Consider a transformation group $G$ operating on a space $X$ and a $G$-invariant function $f$ defined on a $G$-invariant subset of $X$. By imposing suitable conditions on $X$, $G$, $f$ and $A$, the author derives sufficient conditions for extending $f$ invariantly to the whole space, and thus generalizing the classical Tietze extension theorem.
The Extension Problem for Functions Invariant Under a Group

by

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A THESIS
submitted to
Oregon State University

in partial fulfillment of
the requirements for the
degree of
Master of Arts
June 1967
APPROVED:

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Date thesis is presented _MAY 5, 1967_

Typed by Clover Redfern for _Bai-Ching Chang_
ACKNOWLEDGEMENT

The author wishes to express his deep gratitude to Dr. J. Wolfgang Smith for his many helpful assistances during the preparation of this paper.
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THE EXTENSION PROBLEM FOR FUNCTIONS INVARIANT UNDER A GROUP

I. INTRODUCTION

Let $X$ be a topological space and $G$ a topological group operating on $X$ by left action. Let $A$ be a closed subset of $X$ and $I$ denote the closed unit interval. A continuous function $f: A \to I$ will be called $G$-invariant provided $GA = A$ and $f(gx) = f(x)$ for all $(g, x) \in G \times A$. We ask: does such a function $f$ admit a $G$-invariant extension to $X$? In case $X$ is normal and $G$ acts trivially on $X$ the answer amounts to the Tietze extension theorem. When $X$ is normal and $G$ compact the answer is still affirmative, as follows by a more general result of A.M. Gleason (see [1]). Gleason's proof would run as follows: Let $\mu$ be Haar measure on $G$ with $\mu(G) = 1$, and let $f^*: X \to I$ be any continuous extension of $f$. Define $f': X \to I$ by

$$f'(x) = \int f^*(gx) dg,$$

the integral being taken with respect to $\mu$. Since $f$ is $G$-invariant and $f^*$ coincides with $f$ on $A$, so does $f'$. But $f'$ is clearly $G$-invariant on $X$. --It may be of interest to note that the result thus established can also be obtained through a different approach. Namely, when $X$ is normal and $G$ compact it follows that the
orbit space \( X/G \) is normal (see Appendix), and this gives the desired conclusion.

In the present paper \( G \) is assumed to be noncompact, in which case the aforesaid extension property certainly fails. We are in fact indebted to Professor Gleason for a very suggestive example\(^1\) in which the additive group of real numbers operates differentiably on a 2-dimensional cylinder in such a manner that not every \( G \)-invariant \( f \) admits a \( G \)-invariant extension. Nonetheless one can obtain some extension theorems for noncompact \( G \) by imposing suitable conditions on \( X \) and on the action of \( G \), as well as on \( f \). So far as the considerations of the sequel are concerned, the topology of \( G \) will play no role, and what is more, our results apply equally to the case where \( G \) is a pseudo-group of local transformations on \( X \). Although we shall continue to speak as though \( G \) were a transformation group on \( X \), everything can as well be interpreted for pseudo-groups.

A simple extension theorem results from the condition that \( X \) be compact metric, or more generally, compact pseudo-metric, and the transformations of \( G \) be uniformly bounded (see §2). This implies that \( X/G \) is again normal, and consequently, that the extension property holds without restriction on \( f \). It should be noted,

\(^1\)This example has proved to be very helpful, and we shall have occasion to present some variations of it in §6.
however, that apart from this one case our assumptions will not imply normality of the orbit space. For the case of locally compact $X$ one can obtain a somewhat specialized extension theorem by imposing suitable conditions on $X/G$ and $A$ (see §3). The main development of this paper begins with §4 and depends upon the fact that every compact $X$ admits a minimal $G$-invariant equivalence relation $R$ such that $X/R$ is normal. This gives two extension theorems for compact $X$ (§4, §7), and each of these implies a corresponding result for general noncompact $X$ (see §8). While most of our propositions are established by elementary arguments, it may be of interest to note that the main results (Theorem 5 and its corollary) require a somewhat more delicate approach involving transfinite induction.
II. PSEUDO-METRIC SPACES

**Theorem 1.** Let $X$ be compact and $\delta$ a pseudo-metric on $X$. If $\delta(gx, gy) < M\delta(x, y)$ for some $M$ and all $(g, x, y) \in G \times X \times X$, then every $G$-invariant $f : A \to I$ admits a $G$-invariant extension to $X$.

We will first prove that $X/G$ is normal. Let $A_1, A_2$ be closed disjoint subsets of $X/G$ and let $A_i^* = \pi^{-1}(A_i)$, $i = 1, 2$; where $\pi : X \to X/G$ denotes the natural projection. Thus $A_1^*, A_2^*$ are closed disjoint subsets of $X$. Since pseudo-metric spaces are normal, there exist open disjoint subsets $O_1^*, O_2^*$ of $X$ with $A_i^* \subseteq O_i^*$. Moreover, $A_i^*$ and $X - O_i^*$ are now closed disjoint subsets of $X$ for each value of $i$, and since $X$ is compact, this implies $\delta(A_i^*, X - O_i^*) > 0$. Now let $O_i = \{x \in X : \delta(x, A_i^*) < M^{-1}\delta(A_i^*, X - O_i^*)\}$, $i = 1, 2$. Obviously each $O_i$ is open, and therefore $GO_i$ is likewise open in $X$. Since $A_i^*$ is invariant under $G$, one has $GO_i \subseteq O_i^*$, and consequently $GO_1, GO_2$ are disjoint. Since both sets are $G$-invariant, it follows that their projections $\pi(GO_1), \pi(GO_2)$ are likewise open and disjoint in $X/G$. But clearly $A_i^* \subseteq GO_i$, and therefore $A_i \subseteq \pi(GO_i)$, $i = 1, 2$; proving normality of $X/G$.

---

2 This means that $\delta(x, y) = 0$ does not imply $x = y$, although $\delta$ satisfies the remaining conditions for a metric.
Now let $f : A \to I$ be a $G$-invariant function. In accordance with our definition, $A$ is then closed and $G$-invariant, which makes $\pi(A)$ closed in $X/G$. By $G$-invariance $f$ may be factored through $X/G$, giving the diagram

![Diagram](attachment:diagram.png)

and one sees immediately that $h$ must be continuous. By normality of $X/G$, $h$ admits a continuous extension $h : X/G \to I$, and now $h \circ \pi$ gives a $G$-invariant extension of $f$. 
III. LOCALLY COMPACT SPACES

Theorem 2. Let $X$ be locally compact and $X/G$ Hausdorff or regular. Then every $G$-invariant $f : A \to I$ with $A$ compact admits a $G$-invariant extension to $X$.

Since the natural projection $\pi : X \to X/G$ is clearly open, $X/G$ is locally compact. Moreover, $A$ being closed and $G$-invariant, $\pi(A)$ will be closed. Compactness of $A$ also implies compactness of $\pi(A)$. On this account the assertion of Theorem 2 can be referred entirely to the orbit space, and it is easily seen that Theorem 2 is equivalent to the following result:

Theorem 2'. Let $Y$ be a locally compact space and $B$ a closed compact subset of $Y$. If $Y$ is Hausdorff or regular, then every continuous function $f : B \to I$ admits a continuous extension to $Y$.

Let $Y^*$ denote the one-point compactification of $Y$. If $Y$ is locally compact and Hausdorff, then $Y^*$ is Hausdorff (see [2], p. 150), and consequently $Y^*$ is then normal. We will show in a moment that $Y$ locally compact and regular again implies normality of $Y^*$. Assuming, then, that $Y^*$ is normal, we note that $B$, being closed and compact in $Y$, is closed in $Y^*$, so that $f : B \to I$ admits a continuous extension to $Y^*$, and therefore to $Y$.

It remains to prove that $Y^*$ is normal in the case where $Y$
is regular. Moreover, in virtue of the Tychonoff lemma, it will suffice to establish regularity of $Y^*$. To this end let $x \in Y^*$ and $O^*$ denote an open neighborhood of $x$ in $Y^*$. We consider first the case where $x \in Y$. Now $O^* \cap Y$ is an open neighborhood of $x$ in $Y$. But when $Y$ is locally compact and also Hausdorff or regular, the closed compact neighborhood system (see [2], p. 146). Consequently there exists a closed compact neighborhood $O$ of $x$ in $Y$ such that $O \subset O^*$. It follows that $O$ is a closed neighborhood of $x$ in $Y^*$, as was to be shown.

It remains to consider the case $x = \infty$. Letting $V$ denote the complement of $O^*$ in $Y^*$, we note that $V$ is now a closed compact subset of $Y$. By our preceding observation regarding closed compact neighborhoods, every point $y \in V$ admits such a neighborhood $U_y$ in $Y$. By compactness of $V$ one can select a finite family $\{U_i\}$ of these such that $\{U_i^O\}$ covers $V$, where $U_i^O$ denotes the interior of $U_i$. Let $U^O$ denote the union of the sets $U_i^O$ and $O$ the complement of $U^O$ in $Y^*$. It follows that $O$ is a closed neighborhood of $x$ in $Y^*$ with $O \subset O^*$. 
IV. PAIRWISE EXTENDIBLE FUNCTIONS

If \( Y \) is a topological space, \( B \) a subset of \( Y \) and \( f : B \to I \) a continuous function, we say that \( f \) is pairwise extendible if, for every pair \( (x, y) \in B \times B \), there exists a continuous function \( h : Y \to I \) such that \( h(x) = f(x) \) and \( h(y) = f(y) \). Moreover, if \( X \) and \( G \) are given as above and \( f : A \to I \) is continuous, \( f \) is said to be pairwise extendible if, for every \( (x, y) \in A \times A \), there exists a \( G \)-invariant \( h : X \to I \) which agrees with \( f \) on \( x \) and \( y \).

Theorem 3. If \( X \) is compact and \( f : A \to I \) is pairwise extendible with \( A \) closed in \( X \), then \( f \) admits a \( G \)-invariant extension to \( X \).

Again we see that the question may be referred to the orbit space, and that the given assertion is equivalent to

Theorem 3'. If \( Y \) is compact, \( B \) a closed subset of \( Y \) and \( f : B \to I \) pairwise extendible, then \( f \) admits an extension to \( Y \).

To prove this we consider the set \( R \) of all pairs \( (x, y) \in Y \times Y \) such that \( f(x) = f(y) \) for every continuous \( f : Y \to I \).

\(^3\)It suffices, in fact, to assume that \( B \) is compact. Obviously a similar observation applies to Theorem 3.
Clearly $R$ is an equivalence relation on $Y$, and we note that $Y/R$ is compact. To show that $Y/R$ is also Hausdorff, let $\bar{x}, \bar{y}$ denote distinct points in $Y/R$ and let $x, y$ denote representatives respectively, in $Y$. Since $(x, y) \notin R$, there exists a continuous $f: Y \to I$ such that $f(x) \neq f(y)$. Let $O_x, O_y$ denote disjoint open neighborhoods in $I$ of $f(x)$ and $f(y)$, respectively. We now observe that there exists a function $\bar{f}$ such that

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & Y/R \\
\downarrow{f} & & \downarrow{f} \\
I & \xrightarrow{\bar{f}} & I
\end{array}
\]

commutes ($\pi$ being the natural projection), and clearly $\bar{f}$ is continuous. Consequently $\bar{f}^{-1}(O_x)$ and $\bar{f}^{-1}(O_y)$ are disjoint open subsets of $Y/R$ containing $\bar{x}$ and $\bar{y}$, respectively, as was to be shown. Since $Y/R$ is compact and Hausdorff, it is normal.

We show next that $\pi: Y \to Y/R$ is closed. For if $A \to Y$ is closed, then $A$ is compact, and therefore $\pi(A)$ is compact. But since $Y/R$ is Hausdorff, this implies that $\pi(A)$ is closed.

Now consider $f: B \to I$ and let $\bar{B} = \pi(B)$. Our assumption that $f$ be pairwise extendible implies that there exists a function $\bar{f}: \bar{B} \to I$ such that $\bar{f} = f \circ \pi$, and one sees again that $\bar{f}$ must be continuous. But since $\bar{B}$ is closed and $Y/R$ normal, $\bar{f}$ admits a continuous extension $\bar{h}: Y/R \to I$, and now $\bar{h} \circ \pi$ gives an extension of $f$. 
V. THE NORMALIZING TOPOLOGY

Theorem 4. Let \( X \) be compact. There exists then a unique topology \( T \) on \( X \) such that

(i) for all \( f : X \to I \), \( f \) is \( T \)-continuous if and only if \( f \) is \( G \)-invariant\(^4\);

(ii) \( X \) is \( T \)-normal;

(iii) \( T \) is minimal with respect to (i) and (ii).

We first establish existence of \( T \). Let \( R \) denote the equivalence relation on \( X \) consisting of all pairs \((x, y) \in X \times X\) such that \( f(x) = f(y) \) for all \( G \)-invariant \( f \). It follows by the considerations of 4 that \( X/R \) is normal. We now define a subset \( B \subset X \) to be \( T \)-open if it is the inverse image under the natural projection \( \pi : X \to X/R \) of an open set in \( X/R \). It is immediately verified that this gives a topology \( T \) on \( X \). Moreover, a subset \( A \subset X \) is \( T \)-closed if and only if it is the inverse image under \( \pi \) of a closed set in \( X/R \). We proceed to verify property (i) of the theorem.

Let \( f : X \to I \) be \( G \)-invariant. Then there exists (a unique) \( \bar{f} \) such that

---

\(^4\)Here \( G \)-invariance is understood precisely as defined in §1. In particular, this implies continuity with respect to the original topology on \( X \).
commutes, and one sees again that $\bar{f}$ is continuous. Now if $U \subseteq I$ is open, $f^{-1}(O)$ is the inverse image under $\pi$ of $\bar{f}^{-1}(O)$, which is open in $X/R$, and consequently $f^{-1}(O)$ is $T$-open. Hence every $G$-invariant $f : X \rightarrow I$ is $T$-continuous. Conversely, let $f : X \rightarrow I$ be $T$-continuous. Since every $T$-open set is open, $f$ is certainly continuous. To establish $G$-invariance of $f$, we must show that $f$ factors through $X/R$. To this end consider $(x, y) \in R$. Since $f$ is $T$-continuous, $f^{-1}(f(x))$ is $T$-closed, and since $\pi(x) = \pi(y)$, this implies $y \in f^{-1}(f(x))$. Therefore $f(x) = f(y)$, as was to be shown.

It is easy to see that $X$ is $T$-normal. For if $A_1, A_2$ are disjoint $T$-closed sets in $X$, then each $\overline{A_i}$ is the inverse image under $\pi$ of a closed set $\overline{A_i}$ in $X/R$, and $\overline{A_1}, \overline{A_2}$ are again disjoint. Now if $O_1, O_2$ are separating neighborhoods of $A_1, A_2$; their inverse images under $\pi$ will be separating neighborhoods of $A_1, A_2$ with respect to the topology $T$.

Let $T$ denote a second topology on $X$ satisfying conditions (i) and (ii). To complete the proof it will suffice to show that
every $\tau$-closed $A \subseteq X$ is also $\tau^*$-closed. To this end let $B$ denote the complement of $A$ in $X$, and let $X \in B$. We recall now that $X/R$ is also Hausdorff, so that $[\overline{x}]$ is closed, where $\overline{x} = \pi(x)$. Consequently $\pi^{-1}(x)$ is $\tau$-closed and clearly disjoint from $A$. Since $X$ is $\tau$-normal, this implies existence of a $\tau$-continuous $f : X \to I$ such that $f$ takes the value 0 on $A$ and 1 on $\pi^{-1}(x)$. But since $\tau$ and $\tau^*$ both satisfy condition (i), it follows that $f$ is also $\tau^*$-continuous. Therefore, if $J$ denotes the open subset $[0, 1] \subseteq I$, $f^{-1}(J)$ will be a $\tau^*$-neighborhood of $x$ contained in $B$. Consequently $B$ is $\tau^*$-open, as was to be proved.

Thus every compact topological space $X$ with operators has an associated topology $\tau$ defined by Theorem 4, and we will refer to this as the normalizing topology of $X$. The concept of a normalizing topology $\tau$ is obviously meaningful for a compact topological space $Y$ (without operators) as well, in which case $\tau$ is just the minimal normal topology on $Y$ such that $f : Y \to I$ is $\tau$-continuous if and only if $f$ is continuous. If $X$ is a compact space with operator group $G$, it is clear that its normalizing topology $\tau$ is induced from the normalizing topology $\overline{\tau}$ of its orbit space $X/G$ in the following sense: a set $B \subseteq X$ is $\tau$-open if and only if $B = \pi^{-1}(\overline{B})$, where $\overline{B}$ $X/G$ is $\overline{\tau}$-open and $\pi : X \to X/G$ denotes the natural projection.
We now extend the definition of normalizing topology to noncompact spaces as follows: If $Y$ is a noncompact space, let $Y^*$ denote its one-point compactification and $\tau^*$ the normalizing topology of $Y^*$. The normalizing topology of $Y$ is defined to be its relative topology with respect to $\tau^*$. If $X$ is a noncompact space with operator group $G$, its normalizing topology is the topology induced from the normalizing topology of $X/G$. 
VI. INDUCTIVE CHARACTERIZATION OF R

Given a topological space $Y$, we will construct an increasing sequence $\{R_a\}$ of equivalence relations on $Y$ as follows: $R_0$ shall be the diagonal in $Y \times Y$. Now let $\gamma$ be an ordinal and suppose that $R_a$ has been defined for all $a < \gamma$, subject to the conditions that $R_a \subseteq R_\beta$ when $a < \beta < \gamma$. If $\gamma$ is a limit ordinal, we define

\[
R_\gamma = \bigcup_{a<\gamma} R_a
\]

Obviously this gives an equivalence relation on $Y$ and perpetuates the monotonicity condition. Let us suppose then, that $\gamma$ is not a limit ordinal, so that $\gamma = a + 1$ for some ordinal $a$. We set

$Y_a = Y/R_a$ and let $\overline{R}$ denote that equivalence relation on $Y_a$ generated by the set $\rho$ of all pairs $(x, y) \in Y_a \times Y_a$ such that for all neighborhoods $O_x, O_y$ of $x$ and $y$, respectively, $O_x$ meets $O_y$. Now $R_\gamma$ is defined to be the set of all $(x, y) \in Y \times Y$ such that $\pi(x) = \pi(y)$, where $\pi: Y \to Y_a/\overline{\rho}$ denotes the natural projection. Clearly $R_\gamma$ is an equivalence relation on $Y$ and $R_a \subseteq R_\gamma$. It follows by transfinite induction that $R_a$ is well-defined for all ordinals $a$.

Lemma 1. There exists an ordinal $\lambda$ such that $R_\lambda = R_a$.
for all \( a > \lambda \).

We note, in the first place, that it obviously suffices to show \( R_\lambda = R_{\lambda+1} \) for some \( \lambda \). Let \( C \) denote the cardinality of \( Y \), \( a \) the initial ordinal of \( C \) and \( \beta \) the initial ordinal of \( 2^C \). We will prove that \( R_\lambda = R_{\lambda+1} \) for some ordinal \( \lambda \leq \beta \). Let us suppose, then, that \( R_\lambda \neq R_{\lambda+1} \) for all \( \lambda \leq \beta \), and let \( Y \) be well-ordered. For every ordinal \( a \) we can now define a map \( \Phi_a : Y/R_a \rightarrow Y \) by taking \( \Phi_a(z) \) to be the minimal element in \( z \) for every \( z \in Y/R_a \). Let \( Y^a \subset Y \) denote the image of \( Y/R_a \) by \( \Phi_a \). Since \( R_a \subset R_{a+1} \), one has \( Y^{a+1} \subset Y^a \). Moreover, for \( \lambda \leq \beta \) our assumption \( R_\lambda \neq R_{\lambda+1} \) implies \( Y^{\lambda+1} \neq Y^\lambda \), and we will let \( y_\lambda \) denote the minimal element belonging to the complement of \( Y^{\lambda+1} \) in \( Y^\lambda \). The correspondence \( \lambda \mapsto y_\lambda \) then defines an injection of \( B \) into \( Y \), where \( B \) denotes the set of all ordinals \( \lambda \leq \beta \). But since \( B \) has cardinality \( 2^C \), this is a contradiction.

In virtue of Lemma 1 we can associate with every topological space \( Y \) the minimal ordinal \( \lambda \) such that \( R_\lambda \) has the desired property. This \( \lambda \) will be called the length of \( Y \).

**Lemma 2.** Let \( Y \) be a topological space of length \( \lambda \). Then \( R_\lambda \) is the smallest equivalence relation \( R \) on \( Y \) such that \( Y/R \) is Hausdorff.
Given such a relation $R$, we prove that $\forall a \in \mathbb{R}$ for every $a$. We suppose, then, that $\gamma$ is an ordinal and $\forall a \in \mathbb{R}$ for every $a < \gamma$. If $\gamma$ is a limit ordinal, $R_\gamma$ is given by (1), and this clearly implies $\forall a \in \mathbb{R}$. Otherwise $\gamma = a + 1$, and we let $Y_\alpha, \rho$ and $\bar{\rho}$ be defined as above. Now consider points $x, y \in Y$, and let $x_\alpha, y_\alpha \in Y$ and $\bar{x}, \bar{y} \in Y/R$ denote their respective projections. To show that $\forall a \in \mathbb{R}$, it clearly suffices to show that $(x, y) \notin R$ implies $(x_\alpha, y_\alpha) \notin \rho$. Suppose then that $(x, y) \notin R$. Then $\bar{x} \neq \bar{y}$, and since $Y/R$ is Hausdorff, the points $\bar{x}, \bar{y}$ admit separating neighborhoods $\overline{O}_x, \overline{O}_y$ in $Y/R$. But by our inductive hypothesis, $\forall a \in \mathbb{R}$, so that $Y/R$ is a quotient space of $Y$. Consequently $\overline{O}_x, \overline{O}_y$ induce separating neighborhoods of $x_\alpha, y_\alpha$ in $Y$, which implies $(x_\alpha, y_\alpha) \notin \rho$. Thus again $\forall a \in \mathbb{R}$. Since $\forall a \in \mathbb{R}$, it follows now by transfinite induction that $\forall a \in \mathbb{R}$ for every ordinal $a$.

Now let $R$ denote the equivalence relation on $Y$ consisting of all pairs $(x, y) \in Y \times Y$ such that $f(x) = f(y)$ for all continuous $f : Y \to \mathbb{R}$. We shall require the following results:

**Lemma 3.** Let $Y$ be compact and $R'$ an equivalence relation on $Y$ such that $Y/R'$ is Hausdorff. Then $R \subseteq R'$.

To show this, we consider $(x, y) \in Y \times Y$ and suppose $(x, y) \notin R'$. Then $x, y$ correspond to distinct points $\bar{x}, \bar{y}$ in
$Y/\sim$. Since $Y/\sim$ is compact and Hausdorff, it is also normal, and consequently there exists a continuous $\overline{f}: Y/\sim \to I$ such that $\overline{f}(x) = 0$ and $\overline{f}(y) = 1$. Precomposing $\overline{f}$ with the natural projection yields a function $f: Y \to I$ which separates $x$ and $y$, so that $(x, y) \not\in R$.

As an immediate consequence we obtain the desired inductive characterization of $R$ for compact spaces.

**Lemma 4.** If $Y$ is compact, then $R_\lambda = R$.

To show this, we recall (see §4) that $Y/R$ is Hausdorff when $Y$ is compact, so that $R_\lambda \subseteq R$ by Lemma 2. On the other hand, $R \subseteq R_\lambda$ by Lemma 3.

Before proceeding it may be of interest to show by examples that orbit spaces with various lengths $\lambda$ do in fact occur. More specifically, we shall indicate how, for a given denumerable ordinal $\lambda$, one can define a continuous action$^5$ of the additive group $G_1$ of real numbers on the 2-dimensional torus $T^2$ so that $T^2/G_1$ has length $\lambda$. As our starting point we consider the following example, supplied by Professor Gleason: On the real plane $E^2$ we define a vector field whose components at the point $(x, y)$ are $(1, \sin y)$. This generates an action of $G_1$ on $E^2$. Since the vector field

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$^5$With sufficient care the construction will actually give a differentiable action of $G_1$. 

commutes with translation in the \( x \)-direction, this gives also an action of \( G_1 \) on the cylinder \( X^2 \) obtained by identifying points differing by an integer in the \( x \)-coordinate. We note in passing that the subset \( A \subset X^2 \) consisting of the two circles \( y = \pm \pi \) is closed and invariant under \( G_1 \), and that a function \( f : A \rightarrow I \) which assumes distinct constant values on the two circles is \( G_1 \)-invariant, but not extendible to a \( G_1 \)-invariant function \( f : X^2 \rightarrow I \).

Now let \( X \) denote the subset of \( X^2 \) given by \( 0 \leq y \leq \pi \). Clearly \( G_1 \) operates on \( X \), and one sees that the orbit space \( Y = X/G_1 \) is non-Hausdorff. On the other hand, the first quotient space \( Y_1 \) obtained by identifying nonseparated points in \( Y \) reduces to a single point, so that \( Y \) has length \( 1 \).

Let \( \alpha \) be an ordinal and suppose there is given an action of \( G_1 \) on \( X \) such that (i) the orbit space \( Y \) has length \( \alpha \); (ii) \( Y/R_\alpha \) reduces to a point; (iii) the circles \( y = 0, \pi \) constitute orbits with unit velocity. We can then define a new action of \( G_1 \) on \( X \) having properties (i), (ii) and (iii) with \( \alpha \) replaced by \( \alpha + 1 \). To this end we subdivide \( X \) into an infinite sequence of closed overlapping bands given by \( \frac{1}{n+1} \leq y \leq \frac{1}{n} \), \( n = 1, 2, \ldots \); and let \( G_1 \) operate on the \( n \)-th band by identifying this band with \( X \) in the obvious way and applying the given action of \( G_1 \). We note that this defines an action of \( G_1 \) on the region \( 0 < y \leq \pi \) in virtue of condition (iii), and that this action extends
to \( X \) so as to satisfy condition (iii). Under this new action \( Y/R_a \) reduces to a pair of points with indiscrete topology, and \( Y \) has thus length \( a + 1 \). Obviously condition (ii) is likewise propagated.

By finite induction one obtains now an action of \( G_1 \) on \( X \) with \( X/G_1 \) of arbitrary positive length. Let \( \omega \) denote the smallest countable ordinal. We obtain now an action of \( G_1 \) on \( X \) satisfying (i), (ii) and (iii) with respect to \( \omega + 1 \) by the following variation of the previous construction: Subdivide \( X \) into bands just as before, but now define an action of \( G_1 \) on the \( n \)th band by means of the given action of \( G_1 \) on \( X \) corresponding to length \( n \). It is not difficult to see that this gives the desired action of \( G_1 \) on \( X \). It is not difficult to show, by a slight modification of the previous construction, that we can construct examples with \( X/G_1 \) of arbitrary denumerable length.\(^6\)

To obtain corresponding examples for actions of \( G_1 \) on the torus \( T^2 \) we may proceed as follows: \(^7\) Let \( X' \) denote the subset of the cylinder \( X^2 \) given by \( 0 \leq y \leq 2\pi \), and take \( T^2 \) to be the quotient of \( X' \) under the identification \((x, 0) \sim (x, 2\pi)\). We can now extend each previously defined action of \( G_1 \) on \( X \) to \( X' \) such that the circles of constant \( y \) with \( \pi \leq y \leq 2\pi \) constitute

\(^6\)When \( a \) is a limit ordinal, the topology of \( X \) should be changed by identifying \((x, 0)\) with \((x, \pi)\).

\(^7\)When \( a \) is a limit ordinal, see 6.
orbits with unit velocity, and this induces an action of $G_1$ on $T^2$.

It is not difficult to see that the orbit spaces $X/G_1$ and $T^2/G_1$

have equal length.
VII. LOCALLY EXTENDIBLE FUNCTIONS

We return to the extension problem and consider a compact space $X$ with operator group $G$ and normalizing topology $T$. It might seem reasonable to suppose that if $f: A \to I$ is $G$-invariant and $A$ $T$-closed, then $f$ admits a $G$-invariant extension to $X$. But this conjecture is false. We shall presently show by an example that the conjecture remains false even when $X$ is assumed to be normal. As a matter of fact, normality of $X$ seems to become quite irrelevant in the case of noncompact $G$. What is needed is some condition on $f$ which assures that $f$ can be factored through $X/R$, this being a quotient of the orbit space. We have seen before that pairwise extendibility is a condition of this kind. We will now give a second condition which has the advantage of being somewhat more local. More precisely, the condition is local with respect to the normalizing topology, and this is as far as one can go.

**Theorem 5.** Let $X$ be compact, $A$ a compact subset of $X$ and $f: A \to I$ continuous. If $f$ is $G$-invariantly extendible to a $T$-neighborhood of $A$, then $f$ is $G$-invariantly extendible to $X$.

The assertion of this theorem may be referred to the orbit space and thus becomes equivalent to
Theorem 5'. Let \( Y \) be compact, \( A \) a compact subset of \( Y \) and \( f: A \to I \) continuous. If \( f \) admits an extension to a \( T \)-neighborhood of \( A \), then \( f \) is \( G \)-invariantly extendible to \( X \).

The crux of the matter is contained in the following results.

Lemma 5. Let \( B \) denote a \( T \)-open subset of \( Y \) and \( R' \) the equivalence relation on \( B \) consisting of all pairs \( (x, y) \in B \times B \) such that \( h(x) = h(y) \) for all continuous \( h: B \to I \). Then \( R' = R|B^8 \).

The inclusion \( R' \subseteq R|B \) is of course obvious. To establish the opposite inclusion, on the other hand, we require Lemma 4 and must proceed inductively. Clearly \( R_0|B \subseteq R' \), since \( R_0 \) is just the diagonal in \( Y \times Y \). Let us suppose, then, that \( R_a|B \subseteq R' \) for all \( a < \gamma \). If \( \gamma \) is a limit ordinal, then \( R_\gamma \) is defined by Equation (1), §6; and consequently one obtains \( R_\gamma|B \subseteq R' \). It remains to examine the case where \( \gamma = a + 1 \) for some ordinal \( a \).

To this end we consider the diagram

\[
\begin{array}{ccc}
Y & \to & Y/R_a \\
\downarrow^{\pi_a} & & \downarrow^{\pi_a} \\
Y/R & \to & Y/R_a \\
\end{array}
\]

\( R|B = R \cap [B \times B] \), and \( R \) is the equivalence relation on \( Y \) previously considered (see §§4, 5, 6).
where all maps are natural projections, and we let \( \overline{B} = \pi_a(B) \).

Since \( B \) is \( \tau \)-open, it is invariant under \( \pi^{-1} \circ \pi \), which implies that \( \overline{B} \) is open in \( Y/R_a \). Given \( (x, y) \in R_{a+1} \cap [\overline{B} \times B] \), there exists a finite sequence of points \( x_1, \ldots, x_n \in Y \) with \( x_1 = x \), \( x_n = y \); such that \( \pi_a(x_i), \pi_a(x_{i+1}) \) are nonseparated in \( Y/R_a \) for \( i < n \). Moreover, since \( B \) is \( \tau \)-open, each \( x_i \in B \). The points \( \pi_a(x_i) \) belong therefore to \( \overline{B} \), and since \( \overline{B} \) is open, they are pairwise nonseparated in \( \overline{B} \). On the other hand, by our inductive hypothesis that \( R_a \big| B \subseteq R' \), every continuous \( h : B \rightarrow I \) can be factored through \( \overline{B} \), which now implies that \( h(x_i) = h(x_{i+1}) \) for \( i < n \). Consequently \( (x, y) \in R' \), showing that \( R_{a+1} \big| B \subseteq R' \).

We may conclude by transfinite induction that \( R_a \big| B \subseteq R' \) for every ordinal \( a \). In virtue of Lemma 4, this implies that \( R \big| B \subseteq R' \), as was to be shown.

To prove Theorem 5' we note that \( A \) projects to a compact set \( \overline{A} \) in \( Y/R \), and since \( Y/R \) is Hausdorff, \( \overline{A} \) is also closed. By hypothesis, \( f \) may be extended to a \( \tau \)-open neighborhood \( B \) of \( A \), which projects to an open neighborhood \( \overline{B} \) of \( \overline{A} \). In virtue of Lemma 5, \( f : B \rightarrow I \) may be factored through \( \overline{B} \), and consequently \( f : A \rightarrow I \) may be factored through \( \overline{A} \). Since \( Y/R \) is normal, this implies that \( f \) admits an extension to \( Y \).

It can be seen by Gleason's example that local extendibility with respect to the original topology of \( X \) does not guarantee global
extendibility. The following example shows likewise that our condition of local extendibility cannot be replaced by the condition that \( A \) be \( T \)-closed. In the real plane \( E^2 \) we consider the closed line segments \( x_n = [0, \pi] x \left[ \frac{1}{2^n} \right] \), \( n \) being a positive integer, together with \( x_0 = [0, \pi] x 0 \), and we let \( x = \bigcup_{n=0}^{\infty} x_n \). Let \( X \subset E^2 \) be endowed with the relative topology. Since \( X \) is closed and bounded in \( E^2 \), it is compact, and since \( X \) is metric, it is also normal. Noe consider each point \( (x, y) \in x_n \) to be subject to a motion in the \( x \)-direction with velocity \( v = \sin 2^n x \) for \( n > 0 \) and \( v = 0 \) for \( n = 0 \). It is not difficult to verify that this defines a continuous action of \( G_1 \) on \( X \), where \( G_1 \) again denotes the additive group of real numbers. Moreover, one sees that each \( X_n \) is an \( R \)-equivalent class. Consequently \( X_0 \) is \( T \)-closed. The function \( f: X_0 \to I \) defined by \( f(x, 0) = \frac{1}{\pi} x \) is therefore \( G \)-invariant, but clearly it is not \( G \)-invariantly extendible to \( X \).

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9The fact that \( X \) in Gleason's example (see §6) is noncompact is not essential. By a simple modification we can obtain a corresponding action of \( G_1 \) on the torus which shows that local extendibility in the ordinary topology is insufficient also for compact \( X \).
We have given three extension theorems\(^{10}\) for compact \(X\). The first of these depends upon a pseudo-metric structure on \(X\) and does not admit an obvious generalization to the noncompact case. The other extension theorems, however, do admit such a generalization in virtue of the fact that \(X\) can be embedded in the one-point compactification \(X^*\).

Thus if \(X\) is noncompact, we can strengthen the definition of pairwise extendibility in an obvious way: A function \(f:A \rightarrow I\) will be called pairwise extendible if, given \((x, y) \in A \times A\), there exists an \(h:X^* \rightarrow I\) such that \(h|X\) is \(G\)-invariant and \(h\) agrees with \(f\) on \(x\) and \(y\). It is now easy to see that Theorem 3 admits the following generalization:

**Theorem 3*. Let \(X\) be a topological space with operator group \(G\), \(A\) a compact subset of \(X\) and \(f:A \rightarrow I\) pairwise extendible. Then \(f\) admits a \(G\)-invariant extension \(f:X^* \rightarrow I\).**

A similar consideration applied to Theorem 5 gives

**Theorem 5*. Let \(X\) be a topological space with operator group \(G\), \(A\) a compact subset of \(X\) and \(f:A \rightarrow I\) continuous. If \(f\) is \(G\)-invariantly extendible to a \(T\)-neighborhood of \(A\),

\(^{10}\) Theorems 1, 3 and 5.
then \( f \) is \( G \)-invariantly extendible to \( X \).

Thus it turns out that Theorem 5 remains valid after one has dropped the compactness assumption on \( X \). On the other hand, the compactness condition on \( A \) now carries more weight. We will show by an example that compactness of \( A \) is indeed essential for Theorems 3* and 5* alike.

To this end we take \( X \) to be a subspace of the real plane \( \mathbb{E}^2 \) obtained by deleting a punctured circle. More precisely, we delete the circle \( x^2 + y^2 = 1 \), except for the point \( P = (-1,0) \). Let \( v \) denote the vector field on \( \mathbb{E}^2 \) given by

\[
v(r, \theta) = r \sin \theta \frac{\partial}{\partial \theta}
\]

where \( (r, \theta) \) denote polar coordinates. This defines an action of \( G_1 \) (the additive group of real numbers) on \( \mathbb{E}^2 \), and since \( X \) is invariant under \( G_1 \), it defines an action of \( G_1 \) on \( X \). We note that points on the \( x \)-axis are fixed, while the orbits of points off the \( x \)-axis are open semi-circles.

Now let

\[
A = \{(x, 0) \mid 0 \leq x \leq 2\} \cap X
\]

and define \( f : A \rightarrow \mathbb{I} \) by

\[
(2) \quad f(x, 0) = \begin{cases} 1 & \text{for } x > 1 \\ 0 & \text{for } x < 1 \end{cases}
\]
Clearly $A$ is closed and $G_1$-invariant in $X$ and $f$ is continuous.

We claim (i) $f$ is pairwise extendible; (ii) $f$ is $G_1$-invariantly extendible to a $\sim$-neighborhood of $A$; and (iii) $f$ does not admit a $G_1$-invariant extension $f: X \to I$. To show (i), we consider two points $(x_1, 0), (x_2, 0) \in A$ with $x_1 < 1$ and $x_2 < 1$, and define $h: X^* \to I$ by

$$h(r, 0) = \begin{cases} 
1 & \text{for } r > 1 \\
0 & \text{for } r < x_1 \\
\frac{r - x_1}{1 - x_1} & \text{for } x_1 < r < 1
\end{cases}$$

with $h(\infty) = 1$. Clearly $h$ has the requisite properties, and consequently $f$ is pairwise extendible. Moreover, one easily verifies that the complement $B$ of $P$ in $X$ is a $\sim$-neighborhood of $A$.

But obviously $f$ can be $G_1$-invariantly extended to $B$ by (2).

Lastly, it is clear that $f$ does not admit a $G_1$-invariant extension to $X$. 
BIBLIOGRAPHY


APPENDIX

**Theorem:** If $X$ is normal and $G$ compact, then $X/G$ is normal.

For let $\overline{A_1}, \overline{A_2}$ denote disjoint closed subsets of $X/G$ and let $A_1, A_2$ denote their inverse images, respectively, in $X$. By normality of $X$, $A_1$ and $A_2$ admit open separating neighborhoods $O_{1i}$ and $O_{2i}$, and we let $O'_i$ denote the set of all $x \in X$ such that $Gx \subseteq O_{1i}$, $i = 1, 2$. Clearly $A_i \subseteq O'_i$ for each $i$. Moreover, $O'_i$ is open. For given $x \in O'_i$ and $g \in G$, then $gx \in O_{1i}$, and there exist open neighborhoods $G_g \subseteq G$ and $O_g \subseteq X$ of $g$ and $x$, respectively, such that $G_g O_g \subseteq O_{1i}$. Now $\{G_g : g \in G\}$ gives an open covering of $G$, and since $G$ is compact, $\{G_g\}$ covers $G$ for some finite sequence $\{g_j\}$ of elements in $G$. Let $O_x$ denote the intersection of all neighborhoods $O_{g_j}$. Then $O_x$ is an open neighborhood of $x$ with $G O_x \subseteq O_{1i}$. Hence $O_x \subseteq O'_{1i}$, so that $O'_{1i}$ is indeed open. Since $O'_{1i}$ is also invariant under $G$, it projects to an open neighborhood $\overline{O'_{1i}}$ of $A$. But $\overline{O'_{1i}}$, $\overline{O'_{2i}}$ are clearly disjoint, as was to be proved.