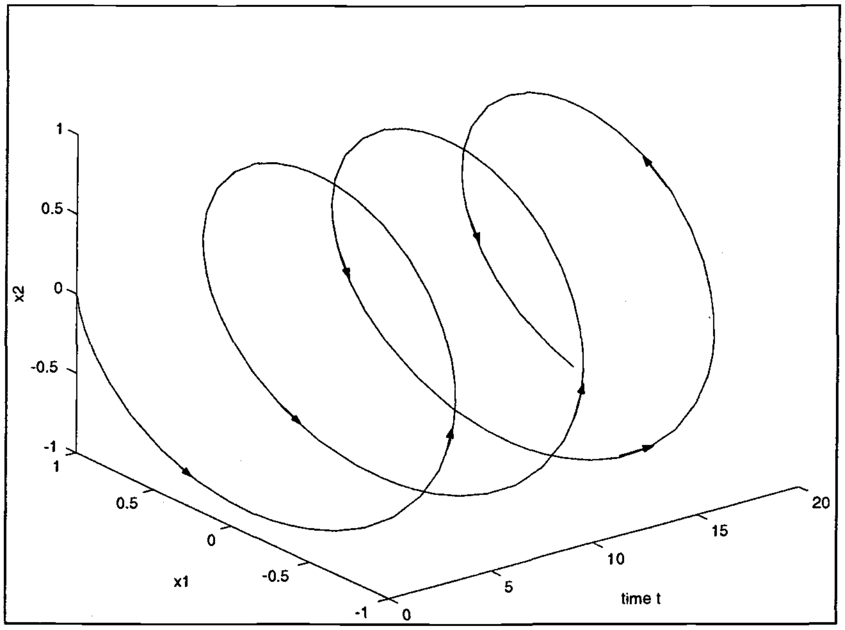

Figure 2.3: Trajectory of the linear harmonic oscillator

We see that the system is exhibiting periodic motion as time progresses. These types of three dimensional pictures can become quite complex, so a two dimensional analog was developed. We say that the projection of trajectories onto the (x_1, x_2) plane are called orbits. The projection of the above helical trajectory onto the (x_1, x_2) plane will produce a circular orbit.

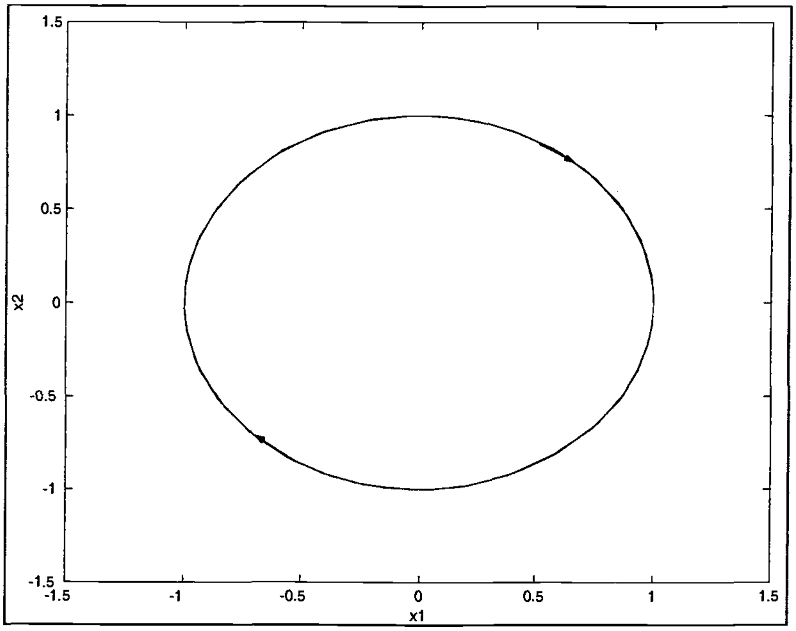

Figure 2.4: Periodic orbit of the linear harmonic oscillator

The flow of a differential equation can then be drawn as a collection of all its orbits, possibly including direction arrows. This resulting two dimensional picture is called a phase portrait and the (x_1, x_2) plane is called the phase plane. The phase portrait for the linear harmonic oscillator is shown below.

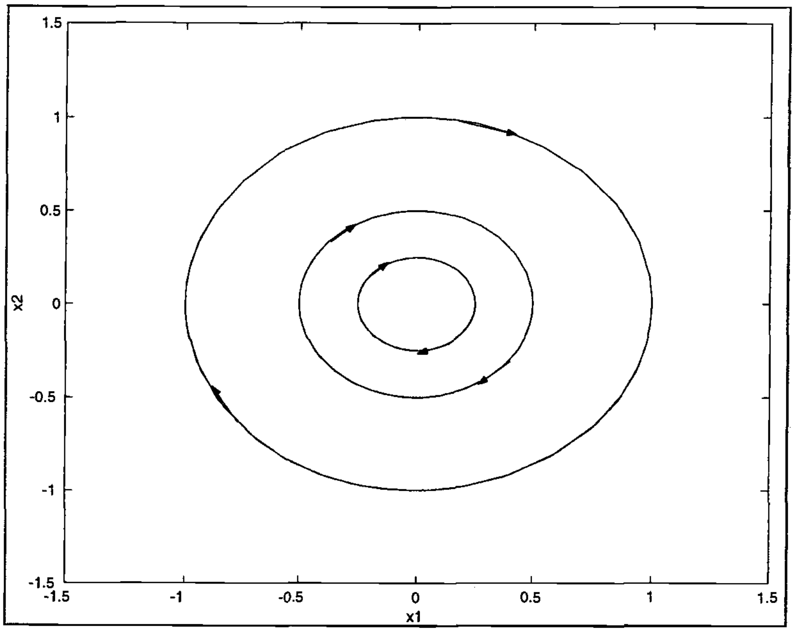

Figure 2.5: Phase portrait for the linear harmonic oscillator

Certain types of orbits play a special role in the theory of systems of differential equations. The simplest type of orbit is an equilibrium point.

Definition 3 *A point* $\bar{x} \in \mathbb{R}^2$ *is called an equilibrium (or critical) point of* $\dot{x} = f(x)$ *if* $f(\bar{x}) = 0$. *In other words, if* $\bar{x} = (\bar{x}_1, \bar{x}_2)$, *then* $f_1(\bar{x}_1, \bar{x}_2) = 0$ *and* $f_2(\bar{x}_1, \bar{x}_2) = 0$.

Another type of special orbit is a periodic orbit.

Definition 4 *A solution* $\phi(t, x_o)$ *of* $\dot{x} = f(x)$ *is called a periodic solution of period* p, *with* $p > 0$, *if* $\phi(t + p, x_o) = \phi(t, x_o)$ *for all* $t \in \mathbb{R}$. *The orbit corresponding to a periodic solution* $\phi(t, x_o)$ *with period* p *is said to be a periodic (or closed) orbit of period* p. *That is, a cycle or periodic orbit of* $\dot{x} = f(x)$ *is any closed solution curve of* $\dot{x} = f(x)$ *which is not an equilibrium point.*

Returning to the linear harmonic oscillator, we immediately see that the only equilibrium point is the origin and that the other orbits are periodic. In fact, we have a family of concentric periodic orbits that encircle an equilibrium point. An equilibrium point with this behavior is called a center. Here, we have only presented a brief introduction to the qualitative theory of systems of differential equations. Further information can be found in [4] and [7].

2.3 VAN DER POL'S OSCILLATOR

In the early twentieth century, Van der Pol was studying electric circuits that for our purposes can be considered to be a series RLC circuit. The behavior of this type of electrical circuit is dependent on the currents through the resistor R, the capacitor C, and the inductor L.

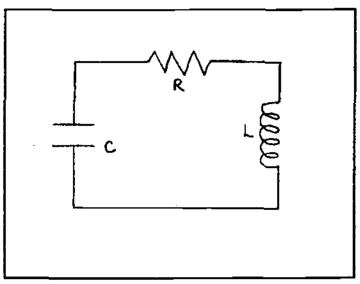

Figure 2.6: RLC circuit

Resistors are devices that control the current level in a particular part of the circuit. They act as a type of bottleneck that regulates how much current is flowing through that portion of the circuit. A capacitor consists of two conductors which have a potential difference between them. If the plates of a charged capacitor are connected together by a wire, charge will transfer from one plate to the other. Just as resistance is a measure of the opposition to current, inductance is a measure of the opposition to the change in current. The inductance of the device is dependent on the geometric features of its interior circuit. A circuit element that has a large inductance is called an inductor. In the case of the series RLC circuit, it is assumed that the wire connecting these three devices has no resistance. We will denote the currents through each of these three devices as i_R, i_C, i_L and the voltage drops across the devices as v_R, v_C, v_L. From Ohm's Law, we know that the voltage across a resistor is a function of the current flowing through it. Also, the voltage and the current across an inductor must satisfy Faraday's Law of Inductance. Thus we have the following relationships

$$\begin{aligned} v_R &= f(i_R) \\ L\frac{di_L}{dt} &= v_L \\ C\frac{dv_C}{dt} &= i_C \end{aligned} \tag{2.21}$$

where the function f depends upon the material from which the resistor is constructed, and L and C depend on the inductor and capacitor respectively.

This RLC circuit must also obey Kirchhoff's conservation laws which state that the sum of the voltage drops across all of the elements around any closed circuit loop must be zero and that the sum of the currents entering any junction in the circuit must equal the sum of the currents leaving that junction. These conservation laws translate into the following relationships

$$\begin{aligned} i_R - i_L + i_C &= 0 \\ v_R + v_L - v_C &= 0. \end{aligned} \tag{2.22}$$

Using these relationships, we can simplify (2.21) to the following system of differential equations

$$\begin{aligned} L\frac{di_L}{dt} &= v_C - f(i_L) \\ C\frac{dv_C}{dt} &= -i_L. \end{aligned} \tag{2.23}$$

By scaling the time variable as $t \longmapsto (CL)^{1/2}t$ and by letting $i_L = x_1$ and $v_C = \left(\frac{L}{C}\right)^{1/2} x_2$, we obtain the following relationships for $\frac{di_L}{dt}$ and $\frac{dv_C}{dt}$

$$\begin{aligned} \frac{di_L}{dt} &= \dot{x}_1(CL)^{-1/2} \\ \frac{dv_C}{dt} &= \left(\frac{L}{C}\right)^{1/2} \dot{x}_2(CL)^{-1/2}. \end{aligned} \tag{2.24}$$

Substituting these into (2.23), we obtain the planar system of differential equations, which is known as Lienard's equation

$$\begin{aligned} \dot{x}_1 &= x_2 - \left(\frac{C}{L}\right)^{1/2} f(x_1) \\ \dot{x}_2 &= -x_1 \end{aligned} \quad . \tag{2.25}$$

In certain types of resistors $f(x_1)$ behaves like a cubic function, i.e. $f(x_1) = \frac{1}{3}x_1^3 - x_1$. Making this substitution into (2.25), we obtain a system of equations that is equivalent to

$$\ddot{x} + \left(\frac{C}{L}\right)^{1/2} \left(1 - x^2\right) \dot{x} + x = 0, \tag{2.26}$$

This is the differential equation of Van der Pol and can be written more generally as

$$\ddot{x} - \mu \left(x_o^2 - x^2\right) \dot{x} + \omega_o^2 x = 0 \tag{2.27}$$

where μ is a positive parameter. If the amplitude of the oscillation, $|x|$, is greater than the critical value $|x_o|$, then the coefficient of $\dot{x}$ is positive and the system would be damped. However, if $|x|$ is less than $|x_o|$ the oscillations of the system would increase. If we require some kind of continuity of the component functions, it stands to reason that there must be some value of $|x|$ for which the amplitude of the motion neither increases nor decreases. This is a result of Lienard's theorem which also contains the necessary continuity requirements.[10] Before we state the theorem we must introduce some terminology.

Consider the system of differential equations

$$\dot{x} = f(x) \tag{2.28}$$

with $f \in C^1(E)$ where E is an open subset of $\mathbb{R}^2$. We introduce the following concepts relating to trajectories.

Definition 5 *Let x_o be the initial condition, i.e. the point from which the flow is originating. A point $p \in E$ is an ω-limit point of the trajectory $\phi(\cdot, x_o)$ of the system (2.28) if there is a sequence $t_n \to \infty$ such that*

$$\lim_{n\to\infty} \phi(t_n, x_o) = p.$$

A point $q \in E$ is an α-limit point of the trajectory $\phi(\cdot, x_o)$ of the system (2.28) if there is a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x_o) = q.$$

Thus, a point p is an ω-limit point of a trajectory if in infinite time the trajectory approaches p. Similarly, a point q is an α-limit point of a trajectory if by running time backwards towards negative infinity, the trajectory approaches q. The collection of these points is called the limit set of a trajectory.

Definition 6 *The set of all ω-limit points of a trajectory Γ, where $\Gamma = \{x \in E \mid x = \phi(t, x_o),\ t \in \mathbb{R}\}$, is called the ω-limit set of Γ and is denoted by $\omega(\Gamma)$. The set of all α-limit points of a trajectory Γ is called the α-limit set of Γ and is denoted by $\alpha(\Gamma)$.*

With the notions of limit sets, we can define the idea of stable and unstable limit cycles which is what was occurring in the case of Van der Pol's oscillator.

Definition 7 *A limit cycle or periodic orbit Γ of a system of differential equations is an orbit of (2.28) which is the α or ω-limit set of some trajectory of (2.28) other than Γ. If a cycle Γ is the ω-limit set of every trajectory in some neighborhood of Γ, then Γ is called an ω-limit cycle or stable limit cycle. If Γ is the α-limit set of every trajectory in some neighborhood of Γ, then Γ is called an α-limit cycle or unstable limit cycle.*

If the other orbits approach a periodic orbit Γ, then Γ is called a stable limit cycle since the trajectories near this periodic orbit will approach it and thus, the system exhibits regular motion as time progresses. If the orbits spiral away from the periodic orbit, then Γ is unstable since the system is not approaching any kind of regular, periodic motion. We are now ready to state Lienard's Theorem which guarantees the existence of a stable limit cycle in Van der Pol's oscillator.

Theorem 8 *Suppose that in the second-order differential equation* $\ddot{x} + f(x)\dot{x} + g(x) = 0$ *the functions* $f(x)$ *and* $g(x)$ *satisfy the following conditions:*

(i) $f(x)$ *and* $g(x)$ *are* $C^{\infty}(\mathbb{R})$, *that is continuously differentiable on* $\mathbb{R}$.

(ii) $g(x)$ *is an odd function with* $g(x) > 0$ *for* $x > 0$ *and* $f(x)$ *is an even function*

(iii) *the odd function* $F(x) = \int_0^x f(u)du$ *has exactly one positive zero at* $x = a$, *is negative for* $0 < x < a$, *is positive and non-decreasing for* $x > a$, *and* $F(x) \to \infty$ *as* $x \to \infty$.

Then the system of differential equations $\dot{x} = y$ *and* $\dot{y} = -g(x) - f(x)y$ *has a unique, stable limit cycle surrounding the origin in the phase plane.*

A proof of this theorem is not presented here, but can be found in Perko.[7] Returning to our example of Lienard's equation, the curve generated by the value of $|x|$, for which the amplitude of the motion neither increases nor decreases, is called a limit cycle. In the figure below, we have produced the phase portrait corresponding to Lienard's equation (2.25). For simplicity, we have taken $L = C = 1$.

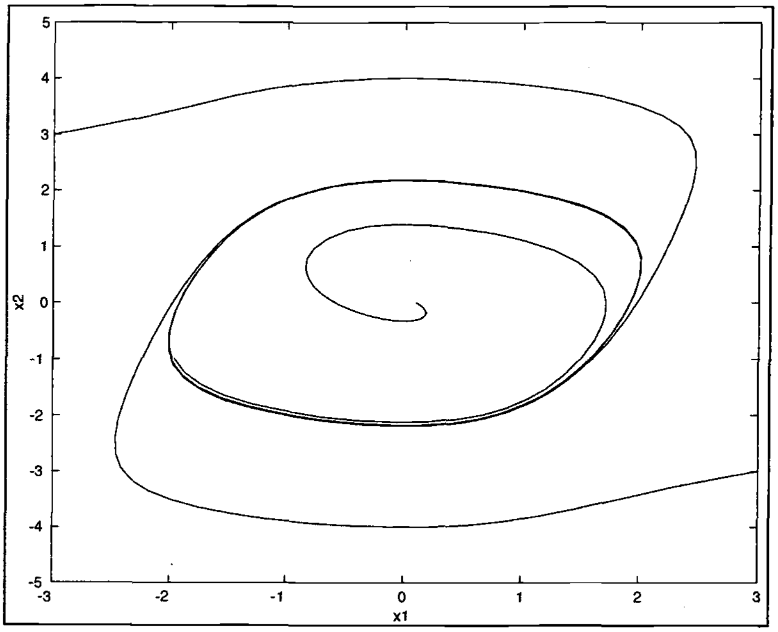

Figure 2.7: Phase portrait for Van der Pol's oscillator (2.25)

Those curves that originate outside the limit cycle will spiral inwards while those emanating inside the limit cycle will spiral outward towards the limit cycle. We say that this limit cycle or periodic orbit is stable since it is the ω-limit set of nearby trajectories. Thus Van der Pol's oscillator will tend to settle into a self-sustained oscillation, that is a periodic orbit.

2.4 DUFFING'S OSCILLATOR

Suppose that the function $v : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. We say that the second-order differential equation $\ddot{x} + v(x) = 0$ or the equivalent system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -v(x_1) \end{aligned} \tag{2.29}$$

is a second-order conservative system. This is a class of conservative system. The terminology results from the fact that the total energy of the system is a conserved quantity. Let the potential energy be expressed as

$$V(x_1) = \int_0^{x_1} v(s)ds \tag{2.30}$$

and the kinetic energy be $\frac{1}{2}x_2^2$. The total energy of the system is given by

$$E(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1) \tag{2.31}$$

and since

$$\frac{d}{dt}E(x_1(t), x_2(t)) = x_2\dot{x}_2 + V'(x_1)\dot{x}_1 = -x_2v(x_1) + x_2v(x_1) = 0, \tag{2.32}$$

the energy is a constant, i.e., it is conserved.[4]

The Duffing equation describes the undamped motion of a mass and nonlinear spring system. The nonlinear spring has a restoring force of the form $F(x) = kx - \varepsilon x^3$. Applying Newton's Second Law of Motion, we obtain

$$m\ddot{x} = kx - \varepsilon x^3. \tag{2.33}$$

For our purposes, we suppose that it is a unit mass and that $k = 1$ and $\varepsilon = 1$. This yields the system of first order differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3. \end{aligned} \tag{2.34}$$

Here, we see this is a conservative system with $v(x_1) = x_1 - x_1^3$, so from the definition above,

$$V(x_1) = \int_0^{x_1} v(s)ds = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4. \tag{2.35}$$

Thus, the total energy of the system is given by

$$E(x_1, x_2) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \tag{2.36}$$

and direct calculation yields that

$$\frac{d}{dt}E(x_1(t), x_2(t)) = 0 \tag{2.37}$$

implying that the total energy of the system is conserved.

Potential functions can be useful in determining the phase portrait of the corresponding conservative system.

Definition 9 *A point $\bar{x}$ is called a critical point of the C^1 function $V : \mathbb{R} \to \mathbb{R}$ if $V'(\bar{x}) = 0$. The value of V at a critical point is called a critical value. A critical point $\bar{x}$ is called nondegenerate if $V''(\bar{x}) \neq 0$.*

Recall that to find the equilibrium point of a system of differential equations, we set $\dot{x}_1 = \dot{x}_2 = 0$. In the case of conservative systems, this will result in $x_2 = 0$. Thus, the equilibrium points of a conservative system must lie on the x_1-axis and that a point $(\bar{x}_1, 0)$ is an equilibrium point if and only if $\bar{x}_1$ is a critical point of $V(x_1)$. We can also establish the following theorem. A proof of this theorem is given in Hale and Koçak.[4]

Theorem 10 *Suppose that $\bar{x}_1$ is a critical point of a potential function V so that $(\bar{x}_1, 0)$ is an equilibrium point of the conservative system (2.29). Then*

(i) $(\bar{x}_1, 0)$ *is a saddle point if* $V''(\bar{x}_1) < 0$;

(ii) $(\bar{x}_1, 0)$ *is a center is* $V''(\bar{x}_1) > 0$.

Now considering Duffing's equation's conservative system (2.34) and its corresponding potential function, we see that $V(x_1)$ has critical points at $x_1 = 0, \pm 1$. All three of these critical points are nondegenerate. We can conclude that the critical point $(0,0)$ is a maximum and thus is a saddle point. The critical points $(-1,0)$ and $(1,0)$ are minima and thus are centers. The phase portrait is shown below.

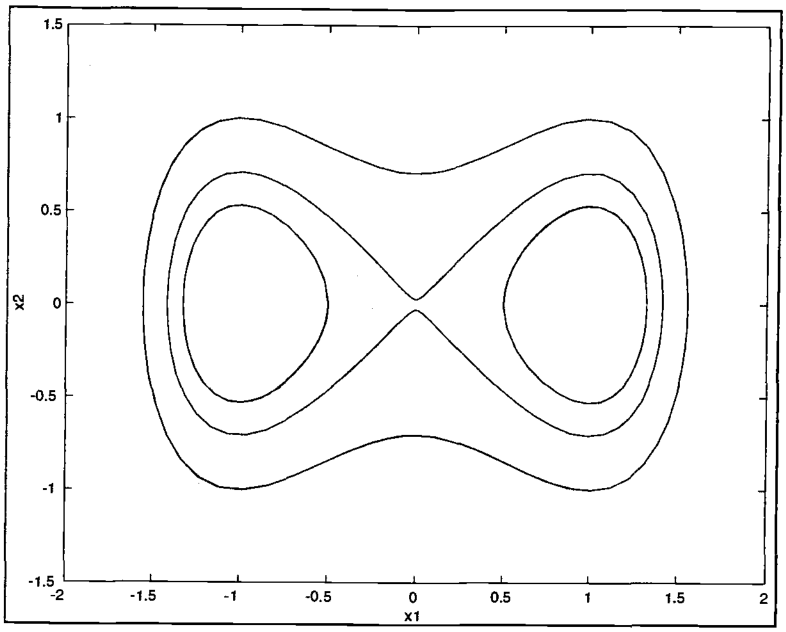

Figure 2.8: Phase portrait for Duffing's equation.

Van der Pol's and Duffing's oscillators are the classical examples of nonlinear oscillatory systems. However, in both cases, the right-hand sides of the differential equations were continuously differentiable. In fact, they were polynomials. In the next section we consider the case where the differential equation may be discontinuous in either the time or the spatial variables.

3 FILIPPOV THEORY AND DRY FRICTION

All ordinary differential equations can be reduced to a system of equations of the form

$$\dot{x} = f(t, x), \tag{3.1}$$

where $x = (x_1, \ldots, x_n)$ and $f = (f_1, \ldots, f_n)$. If the function $f(t, x)$ is continuous in both variables, then the classical method of solution to this problem would be to consider the equivalent integral equation

$$x(t) = x(t_o) + \int_{t_o}^{t} f(s, x(s))ds. \tag{3.2}$$

However, if the right-hand side of (3.1) is discontinuous in t, but continuous in x, then those functions satisfying (3.2) can still be called solutions to (3.1). The points of discontinuity of $f(t, x)$ are taken into account by defining the solution to be such that $\frac{dx}{dt} = f(t, x(t))$ almost everywhere in the interval. Equations of this type arise in control and impulse theory and are known as Carathéodory differential equations. Now we state the precise requirements on $f(t, x)$.[3]

Definition 1 (Carathéodory Differential Equation) *Let x be a scalar or a vector. In the domain D of the (t, x) space, let $f(t, x)$ satisfy the following Carathéodory conditions:*

(i) *the function $f(t, x)$ is defined and continuous in x for almost all t;*

(ii) *the function* $f(t, x)$ *is measurable in* t *for each* x*;*

(iii) $|f(t, x)| \leq m(t)$, *the function* $m(t)$ *being an integrable function.*

Then the equation $\dot{x} = f(t, x)$ *is called a Carathéodory differential equation.*

There are some additional requirements that are placed on the solution $x(t)$. Before we state the formal definition, we recall the concept of absolutely continuous functions.[8]

Definition 2 (Absolute Continuity) *A function* $f(x)$ *is absolutely continuous in* $[a, b]$ *if given an* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if for every finite set of intervals* (x_i, x_i^*) *where* $\sum_{i=1}^{n} |x_i^* - x_i| < \delta$, *then* $\sum_{i=1}^{n} |f(x_i^*) - f(x_i)| < \varepsilon$.

Absolute continuity is needed because we must require f to have a derivative almost everywhere in the interval. If f is absolutely continuous on an interval, then f has a derivative almost everywhere on that interval.[8] We can now state the formal definition for a solution to a Carathéodory differential equation.[3]

Definition 3 *The function* $x(t)$ *defined on an open or closed interval* I *is called a solution of the Carathéodory differential equation* $\dot{x} = f(t, x)$ *if it is absolutely continuous on each closed interval* $[a, b] \subset I$ *and if it satisfies the differential equation almost everywhere on* I.

Now consider an example of such a differential equation whose right-hand side is discontinuous in t. Recall that the signum function is defined as

$$\text{signum}(t) = \text{sgn}(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases} \tag{3.3}$$

and consider the differential equation

$$\dot{x} = \text{sgn}(t). \tag{3.4}$$

In this example, $f(x,t) = \text{sgn}(t)$ which is clearly continuous in x and for almost all t. The function is measurable for each x and is bounded in absolute value by $m(t) = 1$ which is integrable for $t \in \mathbb{R}$. Thus the three conditions for a Carathéodory differential equation are satisfied. If $t < 0$, then (3.4) becomes $\dot{x} = -1$ and if $t > 0$, the differential equation becomes $\dot{x} = 1$. These have solutions of $x(t) = -t + c_0$ for $t < 0$ and $x(t) = t + c_1$ for $t > 0$. Using the requirements of continuity at $t = 0$, we see that

$$x(0) = \lim_{t \to 0^-} (-t + c_0) = \lim_{t \to 0^+} (t + c_1). \tag{3.5}$$

This in turn implies that $x(0) = c_0 = c_1$. Thus the solution to (3.4) is given by

$$x(t) = |t| + c. \tag{3.6}$$

The solution is absolutely continuous. However, the derivative only exists almost everywhere since it does not exist at $t = 0$.

With the definition of a solution to a Carathéodory differential equation, we could investigate the existence and uniqueness properties of such a solution. These ideas are detailed in many books, including Filippov's and we will not state them here. We see that discontinuities in t can be handle without too much difficulty. Unfortunately, the same cannot be said for discontinuities in the spatial variable.

Now we consider the differential equation (3.1), where the right-hand side is discontinuous in the spatial variable, x. First, we will consider an example. Let $\dot{x} = 1 - 2\,\text{sgn}(x)$. Notice for $x < 0$, the differential equation becomes $\dot{x} = 3$ which has a solution of $x(t) = 3t + c_1$. For $x > 0$, we have $\dot{x} = -1$ which has solution $x(t) = -t + c_2$. We observe that as t increases, both solutions reach the line $x = 0$. The direction field of this differential equation indicates that the solutions are flowing towards the line $x = 0$. Thus, we can conclude that once the solutions reach that point, they are prevented from deviating from this line and thus the solution $x(t) = 0$ is obtained. This function does not satisfy the differential equation since $\dot{x}(t) = 0$ but, $0 \neq 1 - 2\,\text{sgn}(0) = 1$. We can see from this example that the concept of a solution to a differential equation must be generalized in the case where $f(t, x)$ is discontinuous in x.

Filippov developed a theory for these types of differential equations. In dealing with this type of equation it is necessary to introduce several concepts. The first of these is that of a set-valued map.[1]

Definition 4 *Let X and Y be two sets. A set-valued map F from X to Y is a map that associates with any $x \in X$ a subset $F(x)$ of Y. The subsets $F(x)$ are called the images or values of F.*

For example, let $X = [-1, 1]$ and Y be the unit circle and consider the map from X into Y given by the function $F(x) = \{\arccos x\}$. Then the point $x = 1/2$ would be mapped into the set $\{\pm 1/\sqrt{2}\}$. The next idea that we will need to introduce is that of a differential inclusion and the closed convex hull of a set.

Definition 5 *The closed convex hull of a set E, denoted by $\overline{conv}E$, is a set which contains E and is the intersection of all closed, convex sets containing E.[8]*

A set is said to be convex if whenever it contains x and y, it contains the line segment $\lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$. Now consider the system of differential equations (3.1). Let f be measurable and the set of discontinuities of f have Lebesgue measure zero. We can define a solution of (3.1) in terms of solutions to an associated differential inclusion. We define this differential inclusion as

$$x'(t) \in \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{conv} f(t, U(x(t), \delta) - N)$$

for almost all t and where $U(x, \delta) = \{y : |x - y| < \delta\}$ and μ denotes Lebesgue measure.[9] The following definition can now be stated.

Definition 6 *A solution of (3.1) is a solution of the associated differential inclusion $\dot{x} \in F(t, x)$, that is, an absolutely continuous vector-valued function $x(t)$ defined on an interval I for which $\dot{x}(t) \in F(t, x(t))$ almost everywhere on I where $F(t, x(t))$ is a set-valued function.[5]*

Given $f(t, x)$, we must establish a method of determining a set-valued function $F(t, x)$. Suppose that $f(t, x)$ is a piecewise continuous function in some domain D and $x \in \mathbb{R}^n$. Let N denote the set of points of discontinuity of $f(t, x)$ and require that N be a set of measure zero. We will define a set $F(t, x)$ in an n-dimensional space for each point $(t, x) \in D$. If the function $f(t, x)$ is continuous at the point (t_o, x_o), then we define the set $F(t_o, x_o) = \{f(t_o, x_o)\}$. If the point (t_o, x_o) is a point of discontinuity of $f(t, x)$, i.e. $(t_o, x_o) \in N$, then we must specify $F(t_o, x_o)$ in some other way. For each point $(t_o, x_o) \in N$, let $F(t_o, x_o)$ be the smallest convex closed set containing all the limit values of the vector-valued function $f(t, \hat{x})$ for $(t, \hat{x}) \notin N$, $\hat{x} \to x_o$, and t constant.

The Filippov solution to the system of differential equations

$$\dot{x} = f(t, x) \tag{3.7}$$

is the solution to the differential inclusion

$$\dot{x} \in F(t, x) \tag{3.8}$$

with $F(t, x)$ constructed on the set of discontinuities of f as above. Some examples are needed to illustrate how to implement these ideas. We noted that dry friction or the stick-slip condition is really two different phenomena. If the mass begins to move away from equilibrium, the stick mode resists this motion. However, if the mass is already in motion, the slip mode will resist this movement. In an oscillating motion, both of these conditions will be present. Thus, dry friction consists of two parts with a non-smooth transition between them. Therefore, the resulting motion of the mass will exhibit non-smooth behavior. Dry friction is modelled using the signum function which is discontinuous at $x = 0$. In general, the coefficient of static friction is greater than the coefficient of sliding or viscous friction. However, we will first look at an example where these coefficients are equal.

Consider the differential equation

$$\dot{x} = -\operatorname{sgn} x \tag{3.9}$$

with $x(0) = 1$. The right hand side of this equation is discontinuous at $x = 0$. The first step is to determine $F(t, x)$ using the foregoing definition. For $x > 0$, the function $f(t, x) = -\operatorname{sgn} x = -1$ and is thus continuous, so $F(t, x) = \{-1\}$ for these values of x. If $x < 0$, $f(t, x) = 1$ and $F(t, x) = \{1\}$ in this interval. This leaves the point of discontinuity $x = 0$. Here, we define $F(t, x)$ to be the smallest convex closed set that contains all of the limit values of $f(t, x)$. In this case, the limit values are $x = -1$ and $x = 1$. Thus the smallest convex closed set is $[-1, 1]$ and the associated differential inclusion to (3.9) is

$$x'(t) \in \begin{cases} \{-\operatorname{sgn} x\}, \text{ when } x \neq 0 \\ \{[-1, 1]\}, \text{ when } x = 0 \end{cases} . \tag{3.10}$$

For $0 \leq t < 1$, we know that $x > 0$ since the initial condition is $x\,(0) = 1$. Thus the solution to (3.10) is the same as that of the differential equation $x'(t) = -1$. This has the solution $x(t) = 1 - t$. Now consider what happens for $t > 1$. For these values of t, initially x is equal to zero. The question becomes what happens to x as time progresses. From the differential inclusion, we know that $x'(1)$ can take on any value between -1 and 1. Suppose that $x'(1) = \varepsilon$, where $0 < |\varepsilon| \leq 1$. Then for $t > 1$, the solution will move away from the point $x = 0$, since the slope of the solution is now non-zero. However, as soon as the solution moves away from the t-axis, the differential equation takes over since $x \neq 0$. If $\varepsilon > 0$, then these values of x will be positive and the differential equation will require that $x'(t) < 0$, which forces the solution curve back down towards the t-axis. If $\varepsilon < 0$, then these values of x will become negative and the differential equation will require that $x'(t) > 0$, which forces the solution curve back up towards the t-axis. So the solution of (3.9) is required to be $x\,(t) = 0$ for all values $t > 1$, and it is $x\,(t) = 1 - t$ for $0 \leq t \leq 1$. We have presented an intuitive argument to reach the conclusion that the solution should be $x(t) = 0$ for $t > 1$. Now a more rigorous proof of this result is constructed.

Suppose that once the solution reached $t = 1$, it jumped up to some point, e.g. $(1, x_o)$, where $x_o > 0$. Since $x_o > 0$, the differential equation applies and will force the solution to have a negative slope and head towards the t-axis. This solution satisfies the differential inclusion almost everywhere, so why is it not a possible solution to the differential equation? Recalling the definition, we see that a solution of the differential equation is a solution of the associated differential inclusion, which means that it is an absolutely continuous vector-valued function which satisfies the differential inclusion almost everywhere. In this case, the solution is not continuous and therefore, not absolutely continuous. So it does not satisfy the definition of a solution and is discounted as a possibility. The same reasoning can be applied to the case where $x_o < 0$. Thus, the only possible solution is $x(t) = 0$ when $t > 1$ and $x(t) = 1 - t$ for $0 \leq t \leq 1$.

Now, we consider an example where the coefficient of static friction is strictly greater than the coefficient of sliding friction. This is the relationship that is more common in nature. For example, when trying to push a table across the floor, once we get it moving, it is easier to continue pushing it than it was to begin the movement. That is, the coefficient of static friction between the table and the floor is greater than that of sliding friction between the two surfaces. Consider the typical differential equation with the friction term

$$\ddot{x} + f(\dot{x}) + x = 0. \tag{3.11}$$

The differential equation in this case must be represented as a differential inclusion. This is a result of the form of the friction term. With static friction strictly greater than sliding friction, $f(\dot{x})$ is modelled by

$$f(\dot{x}) = \begin{cases} [-1.5, 1.5], \text{ when } \dot{x} = 0 \\ -\operatorname{sgn}\dot{x}, \text{ when } \dot{x} \neq 0 \end{cases}. \tag{3.12}$$

Using the change of variables $\dot{x}_1 = x_2$ and $\dot{x}_2 = -f(x_2) - x_1$, the differential equation can be converted into the differential inclusion

$$\dot{x}(t) \in \left\{ \begin{array}{c} \{x_2\} \\ \left\{ \begin{array}{l} \{[-1.5, 1.5]\}\,,\ \text{when } x_2 = 0 \\ \{-\operatorname{sgn} x_2\}\,,\ \text{when } x_2 \neq 0 \end{array} \right. \end{array} \right. \cdot \tag{3.13}$$

We will apply the initial conditions $x(0) = 10$ and $\dot{x}(0) = 20$. Initially, since $\dot{x} > 0$, the differential equation

$$\ddot{x} - 1 + x = 0 \tag{3.14}$$

governs the motion of the mass. Solving this differential equation and applying the initial conditions yields the solution

$$x(t) = 9\cos t + 20\sin t + 1. \tag{3.15}$$

At time $t \approx 1.148$, the mass reaches a point where $\dot{x} = 0$ and $x = 22.932$. At this point the differential inclusion applies and the velocity of the mass may assume any value in the closed interval $[-1, 5, 1.5]$. However, if the mass takes a positive value for $\dot{x}$, the differential equation will then force the mass back down towards the x-axis. The system still has enough energy so that the mass will not permanently come to a rest at this point. Thus, the only possibility is that it will take on a value of $-1.5 < \dot{x} < 0$. When the mass moves away from $\dot{x} = 0$, the differential equation

$$\ddot{x} + 1 + x = 0 \tag{3.16}$$

describes the motion along with the initial conditions $x(1.148) = 22.932$ and $\dot{x}(1.148) = 0$. Solving this differential equation yields

$$x(t) = 7.907\cos t + 17.572\sin t - 1. \tag{3.17}$$

The mass will again have zero velocity at the point $x = 18.269$. This process can be repeated and we will eventually observe that the mass will come to stop in finite time. Its location will be somewhere between $x = 18.269$ and $x = 22.932$. After some time, the mass is unable to continue to overcome the coefficient of static friction and comes to a rest.

For another example, consider the unforced damped oscillator with dry friction which was studied by Martin Šenkyřík.[9] Its equation of motion is given by

$$u'' + 2u' + 3u + \operatorname{sgn} u' = 0. \tag{3.18}$$

Since this is a damped oscillator, intuitively we would reason that the system would come to rest after some amount of time. Šenkyřík showed that Filippov's solution to this system has this desired property. By defining $u = x_1$, this differential equation can be converted into the system of first order equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - 3x_1 - \operatorname{sgn} x_2. \end{aligned} \tag{3.19}$$

The discontinuity occurs when $x_2 = 0$. Therefore, using Filippov's definition, the system of equations becomes

$$\dot{x}_1 \in \{x_2\}$$
$$\dot{x}_2 \in \begin{cases} \{-2x_2 - 3x_1 - \operatorname{sgn} x_2\}, \text{ when } x_2 \neq 0 \\ \{[-3x_1 - 1, -3x_1 + 1]\}, \text{ when } x_2 = 0 \end{cases} . \tag{3.20}$$

At the discontinuity, the entire set of values of $f(t, x)$ are considered as an interval. Since this is a damped system, both x_1 and x_2 will decay exponentially If we let $x_2 \to 0$ in the upper half plane, we see that $x_1 \to -1/3$. However, if $x_2 \to 0$ in the lower half plane, then $x_1 \to 1/3$. Thus, we deduce that since x_2 is alternating at some finite time t_o, $|x_1(t_o)| < 1/3$ and $x_2(t_o) = 0$. For those $t > t_o$, we can conclude that since $x_2 = 0$, we have $\dot{x}_1 = 0$. This implies that x_1 is a constant. Recalling that $u = x_1$, we have that u is constant. Since u represents the position of the mass at any given time, u being constant after some time t_0 means that after a certain point in time the mass has come to rest, i.e. its position is not changing. This is the expected result since the system is damped. In the next section, the solution to this differential equation is plotted.

The Filippov theory of differential equations is more general than that of Carathéodory equations. If the function $f(t, x)$ is continuous in the spatial variable, then $F(t, x) = \{f(t, x)\}$ and the differential inclusion is really just a restatement of the original differential equation, so the Filippov theory reduces to that of Carathéodory differential equations. Further, if the right-hand side is continuous in both variables, the Filippov theory is equivalent to the classical theory of solutions to ordinary differential equations that was investigated in section 2.1. In the next section we will investigate which numerical solver is best suited for finding solutions to these types of problems.

4 DETERMINATION OF THE NUMERICAL METHOD

The system we will be considering is stiff and contains a discontinuity in the spatial variable. There are four ordinary differential equation solvers in Matlab that are specifically designed for stiff systems. The solver ode15s is a variable order solver based on the numerical differentiation solvers and is a multistep method. The ode23s solver is based on a modified Rosenbrock formula of order 2. It is a single step solver and may therefore be more efficient than ode15s for crude tolerances. The ode23t solver implements the trapezoidal rule and is useful for systems that are moderately stiff and do not contain numerical damping. The ode23tb solver is an implicit Runge-Kutta method in which the first stage is a trapezoidal rule and the second stage is a backwards differentiation formula of order two. Several test programs were run to determine which were best suited for our system. The first test system that was considered was the planar system

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= -256x_1^3x_2. \end{aligned} \tag{4.1}$$

This is a stiff system because of the x_1^3 term. With the initial conditions $x_1(0) = 0$

and $x_2(0) = 1$, the exact solution has the form

$$
\begin{aligned}
x_1 &= t \\
x_2 &= e^{-64t^4}.
\end{aligned}
\tag{4.2}
$$

The phase portrait was plotted using all four stiff solvers with the initial conditions $x_1(0) = 0$ and $x_2(0) = 1$.

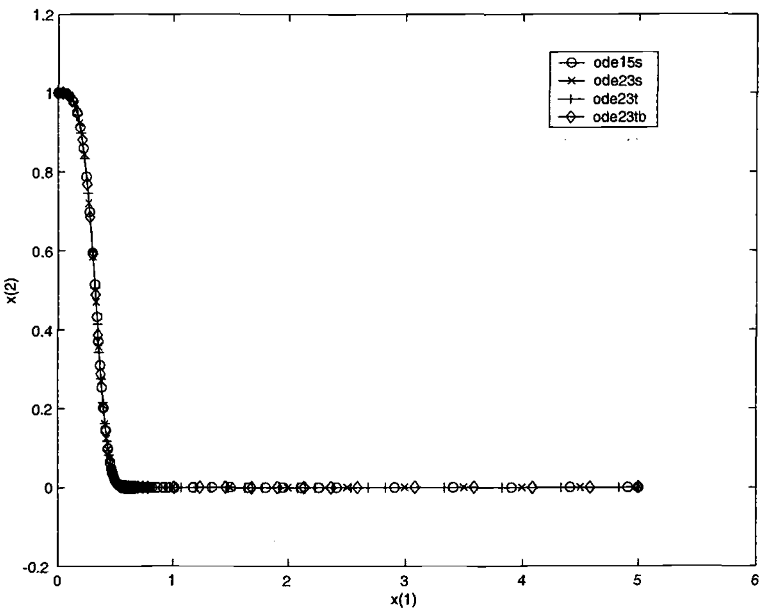

Figure 4.1: The numerical solution of (4.1).

All four of the numerical solvers provide solutions which approximate the exact solution very precisely.

The next test equation used was

$$\dot{x} = -\operatorname{sgn}(x). \tag{4.3}$$

The solution was plotted for the initial condition $x(0) = 1$ and is shown using the ode23s numerical solver. It is the solution that we would accept and thus, conclude that this solver yields an accurate solution.

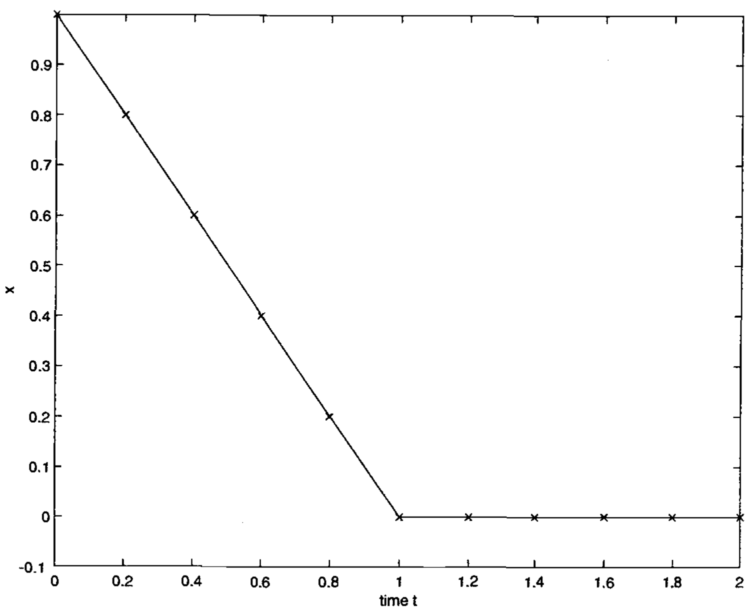

Figure 4.2: The numerical solution of (4.3)

When the ode15s was applied to this system, the solver was unable to meet integration tolerances without reducing the stepsize below the unit round of 10^{-16} for $t > 1$. For $0 \leq t \leq 1$, the solver produced a solution identical to that shown above. The other two solvers, ode23tb and ode23t, were able to solve the system for $0 \leq t \leq 1$. For $t > 1$, the stepsize became smaller than machine precision and thus the solvers were unable to continue past this point.

The only method that was able to accurately solve both test systems was ode23s. One more system was passed to ode23 to ensure its accuracy. In his dissertation, Martin Šenkyřík examined the unforced damped oscillator with dry friction.[9] Its equation of motion is given by

$$\ddot{x} + 2\dot{x} + 3x + \operatorname{sgn}\dot{x} = 0, \tag{4.4}$$

or, with $x = x_1$, by the planar system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - 3x_1 - \operatorname{sgn} x_2. \end{aligned} \tag{4.5}$$

Physical principles would dictate that after some time interval the system would come to rest. Applying the initial conditions $x_1(0) = 200$ and $x_2(0) = -200$, we see the following result

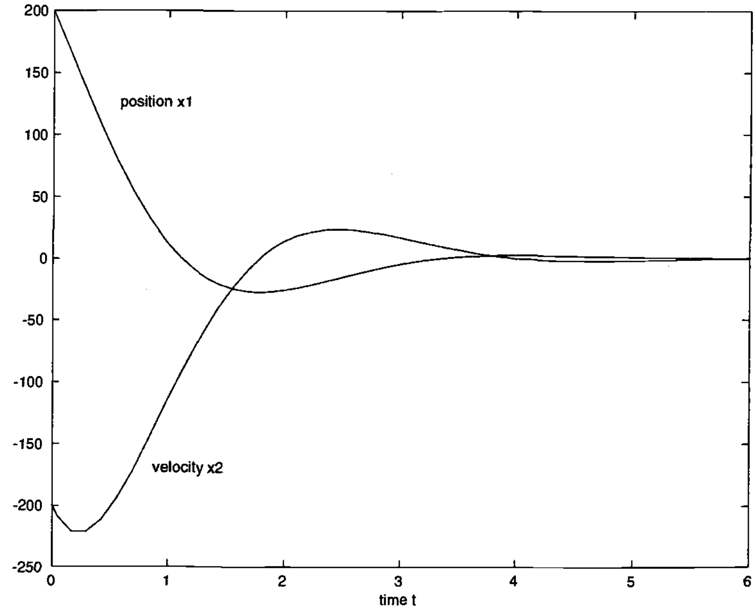

Figure 4.3: Graph of the solution of the unforced damped oscillator (4.5).

The mass has stopped between $t = 5$ and $t = 6$. This is in agreement with Šenkyřík's results.[9] With these three tests completed, we are comfortable that ode23s is providing the Filippov solution to the system of differential equations and thus, will demonstrate the correct behavior of the system in question.

5 ANALYSIS OF COMPONENTS OF THE NONLINEAR OSCILLATOR WITH DRY FRICTION

Before we consider the non-linear oscillator with dry friction, linear and cubic damping effects, and a harmonic excitation term, we will study the behavior of simpler systems in order to better understand the effect that these phenomena have on an oscillatory system. We first consider a system with only the linear and cubic stiffness terms. For the purposes of our system, we set the mass $m = 1$, the linear coefficient $k_1 = 0$, and the cubic coefficient $k_2 = 10,000$. Thus, the equation of motion becomes

$$\ddot{x} + 10,000\, x^3 = 0. \tag{5.1}$$

The associated system of first order equations is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10,000\ x_1^3. \end{aligned} \tag{5.2}$$

We see that this system has an equilibrium point at the origin. By plotting the position x_1, versus the velocity x_2, we can more easily see the motion of this system over time. The initial conditions are set to be $x_1(0) = 0.01$ and $x_2(0) = 0$ and the time interval is $0 \leq t \leq 10$. The result is shown in Figure 5.1.

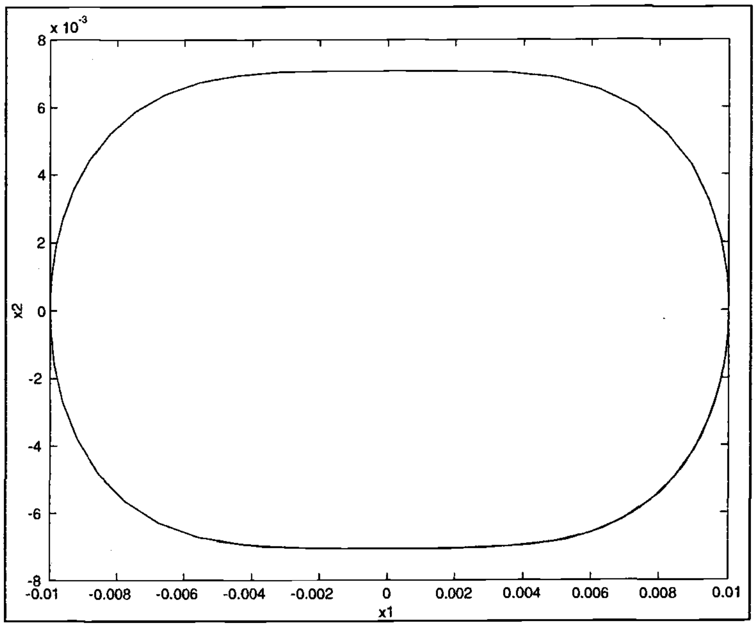

Figure 5.1: Phase portrait for (5.2) with $x_1(0) = 0.01$.

We see that the motion is periodic and the system completes an orbit in approximately ten seconds. If the mass is displaced less initially, it would be reasonable to assume that it would take longer to complete an orbit since the system will have less potential and kinetic energy. Indeed, if $x_1(0) = 0.001$ and $x_2(0) = 0$, we must take a time interval of $0 \leq t \leq 75$ to achieve a complete orbit.

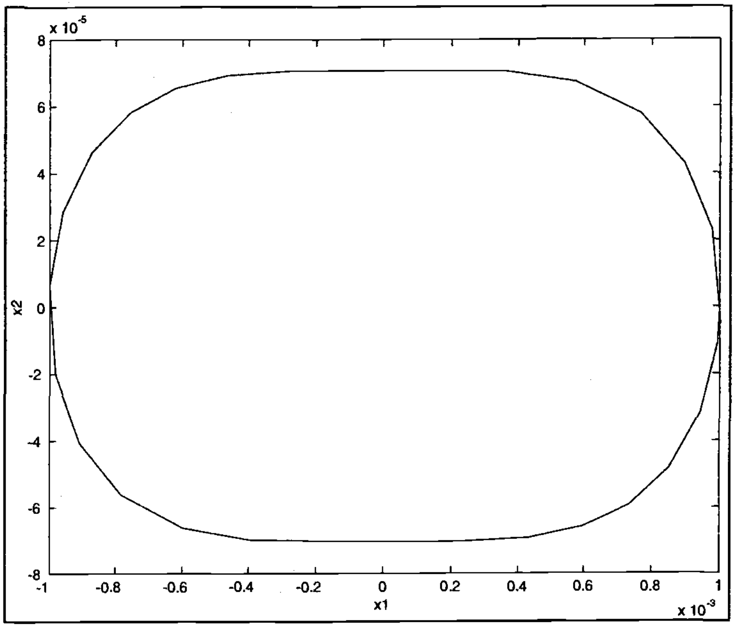

Figure 5.2: Phase portrait of (5.2) with $x_1(0) = 0.001$.

In comparing Figure 5.1 and Figure 5.2, we notice that the scales of the oscillations are quite different. The mass in Figure 5.2 has a much smaller amplitude of motion than that of Figure 5.1. This is a result of the smaller initial displacement. In both of these instances, we have specified that $x_2(0) = 0$. Now, we observe the behavior of the system if the mass is displaced and given an initial velocity. Let $x_1(0) = 0.001$ and $x_2(0) = 0.001$. The system still exhibits periodic behavior, however, the amount of time needed to complete one orbit has decreased significantly (see Figure 5.3). In this instance, we need only take a time interval of $0 \le t \le 20$.

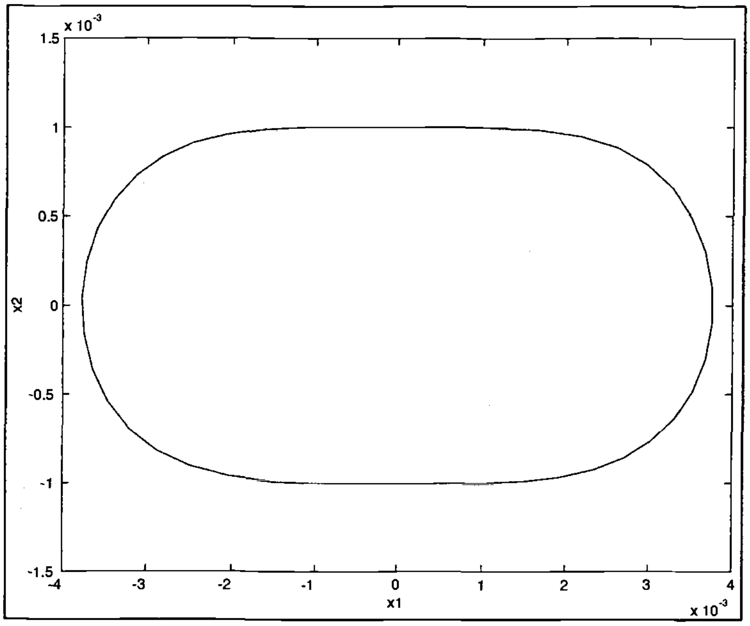

Figure 5.3: Phase portrait for (5.2) with $x_1(0) = x_2(0) = 0.001$.

The initial velocity increases the energy in the system, thus allowing the mass to complete the cycles more quickly.

Consider a system with the stiffness terms as before, but with the addition of a linear and cubic damping term. The coefficients are set to be the same as those of the nonlinear oscillator with dry friction that will be investigated in the next section. The equation of motion for this system is given by

$$\ddot{x} + 10,000x^3 - 0.49(1 - \dot{x}) - 0.196(1 - \dot{x})^3 = 0. \tag{5.3}$$

This system can be transformed into the planar system of first order equations

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10,000x_1^3 + 0.49(1 - x_2) + 0.196(1 - x_2)^3.\end{aligned} \tag{5.4}$$

By setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, we see that the system has a equilibrium (or fixed) point at approximately $(0.04, 0)$. As the system is allowed to oscillate, we observe that the orbit will spiral into the equilibrium point, which is therefore called a stable focus. Below, we have started the mass at the origin and allowed the system to run for 10 seconds.

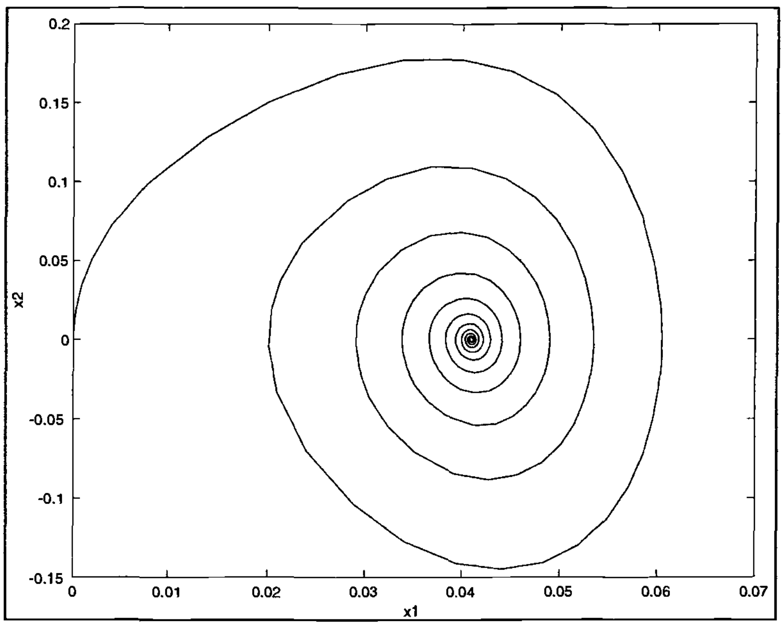

Figure 5.4: Phase portrait for (5.4).

Now we investigate the behavior of the system when the dry friction (stick-slip) condition is added.

The dry friction term is modelled by the discontinuous function $\text{sgn}(\dot{x})$. Since it is dependent on the velocity of the mass compared to the initial velocity of the mass, it has the form $\text{sgn}(v_o - x_2)$.[6] In this system we set the initial velocity v_o, equal to one. Therefore, the equation of motion becomes

$$\ddot{x} + 10,000x^3 - 0.49(1-\dot{x}) - 0.196(1-\dot{x})^3 - 5.88\,\text{sgn}(1-\dot{x}) = 0. \tag{5.5}$$

Converting this second order system to a planar system of first order equations, we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10,000x_1^3 + 0.49(1-x_2) + 0.196(1-x_2)^3 + 5.88\,\text{sgn}(1-x_2). \end{aligned} \tag{5.6}$$

By setting both $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, we observe that the system has an equilibrium point at approximately $(0.09, 0)$. It appears that all orbits are spiraling into the equilibrium point as we would expect, however, theorems do not currently exist to demonstrate this conclusively. The combination of the damping and dry friction terms has resulted in the mass coming to rest at approximately $(0.09, 0)$. Again, we start the mass at the origin in the figure below and allow time to run to 10 seconds.

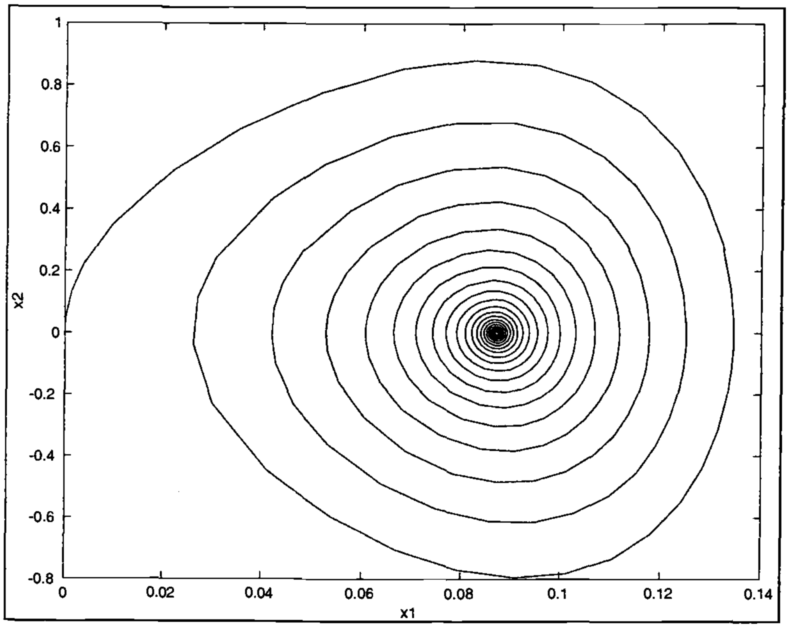

Figure 5.5: Phase portrait for (5.6).

We see that the orbits are decaying quickly towards the equilibrium point. The same behavior is observed for initial conditions other than the origin.

We also investigate the effect of a harmonic excitation term on the system by setting both the linear and cubic stiffness term to zero and removing the dry friction term. The resulting equation of motion is

$$\ddot{x} - 10.5\cos(23t) = 0. \tag{5.7}$$

Converting this to a system of first order equations, we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 10.5\cos(23t). \tag{5.8}$$

Note the system is now explicitly dependent on time, i.e., it is a non-autonomous system. We can choose to plot the phase portrait in either the (t, x_1, x_2) plane or the (x_1, x_2) plane. It may be easier to understand the behavior of the system by comparing these plots. The initial conditions are chosen to be $x_1 = 0.001$ and $x_2 = 0$. Both of these plots are shown below.

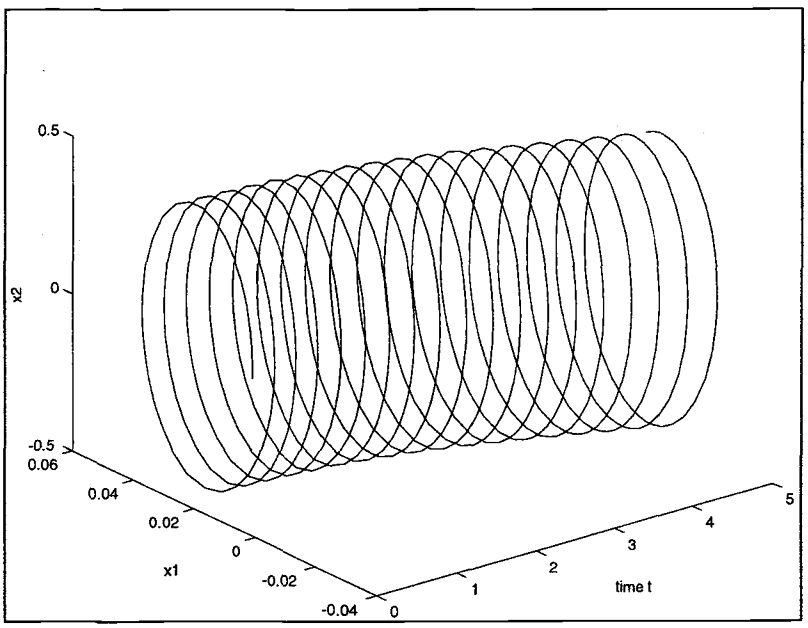

Figure 5.6: Three dimensional phase portrait for (5.8).

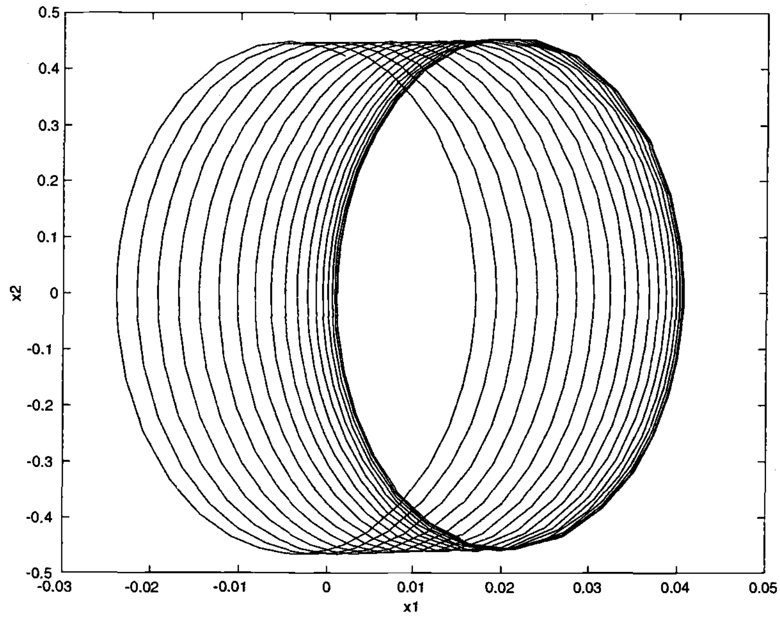

Figure 5.7: Two dimensional phase portrait for (5.8).

The stiffness, damping, and dry friction terms have been eliminated so the system should exhibit some kind of periodic behavior. Indeed, by comparing these two figures we observe that the mass is oscillating back and forth in a periodic orbit whose center point is oscillating as well . This is reasonable since the excitation term is periodic and thus will affect the system by adjusting the position of the oscillation. Similar results would be obtained by choosing different initial conditions.

By introducing the stiffness terms, with the linear stiffness term coefficient set to zero, the equation of motion becomes

$$\ddot{x} + 10,000x^3 - 10.5\cos(23t) = 0. \tag{5.9}$$

We plot the following system of equations with initial conditions $x_1 = 0.001$ and $x_2 = 0$, again in both two and three dimensional phase space.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10,000x_1^3 + 10.5\cos(23t). \end{aligned} \tag{5.10}$$

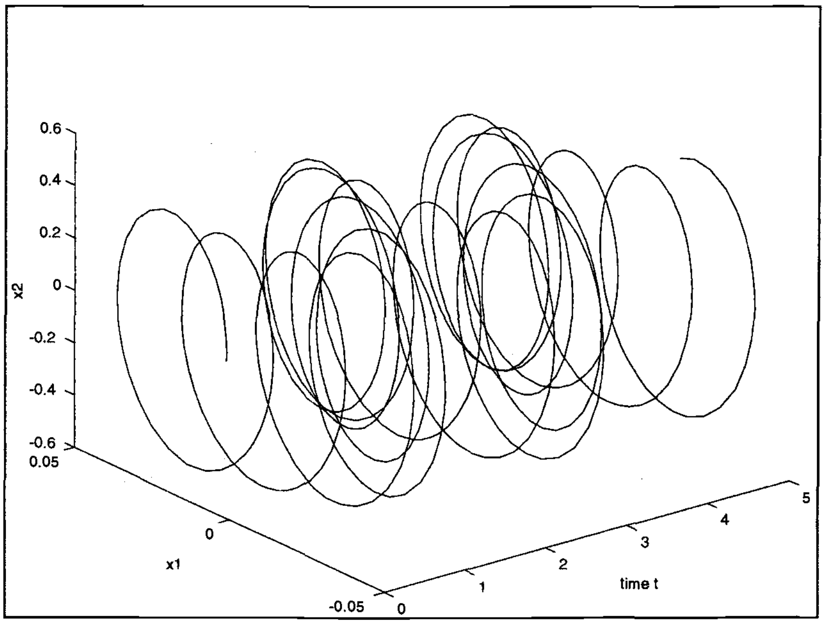

Figure 5.8: Three dimensional phase portrait for (5.10).

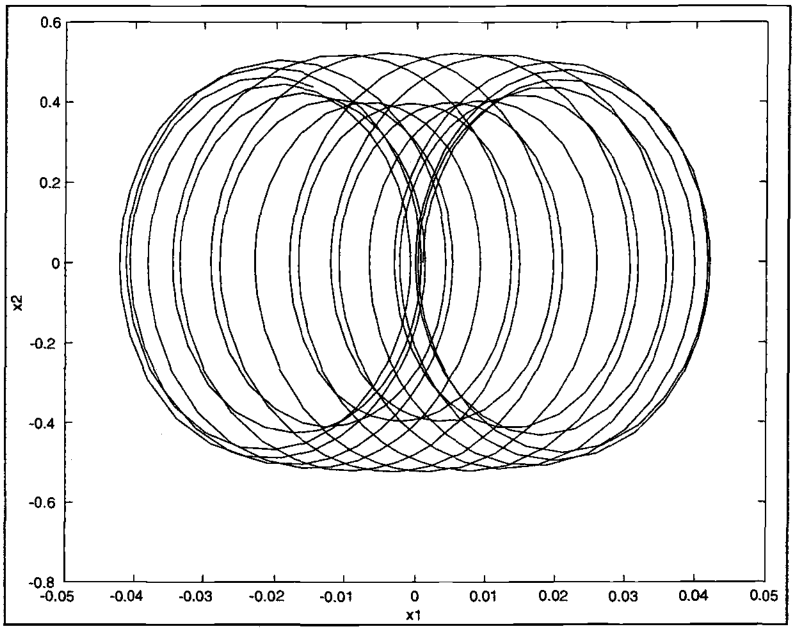

Figure 5.9: Two dimensional phase portrait for (5.10).

It is difficult to discern any kind of periodic motion in this system from either of the phase portraits. The mass is now exhibiting very intricate movement. The combination of the cubic stiffness term and the harmonic excitation term results in the behavior of the system becoming quite complex.

Now we introduce the damping terms into the system above. The equation of motion and system of first order equations become

$$\ddot{x} + 10,000x^3 - 0.49(1-\dot{x}) - 0.196(1-\dot{x})^3 - 10.5\cos(23t) = 0, \tag{5.11}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10,000x_1^3 + 0.49(1-x_2) + 0.196(1-x_2)^3 + 10.5\cos(23t). \end{aligned} \tag{5.12}$$

When the corresponding system of first order equations is plotted with initial conditions $x_1 = 0.001$ and $x_2 = 0$, the following phase portraits are produced.

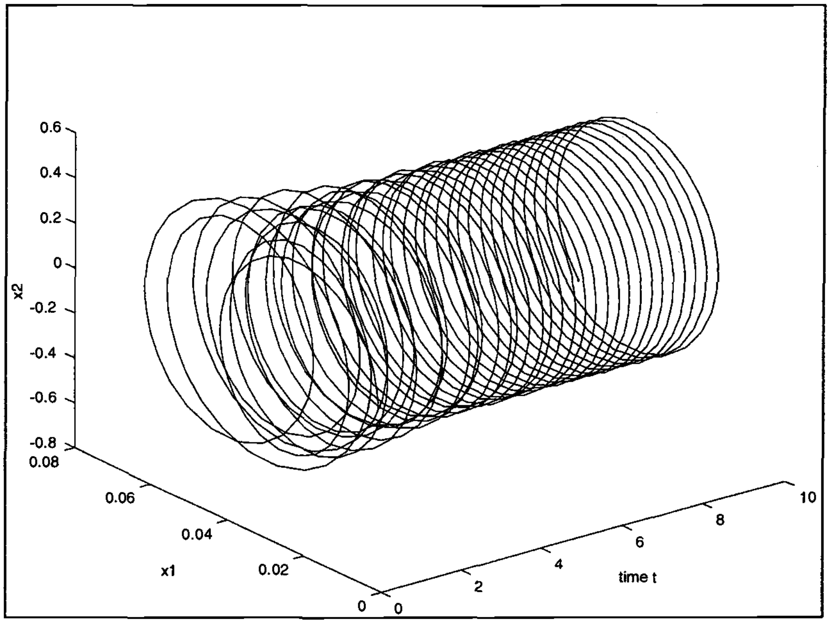

Figure 5.10: Three dimensional phase portrait for (5.12).

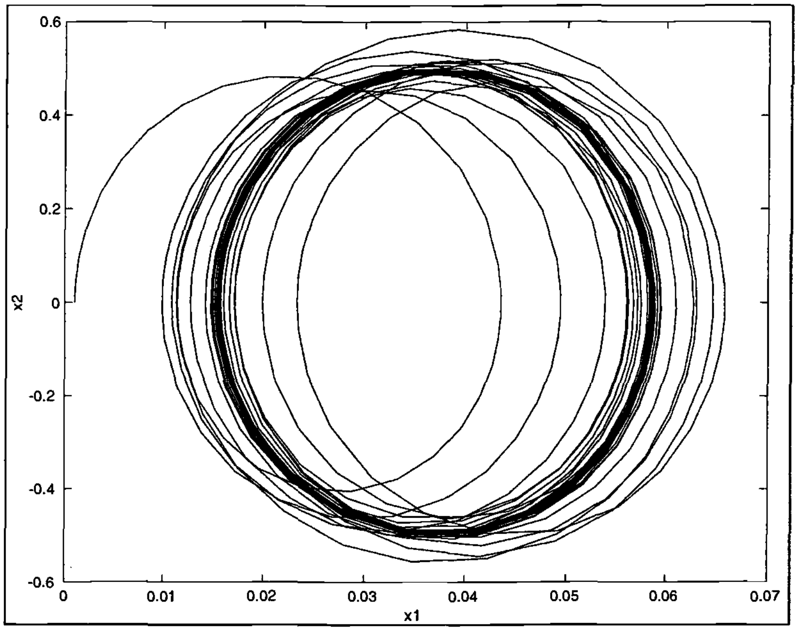

Figure 5.11: Two dimensional phase portrait for (5.12).

The three dimensional plot clearly demonstrates that after some initial oscillations, the system settles into a periodic orbit. This can also be seen in the two dimensional phase portrait since the orbits become more dense in the area of the periodic orbit.

The last combination we will examine is the system with the linear and cubic damping term, the harmonic excitation term, but without the cubic or linear stiffness terms. The equation of motion becomes

$$\ddot{x} - 0.49(1 - \dot{x}) - 0.196(1 - \dot{x})^3 - 10.5\cos(23t) = 0. \tag{5.13}$$

The resulting system of first order equations is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0.49(1 - x_2) + 0.196(1 - x_2)^3 + 10.5\cos(23t). \tag{5.14}$$

The resulting phase portraits are shown below.

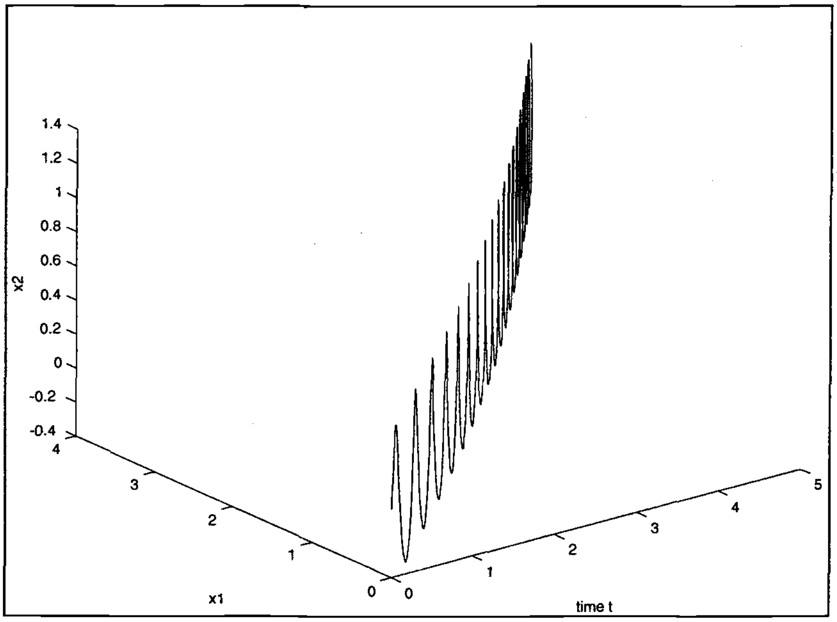

Figure 5.12: Three dimensional phase portrait for (5.14).

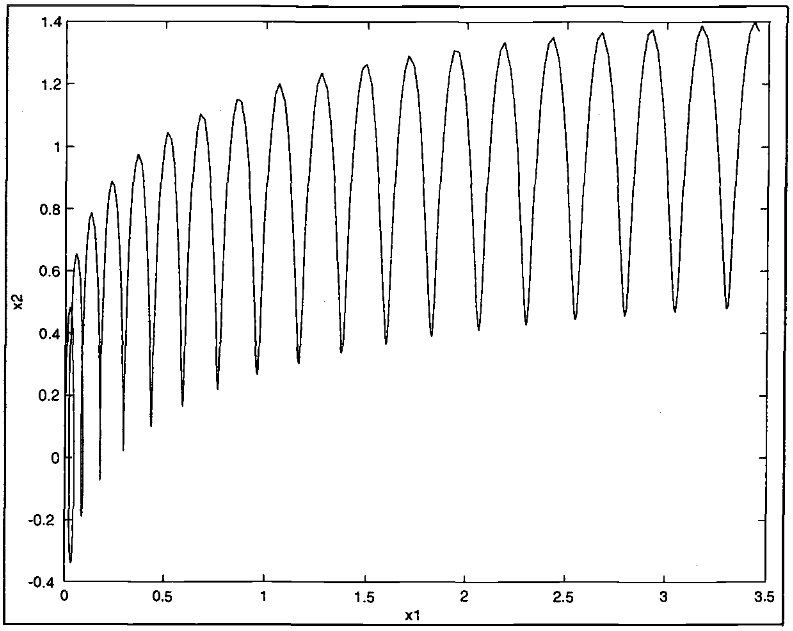

Figure 5.13: Two dimensional phase portrait for (5.14).

We observe that there is oscillatory motion in the system, however, the mass is moving around a point that is itself in motion. This accounts for the progression of the waveform seen above. As time increases, the oscillations approach a certain amplitude, but the frequency of the oscillations is increasing.

The different combinations of the stiffness, damping, dry friction, and harmonic excitation terms have yielded a variety of behaviors of the mass-spring system. We will now look at the system that incorporates all of these parameters simultaneously.

6 NONLINEAR OSCILLATOR WITH DRY FRICTION

The mass-spring system under consideration here combines the linear and cubic damping effects, dry friction, and the harmonic excitation term. This kind of system would arise if we had a single-degree of freedom mass sliding along a moving belt.

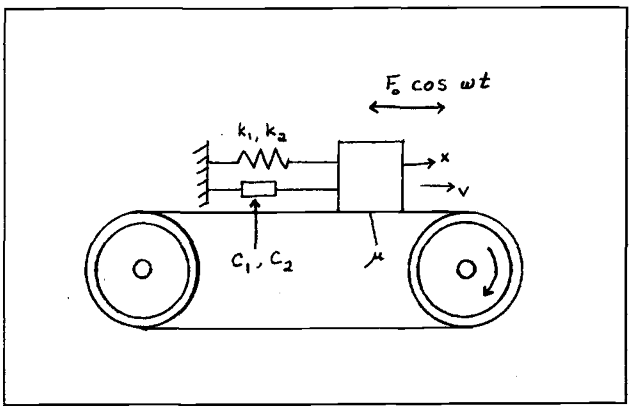

Figure 6.1: Non-linear mass-spring system on a moving belt.

We account for the force of linear and cubic stiffness terms of the spring by $F = -k_1 x - k_2 x^3$. Besides the nonlinear spring, the mass is attached to a nonlinear damper whose force is described by $F = c_1(v - \dot{x}) + c_2(v - \dot{x})^3$, where c_1 and c_2 represent the viscous damping coefficients. Here, v denotes the velocity of the belt. The dry friction, or the stick-slip condition, between the mass and the belt is given by $F = mg\mu\,\mathrm{sgn}(v - \dot{x})$, where $mg\mu$ is the coefficient of dry friction. The mass is also subjected to an external harmonic excitation of the form $F = P_o \cos\omega t$, where ω is the frequency and P_o is the amplitude. Applying Newton's Second Law of

Motion yields the following equation of motion for the mass in this system:

$$m\ddot{x} - c_2(v - \dot{x})^3 - c_1(v - \dot{x}) + k_1x + k_2x^3 - mg\mu\,\mathrm{sgn}(v - \dot{x}) = P_o\cos(\omega t). \quad (6.1)$$

Here, the parameter m represents the mass and the acceleration due to gravity is denoted, as usual, by $g = 9.8\,\mathrm{m}\,/\,\mathrm{s}^2$. The variable x represents the displacement of the mass from its rest position, that is, its position when the belt is not moving and the spring is relaxed.

The second order differential equation (6.1) can be converted in a system of first order equations by making the substitution $x_1 = x$ and $x_2 = \dot{x}$. The system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g\mu\,\mathrm{sgn}(v - x_2) + \alpha g(v - x_2) + \beta g(v - x_2)^3 - \gamma_1 x_1 - \gamma_2 x_1^3 + F_o\cos\omega t, \end{aligned} \quad (6.2)$$

where we follow the notation of Narayanan and Jayaraman and let $\alpha = \frac{c_1}{mg}$, $\beta = \frac{c_2}{mg}$, $\gamma_1 = \frac{k_1}{m}$, $\gamma_2 = \frac{k_2}{m}$, and $F_o = \frac{P_o}{m}$. In his paper, Awrejcewicz claimed that for the parameter values $\alpha = 0.05\,\mathrm{s}\,/\,\mathrm{m}$, $\beta = 0.02\,\mathrm{s}\,/\,\mathrm{m}^3$, $\gamma_1 = 0\,\mathrm{s}^2$, $\gamma_2 = 10000(\mathrm{m\,s})^{-2}$, $\mu = 0.6$, $v = 1\,\mathrm{m}\,/\,\mathrm{s}$, and $F_o = 10.5\,\mathrm{m}\,/\,\mathrm{s}^2$ chaotic motion was observed in the system when a forcing frequency of $\omega = 23.0\,\mathrm{rad}\,/\,\mathrm{s}$ was applied. However, when the phase portrait for this system was plotted using Matlab, a different result was obtained.

The corresponding system is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 5.88\,\mathrm{sgn}(1 - x_2) + 0.49(1 - x_2) + 0.196(1 - x_2)^3 - 10000x_1^3 + 10.5\cos 23t. \tag{6.3}$$

The initial conditions, $x_1 = 0.001\,\mathrm{m}$ and $x_2 = 0\,\mathrm{m}/\mathrm{s}$, were chosen to correspond with Awrejcewicz's investigation. We first look at the three-dimensional phase portrait in which time was allowed to run for 15 seconds.

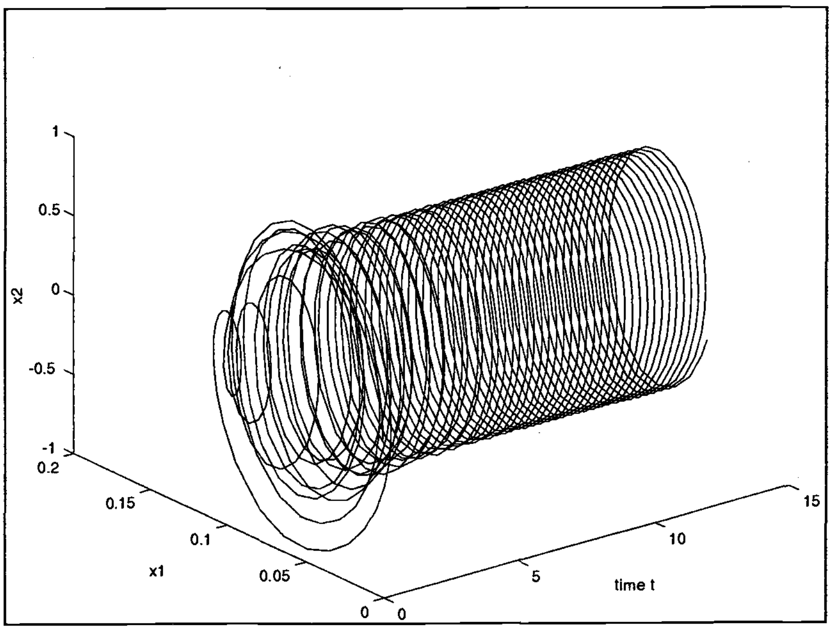

Figure 6.2: Three dimensional phase portrait of (6.3) with $\omega = 23.0\,\mathrm{rad}/\mathrm{s}$.

We immediately see that there is some regularity to the motion. To better observe the behavior of the mass, we consider the two-dimensional phase portrait for the same period of time.

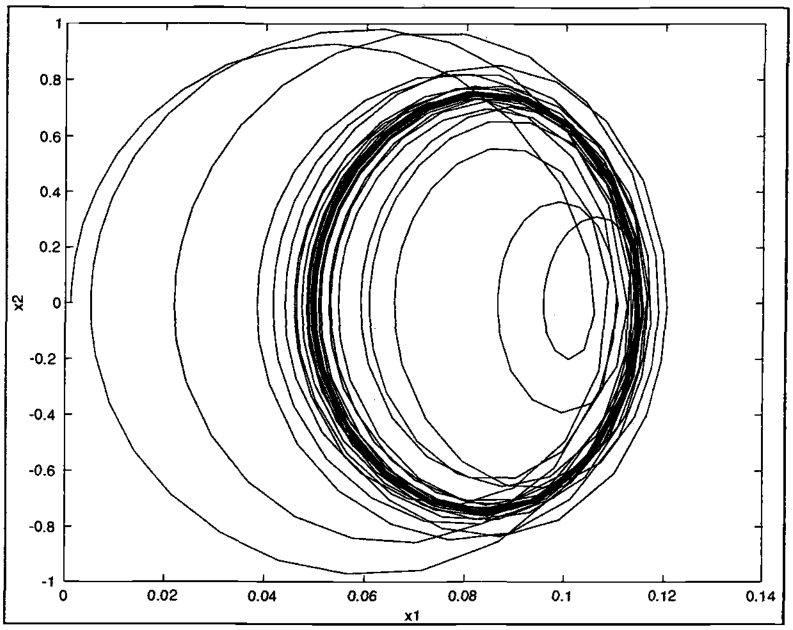

Figure 6.3: Two dimensional phase portrait of (6.3) with $\omega = 23.0\,\mathrm{rad}\,/\,\mathrm{s}$.

The orbits are centering around a particular area, indicating that the mass is spiraling towards a stable orbit. One method of determining whether the system is tending towards a stable orbit is to consider the trajectory coming from inside the orbit. If the orbit is a stable limit cycle, this trajectory should spiral in towards the orbit. Based upon the figure, we chose an initial value of $x_1 = 0.08\,\mathrm{m}$ and $x_2 = 0\,\mathrm{m}\,/\,\mathrm{s}$ and produced the following figure.

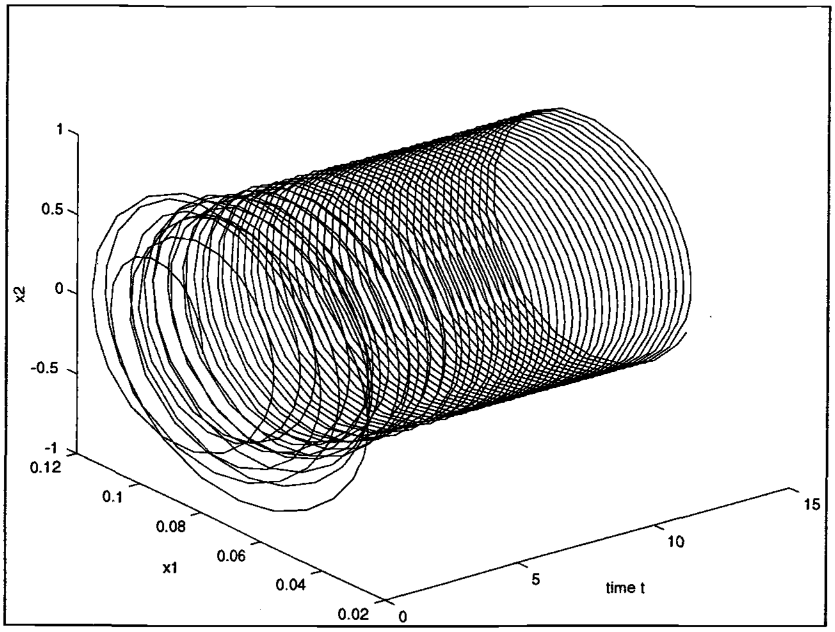

Figure 6.4: 3D phase portrait for (6.3) with initial condition inside suspected stable limit cycle

It appears that starting from outside the orbit also results in the trajectory spiraling in towards the same stable orbit. We also produced a two dimensional version to verify this conclusion.

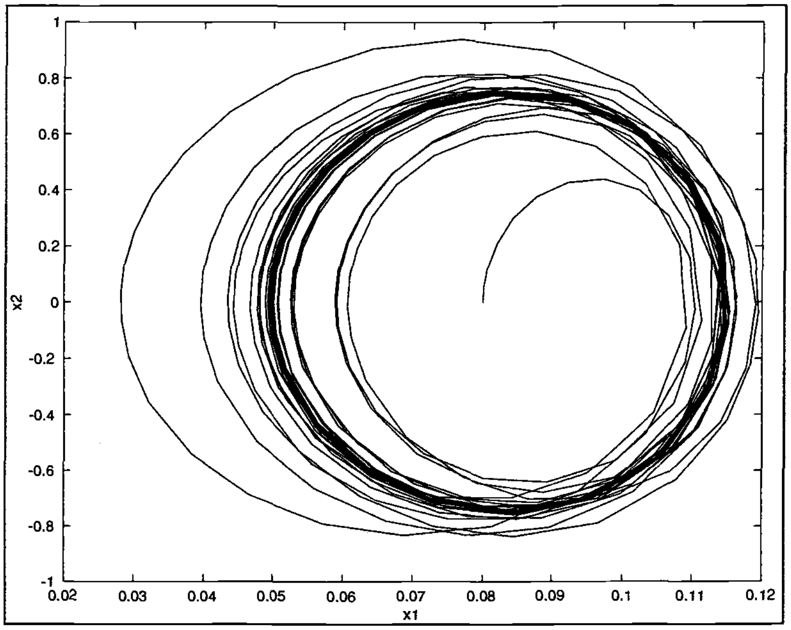

Figure 6.5: 2D phase portrait for (6.3) with initial condition inside suspected stable limit cycle.

These two phase portraits provide compelling evidence that for a value of $\omega = 23\,\text{rad}/\text{s}$, chaotic motion is not observed. In fact, the system is tending towards a stable, periodic orbit.

In the paper by Narayanan and Jayaraman, a wider variety of frequencies were considered. For the case of $\omega = 2\,\text{rad}/\text{s}$, it was found that the system exhibited chaotic motion. When we considered the system with this excitation frequency, these results were not obtained. The motion, while intricate, does appear to become periodic after some length of time. The system was run out to a variety of time intervals and the phase portraits are shown below for the initial conditions $x_1 = 0.001\,\text{m}$ and $x_2 = 0\,\text{m}/\text{s}$.

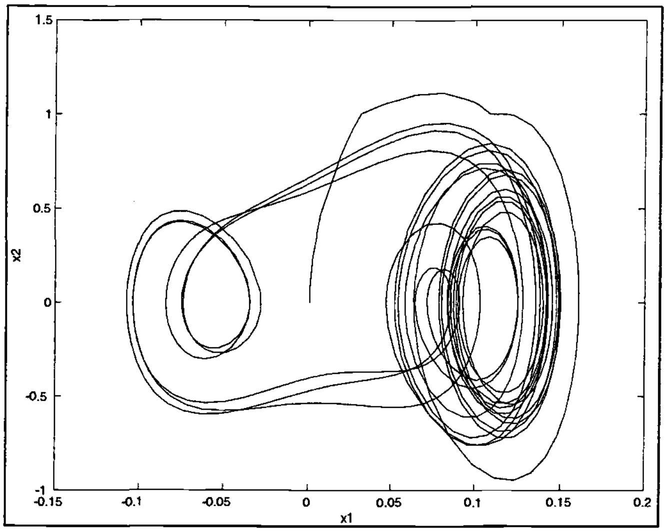

Figure 6.6: Phase portrait of (6.2) with $\omega = 2\,\mathrm{rad}\,/\,\mathrm{s}$ and $t = 10\,\mathrm{s}$.

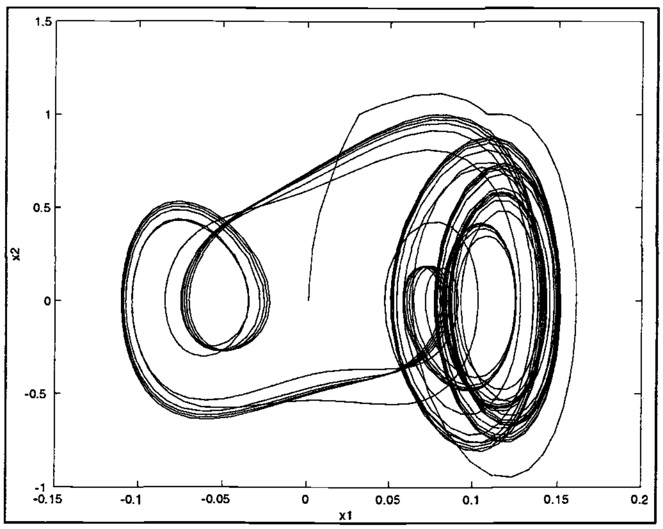

Figure 6.7: Phase portrait for (6.2) with $\omega = 2\,\mathrm{rad}\,/\,\mathrm{s}$ and $t = 20\,\mathrm{s}$.

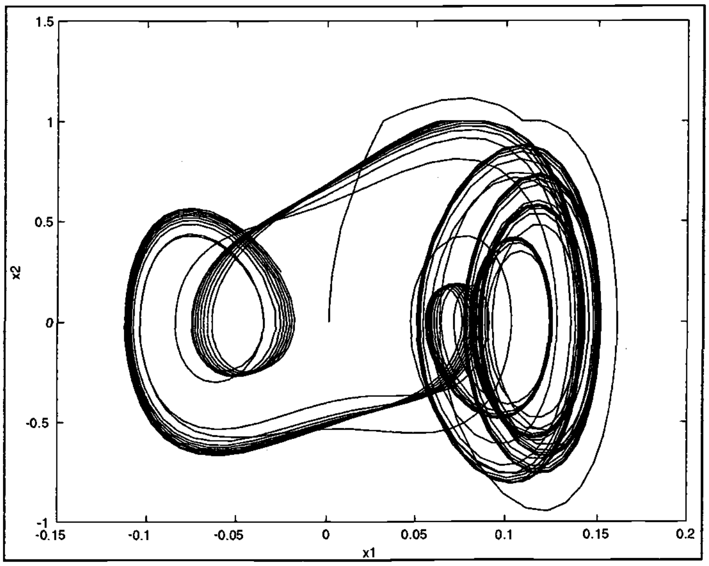

Figure 6.8: Phase portrait for (6.2) with $\omega = 2\,\mathrm{rad}\,/\,\mathrm{s}$ and $t = 30\,\mathrm{s}$.

Since it appeared that some kind of periodic motion may be occurring, the system was run for a longer time interval. While the two dimensional phase portraits looked similar to that for $t = 30$, the three dimensional plot produced a clear conclusion.

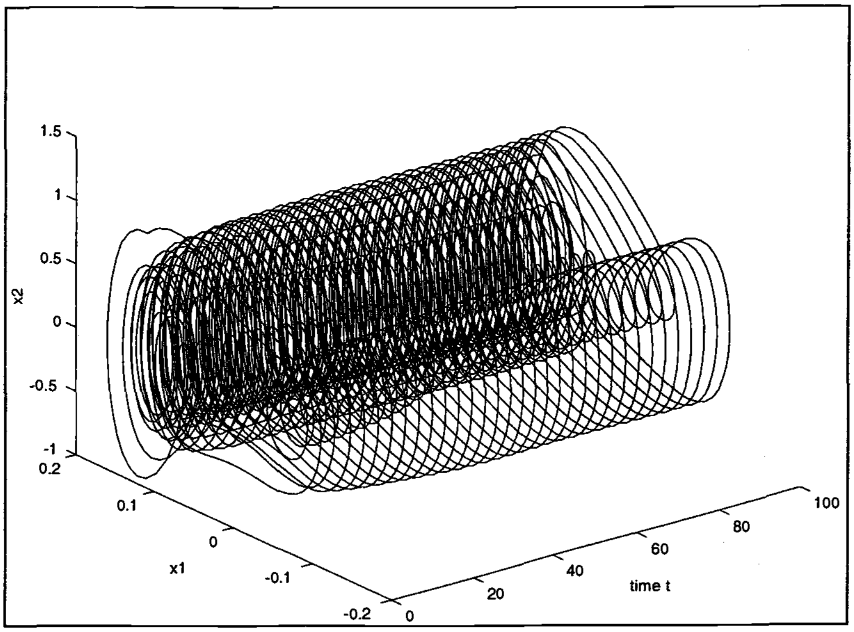

Figure 6.9: Three dimensional phase portrait for (6.2) with $\omega = 2\,\mathrm{rad}\,/\,\mathrm{s}$ and $t = 100\,\mathrm{s}$.

After some initial non-periodic oscillatory motion, the mass settled into a periodic motion. The excitation frequency $\omega = 15\,\mathrm{rad}\,/\,\mathrm{s}$ was also considered with the same initial conditions.

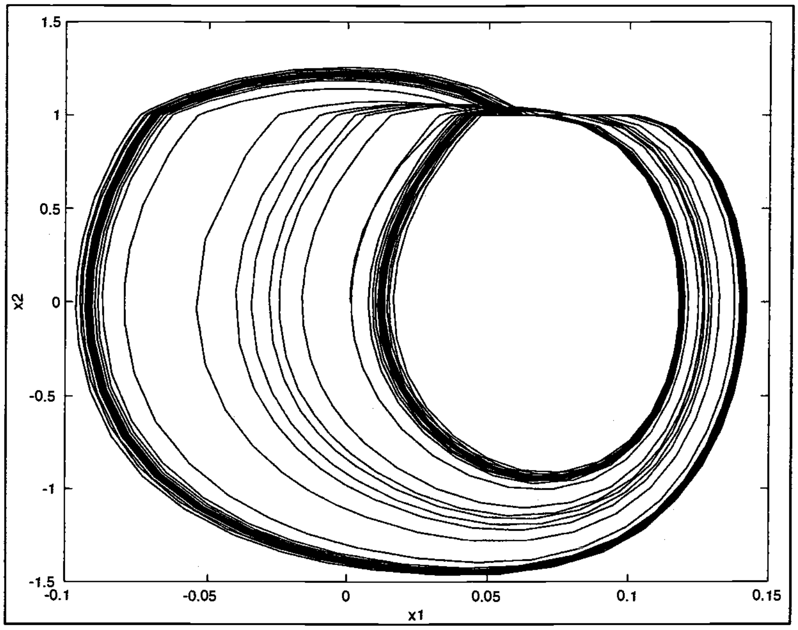

Figure 6.10: Phase portrait for (6.2) with $\omega = 15\,\mathrm{rad}/\mathrm{s}$ and $t = 30\,\mathrm{s}$.

While the results we obtained appear quite similar to those of Narayanan and Jayaraman, there are subtle differences that demonstrate that the mass is behaving differently than they observed.

The logical questions to ask are which of these results is correct and are these systems chaotic? We believe that the results presented here are correct. Narayanan and Jayaraman used a fourth order Runge-Kutta scheme to numerically solve the equations. The Runge-Kutta methods required that $f(t, x)$ be differentiable and therefore continuous. This is not the case with this system. It was also claimed that this system exhibited chaotic motion. It is not immediately clear from the phase portraits that this is the case.

In their paper, they looked at the Lyapunov exponent of the system to make a determination of chaotic motion. We must consider what is meant by chaotic motion. There are generally three components to the definition. First, if a system is to be chaotic, then there are trajectories that do not tend towards fixed points or periodic orbits as $t \to \infty$. Since we are dealing with a deterministic system, that is, a system that does not have noisy inputs or parameters, the irregular behavior must arise strictly from the system's nonlinearity. Lastly, the system must be sensitive to changes in the initial conditions. This means that trajectories that originate close together separate exponentially fast if the system is chaotic.

The Lyapunov exponent is an estimate of the rate of divergence of neighboring trajectories. Consider the differential equation

$$\dot{x} = f(t, x), \tag{6.4}$$

where $f(t, x)$ is continuous and $x(0) = x_o$. The solution of this equation is given by the corresponding integral equation

$$x(t) = x(t_o) + \int_{t_o}^{t} f(s, x(s))ds. \tag{6.5}$$

If we wanted the solution over an entire time interval, we could divide the interval into N partitions and rewrite the integral equation as

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(s, x(s))ds, \tag{6.6}$$

where $n = 0 \ldots N-1$. Starting with the initial condition x_o, we consider a nearby point $x_o + \delta_o$ where δ_o is small. Let δ_n be the separation of these two corresponding orbits after n partitions. If we find that

$$|\delta_n| \approx |\delta_o| \, e^{n\lambda}, \tag{6.7}$$

then λ is called the Lyapunov exponent. Positive Lyapunov exponents represent chaotic motion since the exponential is growing, while negative exponents indicate the trajectories are becoming closer together as time progresses. Returning to the differential equation $\dot{x} = f(t, x)$, let f^n be the resulting composition of f with itself

n times. We can then proceed with the following computation[10]

$$\begin{aligned}
\lambda &= \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_o}\right| \\
&= \frac{1}{n}\ln\left|\frac{f^n(x_o+\delta_o)-f^n(x_o)}{\delta_o}\right| \\
&\approx \frac{1}{n}\ln\left|(f^n)'(x_o)\right| \\
&= \frac{1}{n}\ln\left|\prod_{i=0}^{n-1} f'(x_i)\right| \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\ln\left|f'(x_i)\right|.
\end{aligned} \tag{6.8}$$

If the limit as $n \to \infty$ exists, we say that

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln\left|f'(x_i)\right|\right\} \tag{6.9}$$

is the Lyapunov exponent. If we were considering a system of differential equations, we would have the Jacobian matrix of $f(t,x)$ instead of the first derivative. Thus, the computation of the Lyapunov exponents for a system of differential equations requires looking at the Jacobian of the right hand side, that is the Jacobian of $f(t,x)$. This requires that we differentiate $f(t,x)$; however, in our system this function is not continuous and therefore the derivative does not exist everywhere. Therefore, this method of determining chaotic motion cannot be employed. Narayanan and Jayaraman used this method to conclude that the system exhibited chaotic motion.

In conclusion, chaotic motion for this system cannot be proven at this time. We have discovered errors in the conclusions of Awrejcewicz, Narayanan, and Jayaraman since some of the systems they studied do not exhibit chaotic behavior, but rather contain a stable limit cycle. It is possible that for some excitation frequencies, this system exhibits chaotic motion. Techniques to handle discontinuous systems analogous to that of the Lyapunov characteristic exponents must be developed to conclusively determine if any of the excitation frequencies result in the chaotic behavior of the nonlinear oscillator with dry friction.

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