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The problem of determining a function from its integrals over k-planes, $0 < k < n$, is studied. It is shown that when $k < n/2$, the x-ray transform of an arbitrary square integrable function is in a certain L^p space on a fiber bundle over the Grassmann manifold $G_{n,k}$. If $k \geq n/2$, the x-ray transform of a square integrable function may not exist. A characterization of the range of the transform is given and a generalization of a theorem of Ludwig on the Radon transform is proved. Finally a relationship between a theorem on the null space and a particular iterative scheme that is being used to detect brain tumors is discussed.

The X-Ray Transform

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THE X-RAY TRANSFORM

INTRODUCTION

The problem of characterizing a function on R^n, $n \geq 2$, by means of its integrals over k-dimensional planes, $0 < k < n$, is not a new one. Radon [12] and John [8] proved that a differentiable function f with compact support in R^n is uniquely determined by means of its integrals over the hyperplanes in the space. The Radon transform is defined by

$$R_\theta f(t) = \int_{\langle x, \theta\rangle = t} f(x)\, d\alpha_{n-1}(x)$$

where $\theta \in S^{n-1}$, $t \in R^1$, and α_{n-1} is the (n-1)-dimensional surface area measure in R^n. Considerable literature on the Radon transform exists.

The lower dimensional problem, $k \leq n-2$, has received less attention. However, Helgason [6] and [7], has obtained some interesting results.

In order to motivate the lower dimensional case, consider the following situation. An object in 3-dimensional space is determined by a density function f on the space R^3, $f(x)$ being the density at the point x. An x-ray picture taken in the direction θ provides a function $L_\theta f$ on the subspace orthogonal to θ whose value at a point x of this subspace is the total mass along the line through x in the direction θ. The problem is to recover a density function f from its x-rays $L_\theta f$.

If we restrict ourselves to a 2-dimensional cross section of the 3-dimensional object then the problem is exactly the Radon problem. However, if we wish to reconstruct the object without going to cross sections, the problem is one of integration over lines in R^3 and the Radon theory does not apply.

The above x-ray reconstruction problem has been the object of a great deal of recent mathematical and practical research. For example, see [4] where a bibliography of recent results is given. Within the past year R. Guenther and K. T. Smith [5], have detected brain tumors in a human patient by reconstructing cross sections of the skull from ordinary x-ray data. The research in this thesis stems from the reconstruction problem and was suggested by K. T. Smith.

Let us put the problem in a more abstract setting. If θ is a direction in R^n, $n \geq 2$, x is a point in the (n-1)-dimensional subspace orthogonal to θ and f is an integrable function, then the ordinary x-ray of the object with density function f in the direction θ at x is given by

$$L_\theta f(x) = \int_{-\infty}^{\infty} f(x+t\theta)dt ,$$

provided that the integral exists in the Lebesgue sense. Note that the x-ray of a function f depends on the variables (θ, x) with $\theta \in S^{n-1}$ and $x \in \theta^\perp$. Thus the x-ray of f is a function on the tangent bundle $T(S^{n-1})$ to the sphere.

More generally let π be a k-dimensional subspace of R^n. The x-ray of f in the direction π at the point x'' in $\pi^\perp$ is defined by

$$L_\pi f(x'') = \int_\pi f(x', x'')dx' ,$$

provided that the integral exists in the Lebesgue sense. Here, and in general, once a subspace π is fixed, we write $x = (x', x'')$ where x' and x'' are the orthogonal projections of x on π and $\pi^\perp$. The k-dimensional subspaces of R^n form the Grassmann manifold $G_{n,k}$. The x-ray of f is a function $Lf(\pi, x'')$ on a fiber bundle $T(G_{n,k})$ with base space $G_{n,k}$ and fibers isomorphic to R^{n-k}. When $k = 1$ or $k = n-1$, $G_{n,k} = S^{n-1}$.

Note when $k = n-1$, the x-ray transform is essentially the Radon transform. In fact

$$R_\theta f(t) = L_{\theta^\perp} f(t\theta) .$$

Thus in this case all results on the Radon transform carry over with appropriate change of notation.

The central problem of this thesis is the study of the relationship between a function f and the transformed function Lf. Although one might expect the results to depend somewhat on the magnitude of k there is a surprising difference in the results when $k < n/2$. Indeed, we prove that any square integrable function on R^n is actually integrable over almost every translate of almost every k-space, $k < n/2$, while an example is given of a square integrable function on R^n which is not integrable over any k-plane, $k \geq n/2$. We also show that when $k < n/2$, the x-ray transformation has a natural extension as a closed operator from a domain $D_k \subset L^2(R^n)$ into a certain L^2 space on the fiber bundle $T(G_{n,k})$. Again this is shown to be impossible when $k \geq n/2$. An inversion formula is given for all k.

A characterization of the range of the Radon transform was given by Ludwig [10]. We do the same here for the x-ray transform, giving necessary and sufficient conditions that a function on the fiber bundle $T(G_{n,k})$ be the x-ray of a square integrable function with support in a given compact convex set K. Our proof fails when $k = n-1$, but the result holds from the work of Ludwig. However, we prove more. Indeed, we show that it is possible, in some cases, to get inside the convex hull of the support of f from the knowledge of the x-rays when $k \leq n-2$. Finally, a characterization of the null space of L_π is given and a few remarks are made on the relationship of this theorem to the particular iterative scheme that is being used to detect brain tumors.

1. DEFINITION AND FUNDAMENTAL PROPERTIES OF THE X-RAY TRANSFORMATION

If π is a k-dimensional subspace of R^n the x-ray of the integrable function f in the direction π at the point x'' in $\pi^{\perp}$ is defined by

$$(1.1)\qquad L_{\pi}f(x'') = \int_{\pi} f(x', x'')dx' ,$$

provided that the integral exists in the Lebesgue sense. Here, and in general, once a subspace π is fixed, we write $x = (x', x'')$ where x' and x'' are the orthogonal projections of x on π and $\pi^{\perp}$. The k-dimensional subspaces of R^n form the Grassmann manifold $G_{n,k}$. The x-ray of f is a function on a fiber bundle $T(G_{n,k})$ with base space $G_{n,k}$ and fibers isomorphic to R^{n-k}. For $k = 1$ or $k = n-1$, $G_{n,k} = S^{n-1}$.

<u>Lemma (1.2). If f is integrable on R^n, then for each k-space π, $L_{\pi}f$ is an integrable function on $\pi^{\perp}$ and</u>

$$\| L_{\pi}f \|_{L^1(\pi^{\perp})} \le \| f \|_{L^1(R^n)} .$$

<u>Proof.</u> The proof is an immediate application of Fubini's theorem. From (1.1) we have

$$\|L_\pi f\|_{L^1(\pi^\perp)} \leq \int_{\pi^\perp}\int_\pi |f(x', x'')|\,dx'dx'' = \|f\|_{L^1(R^n)} .$$

<u>Lemma (1.3). If ρ is a locally integrable function of one variable and for fixed $y'' \in \pi^\perp$, $\rho(<x'', y''>)L_\pi|f|(x'')$ is integrable on $\pi^\perp$, then</u>

$$(1.4) \qquad \int_{\pi^\perp} \rho(<x'', y''>)L_\pi f(x'')dx'' = \int_{R^n} \rho(<x, y''>)f(x)dx.$$

<u>Proof</u>. Fubini's theorem gives

$$\int_{\pi^\perp} \rho(<x'', y''>)L_\pi f(x'')dx'' = \int_{\pi^\perp}\int_\pi \rho(<x, y''>)f(x',x'')dx'dx''$$

$$= \int_{R^n} \rho(<x, y''>)f(x)dx.$$

The Fourier transform of an integrable function on R^n is given by

$$\hat{f}(\xi) = (2\pi)^{-n/2}\int_{R^n} e^{-i<x, \xi>} f(x)dx.$$

Thus if $g \in L^1(\pi^\perp)$, where π is a k-space, it is natural to define

$$\hat{g}(\xi) = (2\pi)^{(k-n)/2}\int_{\pi^\perp} e^{-i<x, \xi>} g(x)dx, \quad \underline{\text{for}} \quad \xi \in \pi^\perp .$$

From Lemmas (1.2) and (1.3) we derive immediately a relationship between the Fourier transform of f and the Fourier transform of $L_\pi f$.

Lemma (1.5). For each k-space π and integrable function f,

$$(L_\pi f)^\wedge(\xi'') = (2\pi)^{k/2}\hat{f}(\xi'') \quad \text{for} \quad \xi'' \in \pi^\perp .$$

Proof. The proof is immediate if one takes $\rho(t) = e^{-it}$ in Lemma (1.2).

If V is any $(k+1)$ dimensional subspace of R^n and ξ is an arbitrary point in R^n, then $\xi \in \pi^\perp$ for some k-space π contained in V. Lemma (1.5) tells how to compute $\hat{f}(\xi)$ from $L_\pi f$. Since $f \in L^1(R^n)$ is uniquely determined by its Fourier transform, f is uniquely determined by its x-rays in the directions $\pi \subset V$. Thus we have proved the following result, $G_{n,k}(V)$ being the submainfold of $G_{n,k}$ consisting of the k-spaces contained in V.

Corollary (1.6). For any $(k+1)$ dimensional subspace V of R^n, an integrable function f on R^n is uniquely determined by its x-rays in the directions $G_{n,k}(V)$.

Note that if $k = 1$, then $G_{n,k}(V)$ is a great circle on the sphere S^{n-1}.

The preceding result can be improved upon when f has compact support. First a well known algebraic result is needed.

Lemma (1.7). The vector space of homogeneous polynomials of degree m in R^n is spanned by those polynomials of the form $\langle a, x\rangle^m$.

Proof. The monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with

$$|\alpha| = \sum_{i=1}^{m} \alpha_i = m$$

form a basis for the homogeneous polynomials of degree m. A linear functional A on this vector space can be represented by

$$A\Big(\sum_{|\alpha|=m} a_\alpha x^\alpha\Big) = \sum_{|\alpha|=m} b_\alpha a_\alpha , \quad b_\alpha \in \mathbb{C} .$$

Now

$$\langle a, x\rangle^m = \sum_{|\alpha|=m} \binom{m}{\alpha_1 \ldots \alpha_n} a_1^{\alpha_1} \ldots a_n^{\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

where

$$\binom{m}{\alpha_1 \ldots \alpha_n} = \frac{m!}{\alpha_1! \ldots \alpha_n!} .$$

If A vanishes on all polynomials of the form $\langle a, x\rangle^m$ then

$$\sum_{|a|=m} b_a(a_1 \overset{m}{\cdots} a_n) a_1^{a_1} \cdots a_n^{a_n} = 0.$$

But this is a polynomial in a. Thus $b_a(a_1 \overset{m}{\cdots} a_n) = 0$ for all a. So $b_a = 0$ for all a.

Lemma (1.8). A nonzero homogeneous polynomial of degree m in R^n can vanish identically on at most m distinct hyperplanes.

Proof. The proof is by induction on m. If $m = 0$, then the result is trivial. Suppose that the result has been established for $m < \ell$ and that P_ℓ vanishes on the hyperplanes $\theta_1^\perp, \ldots, \theta_{\ell+1}^\perp$, $\theta_i \neq \pm\theta_j$, $i \neq j$. Without loss of generality we may assume $\theta_1 = e_1$, the direction of the x_1 axis. Then $P_\ell(x) = 0$ whenever $x_1 = 0$. Thus

$$P_\ell(x) = x_1 Q_{\ell-1}(x)$$

with $Q_{\ell-1}$ homogeneous of degree ℓ-1. But $Q_{\ell-1}$ vanishes on $\theta_i^\perp \setminus \theta_1^\perp$ for $i = 2, \ldots, \ell+1$. Since $Q_{\ell-1}$ vanishes on a closed set it follows that $Q_{\ell-1}$ vanishes on $\theta_i^\perp$, $i \geq 2$ and the induction hypothesis implies $Q_{\ell-1} \equiv 0$.

For any integer $k > 0$ let N^k be the set of k-tuples of positive integers $\ell = (\ell_1, \ldots, \ell_k)$ such that $\ell_1 < \ell_2 < \ldots < \ell_k$.

Definition (1.9). Let $\{\theta_i\}$ be a sequence of directions in R^n such that any $(k+1)$ of the θ_i are linearly independent. For each $\ell \in N^k$, let $\pi_\ell = [\theta_{\ell_1}, \ldots, \theta_{\ell_k}]$ be the k-space generated by $\theta_{\ell_1}, \ldots, \theta_{\ell_k}$. The set $\{\pi_\ell : \ell \in N^k\}$ is called a fundamental system of k-spaces for R^n.

Theorem (1.10). If $f \in L^1(R^n)$ has compact support, then f is uniquely determined by the $L_\pi f$ coming from any fundamental system of k-spaces for R^n.

Proof. Suppose $L_{\pi_j} f = 0$ for a fundamental set of k-spaces π_j, $j \in N^k$. Formula (1.4) gives for $y'' \in \pi^\perp$,

$$(1.11) \qquad \int_{\pi^\perp} \langle x'', y''\rangle^m L_\pi f(x'')dx'' = \int_{R^n} \langle x, y''\rangle^m f(x)dx = P_m(y'')$$

where P_m is the homogeneous polynomial defined by the right hand side of (1.11). From the assumption P_m vanishes on $\pi_j^\perp$ for all j. Let $\ell \in N^{k-1}$. For each $j \in N^k$ such that $j = (\ell, j_k)$ the space π_ℓ is contained in π_j. Thus $\pi_j^\perp \subset \pi_\ell^\perp$. Each $\pi_j^\perp$ is a distinct hyperplane in $\pi_\ell^\perp$ by construction of the fundamental system. Since P_m vanishes on $\pi_j^\perp$ for all $j \in N^k$ it follows from Lemma (1.8) that P_m vanishes on $\pi_\ell^\perp$, $\ell \in N^{k-1}$. Proceeding inductively it easily follows from the construction of the fundamental system of k-spaces that P_m vanishes on the hyperplanes $\theta_i^\perp$ where the θ_i

are the directions generating the fundamental system as in Definition (1.9). Now Lemma (1.8) implies P_m is identically zero for all m while (1.11) and Lemma (1.7) imply f is orthogonal to all polynomials. Since f has compact support, $f = 0$ almost everywhere.

Note that if $k = 1$, a fundamental system is contained in any infinite set of directions with distinct entries. Thus if f has compact support, f is uniquely determined by any infinite set of ordinary x-rays.

Remark. No function f is determined by any finite set of x-rays. Indeed if $\theta_1, \ldots, \theta_m$ is a finite set of directions, Ω is any open set in R^n, $\psi \in C_0^\infty(\Omega)$, and we define

$$P(\xi) = \langle \xi, \theta_1 \rangle \ldots \langle \xi, \theta_m \rangle ,$$

then

$$[P(D)\psi]^\wedge(\xi) = i^m P(\xi)\hat{\psi}(\xi) .$$

Thus Lemma (1.5) shows that if $g = P(D)\psi$, then

$$L_{\theta_i} g \equiv 0, \quad i = 1, \ldots, m.$$

The set of all functions $g = P(D)\psi$ with $\psi \in C_0^\infty(\Omega)$ is obviously infinite dimensional. Thus the set of functions with compact support in any open set $\Omega \subset R^n$ having the same x-rays in a finite number of directions is infinite dimensional.

Also if the assumption of compact support is dropped in Theorem (1.10), the theorem fails even for analytic functions. Indeed let $\phi \in C_0^\infty(R^n)$ with $\operatorname{supp} \phi \subseteq B(b, 1)$, $|b| > 1$. There exists an open set $G \subset S^{n-1}$ such that for all $\theta \in G$, $\theta^\perp \cap B(b, 1)$ is empty. If ψ is the inverse Fourier transform of ϕ, ψ is analytic and Lemma (1.5) implies $L_\theta \psi = 0$ for all $\theta \in G$. Thus $L_\theta \psi$ vanishes in a continuum of directions. In fact by choosing b sufficiently large the x-rays of ψ can be made to vanish in all directions outside of an arbitrarily small neighborhood of $b^\perp \cap S^{n-1}$. This is the best that can be said about the vanishing of the x-rays of an integrable function since the Fourier transform of a non-zero function f in $L^1(R^n)$ is continuous and thus nonzero on some open set.

2. LOWER DIMENSIONAL INTEGRABILITY OF L^2 FUNCTIONS

The symbol $L^2(T(G_{n,k}))$ will be used to designate the measurable functions on the fiber bundle $T(G_{n,k})$ which satisfy

$$\int_{G_{n,k}} \int_{\pi^\perp} |g(\pi, x'')|^2 dx'' d\mu < \infty$$

where μ is the finite measure on $G_{n,k}$ invariant under orthogonal transformations [11], and normalized so that

$$\mu(G_{n,k}) = \frac{|S^{n-1}|}{|S^{n-k-1}|} ,$$

the bars denoting the appropriate area measures on the spheres.

The symbol L_π will be used when the x-ray is acting in the direction of a fixed k-space π. The total x-ray transformation will be denoted by L.

<u>Lemma (2.1). If g is nonnegative and measurable on the sphere S^{n-1}, then</u>

$$\int_{G_{n,k}} \int_{S^{n-1} \cap \pi^\perp} g(\omega) d\omega d\mu = \int_{S^{n-1}} g(\theta) d\theta .$$

<u>Proof</u>. The integral on the left defines a continuous linear form on the space $C(S^{n-1})$, (hence a measure on S^{n-1}). This

form is obviously finite and rotation invariant and there is only one such up to a constant factor, namely the integral on the right. The normalization of μ is chosen to make the constant 1. Once the formula is established for continuous functions it extends immediately to nonnegative measurable functions by the standard arguments of measure theory.

Lemma (2.2). If g is nonnegative and measurable on R^n, then

$$\int_{G_{n,k}} \int_{\pi^\perp} |x''|^k g(x'')dx''d\mu = \int_{R^n} g(x)dx .$$

Proof. Using polar coordinates in the integral over $\pi^\perp$ gives

$$\int_{G_{n,k}} \int_{\pi^\perp} |x''|^k g(x'')dx''d\mu$$

$$= \int_0^\infty \int_{G_{n,k}} \int_{\pi^\perp \cap S^{n-1}} r^{n-1} g(r\theta'')d\theta''d\mu dr.$$

The result follows from Lemma (2.1).

In terms of Fourier transforms the operator Λ is defined by

(2.3) $$(\Lambda f)^\wedge(\xi) = |\xi|\hat{f}(\xi) .$$

To avoid cumbersome notation the same symbol is used irrespective of the space R^n, or subspace of R^n in which Λ acts.

<u>Theorem (2.4). The map $(2\pi)^{-k/2}(\Lambda^{k/2}L)$ is an isometry from $L^2(R^n)$ into $L^2(T(G_{n,k}))$.</u>

<u>Proof</u>. It is enough to establish the result on a dense subset of $L^2(R^n)$. If $f \in C_0^\infty(R^n)$ then clearly $\Lambda^{k/2}(L_\pi f)$ is in $L^2(\pi^\perp)$. The definition of Λ, (2.3), the Parseval relation on $\pi^\perp$, Lemma (1.5), and the last lemma give

$$\int_{G_{n,k}} \int_{\pi^\perp} |\Lambda^{k/2} L_\pi f(x'')|^2 dx'' d\mu$$

$$= (2\pi)^k \int_{G_{n,k}} \int_{\pi^\perp} |\xi''|^k |\hat{f}(\xi'')|^2 d\xi'' d\mu = (2\pi)^k \|f\|^2_{L^2(R^n)} .$$

The space of bounded, measurable, functions which vanish outside a compact subset of R^n is denoted by $L_0^\infty(R^n)$.

The next theorem, which is a rather surprising one, is part of a recent paper done jointly with K. T. Smith. See [13] for complete details of the proof.

<u>Theorem (2.5). For $k < n/2$ there is a constant c (depending on k and n) such that if $f \in L^2(R^n)$, then for almost every</u>

<u>$\pi \in G_{n,k}$, $L_\pi f$ is defined almost everywhere on $\pi^\perp$ by an absolutely convergent integral and</u>

$$(2.6)\qquad \int_{G_{n,k}} \|L_\pi f\|^2_{L^q(\pi^\perp)}\, d\mu \le c^2 \|f\|^2_{L^2(R^n)}, \qquad q = \frac{2(n-k)}{n-2k}.$$

<u>Proof</u>. It is sufficient to prove inequality (2.6) for $f \in L_0^\infty(R^n)$. Indeed if this has been done and $f \in L^2(R^n)$, $f \geq 0$, we can choose an increasing sequence of functions f_n with $f_n \geq 0$, $f_n \in L_0^\infty(R^n)$ and with f_n converging to f. Inequality (2.6) for the f_n and the monotone convergence theorem give the desired result for nonnegative f. The result follows immediately for all $f \in L^2(R^n)$.

Assume that $f \in L_0^\infty(R^n)$. It is clear that $L_\pi f \in L_0^\infty(\pi^\perp) \subset L^2(\pi^\perp)$ so that $\hat{f} \in L^2(\pi^\perp)$. According to Lemma (2.2)

$$\int_{G_{n,k}} \int_{\pi^\perp} |\xi''|^k |\hat{f}|^2 d\xi'' d\mu < \infty$$

and it follows that for almost every π,

$$|\xi''|^{k/2} \hat{f}(\xi'') = \hat{g}(\xi'') \in L^2(\pi^\perp).$$

Fix any such π. Since

$$\hat{f}(\xi'') = |\xi''|^{-k/2} \hat{g}(\xi'')$$

and since both $\hat{f}$ and $\hat{g}$ are in $L^2(\pi^\perp)$ the theorem in [13] on the Riesz transform gives that

$$(2.7) \qquad L_\pi f = R_{k/2} g = R_{k/2} \Lambda^{k/2} L_\pi f ,$$

where

$$R_\alpha h(x) = C_1(n, \alpha) \int_{R^n} |x-y|^{\alpha-n} h(y)dy, \qquad 0 < \alpha < n.$$

Now, Sobolev's inequality in the space R^m [14], asserts that

$$\| R_\alpha g \|_{L^q} \leq C \| g \|_{L^2} \quad \underline{\text{for}} \quad \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{m} > 0 .$$

In the present case we have $\alpha = k/2$ and $m = n-k$, so we get from (2.7) and Sobolev's inequality that

$$\| L_\pi f \|_{L^q(\pi^\perp)} \leq C \| \Lambda^{k/2} L_\pi f \|_{L^2(\pi^\perp)}, \qquad q = \frac{2(n-k)}{n-2k} .$$

Squaring and integrating over $G_{n,k}$ and making use of Theorem (2.4) gives the desired result.

Remark (2.8). The function

$$f(x) = \begin{cases} |x|^{-n/2}(\log|x|)^{-1}, & |x| \geq 2 \\ 0, & \underline{\text{otherwise}} \end{cases}$$

is square integrable on R^n but is not integrable over any plane of dimension $\geq n/2$.

3. THE X-RAY TRANSFORM AS AN UNBOUNDED OPERATOR

We shall study the x-ray transformation as an unbounded operator from $L^2(R^n)$ into $L^2(T(G_{n,k}))$. (We will usually write $L^2(T)$ in place of $L^2(T(G_{n,k}))$.) The natural domain would appear to be the set

$$(3.1) \qquad D_k = \{f \in L^2(R^n) : \int_{G_{n,k}} \|L_\pi f\|^2_{L^2(\pi^\perp)} d\mu < \infty\}$$

This turns out to be entirely satisfactory for $k < n/2$, but not for $k \geq n/2$. In the latter case, for example, we have not been able to decide whether the operator is even closable with domain D_k.

<u>Lemma (3.2). If $f \in L^1 \cap L^2$ then $f \in D_k$ and</u>

$$(3.3) \qquad \|Lf\|^2_{L^2(T)} \leq (2\pi)^k(|S^{n-1}| \, \|f\|^2_{L^1} + \|f\|^2_{L^2}) .$$

<u>Proof.</u> Since $f \in L^1 \cap L^2$, it follows that $|\xi|^{-k/2}\hat{f} \in L^2$ and an easy calculation shows that

$$(3.4) \qquad \| \, |\xi|^{-k/2}\hat{f}\|^2_{L^2} \leq |S^{n-1}| \, \|\hat{f}\|^2_{L^\infty} + \|\hat{f}\|^2_{L^2} .$$

On the other hand, Parseval's relation on $\pi^\perp$, Lemma (1.5), and Lemma (2.2) with $g = |\xi|^{-k/2}\hat{f}$ give

$$\|Lf\|^2_{L^2(T)} = (2\pi)^k \int_{G_{n,k}} \int_{\pi^\perp} |\hat{f}|^2 d\xi'' d\mu$$

$$= (2\pi)^k \| \, |\xi|^{-k/2} \hat{f} \|^2_{L^2} .$$

The result follows from (3.4), Parseval's relation on R^n and the fact that $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$.

Let us define

(3.5) $$\tilde{D}_k = \{f \in L^2 : |\xi|^{-k/2}\hat{f} \in L^2\} .$$

Theorem (3.6). If $k < n/2$ then $D_k = \tilde{D}_k$ and for almost every k-space π

(3.7) $$(L_\pi f)^\wedge = (2\pi)^{k/2}\hat{f} \quad \text{a.e. on } \pi^\perp,$$

whenever $f \in D_k$. If $k \geq n/2$, then $D_k \neq \tilde{D}_k$.

Proof. Let $0 < k < n$ and assume that $f \in \tilde{D}_k$. Choose $\hat{f}_n \in C_0^\infty(R^n)$ such that $\hat{f}_n \to \hat{f}$ in L^2 and $|\xi|^{-k/2}\hat{f}_n \to |\xi|^{-k/2}\hat{f}$ in L^2. Let f_n and f be the inverse Fourier transforms of $\hat{f}_n$ and $\hat{f}$ respectively. Since $\hat{f}_n \in C_0^\infty(R^n)$, it follows that $f_n \in \mathcal{S}(R^n)$, (the space of rapidly decreasing functions of Schwartz). Lemma (1.5) and Lemma (2.2) give

$$\int_{G_{n,k}} \| (L_\pi f_n)^\wedge - (2\pi)^{k/2}\hat{f}\|^2_{L^2(\pi^\perp)} d\mu$$

$$= (2\pi)^k \int_{G_{n,k}} \|\hat{f}_n - \hat{f}\|^2_{L^2(\pi^\perp)} d\mu = (2\pi)^k \int_{R^n} |\xi|^{-k} |\hat{f}_n - \hat{f}|^2 d\xi .$$

But the last integral converges to 0 as $n \to \infty$, so, choosing a subsequence if necessary, we have for almost every π,

(3.8) $\qquad L_\pi f_n \to g_\pi \quad$ a.e. <u>in</u> $\pi^\perp$

where g_π is defined by

(3.9) $\qquad \hat{g}_\pi = (2\pi)^{k/2}\hat{f} \quad$ <u>on</u> $\pi^\perp$.

However when $k < n/2$, Theorem (2.5) shows that for almost every π, and again a suitable subsequence,

$$L_\pi f_n \to L_\pi f \quad \text{a.e. } \underline{\text{in}} \ \pi^\perp .$$

It follows that for almost every π

(3.10) $\qquad g_\pi = L_\pi f \quad$ a.e. <u>on</u> $\pi^\perp$.

Hence $f \in D_k$. Moreover (3.9) and (3.10) show that (3.7) is valid.

Conversely suppose $f \in D_k$, $k < n/2$, and define $f_\rho(x) = e^{-\rho^2|x|^2} f(x)$. It is easy to see that $f_\rho \to f$ in L^2 and consequently $\hat{f}_\rho \to \hat{f}$ in L^2. Lemma (2.2) implies (for a suitable sequence of ρ's) that

(3.11) $$\lim_{\rho \to 0} \int_{\pi^\perp} |\xi''|^k |\hat{f}_\rho - \hat{f}|^2 d\xi'' = 0 \quad \underline{\text{for}} \quad \text{a.e.} \quad \pi ,$$

and Theorem (2.6) gives (for a suitable sequence of ρ's) that

$$L_\pi f_\rho \to L_\pi f \quad \underline{\text{in}} \quad L^q(\pi^\perp) \quad \underline{\text{for}} \quad \text{a.e.} \quad \pi,$$

and hence that

(3.12) $$L_\pi f_\rho \to L_\pi f \quad \underline{\text{in}} \quad \mathcal{S}'(\pi^\perp) \quad \underline{\text{for}} \quad \text{a.e.} \quad \pi ,$$

where $\mathcal{S}'(\pi^\perp)$ denotes the tempered distributions in $\pi^\perp$. Since the Fourier transform is a topological isomorphism on $\mathcal{S}'(\pi^\perp)$ and $f_\rho \in L^1$, Lemma (1.5) and (3.12) give

(3.13) $$(2\pi)^{k/2}\hat{f}_\rho \to (L_\pi f)^\wedge \quad \underline{\text{in}} \quad \mathcal{S}'(\pi^\perp) \quad \underline{\text{for}} \quad \text{a.e.} \quad \pi.$$

From this and (3.11) it follows that

$$(L_\pi f)^\wedge = (2\pi)^{k/2}\hat{f} \quad \text{a.e.} \quad \underline{\text{on}} \quad \pi^\perp$$

for any π for which (3.11) and (3.13) hold. Since $f \in D_k$, Lemma (2.2) shows that $|\xi|^{-k/2}\hat{f} \in L^2$, and the theorem is proved for the case $k < n/2$.

To prove the second part of the theorem, we construct a function $g \in L^2$ such that $|\xi|^{-k/2}\hat{g} \in L^2$ but g is not Lebesgue integrable over any plane of dimension $k \geq n/2$. Assume that $k \geq n/2$ and let $f(x)$ be the function in Remark (2.8). Define g_0 by

$$(3.14) \qquad g_0(x) = \begin{cases} f(x), & 2m_k \leq x_k \leq 2m_k+1, \; k = 1, \ldots, n \\ 0, & \underline{\text{otherwise}} \end{cases}$$

where the m_k's run through the integers. (For example, on the line, $g_0 = 0$ on alternate intervals between integers.) Let $e_1, \ldots, e_n$ be the unit vectors along the axes, and put

$$g_k(x) = g_{k-1}(x+e_k) - g_{k-1}(x) .$$

One may easily check that g_n is not Lebesgue integrable over any k-plane. Moreover

$$\hat{g}_n(\xi) = (e^{i\xi_1}-1)\ldots(e^{i\xi_n}-1)\hat{g}_0(\xi)$$

which gives the desired result since

$$(e^{i\xi_1}-1)\ldots(e^{i\xi_n}-1)|\xi|^{-n}$$

is bounded.

<u>Theorem (3.15). The x-ray transform with domain</u> C_0^∞ <u>admits a closure</u> $\bar{L}$ <u>with domain</u> $\widetilde{D}_k$ ($=D_k$ <u>for</u> $k < n/2$ <u>and</u> $\neq D_k$

for $k \geq n/2$).

Proof. First we show that the x-ray transformation admits a closure. Suppose that $f_n \in C_0^\infty$, $f_n \to 0$ in $L^2(R^n)$ and $Lf_n \to g$ in $L^2(T)$. We must show that $g = 0$. Now

$$\|g\|_{L^2(T)} = \lim_{n\to\infty} \int_{G_{n,k}} \|L_\pi f_n\|^2_{L^2(\pi^\perp)} d\mu$$

$$= (2\pi)^k \lim_{n\to\infty} \int_{G_{n,k}} \int_{\pi^\perp} |\hat{f}_n|^2 d\xi'' d\mu$$

$$\leq (2\pi)^k \lim_{n\to\infty} \left(\int_{G_{n,k}} \int_{|\xi''| \leq \epsilon} |\hat{f}_n|^2 d\xi'' d\mu + \frac{1}{\epsilon^k} \int_{G_{n,k}} \int_{|\xi''| > \epsilon} |\xi''|^k |\hat{f}_n|^2 d\xi'' d\mu \right)$$

$\epsilon > 0$,

where Lemma (1.5) and Parseval's relation on $\pi^\perp$ have been used. If $\epsilon > 0$ is fixed and we let $n \to \infty$, Lemma (2.2) gives

$$\|g\|_{L^2(T)} \leq \int_{G_{n,k}} \int_{|\xi''| \leq \epsilon} |\hat{g}_\pi|^2 d\xi'' d\mu .$$

Since ϵ is arbitrary and $g \in L^2(T)$, the right-hand side can be made arbitrarily small. Thus $g = 0$.

Let $\overline{D}_k$ be the domain of $\overline{L}$. If $f \in \overline{D}_k$, then there exists a sequence $f_n \in C_0^\infty$ such that $f_n \to f$ in $L^2(R^n)$ while $Lf_n \to g$ in $L^2(T)$. The proof of the first part of the theorem shows that for almost every π, (and an appropriate subsequence if necessary),

$$(2\pi)^{k/2}\hat{f}_n \to \hat{g}_\pi \quad \text{a.e. } \underline{\text{on}} \ \pi^\perp .$$

But we also know that for an appropriate sequence of n,

$$f_n \to f \quad \text{a.e. } \underline{\text{on}} \ R^n .$$

It follows that

$$\text{(3.16)} \qquad \hat{g}_\pi = (2\pi)^{k/2}\hat{f} \quad \text{a.e. } \underline{\text{on}} \ \pi^\perp \ \underline{\text{for}} \ \text{a.e. } \pi .$$

Since $g \in L^2(T)$, it follows from Lemma (2.2) and (3.16) that $|\xi|^{-k/2}\hat{f} \in L^2(R^n)$. Thus $f \in \tilde{D}_k$.

Finally suppose $h \in \tilde{D}_k$, the first part of the proof of Theorem (3.6) shows that there exists $h_n \in \mathcal{S}(R^n)$ such that $h_n \to h$ in $L^2(R^n)$ and Lh_n converges in $L^2(T)$. Thus it suffices to show that $\mathcal{S}(R^n) \subset \overline{D}_k$. But this is immediate from (3.3). Indeed, if $f \in \mathcal{S}(R^n)$, simply choose $f_n \in C_0^\infty$ such that $f_n \to f$ in $L^1(R^n) \cap L^2(R^n)$, i.e., in $L^1(R^n)$ and in $L^2(R^n)$.

4. AN INVERSION FORMULA

From Theorem (2.4) we know that $(2\pi)^{-k/2}(\Lambda^{k/2}L)$ is an isometry from $L^2(R^n)$ into $L^2(T) = L^2(T(G_{n,k}))$ and it follows that

$$(2\pi)^{-k}[(\Lambda^{k/2}L)^*(\Lambda^{k/2}L)]f = f \quad \underline{\text{if}} \quad f \in L^2(R^n) ,$$

where $(\Lambda^{k/2}L)^*$ denotes the adjoint of $(\Lambda^{k/2}L)$. In this chapter we investigate the adjoint of the x-ray transform, L^*, and give an inversion formula which shows how to recover f from Lf or $\bar{L}f$ whenever $f \in \tilde{D}_k$.

Lemma (4.1). If $g \in L^2(T)$, then g is in the domain of L^* provided that

$$\ell(x) = \int_{G_{n,k}} g(\pi, x'')d\mu \in L^2(R^n).$$

Moreover the above formula defines the adjoint operator, i.e.,

$$\ell = L^*g .$$

Proof. By Theorem (3.15) it is enough to compute the adjoint of the restriction of L to $C_0^\infty(R^n)$. If $f \in C_0^\infty$, then $L|f| \in L_0^\infty(T)$ and the use of Fubini's theorem is justified in the following calculations. If $g \in L^2(T)$ and $P_{\pi^\perp}(x)$ is the projection of x on $\pi^\perp$, then

$$\infty > \int_{G_{n,k}} \int_{\pi^\perp} L_\pi f(x'')\overline{g(\pi, x'')}dx''d\mu$$

$$= \int_{G_{n,k}} \int_{\pi^\perp} \int_{\pi} f(x', x'')\overline{g(\pi, x'')}dx'dx''d\mu$$

$$= \int_{G_{n,k}} \int_{R^n} f(x)\overline{g(\pi, P_{\pi^\perp}(x))}dxd\mu$$

$$= \int_{R^n} f(x)\left(\overline{\int_{G_{n,k}} g(\pi, P_{\pi^\perp}(x))d\mu}\right)dx$$

$$= \int_{R^n} f\bar{\ell}dx .$$

Since the range of L^* must be contained in $L^2(R^n)$ the result follows.

<u>Theorem (4.2). (Inversion Formula) If</u> $f \in \tilde{D}_k$ <u>and</u> $Lf = g$, $k < n/2$, <u>or</u> $\bar{L}f = g$, $k \geq n/2$, <u>then</u>

$$(2\pi)^{-k}\Lambda^{k/2}L^*\Lambda^{k/2}g = f.$$

(Note that the $\Lambda^{k/2}$ on the left is acting in R^n while the $\Lambda^{k/2}$ on the right is acting on each fiber in $T(G_{n,k})$.)

<u>Proof.</u> Let f and g satisfy the hypotheses of the theorem. First we show that $\Lambda^{k/2}g$ is in the domain of L^*. Let $h \in C_0^\infty(R^n)$. Parseval's relation, Theorem (3.6), (3.16), and Lemma (2.2) give that

$$<Lh, \Lambda^{k/2}g>_{L^2(T)} = <(Lh)^\wedge, (\Lambda^{k/2}g)^\wedge>_{L^2(T)}$$

$$= (2\pi)^k<\hat{h}, |\xi''|^{k/2}\hat{f}>_{L^2(T)} = (2\pi)^k<\hat{h}, |\xi|^{-k/2}\hat{f}>_{L^2(R^n)}$$

$$= (2\pi)^k<\hat{h}, \hat{u}>_{L^2(R^n)},$$

where

$$\hat{u} = |\xi|^{-k/2}\hat{f}. \tag{4.3}$$

Since $f \in \tilde{D}_k$, $\hat{u} \in L^2(R^n)$. Thus if u is the inverse Fourier transform of $\hat{u}$, Parseval's relation on R^n gives that

$$<Lh, \Lambda^{k/2}g>_{L^2(T)} = (2\pi)^k<h, u>_{L^2(R^n)}$$

for every $h \in C_0^\infty(R^n)$. According to Theorem (3.15) it follows that $\Lambda^{k/2}g$ is in the domain of L^* and that

$$L^*\Lambda^{k/2}g = (2\pi)^k u.$$

The theorem now follows from the definition of Λ and (4.3).

5. THE SUPPORTS OF f AND Lf

If A is a subset of R^n the support function of A is defined by

(5.1) $$S_A(\xi) = \sup_{y \in A} \langle \xi, y \rangle$$

The support function is convex, homogeneous of degree 1, and lower semicontinuous, with values in $(-\infty, \infty]$. The value $+\infty$ may be obtained when A is unbounded.

Until further notice K will denote an n-dimensional, compact, convex subset of R^n.

Lemma (5.2). If π is a k-dimensional subspace of R^n, then $x+\pi$ intersects K if and only if

(5.3) $$\langle x, \omega \rangle \leq S_K(\omega) \quad \text{for all} \quad \omega \in \pi^{\perp}$$

Proof. If $x+\pi$ intersects K then $(x+y)$ is in K for some $y \in \pi$. But if $\omega \in \pi^{\perp}$ then

$$\langle x, \omega \rangle = \langle x+y, \omega \rangle \leq S_K(\omega).$$

Conversely suppose (5.3) holds. Let K'' and x'' be the projections of K and x on $\pi^{\perp}$. Now for all $\omega \in \pi^{\perp}$

$$\langle x'', \omega \rangle = \langle x, \omega \rangle \leq S_K(\omega) = S_{K''}(\omega) .$$

Thus x'' is contained in each halfspace in $\pi^{\perp}$ which contains K''. Since K'' is closed and convex, x'' is in K''. Thus $x'' + \pi$ intersects K and hence $x + \pi$ intersects K.

The following is an immediate consequence of Lemma (5.2).

Corollary (5.4). *If f is integrable and has support in K, then $L_\pi f(x'') = 0$ whenever*

$$\langle x'', \omega\rangle > S_K(\omega) \quad \textit{for some} \quad \omega \in \pi^{\perp}.$$

More interestingly the converse of Corollary (5.4) holds when $k \leq n-2$.

Theorem (5.5). *Let $f \in L^1$ and $k \leq n-2$. If for every k-space π, $L_\pi f(x'') = 0$ for almost every $x'' \in \pi^{\perp}$ such that $x'' + \pi$ does not intersect K, then f vanishes almost everywhere outside K. The result fails when $k = n-1$.*

Proof. Let H be a supporting hyperplane of K and let $H_+ = \{x: \langle x, \theta\rangle > t_0\}$ be the open halfspace determined by H disjoint from K. Define

$$h(x) = \begin{cases} f(x), & x \in H_+ \\ 0, & \text{otherwise.} \end{cases}$$

If π is a k-dimensional subspace in $\theta^{\perp}$, it is clear that

$$L_{\pi}h(x'') = 0 \ \underline{\text{for}} \ \text{a.e.} \ x'' \in \pi^{\perp},$$

for if $<x'', \theta> \leq t_0$, then $x'' + \pi$ does not intersect H_+. On the other hand if $<x'', \theta> > t_0$, then $(x'' + \pi) \subset H_+$. and

$$L_{\pi}h(x'') = L_{\pi}f(x'') = 0 \ \underline{\text{for}} \ \text{a.e.} \ x'' \in H_+ \cap \pi^{\perp}.$$

Since $k \leq n-2$, Corollary (1.6) shows that h vanishes almost everywhere. Thus f vanishes almost everywhere in H_+. Since H was an arbitrary supporting hyperplane of K, the result follows.

To see that the theorem fails when $k = n-1$ it is more convenient to use the Radon transform notation. Recall that the Radon transform of an integrable function f is defined by

$$R_{\theta}f(t) = \int_{<x, \theta>=t} f(x)d\alpha_{n-1}(x) ,$$

where $\theta \in S^{n-1}$, $t \in R^1$, and α_{n-1} is the $n-1$ dimensional Lebesgue measure. Notice that

$$R_{\theta}f(t) = L_{\theta^{\perp}}f(t\theta).$$

Thus it suffices to construct $f \in L^1(R^n)$ such that f does not have compact support but $R_\theta f(t) = 0$, $|t| < r < \infty$ for all θ.

Let h be a C^∞ function on R^n which is equal to $|x|^{2-n}$ for $|x| \geq 1$, when $n \geq 3$. When $n = 2$, let h be equal to $\log |x|$ for $|x| \geq 1$. Let f be obtained from h by differentiation in such a manner that f is integrable but does not have compact support. If Δ denotes the Laplacian, then

$$\Delta f = \Delta h = 0, \quad |x| > 1.$$

Moreover

$$(R_\theta f)'' = R_\theta(\Delta f) = 0, \quad |t| > 1,$$

primes denoting differentiation with respect to t. But since $R_\theta f$ is an integrable function of t, $R_\theta f \to 0$ as $t \to \infty$. Thus

$$R_\theta f(t) = 0, \quad |t| > 1,$$

and the theorem is proved.

Remark (5.6). A function $g(\pi, x)$ on $T(G_{n,k})$ vanishes in a neighborhood of a point (π_0, x_0) if there exists an open neighborhood $\mathcal{P}$ of π_0 in $G_{n,k}$ and an open neighborhood U of x_0 in R^n such that

$$g(\pi, x) = 0 \quad \underline{\text{for all}} \quad (\pi, x) \in (\mathcal{P} \times U) \cap T(G_{n,k}).$$

The function g vanishes in an ϵ-neighborhood of (π_0, x_0) if U above can be chosen to contain the ball $B(x_0, \epsilon)$. As usual we say f vanishes in a neighborhood, (ϵ-neighborhood), of a subset $A \subset T(G_{n,k})$ if it vanishes in a neighborhood, (ϵ-neighborhood), of each point of A.

If U is a subset of R^n and $\mathcal{P}$ is a subset of $G_{n,k}$, we define

$$\mathcal{P} \boxtimes U = (\mathcal{P} \times U) \cap T(G_{n,k}) \, .$$

Note $\mathcal{P} \boxtimes U$ consists of the pairs (π, x) with $\pi \in \mathcal{P}$ and $x \in (U \cap \pi^{\perp})$.

The idea for the proof of the next theorem comes from Lemma (3.1) in [10]. The importance of the theorem is that it shows that it is possible in some cases to get inside the convex hull of the support of f from a knowledge of its x-rays in dimensions $k \leq n-2$.

Theorem (5.7). Let π_0 be a k-space $k \leq n-1$, and V be an open, connected, unbounded subset of $\pi_0^{\perp}$. If $f \in C_0^{\infty}(R^n)$ and Lf vanishes in a neighborhood of $\pi_0 \boxtimes V$ then f vanishes on $\pi_0 + V$.

Proof. Without loss of generality assume that $\pi_0 = [e_1, \ldots, e_k]$. Choose neighborhoods $N_1, \ldots, N_k$ of $e_1, \ldots, e_k$ such that whenever $w = (w_1, \ldots, w_k) \in N_1 x \ldots x N_k$, then $\pi_w = [w_1, \ldots, w_k]$ in a subspace of dimension k. Now if $y \in R^n$, define

$$\widetilde{L}_w f(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y+t_1 w_1 + \ldots + t_k w_k) dt_1 \ldots dt_k \ .$$

Note

$$\widetilde{L}_w f(y) = \frac{1}{|w_1| \ldots |w_k|} L_{\pi_w} f(y'') \ . \tag{5.8}$$

Since Lf vanishes in a neighborhood of $\pi_0 \boxtimes V$, (5.8) shows that $\widetilde{L}_w f$ vanishes in a neighborhood of each point $(y, w_1, \ldots, w_k)$ with $y \in (\pi_0 + V)$, $w_i = b_i e_i \in N_i$. If $w_i = (w_{i1}, \ldots, w_{in}) \in N_i$ and $y \in \pi_0 + V$ is fixed, then

$$0 = \left(\frac{\partial}{w_{1i}}\right)^{a_1} \left(\frac{\partial}{w_{2i}}\right)^{a_2} \ldots \left(\frac{\partial}{w_{ki}}\right)^{a_k} (\widetilde{L}_w f(y))\Big|_{w=(e_1, \ldots, e_k)}$$

$$= \left(\frac{\partial^{a_1}}{w_{1i}} \cdots \frac{\partial^{a_k}}{w_{ki}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y+t_1 w_1 + \ldots + t_k w_k) dt_1 \ldots dt_k \right)\Bigg|_{w=(e_1, \ldots, e_k)}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{a_1} \ldots t_k^{a_k} \frac{\partial^{|a|} f}{\partial x_i^{|a|}} (y+t_1 e_1 + \ldots + t_k e_k) dt_1 \ldots dt_k \ ,$$

where

$$a = (a_1, \ldots, a_k) \quad \text{and} \quad |a| = \sum_{i=1}^{k} a_i \ .$$

Letting y vary in $\pi_0 + V$ we have

$$\frac{\partial^{|\alpha|}}{\partial y_i^{|\alpha|}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{\alpha_1} \cdots t_k^{\alpha_k} f(y+t_1e_1+\cdots+t_ke_k)dt_1\cdots dt_k = 0.$$

Thus

$$(5.9) \qquad P_\alpha(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{\alpha_1} \cdots t_k^{\alpha_k} f(y+t_1e_1\cdots+t_ke_k)dt_1\cdots dt_k$$

is a polynomial of degree less than $|\alpha|$ in y_i in the domain $\pi_0 + V$. Since y_i was arbitrary $P_\alpha(y)$ is a polynomial in each of its variables separately and it follows that $P_\alpha(y)$ is a polynomial in y in the domain $\pi_0 + V$. If we choose $r > 0$ such that $\text{supp } f \subseteq B(0, r)$ then

$$f \equiv 0 \quad \underline{\text{on}} \quad [\pi_0 + (V \setminus B(0, r))] \subset \pi_0 + V.$$

It follows that the polynomials $P_\alpha(y)$ vanish on an open subset of $\pi_0 + V$. Since V is connected the polynomials vanish identically on $\pi_0 + V$. But then (5.9) shows that for all $y \in V$, the function $f(y+t_1e_1+\cdots+t_ke_k)$ is orthogonal to all polynomials in the variables $t_1, \ldots, t_k$. Since f has compact support, f vanishes on $\pi_0 + V$.

The convolution of two integrable functions f and h is given by

$$f * h(x) = \int_{R^n} f(y)h(x-y)dy$$

and it is well known that

$$(f * g)^\wedge = (2\pi)^{n/2} \hat{f}\hat{g}.$$

Thus it follows from Lemma (1.5) that

(5.10) $$L_\pi(f * g) = (L_\pi f) * (L_\pi g).$$

Theorem (3.6) shows that when $k < n/2$ and $f \in D_k$, formula (5.10) holds almost everywhere on $\pi^\perp$ for almost every π.

<u>Corollary (5.11). Let $f \in L^1(R^n)$ have compact support and</u> V, π_0 <u>be as in Theorem (5.7). Assume in addition that for sufficiently large integers</u> $n > 0$

(5.12) $$V_n = \{x'' \in V : d(x'', \pi_0^\perp \setminus V) > 1/n\},$$

<u>(where d denotes distance), is an open, unbounded, connected subset of V. Then if Lf vanishes in a neighborhood of</u> $\pi_0 \boxtimes V$, f <u>vanishes almost everywhere on</u> $\pi_0 + V$.

<u>Proof</u>. Let $\phi \in C_0^\infty(R^n)$ be nonnegative, have support in the unit ball, and satisfy

$$\int_{R^n} \phi(x)dx = 1.$$

If

$$\phi_\rho(x) = \rho^{-n}(x/\rho), \quad \rho > 0 ,$$

then $\phi_\rho * f$ is in C_0^∞, has support in $B(0,\rho) + \text{supp } f$, and

$$\phi_\rho * f \to f \quad \underline{\text{in}} \quad L^1(R^n) \quad \underline{\text{as}} \quad \rho \to 0 .$$

For $n > 0$ sufficiently large, V_n in (5.12) is an open, unbounded, connected subset of V. Suppose that f has support in $B(0,r)$. Since $\bar{V}_n \cap \overline{B(0,r+1)}$ is compact and contained in V, Lf vanishes in an ϵ-neighborhood of $\pi_0 \boxtimes (\bar{V}_n \cap \overline{B(0,r+1)})$ for sufficiently small $\epsilon(n) > 0$. From (5.10) we have

$$\text{(5.13)} \qquad L_\pi(\phi_\rho * f)(x'') = \int_{\pi^\perp} L_\pi \phi_\rho(y'') L_\pi f(x''-y'')dy'' .$$

If $|x''| \geq r+1$ and $0 < \rho < 1$, then $|x''-y''| > r$ for all $y'' \in B(0,\rho)$ and Corollary (5.4) implies

$$L_\pi(\phi_\rho * f)(x'') = 0.$$

If $|x''| \leq r+1$ and $x'' \in \pi_0^\perp$, Lf vanishes in an ϵ-neighborhood of (π_0, x'') and (5.13) shows that $L_\pi(\phi_\rho * f)(x'')$ vanishes if

$0 < \rho < \epsilon < 1$ and π is sufficiently close to π_0. Now Theorem (5.7) implies that $\phi_\rho * f$ vanishes on $\pi_0 + V_n$. Letting n go to infinity we get the desired result.

Remark (5.14). The assumption of the unboundedness of U is necessary. Indeed consider the function with support in the rectangle $|x_i| \leq 1$, $1 \leq i \leq n$ defined by

$$f(x) = \begin{cases} 1, & 0 \leq x_1 \leq 1 \\ -1, & -1 \leq x_1 < 0 \end{cases}$$

If $\pi = [e_1, \ldots, e_k]$, $0 < r < 1$, and

$$V = \{x \in \pi^{\perp}: |x_j| < r, \ \ j = (n-k+1), \ldots, n\}$$

then $L_\pi f$ vanishes in a neighborhood of $\pi \boxed{x} V$ while f does not vanish on $\pi + V$.

Corollary (5.13). If $f \in L^1(R^n)$ has compact support and for each (n-1)-dimensional subspace π, $L_\pi f(x'') = 0$ whenever $x'' + \pi$ does not intersect K, then f vanishes almost everywhere outside K.

Proof. Set $k = (n-1)$ in Corollary (5.11). In this case the set V is a half line. Thus $\pi_0 + V$ is a halfspace. Corollary (5.11) implies that f vanishes almost everywhere on each halfspace not containing K. Since K is convex we are done.

6. THE RANGE OF L

Ludwig [10], and later Lax and Phillips [9], derived necessary and sufficient conditions that a measurable function $g(\theta, t)$, $\theta \in S^{n-1}$, $t \in R^1$ be the Radon transform of a square integrable function with support in an n-dimensional, compact, convex set. The same is done here for the operator L. Although our proof fails in the case of hyperplanes, Ludwig's theorem establishes the result. Moreover when $k \leq n-2$ Corollary (5.11) gives a characterization of the support of f which, in some cases, is sharper than the convex hull of f.

Lemma (6.1). If $f \in L^1(R^n)$ has compact support and $y \in R^n$, then

$$P_m(y) = \int_{\pi^\perp} \langle y, x''\rangle^m L_\pi f(x'')dx'', \quad y \in \pi^\perp, \tag{6.2}$$

is independent of the choice of the k-space $\pi \subset y^\perp$. Moreover as π varies in $G_{n,k}$ the integrals (6.2) determine a homogeneous polynomial of degree m on R^n.

Proof. Taking $\rho(t) = t^m$ in Lemma (1.3) gives

$$P_m(y) = \int_{\pi^\perp} \langle y, x''\rangle^m L_\pi f(x'')dx'' = \int_{R^n} \langle y, x\rangle^m f(x)dx.$$

The integral on the right is independent of π and obviously defines a homogeneous polynomial of degree m on R^n.

Let $f \in L^1(R^n)$ have compact support. We investigate the polynomials

$$(6.3) \qquad P_m(\xi) = P_m(f, \xi) = \int_{R^n} \langle x, \xi\rangle^m f(x)dx, \qquad \xi \in R^n.$$

These can also be defined for $\zeta \in \mathbb{C}^n$ simply by replacing ξ by ζ in (6.3). Lemmas (6.4) and (6.6) were established by K. T. Smith in an unpublished paper where he gives a new proof of Ludwig's theorem.

Lemma (6.4). If $f \in L^1(R^n)$ has compact support, then the Fourier transform of f extends to a complex analytic function on $\mathbb{C}^n$ with the expansion

$$(6.5) \qquad \hat{f}(\zeta) = (2\pi)^{-n/2} \sum_{m=0}^{\infty} \frac{i^{-m} P_m(\zeta)}{m!}.$$

Proof. The extension of the Fourier transform is given by the Laplace transform

$$\hat{f}(\zeta) = (2\pi)^{-n/2} \int e^{-i\langle x, \zeta\rangle} f(x)dx.$$

The integral converges absolutely for every $\zeta \in \mathbb{C}^n$ even after differentiating under the integral sign, and does determine an entire function on $\mathbb{C}^n$.

On the other hand if the support of f is contained in $B(0,r)$, then from (6.3)

$$|P_m(\zeta)| \leq r^m |\zeta|^m \|f\|_{L^1} .$$

This shows that the series on the right of (6.5) converges absolutely for every $\zeta \in \mathbb{C}^n$ and also determines an entire function. Thus to prove (6.5) it is enough to show it for $\xi \in R^n$, as two distinct entire functions cannot agree on R^n.

If θ is a direction in R^n and $D = (\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n})$ then Taylor's formula gives

$$f(\tau\theta) = \sum_{m=0}^{\infty} \frac{[<\theta, D>^m \hat{f}](0)\tau^m}{m!}$$

while

$$P_m(\theta) = (2\pi)^{n/2} (<x, \theta>^m f(x))^\wedge |_{\xi=0}$$

$$= (2\pi)^{n/2} (<\theta, D>^m \hat{f})(0) .$$

Comparison of the two gives formula (6.5) for $\xi = \tau\theta$.

The following theorem of Smith gives a characterization of the possible sequence of polynomials P_m that can arise from a square integrable function f with support in $B(0, r)$.

Lemma (6.6). Let $\{P_m\}$ be a sequence of polynomials such that

a) P_m is homogeneous of degree m.

b) $P_m(\zeta) \leq r^m |\zeta|^m$ for all m and all $\zeta \in \mathbb{C}^n$.

c) The sum $\sum \frac{i^{-m} P_m(\xi)}{m!}$ (which converges by b) is square integrable on R^n.

Then there is a square integrable f with support in the ball $B(0, r)$ such that $P_m(\zeta) = P_m(f, \zeta)$ for all m.

Proof. Define the entire function $\hat{f}$ by the formula (6.5). According to c), $\hat{f}$ is square integrable on the real space, so it is the Fourier transform of a square integrable f, and the problem is to show that f has support in $B(0, r)$. By virtue of b)

$$|\hat{f}(\zeta)| \leq (2\pi)^{-n/2} ce^{r|\zeta|} \leq c_1 e^{r|\zeta_1| + \ldots + r|\zeta_n|} .$$

By the Paley-Wiener theorem [3] f must vanish outside the cube $Q = \{x : |x_i| \leq r, \ i = 1, \ldots, n\}$. By a rotation of coordinates Q becomes any cube that circumscribes $B(0, r)$, so f must vanish outside the ball.

The Sobolev space H^s, $s \geq 0$, on the fiber bundle $T(G_{n,k})$ consists of those functions $g(\pi, x)$ satisfying

$$\|g\|_s^2 = \int_{G_{n,k}} \int_{\pi^\perp} (1+|\xi''|^2)^s |\hat{g}_\pi(\xi'')|^2 d\xi'' d\mu < \infty .$$

These spaces will be denoted $H^s(T(G_{n,k}))$ or $H^s(T)$.

Theorem (6.7). *Let $g(\pi, x)$ be a measurable function on $T(G_{n,k})$. There exists a square integrable function f with support in K such that for every π,*

$$L_\pi f = g_\pi \quad \text{a.e. } \underline{\text{on}} \quad \pi^\perp$$

if and only if

i) $g \in H^{k/2}(T)$

ii) $\int_{\pi^\perp} \langle y, x\rangle^m g_\pi(x)dx$, $y \in \pi^\perp$, *is independent of the k-space $\pi \subset y^\perp$ and as π varies in $G_{n,k}$ these integrals define a homogeneous polynomial of degree m in R^n.*

iii) $g(\pi, x) = 0$ *whenever $x + \pi$ does not meet K.*

iv) *there exists a constant $c > 0$ such that*

$$\|g_\pi\|_{L^1(\pi^\perp)} < c \quad \underline{\text{for all}} \quad \pi \in G_{n,k} .$$

Proof. When $k = n-1$, this is the theorem of Ludwig [10], and our proof fails. We prove the theorem when $k \leq n-2$. The necessity of the conditions has already been shown. We proceed to establish the sufficiency.

For each π and $\xi \in R^n \cap \pi^{\perp}$, define

$$P_{\pi,m}(\xi) = \int_{\pi^{\perp}} \langle x, \xi\rangle^m \, g_{\pi}(x)dx \;. \tag{6.8}$$

Condition (ii) shows that

$$P_{\pi_1,m}(\xi) = P_{\pi_2,m}(\xi) \quad \underline{\text{if}} \quad \xi \in \pi_1^{\perp} \cap \pi_2^{\perp}.$$

Moreover (iii), (iv), and Lemma (6.4) give that

$$\hat{g}_{\pi}(\xi) = (2\pi)^{(k-n)/2} \sum_{m=0}^{\infty} \frac{i^{-m} P_{\pi,m}(\xi)}{m!} \quad \underline{\text{for each}} \quad \pi. \tag{6.9}$$

Now condition (ii) implies that

$$P_m(\xi) = (2\pi)^{-n/2} P_{\pi,m}(\xi), \quad \xi \in \pi^{\perp}, \tag{6.10}$$

is a homogeneous polynomial of degree m on R^n. We claim that the polynomials P_m satisfy the conditions of Lemma (6.6). We have already shown that (a) holds. Moreover (6.10), (6.9), (i), and

Lemma (2.2) show that

$$(6.11) \qquad \hat{f}(\xi) = \sum_{m=0}^{\infty} \frac{i^{-m}P_m(\xi)}{m!}$$

is square integrable on R^n. Thus (c) also holds. To establish (b) in Lemma (6.6), note that each polynomial $P_{\pi,m}$ has a unique extension to

$$\pi^{\perp}_{\mathbb{C}} = \{\zeta \in \mathbb{C}^n : \zeta = \xi + i\eta, \quad \xi \in \pi^{\perp}, \quad \eta \in \pi^{\perp}\}$$

given by writing ζ in place of ξ in (6.8). If $K \subset B(0,r)$, then (iii) and (iv) imply that

$$|P_{\pi,m}(\zeta)| \leq cr^m|\zeta|^m, \quad \zeta \in \pi^{\perp}_{\mathbb{C}} .$$

Moreover since $P_m = (2\pi)^{-n/2}P_{\pi,m}$ on $\pi^{\perp}$, it follows that the unique extension of P_m to $\mathbb{C}^n$ agrees with $P_{\pi,m}$ on $\pi^{\perp}_{\mathbb{C}}$. Thus if $\zeta \in \mathbb{C}^n$ is perpendicular to a real k-space, then

$$|P_m(\zeta)| \leq |P_{\pi,m}(\zeta)| \leq cr^m|\zeta|^m .$$

But $\{x \in R^n : \langle x, \zeta\rangle = 0\}$ is a subspace of dimension $\geq n-2$. Thus (b) of Lemma (6.6) is satisfied. Now, if f is the inverse Fourier transform of $\hat{f}$, Lemma (6.6) implies that f has support in the ball $B(0, r)$. Consequently, Lemma (1.5), (6.11), (6.10), and (6.9) give that for each π

$$(L_\pi f)^\wedge = (2\pi)^{k/2} \hat{f} = \hat{g}_\pi \quad \underline{\text{on}} \quad \pi^\perp .$$

Thus $Lf = g$. The fact that f has support in K follows from Theorem (5.5).

7. THE NULL SPACE OF L_π

Let K be a fixed compact subset of R^n. We denote the square integrable functions on R^n with support in K by $L^2(K)$.

Lemma (7.1). For each k-space, π, the transformation L_π is continuous from $L^2(K)$ into $L^2(\pi^\perp)$.

Proof. Let $f \in L^2(K)$ and let χ denote the characteristic function of K. Lemma (1.2) shows that for each π, $f(x', x'') \in L^2(\pi)$ for almost every $x'' \in \pi^\perp$. Thus we may use the Cauchy Schwarz inequality in $L^2(\pi)$ and we have

$$\|L_\pi f\|^2_{L^2(\pi^\perp)} = \int_{\pi^\perp} |L_\pi f|^2 dx''$$

$$= \int_{\pi^\perp} \left| \int_\pi \chi(x', x'') f(x', x'') dx' \right|^2 dx''$$

$$\leq \int_{\pi^\perp} \left[\left(\int_\pi \chi(x', x'') dx' \right)^{1/2} \left(\int_\pi |f(x', x'')|^2 dx' \right)^{1/2} \right]^2 dx''$$

$$\leq C(k, K) \|f\|^2_{L^2}$$

where

$$C(k, K) = \sup_{x'' \in \pi^\perp} \int_\pi \chi(x', x'') dx' .$$

Theorem (7.2). The null space $\mathcal{N}_{L_\pi}$ is a closed subspace of

<u>$L^2(K)$ whose orthogonal complement consists of the functions in $L^2(K)$ that are constant in the direction π, or more precisely on the intersection of k-planes in the direction π with K.</u>

<u>Proof</u>. Lemma (7.1) shows that L_π is continuous, so $\mathcal{N}_{L_\pi}$ is closed. Without loss of generality assume $\pi = [e_1, \ldots, e_k]$. A function h is constant on k-planes in the direction π if and only if h is a function of $x'' = (0\ldots 0, x_{k+1}, \ldots, x_n)$ alone. Suppose that $f = \chi h$ where χ is the characteristic function of K and h is a function of x'' alone. If $g \in \mathcal{N}_{L_\pi}$, then

$$<g, f> = \int_{R^n} g\,\overline{\chi h}\,dx = \int_{\pi^\perp} \bar{h} \int_\pi g\,dx'dx'' = 0,$$

since the inner integral is $L_\pi g(x'')$.

Suppose $f \in \mathcal{N}^\perp_{L_\pi}$. It is sufficient to find a function h of x'' alone such that

$$L_\pi f = L_\pi \chi h .$$

Indeed if such an h is found, then

$$L_\pi (f - \chi h) = 0,$$

so $f - \chi h \in \mathcal{N}_{L_\pi}$. But by the above $\chi h \in \mathcal{N}^\perp_{L_\pi}$. So $(f-\chi h) \in \mathcal{N}^\perp_{L_\pi}$. Thus $f = \chi h$.

To find h note that

$$L_\pi f = L_\pi \chi h$$

if and only if

$$\int_\pi f(x', x'')dx' = \int_\pi \chi(x', x'')h(x'')dx' = \delta(x'')h(x'') \tag{7.3}$$

where $\delta(x'') = L_\pi \chi(x'')$.

Define $h(x'')$ by the above formula. We need only show that $\chi h \in L^2(K)$. Now

$$\int_{R^n} |\chi h|^2 dx = \int_{\pi\perp}\int_\pi |\chi h|^2 dx' dx'' \tag{7.4}$$

$$= \int_{\pi\perp} \delta(x'')|h(x'')|^2 dx'' .$$

But since $\chi f = f$, (7.3) and the Cauchy-Schwarz inequality give

$$(\delta(x''))^2 |h(x'')|^2 = |\int_\pi \chi f(x', x'')dx'|^2 \leq \delta(x'') \int_\pi |f|^2 dx' .$$

It follows from (7.4) that

$$\int_{R^n} |\chi h|^2 dx \leq \|f\|^2_{L^2(R^n)}$$

and we are done.

We now describe an iterative scheme due to Kacmarz [2] that is being used to detect brain tumors. In the brain tumor work the method is used in two dimensions to reconstruct cross sections. The method is general and is presented here for functions in R^n and k-planes.

Suppose that K is a compact subset of R^n, that $f \in L^2(K)$, and that the x-rays $L_{\pi_i} f$, $i = 1, \ldots, J$, are known. Let P_i be the projection on the closed plane $f + \mathcal{N}_{\pi_i}$ in $L^2(K)$. Let $f_0 \in L^2(K)$ be an arbitrary initial guess and define

$$f_\ell = P_i f_{\ell-1} \qquad \ell \equiv i \bmod(J).$$

The characterization of the null space in Theorem (7.2) shows that

$$L_{\pi_i} f_\ell(x'') = L_{\pi_i} f(x''), \quad \ell \equiv i \bmod(J),$$

and $f_\ell - f_{\ell-1}$ is constant on planes with direction π_i.

If $P = P_J \cdots P_1$, then a theorem of Amemiya and Ando [1], shows that

$$P^n f_0 \to P_{\mathcal{M}} f_0 \qquad \underline{\text{strongly as}} \quad n \to \infty,$$

where

$$\mathcal{M} = \bigcap_{i=1}^{J} (f + \mathcal{N}_{\pi_i}) ,$$

and $P_{\mathcal{M}}$ is the projection on $\mathcal{M}$.

Thus the iterative scheme of projections converges strongly to the projection of the initial guess, f_0, onto the plane $\mathcal{M}$.

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