AN ABSTRACT OF THE THESIS OF

Karen Astrid Swenson for the degree of Doctor of Philosophy in Mathematics Education presented on November 25, 1997. Title: Middle School Mathematics Teachers' Subject Matter Knowledge and Pedagogical Content Knowledge of Probability: Its Relationship to Probability Instruction.

Redacted for Privacy

Abstract approved: ______________________________

Dianne K. Erickson

As a result of the calls for reform in mathematics education and the ever-changing nature of mathematics, today's teachers face the challenge of teaching unfamiliar content in ways that are equally unfamiliar. In view of this challenge, the purpose of this study was to investigate middle school teachers' subject matter and pedagogical content knowledge of probability and its relationship to the teaching of probability. The study also explored the nature of the instructional tasks and classroom discourse during probability instruction.

Case study methodology was used to examine the knowledge and practice of 4 middle school teachers. A pre-observation interview assessed the teachers' subject matter knowledge of probability. The teachers were then observed as they taught probability. Post-observation interviews further explored teacher knowledge and its relationship to teaching practice. Data sources included interview transcripts, observational field notes, video and audiotapes of classroom instruction, and written instructional documents. Individual case studies were written describing the teachers' background and probability instruction. Cross-case analyses compared and contrasted the cases in response to the research questions.

The results of this study indicate the teachers generally (a) lacked an explicit and connected knowledge of probability content, (b) held traditional views about mathematics and the learning and teaching of mathematics, (c) lacked an understanding of the "big ideas" to be emphasized in probability instruction, (d) lacked knowledge of students' possible conceptions and misconceptions, (e) lacked the knowledge and skills needed to orchestrate discourse in ways that promoted students' higher level learning, and (f) lacked an integrated understanding of the nature of the reform.

One teacher captured the essence of the reform effort in her probability instruction; the other 3 teachers generally fell short of the goal despite their efforts to implement aspects of the reform. Although students were actively involved in exploring probability content through the use of games, simulations, and other hands-on instructional tasks, the cognitive level of the tasks and discourse was limited by the nature of instruction.

The findings of this study have implications for mathematics education reform, preservice teacher preparation, staff development, and curriculum development.

MIDDLE SCHOOL MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE OF PROBABILITY: ITS RELATIONSHIP TO PROBABILITY INSTRUCTION

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented November 25, 1997
Commencement June 1998

Doctor of Philosophy thesis of Karen Astrid Swenson presented on November 25, 1997

APPROVED:

Redacted for Privacy

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

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Karen Astrid Swenson, Author

ACKNOWLEDGMENTS

Although my name alone appears as the author of this dissertation, many others have contributed to its completion. In recognition of that, I wish to extend a heartfelt thank you to . . .

. . . Dianne Erickson (Science and Mathematics Education), my major professor, for being an excellent sounding board and wonderful source of guidance, encouragement, and support;

. . . Norm Lederman (Science and Mathematics Education) for his constructive insights and expertise in educational research;

. . . John Lee (Mathematics) for his time and service through the years, first as major professor for my master's degree and now in support of my transformation into a mathematics educator;

. . . Thomas Dick (Mathematics), a recent arrival to my committee, for stepping in when others moved on;

. . . Clayton Paulson (Oceanography), my Graduate Council Representative, for staying on board for the entire expedition;

. . . Tom Evans (Education), William Burger (Mathematics), Blake Peterson (Mathematics), and Howard Wilson (Mathematics), who at one time or another served on my doctoral committee;

. . . the middle school teachers who opened up themselves and their classrooms to me and allowed me to explore their knowledge and practice;

. . . those who assisted me with the pilot interviews, with transcribing the data, and with editing the dissertation;

. . . family members and numerous friends around the world whose encouragement and prayers have sustained me through this loooong process.

We did it!

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CHAPTER I
THE PROBLEM

Introduction

Probability plays increasingly vital roles in our daily lives, whether it is in interpreting today's AIDS testing results, predicting tomorrow's weather, or taking a risk on a future small business. Although probability had its beginnings in games of chance, it has become a branch of mathematics with wide repercussions in scientific research, business and industry, politics, and practical human activity. As the study of the likelihood of uncertain events, the business community depends on probability to forecast the future state of the market, to predict the anticipated value of the dollar, to determine the rates for insurance, and to monitor the quality of consumer products. Health care professionals rely on probability to make decisions regarding diagnosis and treatment of their patients, and to inform them of the possible risks or side effects of various medications or procedures. Natural and social science researchers apply the theory of probability as they design and interpret the results of their experiments. Even everyday citizens are influenced by probability, perhaps unknowingly, when they accept the claims of advertising, believe the reports of political polls, follow the weatherman's recommendation to carry an umbrella, or purchase a ticket for the state lottery. Because of the ever-increasing number of ways that probability is a part of our everyday lives, it has become an important branch of mathematics for elementary and secondary students to study.

In earlier decades, several influential organizations, including the National Council of Supervisors of Mathematics [NCSM] (1978), the National Advisory Committee on Mathematical Education [NACOME] (1975), the United Nations Educational, Scientific, and Cultural Organization [UNESCO] (1972), the Cambridge Conference on School Mathematics (1963), and the College Entrance Examination Board [CEEB] (1959/1970), recognized the vital role of probability in society and recommended that the study of probability be included as part of the mathematics curricula of elementary and secondary schools. Despite these recommendations and the fact that sections on probability are now found in nearly all mathematics textbook series, Shaughnessy (1992) observes that very little systematic instruction in probability has occurred at the K-12 level. Instead, in many mathematics programs, probability has been treated as a luxury topic, either omitted altogether or included only if time permits. As a result, student performance on national assessments has frequently been disappointing. For example, although performance levels increased on the sixth mathematics assessment of the National Assessment of Educational Progress [NAEP], Zawojewski and Heckman (1997) noted that fewer than 20% of the eighth grade students provided satisfactory responses to basic probability items presented in constructed-response format, suggesting the students "may not have the underlying knowledge about probability that would enable them to explain their reasoning" (p. 220).

In recent years, there have been an increasing number of calls for the study of probability in the schools. The NCSM (1989) repeated its call, including probability as one of the 12 "essential mathematical competencies that citizens will need to begin adulthood in the next millennium" (p. 44). In particular, they suggest

> students should understand elementary notions of probability to determine the likelihood of future events. They should identify situations in which immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make such predictions as election results, business forecasts, and outcomes of sporting events. They should learn how probability applies to research results and to the decision-making process. (p. 46)

The National Council of Teachers of Mathematics [NCTM], in its *Curriculum and Evaluation Standards* (hereafter called the *Curriculum Standards*), has also called for introducing a number of probability concepts throughout the school curricula. At the K-4 level, the *Curriculum Standards* recommends using experiments and real-world examples to introduce students to initial concepts of chance and to develop their probability sense. In grades 5-8, the study of probability is to take on new dimensions as the students

- model situations by devising and carrying out experiments or simulations to determine probabilities;
- model situations by constructing a sample space to determine probabilities;
- appreciate the power of using a probability model by comparing experimental results with mathematical expectations;
- make predictions that are based on experimental or theoretical probabilities; and
- develop an appreciation for the pervasive use of probability in the real world. (NCTM, 1989, p. 109)

This proposal for an increased emphasis on probability is part of a broader call by the NCTM for change in what mathematics is taught. The shift from an industrial to an information society impacted by technology has transformed the aspects of mathematics students need to master. Today, and in the future, mathematical competence means far more than computational proficiency. Instead, basic mathematical literacy, or what the NCTM calls mathematical power, involves an individual's abilities to explore, conjecture, and reason logically and to communicate effectively about mathematics. It includes the ability to apply mathematical methods effectively to solve a variety of nonroutine problems. It involves seeing the interconnections among mathematical ideas as well as applying the growing and ever-changing discipline of mathematics to everyday experiences, both in school and in the world outside the classroom.

In addition to calling for changes in *what* mathematics is taught, the NCTM has called for changes in *how* mathematics is taught. To articulate its vision, the NCTM (1991) prepared a companion document, *Professional Standards for Teaching Mathematics* (hereafter called the *Teaching Standards*). Supported by research findings from cognitive psychology and mathematics education (Case & Bereiter, 1984; Cobb, Yackel, & Wood, 1992; Davis, Maher, & Noddings, 1990; Lampert, 1986; Lesh & Landau, 1983; Schoenfeld, 1987; Steffe & Kieren, 1994), this new vision of teaching is based on the belief that learning occurs as students construct their own understanding of mathematical ideas and concepts by actively assimilating new information and experiences into their already-existing structure of knowledge based on earlier experiences and intuitions. The change in the nature of the mathematical knowledge and abilities expected of students, together with the new understanding of how students acquire such knowledge and abilities, leads the NCTM to call for a new form of pedagogy and a new learning environment in the mathematics classroom.

Two important aspects of the new learning environment emphasized by the *Teaching Standards* (NCTM, 1991) are the mathematical tasks and classroom discourse in which the students are engaged. Teachers are encouraged to select tasks–problems, questions, applications, and projects–that provide students with opportunities to deepen their understanding of the mathematical concepts being studied. The classroom discourse, or how the teacher and students interact with the content and with each other, also conveys messages about the nature of mathematics and mathematical activity. The *Teaching Standards* envisions the classroom as a learning community in which tasks and classroom discourse focus on exploring and making sense of mathematical ideas. Various forms of communication, including oral, written, and pictorial, are involved as the students learn with and from others. Mathematical reasoning is emphasized as the students are encouraged and expected to question one another's ideas and to explain and justify their own ideas.

This new vision of the mathematics classroom also involves changes in the roles of teachers and students. Traditionally, the teacher's role has been to explain a particular procedure or problem to the students, work a few examples on the chalkboard, and then assign practice exercises. The student has been a passive recipient of knowledge who quietly and individually worked on the exercises, getting help from the teacher as needed. In the new mathematics classroom, teachers are to do more than provide direct instruction. They are to be at various times role model, consultant, moderator, and questioner as they select appropriate learning tasks and orchestrate student exploration and discussion. Students, for their part, are to be more actively involved in making sense of mathematical ideas as they interact with each other and with the teacher in solving problems, investigating conjectures, and discussing alternative strategies and solutions.

In summary, the *Teaching Standards* (NCTM, 1991) envisions mathematics teaching and learning that focuses on meaningful mathematical tasks and classroom discourse which deal with mathematical ideas within a learning environment that encourages risk-taking and exploration. Although this vision involves new roles for teachers, their importance would not be diminished. In fact, the *Teaching Standards* claims that "teachers are the key to changing the way in which mathematics is taught and learned" (p. 2). Specifically, the teacher has a key role to play in setting mathematical goals, in creating a classroom environment in which these goals are pursued, and in implementing any proposed curriculum (Romberg, 1988; Shulman, 1986a). More importantly, in their classrooms, teachers are the ones who decide what and how mathematics will be taught and, through their daily interactions with the students, are the ones who shape the students' mathematical experience.

Research has demonstrated that teachers' decisions and actions in the classroom are influenced by their knowledge and beliefs (Clark & Peterson, 1986). The aspect of teachers' knowledge most extensively researched has been their general pedagogical knowledge. This encompasses a body of general knowledge, beliefs, and skills related to teaching, including knowledge and beliefs about learners and how learning occurs, knowledge of curriculum and general principles and strategies of instruction, knowledge and skills in classroom organization and management, and knowledge and beliefs about the aims and purposes of education. In recent years, researchers have begun to focus more attention on other aspects of teacher knowledge which until recently had largely been ignored. Two facets of this emerging picture of teacher knowledge are subject matter knowledge and pedagogical content knowledge.

Although philosophical arguments (Buchmann, 1984), as well as common sense, have long suggested that teachers' knowledge of mathematics may influence their teaching of mathematics, early attempts to explore this relationship were fruitless as a consequence of how *subject matter knowledge* was defined (Ball, 1991). Now armed with an expanded definition of subject matter knowledge, researchers have once again begun to examine how teachers' subject matter knowledge influences instruction (Brophy, 1991). The current conceptualization of subject matter knowledge includes more than just knowledge of the content. It also involves knowledge about the nature of mathematics as well as dispositions toward the subject (Ball & McDiarmid, 1990). In case studies of beginning high school teachers, Steinberg, Haymore, and Marks (1985) found the more knowledgeable teachers offered better explanations, stressed more important ideas, and were less didactic in their instruction. In an elementary study, Stein, Baxter, and Leinhardt (1990) found a fifth-grade teacher with a fragile grasp of mathematical content placed an overemphasis on rules and procedures. Lampert (1985b, 1986, 1989, 1990, 1991), in her own teaching, has demonstrated how a rich understanding of mathematics can contribute to teaching that develops understanding of the concepts and nature of mathematics.

Teacher knowledge research has also begun to explore what Shulman calls *pedagogical content knowledge*. As part of pedagogical content knowledge, Shulman (1986b) includes "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). This knowledge includes the possible examples, illustrations, explanations, demonstrations, and analogies that a teacher might use. It also includes, according to Shulman, an understanding of the preconceptions and misconceptions students may have as well as knowledge of where students may encounter difficulties as they are learning the subject. In conceptualizing what pedagogical content knowledge looks like in elementary school mathematics, Marks (1990a, 1990b) suggested four major areas: instructional aspects of the subject matter, students' understanding of the subject matter, texts and materials for teaching the subject matter, and instructional processes for presenting the subject matter. Carpenter, Fennema, Peterson, Chiang, and Loef (1989) explored one particular aspect of pedagogical content knowledge, namely teachers' knowledge of students' cognitions. They concluded that understanding the knowledge students bring to the topic, the strategies they use in solving problems, and the stages through which they pass in acquiring more advanced strategies allows the teachers to structure instruction so that students can connect what they are learning to the knowledge they already possess.

Statement of the Problem

Thus, there is a growing body of research suggesting that teacher knowledge, including both subject matter knowledge and pedagogical content knowledge, has the potential of influencing what teachers do in their classrooms. Previous research has described teacher knowledge in the areas of place value, whole number operations, and fractions (Ball, 1988a, 1990c, 1991); multiplication and division (Simon, 1993; Tirosh & Graeber, 1989); rational numbers (Post, Harel, Behr, & Lesh, 1991); ratio and proportion (Fisher, 1988); geometry (Mayberry, 1983); functions (Even, 1993); and the concept of zero (Wheeler & Feghali, 1983). No research has considered what subject matter or pedagogical content knowledge teachers possess about probability. Likewise, little is known about the nature of probability instruction.

In this study, probability will be defined as the branch of mathematics dealing with theories of uncertainty, with ways of measuring uncertainty and determining the likelihood of uncertain events, and with the application of techniques involving uncertainty. Probability has been chosen as the focus of this investigation because of its growing importance in the lives of people in today's and tomorrow's world. In addition, the study of probability is a part of the NCTM's calls for reform, not only for its importance but also for its potential to encompass the central goals of the reform effort. Probability instruction offers an excellent setting for incorporating problem solving, reasoning, and communicating about mathematics as well as providing an opportunity for connecting such topics as fractions and decimals to applications of mathematics. Yet, despite its importance and potential benefits, little is known about how probability is or should be taught.

Middle school has been selected as the level for this study because these years are a formative period in the development of probabilistic understanding. After initial experiences with chance in grades K-4, the NCTM sets forth far-reaching goals for the study of probability in grades 5-8. Further, these years are an appropriate place for the study of probability because it is during these years that students are developing skills and cognitive abilities in proportional reasoning which is generally considered foundational to the study of probability (Piaget & Inhelder, 1951/1975). The middle school years are a convenient place to introduce probability because there is generally more room and flexibility in the middle school curriculum than in either the elementary or high school curriculum, allowing the study of probability to be added. It is also a logical place because of its nature as an application of the arithmetic generally reviewed as part of the middle school curriculum.

To be effective in accomplishing the goals of the reform and in developing future citizens who are able to apply an understanding of probability to solve the problems they may encounter, more needs to be known. In order to provide ongoing preparation and support, mathematics educators need to know what knowledge teachers have and what knowledge they need. Mathematics teachers need to know more about what instructional tasks and classroom discourse can be used in the teaching of probability. Therefore, this study will focus on the following questions:

1. What general pedagogical knowledge do middle school teachers demonstrate in the context of teaching probability?
2. What is the teachers' subject matter knowledge of probability?
3. What is the teachers' pedagogical content knowledge concerning the teaching of probability?
   a. What instructional tasks do the teachers use as they teach probability?
   b. What is the nature of classroom discourse during probability instruction?
   c. What is the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability?

Significance of the Study

Until recently, probability has not been viewed as an important part of the mathematics curriculum and teachers have not been prepared to teach it (Garfield, 1988; Shaughnessy, 1992). As a result, probability generally has not been taught at the precollege level. Now, however, the importance of probability has been recognized and the reform proposals call for paying increased attention to probability within the mathematics curriculum, particularly at the middle school level. Combined with a new vision of mathematics teaching that focuses on meaningful mathematical understanding, problem solving, reasoning, communication, and connections, teachers face the challenge of teaching "a mathematics that they never learned, in ways that they never experienced" (Cohen & Ball, 1990, p. 238). As this investigation considers this part of the reform effort, it will potentially contribute to the knowledge base of mathematics education, inform the processes and policies of teacher education, provide information and ideas to middle school teachers, and might, ultimately, improve the teaching of probability in the middle grades.

First, this study will add to the knowledge base of mathematics education by (a) providing a picture of what subject matter knowledge and pedagogical content knowledge teachers possess and (b) painting a portrait of the current practice in teaching probability. The current research focus on teachers' knowledge suggests teaching mathematics for understanding requires teachers whose subject matter knowledge includes a conceptual understanding of the content, as well as an understanding of the nature of mathematics. However, because probability may not have been part of the curriculum when today's teachers were students, many may not have had an opportunity to develop an understanding of the concepts and nature of probability. Because most of the teacher knowledge research has focused on arithmetic or basic geometric knowledge, little is known about teachers' knowledge of probability, especially when it has been learned outside of teacher preparation programs. Further, in today's rapidly changing world, teachers may more frequently be faced with the challenge of teaching new and changing mathematical content. However, it is not known how effectively teachers may gain new subject matter knowledge without the guidance of teacher preparation or staff development programs.

In addition, the teacher knowledge research has shown the value of pedagogical content knowledge, including knowledge of appropriate representations, examples, and explanations to use in presenting the content; awareness of students' common conceptions and misconceptions; and understanding of potential difficulties in the learning process. While research has begun to provide some understanding of the developmental stages of probability knowledge (Piaget & Inhelder, 1951/1975), the intuitions students may possess (Fischbein, 1975), and the potential misconceptions and difficulties students may encounter (Garfield & Ahlgren, 1988; Hope & Kelly, 1983; Shaughnessy, 1981; Tversky & Kahneman, 1974), it is not known how much teachers are aware of these aspects of pedagogical content knowledge or whether this information is considered in developing curriculum or planning instruction. Nor is it known what representations middle school teachers use in presenting the concepts of probability to their students.

Further, not much is known about how probability is currently being taught. Lortie (1975) suggests that teachers' ideas about teaching mathematics are largely shaped by their own experiences as learners of mathematics during the many years of what he calls an "apprenticeship of observation." So one might ask if teachers are using the traditional forms of instruction they most likely experienced and that continue to be so prevalent in teaching basic computation. Or, without the traditional "apprenticeship of observation" experience with probability squeezing them into the mold of traditional instructional formats, are teachers teaching probability in the way envisioned by the mathematics reform movement? The answers to these questions are not known. Therefore, there is a great need to discover what subject matter knowledge and pedagogical content knowledge teachers possess and how probability is being taught.

Second, as this study explores the relationship between teacher knowledge and teacher classroom practice, it will provide guidance to teacher education programs as they seek to prepare teachers to meet the challenge of the NCTM Standards. In the past, one of the obstacles to effective teaching of probability has been the preparation of mathematics teachers (Garfield, 1988). Because many teachers have not had experience with probability in their own backgrounds, prospective and current teachers have a critical need for ongoing preparation and staff development. Meeting this need may involve having more probability taught as part of teacher education or staff development programs, expanding both the subject matter and pedagogical content knowledge of teachers. It may also mean changes in the ways probability is presented to teachers, thereby giving them an opportunity to observe and experience probability instruction as envisioned by the NCTM Standards. By providing a picture of what teachers know about probability and how they teach probability, teacher educators will have a clearer picture of what ongoing preparation will be necessary.

Third, this study will provide useful information and ideas to current and future middle school teachers. Although this study will not be measuring the effectiveness of probability instruction, the portrait painted of current instructional practice may influence middle school teachers to reflect on their own teaching of probability, encourage them to expand their subject matter knowledge and pedagogical content knowledge and, thereby, assist them in teaching a content that may be new to them. In particular, the examples that come out of this study may enrich the teachers' repertoire of representations and applications to use in teaching probability. Further, it will offer them the opportunity to expand their understanding of what intuitive notions students bring to the classroom and what misconceptions need to be addressed as the teacher guides the learning process.

Finally, this study hopes ultimately to facilitate improvement in the teaching of probability in the middle grades as it provides a link between research and practice. As mathematics educators become aware of what knowledge teachers need in order to effectively teach probability, preparation and development programs can be strengthened and updated. And, as teachers become more familiar with what research reveals about the probabilistic thinking of students and methods to affect their thinking about chance events, this knowledge may impact the development of suitable instructional materials and effective methods for teaching probability.

CHAPTER II
REVIEW OF THE LITERATURE

The purpose of this study is to explore the interrelationship between middle school teachers' knowledge of probability and their instructional practices as they teach probability. Two bodies of research inform this investigation and provide the foundation for the theoretical framework. The first portion of this chapter will review the teacher knowledge research. This section will consider, first, how teacher knowledge is conceptualized and, then, will explore the impact of teacher knowledge on mathematics instruction. The second portion of this chapter will focus on the research regarding the learning and teaching of probability. Specifically, this section will look at what is known about students' conceptions of probability, about students' misconceptions and difficulties in learning probability, and about the impact of instruction on students' conceptions and misconceptions.

Research on Teacher Knowledge

Various research approaches have explored the effects of teacher knowledge, generally focusing on subject matter knowledge. The earliest research compiled characteristics of teachers whom others perceived as effective (Medley, 1979). Among these characteristics, students reported that the best teachers knew the subject matter better. Although these results seem intuitively correct, the early studies did nothing to empirically test the influence of teachers' subject matter knowledge on what they did in the classroom or what their students actually learned.

Later research attempted to explore the relationship between certain teacher characteristics (including teacher subject matter knowledge) and student achievement in mathematics. The National Longitudinal Study of Mathematical Abilities, for example, followed 112,000 students from over 1,500 schools in 40 states during the 1960s (Ball, 1991). Twenty teacher characteristics were studied, including years of teaching experience, credits in mathematics, and having a major or minor in mathematics. Overall, Begle and Geeslin (1972) concluded that no single teacher characteristic was consistently and significantly correlated with student achievement. Other studies reported the relationship between student learning and teacher knowledge as measured by achievement tests such as the National Teachers Examination. Their results were also inconclusive. The empirical results were so discouraging, Begle (1979) suggested that "the effects of a teacher's subject matter knowledge . . . seem to be far less powerful than most of us had realized" (p. 53).

Before accepting the counterintuitive conclusion that teacher subject matter knowledge does not influence student learning, the measures of teacher knowledge used must be considered. Are the number of courses in college-level mathematics or the results of a standardized test a reasonable proxy for teachers' mathematical knowledge? In neither case was any attempt made to directly assess teacher subject matter knowledge. Little evidence was presented about how their mathematical knowledge was integrated or whether a relationship existed between the formal mathematics the teachers knew or had studied and the mathematics they taught in the classroom. It is quite possible that any relationships existing between teachers' knowledge and student learning were concealed by the inadequate measures of teachers' subject matter knowledge. Nevertheless, the focus of research shifted away from considering teacher knowledge as an important variable.

In more recent years, as research shifted to teacher thinking and decision making, teachers' knowledge and beliefs began to reappear as potentially significant variables. Based upon the assumption that what teachers know influences what they do in their classrooms, the goal has been to explore the varied facets of teacher knowledge and provide rich descriptions of teachers in action in the classroom. This section will first explore how teacher knowledge is currently being conceptualized by researchers in the field. Then, the research evidence of the impact of teacher knowledge on what teachers do in mathematics classrooms will be considered.

Teacher Knowledge: Its Conceptualization

A number of models of teacher knowledge are currently suggested by researchers in this field. Leinhardt and Smith (1985) consider just two aspects of teacher knowledge: subject matter knowledge and knowledge of lesson structure. Kennedy, Ball, and McDiarmid (1993), as part of the Teacher Education and Learning to Teach (TELT) study, suggest the teaching act is influenced by six domains of knowledge: knowledge of subject matter, knowledge of curriculum, knowledge of the teacher's role, knowledge of pedagogy, knowledge of learning and knowledge of learners.

Researchers at Stanford University (Shulman, 1986a; Wilson, Shulman, & Richert, 1987), as part of the Knowledge Growth in a Profession project, outlined seven components of the professional knowledge base of teaching: subject matter knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and learning, knowledge of educational contexts, and knowledge of educational philosophies, goals, and objectives. In particular, they see pedagogical content knowledge as a separate body of knowledge, though overlapping the domains of content knowledge and pedagogical knowledge. Grossman (1990), a student of Shulman, suggested that these components could be combined into four general areas: subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge, and knowledge of context.

While there are many similarities between these models of teacher knowledge, a great deal of variation exists in how the various components are defined. This section will explore two aspects of teacher knowledge related specifically to mathematics teachers: subject matter knowledge and pedagogical content knowledge.

Subject Matter Knowledge

> That subject matter is an essential component of teacher knowledge is neither a new nor a controversial assertion. After all, if teaching entails helping others learn, then understanding what is to be taught is a central requirement of teaching. The myriad tasks of teaching, such as selecting worthwhile learning activities, giving helpful explanations, asking productive questions, and evaluating students' learning, all depend on the teacher's understanding of what it is that students are to learn. (Ball & McDiarmid, 1990, p. 437)

Although there is general agreement that subject matter knowledge is essential, no consensus exists for what knowing and understanding the subject matter for teaching includes. Current research generally focuses on three dimensions of subject matter knowledge: knowledge *of* the subject, knowledge *about* the subject, and dispositions toward the subject (Ball & McDiarmid, 1990).

The first dimension of subject matter knowledge is knowledge *of* the subject or, in other words, knowing the "stuff" of the field. This dimension is generally what is thought of as subject matter knowledge and has been referred to as content knowledge (Grossman, Wilson, & Shulman, 1989; Shulman, 1986b) or substantive knowledge (Ball, 1991; Wilson et al., 1987). Knowledge *of* the subject refers to knowledge of the factual information and central concepts within the field and the relationships among them. In mathematics, this knowledge includes understanding of particular topics (e.g., decimals and geometry), procedures (e.g., adding fractions and solving equations), and concepts (e.g., place value and zero), as well as the relationships among these topics, procedures, and concepts (e.g., the relationship between decimals, adding fractions, and place value).

In considering content knowledge, it is valuable to make a distinction between procedural and conceptual knowledge, or what Skemp (1987) calls instrumental and relational understanding. Historically this has been recognized as the distinction between algorithmic skill and the underlying understanding, not only knowing "what to do" but "why it works" (Burns, 1986). Procedural knowledge, as defined by Hiebert and Lefevre (1986), encompasses two kinds of information. One part consists of knowledge of the formal language or symbols of the system and familiarity with the accepted conventions of how the symbols can be arranged. The second part of procedural knowledge consists of knowing the rules, algorithms, or procedures for solving mathematical problems. As Skemp suggests, this kind of knowledge involves learning an increasing number of plans, separate from each other, to get from a given starting point to the result. For example, "invert and multiply" is a plan that tells a student what to do for division by a fraction. While this type of knowledge may be useful, it is limited and the steps of various procedures can be confused and interchanged by students.

In contrast, conceptual knowledge or relational understanding is characterized by knowing not only what method works but why. It is characterized as knowledge that is rich in relationships or connecting networks (Hiebert & Lefevre, 1986). For example, a student or teacher may know and be able to apply the various formulas for area of a triangle, rectangle, parallelogram, and trapezoid. If, however, these individual pieces of information are linked and the relationships between the area formulas are known, this knowledge would be an example of conceptual knowledge. Further, such conceptual knowledge of area would allow a student to apply that knowledge in a variety of different contexts and, perhaps, invent mathematically appropriate ways to find the area of a figure without remembering any of the formulas.

Understanding of any subject matter, however, goes beyond knowledge of the facts, procedures, or concepts of a domain. Shulman (1986b) argues that teachers must be able to do more than present the accepted truths in a domain. They must also be able to explain to students why particular knowledge is justified, why it is worth knowing, and how it relates to other ideas. This second dimension of subject matter knowledge, knowledge *about* the subject, encompasses a knowledge about the intellectual fabric and nature of the subject matter, and includes understanding of what Schwab (1964, 1978) calls the substantive and syntactic structures of the subject matter. The substantive structures of a discipline include those explanatory frameworks or paradigms that are used to organize the field, interpret the data, and guide inquiry. The syntactic structures of a discipline are the canons of evidence and proof within the discipline. They are the means by which new knowledge is introduced and accepted into that community (Grossman et al., 1989). In mathematics, for example, some facts are established by convention, others as a result of logical construction. Positive numbers running to the right on the number line or the use a base-ten system of numeration has been established arbitrarily. On the other hand, that division by zero is undefined or that any number to the zero power (e.g., 6^0) is equal to one is not arbitrary but is established by logical argument (Ball & McDiarmid, 1990). Another aspect of knowledge about mathematics is an understanding of what it means to *do* mathematics, knowing the fundamental activities of the field–looking for patterns, making and justifying conjectures, validating solutions, and seeking generalizations.

The third dimension of subject matter knowledge are the dispositions toward the subject that students develop. Students acquire likes and dislikes for particular topics and activities. They develop inclinations to pursue certain questions or problems and to avoid others. In addition, they develop conceptions of themselves as good at particular subjects and not at others. These beliefs and dispositions constitute a critical element of subject matter knowledge for they influence how one understands the subject. In particular, how people feel about mathematics and about themselves as knowers of mathematics interacts with what they understand. These findings shape their participation in and experience of mathematics (Ball, 1988a).

When considering what aspects of subject matter knowledge were important for teachers specifically, Ball (1988a) and her colleagues at the National Center for Research on Teacher Education (1991) identified several characteristics. First, they suggested teachers' knowledge should be correct or in agreement with the accepted knowledge in the field. Second, teachers' knowledge should be explicit–they should be able to explain. While tacit knowledge may be valuable in mathematical activity, Ball maintains it is inadequate for teaching. Being able to "do it" oneself is not sufficient. Rather, Ball argues a teacher must be able to talk about mathematics, about the judgments made, about the meanings and reasons for certain relationships or procedures, not just describe the steps of an algorithm. Third, Ball suggested teachers need to understand the underlying meanings and connections. Treating mathematics as a collection of separate facts and procedures inhibits meaningful understanding and misrepresents the nature of the discipline. Rather than each problem needing a different rule, which has been memorized individually, teachers and students need to see the connected, dynamic nature of mathematics. Finally, with regard to knowledge about mathematics, the researchers proposed teachers need knowledge about the nature of mathematics and justification–explaining, verifying, and proving mathematical propositions.

In summary, subject matter knowledge for teaching mathematics involves knowing the "stuff" *of* mathematics–the facts and procedures–as well as understanding the underlying concepts. It includes knowing the relationships within mathematics and seeing the interconnections between mathematics and other content areas. Further, subject matter knowledge includes knowledge *about* mathematics–how the field is organized, how knowledge grows and is evaluated, and what it means to do mathematics. Finally, it includes the willingness to participate in the mathematical experience.

Pedagogical Content Knowledge

> A second kind of content knowledge is pedagogical knowledge, which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*. I still speak of content knowledge here, but of the particular form of content knowledge that embodies the aspects of content most germane to its teachability. (Shulman, 1986b, p. 9)

Although teachers utilize both general pedagogical knowledge and subject matter knowledge in teaching, research has indicated teachers also draw upon another form of knowledge: knowledge that is specific to teaching particular subject matter (Grossman, 1990). Some refer to this blending of content and pedagogy as content-specific cognitional knowledge (Peterson, 1988) or as subject matter specific pedagogical knowledge (Tamir, 1988). Shulman calls this knowledge specific to teaching content, pedagogical content knowledge.

> Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations–in a word, the ways of representing and formulating the subject that make it comprehensible to others. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (Shulman, 1986b, p. 9)

Although Shulman's terminology is new, the concept of pedagogical content knowledge is not. Dewey (1902/1969) admonished teachers to learn to "psychologize" their subject matter for teaching, to rethink disciplinary topics and concepts in order to make them more accessible to students. In related work, McEwan (1987) describes "pedagogic interpretations," in which teachers consider both their knowledge of subject matter and their knowledge of students' prior knowledge and conceptions as they select appropriate topics and formulate suitable representations of the content.

In an effort to refine the conception of pedagogical content knowledge, Marks (1990a, 1990b) sought to collect a number of examples of pedagogical content knowledge from the elementary mathematics classroom and to formulate a characterization of the sources of pedagogical content knowledge. Because this form of knowledge is specialized and contextualized, Marks (1990b) focused his study on a particular context, specifically the equivalence of fractions as taught in fifth grade. Eight teachers participated in the study, selected from a larger sample of 20 teachers to provide contrasting backgrounds of knowledge and experience.

The data used for this study were a subset of the data gathered as part of the Teacher Assessment Project at Stanford University, a large research project on alternative forms of teacher assessment directed by Lee Shulman. The eight task-based exercises, each taking between 45 minutes and 90 minutes to complete, asked teachers to respond to a semi-contextualized scenario similar to what they might encounter in the performance of their duties as a teacher. The tasks all focused on teaching equivalence of fractions in the fifth grade and included evaluating different kinds of instructional materials, planning a lesson, critiquing a classroom videotape, and diagnosing and remediating students' misunderstandings. The participants and the exercises were both partitioned into two sets and then matched so that responses to four questions for each participant were included in the data. Because the intent was to generate the broadest possible description, the data were pooled from all eight candidates.

Using the constant comparative method described by Glaser and Strauss (1967), Marks analyzed the interview data by an iterative process of coding chunks of text, restructuring the categories, and recoding the data until the process stabilized. The final product was a detailed taxonomy of the three knowledge categories–subject matter knowledge, general pedagogical knowledge, and pedagogical content knowledge.

The first goal of the study was to provide a description and structure for pedagogical content knowledge. The picture that emerged focused on four major areas: subject matter for instructional purposes, students' understanding of the subject matter, media for instruction in the subject matter (i.e., texts and materials), and instructional processes for the subject matter. These areas appeared to be highly interrelated. For example, modifications of a teacher's instructional methods may be based on students' indications of understanding. Figure 1 depicts the four major areas, their interrelationships, and the subcategories within each area.

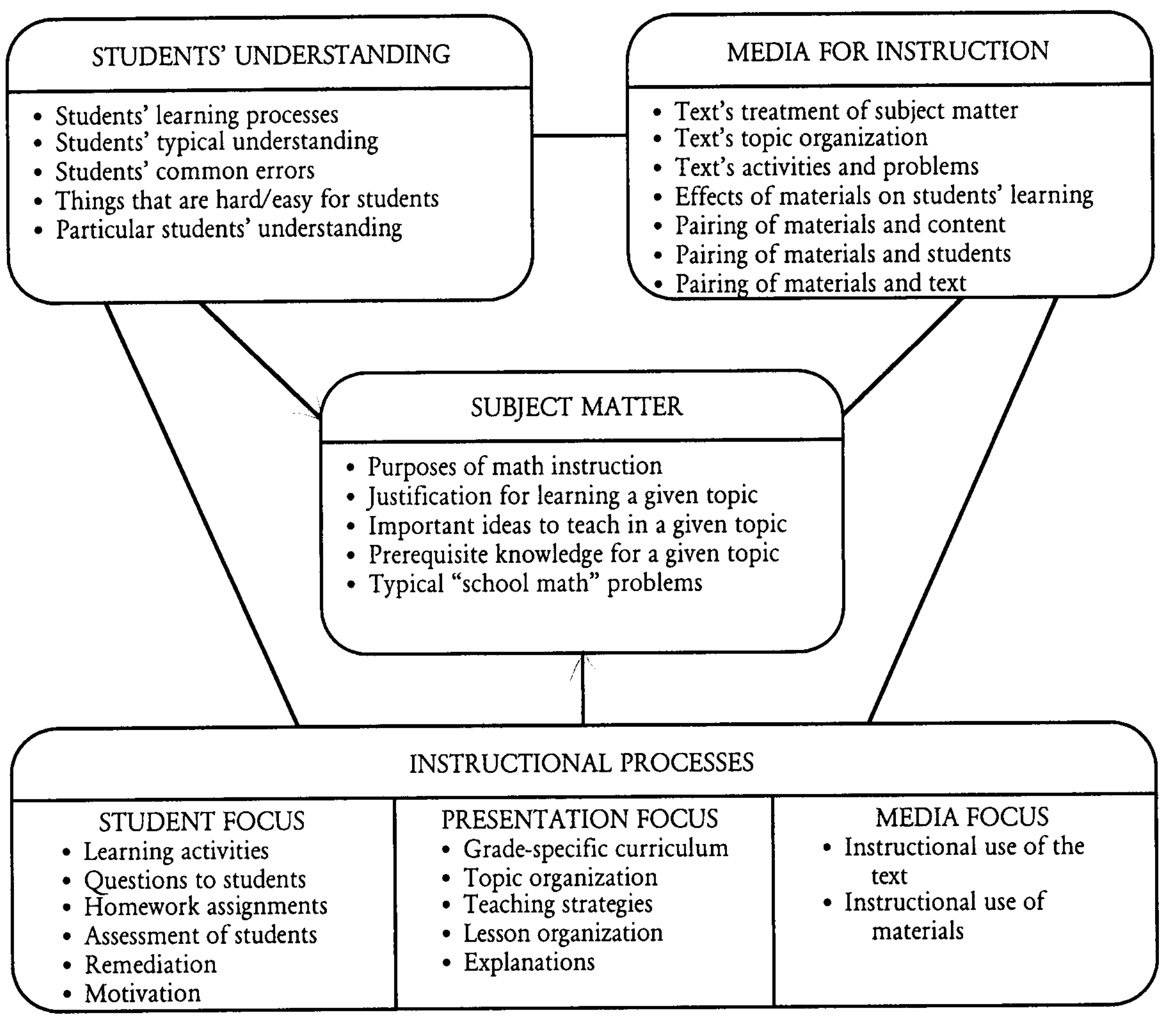


Figure 1. A structure for pedagogical content knowledge. (Marks, 1990b, p. 86)

In order to reach the second goal of forming a characterization of pedagogical content knowledge, Marks began to consider the questions of how it is related to other forms of knowledge and how it might be generated. The commonly accepted view of pedagogical content knowledge is that it is an adaptation of subject matter knowledge for pedagogical purposes. This process has been called transformation (Shulman, 1987), representation (Ball, 1988a), or psychologizing the subject matter (Dewey, 1902/1969). Marks suggests three possible derivations of pedagogical content knowledge: (a) an interpretation of subject matter knowledge, (b) a specification of general pedagogical knowledge, and (c) a synthesis of both general content knowledge and subject matter knowledge or an extension of elements of pedagogical content knowledge.

Clearly, as the commonly held view suggests, certain aspects of pedagogical content knowledge have their origin in subject matter knowledge. Examples include the sequencing of topics for instruction and utilizing pedagogically useful representations of the content. For instance, a teacher in the study suggested explaining the equivalence of 1/3 and 0.333 (it is assumed that what is meant is 0.333...) by saying that, like William being called Bill, they are just different names for the same thing. The process of examining the content for its structure and significance, and transforming it to make it comprehensible to a given group of learners is what Marks calls interpretation.

However, Marks suggests that other facets of pedagogical content knowledge are rooted in general pedagogical knowledge, including teachers' use of questioning strategies and their cognizance of students' learning processes. Although they may study these topics in generic terms, teachers must apply the ideas in particular content areas. For instance, one of the interviewed teachers observed that the teacher in the videotaped lesson should have asked the student to represent 100% on the geoboard in order to check their understanding. Specification is the term Marks uses to describe the process whereby a broadly applicable idea is appropriately employed in a particular context.

Finally, Marks proposes that yet other aspects of pedagogical content knowledge are derived more or less equally from general pedagogical and subject matter knowledge or from previous constructions of pedagogical content knowledge. Examples include the design of learning activities, development of teaching strategies, and awareness of students' misconceptions. For example, while evaluating a textbook, one of the teachers pointed out a diagram showing five balls of different sizes and asking what fraction of them are yellow. Because students generally think of fractions as ratios of areas, and not the number-of-objects ratio intended in the diagram, the teacher suggested students might be confused. Marks contends the knowledge displayed in that assertion represents a synthesis of elements from both general pedagogical knowledge and subject matter knowledge together with an extension of elements of pedagogical content knowledge.

Thus, in characterizing its sources, Marks envisions pedagogical content knowledge as "the offspring of general pedagogical knowledge and subject matter knowledge, a child bearing features of both parents yet distinct from either of them" (Marks, 1990b, p. 181).

And, as a child needs both parents in order to be conceived, the conception of pedagogical content knowledge seems to rely on a sufficient level of subject matter knowledge and general pedagogical knowledge being present.

In summary, pedagogical content knowledge represents a form of knowledge central to the teacher's work. As such, it includes a knowledge of the subject matter that is aware of the important ideas to be taught and the purpose and justification for teaching them. It involves knowledge of the understanding students bring to the learning process and the potential difficulties that await them. Finally, it includes knowledge of the media (text and materials) and instructional processes that will make the subject matter comprehensible to the students.

Teacher Knowledge: Its Impact in the Mathematics Classroom

With the conception of teacher knowledge now being more than courses taken or scores on a standardized test, research in recent years has again begun to explore what teachers know and how that knowledge impacts what teachers do in their classrooms. For this research study, the focus is on the impact of teacher knowledge in the mathematics classroom. This section will explore four research programs that have been conducted in the past 10 years, three of these at the elementary level and one at the secondary level.

Lampert: Impact of an Expert's Knowledge

The first research program, that of Magdalene Lampert, formerly of Michigan State University, demonstrates how a thorough understanding of mathematics can influence what a teacher does in the classroom. Her own writings about her teaching of elementary school mathematics (Lampert, 1985b, 1986, 1989, 1990, 1991) and a case study done by a colleague (Ball, 1991) provide a vivid description of how a rich understanding of mathematics can lead to mathematical instruction that is different from what traditionally happens in the elementary classroom.

Lampert's own understanding of the content of mathematics, as well as the substantive and syntactic aspects of its nature, shape what she is trying to help her students learn. To her, mathematics is a system of human thought, not merely a fixed body of procedures. Because she holds this perspective, she believes that students must have experience in developing meaning in the mathematics they study, in pursuing and verifying their own mathematical hunches and in learning to make mathematical arguments establishing mathematical truth within the context of a discourse community (Ball & McDiarmid, 1990). This stands in stark contrast to how one often encounters mathematics in school, where "*doing* mathematics means following the rules laid down by the teacher, *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher" (Lampert, 1990, p. 32). Approaching research from a sociological and epistemological perspective, her purpose was to examine whether it was possible to bring the practice of knowing mathematics in school closer to what it means to know mathematics within the discipline.

Research design. In the new role of teacher-scholar, Lampert has taught mathematics in fourth- and fifth-grade classes for 6 years and collected data on her own teaching during three of those years. Though never explicitly measured, her background in mathematics is extensive. Further, as a teacher, she has over 10 years of experience in the elementary classroom. As a researcher, she was part of the ongoing research at Michigan State University, where she was a professor.

While the elementary classes she taught have varied from year to year, they generally were typical of the heterogeneous classroom. For example, one year she taught a group of 29 fifth graders, of whom about a third spoke English as a second language. Both a wide range of socioeconomic backgrounds and a wide range of math skill levels were represented in her classes (Lampert, 1989).

In the course of her teaching in the elementary school, data were gathered on both teaching and learning (Lampert, 1991). To describe the teaching taking place, researchers made audiotapes of lessons for 6 months and videotapes of two curriculum units. An observer recorded speech and visual communication occurring during lessons at least three times a week over the 3 years. No further information is provided about how the observation data were collected. From the learners' perspective, notebooks of the students' daily work and homework papers were collected. These data included samples of the writing and drawing students did to represent their thinking.

The description and interpretation of what was happening in the classroom were carried out in three stages. Each day, Lampert described in detailed field notes how the lessons were planned and implemented. In her reflections, she considered the content of the curriculum, the responses and thinking of the students, and the social interactions developing in the classroom culture. The field notes also included an initial analysis of the planning process itself, the implementation of the lesson, and the students' work (Lampert, 1991).

At the second stage, the lessons were analyzed and compared across the entire year. Using triangulation among the different sources of data, themes in the field notes and mathematical and social patterns in the lessons were noted. Finally, the body of data was analyzed collaboratively by educational psychologists, sociolinguists, and mathematicians. In this analysis, each participant used the methodological tools and theoretical frameworks about knowledge drawn from his or her own discipline.

In a blending of the methods of action research and interpretive social science, the data that Lampert had collected on her own teaching were treated as a text to be analyzed in a process she called "textual exegesis." The purpose of the analysis process was to understand the meaning of the "text." Because teaching is a task that can be interpreted in multiple ways and that involves managing multiple and often contradictory goals (Lampert, 1985a), researchers with backgrounds in mathematics and the social sciences were involved in order to consider the data from various perspectives. Lampert (1991) notes the purpose of the analysis was not to verify general propositions about learning or teaching, but to further our understanding of the character of this particular kind of human activity known as teaching.

Results. In the descriptions of her teaching, Lampert provides examples of how she uses her knowledge of mathematics to structure what is done in her classroom. Her goal is to present a realistic picture of the nature of mathematics by helping her students make sense of the mathematics they study, become doers of mathematics, and become their own authorities in determining truth. Within this goal structure, the impact of her knowledge is demonstrated in the choice of representations, the design of activities, and the guidance of classroom discourse.

One of Lampert's objectives is for her students to gain a conceptual understanding of mathematics. As an illustration of how this impacts her teaching, she described a series of lessons in which her students were learning to compare decimal numbers (Ball & McDiarmid, 1990). For example, they were to decide which of 0.0089 and 0.89 was greater or if the numbers were equal. The answer to this question is usually taught as an algorithm, either comparing place by place or adding zeros after the digits to the right of the decimal point until the numbers being compared have the same number of decimal places and then seeing which of the numbers is larger by ignoring the decimal point. Instead of teaching only a procedure where "the correct answer is ascertained through a combination of trust in authority, memory, and mechanical skill" (Lampert, 1989, p. 226), Lampert wanted her students to develop conceptual understanding of place value with decimal numbers. Through the use of various representational tools, including money, the number line, and pieces of a circle, she established a conceptual framework in which the students could explore the meaning of decimal numbers.

The representations chosen were designed to bridge from familiar contexts to the more abstract, allowing the students to make sense of the abstractions. For example, by using the numerical representation of amounts of money and trading among different denominations of bills and coins to begin the series of lessons on decimal numbers, she was using a context universally familiar to the students, yet one inherently containing the essence of the decimal concepts. In such a framework, the students could examine the reasonability of their own assertions as they began to make sense of the mathematical principles.

The impact of her own conceptual knowledge of mathematics on her choice of representations is shown in the following example. In a lesson on multiplication, Lampert (1986) asked her fourth-grade pupils to come up with a story for 28×65. One student suggested 28 glasses with 65 drops of water in each glass. Lampert accepted this proposal, but, feigning laziness, said she did not want to draw 28 glasses so she drew big jugs that held the equivalent of 10 glasses. Using this picture allowed her to represent the decomposition of numbers underlying the multiplication algorithm. While the objects were suggested by a student, Lampert chose specifically how to represent the student's idea in order to emphasize an essential conceptual component of the procedure: grouping by tens.

Her belief that students must be actively involved in constructing their own understandings and meanings also influenced her choice of instructional activities. To achieve the active involvement of her students, she chose to pose problems and use tools which would be meaningful to them. In some cases, the problems posed were within the context of the activities, such as the fifth-grade lesson exploring rates and ratios based on "The Voyage of the Mimi" (Lampert, 1985b). Sometimes, as seen in earlier examples, the students posed the problems. Lampert suggests that placing mathematics problems in such real-life contexts allows all students to participate and learn something from the instruction.

In addition to learning mathematics with understanding, Lampert has a second objective for her students, which is interconnected with the first. She wants her students to learn what mathematics is and how one engages in doing mathematics. As a result, practice takes on an entirely new meaning. Instead of doing repeated examples of the skill being taught, students engage in the practice *of* mathematics. The class activities involve the students in what it means to think about and do mathematics as mathematicians do–looking for patterns, making conjectures, and exploring the validity of their conjectures. In a series of lessons on properties of exponents, for example, Lampert (1990) began by having the fifth-grade students look for patterns in the table of squares of 1 to 100. During the activity, the students invented a way of thinking about the relationships, a way that Lampert could then build upon to take them into new mathematical territory. To answer the questions, "What is the last digit in: 5^4? 6^4? 7^4?" students were challenged to make conjectures and prove them without doing all the multiplication. In the process of answering this question the students were exploring the key ideas behind how exponents work and at the same time learning more about what it means to engage in mathematics.

In their practice of doing mathematics, Lampert has a third objective for her students. She wants to foster a habit of discourse in the classroom in which questions of reasonability about what is done–the assertions made, the procedures invented, and the solutions proposed–are answered by the students themselves. She wants her students to know mathematics as a subject in which legitimate conclusions are based on reasoning, not just what the textbook or the teacher says (Lampert, 1989). In her classroom, she encourages the students themselves to make and test their own ideas and assertions, judging their reasonability within a mathematical framework. When a student in her class asserted 0.0089 was a negative number, for example, Lampert "interpreted his claim as a conjecture whose validity could be judged by the classroom mathematical community rather than as a misconception that she should correct" (Ball & McDiarmid, 1990, p. 438).

Within the context of the classroom community, Lampert assumed a variety of roles. For one, she participated in doing and discussing mathematics with her students. As part of this, she engaged the children in dialogue and continually responded to them by saying, "How do you know that?" or "Why do you think that?" By beginning with problems posed in familiar and often concrete contexts, the students could propose and discuss solutions in a relatively risk-free environment. In the process, the class developed a common language for talking about the topics. With the familiar context and common language to fall back on to provide justifications, the class could then extend their understanding to new mathematical domains.

Secondly, to further their understanding of the concepts, nature, and discourse of mathematics, Lampert modeled the kind of mathematical thinking and activity she expected of her students. When answers to problems were suggested she considered these as hypotheses open for discussion and revision. When an incorrect assertion was made, she could guide the class in exploring the student's assumptions and thinking, leading the student to revise his assertion. As she challenged the students to explain themselves to each other, they could develop the habit of asking themselves if the answer and the procedure they used to arrive at it were reasonable. Further, as an "expert" model, Lampert could bring the mathematical tools of language and symbols into the discussion as additional tools for the students to use.

A third significant role within the classroom community was managing the direction, balance, and rhythm of classroom discourse. Her knowledge of mathematics guided her decisions of which ideas and questions to pursue, and which to redirect. Following the illustration of 28×65, for example, one student suggested "another way of thinking about it." Supporting the view that there may be different ways to approach problems, Lampert gave equal weight to this student's explanation. In another setting, she used an incorrect answer to one question as a stepping stone to the next mathematical idea she wanted to pursue–if $7^4 \times 7^4$ was not 7^5, then what power was it?

Because of her knowledge and understanding of mathematics, Lampert is able to create a rich mathematical environment where students can be actively engaged in mathematics, constructing their own ways to understand mathematics and developing the ability to reason independently about their answers and procedures. Her vision of mathematics and her model of teaching are consistent with what is called for in the *Teaching Standards* (NCTM, 1991). As her own teaching has shown, orchestrating this kind of instruction in an elementary classroom requires that the teacher draw simultaneously on her understanding of the content and concepts, as well as her understanding of the substantive and syntactic aspects of the nature of mathematics. It also requires an appreciation of how mathematics can be used in a variety of situations and a confidence in one's own ability to see mathematics as a set of ideas that make sense.

Leinhardt: Impact of Expert and Novice Teachers' Knowledge

A second line of research focusing on the impact of teacher knowledge involves investigations of expert and novice teachers. Gaea Leinhardt and her colleagues at the Learning Research Development Center at the University of Pittsburgh have observed and interviewed expert and novice teachers with the goal of providing in-depth descriptions of the mental structures of skilled teachers. This work is based on the belief that teaching is a "complex cognitive skill amenable to analysis in a manner similar to other skills described by cognitive psychology" (Leinhardt & Greeno, 1986, p. 75).

The skill of teaching, according to Leinhardt and her colleagues, is determined by at least two fundamental systems of knowledge: lesson structure and subject matter. Knowledge of lesson structure involves the knowledge required for constructing and conducting lessons (Leinhardt & Greeno, 1986). This form of knowledge includes general plans for coordinating lesson segments, interacting with students, and organizing instruction within a day and within a unit. Subject matter knowledge is defined as knowledge of the content to be taught (Leinhardt & Smith, 1985). This knowledge, in their view, does not necessarily include knowledge of more advanced mathematics, but includes a depth of knowledge about topics particular to school mathematics, such as subtraction with regrouping, multiplication of multidigit numbers, and equivalence of fractions. In this form of knowledge, Leinhardt, Putnam, Stein, and Baxter (1991) include

> knowledge about ways of representing and presenting content in order to foster student learning or construction of meaningful understanding. It also includes knowledge of what the students bring to the learning situation, knowledge that might be either facilitative or dysfunctional for the particular learning task at hand. This knowledge of students includes their strategies, prior conceptions (both "naive" and instructionally produced), misconceptions students are likely to have about a particular domain, and potential misapplications of prior knowledge. (p. 88)

These aspects correspond to those features Marks included as part of pedagogical content knowledge.

Using the methodology of cognitive science, the researchers first hypothesized a model of the mental structures involved in teaching. According to their model, teaching is a cognitive skill that involves such features as (a) an agenda or master plan, (b) activity segments, and (c) routines (Leinhardt & Greeno, 1986). The agenda is the dynamic master plan that assembles the goals and actions and organizes the action segments of the lesson.

The activity segments, characterized by their own plans, include such actions as checking homework, presenting new material, and assigning independent seatwork. Routines are relatively low-level activities, such as handing out papers, that are performed frequently using patterns known by both teacher and students.

Research design. To explore the validity of their hypothesized cognitive model of teaching, Leinhardt and her colleagues have conducted a series of investigations studying the knowledge of expert and novice elementary mathematics teachers. These studies have focused on subject matter knowledge (Leinhardt & Smith, 1985), knowledge of lesson structures (Leinhardt & Greeno, 1986), the development of an expert explanation (Leinhardt, 1986, 1987), expertise in instructional lessons (Leinhardt, 1988), and elements of expert mathematics lessons (Leinhardt, 1989).

For their research, expertise was defined by the success of the teachers' students on standardized achievement tests. The experts were initially identified by choosing teachers in grades 2 through 4 whose students' achievement growth scores (averaged residual gains) were in the top 15% for at least 3 years in a 5-year period. From this group, teachers were chosen if the final achievement was in the top 20%. These teachers had also been confirmed as experts by local supervisors and principals. The novices were preservice teachers who were student teaching during their last semester of teacher education. These student teachers had been recommended by their supervisors as among the top teachers in their cohort. They had been teaching in their fourth-grade classes for at least 4 weeks, and had full responsibility for the mathematics instruction of their students.

During the study, extensive data were collected on the teachers. They were observed for approximately one fourth of the mathematics classes they taught during a 3-month period. Observations included at least (a) three classes with open-ended notes, (b) 1 week of consecutive classes, (c) a complete day once during that week, and (d) 3 separate days in which pre- and post-interviews were conducted about the teacher's plans. In addition, up to 10 hours of instruction were videotaped. Of these, preclass planning interviews and postclass interviews using stimulated recall based on the videotapes were held for three to five classes. Teachers were also interviewed about their subject matter knowledge and, in some cases, completed a card sort of mathematical topics. Transcripts of the observations and videotapes became one data base and transcriptions of the interviews became a second.

The analyses performed with the data varied somewhat, depending on the focus of the particular investigation. In some cases, semantic nets were developed for the mental structures of the teacher or the activity segments of the lessons. These semantic nets were then compared to hypothesized models. In other cases, the transcriptions were analyzed for the presence or absence of certain characteristics, or to determine if there were any consistent patterns. Intensive samples were subjected to more detailed analysis and interpretation in some instances.

Results. The proposed cognitive model proved to be powerful in characterizing teaching and in differentiating expert and novice teachers. As the research progressed, other key sites were identified where teachers' subject matter knowledge impacted the instructional process. These key sites included explanations and what the researchers called "curriculum scripts." The discussion of the results will first look at the general differences between expert and novice teachers. Next, the discussion will look through some key windows to see the impact of teachers' subject matter knowledge on their agendas, curriculum scripts, and explanations. Finally, an extension of the original research reveals some of the difficulties that result from a lack of sufficient subject matter knowledge.

As a result of their observations, Leinhardt and her associates found expert teachers used many complex cognitive skills to weave together elegant lessons that build upon and advance material introduced in earlier lessons. Expert lessons also exhibited a highly efficient internal structure characterized by smooth transitions, by a clearly visible system of goals, and by effective use of well-practiced routines. Expert teachers constructed lessons around a core of activity segments which were more consistent over time and for which the teachers had schemata to activate for efficient movement from teacher control to individual seatwork. The presentation of content by experts was frequently characterized by logical explanations connected with prior knowledge and by careful use of multiple representations.

In contrast, novice teachers' lessons were characterized by lesson segments that were erratic in pattern and length, by lesson structures that were fragmented with long transitions between lesson segments, by goals that were ambiguous and often abandoned, and by routines that were not known and applied. Novice lessons displayed constantly changing patterns including the scenarios of jumping from presentation to practice, of failing to complete a coherent lesson, and of beginning a class in the middle of an uncompleted presentation without appropriate review or recapitulation. The lack of instructional patterns and schemata increased the cognitive load of the novice teachers, making it more difficult to access and utilize their subject matter knowledge during instruction.

To explore the impact of subject matter knowledge on lesson structure more specifically, the discussion will review the instructional processes of teachers through three key windows, namely agendas, curriculum scripts, and explanations. Leinhardt and Greeno (1986) hypothesized that the conduct of a lesson was based on a dynamic operational plan that they called an agenda. These agendas were not formal written lesson plans, but rather mental plans containing the overall goals and actions for the lesson. Though both experts and novices had relatively brief agendas, the experts' agendas were richer and more structured. For example, before teaching a lesson on fractions, three expert teachers and two novice teachers were asked what was going to happen in the lesson (Leinhardt, 1988). The experts provided a richer and more detailed plan organized around the mathematical content to be taught. The expert teachers described more separate instructional actions as well as more explicit references to student actions, revealing they maintained parallel plans for themselves and the students. Also, they mentioned test points within the lessons more frequently, demonstrating an understanding of critical points in the learning of the content. Finally, experts provided a sense of instructional logic or flow within their agendas. In contrast, the novice teachers described their plans in more general terms, often just giving the title of the lesson or the topic to be covered. In one particular instance, the novice teacher lacked any usable plan. After teaching a lesson on simplifying fractions that had failed, she had retaught the lesson and given a homework assignment. Her entire plan for the following day was to go over the homework and perhaps go on, though what to go on to was not specified. This teacher is contrasted with an expert who searched for a different approach to use in reteaching a lesson that had failed.

Like agendas, curriculum scripts offer a window into teachers' use of subject matter knowledge in teaching. The concept of curriculum script emerged from Putnam's (1987) research on tutoring in which he found that teachers generally use a limited set of teaching actions when presenting content and responding to student errors. Leinhardt et al. (1991) define a curriculum script as a "loosely ordered set of goals and actions that a teacher has built up over time for teaching a particular topic" (p. 84). In providing structure to the content of the lesson, these scripts provide cohesiveness and include sequences of ideas or steps to be introduced, representations to be used, markers for likely sites of difficulty, and sketches of explanations to be given. Because curriculum scripts represent the transformations of teachers' subject matter knowledge into a form that is accessed during teaching, Leinhardt et al. found teachers' scripts an especially fruitful site for exploration. With rich and flexible knowledge of a particular domain, expert teachers often had curriculum scripts whose overall goals were clear but whose subgoals may have been organized as a network permitting more flexible responses to student input. Because of the uncertainty in where such flexible lessons might go, however, these scripts required more accessible subject matter knowledge on the part of the teacher. The lack of accessible knowledge or well-developed curriculum scripts led to problems for the novices. In one lesson, the student teacher ran into serious difficulties when trying to introduce a trick for multiplying by 9 that her supervising teacher had shown her. Because she was borrowing the script and had not integrated it into her own knowledge, she was unable during the lesson to access the knowledge of how to do the procedure herself, how to present it to the students, or how to fit it into the overall lesson.

Explanations offer a third window into teachers' subject matter knowledge. The ways in which teachers design explanations–the examples they select, the representations they use, the demonstrations they perform, the experiences they arrange for students–reflect knowledge of how to teach subject matter. Based on their work with expert elementary mathematics teachers, Leinhardt et al. (1991) have suggested certain goals that are common to good explanations. First, good explanations are built on subskills and representations that are familiar to the students. In one study, experts used commonly known representations 88% of the time compared to 25% for novices (Leinhardt, 1988). Second, the problem or issue being explained is clearly identified. For instance, to begin a lesson focused on negative numbers, one teacher presented the students with the unfamiliar problem of 50 - 70 (Baxter, Stein, & Leinhardt, 1991). Third, when multiple representations (i.e., numerical and concrete) are used, expert teachers tend to make clearer connections between the representations and to complete the various aspects of the demonstrations more often than novices. In one expert lesson on subtraction with regrouping, for instance, the semantic net was richer and denser with the written representations connected to the use of bundled sticks and felt strips (Leinhardt, 1987). Fourth, the conditions of use are generally specified. In the lessons on subtraction, the expert teacher had students note which problems did not require regrouping and which were "foolers" requiring regrouping. Finally, the researchers suggested good explanations identify the undergirding principles. In contrast, they suggested novices' explanations are often incomplete and disjointed to a student first learning the material though they may be perfectly intelligible to someone who already knows the topic (Leinhardt & Putnam, 1986).

In an extension of the original series of investigations, Stein et al. (1990) explored the relationship between an experienced teacher's knowledge of mathematics and his instructional approach to teaching a fifth-grade unit on functions and graphing. This teacher had 18 years of teaching experience, primarily at the upper elementary level. Although he had not been identified by the same procedures as the experts in the earlier studies, he had been nominated by the mathematics resource teacher and the mathematics curriculum consultant as an exceptional fifth-grade teacher. The teacher was observed and videotaped as he taught a 25-lesson unit on functions and graphing. The teacher was also interviewed about his subject matter knowledge and asked to complete a card sort task. When the results of the interview and card sort task were compared to the results of a mathematics educator on the same interview and card sort task, the teacher's subject matter knowledge of functions and graphing was found to be missing some key mathematical ideas.

Stein et al. (1990) concluded that the teacher's limited subject matter knowledge resulted in a narrowing of instruction in three ways. First, the teacher's incomplete understanding limited the degree to which he could help the students establish a groundwork for future mathematical learning. For example, the teacher's definition of a function lacked the key ideas that a function can exist without a rule and that each input number has a unique output number corresponding to it. These very important ideas were also absent from his classroom approach, leaving students with an inadequate foundation for future work with functions. A second narrowing of instruction was the overemphasis of a limited truth. In comparing a function machine to a computer, the teacher described three major components: an input, a program or function, and an output. The organizing feature of the lesson was what the teacher called the "two-out-of-three" rule: If two of the three pieces of information are known, the third can be determined. This overutilization of a somewhat inaccurate procedural rule presented a limited picture of functions. Finally, the teacher's limited subject matter knowledge led to missed opportunities to foster meaningful connections between key concepts and representations. In particular, his knowledge lacked the idea that mathematical relationships can be represented in several different ways. The only connection made between functions and graphing was a "self-correcting" connection: Graphs can be used to self-correct one's solutions to function-machine problems where straight lines indicate correct answers (which is not necessarily true). In fact, because he saw few connections between the numerical representations of functions and graphs, the teacher suggested "teaching graphing as a separate topic might be a better way to go" (Stein et al., 1990, p. 651).

In conclusion, Leinhardt and her colleagues suggested teachers' knowledge impacts both the content and the processes of instruction, affecting both what they teach and how they teach. The two aspects of teachers' knowledge, namely lesson structure knowledge and subject matter knowledge, are closely intertwined. Subject matter knowledge which can be accessed during lesson formulation and implementation supports lesson structure knowledge by providing the content to be taught. In summarizing the expert-novice comparison, the researchers concluded teachers with more explicit and better organized knowledge tend to provide instruction characterized by conceptual connections, appropriate and varied representations, and active and meaningful student discourse. On the other hand, teachers with limited knowledge were found to portray the subject as a collection of unrelated facts; to provide impoverished or inappropriate examples, analogies, and/or representations; and to emphasize seatwork assignments and/or routinized student input instead of meaningful dialogue.

Carpenter: Impact of Knowledge of Students' Cognitions

A third research program, conducted by Thomas Carpenter and his colleagues at the University of Wisconsin, has focused on one aspect of pedagogical content knowledge, namely, teachers' knowledge of students' understanding. They suggest pedagogical content knowledge includes

> knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions about the topic that they may have developed, and the stages of understanding that they are likely to pass through in moving from a state of having little understanding of the topic to mastery of it. It also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of instructional strategies that can be used to enable students to connect what they are learning to the knowledge they already possess, and knowledge of

> instructional strategies to eliminate the misconceptions they may have developed. (Carpenter, Fennema, Peterson, & Carey, 1988, p. 386)

Further, they believe the influence of this knowledge of students' understanding and misunderstanding should be evident in classroom instruction and impact subsequent student learning.

Carpenter and his colleagues have explored the implications of this knowledge on the teaching and learning of addition and subtraction in first-grade classrooms. This content was chosen because empirical research has identified a variety of strategies children use to solve addition and subtraction problems as well as the major levels through which children pass in acquiring more advanced strategies (Carpenter, 1985, 1986; Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1983, 1984; Riley, Greeno, & Heller, 1983). Initially, when young children solve such problems, they use fingers or physical objects to directly model the actions or relationships in the problems. As children advance in their problem-solving skills, direct modeling gives way to counting strategies such as counting on and counting back. As children continue to advance, they begin to use derived number facts and eventually memorized facts.

In the first of a series of studies, Carpenter et al. (1988) used questionnaires and interviews to measure 40 first-grade teachers' knowledge of students' knowledge and cognitions. In general, the first-grade teachers were able to identify many of the critical distinctions between addition and subtraction word problems and the kinds of strategies children use to solve such problems. However, Carpenter and his colleagues concluded the teachers did not have a sufficiently rich knowledge base nor one that was coherently organized enough to influence their instructional decisions. In particular, they did not link their knowledge about problem types and solution strategies to the processes by which students develop understanding.

As the researchers continued to explore the impact of student cognitions on instructional decisions, their emerging view of the learner was consistent with the growing emphasis on cognitive theory. In a related study, Peterson, Fennema, Carpenter, and Loef (1989) found a significant positive correlation between students' problem-solving achievement and teachers' beliefs that instruction should build on children's existing knowledge and that teachers should help students to construct mathematical knowledge instead of passively absorbing it. These findings, combined with the results of their earlier study, led Carpenter and his colleagues to explore whether providing teachers with the explicit, highly principled knowledge about children's cognitions derived from research might influence teachers' instruction and subsequently affect students' achievement. Would such research-based knowledge improve the teachers' ability to assess their own students? With their resulting knowledge about their students, would they be better at matching instruction to their students' knowledge and problem-solving abilities? And would this facilitate more meaningful learning and problem solving?

Because their research was based on assumptions fundamental to the cognitive research on children's learning, they called their program Cognitively Guided Instruction (CGI). The underlying assumptions guiding their model of instruction were that (a) instruction should develop understanding by building relationships between skills and problem solving, with problem solving as the organizing focus, (b) students should be actively involved in constructing their own knowledge, (c) instruction should be based on what each student already knows, and (d) teachers need to continually assess not only whether a student can solve a particular problem, but also how the learner solves the problem (Carpenter et al., 1989).

Research design. The participants in the experimental part of the study were 40 teachers (39 women and 1 man) from 24 schools, including 2 Catholic schools and 22 public schools, located in Madison, Wisconsin, and four smaller nearby communities. Thirty-six of the teachers taught in first-grade classrooms and four taught first/second-grade combinations. All the teachers in the sample had volunteered to participate in a month-long mathematics inservice program in the summer and to be observed during their classroom instruction in mathematics during the following year. The teachers had been teaching at the elementary level for an average 10.90 years and at the first-grade level for an average 5.62 years. Two of the teachers had just completed their first year of teaching. Thirty-four of the teachers had participated in inservice courses in the last 3 years. Although nine of these teachers had participated in a mathematics inservice, none of the teachers reported receiving any training in which recent research in addition and subtraction was discussed.

Half of the teachers (n = 20) were randomly assigned by school to the treatment group. These teachers participated in a 4-week workshop designed to familiarize them with research findings and give them an opportunity to think about and plan instruction based on that knowledge. They had the opportunity to learn about the addition/subtraction framework and the instructional principles derived from cognitive research and to discuss the instructional implications with other teachers. They were not provided with instructional materials to use in their classes, but were encouraged to design their own instructional programs in line with their beliefs and teaching styles.

The other teachers (n = 20) served as a control group. They participated in two 2-hour workshops focused on nonroutine problem solving. Because there was no intent to provide a contrasting treatment, these workshops were not designed to be comparable to the CGI workshops in duration or extent of coverage. Instead, they provided a sense of participation in and reward from the project.

From November through April of the following school year, each teacher and class was observed for four separate week-long periods (a minimum of 16 days). Two coding systems were developed for the study, one focused on the teacher and another focused on students. The observation categories were based on the research literature and the purpose of the study. During each observation period, two observers coded together, using a 60-second time-sampling procedure in which they observed for 30 seconds and then for the next 30 seconds, coded the observed behavior and activities. The teacher-observer, in each time interval, checked one subcategory within each of the following categories: setting, content, expected strategy, and teacher behavior. To aid in understanding the private interactions between teacher and students, the teacher wore a wireless microphone during observations and the observer listened to the teacher through earphones. In each class, 12 first-grade students (six boys and six girls) were randomly selected as target students for observation. Using a similar time-sampling procedure, the student-observer rotated through the 12 target students during the class period, using a different order each day. The student observer used the following major categories for recording observations: setting, content (including nonengaged with content), strategy used, and lesson phase.

In September and October, observers had been hired and received 2 weeks of training following procedures used previously by Peterson and Fennema (1985). The training involved coding transcripts and videotapes for 1 week and then practice coding in first-grade mathematics classrooms not part of the study for a second week. At the end of the training, observers who achieved unspecified criterion levels on a written test of content knowledge and a test of coding ability were judged sufficiently skilled to begin actual classroom observation. During the study, interobserver agreement was estimated by having a reliability observer code with the assigned observer at specified times. The reported estimates of interobserver agreement ranged from 82 to 100%.

Near the end of the instructional year, teachers' knowledge of their students was assessed using three separate measures. First, teachers were asked to predict the strategy that each target student would use to solve each of five items on the students' number facts interview (knowledge of number-fact strategies). Second, teachers were asked to predict the strategy each target student would use to solve each of six problems on the students' problem-solving interview (knowledge of problem-solving strategies). Third, teachers were asked to predict whether each target student would correctly answer eight specific problems on the written tests (knowledge of problem-solving abilities). The teachers' predictions were matched with students' actual responses to obtain measures of teachers' knowledge of their students' performance. The Cronbach alphas for the teachers' number fact strategies, problem-solving strategies, and problem-solving abilities tests were 0.57, 0.86, and 0.47 (n = 40), respectively (Carpenter et al., 1989).

Results. Among the results reported, Carpenter et al. (1989) described differences in the knowledge of CGI and control teachers and examined how CGI and control classrooms differed in terms of the content, activities, behavior, learning, and instruction observed. In comparing teacher knowledge, means, standard deviations, and t-tests between groups were calculated for scores on the three tests of teachers' knowledge. There were no significant differences in knowledge of students' problem-solving abilities–in the teachers' predictions of success on the complex and advanced problems. However, the CGI teachers were significantly more accurate at predicting number-fact strategies and problem-solving strategies. In particular, control-group teachers overestimated the use of number fact recall by their students by factors of two or three to one.

In drawing conclusions from the classroom observations about how the classrooms differed, means, standard deviations, and t-tests between groups were computed for each of the 27 categories on the teacher observation system and for each of the 27 categories on the student observation system. Although this number of t-tests introduces a greater possibility of error, the results between the two observation systems paint a consistent picture contrasting the CGI and control classrooms. In particular, although results did not show differences in the proportion of time devoted to addition and subtraction content, significantly different content emphases were revealed. During addition and subtraction instruction, CGI teachers spent significantly more time on problem solving and significantly less time on number facts than did control teachers. In addition, CGI teachers more often posed problems to students, more frequently listened to the processes students used to solve problems, and more regularly encouraged and discussed alternative strategies and solutions. As a result of the inservice sessions, the CGI teachers seemed to provide instruction more compatible with the assumptions of cognitive research.

In a case study, Fennema, Franke, Carpenter, and Carey (1993) provided an in-depth look at one teacher who had been part of the project. The teacher, known to her students as Ms. J., had been identified as an expert CGI teacher based on observations of her teaching and assessment of her students' learning. Her teaching, in general, exemplified the goals of the CGI project. Problem solving was consistently the focus of her mathematics lessons, including the wide variety of addition and subtraction problems. She listened to the solutions of her students with an eye toward understanding their thinking. The activities and classroom discourse were largely directed by the students as they wrote their own problems and discussed different solution strategies. An unexpected response often formed the basis for more problem solving or provided direction for further exploration.

During her third year in the project, Ms. J. moved to teaching second grade and was observed during her instruction of fractions. In contrast to her background in addition and subtraction, Ms. J. had not studied fractions for a number of years and reported her knowledge of fractions was not adequate. When measured, her knowledge of fractions was found to be limited in content, but rich in pedagogy (Lehrer & Franke, 1992). Although her instruction of fractions included some elements of CGI including posing problems and challenging her students to justify their answers, Ms. J.'s methods of instruction for fractions were quite different from those for addition and subtraction. When fractions were taught, only one basic type of problem was used: part-whole. The classroom discourse was also much different during fraction instruction.

> Ms. J. directed the students' interactions to a much greater extent. . . . When an unanticipated response was given during a lesson on fractions, Ms. J. listened carefully to the child, but did not act on what the child had said. She usually did not follow up on student responses by asking specific clarification questions, nor did she attempt to pose another problem that would build on the response. Overall, there was less discussion and less mathematics occurring during lessons involving fractions than during problem solving involving addition and subtraction. (Fennema & Franke, 1992, p. 149)

The researchers attribute the difference in Ms. J's classroom behavior at least in part to the differences in her knowledge of the two content areas.

In conclusion, Carpenter et al. (1989) suggest providing teachers access to research-based knowledge about students' thinking and problem solving can influence teachers' beliefs about learning and instruction, their classroom practices, and their knowledge about their students. In particular, teachers who have and use knowledge of their students' thinking can make more informed instructional decisions as they structure instruction so that students can connect what they are learning to the knowledge they already possess.

Shulman: Impact of Different Levels of Knowledge

The fourth and final research program to be discussed in this section is the Knowledge Growth in Teaching project conducted by Shulman and his colleagues at Stanford University. In this project, the researchers focused primarily on "how teachers learn to transform their own understanding of subject matter into representations and forms of presentation that make sense to students" (Shulman & Grossman, 1988, p. 1). Participants in the study were novice secondary teachers representing the subject areas of English, social studies, biology, and mathematics. Reported results include case studies of three of the mathematics teachers (Haymore, 1987a, 1987b; Marks, 1987), a cross-case analysis of the mathematics teachers (Steinberg et al., 1985), and a case study of one of the teachers who was misassigned to teach a remedial mathematics class (Ringstaff, 1987).

Research design. The participants were student teaching or interning as part of a graduate year of teacher education. Each had already completed a bachelor's degree in the subject to be taught or had satisfied requirements by examination. Twenty-one student teachers participated during the first year of the study. Twelve of the participants were followed into their first year of full-time teaching.

To provide baseline information, the researchers conducted a series of interviews directed at developing intellectual biographies of the teachers. For the mathematics teachers, the series of semistructured interviews focused on the participants' intellectual histories, general knowledge of mathematics, specific knowledge about algebra, general pedagogical knowledge and specific knowledge of mathematics pedagogy. Through the use of a variety of tasks, including free association and card sorts, the researchers attempted to ascertain the teachers' conceptions of mathematics and their knowledge of its substantive and syntactic structures.

In addition to the series of knowledge interviews, the researchers conducted a series of planning-observation-reflection cycles. Prior to the observation, the researchers talked with teachers as they prepared to teach a particular piece of subject matter. These interviews focused on what the teachers knew about the content and what they wanted the students to learn about the content. The lesson was then observed as it was taught. After the observation, teachers were asked to reflect on the lesson, student performance, and their own teaching in an effort to detect changes in the teacher's knowledge of subject matter and pedagogy as well as the perceived sources of those changes. Teachers followed into their first year of full-time teaching were observed in a similar fashion.

Results. The participants who were teaching mathematics represented the approximate mean and both extremes of mathematical knowledge that a group of secondary teacher candidates might possess. Joe had completed course requirements for a PhD in mathematics and had worked for two years on a dissertation before quitting. His knowledge of mathematics was extensive and his conceptualization of mathematics was comprehensive and rich in interrelationships. Two other participants, Scott and Sharon, had studied a moderate amount of mathematics as part of a double major in math and English and a major in science respectively. Their conceptualization of the structure of mathematics was less complete than Joe's, focusing primarily on the school subjects with the arithmetic operations as the central core. At the other extreme were Laura and Lewis. Laura, who had studied a very limited amount of mathematics as a foreign language major, had a limited view of mathematics and lacked an understanding of fundamental ideas. Lewis, one of the teachers followed into their first year of teaching, was credentialed as an English teacher but was misassigned to teach a remedial mathematics class. Although he had passed the CBEST, a basic skills test required of teachers in California, Lewis did not appear to have mastered even the basic mathematical skills required of his students.

The influence of teachers' subject matter knowledge on their classroom instruction was seen in a number of ways. Shulman and Grossman (1988) concluded that "prior subject matter knowledge and background in a content area affect the ways in which teachers select and structure content for teaching, choose activities and assignments for students, and use textbooks and other curriculum materials" (p. 12). First, the researchers documented different approaches to selecting and structuring the content for students. They discovered a cluster of teaching behaviors which seemed to be characteristic of teachers at each level of subject matter knowledge. The teachers with the lowest level of knowledge were more rule-based in their teaching, often because they did not have enough mathematical knowledge to explain to their students anything except algorithms and procedures. On the other hand, the teachers with greater mathematical knowledge used more conceptual teaching strategies (Steinberg et al., 1985). In particular, they were more likely to explain to students why certain procedures do or do not work, to relate one concept to another and to the "big picture," and to show applications of the material studied. In addition, these teachers engaged their students in more active problem solving and used more abstract forms and language, yet with understanding.

Second, the teachers' level of knowledge influenced their choices of activities and assignments for the students. This was particularly evident in their motivation for using problem-solving activities and their approaches to adapting to different student abilities. Joe, who saw problem solving as central to mathematics instruction, frequently engaged his students in problem-solving activities, including some fairly challenging problems. These activities were designed to have students actively explore mathematical concepts more deeply, develop problem-solving skills, and see applications of real mathematical content. Although Scott also provided his students with opportunities to solve applications problems and creative word problems, his motivation was to make the class "fun" and more interesting to students. Sharon, on the other hand, chose activities based on what would ensure success for her students.

The teachers' approaches to adapting lessons and activities for the ability levels of their students also seemed to be related to their own knowledge level. With his richer mathematical background, Joe was able to make adjustments in both content and pedagogy to meet the students' differing needs and abilities. In changing the pace of instruction, he, nevertheless, maintained his focus on problem solving. On the other hand, Laura, one of the less knowledgeable teachers, involved the lower ability students in less problem solving and reported teaching less about "why" procedures worked, stressing instead just how to do the procedures. Further, Sharon's lack of a deep content knowledge hindered her efforts to anticipate student difficulties and to provide alternative explanations as needed.

A third aspect impacted by teachers' knowledge was their evaluation and use of textbooks and other curricular materials (Reynolds, Haymore, Ringstaff, & Grossman, 1988). Teachers who lacked confidence in their knowledge found few things wrong with their textbooks and were quite willing to use the curricular materials "as is." For example, Lewis, the first-year teacher misassigned to teach a remedial mathematics class, did not use any materials except the textbook and, in using the text, he rarely deleted, added, or reorganized the material. He explained, "The text works pretty good. . . . I love the book. I think it's a great book. It's very readable. Very good. It's not above their level" (Ringstaff, 1987, p. 8). In contrast, teachers who had more confidence and competence in mathematics drew on their subject matter knowledge as they evaluated and modified the curricular materials.

They were more likely to see mistakes in the content of the materials, limitations in the approach to the content, or questions about the ordering of topics within the materials. Also they were more willing to modify the texts or curricular materials accordingly. For example, Joe rejected some content in the textbook because he found it was incorrect or incomplete or because he felt it was unimportant. As a result, he made corrections to the text, supplemented some material and omitted other material.

In conclusion, Shulman and his colleagues identified several aspects of teaching impacted by teachers' subject matter and pedagogical content knowledge. Teachers with a greater mathematical knowledge base from which to draw were able to apply more conceptual teaching strategies. For example, they more frequently provided explanations of why procedures worked. In addition, the activities in their classrooms were designed to actively involve students in exploring mathematical concepts more deeply, developing problem-solving skills, and applying the mathematical ideas being studied. Their background knowledge also allowed them to anticipate and meet the needs of their students and to supplement curriculum materials as needed and appropriate.

Conclusions

The research programs reviewed in this section have revealed the benefits of a deep and broad subject matter knowledge and a rich pedagogical content knowledge. They also have illustrated what limitations may result from weak subject matter knowledge or pedagogical content knowledge. First, Lampert provided an example of how a teacher's subject matter knowledge and pedagogical content knowledge can enrich the instruction students experience. The impact of her knowledge was demonstrated in how she chose to represent both the content and nature of mathematics as well as in the design of instructional activities and the orchestration of classroom discourse. In her classroom, mathematics was represented as a system of human thought in which students could develop meaning and participate in doing mathematics. In this context, Lampert's knowledge provided her with a repertoire of representations for the content which emphasized essential conceptual components as they bridged from familiar contexts to the more abstract contexts of the discipline. Further, her knowledge enabled her to choose instructional activities that would engage her students in looking for patterns, making conjectures, and establishing legitimate conclusions based on reasoning. Finally, the breadth, depth, and flexibility of her knowledge guided her decisions about the direction, balance, and rhythm of classroom discourse, permitting her to create an effective learning environment.

In the second research program, Leinhardt's study of expert and novice teachers provided contrasting pictures of the impact of teacher knowledge. The researchers concluded expert teachers generally had more explicit and better organized knowledge making it more accessible during the process of instruction. As a result, their presentation of content was frequently characterized by logical explanations connected with prior knowledge, by careful use of multiple representations, and by active and meaningful student discourse. On the other hand, the lack of instructional patterns and schemata in the knowledge of novices made it more difficult for them to access and utilize the necessary knowledge during instruction. In addition, their sometimes incomplete understanding led to an overemphasis on rules and procedures as well as missed opportunities to foster meaningful connections between key concepts and representations.

Third, Carpenter's research program demonstrated how teachers' knowledge of students' understanding impacts classroom instruction. As a result of being provided with background knowledge about how students develop understanding, the teachers used methods of instruction more compatible with the assumptions of cognitive research. In particular, they spent significantly more time in problem solving, developing skills in that context. They more frequently listened to the processes students used to solve problems, allowing the teachers to continually assess what problems the students could solve as well as how they solved the problems. By regularly encouraging and discussing alternative solution strategies, the teachers were allowing students to build upon what they already knew as they were ready. As a result of these methods of instruction, students were actively involved in constructing their own knowledge, guided by teachers who had an understanding of how such knowledge generally develops.

Finally, Shulman and his colleagues found the depth and character of the teachers' subject matter knowledge and pedagogical content knowledge influenced both the substance and the style of instruction. Those teachers with a stronger and richer foundation of knowledge stressed more important aspects of mathematics and employed more conceptual teaching strategies. Their knowledge provided the basis for explaining the procedures and concepts of mathematics, for evaluating and modifying curricular materials, and for making adjustments in the content and pedagogy for students of different abilities. The teachers who lacked a depth of knowledge were, as a result, unable to focus on more than how to do the procedures, nor were they able to modify curricular materials as needed or provide a variety of representations and explanations to meet the needs of their students.

Thus, teachers with depth and breadth to their subject matter knowledge and a richness in their pedagogical content knowledge were more likely to provide mathematics instruction focused on the development of conceptual connections, problem-solving skills, and reasoning abilities. In particular, the teachers made use of appropriate and varied representations as the students were involved in active and meaningful discourse. Because an important aspect of teacher knowledge is the knowledge they have about students' prior conceptions of the content and about effective instructional strategies, this chapter turns next to what information research provides about these aspects of the learning and teaching of probability.

Research on Learning and Teaching of Probability

The investigation of probability knowledge and instruction has been conducted by researchers from around the world representing a variety of disciplines. In the last 15 years, the major contributors have been cognitive psychologists and mathematics educators from Europe. The study of stochastics, as probability and statistics are called in Europe, has received less attention from mathematics educators in North America because very little probability has been part of the K-12 mathematics curriculum. However, with the increasing number of calls by the NCTM and other organizations for teaching probability and the growing efforts to include more probability in the curriculum, there has been an expanding interest in and need for research in the learning and teaching of probability.

However, as psychologists and mathematics educators pursue research in the field of probability, they have different perspectives and different research agenda. The interest of the cognitive and social psychologists in the subject of probability comes from their desire to understand reasoning in situations of uncertainty and from their "concern about how doctors, judges, financial advisors, military experts, political advisors, and others make crucial decisions (perhaps with lives at stake) in situations where the information is probabilistic, at best" (Shaughnessy, 1992, p. 469). Their purpose has been to observe and describe how subjects make judgments and decisions in such situations. Their investigations of people's conceptions and intuitions of probability have provided a theoretical framework describing many of the misconceptions and judgmental biases that appear in the reasoning process. While psychologists have played a major role in building theory in the field of judgment and decision making under uncertainty (Kahneman, Slovic, & Tversky, 1982; Nisbett & Ross, 1980), they have been less concerned with changing conceptions and beliefs about probability. On the other hand, mathematics educators are not content just to observe the difficulties people have in reasoning in probabilistic situations. Their goal is to improve students' knowledge of probability, to influence their conceptions and beliefs about probability, and to change or remove their misconceptions.

In reviewing the research regarding the learning and teaching of probability, this section will consider the contributions of psychologists and mathematics educators as they have sought to answer the following questions:

1. What intuitions and conceptions of probability do children possess and how do they develop?
2. What misconceptions and difficulties in learning probability do teachers face in teaching probability at the late elementary or middle school level?

3. What is the impact of instruction in developing proper conceptions of probability and overcoming the misconceptions?

First, however, before addressing these questions, it is necessary and helpful to consider the nature of probability.

Nature of Probability

The present concept of probability is a rather recent development, evolving around 1660 when Pascal, Huygens, Leibniz, and Fermat independently applied probabilistic ideas to such diverse phenomena as games of chance, legal decisions, and annuities. Hacking (1975), in *The Emergence of Probability*, attributes this relatively late development to the dual meaning historically associated with the word *probability*. This duality was grounded in the difference between knowledge demonstrated deductively from first principles (*scientia*) and beliefs testified to by men of authority or through God-given signs (*opinio*). The word probability was originally associated with the latter so that a "probable" belief or circumstance was one "approved" by some authority. The duality was also the result of differences in what constituted "acceptable evidence for truth." Prior to the 17th century, evidence other than deductive argument was not really considered evidence at all. As the idea of experimental evidence began to be accepted in the 17th century and as frequency data permitted degrees of belief separate from opinion, probability came to refer to the tendency of certain phenomena (like tossing coins or dice) to produce stable frequencies over many repetitions. Approval of authority gave way to approval of data. Therefore, as the mathematical version of probability began to emerge, the word probability took on two meanings. It indicated both "degree of belief" and "calculations of stable frequencies for random events."

The continuing ambiguity about the notion of probability is evidenced in several schools of thought regarding the nature of probability (Hawkins & Kapadia, 1984; Konold, 1991; Shaughnessy, 1992). The first view, generally referred to as the classical or *a priori* interpretation of probability, is based on the assumption of equally likely outcomes. The probability of an event, sometimes called the *theoretical* probability, is defined as the ratio of the number of outcomes favorable to that event to the total number of equally likely outcomes. Obviously, this interpretation is limited to experiments with objects such as coins, dice, and spinners where all outcomes are equally likely. Konold points out this interpretation is also "logically flawed in that its definition of probability is circular: Probability is defined in terms of equally likely alternatives, yet what can be meant by 'equally likely' other than 'equally probable' " (p. 142)?

The second view is the frequentist interpretation, which defines the probability of an event as the limit of the observed relative frequencies of that event in repeated trials. This interpretation, which is sometimes referred to as *empirical* or *experimental* probability, can be applied to events composed of nonequally likely outcomes. However, it is limited to experiments such as tossing coins or dice or drawing balls from urns where "identical" trials can be repeated indefinitely.

The third view is the subjective or intuitive interpretation, which is the 20th century equivalent of *opinio*, or degree of belief. According to this view, probability judgments are expressions of personal belief or perception that are based on a variety of sources of evidence and skills in processing that evidence. The meaning of the probability value in a subjectivist interpretation can be thought of in several ways. One of the most common is to describe the value as a measure of a person's belief in what a fair bet would be. Thus, a person estimating the chance of rain at 60% would bet $6 to win $10 if it did rain as quickly as he or she would bet $4 to win $10 if it did not rain (Konold, 1991). Although based on personal belief, theorists have formalized the subjectivist interpretation by applying various adjustment mechanisms. These mechanisms lead to the revision or "calibration" of initial probabilities on the basis of new information such as results of actual trials.

Hawkins and Kapadia (1984) identify a fourth kind of probability that they call formal probability. This probability, which is sometimes known as *objective* or *normative* probability, is calculated precisely using the mathematical laws of probability. Not surprisingly, the mathematical basis reflects assumptions made in the classical or frequentist approach.

Though theorists may disagree whether or not some event ought to be assigned a probability and argue over the interpretation of the probability, the various schools of thought generally derive identical probabilities for events they agree are probabilistic. Take, for example, the flipping of a fair coin. According to the classical interpretation, the probability of obtaining a tail would be 0.5 because the ratio of favorable outcomes to total number of equally likely outcomes is 1 to 2. For the frequentist, the probability of a tail would be 0.5 if the limit of the relative frequency of tails approaches 0.5 as the number of trials approaches infinity. According to the subjectivist interpretation, different people could validly assign different values to the probability of tossing a tail, reflecting their beliefs about the fairness of the coin, the character of the person tossing the coin, or the technique used in tossing the coin. However, as these values are revised or calibrated on the basis of enough data about the actual occurrence of the event, the various subjective probabilities held by different people would all begin to converge on the frequentist's limit. Undoubtedly, the formal probability would also have the same value.

Though the end results may be the same, the somewhat different interpretations of probability are more than just a philosophical argument. These different interpretations have important implications when considering the research on learning and teaching probability and may, in fact, explain many of the discrepancies and inconsistencies in research findings. These different views may also significantly affect how one might develop the ideas of probability in the classroom.

Developing Knowledge and Conceptions of Probability

A number of psychologists and math educators have explored and attempted to describe the processes whereby understanding of the concepts of chance and probability is gained. The definitive texts on the development of probability cognition are the classics by Piaget and Inhelder (1951/1975) and Fischbein (1975). More recently, others have contributed empirical evidence to the discussion, including Falk (1983) and Green (1979, 1983a, 1983b, 1988).

Piaget: Developmental Stages

Piaget and Inhelder (1951/1975), in their classic text *Le Genèse de l'Idée de Hasard chez l'Enfant* [*The Origin of the Idea of Chance in Children*], are often credited with initiating research dealing with the development of the concepts of probability. From their perspective as developmental psychologists, they systematically analyzed the probability concept in children from preschool ages to adolescence and formulated a theory to explain its development.

As in other Piagetian studies, the clinical method was used and the data consisted of protocols derived from presenting children with a variety of ingenious tasks designed to yield "chance" outcomes. The materials used in the experiments included a tray designed to produce a random mixture of balls; boxes devised to generate normal, skew, and uniform distributions; urn and spinner tasks planned to permit not only random results but also rigged or "miraculous" outcomes; and situations created to elicit combinations, permutations, and arrangements of elements. These materials were used to probe the children's thinking and question them about their understanding of what they experienced.

From their analysis of the protocols, Piaget and Inhelder concluded an understanding of probability concepts is acquired in stages corresponding to the familiar preoperational, concrete operational, and formal operational periods. In the preoperational stage, generally under 6 or 7 years of age, Piaget and Inhelder claim children are unable to distinguish between caused events and chance events. At this stage of development, the child lacks the ability to construct such logical relations as cause and effect and, without an understanding of caused events, the child has no frame of reference for identifying events that are due to chance. According to Piaget and Inhelder, children at this age believe random or chance events are subject to the same deterministic order which controls predictable events.

In the concrete operational stage, from age 7 to about 11, children are able to distinguish between two classes of events, one governed by the laws of cause and effect and thereby predictable and another characterized as random, unpredictable, and subject to chance. Although they supposedly do not have the skills to make an abstract model of a probability task, they are at least able to estimate the relative probability of alternative outcomes if the task is not too complex.

In the formal operational stage, beginning about age 12, children begin to develop facility with the arithmetical tools of combinations and permutations. With these tools, the set of all possible chance outcomes for a given situation can be conceptualized and, sometimes, calculated. In addition, children begin to understand probability as the limit of relative frequency. Because they see ratio and proportion as central to an understanding of the probability concept and because these concepts are not available until the level of formal operations, Piaget and Inhelder conclude fundamental probabilistic notions are not constructed until the level of formal operations.

The approaches and conclusions of Piaget and Inhelder have engendered much debate and controversy for a variety of reasons. First, their approach to probability is clearly one that is classical and more formal, based on a priori notions and proportional reasoning. Their chapter headings mention normal curves and uniform distributions as well as permutations and combinations, "hardly the ideas teachers would mention when considering 'the origin of the idea of chance in children' " (Kapadia, 1985, p. 262). In particular, some of the tasks became games of comparing fractions rather than an exploration of the ideas of chance. A second criticism suggests that, because the research was lacking in proper experimental controls, it is difficult to make unambiguous interpretations (Hawkins & Kapadia, 1984). The high degree of verbalization necessary from the children and the number of variables which may be altered by accident or design may have confounded the conclusions drawn. In particular, the use of different tasks with different age groups raises the question about the equivalence of the experimental situations and begs the question it proposes to answer–namely, that there are different levels. Finally, in considering what concepts children "spontaneously" develop, they have ignored the potential role that a teacher may play in facilitating the development of ideas about chance. Despite the criticism, Shaughnessy (1992) suggests that "Piaget's descriptions of what children do and know at various stages are quite in line with the results of more recent research by cognitive psychologists" (p. 479). It may be more accurate, however, to say their conclusions are true for the development of the more formal ideas of probability.

Fischbein: The Role of Intuition

A second major contributor to the discussion about the development of probability understanding in children is the psychologist Fischbein. In his book *The Intuitive Sources of*

Probabilistic Thinking in Children, Fischbein (1975) reviews the research literature available at the time and reports the results of his own investigations with colleagues (Fischbein, Barbat, & Mînzat, 1971; Fischbein, Pampu, & Mînzat, 1967, 1970a, 1970b). Fischbein (1975) and his colleagues make a distinction "between the *concept* of probability as an explicit, correct computation of odds and the *intuition* of probability as a subjective, global evaluation of odds" (p. 79) and suggest an intuition of chance emerges at the preoperational level.

According to Fischbein, intuition or intuitive knowledge is a type of cognition central to intelligent behavior. In his more recent book, Fischbein (1987) describes several characteristics of intuitions. First of all, intuitions are accepted as being immediate and self-evident, without the need for formal or empirical proof. Intuitions, once established, are very robust and exert a coercive effect on the individual's ways of reasoning. As a theory, an intuition implies an extrapolation beyond the data on hand, but with a feeling of certainty. Finally, an intuition that is accepted as self-evident is also accepted globally as a structured, meaningful, unitary view of a certain situation. Fischbein (1975) distinguishes between the primary and secondary origins of intuitions. Primary intuitions are those which develop as a result of normal everyday experience apart from and independent of any systematic instruction. On the other hand, secondary intuitions are those which are systematically constructed during the instruction process.

To explore their hypothesis about the existence of probabilistic intuitions and to investigate the role of these intuitions in the development of probabilistic understanding, Fischbein and his colleagues at the Institute of Psychology in Bucharest conducted a series of four investigations. The designs of these investigations will be described initially and then the combined results will be discussed.

In the first study, Fischbein et al. (1967) tried to determine children's ideas about chance by considering situations where chance occurs in the simplest possible form. The subjects were children aged 6 to 14 years, subdivided into five age groups. The experimental materials consisted of inclined boards on which a system of progressively forked channels had been constructed with thin strips of wood. Five different boards were used. In the simplest case, the first layout contained two equiprobable channels. The second layout contained eight equiprobable routes and the third layout had four equiprobable but asymmetrical routes. The last two layouts contained channels that were not equiprobable. In individual interviews, the children were asked to imagine a marble had been released in the main channel and were to say where they thought the marble would come out at the bottom. If more marbles were dropped, the children were to decide if they would come out at each place the same number of times or if the marbles would come out of some channels more often that others.

In a second experiment, Fischbein et al. (1970a) used marble tasks similar to Piaget to explore whether or not children have an intuition of relative frequency. The subjects were

180 school children in Bucharest, 60 at each of three age levels: preschool (age 5 to 6), third grade (age 9 to 10), and sixth grade (age 12 to 13). The children, interviewed individually, were asked to decide from which of two boxes of black and white marbles they would be more likely to draw a marble of a given color. They were also given the option of determining the chance was the same from either box. The 18 problems were divided into three categories: (a) pairs of boxes with an equal number of marbles of one color in the two boxes (e.g., 1W, 2B and 5W, 2B) or pairs of boxes where one box had the same number of white and black marbles (e.g., 2W, 2B); (b) pairs of boxes with no restrictions imposed; and (c) pairs of boxes where the ratios to be compared were equal.

In a third study, Fischbein et al. (1970b) investigated the ability of children and adolescents to handle permutations, a concept determined by Piaget to be at the level of formal operations. Their subjects were again pupils from public schools in Bucharest, 20 each at fourth grade (age 10 to 11), sixth grade (age 12 to 13), and eighth grade (age 14 to 15). The materials consisted of small cards on which letters and numbers were printed and colored geometric shapes cut from cardboard. In individual interviews, students were shown an example of a permutation and then were asked to estimate how many permutations were possible with three, four, and five objects. Next, the researchers investigated the effect of what they called *instruction by programmed discovery* upon the combinatorial ability of the students. Within the individual interviews, the interviewer used tree diagrams to systematically guide the discovery process.

In a fourth investigation, Fischbein et al. (1971) were seeking to discover what intuitional biases might exist corresponding to fundamental concepts and properties of probability. They were also interested in exploring whether these intuitional biases facilitated the acquisition of probabilistic understanding during instruction. The following aspects were the focus of the study: probability as a measure of chance, the multiplication law of probabilities in the case of the intersection of independent events, and the addition law of probabilities in the case of mutually exclusive events. The subjects were also students from schools in Bucharest, including 20 sixth-grade students (age 12 to 13), 20 eighth-grade students (age 14 to 15), 20 tenth-grade students (age 16) from representative schools, and 20 tenth-grade students (age 16) from schools which emphasize mathematical preparation. None of the subjects had any prior knowledge of probability theory.

In some of the simplest experiments, almost all of the children, including those of preschool age, demonstrated they could distinguish between certainty and uncertainty, between predictable and unpredictable. For example, in considering the probability of the two equally likely channels, a preschool child responded, "I think it will be the same, because the paths are the same. They both turn a corner, and then go straight" (Fischbein, 1975, p. 163). As the arrangement of the channels became more complicated, however, the number of correct responses declined for all ages, as the intuition of equiprobability was thwarted by the increasing complexity of the system. In addition, for the simpler cases of the marble task, those involving only a comparison of two terms, the preschoolers were able to perform above the chance level. Fischbein concludes from this that "preschool children can correctly understand and cope with situations involving chance" (p. 186).

The data from these investigations also confirm the suggestion that this understanding of probabilistic situations increases with age and, to some extent, support Piaget's proposed developmental stages. Particularly in the cases of the boards with routes which were not equally probable and in the marble tasks involving comparison of more than two terms, the number of correct responses increased from younger to older. Different solution strategies were used at different age levels as well. On the marble tasks, for example, before any of the students were shown a strategy, the preschoolers and the third graders focused primarily on a simple binary comparison (e.g., "I chose this box because it had more black marbles than the other one."). On the other hand, most of the sixth graders made their decisions based on explicit comparison of the ratios (e.g., "In this box there are three times more black than white and in this box only two times as many."). In addition, the differences due to age were evident in the subjective estimations of the number of permutations. Again, at the sixth-grade level (age 12), Fischbein (1975) finds a threshold or leap "that spectacularly fits the stages indicated by Piaget" (p. 199).

There were also some unexpected results in comparing age groups. The most surprising finding was that, for the board layouts with equiprobable routes, the number of correct responses decreased from the younger to the older students. In addition to being more frequently incorrect, the responses became more erratic and more hesitant as age increased. The older students more frequently opted for a determined route and "offered obviously confabulated causal explanations to justify their choices" (Fischbein, 1975, p. 72). As an explanation, Fischbein suggests the tendency to select one particular route is a result of the overemphasis in the schools on deterministic interpretations of phenomenon: "The child is taught that explanation consists in specifying a cause; that a scientific prediction must be a certainty; that ambiguity and uncertainty are not acceptable in scientific reasoning, and so on. Even if all this is not explicitly stated it is implied in all that is taught in schools" (p. 71).

As part of the interview in three of the studies, some of the students received a short but systematic explanation of how the problems could be solved. In the comparison of odds task, for instance, the children were shown a practical technique which required no knowledge of fractions yet demonstrated the concepts of proportionality and relative frequency. For example, in comparing a box containing 4B and 1W with another containing 8B and 2W, the child would separate the second box into two groups of 4B and 1W. Because four out of every five marbles in each box are black, the probabilities of selecting a black are equal in both boxes. If, however, there were 4B and 1W in one box and 9B and 2W in the second box, this grouping process would reveal an excess of black in the second box. Although the brief systematic instruction did not produce essential changes at the preschool and sixth-grade level, it did impact the third graders. They were then able to estimate chances by comparing the ratios correctly, giving responses comparable to the 12-year-olds. Fischbein (1975) suggests from this that "the mental mechanisms necessary for the active understanding of proportionality are already present at the level of concrete operations and can be brought into play by means of brief instruction" (p. 91).

In the last two studies, Fischbein et al. (1970b, 1971) used a method of investigation they called learning by programmed discovery, combining certain features of programmed learning and of teaching through discovery. Each stage of the investigation involved a standardized sequence of questions formulated in such a way that the child could give a meaningful answer, based either on intuition or on transfer of prior knowledge. If the child did not give a correct response, progressively more general, auxiliary questions were asked to discover the primary intuitions which may not have been able to respond to the more specialized initial question. Besides being a tool used by the researchers to discover the abilities and difficulties of the children, the sequence of questions was intended to lead the learner to the knowledge of how to solve the problems. In the case of the permutation problems, the children were shown how to inventory the possible arrangements using a tree diagram. And, although there had been differences between age levels initially, these differences diminished as a result of the instruction. At the end, even the fourth graders (age 10) were able to understand and use the tree diagrams to help them respond correctly to combinatorial problems. Fischbein suggests these findings provide additional evidence that, as a result of systematic instruction, children at the concrete operational stage can obtain the body of knowledge and mental skills necessary for solving problems usually thought to be at the formal operational level.

Finally, Fischbein and his colleagues found in some situations there was a natural intuitive foundation upon which instruction could build. For example, in developing the basic axioms of probability, the researchers found a favorable intuitive bias for (a) the concept of chance, (b) the concept of a measure of chance, (c) the use of the values 0 and 1 to denote impossible and certain events respectively, and (d) the quantification of chance as the ratio of the number of favorable outcomes to the total number of equally possible outcomes. In other cases, however, they encountered situations where intuitions were either absent or contradictory to the ideas being developed. For example, intuition was unable to grasp the rapid increase in the number of possible permutations as the number of objects increased. In the case of the multiplication law for the intersection of independent events, intuition facilitated an understanding that chances are reduced as more conditions are imposed. However, the calculation of multiplying probabilities appeared spontaneously only in a limited number of students, particularly those with a stronger mathematical background. In cases where the addition law would apply, the students had difficulty in making an inventory of the elementary outcomes which constituted the event and in understanding such an inventory was necessary in the first place. In fact, for some students, the need to make an inventory of possibilities was counter-intuitive.

In a more recent study, Fischbein, Nello, and Marino (1991) have further explored the factors affecting probabilistic judgments in children and adolescents. The subjects were 618 students from six schools in the region of Pisa, Italy. They represented three groups: 211 elementary students (aged 9 to 11), 278 junior high students (aged 11 to 14) without prior instruction in probability, and 130 junior high students (aged 11 to 14) with prior instruction in probability. In the usual classroom setting students completed one of two forms of a questionnaire asking them to solve 14 probability problems and to explain their solutions. The items on the two forms were parallel, addressing the same type of probability problems but using different embodiments. No information about validity or reliability of the questionnaires was provided. Results to six pairs of questions were presented.

The main topics considered were (a) types of events, including impossible, possible, and certain; (b) the role of different embodiments of the same mathematical structure; and (c) compound events. Although a majority of students at both age levels could adequately identify impossible, possible, and certain events, some surprising difficulties were discovered. The questions which presented the most difficulty were those referring to "certain" events, which were interpreted as "unique" or decomposed in the child's mind into a number of possibilities. On the other hand, "impossible" was identified with "rare" or "uncertain." The researchers concluded that children do not necessarily have an appropriate intuitive understanding of these terms.

To determine if the student would recognize identical probabilistic structure from different embodiments, the students were asked to compare, for example, the probability of obtaining three heads tossing a coin three times or by simultaneously tossing three coins. Again, a majority recognized the probabilities would be the same, but many thought the outcome could be controlled by the individual in one setting more than the other.

In considering the results of several items involving compound events, Fischbein et al. (1991) concluded "children develop a natural, intuitive tendency to evaluate the probability of a compound event on the basis of the corresponding sample space" (p. 546). They, however, discovered several obstacles that interfered with a correct evaluation. These included: (a) no natural intuition to consider order in outcomes (e.g., 5, 6 and 6, 5 on two dice); (b) the tendency to forget limitations in the experiment (e.g., considering numbers such as 7, 8, etc. in a dice game); (c) lack of a systematic technique for producing all possible outcomes in a sample space; (d) reliance on how easy it is to produce some outcomes (the "heuristic of availability" which will be discussed later); and (e) misunderstanding of the idea of chance leading to the conclusion that two events have equal chances because they are both chance events.

According to Fischbein (1975), the stage model of the development of the probability concept suggested by Piaget had two serious deficiencies. It is to these deficiencies Fischbein and his colleagues have directed their research. First, they postulated and provided some evidence children, even at the preoperational stage, have an intuitive understanding of chance and relative frequencies. Second, rather than focusing only on the spontaneous development of concepts as Piaget had, Fischbein and his colleagues investigated the interaction of intuition and instruction on the formation of probability concepts. Their results suggest that children's intuitions can be influenced by systematic instruction, allowing the children to advance through the stages more rapidly. Their research also begins to point out difficulties that need to be overcome as part of instruction.

There is room, however, to question some of the tasks Fischbein and his colleagues used in their investigations. One can wonder how valid the tasks may be for considering the intuition of chance. If the students give responses that fail to demonstrate an understanding of chance, is it because they lack such an understanding or because they lack familiarity with the materials or context of the task? And, as was true of Piaget, some of the tasks focus more on the formal conceptual aspects of probability than on more intuitive notions. Intuition is clearly a difficult construct to characterize and measure.

Green: Empirical Survey of 11- to 16-Year-Olds

A number of researchers have followed up the work of psychologists Piaget and Fischbein, using similar tasks in their subsequent investigations. The most extensive study was the Chance and Probability Concepts Project conducted by Green (1979, 1983a, 1983b). In an effort to survey the intuitions of chance and concepts of probability of adolescents, Green tested nearly 4,000 English school pupils aged 11 to 16. Results have been reported for a sample of 2,930 students stratified according to grade level and intellectual level.

The researchers developed a Probability Concepts Test over a 2-year period (1978-1980) including six pilot versions before its final form. The test consisted of 26 questions, subdivided into 58 items. Most of the items were in a multiple-choice format, although students were asked to give reasons for their choices on some of the items. The questions, which reportedly did not require any formal knowledge of probability, examined students' notions of chance, randomness, probability as ratio, and expected value. Tasks included visual representations of randomness, spinners with area models of probability, Piagetian marble tasks, and tree diagrams. The test also explored the students' verbal understanding and use of such phrases as "unlikely," "certain," "equally likely" and "influenced by chance."

Despite the obvious efforts in developing this test, no reports of validity or reliability were given in available sources.

Results reported deal with the students' ability to (a) use the language of probability, (b) determine simple probabilities, (c) recognize randomness, (d) solve comparison of odds problems, and (e) interpret tree diagrams and apply the multiplication principle. Green reported the students' understanding and verbal ability were inadequate in using the common language of probability. For example, "certain" and "highly probable" were often confused as were "impossible" and "less probable." Students freely associated a 50% probability with either outcomes that might or might not happen or with equally likely outcomes, even when more than two outcomes existed. Students also displayed difficulties in determining simple probabilities. In an apparently trivial question of recognizing that the two sides of a counter tossed in the air are equally likely, the percentage of students answering the problem correctly ranged from 45% in Year 1 to 74% in Year 5. In other questions, students were distracted by noncontiguous regions on the area model spinners. In addition, Green found a concept of randomness was almost totally lacking in the students. In particular, the pupils were unable to distinguish randomly distributed snowflakes in a rectangular grid from nonrandom distributions or to pick out random sequences of 0s and 1s from hand-manufactured sequences. Further, Green concluded that, although the ratio concept is crucial to a conceptual understanding of probability, items requiring the ratio concept were poorly done. For example, only 33% of the subjects recognized the box with 6 black and 2 white counters had the same chance to yield a black counter as the box with 3 black counters and 1 white counter. And, although computing ratios improved markedly as a function of age and intelligence, the majority of students continued to use other strategies such as counting and differencing, though these were not applied consistently. Finally, Green's findings suggest tree diagrams and the multiplication principle were not understood, even by the Year 5 students who had encountered these ideas in school.

The data from selected questions were used to determine the students' level of Piagetian development. From these results, Green (1983a) concluded that "most English school children do not achieve the level of formal operations" (p. 779), even by the age of 16. In the face of this evidence, Green recommended an extensive program of class-based activities is needed to provide the weight of evidence and experience necessary to eliminate the erroneous thinking exhibited by the students.

In a second study, Green (1988) explored the understanding of the concept of randomness in 1600 primary school children aged 7 to 11. One of the questions used was a raindrop problem similar to a task used by Piaget. In considering how the first raindrops would fall on a tiled roof (16 drops on a roof with 16 square tiles), the students were asked to select the best picture from among three pictures: a regular symmetrical pattern, a semirandom pattern, and a random pattern. With the increase in age, there was a decline in those preferring the regular pattern, but there was no appreciable increase in the number choosing either the random or semirandom patterns.

Included in the test was another item structurally similar, yet set in a different context. In this situation a random selection of counters numbered 1 to 16 was being modeled. As the numbers are drawn (and drawn numbers are replaced each time), an x is put in the square with the corresponding number on a grid of 16 squares. After a brief experiment modeling this problem, including at least one duplicated number, the students were asked several questions of what they think might happen if the experiment were continued. Another question presents them with eight pictures drawn by children who have played the game drawing 16 counters. For each picture, the student is to determine if the student actually played the game or made up the results. Three of these patterns are equivalent to the regular, semirandom, and random raindrop pictures. In this setting the students demonstrated a high facility for recognizing and rejecting the regular pattern as well as identifying those that were more random. Green (1988) concludes this suggests "young children do have a sound conceptual awareness of randomness" (p. 291) in contrast to the conclusions of Piaget and of his own earlier study.

The conflict in the results between this problem and the raindrop problem or the similar snowflake problem used in Green's study of 11- to 16-year-olds points out several potential difficulties. First of all, the setting of the problem seems to have made a difference. For example, rather than being unable to recognize a random pattern, perhaps the students did not believe raindrops or snowflakes fall in a random pattern. Also the format of the items may be responsible for the difference. In one case, the students were to make a forced choice among three pictures (or none of them) and, in the other, they were to respond to each item. Green (1988) suggests these results "call into question the validity of Piaget and Inhelder's findings" (p. 291). They may also call into question what Green himself has concluded as well.

Conclusions

The purpose of this section has been to explore what intuitions and conceptions children may possess about probability and to describe how these might develop. Each of the researchers reviewed in this section would provide a slightly different answer to the question.

Piaget suggests the acquisition of probability concepts depends on the development of cognitive structures and abilities and, therefore, is related to age and maturation. Because some of the related concepts, particularly permutations, combinations, ratio, and proportion, are not available to students until the stage of formal operations, Piaget concludes it is not until this stage that students acquire an understanding of probability concepts.

Fischbein, on the other hand, proposes students have an intuitive knowledge of probability as early as the preoperational stage. Further, Fischbein suggests these intuitions can facilitate the acquisition of formal understanding as they provide a foundation on which instruction can build. However, these intuitions may also be obstacles that need to be overcome or they may be lacking altogether.

Even though these conclusions appear to be contradictory, perhaps both Piaget and Fischbein are correct given the sphere of their investigations. Although they used many of the same conventional experiments, Piaget was exploring the development of more formal "a priori" ideas of probability and Fischbein was seeking to discover more informal subjectivist notions of probability. Piaget was observing the lack of completed concepts, but Fischbein was looking for the existence of partially formed precursors of probabilistic knowledge. While Piaget focused on the "spontaneous" development of concepts, Fischbein was considering how instruction can impact the development of concepts.

Whether one takes the view of Piaget or Fischbein or a blending of the two, the results of Green's rather extensive survey might raise concerns. Though Piaget suggests certain formal operational tasks and cognitive structures are generally acquired by the age of 12, Green's results suggest many had not acquired such understanding even at the age of 16. Fischbein suggests understanding of probabilistic tasks can be acquired even earlier with systematic instruction. It was not clear what instruction, if any, the students in Green's survey had received, although at least some was implied in certain instances. Yet the students, in general, lacked an understanding of the concepts presented in the survey. These differences may just represent a discrepancy between potential and actual understanding, or they may reflect unreasonable expectations and conclusions from the research.

Further doubts and concerns arise as one considers the nature of the tasks used in these assessments. Concerns have already been expressed about whether some of the tasks measure a student's familiarity with chance or familiarity with the context or materials used. Other difficulties arise because of the students' lack of verbal abilities, particularly their difficulties in using the common language of probability. But other concerns can also be raised about one task used in all the investigations considered here: the comparison of odds task with marbles. In a study similar to the others, Falk (1983) found children used a variety of strategies to select the urn most likely to yield a marble of the "payoff" color. Some of the students consistently selected the urn with the greater absolute number of payoff color elements. Others chose the urn where there were absolutely less elements of the nonpayoff color. As students began to realize the need to consider both quantities simultaneously, some computed the difference and selected the urn with a greater difference in favor of the payoff color. Finally, some based their choice on the comparison of ratios. Because each of these strategies can result in correct solutions in certain situations, it raises the question whether correct responses to a task such as this validly reflect an intuition or an understanding of relative frequency. And because the easier cases provide fairly obvious solutions and the more difficult cases become questions of comparing fractions, it is difficult to determine what knowledge of probability is demonstrated. To say the least, the concepts of proportion and probability are very elusive and difficult to measure.

So what can be said in response to the question asked in this section, namely, "What intuitions and conceptions of probability do children possess and how do they develop?" Almost certainly students are not *tabulae rasae* as they come into the classroom. As a result of experiences with games of chance and other uncertain circumstances, they no doubt have some intuitive or subjective beliefs about probability. Just what these beliefs and understanding of probability are, however, is less certain because of a need for more valid assessment tasks and instruments. In developing more formal knowledge of probability, instruction appears to have an important role to play as students develop the necessary cognitive structures. In some cases, the instructional process may be able to build on the initial intuitions the students possess. In other cases, these intuitive beliefs may be obstacles that need to be overcome. The next section considers specifically what some of these misconceptions and difficulties may be.

Misconceptions and Difficulties in Learning Probability

The phenomena of misconceptions and errors in probabilistic reasoning have been the focus of a great deal of research. Again, much of this research has been done by psychologists, who are interested in learning how people perceive, process, and evaluate the subjective probabilities of uncertain events. Among them, psychologists Kahneman and Tversky (1972) have concluded "people do not follow the principles of probability theory in judging the likelihood of uncertain events" (p. 431). They suggest instead people who are statistically naive rely on a limited number of judgmental heuristics to reduce their judgments to simpler ones (Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1973, 1974). During the past 20 years, considerable evidence which supports the use of these heuristics has been found (Bar-Hillel, 1980; Kahneman et al., 1982; Shaughnessy, 1977, 1992; Tversky & Kahneman, 1983). Sometimes these heuristics yield reasonable estimates; other times they lead to severe and systematic errors and biases.

These psychologists have provided a theoretical framework to guide mathematics educators in their study of the learning of probability. The purpose of this section, however, is not to provide an in-depth analysis of the psychological aspects of people's thinking processes. Instead, with teachers in mind, the purpose is to provide examples of the various heuristics and biases and to briefly report the empirical evidence of the use of these heuristics. These heuristics should be of interest to teachers because they lead to the misconceptions and difficulties which teachers may need to overcome in themselves and in their students. In addition to the biases that result from these heuristics, there are other difficulties students encounter as they learn probability. Knowledge of these difficulties may also be useful to classroom teachers.

Much of this research has involved students at the college level, frequently those in psychology classes. However, this section will focus on examples from elementary and secondary students or college students in mathematics classes when available. The applications of probability in a person's everyday life generally do not involve flipping coins and tossing dice, but rather decisions about what stock to buy or what medication to prescribe. In such situations, however, there may be no objective or normative result with which to compare subjects' responses. Therefore, for the research on misconceptions, more traditional probability questions have generally been used.

Several sources of misconceptions and difficulties will be considered in this section. First, the heuristic of representativeness will be described. The biases resulting from this heuristic include the negative recency effect or gambler's fallacy, a neglect of sample size, and the base-rate fallacy. Second, the availability heuristic will be discussed. The third misconception will be the conjunction fallacy. Finally, difficulties encountered with conditional probabilities and independence will be discussed.

Representativeness

According to the representativeness heuristic, the likelihood of an event or a sample is determined by the degree to which it "is similar in essential characteristics to its parent population" or "reflects the salient features of the process by which it is generated" (Kahneman & Tversky, 1972, p. 431). By this definition there are two ways in which an event or a sample may be judged to be representative.

First, it may be representative of the major characteristics of its parent population. In particular, people believe a sample should reflect the distribution of the parent population. For example, Kahneman and Tversky (1972) asked the following question of students in grades 10, 11, and 12 of college-preparatory high schools (ages 15-18) in Israel:

> All families of six children in a city were surveyed. In 72 families the exact order of births of boys and girls was G B G B B G. What is your estimate of the number of families surveyed in which the exact order of births was B G B B B B? (p. 432)

Although the sequences are equally likely from the normative point of view, the sequence with five boys and one girl does not appear representative of the near 50-50 distribution of boys and girls in the general population. In this case, 75 of the 92 students questioned judged B G B B B B to be less likely than G B G B B G ($p < 0.01$ by sign test). When a similar question was given to college students prior to a course in probability and statistics, 50 out of 80 students judged B G G B G B to be more likely than B B B B G B for the same reason (Shaughnessy, 1981).

In addition to being similar to its parent population in order to be representative, people feel an event should also reflect the properties of the process by which it is generated. In other words, it should appear random. The general properties of irregularity and local representativeness seem to capture the intuitive notion of randomness. For example, in a similar question about birth order, Israeli high school students viewed B B B G G G as significantly less likely than G B B G B G, presumably because the former does not appear representative of the random process of having children (Kahneman & Tversky, 1972). Even when students were given the option that the sequences were equally likely, 28 out of 80 college students selected B G G B G B as more likely than B B B G G G (Shaughnessy, 1977).

Some irregularity is expected, not only in the order of outcomes, but also in their distribution, as shown in the following problem:

> On each round of a game, 20 marbles are distributed at random among five children: Alan, Ben, Carl, Dan, and Ed. Consider the following distributions:
>
> | | I | | II |
> |---|---|---|---|
> | Alan | 4 | Alan | 4 |
> | Ben | 4 | Ben | 4 |
> | Carl | 5 | Carl | 4 |
> | Dan | 4 | Dan | 4 |
> | Ed | 3 | Ed | 4 |
>
> In many rounds of the game, will there be more results of type I or of type II? (Kahneman & Tversky, 1972, p. 434)

Although the uniform distribution of marbles (type II) is objectively more probable than the nonuniform distribution (type I), it appears too lawful to be the result of a random process. A significant majority of students in Kahneman and Tversky's studies, 36 of 52 ($p < 0.01$ by sign test) viewed distribution I as more probable because it was more representative of random allocation. Therefore, in a purely random allocation of marbles, they expected each child to get approximately, but not exactly, the same number of marbles. Thus, it seems people have an intuitive notion that chance, though unpredictable, is essentially fair.

In addition to irregularity, people seem to expect the essential characteristics of the parent population are represented not only globally in a representative sample but locally in each of its parts as well. However, to be locally representative, a sample will deviate systematically from chance outcomes, containing too many alternations and too few clusters. An example of this comes from Green's survey of 3,000 pupils aged 11 to 16 discussed earlier. In one question, two girls have been asked to toss coins and record each time whether the coin landed heads or tails. One girl tossed the coin, a second cheated and just made up the sequence. The students were asked which girl they thought cheated and how they could tell. For all ages, more pupils incorrectly concluded Susan had cheated because her results varied too far from the expected 50-50 proportion and because her sequence contained runs that were too long (Green, 1983a).

In conclusion, a representative sample is similar to the population in essential characteristics and reflects randomness as people see it. In addition, all its parts are representative and none is too regular. Sometimes the use of the representativeness heuristic yields appropriate results because in some cases events which appear more representative are also more probable, and vice versa. In other cases, however, representativeness leads to serious errors because factors that should affect judgments of probability are ignored. Among these possible errors are the negative recency effect or "gambler's fallacy," the neglect of sample size, and the base-rate fallacy.

Negative recency effect or "gambler's fallacy." An extension of the idea of local representativeness leads to what is called the negative recency effect or the gambler's fallacy. After observing a run of heads, for example, many people erroneously believe a tail is now due, presumably because that occurrence will result in a more representative sequence than another head. Evidence of this belief is seen in the responses to the following question on the fourth NAEP (Brown & Silver, 1989, p. 25):

> If a fair coin is tossed, the probability it will land tails up is 1/2. In four successive tosses the coin lands tails up each time. What happens when it is tossed a fifth time?
>
> A. It will most likely land heads up.
> B. It is more likely to land heads up than tails up.
> C. It is more likely to land tails up than heads up.
> D. It is equally likely to land tails up or heads up.

Although approximately half of the 7th graders and 11th graders answered the question correctly, 38% of the 7th graders and 33% of the 11th graders believed a head was most likely or more likely on the next toss. On a similar question, Green (1983a) found, overall, 12% of the students aged 11 to 16 expected a tail to be more likely after a run of five heads.

According to Tversky and Kahneman, a misunderstanding of the fairness of the laws of chance is at the heart of the gambler's fallacy. The fairness of the coin implies, in the mind of the gambler, that any deviation in one direction will soon be canceled out by a corresponding deviation in the other direction. As Tversky and Kahneman (1971) point out, however, a coin has neither memory nor moral sense and, therefore, "cannot be as fair as the gambler expects it to be" (p. 106). Gamblers, though, are not the only ones susceptible to this kind of reasoning. Consider the following example:

> The mean IQ of the population of eighth graders in a city is *known* to be 100. You have selected a random sample of 50 children for a study of educational achievements. The first child tested has an IQ of 150. What do you expect the mean IQ to be for the whole sample? (Tversky & Kahneman, 1971, p. 106)

A surprisingly large number of people expect the IQ for the sample to remain at 100. Again, this is based on the belief that a random process is self-correcting and that "errors cancel each other out." However, the remaining 49 students, as a sample of the population, would be expected to have a mean of roughly 100. Adding these scores and the known 150 score together yields 5,050 total points, for a mean IQ score of 101 for the sample of 50 students. Thus, rather than deviations being canceled out, they are merely diluted.

Neglect of sample size. The idea of local representativeness and the view of chance as a self-correcting process leads people to assume any sample, no matter its size, should have the same characteristics as the original population and, also, as any other sample. This assumption mistakenly leads to what Tversky and Kahneman (1971) call the "law of small numbers." The law of large numbers states that for large samples, the statistics of the sample become less variable and will, in fact, be highly representative of those of the population from which the sample was selected. People's intuitions about random sampling seem to assert the law of large numbers applies to small numbers as well, or, in other words, the size of the sample does not matter.

One problem that has been asked in various forms is the maternity-ward problem first used by Kahneman and Tversky (1972, p. 443):

> A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50% of all babies are boys. The exact percentage of baby boys, however, varies from day to day. Sometimes it may be higher than 50%, sometimes lower. For a period of 1 year, each hospital recorded the days on which more than 60% of the babies born were boys. Which hospital do you think recorded more such days?
> (a) The larger hospital
> (b) The smaller hospital
> (c) About the same (i.e., within 5% of each other)

Of 50 college undergraduates responding to this question, 24% selected the larger hospital, 29% selected the smaller hospital (the correct answer), and 56% thought the results would be about the same at both hospitals (Kahneman & Tversky, 1972). In response to a similar question, 60% of the 80 undergraduates surveyed felt the distribution of the boys in a sample of babies would be the same regardless of the hospital size (Shaughnessy, 1981). In an unspecified sample of 40 subjects, Bar-Hillel (1982) also reported 60% chose the response "about the same." However, as Bar-Hillel varied the proportion of boys to 70%, 80%, and 100% with other subjects, an increasing number of them began to recognize such nonrepresentative samples were more likely in smaller samples, i.e., that 15 boys in a row is more likely than 45 boys in a row.

Green included a similar question in his investigation of the probability knowledge of students aged 11 to 16 (Schrage, 1983, p. 353):

> Which of the following results is more likely?
> (a) Getting 7 or more boys out of the first 10 babies born in a new hospital.
> (b) Getting 70 or more boys out of the first 100 babies born in a new hospital.
> (A) They are equally likely.
> (B) 7 or more out of 10 is more likely.
> (C) 70 or more out of 100 is more likely.
> (D) No one can say.

Of the students in Green's study, 25% reported they were equally likely while 61% thought no one could say. When Schrage asked the same question of 153 of his education students at the University of Dortmund, Germany, except with answer (D) omitted, 60% believed the two events were equally likely, each with a probability of 7/10. Even among 17 students who had taken a stochastics course in school, only one gave the correct answer (B).

Tversky and Kahneman (1971) found such misconceptions were not limited to naive subjects. A study of experienced research psychologists revealed their intuitions were also based on the law of small numbers. Participants attending meetings of the Mathematical Psychology Group and the American Psychological Association were asked to respond to the following question:

> Suppose you have run an experiment on 20 subjects, and have obtained a significant result which confirms your theory ($z = 2.23$, $p < 0.05$, two-tailed). You now have cause to run an additional group of 10 subjects. What do you think the probability is that the results will be significant, by a one-tailed test, separately for this group? (Tversky & Kahneman, 1971, p. 105)

Of the 84 people who responded, the median answer in both of the groups was 0.85, although a much more reasonable estimate is 0.48. The responses of these psychologists reflect the expectation that a valid hypothesis about a population will be represented by a statistically significant result in a sample, regardless of its size. As a consequence of this bias, researchers run the risk of putting too much faith in the results of small samples and of overestimating the replicability of such findings.

Base-rate fallacy. Another phenomenon often attributed to representativeness is the base-rate fallacy (Bar-Hillel, 1980; Tversky & Kahneman, 1974, 1980, 1982a). In this case, information about the frequency of the outcomes within the population is ignored in favor of judging probability on the basis of the representativeness of other characteristics.

One approach often used in the research is to provide a sketch of a person and ask the subject to assess the probability the person described is from a particular profession. Consider, for example, the following problem (Tversky & Kahneman, 1980, p. 61):

> A panel of psychologists interviewed a sample of 30 engineers and 70 lawyers, and summarized their impressions in thumbnail descriptions of those individuals. The following description has been drawn at random from the sample of 30 engineers and 70 lawyers.
>
> > "John is a 30-year-old man. He is married and has two children. He is active in local politics. The hobby that he most enjoys is a rare book collection. He is competitive, argumentative, and articulate."
>
> Question: What is the probability that John is a lawyer rather than an engineer?

An unspecified group of 85 subjects responded to this question while another group of 86 subjects responded to a similar question, but which had an original sample of 70 engineers and 30 lawyers. The median answer was 0.95 in both groups, irrespective of the base-rate frequency of engineers and lawyers. Tversky and Kahneman suggest the subjects apparently evaluated the likelihood the description belonged to a lawyer rather than an engineer by the degree to which the description was representative of stereotypes of the two professions, with little or no regard for the frequencies of the categories within the sample. In making their judgments, the subjects, therefore, neglected the reliable quantitative information of the base rates in favor of more subjective stereotypes.

Tversky and Kahneman suggest the neglect of base-rate information appears to be a more general phenomenon, occurring even when probability assessments are not mediated by representativeness. The following example, which has come to be known as the Taxi problem, has been widely studied (Tversky & Kahneman, 1982a, p. 156):

> A cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given the following data:
> (i) 85% of the cabs in the city are Green and 15% are Blue.
> (ii) A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.
> Question: What is the probability that the cab involved in the accident was Blue rather than Green?

Of the supposed several hundred subjects who have been given slightly different versions of this question, the modal and median response has been 80%, coinciding with the credibility of the witness. Thus, an overwhelming majority of the subjects ignored the relevant base-rate information. The formal solution to the problem, however, finds the probability the cab was blue as only 0.41 (see Figure 2). Therefore, despite the witness' report, the hit-and-run cab is more likely to be green than blue, because the base-rate is more extreme than the credibility of the witness.

		Witness Says (80% accurate)	
		Blue	Green
Cabs	150 Blue	120	30
	850 Green	170	680
		290	710

Sample of 1,000 cabs

Probability it was a Blue Cab, given that the witness said it was blue, is 120/290 = 0.41

Figure 2. Contingency table for the Taxi problem. (Shaughnessy, 1992, p. 471)

In the absence of other data, the base-rate data seemed to be used appropriately. For example, when item (ii) was omitted, most subjects viewed it as a sampling problem in which a single cab is being selected from the population of cabs in the city. In this case, they correctly answered that the probability the cab was blue is 0.15. However, when additional information about the hit-and-run cab is introduced, namely the witness' report, the base-rate no longer is considered relevant.

Availability

Besides representativeness, availability is a second heuristic people use in subjectively judging the probability of an uncertain event. In this case, predictions about the frequency of a class or the likelihood of an event are based on how "available" instances of that class or event are or, in other words, the ease with which particular instances can be constructed or brought to mind. For example, if someone is asked to estimate the local crime rate or the probability of having a heart attack, personal experience or that of their relatives or acquaintances may influence the probability estimates. One who has been a crime victim may estimate a higher crime rate than someone who has not and knows no one who has. And one who has many friends or relatives who have had heart attacks may estimate a high probability of having a heart attack.

Even though instances of large classes or events may be recalled better and faster than instances of less frequent ones or likely occurrences may be easier to imagine than unlikely occurrences, the use of availability may also lead to predictable biases. First, there is a risk of bias because judgments are based on one's own limited experience. Also, the ease of constructing or retrieving outcomes may be affected by factors other than frequency or probability. Thus, because people tend to believe outcomes that can easily be called to mind will be more likely to occur, probability estimates can be biased.

Tversky and Kahneman (1973) provide examples of tasks in which subjects seemed to base their likelihood estimates on how easy it is to construct instances. Shaughnessy has used similar tasks with undergraduate students prior to a course in probability and statistics. In particular, the following task was used in both studies (Shaughnessy, 1977, p. 312):

> Consider the grids below.
>
> | Grid A | Grid B |
> |---|---|
> | XXXXXXXX | XX |
> | XXXXXXXX | XX |
> | XXXXXXXX | XX |
> | | XX |
> | | XX |
> | | XX |
> | | XX |
> | | XX |
> | | XX |
>
> Are there
>
> (a) more paths in grid A?
> (b) more paths in grid B?
> (c) about the same number of paths in each?
>
> (Note: A 'path' was carefully defined for the subject as a sequence of line segments starting from the top row running down through each row to the bottom row, meeting one and only one symbol in each horizontal row of the array.)

Although there are an equal number of paths in each grid, because $8^3 = 2^9 = 512$, a majority of students believed more paths were possible in grid A. Tversky and Kahneman reported 46 out of 54 respondents saw more paths in grid A than in grid B ($p < 0.001$ by sign test) with median estimates of 40 paths in grid A and 18 paths in grid B. Likewise, Shaughnessy reported 53 out of 80 undergraduates favored grid A, giving such reasons as, "It is easier to draw a path in grid A" or "There are more *X*s in grid A" (p. 94).

Another question explored the impact of availability on the estimates of combinatorial tasks. Tversky and Kahneman (1973) asked subjects, who were students in the 10th and 11th grades of several college-preparatory high schools in Israel, to "estimate the number of possible committees of *r* members that can be formed from a set of ten people" (p. 214). Four groups of students (total $n = 118$) evaluated the following values of *r*, respectively: 2 and 6; 3 and 8; 4 and 9; and 5. Tversky and Kahneman hypothesized that small committees would be more available than large committees because they would be easier to construct. For example, ten people can be divided into five disjoint committees of two but no more than one disjoint committee of eight. Further, because committees of eight share overlapping members, they are more difficult to imagine. As predicted, the estimates of the number of possible committees decreased with their size.

Shaughnessy (1981) asked the following version of the question of undergraduate students (p. 94):

> A person must select committees from a group of 10 people. (A person may serve on more than one committee.) Would there be
> (a) more distinct possible committees of 8 people?
> (b) more distinct possible committees of 2 people?
> (c) about the same number of committees of 8 as committees of 2?
> Give a reason for your answer.

In response, the students overwhelmingly believed committees of two would be more numerous than committees of eight, with 47 choosing committees of two, 7 choosing committees of eight, and 19 thinking there would be the same number of each. There are, in fact, the same number of committees of two as there are committees of eight, because for each committee of two chosen, a "non-committee" of eight is left out.

Fischbein et al. (1991) have also suggested availability may be a factor in the intuitive evaluation of the magnitude of a sample space. When students aged 9 to 14 were asked to compare the possibilities of getting various sums in a dice game, the students were far more successful with sums where the number pairs were more "available." For example, 50.5% of the elementary students and 67.7% of the junior high students with prior probability background correctly determined the sums of 2 and 12 are equally likely. On the other hand, only 30.4% of the elementary students and 38.5% of the junior high students correctly recognized sums of 3 and 11 as equally likely. With other possible sums, the students seemed to feel the larger number of the two would have more possible pairs and, therefore, be more likely. These examples have shown how the heuristics of representativeness and availability may lead to biases and errors in probabilistic judgments.

The Conjunction Fallacy

There are also other areas of difficulty in the application of probabilistic reasoning. One of these difficulties is a violation of the conjunction rule of probability, or what Tversky and Kahneman (1983) call the conjunction fallacy. According to the conjunction rule of probability, the probability of a conjunction, A and B, cannot exceed the probabilities of its constituents. For example, consider a ball selected from a container of numbered balls of various colors. Because the even-numbered red balls are a subset of the red balls, there can be no more even-numbered red balls than there are red balls. In particular, if 10 out of 24 balls are red, then there can be no more than 10 even-numbered red balls. Thus, the probability a selected ball is red and even-numbered is no greater than 10/24, or no greater than the probability the selected ball is red. Likewise, the probability a selected ball is red and even-numbered is no greater than the probability the selected ball is even-numbered.

In a study exploring the conjunction fallacy, Tversky and Kahneman (1982b) gave subjects the following personality sketch of Linda and asked the subjects to rank the associated eight statements by the degree to which Linda resembles the typical member of that class (p. 92):

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.
>
> Linda is a teacher in elementary school.
> Linda works in a bookstore and takes Yoga classes.
> Linda is active in the feminist movement.
> Linda is psychiatric social worker.
> Linda is a member of the League of Women Voters.
> Linda is a bank teller.
> Linda is an insurance salesperson.
> Linda is a bank teller and active in the feminist movement.

In this case, one should conclude in applying the conjunction rule that the statement "Linda is a bank teller and active in the feminist movement" is less likely than either "Linda is a bank teller" or "Linda is active in the feminist movement." The subjects who were given this question included statistically naive undergraduate students from the University of British Columbia and Stanford University, graduate students with some knowledge of probability and statistics from Stanford University, and statistically sophisticated graduate students in the decision-making program of the Stanford Business School. In response to this problem, 85% of the statistically sophisticated group, 90% of the knowledgeable group, and 89% of the statistically naive group violated the conjunction rule, judging "Linda is a bank teller and active in the feminist movement" as more likely than "Linda is a bank teller."

Tversky and Kahneman suggest that, in such situations, the representativeness and availability heuristics may be involved in making the conjunction seem more probable. In particular, the conjunction may be more representative than one of its constituents or instances of the more specific category may be easier to imagine than the more general category. Other explanations for the difficulties have been offered as well, including difficulties that arise from the language of the problem. For example, the following problem was given to 115 undergraduates at Stanford University and the University of British Columbia (Tversky & Kahneman, 1983, p. 305):

> A health survey was conducted in a representative sample of adult males in British Columbia of all ages and occupations. Mr. F. was included in the sample. He was selected by chance from the list of participants. Which of the following statements is more probable? (check one)
> A. Mr. F. has had one or more heart attacks.
> B. Mr. F. has had one or more heart attacks and he is over 55 years old.

The number of people who are over 55 years of age and have had a heart attack would be no more than the number of people in the entire sample who have had a heart attack, meaning the conjunction in statement B cannot be more likely than statement A. Nevertheless, 58% of the students selected answer B. Shaughnessy (1992) suggests the students may be confusing "has had one or more heart attacks *and* is over 55 years old" with "has had one or more heart attacks *given that* he is over 55 years old," thus confusing conjunctions and conditional probabilities. For whatever reason, the conjunction rule appears to be one principle of probability frequently violated in making subjective estimates of probability.

Difficulties with Conditional Probability and Independence

Two other aspects of probability that present difficulties to students are conditional probabilities and the related concept of independent events. In the case of conditional probabilities, a particular condition or known information affects the possible outcomes and reduces the sample space of an experiment. Falk (1988) outlines three misconceptions or fallacies that arise as students' ideas of conditional probabilities have been probed.

The first difficulty arises in interpreting conditionality as causality. Conditional probability applied in "forward looking" situations lend themselves fairly naturally to a causality argument. However, it is more difficult for students to infer cause for an event conditioned on an a posteriori event or make a "backward inference" that reverses the time axis. The following example has been referred to as the "Falk Phenomenon" in the literature:

> An urn has two white balls and two black balls in it. Two balls are drawn out without replacing the first ball.
> 1) What is the probability that the second ball is white, given that the first ball was white?

> 2) What is the probability that the first ball was white given that the second ball is white? (Shaughnessy, 1992, p. 473)

The first question is a straightforward conditional probability. If a white ball is drawn the first time then there are one white and two black balls remaining. The probability of now drawing a white ball is 1/3. When presented with the reverse problem, Falk (1988) reports students argue that "conditioning the probability of an outcome of a drawn event on an event that occurs later is not permissible" (p. 292). Reflecting their causal reasoning, they suggest the color of the second ball cannot influence the color of the ball drawn first. To overcome this difficulty in reasoning, Falk suggests physically simulating the problem. One ball is drawn and set aside. A second ball is drawn and shown to be white. The students then are to determine the probability the hidden ball is also white.

A second difficulty students have is in determining what the conditioning event is. As an example, Falk (1988) provides the following teaser:

> Three cards are in a hat. One is blue on both sides, one is green on both sides, and one is blue on one side and green on the other. We draw one card blindly and put it on the table as it comes out. It shows a blue face up. What is the probability that the hidden side is also blue? (p. 293)

The most common response is 1/2, as people rule out the double-green card. Of the two equally likely cards remaining, only one has blue on the other side. Falk argues instead "the probability of the target event should be conditioned on the immediate event given as datum in the problem and not on some inferred event" (p. 294). In this case, the outcomes of the experiment are the six equally likely faces of the cards. Of the three blue that could be shown on the first draw, two have blue on the opposite side, so the probability of the second side also being blue is 2/3.

A third difficulty with conditional probabilities is the confusion between a conditional event and its inverse. As an example, Falk (1988) cites the confusion often surrounding the interpretation of test results in medical contexts. In these situations the probability of disease given a positive test result is erroneously equated with that of a positive result given the disease. For example, given a 1% frequency of cancer within a particular group and an 87% accuracy rate of mammography (Eddy, 1982), the probability a woman tests positive given she has cancer is 0.87 while the probability she has cancer given a positive test is only 0.06 (see Figure 3).

Although Falk and others have written about difficulties with conditional probabilities, little empirical research has been reported dealing with students' beliefs and intuitions in cases of conditional probabilities. One exception is a study by Pollatsek, Well, Konold, Hardiman, and Cobb (1987). In one experiment, 86 undergraduates taking lower division psychology courses at the University of Massachusetts were given a series of six questions in which they were asked which, if either, of two conditional probabilities was larger. In a second experiment, 120 undergraduates were recruited from sections of an introductory psychology course designed for majors. Given similar problems with paired statements worded either in terms of probabilities or in terms of percentages, the subjects were asked to estimate both conditional probabilities. In a final condition, they were asked to calculate two conditional probabilities from data that were presented. From the results, Pollatsek et al. concluded that "statistically naive college students are capable of grasping the concept of conditional probability and its directionality" (p. 267). They suggest some of the difficulties encountered may have been the result of misunderstanding the wording used to express the attributes, lack of real-world knowledge on which to base the estimates, or a confusion between the concepts of "independent events" and "equally probable events." The most common error on the final part of the questionnaire was a confusion between conjunctions and conditional probabilities, taking "the percentage of green-eyed people who have brown hair" to mean "the percentage of people who have both green eyes *and* brown hair" instead of "the percentage of people who have brown hair *given that* they have green eyes."

		Mammography Results (87% accuracy rate)		
		Positive	Negative	
Cancer	100 have cancer	87	13	Sample of 10,000 women
	9,900 do not have cancer	1,287	8,613	Probability patient has cancer, given a positive test, is 87/1,374 = 0.06
		1,374	8,626	

Figure 3. Contingency table for the Cancer problem.

The final area of difficulty to be discussed, students' misunderstanding of independent events, may be closely related to their understanding of conditional events. Mathematically, two events A and B are considered to be independent if the probability both occur is equal to the probability A occurs times the probability B occurs, or $P(A \cap B) = P(A) \times P(B)$. Kelly and Zwiers (1988) believe, however, that it is more intuitive for students to link the ideas of independence and conditional probability. In this case, events are independent "when the occurrence (or nonoccurrence) of one of the events carries no information about the occurrence (or nonoccurrence) of the other event" (Kelly & Zwiers, 1988, p. 97). In other words, there is no difference between the probability of event A occurring and the probability of A occurring given B occurs.

Aside from the formality of how to present the idea of independence, Kelly and Zwiers (1988) point out two particular difficulties students encounter. First, students have difficulty determining when events in the real world are independent or dependent. A second common misunderstanding involves interpreting a dependent relationship between events as a causal relationship. These difficulties are evident in the number of students who make decisions based on the gambler's fallacy described earlier, believing how a coin may land depends on what has happened in earlier tosses.

Conclusions

In an earlier section, Fischbein's exploration of students' intuitions and Green's survey of students' understanding of probability began to reveal some of the errors students make and difficulties they face. This section has provided further examples of misconceptions and difficulties that occur in probabilistic reasoning. Garfield and Ahlgren (1988) suggest these difficulties fall into four categories. Two of these areas of difficulties may be of a mathematical nature. In particular, students may lack the necessary mathematical background or skills such as an understanding of ratio and proportion. Other probability concepts may be difficult because of the students' inexperience with any concepts like those encountered in probability. This difficulty may account for some of the trouble students have with independent events, conditional probability, and conjunctions. Other difficulties may be of a linguistic nature because the everyday expressions of probability are ambiguous, leading to different interpretations of terms such as *unusual*, *improbable*, or *possible*. Finally, many of the examples in this section suggest other misconceptions about probability are of a psychological nature. This is particularly evident when the concepts interfere with intuitive ideas the students already have.

Much of the evidence for these misconceptions and difficulties was the result of studies done at the college level. These data, nevertheless, have implications for this investigation for two reasons. First, teachers have come through a college experience and they may fall prey to some of these same misconceptions. No one, except for Schrage (1983), has asked questions of teachers or teacher education students. Because they are part of the general college population who might take an introductory psychology class, their results may be similar to the examples given here. It certainly would be of interest to know if teachers' probabilistic reasoning includes some of these misconceptions.

In addition, these results have implications for the middle school level. The studies and assessments that have been done with younger students, though few in number, have revealed similar misconceptions and difficulties at that level. This finding was particularly true of the biases resulting from the heuristics representativeness (where judgments are based on how "typical" an occurrence is) and availability (where judgments are based on how easily instances can be brought to mind). Because many of these misconceptions, particularly those based on the intuitive heuristics, have their foundation in the early experiences students have with probability, the middle school may have an important role to play in building a more appropriate intuitive foundation for probabilistic reasoning.

Some concern can be expressed about the design of this body of research. In most cases, the methodology involved forced-choice tasks with little or no attempt to obtain explanations for the subjects' choices. The one exception is the study done by Shaughnessy (1977) where he did ask students to provide reasons for their solutions. Because it is thinking that is being explored, it would be essential to use instruments or methodology that actually probe the subjects' thinking. Further, in addition to the ambiguity of the common language of probability, many of the problems were laced with contextual traps that hinder getting a clear picture of the subjects' understanding of probability. Finally, results have generally been presented as individual examples of various proposed difficulties without consideration of the validity and reliability of the tasks or the instruments in general. In some cases, researchers have offered alternative explanations for the apparent thinking of subjects. For example, Konold (1989a) has suggested that people reasoning according to what he calls the outcome approach believe that the goal in questions of uncertainty is to correctly predict what the next outcome will be, rather than estimate what is likely to occur. Further, in what is called the fundamental attribution error, Nisbett and Borgida (1975) and Ross (1977) suggest that people attribute more significance to dispositional factors and less to situational variables. People using such reasoning would attribute undue accuracy to the witness in the Taxi problem, for example, and ignore the base-rate data.

This section has attempted to answer the question, "What misconceptions and difficulties in learning probability do teachers face in teaching probability at the late elementary or middle school level?" Examples have offered some evidence students sometimes base their decisions on mistaken applications of such judgmental heuristics as representativeness and availability. Further illustrations indicate difficulties with the language of probability and such concepts as independent events, conditional probabilities, and conjunctions. The methodology of the research does not permit conclusive findings, nor does it suggest how widespread these misconceptions are, and alternate explanations do exist. Nevertheless, the research presents evidence these misconceptions and difficulties do exist as something teachers may face in seeking to develop an understanding of probability through instruction. The impact of instruction on developing understanding of probability and overcoming these misconceptions and difficulties will be considered in the next section.

Impact of Probability Instruction

As has been suggested earlier, the teaching of probability has been very limited at the elementary and secondary level. As a result, there have been few studies exploring the impact of probability instruction. One exception was the period of the 1960s and early

1970s when reforms gave more attention to probability and several studies, primarily dissertations, investigated the teaching of probability. A number of these were feasibility studies (Doherty, 1965; Jones, 1974; Leake, 1962; Leffin, 1971) undertaken to determine whether probability topics could be taught to elementary or middle school students. Most of these studies concurred with Fischbein's later claim that elementary and middle school students already possess some notions of probability before instruction. They also concluded it was possible to teach probability to children in the upper elementary and middle school years. Other studies compared the effects of different approaches to teaching probability (Gipson, 1971; McLeod, 1971) or attempted to find relationships between success in probability and other variables such as attitudes or computational abilities (Mullenex, 1968). Many of the units taught as part of these studies were too short for significant changes to be observed in attitudes or skills.

In addition to being conducted over 20 years ago, these studies reflect the assumptions of a different paradigm of mathematics education. Built upon the foundation of behaviorism, the focus of instruction was on satisfying performance objectives established by task analysis rather than attention to meaningful understanding and student thinking. The focus of instruction was on a more formal development of probability involving points in a finite sample space and probability distributions. In addition, the model of instruction was quite different from the model currently envisioned by the NCTM. As a result of these differences, these studies will not be considered in detail here. Instead, this section will focus on studies influenced by the developments of cognitive psychology, studies that consider the impact of instruction on students' conceptions and misconceptions of probabilistic ideas. Some of the instruction attempting to overcome difficulties has focused on individual or small group treatments. Because the interest of this investigation is in classroom instruction, this section will consider only those efforts that take place in a classroom setting. In particular, two studies will be discussed. The first is a study focusing on an instruction program designed for students in grades 5, 6, and 7. The second study looks at a course attempting to overcome the misconceptions of college students.

Fischbein and Gazit: An Elementary Study

Fischbein and his colleagues were among the first to suggest children's intuitions and conceptions of probability can be influenced by systematic instruction. In one study discussed earlier, Fischbein et al. (1970a) reported children, particularly 9- and 10-year-olds, improved their predictions of outcomes following instruction on Piagetian probability ratio tasks. Similarly, improvement was noted when instruction on combinatorial estimation tasks was provided (Fischbein et al., 1970b). However, the instruction in these earlier studies was in the form of learning by programmed discovery within the context of one-on-one interviews with the subjects, and does not reflect classroom practice.

In a more recent study, Fischbein and Gazit (1983, 1984) have explored the effects of a classroom teaching program in probability developed for students in grades 5, 6, and 7. The purpose of their study was twofold. First, they wanted to determine the capacity of children 10 to 13 years old to correctly and productively assimilate the basic concepts of probability and the strategies for determining probability. Second, they wanted to evaluate the indirect effect of a course in probability on the subjects' intuitive interpretations and probabilistic misconceptions.

The teaching program was based on two assumptions. First, they suggested the teaching process needs to take into account the intuitions students already possess. Their earlier research had indicated students do have probabilistic intuitions as they enter the classroom, some that facilitate acquisition of more formal concepts and others that inhibit the process of learning and understanding. Second, they believed verbal explanations alone are insufficient for modifying intuitions. They claim students, instead, need to be actively involved in probabilistic situations in order to develop an appropriate intuitive background for probability concepts along with the corresponding formal knowledge.

The program consisted of 12 lessons intended to teach the following concepts: certain, possible, and impossible events; chance and ways of quantifying chance outcomes; calculation of probability when the possible outcomes are equally likely; probability and relative frequency and the relationship between them; procedures for counting outcomes; and simple and compound events and their probability. In accordance with their assumptions, the lessons involved the students in activities with dice, coins, and marbles in which they were predicting, recording, and counting up different sets of outcomes as well as calculating both empirical and theoretical probabilities.

The teaching program was attended by 70 students in grade 5, 160 in grade 6, and 55 in grade 7. Parallel control classes included 50 students in grade 5, 200 in grade 6, and 55 in grade 7. Both the experimental and control groups included two grade 5 classes, five grade 6 classes, and two grade 7 classes. Although never stated, it is assumed the study was done in Israel.

The assessment of the effects of the teaching program was obtained by means of two questionnaires. Questionnaire A, administered only to the experimental classes, was devised to satisfy the first goal of the study, that of assessing the extent to which the students in the experimental program had assimilated and were able to use the concepts and procedures they had been taught. The nine, open-ended questions asked the students to provide examples of various kinds of events and to determine probabilities in different settings. Questionnaire B was devised to meet the second goal of the study, namely, to assess the indirect effect of instruction on the students' intuitively based misconceptions. This questionnaire was administered to both experimental and control classes. Among the eight open-ended questions were ones focusing on such misconceptions as representativeness, the negative recency effect, the notion of a "lucky" choice, and the superstitious belief the probability of an event may be influenced by particular behavior. Questionnaire B also explored students' understanding of proportional reasoning. No validity or reliability information was provided about the questionnaires.

The results of Questionnaire A, assessing how well the concepts taught had been understood, showed distinct differences by grade level, particularly between the fifth graders and the sixth and seventh graders. For the fifth graders, most of the concepts covered in the program turned out to be too difficult. These students were able to give correct examples for certain, possible, and impossible events, but generally gave low percentages of correct answers for the other concepts, particularly on procedures for calculating probability and on simple and compound events. Fischbein and Gazit (1983) suggest 10- to 11-year-olds "seem not to be intellectually ready to assimilate the basic concepts and procedures of probability reasoning" (p. 749). On the other hand, 60–70% of the sixth graders and 80–90% of the seventh graders were able to correctly answer most of the items on Questionnaire A. Another explanation may be the instructional approach was not appropriate or effective at the fifth-grade level.

Two items turned out to be especially difficult, even for the seventh graders. The first was calculating probabilities of compound events involving the sums of the numbers on two dice. On this item many students added 6 + 6 to find the number of all possible outcomes instead of multiplying. A second difficulty or systematic bias arose in providing examples of simple and compound events when rolling one or two dice. For some reason, it was easier for the students to give examples of simple events when rolling one die and to give examples of compound events when rolling a pair of dice.

When the data from Questionnaire B for the experimental and the control classes were compared by chi square tests, the results were mixed. Based on the responses to the questions dealing with representativeness, the negative recency effect, and other common misconceptions, Fischbein and Gazit (1984) concluded that "in grades 6 and 7 the teaching programme had an indirect positive effect on the respective intuitive biases" (p. 22). On these questions, more experimental students selected correct answers than did control students and many of these differences were judged significant.

On the other hand, on the two items related to proportional reasoning, the experimental classes, in general, performed worse than the control classes. Fischbein and Gazit (1983) argue that although computation of probability may require ratio comparisons, "probability, as a specific mental attitude, does not, necessarily, imply a formal understanding of proportion concepts" (p. 750). Further they suggest that by emphasizing specific probability concepts and procedures in systematic instruction, they may have disturbed the subjects' proportional reasoning ability, still fragile in many adolescents. If nothing else, it points out a potential danger teachers must be aware of.

It is evident Fischbein and Gazit believe instruction can improve students' conceptual and intuitive understanding of probability. However, errors or limitations in their research design leave room for doubt. In particular, they did not assess students' conceptions or intuitions prior to the instruction and they argued against having the control classes respond to Questionnaire A. As a result, they missed an opportunity to actually judge the impact of instruction. In addition, the lack of reliability and validity information on the questionnaires and the ambiguity in some of the questions and some of the students' responses leaves further room for doubt. There certainly is the need for further research on the impact of instruction in probability at the elementary/middle school level.

Shaughnessy: A College Study

Shaughnessy (1977) conducted an intensive 12-week teaching experiment at the college level, exploring whether an activity-based approach to teaching elementary probability and statistics would be more effective than the conventional lecture approach in overcoming some of the students' misconceptions about probability and reducing their reliance on such heuristics as representativeness and availability. The experimental activity-based course was constructed as an alternative to the usual lecture format of a finite mathematics course designed primarily to serve the needs of students majoring in business, agriculture, or biology.

The subjects in the study were 80 undergraduate students, 48 men and 32 women, registered in four sections of the finite mathematics class at Michigan State University. Of the 80 subjects, 51 were freshmen and 45 were either business or accounting majors. Personal background information indicated the subjects generally did not have strong backgrounds in mathematics. In addition to completing the prerequisite College Algebra course, 51 of the students had also taken at least one of the two remedial college courses corresponding to high school algebra. Only seven students in the sample indicated they had any prior exposure to probability.

Of the seven sections of the finite mathematics class scheduled in the Spring term of 1976, four were randomly selected for the study. Two sections, each containing 20 students, were assigned to the experimental activity-based course taught by the researcher. The other two sections, containing 26 and 14 students respectively, were assigned to the lecture-based course. No information was given about who taught these sections or even if the two sections had been taught by the same professor. Although different textbooks were used, the mathematics content of the experimental and lecture courses was quite similar. Both sections studied probability models, counting principles, and game theory. In place of a segment devoted to linear programming in the lecture course, the experimental course introduced some elementary statistics.

The researcher developed a series of nine activities in probability, combinatorics, game theory, expected value, and elementary statistics that served as the focal point of the experimental classes. As they worked together on these activities in groups of four or five, the students were actively involved in performing experiments as they gathered, organized, and analyzed data. As a final result, the groups were expected to state a conclusion in the form of a mathematical principle or mathematical model. This approach involving small groups, problem solving, and model building was used because Shaughnessy (1977) felt the transition from misconceptions to the use of appropriate probability models might be facilitated if students were encouraged to experience probability as "a *process* of describing observed experimental phenomena more and more accurately, rather than as a *system* of rules, axioms, and counting techniques that must be learned and applied to problems" (p. 300). Reading assignments and problem assignments from several texts were used to supplement and reinforce the class activities. In addition, the students were to keep a daily journal of all their class experiments and homework problems, including personal comments on how they felt and what they had learned.

The researcher served as the instructor for the experimental sections. Although he provided some instruction more formally, most of the interaction with the groups was intended to encourage them to work problems out for themselves. At various times, the instructor played the role of organizer, diagnostician, devil's advocate, or critic. While the students worked on the activities, the instructor circulated among the groups, clarifying questions and assisting groups as needed. Assistance generally came in the form of questions intended to guide the students from what they knew back to their original question. The researcher was the one responsible for reporting what instruction had been given in the experimental sections. In particular, no observations were done or videotapes made as verification of the teaching techniques reported.

Students' knowledge of probability concepts and their reliance upon the heuristics of representativeness and availability were assessed both by written responses and by taped interviews before and after instruction. The instruments, developed by the researcher, included many tasks similar to or the same as tasks used by Kahneman and Tversky (1972; Tversky & Kahneman, 1973) to assess reliance on the judgmental heuristics. The "either-or" forced response format was relaxed, however, and students were asked to supply a reason for each of their responses in an attempt to gain some insight into the thinking processes of the students. Because the students had only 50 minutes to answer about 20 questions and supply a reason for their answer in each case, one might wonder if time were a factor. No validity or reliability information was provided.

Data from the pretests supported the claim of Kahneman and Tversky that statistically naive college students do rely upon representativeness and availability to estimate the likelihood of events. The posttest results revealed mixed success in overcoming these misconceptions. Although Shaughnessy (1977) concluded the "experimental activity-based classes were more successful at overcoming reliance upon representativeness ($p < 0.05$, df = 2) and tended to be more successful at overcoming reliance upon availability ($p < 0.19$, df = 2)" (p. 308), the success in the case of availability is questionable. In any event, no further information was presented about the analysis of the data, other than comparing the pretest and posttest responses for seven sample problems.

Shaughnessy draws several conclusions from his study. First, he suggests college students are capable of discovering some elementary probability models and formulas for themselves as they are actively involved in conducting probability experiments. Second, even though the activity-based classes had some success in overcoming reliance on the heuristics, it appears intuitive beliefs and misconceptions are not easily changed. Shaughnessy (1992) concludes "it is very difficult to replace a misconception with a normative conception, a primary intuition with a second intuition, or a judgmental heuristic with a mathematical model" (p. 481).

Finally, in reflecting back on the study's success in overcoming misconceptions, Shaughnessy (1992) attributes most of the success to the instructional model that placed the students in a position of having to reconcile the dissonance between their misconceptions and their observations. Shaughnessy describes the instructional process as follows:

> Students first had to "buy into the task" by *making a guess* for the outcome of the experiment. This made it their problem, set the hook so to speak. Next, they had to carry out a structured task, *gathering and organizing* their data. Then students answered questions solely based on their data, after which [they] compared their experimental results to their initial guesses. Misconceptions were *explicitly confronted* with experimental evidence. Finally, [they] *built a theoretical probability model* that might account for the experimental data they had collected. Students then compared all three pieces of information: their initial guesses, the experimental empirical results, and the results predicted by the model. (pp. 481-482)

Many of the details of this dissertation study have not been reported in subsequent journal articles so caution must be exercised in interpreting the results. Nevertheless, the conclusions from this study are consistent with more recent efforts to influence students' prior intuitions and misconceptions about probability. Konold (1989b) used a computer modeling intervention and Garfield and delMas (1989) used a computer tutorial program in their attempts to influence misconceptions. Although there was some success, large numbers of students persisted in their misconceptions. In another study, delMas and Bart (1987) provide support for Shaughnessy's approach of confronting misconceptions. As students were forced to record their predictions and to compare their predictions with actual experimental results, they were more likely to employ the frequentist model they were being taught. Unfortunately, information available from these studies is too limited to present any further review.

Conclusions

The evidence addressing the question about the impact of instruction in developing proper conceptions of probability and overcoming misconceptions is obviously very limited. There is a definite need for more research exploring this question at all educational levels.

Though the studies have been limited, some patterns may tentatively be observed. First, students bring prior conceptions with them and these frequently involve misconceptions. These misconceptions prove to be very persistent and difficult to overcome, even with instruction.

The two instructional programs reviewed in this section have actively involved the students in conducting probability experiments and calculating empirical and theoretical probability, concluding verbal presentations alone are insufficient for developing understanding and overcoming misconceptions. The instructional format that achieved some measure of success placed students in a position of confronting their misconceptions and inadequate preconceptions. Interestingly enough, this approach incorporated each of the various interpretations of probability. Students confront the subjective notions of probability they possess by comparing them to empirical or frequentist results with the ultimate goal of building formal and normative probability models.

Conclusions and Recommendations

The vision of mathematics teaching and learning outlined in the NCTM Standards focuses on opportunities for students to deepen their understanding of the concepts and nature of mathematics and to develop skills in problem solving, reasoning, and communicating about mathematics. Two important factors influencing students' learning opportunities are the instructional tasks teachers choose and the classroom discourse they orchestrate. The teacher knowledge research reviewed in this chapter suggests these aspects of teachers' decisions and actions are impacted by the many facets of teachers' subject matter and pedagogical content knowledge.

First, teachers' subject matter knowledge, including both their knowledge of the content and an understanding of the nature of mathematics, influences what they do in their classrooms. Content knowledge is, of course, essential to teaching, for no one can teach that which he or she does not know. But, more importantly, those teachers who themselves have only a procedural knowledge of the content can do no more than teach procedures to their students. On the other hand, the research suggests teachers who have connected, explicit, and conceptual knowledge of mathematics are more likely to choose activities and orchestrate discourse aimed at developing the students' conceptual understanding and problem-solving ability.

Understanding of the nature of mathematics was also seen to impact instructional tasks and classroom discourse. Lampert provides a particularly clear example of this impact. Because of the depth and flexibility of her knowledge, she is able to create a classroom environment where students experience the tasks and discourse of mathematics in a relatively authentic manner. The students have an opportunity to experience mathematics as a system of human thought where they can make sense of what is being explored, they can participate in "doing" mathematics, and they can establish mathematical truth by their own reasoning. The depth of her knowledge allows her to choose representations and tasks that help the students grasp the meaning of the concepts. The flexibility of her knowledge permits her to manage the direction, balance, and rhythm of the classroom discourse in such a way as to develop student understanding. In contrast, teachers with a limited view of the nature of mathematics generally portray it as a collection of unrelated facts and procedures and emphasize tasks and discourse aimed at practice and memorization of these facts and procedures.

The teacher knowledge research programs also explored a form of knowledge that is more than the knowledge of the content or nature of mathematics. By its very nature, pedagogical content knowledge is a blending of the processes of teaching and of learning. A review of the definitions suggested by the researchers reveals at least two common threads. First, from the teaching perspective, pedagogical content knowledge includes representations of the subject matter and instructional strategies for presenting the subject matter in ways accessible to the learner. Second, from the learning perspective, pedagogical content knowledge includes knowing about the background conceptions and misconceptions of the learners and the learning process through which learners may pass in obtaining understanding of the content. Thus, pedagogical content knowledge represents the bridge between teacher and learner.

The expert teachers in Leinhardt's studies and the more knowledgeable teacher in Shulman's study provide examples of teachers whose knowledge of representations and instructional strategies enrichs the learning experience. These teachers had a richer repertoire of representations and explanations from which to choose as they designed instructional tasks. In addition, their use of logical explanations and multiple representations enabled them to connect new material to the prior knowledge of students. The impact of this aspect of pedagogical content knowledge is evident as well in Lampert's choice of tasks and activities centered on representations that are meaningful to the students yet embodying the essence of the mathematical concepts. On the other hand, the teachers who lacked a rich repertoire of representations and strategies presented a limited form of knowledge to the students as a result of their impoverished or inappropriate examples and/or representations.

In building the bridge over which the student gains access to the content, an important aspect of pedagogical content knowledge is an understanding of the potential conceptions and misconceptions students may have and how understanding can be developed from this beginning point. The teachers in Carpenter's study who became sensitive to students' thinking were more likely to design tasks and discourse with developing students' understanding in mind. They more often posed problems and listened to the processes the students used to solve the problems, as a means of learning more about the students' thinking. They encouraged the sharing of many different approaches to the problems aimed at addressing the different stages in the developmental process where the students might be. In contrast, teachers without this aspect of pedagogical content knowledge were hampered in their attempts to anticipate student difficulties and to present instruction to overcome these difficulties.

Another part of the NCTM vision for reform in mathematics education is an increased emphasis on content such as probability. However, because little systematic probability instruction has occurred at the K-12 level until recently, research studying the teaching and learning of probability has been quite limited, particularly in North America. The research reviewed in this chapter begins to paint a limited picture of how understanding develops, of what misconceptions and difficulties students and teachers face, and of what impact instruction has in developing understanding and overcoming misconceptions.

As a result of experiences with games of chance and encounters with uncertain circumstances in their everyday lives, Fischbein proposes students have intuitive notions of probability prior to instruction. Some of these intuitions facilitate learning and the acquisition of the more formal concepts of probability Piaget associates with the development of cognitive structures. Other of these intuitions can inhibit the development of a normative understanding of probability, resulting in some of the inadequate knowledge demonstrated on Green's survey of nearly 3,000 students aged 11 to 16.

The research of cognitive psychologists, particularly Kahneman and Tversky, and mathematics educators, namely Shaughnessy, has demonstrated at least some of these intuitive ideas are misconceptions in conflict with the assumptions and results of accepted principles of probability. Various examples have shown how reliance on such judgmental heuristics as representativeness and availability can lead to systematic biases in probabilistic reasoning. Other difficulties of a more mathematical nature are encountered with the language of probability and such concepts as independent events, conditional probabilities, and conjunctions.

The research exploring the impact of probability instruction has been especially limited. What research there has been seems to suggest these misconceptions and difficulties are difficult to overcome, even with systematic instruction. In particular, the misconceptions are not easily changed or replaced with normative mathematical models. Shaughnessy (1977) had some limited success with an instructional model that placed students in a position of having to reconcile the dissonance between their misconceptions and their observations. This model of instruction is consistent with the results of conceptual change research in science learning which has focused on the process of accommodation, or the change that occurs when old cognitive structures and beliefs must be replaced or reorganized because they are unable to incorporate new knowledge. Posner, Strike, Hewson, and Gertzog (1982) suggest accommodation is not likely to occur until the learner is no longer satisfied with existing conceptions, has a preliminary understanding of a new conception, recognizes the new conception is plausible, and sees the potential of the new conception to deal with future problems.

Therefore, as teachers face the challenge of teaching a content that may be new to them in meaningful ways and as mathematics educators face the task of preparing teachers to meet the challenge, there are many unanswered questions. The teacher knowledge research has explored the knowledge teachers have about a wide spectrum of mathematical content, but probability has not been one of those topics. What do teachers know about the content and nature of probability? What pedagogical content knowledge do they have about teaching probability? In particular, to design instruction aimed at overcoming misconceptions and developing a proper understanding, teachers may benefit from knowledge of possible student misconceptions and difficulties and from knowledge of effective instructional representations and strategies to employ in their classrooms. It is not known, however, to what extent teachers have knowledge about how understanding of probability is developed or what possible difficulties students may encounter. Finally, a more complete understanding is needed about what teachers do as they teach probability—the instructional tasks they choose and the classroom discourse they orchestrate—and how their instruction is impacted by or related to the knowledge they possess.

CHAPTER III
DESIGN AND METHOD

Overview

This investigation of middle school mathematics teachers' knowledge of probability and their instructional practice while teaching probability utilized a case study research design. This design was selected because it is particularly well suited for addressing those aspects of educational phenomenon where understanding is sought for the purpose of improving practice and extending the knowledge base of the field (Merriam, 1988). Because teachers' knowledge of probability as well as their instructional practice while teaching probability is largely unknown, the exploratory nature of the case study design makes it an appropriate approach for answering the questions of this study:

1. What general pedagogical knowledge do middle school teachers demonstrate in the context of teaching probability?
2. What is the teachers' subject matter knowledge of probability?
3. What is the teachers' pedagogical content knowledge concerning the teaching of probability?
    a. What instructional tasks do the teachers use as they teach probability?
    b. What is the nature of classroom discourse during probability instruction?
    c. What is the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability?

To gain a richer description of teacher knowledge and a deeper understanding of any possible relationship between teacher knowledge and teacher classroom practice, multiple cases were explored. Using purposeful sampling techniques, teachers were selected to provide a range of similar and contrasting cases to investigate. The selection of the teachers for the final sample was based on the teachers' subject matter knowledge, their educational background, the number of years they had been teaching, and the nature of what they planned to do for their probability instruction.

Because of the exploratory nature of the study, qualitative data collection and analysis procedures were utilized. Data sources included teacher interviews, classroom observations, researcher journals, and documents such as class handouts, homework assignments, projects, and tests. Data collection was divided into three phases. In Phase I, an interview assessed the teachers' subject matter knowledge of probability and inquired about the teachers' plans for probability instruction. From this assessment of teacher knowledge and information about planned activities, teachers for the remainder of the study were selected. In Phase II, the selected teachers were observed as they taught probability lessons.

Additional interviews in Phase III further explored teacher knowledge and its relationship to teaching practice.

Data analysis was an ongoing process. As the data were being gathered, emerging themes and possible categories were noted. Once the data for all the teachers had been collected and organized, they were coded using categories that emerged from the data. The final analysis proceeded in two stages. Initially, individual case studies were written. Then, the researcher did a cross-case analysis comparing and contrasting the range of cases. Triangulation between various data sources was conducted to confirm emerging patterns, themes, and conclusions.

This chapter will first discuss the participants of the study, including both the teachers and the researcher. The next two sections will describe the procedures that were used for collecting and analyzing the data, respectively. These sections will be divided into the phases or stages that occurred in the investigation. Finally, the chapter will conclude by looking at the nature of the contributions potentially resulting from the study.

The Participants

The Teachers

Teachers in today's middle school classrooms have a variety of educational backgrounds relating to probability, ranging from none at all to extensive course work and learning experiences. Because probability has generally not been included in the K-12 mathematics curriculum until recently, many teachers may not have encountered probability as a part of their own elementary, middle, or high school experience. Their encounters with probability as part of their teacher education programs may also have varied from none to a course dedicated to the topic. Prospective elementary teachers currently being certified in the state complete a year-long sequence of mathematics education courses, of which 2 or 3 weeks are typically spent on probability. Other prospective teachers completing an emphasis in mathematics or completing special middle school programs often take at least one course in statistics and probability as part of their program. Further, several inservice or staff development opportunities are available for teachers who want to increase their knowledge of probability. Teachers may also gain knowledge of probability as they study a variety of resources while planning to teach this newly required subject. Although taking classes and other such learning experiences may not necessarily translate into knowledge or understanding, the important point is the teachers have had opportunities to learn the concepts and processes at the heart of probability. Because teachers' educational experiences with probability have been varied, it is anticipated their knowledge of probability may also be varied.

Rather than random sampling, the subjects for this study were selected using a purposeful sampling method. According to Bogdan and Biklen (1992), this "research procedure ensures that a variety of types of subjects are included" (p. 71). Although the method neither tells how many nor in what proportion the types appear in the population, exploring the variety of particular subjects chosen is justifiable in situations where the purpose of the research is to gain insight and understanding about the aspects of educational phenomenon under investigation and to discover the implications of and relationships linking those aspects.

Because one of the purposes of the investigation was to explore how probability is being taught, only teachers who would be teaching probability were considered as possible participants. In addition, because researchers (Berliner, 1986; Feiman-Nemser, 1983) have suggested it takes a minimum of 3 years for beginning teachers to master the mechanics of teaching and classroom management and to begin focusing on whether students are learning the content being taught, only teachers with at least 3 years of teaching experience were considered for this study. Further, the sample was one of convenience chosen from within a geographical region that would be easily accessible to the researcher.

In an effort to obtain a final sample of middle school teachers who had a varied background knowledge of probability, a larger group of approximately 10 teachers was sought for the first phase of the study in which the teachers' knowledge of probability would be assessed. Participants in the initial sample were selected to include those who had different educational backgrounds relating to probability and different levels of teaching experience.

Potential participants for the study were selected from four sources: (a) teachers recommended to or known by the researcher from middle schools in nearby communities; (b) teachers who had graduated from a nearby private college and who were known to be teaching middle school in the surrounding geographical area; (c) teachers who had completed the middle school mathematics program at a nearby state university and who were teaching in the local area; and (d) teachers who had taken a probability class for middle school teachers at yet another nearby state university and who taught within an accessible distance. Although not part of the selection criteria, each of these categories provided one of the four middle school teachers presented in this research study.

Initial contacts with teachers were made during winter and spring 1995. At that time the teachers were given a brief explanation of the purpose of the study and were asked if they would be willing to participate in the study. Teachers were told they would be interviewed and observed as part of a study exploring the teaching of probability at the middle school level. Because probability is a relatively new area within the curriculum, it was hoped teachers would view participation in this study as an opportunity to share what they were doing and, in the process, any concern about critical evaluation would be alleviated.

In addition to explaining the general intent of the study, the teachers were given a letter describing the types of data to be collected and the time commitments involved in being part of the study. The letter also assured the teachers that all data collected as part of the study would remain anonymous. A written response form allowed the teachers to confirm their willingness to participate in the study. All requirements for research involving human subjects and all district guidelines were followed in contacting teachers and in dealing with them during their participation in the study.

A sample of 10 teachers who were willing to participate in the study was obtained from the sources described previously. However, before the pre-observation interview was conducted, three of the teachers withdrew. These teachers had decided to use their probability units for professional evaluation purposes and did not want the added pressure of being involved in the study.

The pre-observation interview, the contents of which will be described later, was conducted with the remaining seven teachers. The researcher had anticipated selecting the teachers for the in-depth part of the study on the basis of their knowledge of probability. However, the data from the pre-observation interviews did not reveal a great amount of variation in the teachers' probability knowledge. As will be discussed in a later section, this lack of variation in probability knowledge did not seem to be the result of a deficiency in the interview instrument, but rather seemed to reflect the nature of middle school teachers' probability knowledge. Because there did appear to be significant variation in other characteristics of interest, the researcher decided to select a sample from the seven remaining teachers without seeking to enlarge the original sample further. Five of the seven teachers were selected to participate in the second and third phases of the investigation.

The selection of this smaller sample was based on a number of factors. First, the teachers were selected to incorporate what variation in probability knowledge the interviews had revealed. Second, the nature of the teachers' educational background was considered. In particular, one teacher was selected because he had completed a secondary mathematics education program, in contrast to the other teachers who had completed elementary or middle school teacher education programs. The other four teachers were equally divided between elementary and middle school programs; two had completed elementary education programs, two had completed programs specifically designed to prepare middle school mathematics teachers. Further, the sample was selected to include teachers who had a variety of learning experiences related to probability and/or the teaching of probability. Some of the teachers were primarily self-taught while others had taken some course work in probability.

A third factor used in selecting the final sample was the number of years the teachers had been teaching. The years of experience for the seven teachers in the original sample ranged from 3 years to 27 years. Although barely meeting the minimum experience standard set for this study, the teacher with 3 years experience was included in the study because he was also the teacher with a secondary mathematics education background. The other teachers selected as part of the smaller sample had 6, 14, 22, and 27 years of teaching experience.

Fourth, the nature of the probability instruction the teachers expected to give was also considered in making the selection of the final sample. For one teacher, playing games and analyzing them was going to be the focus of his probability unit. Another teacher expected to use simulation activities from the Math and the Mind's Eye materials, *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993). A third teacher planned on using a pattern of prediction, experimentation, and analysis in her class' exploration of probability activities. A fourth teacher intended to emphasize experimental and theoretical probability as he had his class do activities from the Middle Grades Mathematics Project materials, *Probability* (Phillips, Lappan, Winter, & Fitzgerald, 1986), and from other materials. The fifth teacher intended to implement the Project PASS (Promoting Achievement and Student Skills) materials he had obtained by a grant. These materials, developed jointly by GTE Corporation and the National Football League, included a section on probability and statistics and involved extensive use of video and computer technology. In the end, only 2 of the 19 lessons taught by the fifth teacher dealt with probability; the rest of the lessons dealt with statistics. In addition, the researcher was allowed to observe only one of these lessons because a substitute teacher was presenting the other. Therefore, because of lack of data dealing with probability, the data for the fifth teacher were not considered in the analysis of this research study.

In addition to reflecting a variety of educational backgrounds, years of teaching experience, and nature of anticipated probability instruction, the four middle school teachers who are the focus of this research study also reflect a variety in gender (two females and two males). The selection also was affected by scheduling logistics, including teacher location and anticipated timing of their probability units. The selected sample permitted the researcher to maximize opportunities to actually observe as many lessons as possible.

A summary of the background characteristics of each teacher and his or her school is presented on the following page. Pseudonyms are used to assure the anonymity of the study participants.

The Researcher

Because the data were collected and analyzed from the perspective of the researcher, it is valuable to explain what that perspective is. The researcher has entered the field of mathematics education and teacher education by a somewhat unconventional route. The researcher has a bachelor's degree in mathematics from a state university in the western United States and a master's degree in mathematics from another state university, also in the western United States. In the course of her master's program, the researcher discovered an interest in teaching. For 5 years following the completion of her master's degree, she remained at the university as an instructor in the mathematics department, responsible primarily for teaching precalculus mathematics classes. Then, with no education course work except a mathematics methods course and a field experience in a local high school, the researcher accepted a high school teaching position at an American school located in West Africa. During the next 3 years, she taught General Math, Algebra I, Geometry, Algebra II, and Trigonometry/Calculus.

Summary of Background Characteristics of the Teachers and Their Schools

	Mr. Trackman	Mrs. Books	Mrs. Talent	Mr. English
Years of teaching experience	3 years	6 years	14 years	27 years
College major(s) and/or program	Mathematics education	Elementary education/ Middle school mathematics program	Double major: Elementary education & mathematics	Language arts education/ Elementary education program
Teaching license	Secondary math	Elementary	Elementary	Elementary
Math endorsement	Advanced math	Basic math	Basic math	Basic math
Probability background	Upper division probability course in college	Probability & Statistics course in middle school mathematics program	Self-taught until taking recent course focused on teaching probability	Self-taught except for portion of Middle Grades Math Project course
Grade level	6th	6th	7th	7th
Class observed	General Math	Self-contained gifted class	Advanced Math	Pre-Algebra
Class Size	22 students	24 students	28 students	34 students
School location	Inner city	Inner city	Rural	Suburban
Student population (grades)	900 students (6-8)	700 students (6-8)	350 students (7-8)	425 students (6-8)
Ethnic background of students	50% Caucasian, 30% Asian, 10% African American, remainder Hispanic & Russian	75% Caucasian, remainder Asian, Hispanic, & African American	Most Caucasian, some Hispanic & Russian	95% Caucasian, remainder Hispanic, Asian, or African American

Upon returning to the United States, the researcher enrolled in a doctoral program in mathematics education with the goal of gaining a stronger foundation to support her teaching. At the same time, she also satisfied the requirements for teaching certification in that state. In the course of her doctoral program, the researcher began working with two mathematics faculty members involved in the preparation of preservice elementary teachers. This association led to an opportunity to collaborate with the professors on the preparation of a textbook for prospective elementary teachers and an accompanying student resource handbook. From this experience developed a growing interest in the subject matter knowledge, particularly the conceptual knowledge, necessary to be an effective elementary or middle school mathematics teacher.

While working on her doctoral program, the researcher spent 7 years as an assistant professor in mathematics education at a small liberal arts college in the western United States. Her responsibilities included teaching content and methods courses for preservice elementary and secondary teachers as well as supervising their student teaching experiences in the classroom. As a result of her supervision experiences, the researcher developed skills in observation and evaluation of classroom instruction.

The researcher does not remember when she first encountered probability as a student. She took a statistics course as an undergraduate and may have been introduced to probability at that point. She had her first experience teaching probability as an instructor at the university when she taught a discrete mathematics course designed for biology, business, and social science majors. More recently she has had extensive experience teaching the probability component of the Mathematics for Elementary Teachers sequence.

To address possible researcher biases introduced in the process of collecting and analyzing data, the researcher answered the interview questions herself and these results are reported as part of the study. Other efforts to deal with researcher biases and questions of validity will be discussed in the descriptions of the data collection and data analysis processes.

Data Collection

The data collection process was divided into three phases. During the first phase, the original sample of teachers was interviewed to assess their subject matter knowledge of probability. Following the selection of the teachers whose cases would be explored, these teachers were observed teaching probability during the second phase of the study. The third phase involved follow-up interviews with the teachers who had been observed. The details of these data collection phases will now be discussed, in the order in which they occurred in the investigation.

Phase I: Pre-Observation Interview

The initial data collection phase of the study was a semi-structured interview with each of the participating teachers. These interviews, approximately 1 hour in length, were audiotaped and transcribed. The focus of the interview was to explore the teachers' personal teaching background and to develop a profile of the teachers' knowledge of probability.

The initial part of the interview was designed partly to establish rapport with the teachers, beginning with questions that would help the participants feel at ease and comfortable with the interview. The teachers were then asked to give their mathematical autobiography, describing their elementary, middle, and secondary school experience as well as their experience in college. The interview next focused on the teachers' particular educational experiences with probability. Finally, the teachers were asked more specifically about the probability lessons or unit they would be teaching. (See Appendix A: Background Questions for a listing of the corresponding interview questions.)

The second part of the pre-observation interview was designed to assess the teachers' subject matter knowledge of probability. Questions were designed to probe the limits of their knowledge. However, so as not to sensitize the teachers to the nature of the study and possibly cause them to change their teaching practice, no questions relating probability knowledge to teaching were asked. Instead, the interview involved content questions set in the context of problems. A table of specifications was used to insure key objectives were covered, including those recommended by the *Curriculum Standards* (NCTM, 1989) for grades 5-8. To establish content validity, the interview questions were reviewed by five mathematics educators. Questions approved by at least 80% of the reviewers were accepted; other questions were modified or replaced as necessary. In response to reviewers' comments, several easier questions were added at the beginning of the interview and the order of the questions was rearranged. (See Appendix A: Probability Questions for a listing of the questions used in this portion of the interview.)

The interview procedures and questions were pilot tested with four teachers who were not part of the study. These pilot interviews were conducted for three purposes. The first purpose was to determine the clarity of the questions and procedures. The interview questions were read to the participants. In addition, the questions were displayed on separate sheets to which the teachers could refer. The written copy proved very beneficial and some adjustments were made in how questions were displayed in response to the experience in the pilot interviews.

The second purpose of the pilot interviews was to determine how long the interview would last with the number of questions proposed. Although there was considerable variation in time required to complete the interview, the results of the sample interviews confirmed the interview could reasonably be conducted in about 1 hour.

The third, and perhaps most critical, purpose of the pilot interviews was to determine the potential of the probability questions to reflect differences in teachers' probability knowledge. In the process, the sample interviews also provided an opportunity for the researcher to discover and anticipate possible answers to the probability items. To accomplish the purpose of determining the questions' potential to reflect knowledge differences, teachers with diverse educational backgrounds were sought as subjects for the pilot interviews. Because of the critical nature of this issue, the backgrounds and general responses of these teachers will briefly be discussed.

Although none of the four teachers involved in the pilot interviews were currently teaching middle school mathematics, the teachers had educational backgrounds comparable to the teachers involved in the study. In particular, the teachers for the pilot interviews were licensed to teach middle school mathematics.

The first teacher participating in the pilot interview was at the time teaching second grade after 3 years teaching fifth grade. She had completed an elementary education program and taken additional classes to earn a mathematics endorsement. Although she had been introduced to probability as part of two classes she had taken, the interview revealed she could recall very little of the probability she had studied. For example, in responding to the Two Coins problem (probability question #2), she could correctly identify the probability of tossing two heads as 1/4, the probability of one head and one tail as 1/2, and the probability of two tails as 1/4, but she could provide no explanation for how those theoretical results were obtained. She had similar difficulties with many of the other questions.

The second pilot interview was conducted jointly with a husband and wife who had both previously been middle school mathematics teachers. After teaching for 1 year the husband had returned to school to earn an engineering degree. After 5 years of teaching, including 2 years teaching middle school mathematics, the wife had left the classroom to be at home with her preschool children. It had been 2 years since either had taught middle school mathematics. The two were able to provide correct answers to nearly all of the probability questions. In addition to explaining their solutions, they added comments related to teaching the various aspects of probability.

The participant for the third pilot interview had gone on to complete a master's degree in mathematics after completing a double major in mathematics and mathematics education as an undergraduate. He held a secondary teaching license (grades 5-12) for advanced mathematics. At the time of the interview he was teaching two classes at a local community college and substitute teaching in the public schools. This subject answered all probability questions correctly and provided clear, succinct explanations in support of his answers. He used the vocabulary of probability accurately and extensively (e.g., identifying the outcomes in the One Die problem [probability question #1] as *equally likely*). His approach to many of the problems was more formal. For example, he used the formula $P(A|B) = P(A \cap B)/P(B)$ to find the conditional probability in the Cancer problem (probability question #10). In addition, on the Weather problem (probability question #9), he went beyond the scope of what the researcher had anticipated, suggesting one could use a table of binomial probabilities to evaluate the weather forecaster's prediction, if such a table were available.

A number of conclusions were evident from the responses given by the teachers in the pilot interviews. First, the initial questions had a degree of difficulty which could be addressed by teachers with a minimum of probability knowledge. For most teachers, the simpler beginning questions provided an opportunity for the teachers to overcome any initial uneasiness they might have had about this portion of the interview and to build some confidence in their knowledge. Second, the questions could be approached using a variety of strategies, from very basic to more advanced and formal. Third, the questions provided a number of opportunities to explore the intuitive insights of the teachers, particularly when the questions went beyond what they knew how to solve. Finally the pilot testing provided evidence that variations in teachers' probability knowledge could be detected by the interview questions.

Following the pilot testing of the interview questions, the pre-observation interview was conducted with the teachers participating in the study. When the interviews were administered, the teachers were asked to think aloud as they completed the probability problems. As needed, the researcher asked questions to clarify the meaning of their answers or probe their understanding. Any written work done by the teachers during the interview was also collected. Criteria used in evaluating the teachers' responses, in assessing teacher knowledge, and in selecting the teachers to be observed included the following: (a) the correctness, explicitness, and connectedness of their understanding (Ball, 1988a); and (b) whether their knowledge was procedural or conceptual. Using the information collected from the interviews and the criteria described earlier, the sample of teachers for the in-depth part of the study was selected.

Phase II: Classroom Observations

Data Collection Procedures

The informants selected for in-depth analysis were observed as they taught probability. The researcher took a holistic approach to the classroom observations, considering various aspects of pedagogy in general as well as focusing on the content being taught. Instructional tasks and classroom discourse were among the phenomena explored. In addition to being an integral part of teaching, these aspects receive special attention as part of the *Teaching Standards* (NCTM, 1991).

Some variation existed in the structure of probability instruction planned by the teachers. Three of the teachers provided a clearly defined and distinct unit on probability. For these teachers, the researcher observed as many of the probability lessons as possible and videotaped those she could not observe. Of the 10 days in Mr. Trackman's probability unit, 7 days were observed and videotaped, 1 day was videotaped only, and 2 days were neither observed nor recorded because a substitute teacher was teaching. Of the 15 days Mrs. Talent taught probability, 12 were observed and videotaped while 3 were videotaped only. Of the 17 days in Mr. English's probability unit, the first 12 days were observed and videotaped. He then asked that the remainder of the unit not be observed or videotaped because the pressure of being observed was impacting his other responsibilities. In this case and in the others, the researcher felt the data gathered provided an adequately complete picture of the teachers' probability instruction.

The fourth teacher, Mrs. Books, integrated and interspersed her probability lessons with instruction of other mathematical topics throughout the school year. Unfortunately, two probability activities had been done earlier in the year, before this study had begun, and therefore were neither observed nor recorded. Of the seven lessons taught during the time frame of the study, the researcher observed five lessons. A sixth lesson, a 15-minute discussion, was audiotaped. The seventh day was neither observed nor recorded because the students were involved in finishing a final writing assignment. As with the other teachers, the researcher believed sufficient data to represent the teacher's probability instruction were collected.

In addition to observing probability instruction, the researcher observed and videotaped at least two lessons in each classroom before probability instruction began. This was done for two purposes. First, it was hoped this process would desensitize the class and the teacher to the presence of the researcher and allow the class to proceed as normally as possible. Second, these observations provided a picture of the teacher's general instructional practices.

To be as unobtrusive as possible, the researcher arrived and set up the equipment before the observation period began. When possible, the researcher briefly and informally interviewed the teacher about what he or she was going to do in the lesson. During the lesson, the researcher took a seat near the side or back of the classroom and remained through the end of the class time. The researcher did not participate in classroom activities, nor was she available to answer questions. If time permitted following the lesson, the researcher conducted an informal post-observation interview with the teacher. Otherwise, arrangements were made with the teacher to discuss the lesson at a later time.

Data Collection Instruments

Data collected during the classroom observations included videotapes of the lesson activities, audiotapes of teacher-student discourse, field notes, and supporting documents. These were supplemented by notes from informal interviews with the teachers and a researcher journal.

Videotapes of the lesson activities. A video camera was used to record the lesson activities and classroom discourse. The camera was set up in a position to be as unobtrusive as possible but where it could record the best view of the class activities. The videotapes were reviewed and used together with data from the audiotapes and field notes to provide a record of the class period.

Audiotapes of teacher-student discourse. During the observed lessons the teacher wore a remote microphone to record verbal interactions between the teacher and students. The information was picked up by a receiver attached to a tape recorder and recorded. The audiotapes were transcribed to provide the text of the classroom dialogue. Notes were added to the transcripts from observations of the videotapes and field notes.

Field notes. The researcher recorded field notes during the observations to supplement the video and audio recordings. The descriptive part of the field notes included (a) descriptions of the physical setting and verbal portraits of the people involved, (b) an outline of the instructional activities used, and (c) a record of the nature and substance of interactions between the teacher and students or between students. These notes also included the teacher's explanations of topics and assignments, their use of representations, their references to student learning, and their responses to student questions and comments. The descriptive part of the field notes and the transcripts of the audiotapes were merged and supplemented by data from the videotapes to provide the most extensive narrative record of the class period possible.

The field notes were also of an introspective nature as the researcher began to reflect on what was happening in the classroom. These reflections took the form of labeled "observer's comments" recording the researcher's impressions, feelings, reactions, hunches, and initial interpretations.

Researcher journal. The reflective part of the field notes was a starting point for the researcher journal containing the personal reflections of the researcher throughout the study. In addition to the comments recorded during the class time, the researcher also took time afterward to reflect on the day's observations. These reflections included such things as jotting down additional information, recording general impressions, or speculating on emerging categories or patterns. As suggested by Bogdan and Biklen (1992), the journal also included reflections on analysis, method, ethical dilemmas and conflicts, and the observer's frame of mind as well as points of clarification.

The researcher journal was also used as a tool for protecting against researcher biases. In the process of reflecting on the developing research, the researcher sought to clarify her assumptions and theoretical orientation and to consider whether these were influencing her interpretation of the data.

Supporting documents. Documents collected as part of the observation sessions were coded within the sequence of the lesson and cross filed with other documents collected as part of the study. These documents included copies of the lesson plans, overhead transparencies, corresponding textbook pages, class handouts, homework assignments, projects, tests, and other supplementary materials.

Informal interviews. Where possible, the researcher informally interacted with the teacher before and after the classroom observations. If time was not available immediately following the lesson, other arrangements were made or the teacher was contacted by phone. Before the observations, the teacher was briefly asked, "What are you going to do today?" As a follow-up to the observations, the teachers were asked for their reaction to the lesson. Additional discussion included such questions as goals and objectives for the lesson and whether they thought these goals were reached, reasons for choosing the activities they used, and any discoveries they made during the lessons. All interaction surrounding classroom observations was documented as a separate data source for later analysis.

Phase III: Post-Observation Interview

Following the observation of the series of lessons, a semi-structured interview of approximately 1 hour in length was conducted with each teacher. This interview took place within 2 weeks of the completed observations. These interviews were audiotaped and transcribed.

The purpose of this interview was to explore the teachers' pedagogical content knowledge related to teaching probability. The classroom observations and informal interviews with the teachers had already provided preliminary information about their pedagogical content knowledge. Some of the questions in this post-observation interview followed up and built upon the data previously gathered for each teacher. In that respect, this interview varied somewhat in its focus from teacher to teacher. Some of the questions related specifically to what the individual teacher had done; other questions sought more general information. In particular, this interview sought to obtain a broad picture of the teachers' pedagogical content knowledge, according to the structure proposed by Marks (1990b). This structure included the following categories: instructional aspects of the subject matter, students' understanding of the subject matter, texts and materials for teaching the subject matter, and instructional processes for presenting the subject matter. Of special interest was the teachers' knowledge of students' thinking, including possible conceptions and misconceptions and whether these considerations were involved in planning instruction. (See Appendix B: Pedagogical Questions and Misconception Questions for a list of questions used in the post-observation interview.)

Data Analysis

By its very nature, analysis of the qualitative data was an ongoing and iterative process. The analysis process, in fact, was begun during the process of collecting the data. As the data were gathered for each teacher, the materials were organized chronologically and coded according to category (e.g., interview, observation, notes). Initial analysis of the data was done first as observer's comments and then as entries in the researcher journal. These entries were used to record insights about the data and to speculate about patterns observed.

After all of the data had been collected, the next step of the analysis was the preparation of the case record for each teacher, with the goal of bringing the data together in an organized fashion. Transcripts of the audiotapes were made, checked, and corrected. The videotapes were then reviewed to verify and/or supplement the transcripts from the audiotapes. Narrative comments from a review of the field notes and videotapes were added to the transcripts to complete the record of what had occurred in the classroom (e.g., indications of whether the teacher had been talking to the whole class or to a small group). Finally, information that had been written on the overhead was inserted at the corresponding points of the transcripts. The final transcripts thus provided a synthesis of the audiotapes, videotapes, and field notes. These transcripts of the probability lessons combined with the interview transcripts and supporting documents constituted the case record for each teacher.

After the case record was completed, the next step of the analysis was a thorough reading of the data in order to generate preliminary coding categories. These preliminary categories were altered and refined where necessary as the analysis of the data continued. In some cases, categories relating to the research questions or originating from research literature were initially considered, but categories arising from the data proved to be most relevant and best fitted to the data.

Throughout the period of data analysis, the researcher utilized various techniques in the process of making sense of the data. Coding the data and clustering by conceptual grouping helped the researcher see connections and discover emerging patterns and themes. The practice of writing conceptual memos and summarizing emerging issues served both as a strategy to reduce the large amount of data and as a stimulus for further analysis. Displays of the data permitted a systematic comparison across data sets. These techniques were all part of the analysis process which Huberman and Miles (1994) point out "is sequential and interactive. Displayed data and the emerging written text of the researcher's conclusions influence each other. The display helps the writer see patterns; the first text makes sense of the display and suggests new analytic moves in the displayed data; a revised and extended display points to new relationships and explanations, leading to more differentiated and integrated text, and so on" (p. 433).

Various measures were undertaken during the process of data analysis to insure validity of the findings. As mentioned earlier, the researcher journal was used as a tool for clarifying the researcher's assumptions and theoretical orientation and, thereby, protecting against researcher biases. The ongoing speculation about emerging patterns was used as the basis for questions during informal interviews with teachers, providing an opportunity to check validity with the subjects from whom the data were gathered. Triangulation provided another protection as multiple sources of data were used to confirm the emerging findings. Where conflict arose between data sources, preference was given to what had been done rather than what had been said. As a further protection, the researcher provided an audit trail, describing how the data were gathered, how categories were determined, and how the findings were derived from the data. Finally, by including the most important remarks in the teacher's own words, the researcher has provided the reader the opportunity to determine the validity of the interpretation or to make alternative interpretations based on the actual data. The subsequent analysis of the data occurred in two stages which are described next.

Individual Case Studies

In order to remain true to the intent of case study research, the researcher analyzed the entire case record for each individual at one time. These data included transcripts from the two knowledge interviews, narrative accounts or transcripts of each lesson taught, and the supporting documents corresponding to the particular teacher.

After all of the data had been coded for a particular teacher, a first draft of the case study was written. One of the goals of the first draft was to tell the story of each of the probability lessons. These stories were intended to serve as a data reduction technique as well as an analytic tool. As the individual case studies developed, decisions about which lesson vignettes to include were based on how representative the lesson was of the teacher's instruction, whether the lessons reflected the variety of tasks and discourse included in the unit, and whether the activities had been used in other classrooms. Activities that had been used in multiple classrooms were included for the purpose of comparison between the teachers. Decisions about what to include in the case studies were also determined by how the information related to the questions being explored. Ultimately, the purpose of these case studies was to provide a rich portrait of the knowledge and practice of each teacher and to interpret the case with reference to the research questions of this study.

Cross-Case Analysis

The next stage of analysis, involving cross-case analysis, looked across the range of teachers for patterns and themes related to the teachers' knowledge and classroom practice. While working at this level of the analysis, the researcher referred to the original data as well as the individual case studies in an attempt to find confirming and disconfirming evidence. Triangulation between the various data sources was conducted to validate emerging patterns, themes, and conclusions. Where appropriate, tables and other displays of the data were created by the researcher to test the researcher's interpretations, to search for new or different patterns, and to stimulate further analysis and interpretation.

Conclusion

Through the case studies and cross-case analysis, this investigation has attempted to serve several purposes. The first purpose has been to define and to describe more carefully the nature of middle school teachers' subject matter knowledge and pedagogical content knowledge regarding probability and the teaching of probability. A second purpose has been to provide descriptions of the teaching of probability as a relatively new part of the middle school mathematics curriculum. The third and most central purpose has been to investigate the relationship between what a teacher knows about probability, including both their subject matter knowledge and pedagogical content knowledge, and what a teacher does when teaching probability.

The potential contributions resulting from this study, therefore, are both descriptive and interpretive. The rich descriptions of individual teacher's knowledge and classroom practice serve as examples, potentially enriching the pedagogical content knowledge and instructional practice of other middle school teachers. These descriptions also identify possible variables or hypotheses to guide future inquiry. The interpretive results include the cross-case exploration of the relationship between teacher knowledge and teacher practice with its implications for developing both teachers' subject matter knowledge and pedagogical content knowledge as well as their instructional practice. In the end, as the results of this study are shared with the mathematics education community, it is hoped they will add to the knowledge base of the field, inform teacher education, guide middle school teachers, and, ultimately, improve the teaching of probability in the middle school.

CHAPTER IV
RESULTS

The first objective of this chapter will be to present case studies of the four middle school mathematics teachers participating in this investigation. Then, the second objective of the chapter will be to address the three research questions:

1. What general pedagogical knowledge do middle school teachers demonstrate in the context of teaching probability?
2. What is the teachers' subject matter knowledge of probability?
3. What is the teachers' pedagogical content knowledge concerning the teaching of probability?
   a. What instructional tasks do the teachers use as they teach probability?
   b. What is the nature of classroom discourse during probability instruction?
   c. What is the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability?

To meet this objective, the sections of cross-case analysis will explore the teachers' general pedagogical knowledge, subject matter knowledge, and pedagogical content knowledge.

Case Studies: Portraits of the Teachers and Their Probability Instruction

The presentation of the data will begin with portraits of the four middle school teachers who were observed as they taught probability. These portraits will include information about the teachers' background, the school and classroom environment, the background of the probability unit, the probability unit itself, and the evaluation of the probability unit. The order in which the teachers will be presented is based on their years of teaching experience. The first teacher to be introduced will be Mr. Trackman who had been teaching for 3 years. Following him, Mrs. Books with 6 years of teaching experience and Mrs. Talent with 14 years of teaching experience will be presented. The case studies will conclude with Mr. English who had been teaching for 27 years.

In addition to the differences in their years of teaching experience, the teachers' knowledge and instructional practice had potentially been influenced by a number of different opportunities and factors. In particular, with a secondary mathematics education preparation, Mr. Trackman had a more extensive background in mathematics than the other teachers. In her case, Mrs. Books' teacher education program for middle school mathematics teachers had provided her with some unique opportunities to experience a different way of learning mathematics. In addition, Mrs. Talent had been exposed to the efforts for reform in mathematics education at the state and national levels. Finally, Mr. English's 27 years of teaching experience in itself potentially impacted his knowledge and instructional practice. How these opportunities and factors actually influenced the teachers will be considered in the investigation of the teachers' knowledge and instructional practice.

Mr. Trackman: Influence of Mathematical Background

With 3 years of teaching experience, Mr. Trackman is a novice in terms of teaching. However, he has the most extensive mathematical background of the four middle school teachers observed, having taken a mathematics education major in college and having earned a secondary teaching license in mathematics.

Teacher Background

School experience. Mr. Trackman had grown up and attended schools in the same inner-city neighborhood where he now taught middle school. He recalled "kind of hating mathematics" throughout elementary school, relating he was always bored because it was so easy. As he moved on to high school, he continued to find mathematics "a breeze," admitting he often did nothing until the period before the assignment was due. His experience with mathematics, however, changed when he entered college. Although he had taken calculus in high school, he confessed he was lost in his college calculus class, "just holding on by the skin of my teeth." And even though he had earned A's in high school mathematics, he received no grade higher than a C in his mathematics classes during his freshman year in college.

Teacher preparation. Mr. Trackman majored in mathematics education at a private liberal arts college in the Northwest. The requirements of this major included 45 semester hours of mathematics and led to an advanced mathematics teaching license for grades 5-12. The requirements and courses taken in the mathematics education major were largely the same as those for the standard mathematics major. Most of his grades in mathematics classes were C's and, in order to raise his grade point average to satisfy the requirements of the teacher education program, he repeated at least two of his mathematics classes.

Professional experience. Following graduation, Mr. Trackman accepted his current teaching position, where he had taught for the past 3 years. Mr. Trackman also served as an assistant track coach at his college alma mater where he had competed in the decathlon. Because of his coaching, he had not had time to attend workshops or conferences to extend his mathematics or teaching background, except for a "couple classroom management classes." Mr. Trackman further explained he had spent his summers "recovering from spending all year having no time to do anything." Now that he had received tenure, Mr. Trackman intended to "take next spring off as a study leave and maybe even a year and a half and just try and get my master's." He wanted to get his master's and figured "the easiest thing to get it in is probably education." Although he was not aware of what would be required, he indicated he would take some mathematics education classes if "there's math that I have to take."

Probability background. Mr. Trackman did not remember any specific study of probability in his own school experience, but he emphasized that "figuring numbers out has always been my life." Games and sports were the opportunities for learning probability Mr. Trackman could recall. He suggested that because of games, he had "always had fun with probability." Further, he indicated,

> I've always loved . . . baseball, and probability is throughout baseball. Numbers, statistics, I can't get enough of. I love baseball for that matter . . . to be able to look at a newspaper everyday and look at thousands of statistics. . . . And the statistics and probability, I have always been trying to figure out things. What is my chance of getting it this time? I always had a running percentage of what my free-throw percentage was in . . . during games. I was always trying to [get] that probability up. . . . But, working with probability, I'm always figuring out what the probability of the next turn is going to be.

In college, Mr. Trackman took an upper division calculus-based probability course covering the first half of a textbook written for a year-long study of mathematical statistics. The topics covered in the probability course included combinatorics and counting methods, the axiomatic foundation for probability space, continuous and discrete random variables, mathematical expectation, and the limit theorems. As part of the course, the students explored applications of the theory to simple games. Tree diagrams were introduced, although the diagrams were not emphasized in the text. Pascal's triangle was presented in the context of the binomial theorem. Mr. Trackman recalled "doing a lot of actuary stuff" as part of the course and suggested the tests they took were actuarial tests. While taking the course, he considered "maybe even becoming an actuary," but decided to finish the education major he had begun.

As part of his mathematics education major, Mr. Trackman was required to take one semester (4 credits) of the Mathematics for Elementary Teachers sequence. The second semester in the sequence included 3 or 4 weeks focused on probability. Instead of taking the second semester as most majors did, Mr. Trackman chose to take the first semester. As a result, he had not taken a college course that focused on probability from the perspective of teaching probability to middle school students.

School and Classroom Environment

Mr. Trackman taught at the largest middle school in a large metropolitan school district in the Northwest. This inner-city school had approximately 900 students in the sixth, seventh, and eighth grades. About 50% of the students were Caucasians, 30% Asian, 10% African Americans, and the remaining 10% of Hispanic or Russian descent.

Mr. Trackman taught sixth-grade general mathematics six periods a day. He admitted, "About fifth period it gets kind of . . . monotonous, and I have to regroup." Fortunately sixth period was his planning period so seventh period was not so bad. Mr. Trackman worked closely with the teacher who taught the other six sections of sixth-grade mathematics. Both teachers were in their mid- to late-20s and shared an interest in sports. His colleague had graduated from a state university with a major in physical education teaching, but also had earned a mathematics endorsement. In addition, this colleague had taken a probability course at a local state university based on the Math and the Mind's Eye probability materials (Shaughnessy & Arcidiacono, 1993).

Each of the classes contained a heterogeneous mixture of students. Prior to assigning students to the different periods, Mr. Trackman and his colleague received cards with the students' test scores and their rating by the fifth-grade teachers. Using this information, Mr. Trackman indicated "what we try and do is break them up male and female, and then we break them up high, medium, low, and then we try to split them up into different classes. We try to make it as heterogeneous as possible . . . sort of a jumble in all the classes." Two different sections were observed over the course of this study, each for a period of 4 days. Although fifth period generally had three or four students more than seventh period, the two class periods appeared to be equivalent otherwise. The class sizes ranged from 18 to 22, with a few more girls than boys. The classes observed included several students who were on the honor roll, some who were involved in the TAG program, and one who had been a Math Olympiad winner. The classes also included at least one student, perhaps more, who did not know their multiplication facts.

Mr. Trackman's classroom had previously been the teachers' lounge. As a result, it was carpeted and had its own sink and bathroom. Mr. Trackman liked having the carpet because it meant the noise "doesn't get as bad." Cabinets with counter-top space for storing student portfolios lined one side wall. Four full-length windows on the other side wall opened onto an interior courtyard. However, because the shades were frequently pulled down to allow students to read material on the overhead screen, lighting in the classroom was generally subdued. With the students' desks grouped two or three abreast and arranged in rows, the classroom could accommodate 26 students. The teacher's desk was in the left rear corner of the classroom.

Bulletin boards and posters on the walls was an expectation of the school, one considered during a teacher's evaluation. Several posters with a sports theme were hung around the classroom. Other posters presented the history, multi-ethnicity, and applications of mathematics. Also on display were student-drawn graphs and newspaper articles relating to mathematics. Classroom rules and consequences were posted on a front bulletin board next to the one panel of white board. Displayed on the white board each day was the scoring scale for the homework assignment, the number of school days left in the year, and a mathematical expression for the current date. For example, on one day, the date read "May $2 \times 3 \times 4$." A screen for the projection of material from the overhead pulled down over the front board.

Background of the Probability Unit

Setting educational goals. As a sixth-grade mathematics teacher, Mr. Trackman spoke of the following purposes for education: (a) getting students to have fun with mathematics, thereby overcoming the negative attitudes many of them had developed; (b) having students realize that mathematics "is out there everywhere"; and (c) teaching students how to learn. Mr. Trackman suggested probably two thirds of the students coming into his classes had negative attitudes toward mathematics "because most elementary teachers . . . didn't like math and . . . that attitude gets conveyed to the students." To break the negative attitudes students have already developed, Mr. Trackman indicated he spends "a lot of time just getting them to have fun with math and realize that it's not that big of a deal. It's not that big of a mountain; it's just a mole hill." In addition to overcoming the negative attitudes many students have picked up in elementary school, Mr. Trackman also suggested he spends "at least one third to one half the year mopping up mistakes that elementary teachers make or reforming misconceptions that [students] have been given."

A second goal of Mr. Trackman was to show the students where they are going to use mathematics. To accomplish this, he showed the students the Jaime Escalante video, "Math, Who Needs It?" just before Christmas. Throughout the year, he thought "it's really neat when [the students] realize that the logic that's behind chess, the logic that's behind philosophy, the logic that's behind a lot of different things is the same kind of logic, the if-then procedures that you go through in computers and math and all the different proofs in geometry." In addition, one of the objectives he tried to convey "in just about every unit," including the probability unit, was that "they are going to use this some day."

Mr. Trackman identified a third goal of getting the students "to learn how to learn." Because of the "wide range of learning abilities and learning styles and levels that they are at, as well as enthusiasm for math," he suggested, "if you can teach them how to teach themselves how to learn, then you're in a better situation for them at least picking it up on their own in the future."

Mr. Trackman explained he taught probability basically because he was "required to teach it." He did not "think kids should even be doing probability in sixth grade," stating the "kids are [not] really ready to understand a whole lot of concepts of the probabilities versus the chance versus . . . all the different ways of looking at . . . odds." But given that he had to teach it, Mr. Trackman's conceptions of the purposes for teaching probability paralleled his understanding of the aims and purposes of education. Among his objectives, Mr. Trackman wanted "to make it fun" and "to play games." He also felt it was important for the students "to come up with some sort of understanding of probability" and for them to grasp that "probability is everywhere."

Because Mr. Trackman had "always had fun with probability," he saw probability as a good way to make mathematics fun. In particular, he pointed out that in his sixth-grade class, "we play a lot of games . . . and that's what our probability is." He suggested "they do a lot more extensive work in the seventh grade," including a traffic signal project based on actuarial data. Playing the games, in his mind, satisfied the request of the seventh-grade teachers to get the students "excited about probability" in preparation for that project.

Mr. Trackman explained further that "a lot of what we do with our probability is just the introduction of it, because they don't get a whole lot, if anything, from the elementary teachers." As the students were playing the games in the unit, Mr. Trackman expected they would begin to develop introductory concepts on their own. One of the goals Mr. Trackman identified in teaching probability was

> to get each one of them at one point to come up with . . . something that has to do with probability, whether they figure out . . . well if I roll the dice it's going to be 1 out of 6 chances I'm going to get a 5. If that's all it is, then that's great. If they can figure that out on their own, whereas some kids, you know, figure out that . . . they figure out that the opportunity to roll snake eyes is 1 out of 36. Then you're getting more advanced opportunities, but at least something, get them to come up with something on their own, before I have to tell them.

As in all the units taught, Mr. Trackman reported that one of his objectives was to help students realize probability is something they will use. He pointed out probability opens up "a new wing of mathematics," showing students mathematics "is not just addition, subtraction, multiply, and divide." In particular, he explained even a rock star is going to use probability, giving as an example the technician, in Jaime Escalante's video, trying to determine which combination of all the different knobs on the sound board might produce the best results for a singer's voice.

Designing mathematics instruction. Mr. Trackman recognized the students in his classroom had a "wide range of learning abilities and learning styles." Because of his own experiences with mathematics, Mr. Trackman felt he had an understanding of "every kid in the class," suggesting, "I learned what the kids that are way ahead of everybody feel like, and [I] learned [what] kids that are just barely hanging on, working their tails off, barely hanging on [feel like]. I also learned what the lazy kids feel like, 'cause I was lazy for a couple years." Mr. Trackman indicated this experience and understanding impacted how he responded to students. For example, when a student looked at him with a blank stare that said "I don't know," Mr. Trackman suggested he tried to find out where he had lost the student and provide a different explanation beginning from that point rather than "just doing the whole thing from a different perspective" and possibly repeating what the student already knew.

In Mr. Trackman's view, mathematics is based on logical step-by-step procedures. In reflecting on his own experience with calculus, Mr. Trackman observed, "I had always wished that there was some way that I could do it . . . just by taking it step by step by step." And he suggested "many people like math because they are logical thinkers and . . . that's why they like math and usually don't like English and those that like English usually don't like math."

Mr. Trackman observed as well that in mathematics there was "the security in knowing that there is a right answer." He did not like open-ended questions because "all of a sudden there's not a right answer anymore." As an example, he suggested "you can . . . say in an open-ended question that 3 plus 4 is . . . 5 and not get it wrong, because you can show that if you travel south 3 miles, you travel . . . east 4 miles, you've actually, as the crow flies, only gone 5 miles. Have you traveled 5 miles? Have you traveled 7 miles?" The "truth" or correctness of an answer, however, was not necessarily established by logical arguments in Mr. Trackman's class, but rather was determined by sources such as the teacher or the textbook or other materials. For example, in trying to build students' confidence in their responses, Mr. Trackman reported he was in the habit of asking, "Are you sure?" when students answered questions. He explained, "Then they are . . . by the end of the year, [they are] able to explain if you have this and have this, then it has to be this. And this is what you told us. This is what we learned from the book. This is what we discovered from this project or that project."

Concerning the process of learning mathematics, Mr. Trackman argued for the importance of prerequisites and rote learning. "There is so much in math that, without that foundation, without that rote learning, you can't do it." For example, he suggested "kids are lost when they come to sixth grade and can't do their times tables." Throughout the year, his students took weekly times table tests until they scored 90% or better. Describing how he worked with one student having trouble with his times tables, Mr. Trackman explained, "I would sit back here at my desk with him . . . and I would go over and go over his times tables and go over and go over and go over and give him assignments, 'Okay, today we're just going to focus on the twos and I want you to come back and have written them down.' " In addition to his emphasis on prerequisites and rote learning, Mr. Trackman also had an expectation of student exploration, particularly in the probability unit. As the students played the probability games, he wanted them to discover the concepts for themselves.

As part of his general instructional practices, Mr. Trackman incorporated the use of warm-ups, projects, and textbook assignments. Mr. Trackman reported the warm-ups involved "a critical thinking kind of thing" and were intended to provide a link between the previous day and the new material. In particular, the warm-ups often presented the students with "stuff . . . they had never seen before." The students were encouraged to "at least try . . . to come up with something, to brainstorm anything possible, to just throw something down or guess . . . and as long as they had something down they got the necessary points that they needed."

Mr. Trackman also suggested "we try to keep some sort of project going on throughout the year." In one project, the students assembled a geometry portfolio, including a creative cover, illustrations of geometric terminology, examples of symmetry, and a town map drawn to scale. According to Mr. Trackman, two of the projects done during the year dealt with probability and statistics.

> The first project we do every year . . . is keeping track of statistics and figuring out probabilities of NBA players 'cause it's all right there in the newspaper and so they can all look through. We teach them . . . we spend about a week teaching them how to read the paper and, you know, the girls always say, "I don't understand basketball. I don't know basketball." And we like . . . "You don't have to understand it, all you have to know is how to read the paper and know what these things mean . . . know that P means personal foul and then you can keep track of personal fouls. You don't have to know what a personal foul is, but you just have to know what the paper is saying." And then they put together a big presentation with charts and graphs and all kinds of stuff like that.

A week or two before the probability unit began, the students had completed an M&M's project at Outdoor School. Mr. Trackman described this project:

> We do a statistics and probability [project] with M&M's, figuring out what is the most . . . they guess how many greens are going to be in there, how many yellows, how many reds. They figure out the percentages compared to what they guessed as far as how many are in there. They fill out their graphs, and then they open up the M&M's and figure out how close they were. And with the advanced classes, or the advanced kids, we even have them figure out scientific error on that.

Although Mr. Trackman recognized the recommendations to do otherwise, he admitted he used the textbook quite extensively. "We do a lot of stuff out of the book. . . . I like to do that." For the students, Mr. Trackman saw the book "as something that's . . . stable in their life. You know, they come in, and they have their warm-up, they have their book assignment. Okay, then if they have extra time in class, they work on their project." And although they occasionally did games and hands-on activities, these were "never really for assignments."

Daily classroom activity followed a fairly consistent pattern. In describing a typical day, Mr. Trackman outlined primarily four segments. For the first 2 or 3 minutes of class, the warm-up would be displayed on the overhead screen. Although it was turned off after the first few minutes, the students continued working on the warm-up. Mr. Trackman next went over the warm-up which often served as "the lesson of the new day's stuff." Then the warm-up was put away and the class corrected the assignment from the previous day. Scores of 0 to 4 were assigned, based on the number of items correct. As these scores were reported orally to Mr. Trackman, he entered them into the computer. Finally, for the remaining 15 to 20 minutes of the class, Mr. Trackman indicated "basically I'd just turn them loose" to work on their assignments. He felt he "had given enough instruction so that the kids, the super bright third, they were going to go and run with it and they didn't want to wait around for any more instruction." He usually had an aide in the classroom, whom he asked to monitor the "kids in the middle." That gave him 20 minutes to circulate and "make sure that the lower level kids understood it."

Mr. Trackman also followed a weekly pattern of testing. Once a week the students had a brief quiz instead of the warm-up. Then, each Friday the students took a test rather than have an assignment. These 10-problem tests, each worth a total of 10 points, were taken "straight from assignments. They were the exact same problems that they had done during the week." Mr. Trackman indicated he liked to test the students more frequently and over smaller units so that they did not "have to worry about remembering what they did 6 weeks prior." This, he explained, helped "break their fear of tests."

Many of these general patterns and routines, however, were not followed in the probability unit. First, there was no warm-up to begin the class period. And because the bell was generally not heard, it was not always clear when the period actually began. Second, Mr. Trackman did not include weekly quizzes and tests in the unit. Instead, a test at the end of the unit was used for evaluation. Third, Mr. Trackman indicated probability was one time they do more games and activities instead of relying entirely on the textbook.

Creating the learning environment. Mr. Trackman used different groupings of students depending on the nature of the activities. Students sometimes worked individually.

At other times they worked in groups of two or three. Even when students were not assigned to work together, many still worked with those around them. Students were grouped into pairs or threesomes by the arrangement of the desks. However, Mr. Trackman did not utilize these groupings when arranging students for activities. When groups were called for, students were allowed to move around and form their own groups.

The class period lasted 43 minutes. In the pre-observation interview, Mr. Trackman suggested "we cram as much as we can into it as possible." When schedule changes shortened periods to 27 minutes, as it did on one occasion during the probability unit, Mr. Trackman felt "it's just a wasted day. It really is a waste. If you want to teach something, by the time you get done teaching it, the bell's rung and they don't have any time to ask questions of you and so what you end up doing . . . what I'll do sometimes on those days is I'll have quizzes on those days or . . . you have a movie." To help make efficient use of the time they had, Mr. Trackman had established certain routines at the beginning of the school year. He explained, "We spend the first week role playing so that we . . . so that they know, when this [e.g., warm-up or quiz] is up on the board, this is what they are supposed to do. And so there is not a whole lot of time wasted explaining, 'We've got to do this again. When quiz is up there, then this is what you do. This is how you deal with that.' "

Mr. Trackman admitted discipline was one area he continued to work on. As he pointed out, "I spent the first year and a half [in this position] just trying to learn discipline. And, uh, well, re-learn discipline. I didn't have a problem with it in the high school setting [where I student taught], but middle school is . . . a different bird." For assistance, he had attended a "couple classroom management classes" as well as having one of the other teachers in the school serve as a mentor for him.

On a bulletin board at the front of the classroom, Mr. Trackman had posted the Ten Commandments for classroom behavior:

I. You shall not talk unless called upon.
II. You shall be respectful 2 all adults in this classroom.
III. You shall be polite 2 thy neighbor.
IV. You shall call me Mr. Trackman & adults what they wish.
V. You shall not touch others in a disruptive manner.
VI. You shall do all homework.
VII. You shall raise hands 2 ask ONLY relevant questions.
VIII. You shall not put others down, esp. Moms!
IX. You shall do nothing questionable, & if you aren't sure, DON'T.
X. You shall not take books out of the classroom!

In addition, the following "How 2 B-Attitudes" were posted:

There will B . . .
no gum or candy
no eating! for students

no bathroom breaks or use of pass in first 30 or last 5 minutes of class
no leaving seats w/o permission
no moving of desks
no throwing of anything

You will B . . .
on time!
kind 2 others!

Finally, the consequences (wrath) for breaking these rules were also posted on the bulletin board:

Step	Wrath	Points lost
1	Warning	-
2	Check	1
3	Time out	3
4	Stay after school	4
5	Stay after & call home or work	5
6	Referral	7
7	Principal	10

Mr. Trackman pointed out teaching probability presents some unique instructional problems. In particular, he explained the "noise level [goes] through the roof" when the students are playing games. Further, he suggested you have students fighting over who won or objecting when they have to work with students of the other gender. And, at the end of the year, "everybody . . . has spring fever." Mr. Trackman observed that after the probability unit was over "we gave them a couple book assignments and it was . . . a big relief because things were back in order again."

Planning the probability unit. The unit covering probability was taught in late May and early June, the last instructional unit before preparation began for the final exam. Mr. Trackman indicated it had been scheduled at the end of the year "because of the standardized tests we take." Although at one time probability had been taught earlier in the year, the teachers had reorganized the curriculum to make sure they covered geometry and integers, content they knew to be included on the standardized tests. As a result, probability "just got pushed because there was not anything on probability on the test."

When asked what factors he considered in planning the unit, Mr. Trackman explained, "Getting it to a point where they didn't have to take home homework. I didn't want them to have to take home homework . . . because at the end of the year they don't do it anyway. Not necessarily that we're trying to enable them, but it becomes a losing battle." He also suggested the students' background and their level of maturity influenced what he and his colleague chose to do in the unit.

Mr. Trackman felt the students "don't get a whole lot, if anything, from the elementary teachers" with respect to probability. He observed, "I don't think there is any investigation [of probability] in the earlier grades." As a result, he did not feel the students had many notions about probability coming into the sixth grade other than "flipping coins. That's about it. They realize that that's 1 out of 2." And, according to Mr. Trackman, their idea of fairness was "if they lose, it's not fair."

In addition, Mr. Trackman explained the group of students he and his colleague currently were teaching were "a little bit more immature than in [the] past." He pointed out this immaturity impacted what they chose to do in the unit.

> With this group, because of the relative immaturity, we thought, let's go more with things they're used to as opposed to teaching them new games. . . . There were certain games . . . we had probably 15-20 games that we were going to play, but there were certain games we had played year after year that we knew just weren't going to work because of the immaturity of this group. . . . This year, this group didn't really have the attention span to have a lot of rules and be able to pick them up right away.

In particular, the activities to be used on the first 3 days of the unit were selected, according to Mr. Trackman, because the students were familiar with playing rock, paper, scissors, with rolling dice, and with flipping coins.

The textbook used in the class was Silver Burdett & Ginn's *Mathematics* (Orfan & Vogeli, 1988). Statistics and probability were both covered in chapter 13, followed only by a chapter on integers. The probability portion of the chapter included the following sections: "Experiments and Outcomes," "Probability and Relative Frequencies," "Probabilities of Events," and "Tree Diagrams and Compound Events." A final section, "Problem Solving: Skills and Strategies Review," also dealt, in part, with probability. Mr. Trackman chose to assign three of the textbook sections "mainly . . . because I [will have] a substitute." He was going to omit the other two sections. In particular, Mr. Trackman was going to omit "Probabilities and Relative Frequencies" because he felt it was confusing to the students. He was not going to include the section "Tree Diagrams and Compound Events" for two reasons. First, time was a constraint. Mr. Trackman felt "it would have taken probably another 3 or 4 days" and they needed to move on and prepare for the final exam. Second, Mr. Trackman admitted, "I just didn't really feel comfortable in teaching it. I wasn't sure if I could make it . . . I guess relay it in a way that I would want them to receive it and . . . I just didn't like it, so that was the main thing." In general, because Mr. Trackman did not feel the students were ready to understand the concepts of probability, he suggested the unit on probability was "the one time we really do get away from the math book." He explained they chose instead to play games to help the students develop the concepts on their own.

The games and activities had come from a variety of sources. In addition to the three assignments given from the textbook, Mr. Trackman adapted another page from the textbook for the Coin Tossing Exploration. Other sources for the games and activities included *Problem Solving in Mathematics* for grade 6 (Lane County Mathematics Project, 1983b), *Dealing with Data and Chance* (Zawojewski, 1991) from the NCTM Addenda Series for Grades 5-8, and the *Student Resource Handbook* (Swanson & Swenson, 1988) which accompanied the Mathematics for Elementary Teachers textbook he had used in college. The other sixth-grade teacher had suggested the final two activities using dice.

The Probability Unit

The probability unit extended over a 10-day period, of which 8 days were observed. The other days, Days 4 and 5, were not observed because a substitute teacher was teaching on those days. The unit was loosely divided into three segments, with a series of textbook assignments set between two activity-oriented segments (see Figure 4).

Monday	Tuesday	Wednesday	Thursday	Friday
Day 1 "Paper, Scissors, or Rocks"	Day 2 "Is This Game Fair?"	Day 3 Coin Tossing Exploration	Day 4 "Experiments & Outcomes"	Day 5 "Probabilities of Events"
Memorial Day	Day 6 "Problem Solving"	Video from track meet	Day 7 Owl & Oyster Riddle	Day 8 Dice Sums
Day 9 "Pig"	Day 10 Review & Unit Test			

Figure 4. Calendar for Mr. Trackman's probability unit.

This section will begin with a description of Mr. Trackman's introduction to the unit. Then, as a sample of the probability unit, Mr. Trackman's presentation of three activities will be described. These activities include two based on dice sums and one which involved tossing coins. This section will conclude with a description of the unit test used for evaluation. (Note: Titles enclosed in quotes are the titles assigned to the activities in published materials; titles not in quotes are titles assigned to the activities by the researcher.)

Introduction to the unit. After settling the students down and getting their attention, Mr. Trackman began the probability unit by asking them "where probability is used in our lives." The students first suggested probability happened in math class, in jobs such as accounting, and in such things as sports cards. Mr. Trackman then pointed out "sports has probability throughout." He provided the example of a baseball manager making a pitching change late in the innings to bring in a right-handed pitcher to face a strong right-handed batter, decreasing the probability of the batter hitting the ball. As further examples, Mr. Trackman mentioned the Preakness Stakes and Super Bowl where people gamble on the results. He also pointed out games the students play, such as Monopoly, also involve probability. Indicating they were going to investigate "games and making games fair" for the next week, Mr. Trackman explained the rules for the first game, "Paper, Scissors, or Rocks," which the students were to play for the remainder of the period.

"Is This Game Fair?" After analyzing the "Paper, Scissors, or Rocks" game the following day, Mr. Trackman explained that next they were "going to have a worksheet . . . that involves a game of dice. . . . No, not Andrew Dice Clay." He then went on to explain the rules for "Is This Game Fair?" taken from *Dealing with Data and Chance* (Zawojewski, 1991), a part of the NCTM Addenda Series for Grades 5-8. He pointed out the game was played with two participants, a player and an opponent, who each start with 10 points. He would give each person a die so when they paired up, they'd have two dice between them. Then he explained "the player . . . the person you decide to be the player will be . . . the one that rolls the dice . . . once they roll, roll the dice, you're gonna determine whether or not they have a 7 or not," and circle the "yes" or "no" on the worksheet. To explain the scoring for the game, Mr. Trackman gave them sample results for 10 rolls (see Figure 5). He then explained whenever the player, who was the one rolling the dice, obtained a sum of 7, three points were transferred from the opponent to the player. For any sum other than 7, the player gave one point to the opponent. He then scored the sample he had shown them.

		Y	N	N	N	N	N	Y	N	Y	Y
P	10	13	12	11	10	9	8	11	10	13	16
O	10	7	8	9	10	11	12	9	10	7	4

Figure 5. Results given by Mr. Trackman as a sample for "Is This Game Fair?"

Mr. Trackman then lifted the screen to reveal the directions he had written on the white board.

- Play 'till one has 20
- Play only 10 turns
- Determine which is fair

In addition, he gave the students these instructions:

> I want you to play until you get to 10, 10 tries. That's one of the things right there. And then, I want you to also play it until one person has 20 [*pointing to the final box*] and the other one has 0. And then, if you're done with that, and you still have time in class, switch players. The other person be the dice roller. Okay. . . . And then you have to determine if it's fair. If it's not fair, maybe you can rescore it.

With just over 20 minutes remaining in class, Mr. Trackman handed out the worksheets and the dice, as the students paired up and began playing the game. When students asked questions clarifying the rules, Mr. Trackman directed them to the rules given on the worksheet, suggesting "that's why there's writing up there, so you read it."

When one group of girls finished playing the game for 10 rolls, Mr. Trackman reminded them "you have to play it . . . 20 times, or play it 'til one person gets 20 and the other has 0 . . . points." He circulated among the pairs to make sure they were following directions and then returned to his desk. When one pair of girls came back to his desk saying they were done, Mr. Trackman pointed out, "Now you just have to . . . play it just to 10. Just do it 10. Okay. And then switch players. Keep playing until you hear the bell." About 3 minutes later the girls returned to his desk reporting, "We played again." Mr. Trackman explained, "Getting done fast is not the objective. Getting done . . . just gathering data, we're gathering data and we'll analyze it tomorrow." With 8 or 9 minutes left in the class period, many students seemed to think they had completed the assignment and began to talk with each other, even though Mr. Trackman reminded them, "You're not done . . . until the bell rings."

After spending about 5 minutes at the beginning of the period on Day 3 handing out the weekly school newsletter and discussing some of the items, Mr. Trackman began a follow-up to "Is This Game Fair?" with a general description of the analysis process.

T: Okay. A couple things we're going to go . . . we're just going to go over some numbers. Part of, part of doing experiments . . . involves figuring out ways to analyze the data. Okay? . . . As long as you have a way, and you got . . . sometimes you've got to come up with it on your own. How are we gonna, what's one way we can analyze this data? Okay. We had, we had, you flipped it, er, rolled it, yeh, you rolled the dice . . . and one person wins, and we played it with 10, ah, rolling 10 times. We played it going 20 times, er, until people got to 20. What are a couple ways you might be able to analyze this data?
S: Analyze it?
T: Analyze it. Compare . . . data . . . between . . . just between your group and all the groups.

To begin the analysis of the experimental data, Mr. Trackman asked the students for "one thing that you all did the same." Seemingly confused by the question, the students gave responses such as, "We all rolled the dice," or "We all had names on our paper." After pointing out they all had played the game 10 times, Mr. Trackman attempted to gather some data.

T: So, how many times did the player win?
Ss: [*various responses*]
T: How many times did the player win? How many players here won?
S: I don't know.

T: How many people that were the player in their game won? Okay. Two, three, four. You got four players. All right. The opponent? . . . How many opponents won? Okay. One, 2, 3, . . . 8. Okay. That's about the way it should have turned out. That's the way it's been about turning out. Well, that's one way of analyzing the data. You just come up with stuff, you start writing down numbers. When you get it all written down . . . you start to notice a trend . . . and stuff like that.

Mr. Trackman then moved on to collecting data from the second version of the game. In addition to playing the game 10 times, Mr. Trackman had instructed the students to play the game until one person reached 20 points. Continuing with the analysis on Day 3, Mr. Trackman asked the students, "What were some of the lengths of the games?" When the first pair of students reported their game had lasted 10 rounds, a lengthy discussion ensued about whether they had followed the directions given. To these students at least, the directions from the previous day had not been clear. When Mr. Trackman suggested the discussion was over and moved on to gather other results, only four other pairs reported values. The data reported included 10, 26, 19, 16, and 22. Then, when Mr. Trackman asked whether the player or the opponent had won each of these games, the students reported the first three games had been won by the opponent and the player had won the other two games. A brief discussion followed about what they could conclude.

T: Okay. . . . So we've got some, we've got some . . . . There's nothing necessarily we can notice from all that. One thing we can . . . yes?
S: One thing we can notice is that the opponent . . .
T: Yes. The opponent. It looks like he's gonna win. So does that mean this game is fair?
Ss: No!
T: Probably not. Probably not. We'll look over that in a little bit.

Mr. Trackman then proceeded to find the average of the data reported, comparing this period's result to the results from earlier periods.

T: Uh, you got 26, 36, 55, 61, 71, 80, 93 [*adding up the data*] . . . 93 in five games . . . plus about . . . oh, 18.6 [*giving the average*]. . . . We got 18.6. . . . First period had 18.6. Second period had 17.2. . . . Third period had 34 something . . . .
Ss: What?!
T: Yeh, they . . . had some long games. They were like 58, and 50 . . . had a couple of 40s. They went forever, it seemed like. . . . Fourth period I don't remember. Fifth period was 20 something. So . . . except for third period . . . they were all pretty much about the same. The length of the games would take probably around 18 to 20, somewhere in there.

After summarizing the experimental data in this way, Mr. Trackman next moved on to consider a theoretical analysis of the game with the students.

T: Okay. That's just some . . . ways we can look at the information. Now, let's . . . try and determine . . . one way of looking at it . . . apart from the numbers that you got. Okay. What are the possibilities? What possible numbers . . . if you roll the dice . . . what, how many possible numbers could you get?
Ss: Six . . . 12.
T: We're rolling . . . two dice. How many different numbers, totals of the two dice, added, sums, how many different totals are there?
Ss: Twelve . . . a lot . . . 36 . . . 24.
T: What's the lowest number you can get?
Ss: One . . . 0 . . . 2.
T: What's the highest?
Ss: Twenty-four . . . 24 . . . No [*laughter*] . . . 12 . . . 99.
T: How many possible numbers are there?
S: Eleven.
T: Eleven. Thank you. There's 11 possible. . . . You don't have 1. You've got 2 through 12. That's right. Wonderful.

Having concluded there were 11 possible sums, Mr. Trackman next directed the students' attention to the ways each sum could be obtained. After establishing there was one way to get a sum of 2, namely "snake eyes," Mr. Trackman proceeded to consider a sum of 3.

T: How many different ways can we get 3?
Ss: One and 2 . . . 2 and 1.
T: One and 2 . . . 2 and 1. Let's try it this way. Okay. There's a couple ways you could possibly do this. . . . Couple ways you could possibly do that. You could either . . . try it with only 1 and 2, or you could try it with 1 and 2 and 2 and 1. We're gonna try it with just the one possibility, 'cuz really you got 1, you got a 2. It doesn't matter which die . . . okay? . . . We're going to try it this way, and see what, just kind of see what we can see. It will also save a little time 'cuz we're running a little behind.

Although Mr. Trackman was incorrect in suggesting they could ignore the second pair, he continued with the other possible sums, listing the different combinations as the students suggested them. In listing the combinations, Mr. Trackman had written "·" between each pair of numbers instead of "+," confusing at least one of the students.

S: Doesn't the dot mean times?
T: No, I'm just . . . .
S: There like, um . . . .
T: Don't worry about . . . You're right. You're right. Absolutely right. It does mean "times."

As the list grew, they came to the possibilities leading to a sum of 8. After listing 4 and 4, 2 and 6, 3 and 5, one student suggested 7 and 1.

T: Seven and 1? . . . How can you get 7 and 1?
S: Oh, no.
T: I've gotten somebody that's . . . every period. . . . When was the last time you rolled one dice and got a 7?

After listing all of the possible combinations (but not including the reversals of the pairs), Mr. Trackman proceeded with the analysis of the game.

T: Okay. We got that. How many different ways are there on this? We've listed how many different dice combinations?
Ss: [*counting out loud together*] 1, 2, 3, 4, 5, . . . 19, 20, 21.
T: Twenty-one. How many of those are gonna come up so that the player will get three points? Meaning 7?
S: Three.
T: Okay. Now we might be able to come up with . . . we could also explore the possibility of 4 and 3, uh, 5 and 2, and 1 and 6, er, 6 and 1, I mean. Anyway . . . but we're not. We're not gonna worry about that this time. Okay. For the sake of time. Anyway, there's three possibilities. How many possibilities are there the other way?
S: Um . . . 18.
T: Eighteen. Okay. So they get one point for each one. . . . So we multiply theirs by the 3 that they get . . . they're looking at really only getting nine points. So, usually, the opponent should win.

As this analysis was concluded, one student made a conjecture about making the game fair.

S: Yeh, but what if you times it by 6?
T: What if we times it by 6? That would be something we could investigate if we had more time. But we don't. And that . . . our little theory through this . . . that's part of the trial and error . . . that probability involves. Sometimes you just kinda, you're not sure about things, so you just kinda do it. You go with the theory and then you try it out. You see if it really works. . . . In this little theory, it looks like they should be even.

At this point, Mr. Trackman had the students hand in their papers.

Coin Tossing Exploration. In the 20 minutes remaining after the analysis of "Is This Game Fair?" Mr. Trackman moved on to the next activity, one using coins. Having drawn the grid of circles (see Figure 6) on the overhead as the students handed in their papers, Mr. Trackman orally gave the instructions for the activity. Beginning from the circle marked "Start," the students were to travel through the grid to the circles marked "Stop" by flipping their coins. If the coin landed heads, they were to follow the upward arrow. If it was tails, they were to follow the downward arrow. Each trip through the grid involved six tosses of the coin. He showed them one sample path through the grid and then demonstrated how they were to record tally marks by the circles marked "Stop" to record the experimental results for the 25 trials they were to do. Mr. Trackman next showed them the following questions they were to answer on the back of their sheet after they had completed the activity:

- Which stop has the highest probability? Why?
- Which stop had the most landed on?
- Are these results the same as the probabilities? Why or why not?
- Figure the probability of each stop.

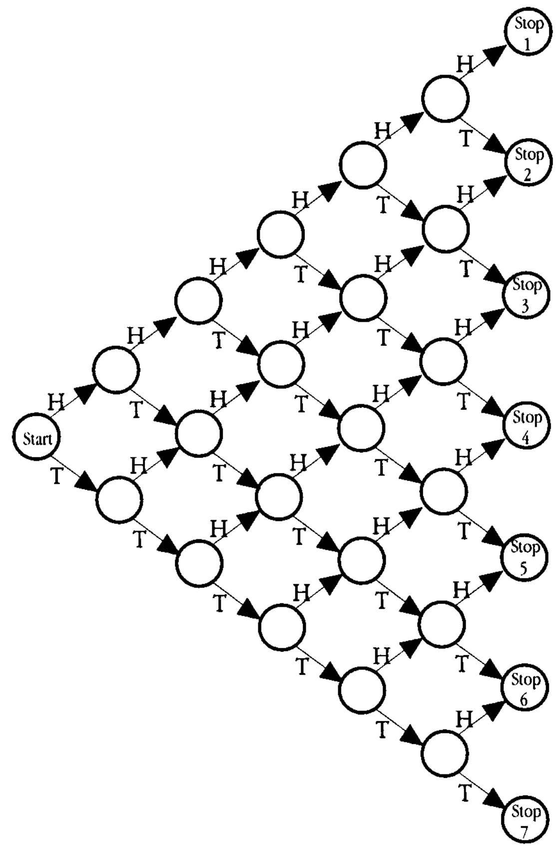


Figure 6. Grid of circles for the Coin Tossing Exploration.

In explaining these questions, Mr. Trackman pointed out that for the first question the students were to predict which "do you think has the most chance," without "looking at your data." In the second and third questions, the students were to report their experimental results and compare those to their predictions. A more extensive discussion followed about the final question.

> T: Then I want you to figure out the probability of each stop. Now this is where I will cease to help you. I want you to come up with some way of figuring out what the chance, what fraction . . . of a chance there is for each of these.
> S: How are we supposed to do that?
> T: That's what you want to come up with. I want you to try and come up with some way of figuring that out.
> S: That's easy.
> T: Okay. Well, there's . . . one thing to remember. What's the chance of it . . . starting right here [*at the "Start" circle*]?
> S: Uh, 25 out of 25.
> T: Or 1. . . . The chance is 1 out of 1. Okay, now I'm gonna get you started here. What's the chance of it landing here [*on one circle in the next row*]?
> S: One in 2.

T: One out of 2.
S: One in 3 . . . 1 in 4 . . . [*as a student in the back of the classroom thinks he's found a pattern*].
T: What is the chance . . . [*pointing to the other circle in the next row*]?
S: One out of 2.
T: One out of 2. If you'll notice, these add up to . . . 1 out of 1. The chance for this is gonna be 1 out of 1, all of . . . your fractions should add up to 1 over 1. Okay?
S: All of them?
T: Yep. When you add all those up you should have 1 over 1. Okay. And as you go, you . . . may begin to notice a pattern. The pattern will help you to find this out. These [*pointing to the "Stop" circles*] are the ones we're looking for.

Now, with about 15 minutes left in the period, Mr. Trackman handed out the worksheet and plastic pennies the students were to use for the activity. While the students were working on the activity, Mr. Trackman attended to such tasks as changing the date on the front board and preparing instructions for the next day's substitute teacher. What interaction there was with students generally involved clarification of the instructions for the activity or of the questions the students were to answer. However, when one girl came back to the teacher's desk with a question about finding the probabilities, Mr. Trackman began to fill in values for the early part of the triangle.

Because Mr. Trackman was absent the following day, there was no follow-up to the Coin Tossing Exploration. The students who had finished the assignment turned in their papers for Mr. Trackman to look over. He later reported to the researcher that a number of students had not turned the page in, but the ones who did "were A's and B's, all of them."

Dice Sums. Mr. Trackman had taken personal leave for the next 2 days of the unit so that he could accompany the college track team he helped coach to their national meet. On these days, he planned for the substitute teacher to assign two sections from the textbook, "Experiments and Outcomes" and "Probabilities of Events." The first section introduced the terms *outcomes* and *equally likely* and defined *probability* as the ratio of the number of favorable outcomes to the number of possible outcomes. The second section defined the term *event* and instructed the students that in order to find the probability of "two events that cannot occur at the same time," they should "find the sum of the probabilities of the outcomes." In the exercises for this section, the students were to find the probabilities of various events involving *or*, *not*, or combinations of outcomes.

After the 3-day Memorial Day weekend, the probability unit continued with a third textbook assignment, one entitled "Problem Solving." Given information on how often in the last 40 years it had rained on the day scheduled for the class picnic and an alternate date, the students were to find the probability it would rain on those days and the probability it would not rain. Another set of questions asked the students to identify the relative frequency of rainy days and of sunny days in the month of April based on the observations displayed on a calendar.

Because Wednesday of the second week was a late start day with shortened periods, Mr. Trackman showed the students a video from the national track competition, with the justification that the students were doing track in their physical education classes. As a follow-up to the textbook assignments, Mr. Trackman gave the students a worksheet assignment on Day 7. In order to discover the letters needed for the solution to the riddle, "What do you get when you cross an owl with an oyster?" the students had to find probabilities for various outcomes, including several joined by *and* or *or.*

On Day 8, after briefly correcting the worksheet from the previous day, Mr. Trackman asked the students to get out a sheet of paper for the game they were going to play. This game, a variation of the River Crossing game introduced in the Math and the Mind's Eye materials, *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993), had been suggested to Mr. Trackman by the other sixth-grade teacher. Mr. Trackman had decided to play the game as a whole-class activity, with the boys competing against the girls. The running score from earlier periods was displayed on the front white board.

Mr. Trackman had the students write the numerals 2 through 12 along the left-hand column of their papers. He then explained that beside the numerals they were to mark 11 *X*s on the sums they felt were most likely to occur when the dice were rolled. As Mr. Trackman rolled the dice, they were to cross off their *X*s as the sums occurred. The first to cross out all their *X*s would be the winner. Mr. Trackman marked some *X*s as a sample and rolled the dice a couple times to demonstrate when to mark off the *X*s. Some of the students were confused about how many *X*s to make, what to mark off, and how Mr. Trackman was going to check the results of the winner, but once the game began, they seemed to catch on.

As they were ready to start the game, Mr. Trackman observed one student was chewing gum, a common infraction in the classroom. This observation led to the following dialogue:

T: All right, if I see gum in your mouth again in this class . . .
S: Yes?
T: Then you're going to start cleaning gum off the bottom of the desks.
S: Ugh.
T: And that's every day. And you've tried to get away with it every day.
S: I'm not very good at it, huh?

With a reminder that the game, like Bingo, would go until "somebody gets all of theirs done," Mr. Trackman began rolling the dice and tallying each outcome on the front board. For the first three rounds of the game, the students were encouraged to discover for themselves the pattern of dice sums without input from the teacher. The students were quick to observe a sum of 7 occurred quite often. When in round 3, no 7s were occurring, the students began to call for them. At one point, Mr. Trackman stopped rolling the dice to ask, "Does anyone not have any [*X*s] on 7?" He went on to observe, "The dam . . . is ready to break loose."

Before the fourth round, Mr. Trackman announced he was going to play, writing his *X*s on the front white board. His *X*s were concentrated between 5 and 9, with four *X*s marked on 7. Mr. Trackman similarly played the fifth round with the students.

As the time was getting near the end of the period, Mr. Trackman suggested they might get one more game in. Just before this sixth round, he presented the correct theoretical possibilities to the students (see Figure 7[a]). "There's 1 real possibility for 2, there are 2 for 3, there are 3 for 4, . . . and there's 4, 5, 6, 5, 4, 3, 2, 1 possibilities" (as he wrote down the number of outcomes for the other possible sums). Unlike the earlier game based on dice sums, Mr. Trackman here was correctly considering all possible pairs. The sixth round was quickly finished and the students were dismissed with the suggestion that "we'll continue the investigation . . . on Monday."

2	–	1
3	–	2
4	–	3
5	–	4
6	–	5
7	–	6
8	–	5
9	–	4
10	–	3
11	–	2
12	–	1

(a) Summary of dice sums outcomes.

1, 1	2, 1
1, 2	2, 2
1, 3	2, 3
1, 4	2, 4
1, 5	2, 5
1, 6	2, 6

(b) Partial list of dice outcomes.

Figure 7. Displays used by Mr. Trackman for dice outcomes.

As class began the following Monday, Mr. Trackman announced the overall score from all the periods. The guy's team had come from behind to win by one point. When the students asked if they could play the game again, Mr. Trackman responded, "Maybe if you're good, someday we'll play it again . . . maybe next week."

Mr. Trackman then mentioned, "We're probably gonna have a test tomorrow" before going on to explain, "Now I can give you a chapter review assignment or we can . . . do another game that will help you to understand a little bit more." He was not surprised when the students wanted to play the game, although he suggested they had to go over what they had done on Friday first.

Mr. Trackman wrote down the numerals 2 through 12 in a column and asked the students, "How many different ways total, if you roll the dice, how many different possibilities could come up from rolling the dice?" The students gave a variety of responses including 12 or 24. When Mr. Trackman asked how they got 24, one of the students suggested 12 plus 12. Another student then suggested 36 which he explained was 6 times 6. Mr. Trackman then began to list a partial array of the dice pairs, as in Figure 7(b). "We can have a 1 and a 1, a 1 and a 2, 1 and a 3, 1 and a 4, 1 and a 5, 1 and a 6. . . . Then you could have a 2 for the first die. . . . And you could continue on, and you'd have . . . six different numbers up here, and . . . there would be six different possibilities with each first number. So you'd get 6 times 6 equaling 36." With only the partial array written, Mr. Trackman went on to ask how many different combinations would add up to 2 or to 3, to which the students gave correct responses. When he asked how many combinations would add up to 4, students gave a variety of responses. Some suggested two ways, the number visible in the partial array. When Mr. Trackman asked one student why they had suggested 3, that student pointed out that it went in a pattern. Mr. Trackman argued that the pattern could be 1, 2, 4, 8. However, another student explained, "If you had a fourth one, it would be 4 and 0, so . . . it's only 3." Mr. Trackman then asked if the students were saying the pattern was going to go 4, 5, 6, 7, 8 . . . . Having been given the pattern on Friday, one student suggested, "No. . . . It'll go up to 6 and it'll go back down." Mr. Trackman suggested they check that out. After observing there were one, two, and three possibilities for sums of 12, 11, and 10, respectively, Mr. Trackman completed the rest of the list.

As they completed the analysis, one student asked, "If you rolled the dice 36 times, would there be that many by each?" They briefly discussed the question.

T: Good question. In theory, yes. . . . How often would that, do you think that would happen?
S: Never.
T: Probably never. You know, you'd be real, actually you'd probably be more lucky to get it to land like that if you rolled it 36 times than you would be any other thing.

After showing the students a couple different strategies he had seen the students using, Mr. Trackman proceeded to ask what the probability was for each of the sums. When they had written down each of the probabilities, he asked, "If you added all those fractions up, what should it add up to?" After one or two incorrect answers, one student responded, "36," which Mr. Trackman summarized as "36 . . . 36ths" or one whole. The remainder of the class period was spent playing another game using dice.

Unit test. For the first 10 minutes of class on Day 10, Mr. Trackman provided a brief review before giving the students the unit test. The questions for both the review and the test came exclusively from the textbook material. In addition to probability, the chapter of the textbook also covered statistics, which the class had studied earlier in the year. However, because Mr. Trackman had not tested them at that time, he included statistics on the probability unit test. As review prior to the test, Mr. Trackman gave the students three questions. In the first, the students were to distinguish between a bar graph and a line graph and identify which "might show change better." In the second, the students were to interpret a circle graph divided into four equal regions, giving the percentage corresponding to each section and the number of students from a class of 28 that each section would represent. In the final review question, the students were to identify the probability of selecting each of the letters M, T, and A from MATT, the nickname of one of the students.

The chapter test contained 36 questions, which Mr. Trackman had cut and pasted together from the Chapter Review and Chapter Test questions given in the textbook. The first 16 questions covered statistics, asking the students to read, interpret, and apply information from bar, line, and circle graphs. Nineteen of the remaining 20 questions applied to probability. The first eight questions asked the students to find the probability of choosing a letter or combination of letters from the word SEASONAL. The next eight items related to a spinner divided into eight equal portions and labeled 1, 1, 2, 2, 3, 3, 4, and 5. The students were to find the probability of such events as spinning a 3, a 2 or a 5, or an odd number. The remaining three questions involved applications of probability. Two of these questions applied the notion of probability as a relative frequency. For example, given that 40 of 50 people surveyed preferred warm weather to cold, the students were asked to "predict how many people out of 200 prefer warm weather." In the final question, a bonus question, the students were to find the probability that Larry will have eggs and juice for breakfast if he could "choose pancakes, eggs, cereal, or toast, and milk or juice."

Evaluation of the Probability Unit

Evaluating student learning. Mr. Trackman seemed to rely on homework and tests as evidence of student learning. For example, he pointed with pride to the improvement his students had made each year on the standardized tests they took. And when asked by the researcher how the students had done on the Coin Tossing Exploration, Mr. Trackman reported the grades the students had received on the homework.

Rather than weekly quizzes and tests during the probability unit, as was his custom during the rest of the year, Mr. Trackman gave the students one test at the end of the unit. And rather than testing over all of the content and activities, the test covered only the material from the textbook assignments.

Evaluating unit effectiveness. In reflecting on the unit, Mr. Trackman concluded it had been "very effective" in meeting his goals, explaining, "Our objective was to make it fun and I think we made it fun. Our objective was to play games and we played games." In particular, he noted that on the last days of the school year, after the unit had been completed, the students had wanted to play some of the probability games. This desire to play the games suggested to him they had been successful in getting the students "excited about math." In addition, as a result of the unit, Mr. Trackman felt the students were realizing that they were going to use probability some day and that probability is everywhere.

Pointing out there may have been "a little bit too much down time . . . some of the days," Mr. Trackman suggested there might be some things they would "do different next year to make it run a little bit smoother." He also mentioned the possibility of doing some probability out of the textbook first. He wondered how that might impact the games they played. The teachers were also talking about "doing it earlier in the year so that we can tie fractions into it later and tie decimals into it later . . . as opposed to an after-the-fact thing."

Mrs. Books: Influence of Learning Experiences

With 6 years of teaching experience, Mrs. Books is the next teacher to be introduced. The mathematics classes she took as part of her teacher education program had given her opportunities to develop a new perspective about learning mathematics.

Teacher Background

School experience. Mrs. Books described her own elementary mathematics experience as "pretty run-of-the-mill" until they began to study decimals in sixth grade. As she recalled, "that was the first time I can remember being really confused about math" because it "did not make any sense." In seventh grade Mrs. Books was moved into an accelerated track, which put her a year ahead of most of her junior high peers. Looking back, she admitted that "probably was not the best arrangement" for her because "I just ended up getting so lost and was so shy that, to ask a question in class, when most of the people in there were a year ahead of me, was really hard."

Although it was a struggle, she continued on the accelerated track, doing the best she could in 2 years of algebra, 1 year of geometry, and 1 year of trigonometry and math analysis. She remembered "feeling terribly lost with math" and pointed out, "I'm not sure how I ended up passing the trig[onometry] and math anal[ysis]. I think it was because I did my homework, and I took the tests and, you know, because of effort we ended up with a C."

Teacher preparation. In part because of her struggle with mathematics, Mrs. Books "chose not to go on to college" after high school. Instead she obtained some business college training and began working as a bookkeeper. About 10 years later, as she explored the possibility of becoming a school secretary, she discovered she enjoyed working with children. She then began thinking about becoming an elementary teacher and considered going on to college. Mrs. Books decided she wanted to take the Mathematics for Elementary Teachers class as her first course at a local community college because she felt mathematics would be the deciding factor of whether she could be successful or not. As she was studying for some pretests that would allow her to enter the Mathematics for Elementary Teachers class, she discovered that "for the first time algebra started making sense to me."

This experience of newfound understanding continued as she took the Mathematics for Elementary Teachers sequence. According to Mrs. Books, they had a "wonderful instructor . . . who really pushed us hard" and "things started coming together." At the community college, Mrs. Books went on to take finite mathematics, calculus, and statistics. To complete her degree she went to a nearby state university. Because she had taken quite a bit of mathematics, she wanted to have mathematics as her emphasis. Somewhat by accident she discovered the university had a special program for preparing middle school mathematics teachers, a program in which she then became involved.

As part of the middle school program, Mrs. Books took seven additional mathematics classes designed specifically for middle school teachers. Many of the professors who taught these courses also had been involved in writing the curriculum materials that were used in the classes. Because the prospective teachers were involved in doing the activities, Mrs. Books reported she was exposed to "a whole new way of learning. It was very much the visual approach, hands-on, a lot of explaining your thinking which was very different from the courses that I had just taken where it was . . . get in there and practice until you perfect this formula." But, according to Mrs. Books, the program not only involved prospective teachers in a new way of learning mathematics. She emphasized the professors also served as role models of a new way of teaching mathematics.

Mrs. Books credited the middle school program, particularly the modeling of the professors, for changing her thinking about mathematics and influencing her approach to teaching mathematics. She recognized these differences as she compared herself to two teachers with whom she taught for her first 3 years after completing the program. All three teachers had taken their Mathematics for Elementary Teachers classes from the same professor and had taken the same curriculum courses from the same professors. The only difference in their education had been their emphasis (mathematics, language arts, etc.). But, as Mrs. Books observed, "I was the only one out of the three [who] was comfortable leaving the [mathematics] textbook behind and having the kids interact. You'd walk into the other classrooms and it was a different atmosphere. The page numbers were written on the board." In explaining the differences, Mrs. Books again emphasized "it was the professors [in the middle school mathematics program] who modeled a very different way of learning [rather] than any of my other course work."

Professional experience. Upon completion of her teacher preparation program, Mrs. Books obtained an elementary teaching certificate with a middle school mathematics endorsement. For the next 3 years, she taught fourth grade at an inner-city elementary school. For the past 3 years, she had been teaching sixth grade at a middle school in the same area. Although mathematics had been her specialty in college, she emphasized she was a generalist as a teacher, particularly because she was teaching a self-contained sixth-grade class and was responsible for teaching "everything throughout the day, including physical education."

In addition to teaching, Mrs. Books had continued to work with the professors from the middle school mathematics program. She had provided a classroom setting for trying out new curriculum materials as they were written and revised. She had also led a number of workshops for elementary teachers, working to help them gain a new understanding of mathematics and a new vision for teaching mathematics.

Probability background. Mrs. Books had no memories of studying probability as part of her own school experience, suggesting it was not part of the curriculum at the time she was in elementary, middle, or high school. Because she had taken a statistics class at the community college, the probability and statistics class in the middle school program was waived for her. However, she "did go back and take the probability course later because I knew it would have a different approach than what I had done previously." Mrs. Books pointed out elements of statistics and probability had also been woven into her Finite Mathematics, Mathematics for Elementary Teachers, and Research in Education classes as well as some "different workshops and things that I have gone to."

School and Classroom Environment

The sixth-grade class Mrs. Books taught was part of a magnet program for gifted students. She and a colleague were responsible for approximately 50 students. The students were divided into two classes with some students being exchanged for different parts of the curriculum. During her math time, Mrs. Books was instructing 24 students.

The students in the magnet program were selected by means of a three-pronged test covering creativity, mathematics, and language, including verbal and nonverbal. To participate in the program students needed to pass the creativity portion and score well in either mathematics or language. Mrs. Books pointed out that for some students, their area of expertise may not be mathematics. In addition, she suggested some of the students were new entries into the program and may not have been challenged in mathematics previously.

Altogether the student body of the middle school included slightly over 700 students in grades 6, 7, and 8; as a whole, the school drew students from lower socio-economic levels. Mrs. Books estimated "probably not more than a quarter of the student body" were minority students. Within that group, there was a "mix . . . of Asians, Hispanic, and African American, [with] probably more Hispanic than African American." In the magnet program, Mrs. Books admitted her population was skewed. Although 20% to 25% of her class were minority, that group was predominantly Asian.

The two sixth-grade teachers in the magnet program did their overall planning together. However, they each taught their classes independently and exercised some freedom in carrying out the plans. During "Cereal Boxes," one of the activities observed, Mrs. Books' colleague gave her the opportunity to teach the lessons to both classes, one after the other. Mrs. Books observed, "That was fun for me. I have never been able to teach the same lesson . . . to two different groups so close together." Although it was instructive to see what different things the students came up with, Mrs. Books admitted, "I still do not choose to teach all math."

The fact the classes were self-contained permitted some flexibility in scheduling. Mrs. Books generally allotted 45 minutes for mathematics. However, the times for the observed lessons ranged from 15 minutes one morning when an assembly interrupted the math time to nearly 60 minutes another morning. The flexibility included having some time in the afternoon for the students to finish conducting their simulations on the day when the morning mathematics time had been interrupted.

Mrs. Books and her colleague shared a large open classroom, equivalent in area to approximately three usual classrooms. Their classroom, at the end of a hallway and up some stairs, was somewhat set apart from the rest of the middle school classrooms. The large area was partially divided in the middle by the teachers' desks, some bookcases, and a comfortable couch students could use during reading times. In one part of Mrs. Books' portion of the space, a carpeted open area with a wooden rocking chair provided a location where students could gather around her during special lessons. Adjacent to this open area were six round tables where the students worked in groups of four. An overhead was available near the tables. Shelves and containers along one wall provided storage for math manipulatives and other materials. Bins in the rear portion of the classroom were assigned to students for their notebooks and other school supplies. White boards were available near the open area and the area with the tables. Bulletin boards displaying academic material or student work filled most of the remaining wall space. On one bulletin board near Mrs. Books' desk, the following school principles were prominently displayed: "We respect each other. We strive to excel. We appreciate individual differences." These principles set the tone for the classroom.

Background of the Probability Unit

Setting educational goals. As one of her primary purposes in teaching mathematics, Mrs. Books wanted to help her students "construct a conceptual base of mathematics." In her mind, this goal involved more than teaching algorithms. It included helping them to see what mathematics is, to recognize where it occurs in the "bigger world," and to develop an understanding of "what's going on." In addition, she wanted the students to be able to pull together their understanding from different areas of mathematics, and combine that understanding with their reasoning abilities, to solve problems in unfamiliar situations. She also felt it is important for students to become more skilled in both written and verbal communication about mathematics. She concluded she wanted her students to become individuals who are mathematically literate, able to solve problems, and make mathematically-based decisions in their everyday world.

When asked to justify why she taught probability, Mrs. Books first cited a number of official documents that call for the study of probability in the mathematics curriculum. These documents included the *Curriculum Standards* (NCTM, 1989), the levels tests given by the district, and the essential learnings of the district's Scope and Sequence. She provided a number of other reasons, as well, pointing out "there are so many things in life that deal with probability and statistics." She also was concerned that "it is such an area of confusion for so many of us," explaining there are misconceptions and "pieces that are still being constructed." In particular, she believed "there's so much of that subjective [probability] that continues to sway our decisions, even though that theoretical is lying right there," perhaps contradicting the subjective notions.

In outlining her specific goals for their study of probability, Mrs. Books pointed out she chose "lessons that hit upon a variety of topics . . . so that the students were aware of a lot of different things." In particular, she wanted the students to become familiar with the different measures of central tendency, to learn how to set up a simulation, and to be aware of bias. She also wanted to give them exposure to "some of the language that they're going to hear." Although Mrs. Books did not expect them to become experts in these areas, she "wanted them to have some start of constructing those understandings."

Designing mathematics instruction. Mrs. Books' views of the nature of mathematics and of the process of learning mathematics were perhaps best displayed in a bulletin board she and the students had constructed at the beginning of the school year, which was still on display. The central theme was the idea that "there is a mathematician within each of us." From this thought, displayed in the center of the bulletin board, yarn emanated out to six other statements:

- Mathematics is a fascinating world of its own.
- The world of mathematics has many connections to other worlds.
- Disequilibrium is a sign of new learning.
- Learning math is a social activity.
- Learning math is an ongoing process.
- Experiences with models for math concepts help us understand, invent, and remember important math ideas.

While working in groups corresponding to each of these statements, the students had added insights or interpretations of their own.

In acting on this view of learning mathematics, Mrs. Books incorporated a variety of opportunities for students to explore what they knew and to extend the boundaries of their understanding. She reported she frequently had the students do "journal writes" where the students were asked to describe what they already understood about a topic or to explain some new understanding they may have. Rather than direct instruction, Mrs. Books guided the students' learning by asking "thought-provoking questions . . . that were not leading but would allow them to bring out some new understandings." And because she recognized students may be at different places in the process, she worked extensively within the small-group structure where she could ask "that next question" from which a particular group could learn.

The general format and instructional strategies used during the probability unit were no different from those used in other mathematics lessons. Because Mrs. Books felt learning and memory are not only impacted by seeing and hearing, but by doing, manipulatives were used extensively throughout mathematics instruction. Problems frequently set the stage for daily explorations of mathematical concepts. The students generally started working individually on the problems. As students had questions or were pushing their boundaries of understanding, they would share insights with one another or with the class as a whole. Additional problems or a follow-up activity gave the students opportunities to build upon the shared insights and observations and bring together what they had talked about during the exploration.

<u>Creating the learning environment</u>. Mrs. Books recognized creating a learning environment such as she had in her classroom is not an easy task. She emphasized one needs to "set [the] standards at the beginning of the year with, 'We are mathematicians. This is

how mathematicians operate. These are some things that will help us move forward.' " In particular, at the beginning of the year, there had been "a lot of direct instruction" and group discussion about such questions as,

> What should it look like when you have a mathematician sharing?
> What do the rest of us do?
> What would we not do?
> What does it look like if somebody is asking a question in a non-threatening way?
> How do we care for our materials?
> What kind of voice level is appropriate if we're going to be talking about something?

Mrs. Books further emphasized the need to follow through throughout the year with the standards established at the beginning of the year, which she felt she was able to do because she had embraced the constructivist philosophy of learning. Again she recognized the influence of the professors who by their modeling had steeped her in that kind of learning environment. She observed her students also picked up on the modeling she did, to the point of imitating her with statements such as, "I have a question for you," or "And how were you thinking about that?"

Although Mrs. Books was currently teaching a class of gifted students, she pointed out the same approach was feasible with students at any level, and, in fact, was the approach she used before moving into the gifted program. She observed in either setting, "you have some of the same difficulties: bored kids not wanting to listen to each other [or] where kids just practice all the time and think [that's] the level of engagement."

Planning the probability unit. Rather than teaching a separate unit on probability, Mrs. Books incorporated a number of probability activities throughout the year. In explaining why she presented the material in this way, she suggested she did not want to shove it off to the end of the year, where it frequently appears in textbooks, and then perhaps not have time to cover it. More importantly, she thought "there are so many different areas of mathematics to explore and students are attracted to different areas," much like an artist who prefers certain mediums. Therefore, Mrs. Books believed incorporating a variety of different topics throughout the year "presents more of a well-rounded picture" of what mathematics is, and gives students the opportunity "to keep in touch with themselves as a mathematician in an area [for which] they have a passion." In addition, she pointed out when you "have time away from something, your mind continues to work on it. And something that wasn't clear the first time . . . when we look at it in a new situation, at a different time, all of a sudden that piece falls into place."

Although Mrs. Books believed the students should have received some instruction in probability before coming to sixth grade, she recognized probably only half of them did have any prior experiences. As a result, she felt their background knowledge was limited to simple situations and subjective notions of chance. Mrs. Books identified the "tie-in with the subjective" as one of the difficulties students face with probability. She added "they struggle with what they believe in their mind" versus what the experimental and theoretical evidence shows. Further, she suggested the students "know that this is theory, this is what should happen, yet it's that gut feeling" that sometimes stands by another outcome.

Although the students were not using a specific textbook, Mrs. Books was generally following the *Visual Mathematics Course Guide* (Bennett & Foreman, 1995). For the probability lessons, Mrs. Books had chosen to use four activities from the related Math and the Mind's Eye materials, *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993). With the background of the students in mind, Mrs. Books had used two of the activities, "Sampling, Confidence, and Probability" and "Experimental and Theoretical Evidence" (the River Crossing game) earlier in the year in order to introduce some of the basic notions of probability and to provide an opportunity for the students to begin looking at some of their subjective notions. She decided not to do a second sampling activity "because it was similar to the first" and she wanted to show the students "a variety of different situations" in the time that was available. Further, she decided not to do the two games of checkers because those were more "pure" probability and she "wasn't quite confident on how much at this point, with their experiences, [the students] would be able to get" from those activities. She was also aware the students might have an opportunity to do those activities the following year in their seventh-grade class. Mrs. Books chose to do the remaining two activities, "Cereal Boxes" and "Monty's Dilemma," because she felt both activities would really get the students involved. She pointed out, in particular, that the disequilibrium between initial guesses and experimental results in "Monty's Dilemma" generally creates "a lot of interest."

In these last two activities, Mrs. Books explained she deviated from the lessons as they were laid out because she wanted to emphasize a different aspect. In particular, the lessons specified "how to carry out the experiment," but she wanted the students to "struggle through" designing their own simulation. She pointed out "by making that decision it opened up that whole area of bias that would not have been a major piece had I chosen to tell them what to do."

The Probability Unit

Two of the activities, "Sampling, Confidence, and Probability" and "Experimental and Theoretical Evidence" (the River Crossing game), had been taught in late fall and early spring before observations were made for this study. The final two activities, "Cereal Boxes" and "Monty's Dilemma," were observed in early June. Several days were spent on each activity, as outlined in Figure 8. As a sample of the probability instruction, the two observed activities will be described.

"Cereal Boxes"

Wednesday	Thursday	Friday
Introduce problem, Initial predictions, Design simulations	Conduct simulations	Analyze data with box plots, Assign follow-up

"Monty's Dilemma"

Monday	Tuesday	Wednesday	Thursday
Introduce problem, Initial predictions, Design simulations	Discuss bias, Finish conducting simulations	Review data, Assign letter	Complete letters

Figure 8. Calendar for Mrs. Books' probability unit.

"Cereal Boxes." Before presenting the new problem, Mrs. Books had the students recall the steps of the process they had used in answering the question in the River Crossing game done earlier in the year. In particular, the students had started off by making a subjective estimate or an initial guess. They had then experimented by playing the game a number of times, refining their guesses as they did so. Finally, they had looked at the "theoretical piece . . . where you took your results and you looked at it through arithmetic and decided what that was going to look like."

Following this introduction, Mrs. Books presented the students with "Cereal Boxes," a problem from *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993).

> General Mills company once included bike racing stickers in their boxes of Cheerios. There were five different stickers. Each box contained just one of the five stickers. How many boxes of Cheerios would you expect to have to buy in order to collect all five stickers?

Before the students made their guesses, Mrs. Books encouraged them to think of "questions that would need to be answered before you make that guess." The students asked a variety of questions clarifying their understanding of the problem and determining common assumptions about the problem. In particular, as a result of their questions, it was decided every box in the store would have a sticker in it, but no indication would be on the box of which sticker was inside. Further, it was assumed that there would be a random mix of the stickers at any time and that the five stickers would occur an equal number of times. Finally, the assumption was made that the shelves would continue to be restocked, maintaining the random and even mix of stickers.

With the assumptions clarified, Mrs. Books next had the students write down their guesses and a brief rationale for their guess, without sharing these with anyone else. Then, in a round robin fashion, she had the students share their guesses with the whole class, as she wrote them on the overhead. Mrs. Books expressed surprise that the students at some of the tables were in agreement even though they had supposedly not shared their guesses with one another. The class then summarized their guesses by observing that the mode was 10 and the range was from 5 to 50.

At this point, Mrs. Books moved on to the task of having the students design a simulation, although she did not explain why a simulation was necessary or important in this case. She reminded the students that the simulation should meet the assumptions of their earlier discussion. She also cautioned them that the simulation should be as fair as it could be, without the results being "biased or skewed." A variety of math materials was available for the students to use, including tiles, dice, beads, spinners, egg cartons, sacks, and cups. As the students at each table discussed how they might simulate the problem, Mrs. Books circulated among the tables asking students to explain their ideas to her.

After taking approximately 5 or 6 minutes to discuss their ideas within their groups, each table was asked to share their ideas with the class. Mrs. Books reminded the students of their responsibilities as they listened to their fellow classmates, asking them to think about "mathematically if their design is going to work," and to consider how the "information that you glean from them [might] cause you to want to maybe make some changes in how you would run your own simulation." Most groups shared plans which concretely matched the selection procedures, using beads as the stickers and cups or sacks as containers from which the beads were selected. Some of these groups would have five beads in the container; others would use a multiple of five beads. As another strategy, some groups suggested using a modified shell game with beads covered by cups. Even though they did not share the idea of using dice, Mrs. Books recalled one group had been discussing that possibility and asked the group to explain that idea. In particular, the students discussed what would happen if the die landed on the sixth side when they only had five stickers. Although the students suggested they would roll the die again, not counting the 6 as a trial, not all students were convinced the probability would be the 1/5 they thought it should be. Mrs. Books left that as a point for the students to think about, without explicitly mentioning the equally likely nature of the outcomes.

To this point, Mrs. Books had generally been following the recommendations of the instructional materials for teaching the lesson. However, she chose not to follow the recommendation to have the students use dice to carry out their simulation. Instead, she wanted to allow the students to use their own simulation designs to collect data. With a final caution that small beads could get lost underneath the flaps of the sacks, Mrs. Books gave the students a sheet on which to tally their results and explained how they were to record their results on the line-plot of class data. For the approximately 16 minutes remaining of class time, the students began to conduct their simulation. Most followed through on their plans to select beads from a container or to conduct some version of a shell game.

As the students were conducting their simulations, some individually and others in pairs, Mrs. Books continued to circulate among the groups. She generally asked questions to clarify how the students were doing their simulations or to guide their decisions. With one pair of boys who had a fairly complicated scheme of choosing numbers, she asked if they thought they were getting a random mix. And with another group, when she noticed they had drawn a bead and set it aside, she asked, "How come this bead's laying here?" Without giving the students any answers, Mrs. Books used questions to guide them in considering whether the beads should or should not be replaced. One student suggested you would not replace a box once it was opened, but other students realized the bead should be replaced because the boxes were being restocked. Even at the end of the discussion, Mrs. Books gave no indication of what was correct. She instead asked if replacement made sense and if they agreed with one student's observation that "you need those same odds every time."

Mrs. Books raised the question of replacement with the whole class as the lesson began on the second day. After describing two experimental methods, one with replacement and the other without, she asked the students to decide individually "which of those two styles is going to give you the most accurate information based on the conditions that we put on our experiment yesterday." A show of hands indicated a few who felt replacement did not matter and many who felt the bead needed to be put back in the container. Mrs. Books then asked one from the latter group to explain why they felt the beads needed to be put back in. After this student referred to the assumption that the shelves were being restocked, another student pointed out if you selected from a container with only five beads and did not replace them, then you would "automatically get all of them" in five draws. This observation led to a related question, "If you put three of each color in or 100 of each color in, what is the maximum draws that you would have to get all five?" After briefly considering this question with the students, Mrs. Books returned to the issue of replacement. She next asked one of the students to explain why she felt replacement did not matter. When that student said she had changed her mind, Mrs. Books asked, "And what was the deciding piece for you? What helped you?" The student suggested it "just kind of clicked" when the students had talked about always restocking the shelves.

Before allowing the students to continue the process of gathering data, Mrs. Books discussed with them what they could do once they had completed gathering the data.

Students suggested such ideas as finding the average, median, mode, or range. Mrs. Books also asked if anyone was going to change the "style of their experiment from what they started yesterday." Some groups suggested they were going to replace the beads they had drawn out, while several other groups suggested they were going to roll dice. When asked why they were going to use dice, the students generally replied that they just wanted to try a different way. The students were given approximately 25 minutes to continue conducting the simulation and to record their results on the line-plot of the class data. As she had the day before, Mrs. Books circulated among the tables during this time, asking questions. Besides checking the details of the simulation designs, Mrs. Books also asked the students if there were "any pieces of unexpected data coming up." For one group, Mrs. Books raised a question about something that had happened when she taught the lesson to the other class. One group, who had been drawing slips of paper, drew the numbers 1 through 5 in order. Mrs. Books wondered, "What's the probability of that happening?" With another group she asked if the roll of the die was random or if they knew how to flip it so it would land a particular way. When one group reported they were surprised when they had to roll 28 times before getting all five stickers, Mrs. Books asked how that number might impact the different averages they were to look at.

As the students continued to record their data on the line-plot (see Figure 9), the number of 5s was going off the chart, a result that troubled one of the students. She did not think people would have gotten 5 all that often. At Mrs. Books' encouragement she began to investigate the other tables to find out where all the 5s were coming from. This question was raised later with the whole class as they realized the mode of their class data was 5. A couple students suggested they agreed with the first student's concern.

As class time was coming to an end and students were putting away their materials, one group of boys began to explore a related problem. They were wondering how many complete sets of stickers they would obtain if they took the container that was full of dice and dumped it on the rug. Mrs. Books suggested the students could come in during lunch time, if they wished, to try their experiment and report their results to the class.

As the third day began, one student again raised the question of the number of 5s in the class data. She had looked at the results for the other class and observed "their 5 plots were a lot lower than ours." Mrs. Books commented the other class had fewer data overall because the students had worked more as tables. Many students suggested, however, the data from the other class was more representative of what they expected to find. After observing that there were two modes for that data, 7 and 11, Mrs. Books reminded the students such a situation was called bimodal.

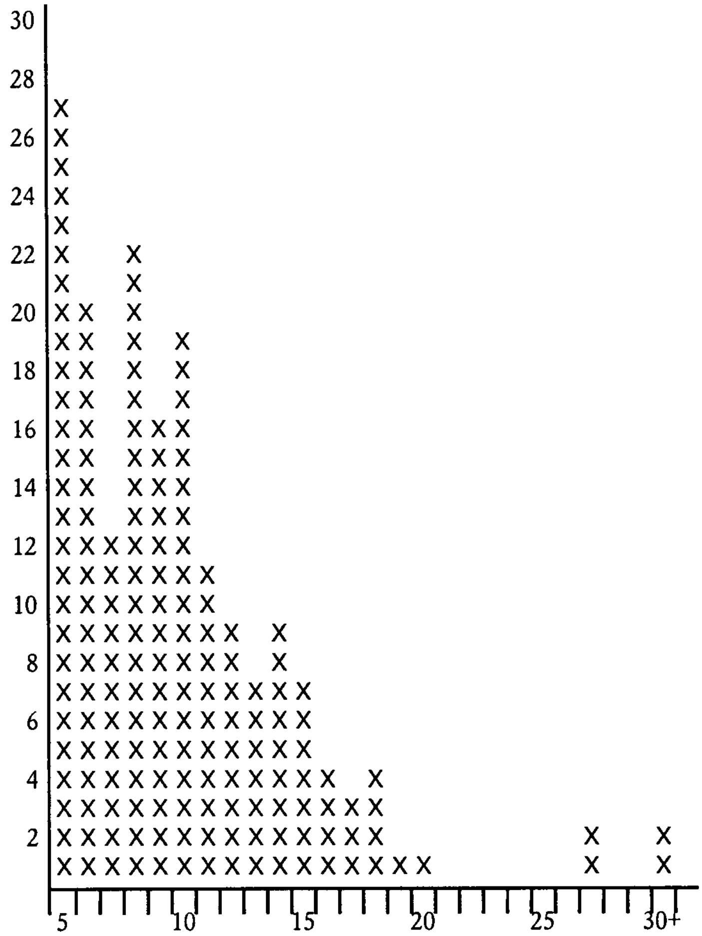


Figure 9. Line-plot of "Cereal Boxes" data for Mrs. Books' class.

Mrs. Books then asked the students what type of information they could "glean from looking at a line-plot." Students suggested one easily could find the mode, median, and range. Mrs. Books also observed the line-plot provided enough information to find the arithmetic average or mean. Other students pointed out you could see where the "basic core" of the data was as well as seeing any data that "stick out there."

Mrs. Books proceeded to introduce "another style of recording information," the box-plot. She asked the students, "How many boxes of cereal would you tell somebody that they would need to buy in order to be 90% certain [of having] gotten all five stickers?" After students shared their responses to that question, she asked what number they would give to be 50% sure. She then asked them to write down their 90% and 50% prediction and discuss their predictions at their tables.

Using a smaller set of data as a sample, Mrs. Books demonstrated how to construct a box-plot. After agreeing there were 19 pieces of data, she asked the groups to determine a strategy for "finding out where 90% of that data is." When one group suggested dividing 19 by 0.9, giving a result of 21.1, Mrs. Books encouraged the class to "think about this." By asking how many 90% of 10 and 90% of 9 were, she led the class to mentally estimate the result to be about 17. One student then suggested they multiply by 0.9 instead, and the class agreed this result of 17.1 was more reasonable. Beginning at the lower end of the data, Mrs. Books counted up to the 17th piece of data and drew a line at that point which was at the value of 16. She then asked the students if "90% of all our data would be on [the left] side of that line." Some students were confused, thinking they were looking for a given value rather than an interval of values. Mrs. Books suggested they "keep looking at this," as she proceeded to guide the class in finding the lowest value and the median. Using these values, Mrs. Books drew the box-plot, introducing the idea of a whisker reaching to the data that were "sticking out there" and called outliers. Mrs. Books then concluded that "if we went to the store and we bought 16 [we] would be 90% sure that we'd get them [all]." In response to the question, "But the most likely number that we would have to buy is what?" one student replied 11, the median, and Mrs. Books seemed to agree.

Mrs. Books then proceeded to provide further examples of box-plots, drawing both an 80% box-plot and a 50% box-plot. After calculating that 80% of 19 pieces of data was 15.2, Mrs. Books drew the 80% line at 15 rather than counting up to the 15th piece of data. Similarly, on the 50% plot, she drew the line at 9.5 or 50% of 19 rather than counting up to the 10th piece of data. When this line happened to fall below the median of 11, she observed, "I don't know that I've ever seen that." Because the median represents the value which has 50% on either side of it, the line with 50% of the data below it should have fallen at the median value, not below it. Mrs. Books, however, did not realize her mistake.

She then gave the students the assignment of combining their data with one other person's data and drawing a 90% box-plot for the combined data. As the students worked on the assignments, Mrs. Books circulated among the tables. At least two of the students had been confused by her error on the 80% and 50% box-plots and needed clarification to correctly consider the number of pieces of data. In addition to questions helping the students construct their box-plots, Mrs. Books asked whether the students had whiskers or outliers. Several students were disappointed they did not have whiskers on their boxes. This reaction led Mrs. Books to ask why some data did not have whiskers.

Towards the end of the class time, Mrs. Books directed the students' attention to the student box-plots that had been drawn on the overhead. At this time, she provided a more complete definition of outlier, suggesting it would be any data that is more than a box-length away from the box. In providing the definition she inaccurately referred to the length of the box as the range of the data.

In conclusion, Mrs. Books had the students consider the kind of data they could "gather from the box-plots as opposed to the line-chart [*meaning line-plot*]" they had drawn earlier. Given a simulation with 10,000 pieces of data, some students suggested they would prefer to see a line-plot while others preferred a box-plot.

To conclude the activity the students were given a homework assignment and an extra-credit assignment. As homework they were to respond to two questions (Shaughnessy & Arcidiacono, 1993):

1. Now that you have completed the cereal box activity, how many boxes would you need to buy to collect all 5 stickers? Explain your reasoning.
2. We have used both line-plots and box-plots to visualize the data from the cereal box simulation experiment. What are some advantages of each of these types of plots? What are some disadvantages? Explain.

On this assignment, Mrs. Books reminded the students she was looking for a "comprehensive" explanation, a paragraph or two that showed their thinking. The extra credit assignment asked the students to do the same type of thing if there were six stickers in the cereal boxes.

The lesson on the third day had lasted nearly 1 hour, although as it began Mrs. Books had indicated they would spend 30 minutes on the lesson. Afterward, she observed she had been surprised that the box-plots had taken so long to complete. Some students, however, had finished before the end of the hour and had been wandering around with nothing to keep them busy.

"<u>Monty's Dilemma</u>." In introducing the next problem, Mrs. Books pointed out "it is a problem that people are curious about" and one that "generates a lot of conversation," even among university professors. She said she had seen it written up in several magazines and even discovered a similar problem on the Internet. She jokingly suggested, "This is a tough problem, especially for those of you that spent your early years at home watching 'Let's Make a Deal.' " From their responses, it seemed several students were familiar with the show. She then presented to the students the following problem called "Monty's Dilemma" (Shaughnessy & Arcidiacono, 1993):

> There is a TV game show in which the contestant is asked to choose one of three doors. Behind one of the doors is a whopping big prize and behind the other two doors are gag prizes.
>
> After the contestant chooses one of the doors, the game show host reveals what is behind one of the other two doors, always showing a gag gift. Then the contestant is presented with the following dilemma: Would you like to keep the door you chose or switch to the other (still veiled) door?

As Mrs. Books had done with the earlier activities, she began by having the students consider the subjective probability. She asked the students to "write down what you would be most inclined to do, and why you would be most inclined to do that," without showing it to anybody else. Most students were familiar enough with the game to understand the problem. Some students were not clear about what was shown and what was not shown; they asked questions and Mrs. Books provided clarification. After the students had taken 2 or 3 minutes to write down their subjective responses to the problem, Mrs. Books had them share these responses with the other students at their table. Then Mrs. Books suggested they try to reach consensus across the room. She asked those who were "going to be proud and stick with that door that you first chose" to stand. Most of the students stood, although there were three or four who said they would switch.

Mrs. Books next brought up some of the ideas she had heard from the conversations at the tables, saying, "The other thing that came up, that I heard a lot as I wandered around . . . there came up 50-50, equal chance, 33 and a bit. What are those things? How come those are things that you're talking about? How might they be helping you make your decision?" Some students suggested "when you take one of the doors out, it could be . . . either of the other two doors" and that 50-50 meant either door could be half and half. Other students pointed out that because there were three doors originally, "you have a 33% chance of picking the right one . . . on your first chance."

After summarizing their thinking, Mrs. Books moved on to summarize the Stick and Switch strategies and add a third approach, the Flip strategy (Shaughnessy & Arcidiacono, 1993).

> Let us pose a mathematical (probabilistic) problem from this dilemma. Which of these three strategies is most likely to lead the contestant to the winning door?
>
> 1) Just stay put and keep the original door you chose, after the door to the gag prize is opened (STICK strategy).
>
> 2) Choose again by randomly selecting a door from the remaining two closed doors (FLIP strategy).
>
> 3) Choose again by switching from the door you chose to the other closed door (SWITCH strategy).

When asked how they might simulate a random choice between two doors, the students offered a variety of suggestions including flipping a coin or playing "eeney, meeney, miney, moe."

Mrs. Books again had the students reflect on the strategies, choose the one they would use, and write down "a statement as to why you think that that strategy is the one that's going to get you that whopping big prize." The students were encouraged to "use some mathematical terms . . . to help communicate to somebody else why you believe that."

When they had finished writing down their choice, the students were asked to share their choices and reasons at their tables. As the students were discussing, Mrs. Books circulated among the tables. After a minute or two of sharing at the tables, Mrs. Books suggested as the next step they wanted to collect some data to see which choice might be the best. She asked the students to discuss at their tables how they might devise a simulation. In less than a minute she gathered the students back together to share their ideas. One group had an idea "kind of like the shell game," putting slips of paper under the cups indicating whether it was the big prize or gag prize. Most groups had ideas similar to this group, although another group wondered if recordings of the program could be reviewed and the choices of the contestants and the results observed. Mrs. Books offered two further suggestions she had seen used. One was to roll a die, with the big prize always behind two of the numbers. The other suggestion was to use a spinner divided into three equal sections, although one student recalled using spinners with paper clips had not worked well when tried previously. With those possibilities to think about, Mrs. Books offered a final word of advice: "As you design your experiment, you need to try to rule out any bias where somebody may have some information that they're using, that it really is a true random pick when you start picking one of the three doors. And that, when you do the Flip, it is random." With an explanation of what materials were available and the demonstration of a recording sheet they were to use, Mrs. Books turned the class free to work on the assignment of individually doing 100 trials for each of the three strategies. A number of students raised questions about how they could do their simulations alone, because "if you know which are the bad prizes, you can always pick the good one." For those students whose design necessitated two people being involved, Mrs. Books instructed them to do 200 trials of each.

As the students began to conduct their simulations, Mrs. Books circulated among the tables questioning students about their designs. Although a few students were trying something with dice, most were concretely acting out the situations with cups marked prize or gag or with cups covering beads or slips of paper indicating the prize doors. Mrs. Books indicated to the researcher she had chosen not to use the spinner recommended by the instructional materials because she was not sure the students were ready for that abstract a representation. She was perhaps a little surprised at how concrete their simulations were, commenting they "really like the cups."

But as at least one student was beginning to realize, acting out the situation was going to take a long time, particularly if they had to do 100 trials for each strategy. One pair of students, however, seemed to have found a way to overcome that difficulty. As Mrs. Books passed by, they suggested, "Now all we have to do is add up these." Noticing they did not have 100 tallies, Mrs. Books discussed with them what they had done.

T: So, how did you do this without the tallies?
S: All you have to do is do it five times, and then . . . multiply it.
S2: Wouldn't that be much easier, and then you have 100 times?
T: Why do we do things more than five times and multiply it?
S: 'Cuz you get more randomness?
T: Aah. If we had stopped after five simulations on "Cereal Boxes," would we have had good data?
S: No.
T: Why not?
S: Next time you could have gotten a 28 or something.
T: Hmm. So, does doing it five times and multiplying it, how does that impact your data?
S: I think we should just do it 20 times, then multiply it.
T: Is that going to be random?
S: Yeh.
T: When you report to your audience to tell them which strategy is best, . . . are you able to tell them what 20 trials is like or what 100 trials is like?
S: Twenty.
T: 'Cuz does multiplying it by 5, does that really give you what 100 trials would look like?
S: I think so.
T: You think so?
S2: Let's keep doing it.
T: That would be interesting.

Thus realizing the error of their approach, the boys began conducting additional trials for the different game-playing strategies.

In addition to the concern about time involved in conducting the simulation, Mrs. Books had a concern about the possibility of bias. When an assembly unexpectedly shortened the math period on the second day, Mrs. Books decided to have a 15-minute discussion with the class about randomness and bias.

T: There's a question for you, and that is, that's the whole issue of randomness . . . and bias. And I notice that different things were going on in the classroom. Some of you were choosing to use the dice to conduct your experiment. Some of you were doing kind of like a shell game, where you had one cup that had the big prize in it, and the other two were the gag gifts, and you were doing some shuffling, and another person was turning over the cup. I would like for us to talk a little bit about types of bias that might be built into either one of those. So, it's really doing some thinking. Are they as random as what we would hope that they would be? Is there another way that we might guarantee that we have that randomness? Can anyone think of anything that could be biased with either the dice or the type of shell game that we used?
S: What do you mean by bias?

T: Bias means that it's not going to be truly random. That there's something that's going into the factor that is going to shift the results away from what we would see if it was truly without bias and truly random.

One of the students who had wanted to do it five times and multiply those results pointed out their approach would not give you "the randomness like . . . 100 trials." Other students pointed out with the cups there might be some way of distinguishing between the cups, giving clues as to which was the big prize.

Mrs. Books went on to ask, "Are you as individuals, do you think that you are truly random when you go to select . . . so that one third of the time you'll pick 1, one third of the time you pick 2, one third of the time you pick 3? Or do you think that there is some bias in how you select?" In response, one student offered the following caution: "The thing that I did is that I had the cup . . . if I'd gotten the prize before, then I'd pick the same cup, and sometimes I'd get it right and other times I would have lost. And so, in that way, if you're the person that's switching around the cups, don't leave the prize in the same spot."

After realizing there could be some bias present in how they were doing their simulations, Mrs. Books guided the class in considering other ways of simulating the choice of the doors. One student suggested slips of paper drawn from a paper bag and Mrs. Books asked about using colored tiles in a similar fashion. Initially several students were concerned you would "know what you picked." However, as Mrs. Books and the students acted out the different strategies, many of the students began to realize knowing the initial result did not matter. Guided by Mrs. Books' questions, the students began to shift their thinking from acting it out to reasoning it out.

T: Okay. Imagine you've picked a gag prize. They're automatically going to show you the other gag prize, right? You have the gag prize out here, and the door that you haven't seen has the big one, right? And so, you flip your coin. If it says, "Stay," then you have the gag prize, right? If you flip a coin and it says, "Switch," now you know that you have the winning one. Okay, now let's look at . . . that third one is that you automatically switch. So if you pulled a gag one out of the sack . . .
S: You have to switch.
T: Have you won or lost?
Ss: Lost (several students in unison).
T: If you have to automatically switch?
S: No, you've won.
T: If you pull the losing one out of the bag and you have to switch . . .
S: You've won.
T: You've won. If you pull the winning one out of the bag and you have to switch, you've lost.

When questioned, some students believed this approach would have less bias. Others were still concerned it would have "more bias because when you pull it out, it's easier to see" which prize you have chosen. As they acted out the process again, the students seemed to realize the decisions were based on the strategy or the random flip, not on the knowledge of what the first selection was.

In a similar way, Mrs. Books asked the students whether using dice or a spinner would give biased or fairly unbiased results. Some students expressed concern that things such as a slanted table or knowing how to spin it so it lands in a particular place might lead to biased results from the spinner. Mrs. Books observed it would be interesting then to compare their results with what the other class obtained because the other class had used spinners (as recommended by the instructional materials).

Mrs. Books concluded the questions they had been discussing were important questions to consider. "As you're doing any type of simulation, you have got to, if you're reporting back to your client . . . or to a group . . . you have to tell them what things you did to try to rule out bias." As the students continued collecting data, they were encouraged to "try to rule out any possible contamination or bias that you have in your experiment." With these new ideas and possible strategies to think about, the students finished conducting their simulations that afternoon.

As the students were conducting their simulations, it appeared many of them had moved from concretely acting out the situation to more abstract representations that yielded more random results. Many were drawing tiles from paper bags, one suggestion made during the discussion. Another student was rolling two dice, one determining which door contained the big prize and the other indicating the initial choice of the contestant.

As the students had also become more familiar with the problem, several had begun making some observations. For example, one pair reported that "as soon as Jeremy and I started to do the Stick, we realized there's 33 . . . . It was simple. And then the Switch, we realized that . . . if you got the prize the first time, the real prize, you'd lose. That was the only way you could lose." So, although they had thought that each strategy would have been 50-50 at the beginning, they had changed their minds after the first few trials.

Mrs. Books began the class on the third day by asking, "How many of you did change your experiment based on the discussion that we had about bias yesterday morning?" When five students indicated they had made changes, Mrs. Books asked them to explain how they thought they had ruled out bias. As they reported, Mrs. Books asked them to stand and reminded them they were talking to their classmates, not just to her.

Three of the students had been using cups that in one way or another provided clues to which one was the big prize. By switching to choosing tiles from a bag, they felt they had removed bias. Another student who also decided to draw prizes from a bag felt she had ruled out bias in hers except for the Switch strategy. In that case, it appeared she was uncomfortable because the results were not what she expected, rather than "thinking that something was rigged." After another student explained how she clarified some confusion she had encountered, Mrs. Books asked, "How did you feel when it finally came together? Was that pretty good?"

The class next moved to consider the experimental results which had been recorded on a transparency (see Figure 10). As the students examined the results, they were asked if they thought the data were "pretty consistent with what we should see" or if they thought some data might be "suspect" or biased.

STICK		FLIP		SWITCH	
Won	Lost	Won	Lost	Won	Lost
38	62	56	44	60	40
31	69	63	44	70	30
79	121	100	100	127	73
27	73	49	51	68	32
37	63	44	56	59	41
43	67	48	52	59	41
32	68	50	50	68	32
36	64	47	53	66	34
41	59	42	58	27	73
36	64	55	45	51	49
41	59	47	53	65	35
27	73	62	38	35	65
100	0	55	45	70	30
30	70	54	46	74	26
47	53	46	54	61	39
29	71	35	65	66	34
30	70	37	63	66	34
44	56	55	45	60	40
40	60	37	63	66	34
44	56	56	44	34	34
31	69				

Figure 10. Display of "Monty's Dilemma" data for Mrs. Books' class.

In looking at the data for the Stick strategy there was consensus among the students that there were some problems. Mrs. Books asked students to indicate which data they thought were a problem. After one student pointed out 100 wins and 0 losses was "a little bit unlikely," Mrs. Books asked the other students to rate the likelihood of that result by showing one to five fingers (1 being a little bit unlikely and 5 being very unlikely). She then asked the student who had reported that result if he felt it was unlikely. The student agreed it was and explained he had made tallies whenever he rolled a 1 or 2 which represented the winning door, but admitted he had not tallied the 3, 4, 5, or 6. When asked how many wins he thought he would have gotten if he had tallied every roll, he suggested "probably around 30." Mrs. Books realized then the student had been trying to tell her that before class began, but she had not had time to look at it. She thanked him "for coming up and sharing that one" because it gave them "something to look at to see if we thought there was some bias there."

One student then suggested, "There's supposed to be one-third chance of winning on the Stick, theoretically." When asked to clarify what he was talking about, the student went on to explain "if you're sticking, there's two chances you could lose, and there's one chance that you can win . . . out of the three" because "it doesn't really matter . . . that they unveil one, that they take one of the nonprize ones away because they're going to be sticking with [the first] one, so it's still a one-third chance."

Mrs. Books then asked the class if the data, other than the 100 and 0, supported this suggestion. In further consideration of the data, she asked the students to raise their hands if they thought the results were reasonable if out of 100 trials the number of wins was 34, 33, 32, 31, . . . , 25. When she stopped at 25, one student still had his hand raised. She then asked a similar series of questions about their expectations of the results if they combined all the data into 2,100 trials or if they were to consider the results of 5,000 trials.

In a similar fashion they considered the results for the Flip strategy and the Switch strategy. In addition to questioning results that seemed quite discrepant from expected results, some students also questioned results that were exactly as expected. For example, one student questioned the Flip results of 100 and 100 (combined data of two students) "because it's very unlikely that you'll end up half and half the whole way. . . . I can understand even for 50-50, but if you went a whole 'nother 100 times . . . I just couldn't see 100-100."

In response to the suggestion of an addition error in one of the results, one student explained he had done more than 100 trials by accident. As this exchange began to escalate into an argument, Mrs. Books thanked both of the students, the one for his explanation and the other "for looking at the data carefully."

After reviewing the data for all three strategies, Mrs. Books asked the students to recall their initial choices. Fifteen indicated they would have used Stick and three would have used Switch. She then asked if any of them "would do any modification based on your trials." By a show of hands, no one indicated they would stick, one indicated he would flip, and the rest indicated they would switch. When the one student was asked why he would flip, he suggested "it would be funner [sic]."

As the final assignment of the activity, the students were asked to "write a letter to [the researcher] discussing your initial prediction and your reason. Then discuss how you conducted your experiment, any bias, your results, [and] how you would play the game. Use mathematical language to communicate." The students had about 6 minutes to begin working on the assignment and time the next day to finish up their letters.

Evaluation of the Probability Unit

Evaluating student learning. Mrs. Books explained much of the day-by-day assessment of the students' learning had been done through observations. As she went around to the tables and talked with the small groups, she was able to notice "where individual students were struggling . . . or how they were approaching things and what level they seemed to be thinking at." The letters written at the end of Monty's Dilemma and other such follow-up activities helped complete the picture of what students were learning.

Evaluating unit effectiveness. In reflecting on what the students had learned from the probability activities, Mrs. Books observed the students discovered that probability is a "whole big area." As she had hoped, the students had begun to recognize "where some of those probability things come up" in their daily lives. In particular, some of the students observed the assumptions they made in the "Cereal Boxes" activity were not realistic from a marketing standpoint. They realized companies often send different prizes to different areas of the country, making it more difficult to obtain a full set. They also suggested because of "how things come off the line," it would be more likely to have a "whole slew of one type of card . . . boxed together."

Although there had been "very little direct instruction," Mrs. Books thought the students "learned a lot about how to set up a simulation" and realized more about the danger of bias. In fact, Mrs. Books believed the students might question teachers in the future who gave them a specific way to do an experiment. As a result of checking their own bias, she felt they would question the accuracy of certain approaches and "come up with some other ways . . . that it could be done."

Mrs. Books concluded the probability activities had been effective in meeting her goals. She observed students had begun to pull different pieces together. In particular, she pointed out the "level of discussion that I was able to have on bias with 'Monty's Dilemma' . . . was there because of prior experiences with bias." Mrs. Books felt the students also had begun to see "how mathematics [is] to be used in probability" and to make connections between probability and fractions and the area model. However, she observed "a big idea that I don't think is there but . . . I would like for them to continue to work on is that . . . confidence, how confident are you . . . of your answer? . . . [I would like] for them to struggle some more with that."

Mrs. Talent: Influence of Reform Efforts

The next teacher to be introduced is Mrs. Talent, who had been a mathematics teacher at the middle school level for the past 14 years. As a result of course work and other professional opportunities, Mrs. Talent was becoming aware of aspects of the reform happening in mathematics education.

Teacher Background

School experience. Mrs. Talent recalled her early experiences with mathematics as being very confusing and stressful, suggesting she "probably came through the new math of the 60s." In particular, she remembered "sitting at the kitchen table, practically in tears," as she tried to complete worksheets picturing bundles of sticks. Because she did not understand "what was going on" and her parents could not help her, she recalled becoming very frustrated, concluding that she "was never any good at math."

In preparation for an algebra program her school was starting in eighth grade, Mrs. Talent was tested when she was in seventh grade. The scores from that test reportedly fell into two fairly distinct groups, with the students in the higher group being assigned to the algebra class. The parents were later told that two students who had scored in between the two distinct groups also were put into the algebra class. Mrs. Talent had always figured she was one of those two students, because, to her the algebra class was "a fog." Although she "muddled through" with a C grade, she did not feel she had been ready for algebra. She recalled not really having "a clue what was going on" in the algebra class, even though she "could kind of manipulate the symbols."

However, when Mrs. Talent was passed on to the next level and took Geometry as a high school freshman, things "really clicked" for her. From then on, she had done "really well" in mathematics. She went on to take Algebra II, Trigonometry, and Calculus, receiving A's in each of those classes. Her exceptional performance earned her a math award as a senior and a partial college scholarship based on mathematics. Because of her mathematics background, she recalled that "everyone was pushing me towards engineering."

Teacher preparation. However, for as long as Mrs. Talent could remember, she had always wanted to follow in the footsteps of her sister (who was 14 years her senior and a kindergarten and first-grade teacher) and become a teacher. Even when Mrs. Talent was quite young, she had held school in her basement, writing workbooks, textbooks, and even teachers' editions for her classes.

To pursue her dream of becoming an elementary school teacher, Mrs. Talent attended a private teachers' college in the Midwest, which specialized in preparing elementary school teachers. She completed a double major in elementary education and mathematics. As part of the mathematics major, Mrs. Talent recalled taking college algebra, calculus, statistics, and problem solving. When she "ran out of math classes to take," other classes had been created for her to do by independent study. In one of these classes, she had written a unit for teaching the concept of area.

Professional experience. After completing her teacher education program, Mrs. Talent moved to the Northwest where she had been teaching for the past 14 years. During her first year, she worked in a talented and gifted program for fifth and sixth graders. In this program she serviced 12 different schools in a metropolitan district on "an itinerant-type schedule." She then spent 7 years teaching seventh- and eighth-grade mathematics at one middle school in a smaller, nearby town before moving to her current position where she had been teaching seventh- and eighth-grade mathematics for the past 6 years.

Mrs. Talent was in the process of completing a master's degree in mathematics education. In addition to summer classes, she had been taking one course a quarter through a nearby state university's long-distance education program. From a location in her area, she was connected by satellite to the professor and other students in the class. Some of these classes focused on teaching algebra, geometry, and probability at the middle school level.

In addition to her teaching responsibilities, Mrs. Talent was becoming involved in leadership roles in the mathematics education community. She participated in a special program communicating with teachers and parents about statewide reforms in mathematics education. Further, one night a week she was teaching a mathematics class for instructional assistants offered by a local community college.

Probability background. Mrs. Talent did not remember "learning very much probability at all, anywhere, ever" until she taught it to herself. She was quite certain she had not taken a course in probability as part of her preservice teacher education program. She remembered taking a statistics class, describing that as "probably the hardest class I took." But if she had studied any probability, she could not recall when or where or anything they had covered.

Learning that probability was an area she was supposed to be teaching and recognizing she was not "strong in it," Mrs. Talent began to go to "all the workshops that had anything to do with probability" at regional mathematics conferences. At these workshops, she picked up many activities and became excited about the activities because they were fun. She would try the activities out on her students, but she realized this method "was real disconnected." In addition, she began ordering books and other materials related to probability. In the last 10 years, she had built on what was in the students' textbook until she had come to a point of no longer using the textbook, suggesting, "I just kind of wing it now with probability and I still don't feel that I'm great at it, but I kind of know where I want it to go now and what I want them to be able to do. And I've done enough reading and work on my own that I think I can get them there."

As part of her master's program, Mrs. Talent had taken a course on teaching probability at the middle school level. Mrs. Talent suggested the class had given her "an overview of different ways to approach probability." In particular, they had done a number of experiments with dice and coins, considered whether games were fair or unfair, and discussed ways of analyzing experiments with tree diagrams and other strategies. Mrs. Talent reported that, for her, the class helped "solidify all the little pieces I had pulled together for myself and told me that I was on the right track and I was doing the right thing because a lot of what we did in the class I was already doing."

School and Classroom Environment

Mrs. Talent had been teaching for the past 6 years in a middle school located in a small rural community. The student population included only seventh and eighth graders of which there were 350 students total. Although welcoming instructions to visitors on the front door of the school were printed in Spanish and Russian as well as English, Mrs. Talent reported most of the middle school students were Caucasians. In particular, she pointed out the few Russian and Hispanic students who attended the school came from families who had been in the area for a number of years and spoke English quite well. However, all different economic and social backgrounds were represented in the school population.

Assignments of students to particular classes were based on a series of tests given to the incoming sixth graders. The students scoring above a certain level qualified for pre-algebra. The next group was assigned to the advanced math classes. The remaining students went into general math or remedial math classes.

The class observed for this study was one of the two advanced math classes Mrs. Talent taught. Observations were made during her seventh period class. The class of nearly 30 students was predominantly seventh graders, but included a few eighth graders, some of whom were repeating the class. There was a pretty even breakdown between boys and girls in the class. Mrs. Talent suggested the class was "supposed to be advanced, but they're not." She instead described the class as "the on-grade-level class." The attendance in the class fluctuated considerably, in part because of a number of end-of-the-year special activities.

The classroom felt uncrowded due to its ample size. The desks were generally arranged in six rows, with five desks in each row. The left side of the classroom was open, leaving a clear walkway between the door and the teacher's office in the back left corner. Windows stretched the length of the back wall. The desks were set up facing the screen for the overhead projector on the wall opposite the windows. Mrs. Talent could dim the lights along that wall to make whatever was shown on the overhead screen more visible. Along the wall below the screen were built-in cabinets and drawers for storage. On the counter top were separate baskets for students to hand in papers from days they were absent or to hand in redone, extra credit, or late papers. This counter also included a sink and drinking fountain. At one end of the counter were puzzles such as Towers of Hanoi to challenge the students when they had spare minutes. Over the puzzle corner hung pockets for the classroom set of calculators. Above the remaining counter space was a bulletin board with colorful posters as well as weekly print-outs of the students' homework scores. A poster giving a decimal approximation of pi hung the length of the wall above the bulletin board. A blackboard covered part of the wall on the right side of the classroom, on which was posted the date and lunch menu. Bulletin boards covering the remaining wall space contained posters with interesting mathematical facts or problems. A poster giving the behavior expectations for the class was prominently displayed on the left wall. A table against the back wall held files for the students' portfolios. Mrs. Talent used an old wooden file cabinet as a stand for the overhead projector and as storage for worksheets.

Background of the Probability Unit

Setting educational goals. In describing the purposes for mathematics instruction, Mrs. Talent recognized the school district's principal mathematics goal was that students score well on standardized tests. Although recognizing she was accountable to the school district, having the students score well on standardized tests was not her "main thing." Mrs. Talent suggested her goals depended somewhat on the level of the class she was teaching. For the students who are in general math and often below grade level and fearful of mathematics, her purpose was "to get them turned on to it and if, in the process, they learn content, that's great." However, she emphasized that she does not just play games all year, but, whenever possible, she presents the content using activities that are less threatening to the students. On the other hand, for the upper level students, Mrs. Talent's goal was to push the students a little bit, particularly in the area of problem solving. At the same time, she aimed to hone and fine tune their mathematical skills.

As justification for teaching probability at the middle school level, Mrs. Talent emphasized the importance of an "awareness of what it is and that the things that maybe appear to be completely up in the air are not." Along with this awareness is the ability to analyze situations mathematically to see what options might be available and to assess the "different weights on things." In addition, Mrs. Talent pointed out probability instruction builds the "awareness that probability is out there . . . everywhere" as the students experience different ways probability can be used. She further argued that people need an understanding of probability "to make good choices" and sound decisions in many areas of their lives.

Mrs. Talent outlined a number of goals she had for her students in the study of probability. Her "number one" goal was that the students would "be able to make good decisions based on being able to analyze information." She also wanted to be able to give the students an unfamiliar task and have them apply what they had learned to analyze the situation. This goal involved being able to look at a game or situation and decide whether it was fair or not. It also involved being able to conduct an experiment and analyze the results. Finally, the students were also expected to be able to analyze certain situations mathematically using such strategies as tree diagrams.

Designing mathematics instruction. Mrs. Talent's decisions about mathematics instruction were generally based on her knowledge of middle school students. In particular, she recognized that "you have to keep them moving and you have to keep a variety or you are going to lose them." She accomplished this goal through a mix of hands-on activities and direct instruction. She also recognized that the students were concrete thinkers, which she considered in the representations she chose to use.

The level of the students also impacted instruction. For example, in the general math classes, Mrs. Talent explained she covered the material more slowly, modeled the thinking processes more carefully, and held "their hand a little bit more." She was also careful to "read the signs" so that she did not push the students past their frustration level.

With the exception of the algebra classes, Mrs. Talent reported she had not "handed out textbooks" for the past 2 or 3 years. Instead she utilized hands-on activities and worksheets to present the content and to give students an opportunity to practice the skills being learned.

A typical class began with five questions shown on the overhead as warm-ups. The content of the warm-ups varied from day to day, but generally it included a review of skills with fractions, decimals, or percents; an application of the geometric area or volume formulas; and sometimes a question from the current material. One day a week, the warm-ups were given orally and the students were to do the problems mentally. After the warm-ups, Mrs. Talent reminded the students about what they had done the day before. They then would correct and/or hand in their homework assignment. With this reminder of what they had done previously, Mrs. Talent would move into the lesson for that day. This lesson might be a continuation of what they had been doing or it might build upon what had been done in earlier lessons. As part of the lesson, the students would be involved in exploratory activities or in completing worksheets for practice. Whatever part of the daily assignment was not completed in class was to be completed as homework, although Mrs. Talent avoided giving homework assignments on Fridays. Mrs. Talent used this same general format in teaching the probability unit.

Creating the learning environment. The desks in the classroom were usually arranged in rows. However, on some occasions, to facilitate group work, Mrs. Talent arranged the desks in groups of two or three so the students could work together. The students had the opportunity to choose where to sit and with whom to work, but they were encouraged to work with different people at different times.

The class period was 43 minutes long except for two days during the probability unit when the period was shortened to 32 minutes to allow for special assemblies. On days when the class period was the usual length, the first 10 to 14 minutes were generally devoted to the warm-ups. Mrs. Talent used the time students were working on the warm-ups to hand back papers. Then as a class they went over and answered the questions on the warm-up. When there had been homework, 2 to 11 minutes were spent correcting and/or discussing the assignment. On some days, the learning activities involved the rest of the class period. On other occasions, particularly when the assignment was a worksheet, Mrs. Talent spent between 2 and 14 minutes presenting an explanation of the content in the assignment or modeling how to complete the worksheet. On these days, the remainder of the time, from 10 to 17 minutes, was work time.

Class rules and expectations, posted along one wall, formed an acrostic on the word CARE, and were stated as follows:

Cooperation
- be helpful and supportive to adults and peers
- demonstrate safe and ethical actions

Attitude
- show respect to others in the class
- offer positive feedback to self and others

Responsibility
- follow student behavior guidelines in the handbook
- bring paper, pencil, homework to class each day
- be on time and ready to work when class begins

Effort
- participate in all activities and discussions
- ask for and get help if needed

Because Mrs. Talent used hands-on materials as a regular part of her mathematics instruction, certain routines had been established for handing out materials and for collecting them at the end of the period. Mrs. Talent often had the materials precounted and sorted ready for distribution, which either she or a student handed out. She also used a variety of routines for collecting materials, depending on how much time was left at the end of the period. On at least one occasion, the students put their materials in the containers as they left the classroom. At other times, they left the materials on the corner of their desks to be collected.

Mrs. Talent had also established certain expectations and routines about turning in assignments. If papers were not completed satisfactorily (e.g., showing a tree diagram, giving probabilities, and stating if the game was fair or unfair), the assignment was returned to the students to be redone and handed in again. If assignments were not completed when due, they could be handed in late, but students were not to fill in answers as they were read when the papers were corrected. Baskets were available on the front counter for students to hand in assignments.

Planning the probability unit. The probability unit was taught in May, finishing about 2 weeks before the end of the school year. On some occasions, Mrs. Talent had taught probability earlier in the year, but she indicated she preferred to teach it at the end of the year because the interest and activity level of the unit kept the students involved. She also saw the probability unit as a good opportunity to apply fractions and percents that the students had studied during the year.

Mrs. Talent had checked with the sixth-grade teachers to find out what activities the students might have done already. In the process, she learned only one sixth-grade teacher had taught any probability during the prior year. However, despite their lack of prior instruction in probability, Mrs. Talent believed the students had some basic ideas about probability. In particular, she felt they had an awareness of games as a result of previous experience playing games. In addition, from the work students had done with fractions, she believed they could look at a spinner and identify the chances of landing on different sections. Mrs. Talent also believed the students had a general sense of what "a 1 out of 3 chance" meant.

The textbook previously used in the advanced math classes was *Mathematics Unlimited* (Fennell, Reys, Reys, & Webb, 1988). Mrs. Talent reported the textbook included four or five pages covering probability which she followed when she first taught probability. In recent years, as she had learned more about probability and had collected more resources, she no longer used the textbook materials except for two practice worksheets. The games and worksheets she selected came from a variety of sources including the Middle Grades Mathematics Project book, *Probability* (Phillips et al., 1986) and the Mathematics Resource Project materials (Hoffer, 1978). For the two opening activities, Mrs. Talent had developed activities based on "Cereal Boxes" and "Monty's Dilemma" from the Math and the Mind's Eye materials, *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993).

In planning the probability unit, Mrs. Talent identified two factors she had considered in particular. First, she tried "to keep in mind, 'Why am I doing this activity?

Where am I going with it? What's this laying the framework for?' " Second, Mrs. Talent considered how the students might react to the activities. For example, she avoided any activities she thought they might not "buy into" or that they might think were "stupid." Other factors also influenced her planning: choosing a variety of activities, including games with dice and coins; trying not to do too much as she had done the last time she taught probability; and incorporating the various aspects of probability such as fair/unfair games, sampling activities, and permutations and combinations.

Mrs. Talent outlined a general sequence she expected to follow with the probability activities. First, she wanted to have the students predict what might happen or predict whether the game would be fair. The second step was to do the activity and gather information. After considering what their data might tell them, Mrs. Talent wanted the class to analyze the activity or game mathematically if it was possible to do so.

The Probability Unit

The probability unit was taught during the last 3 1/2 weeks of May (see Figure 11). The first 11 days of the unit involved the presentation of material in the form of activities and worksheets. During the final 4 days of the unit, the students were involved in completing both individual and group tasks, applying what they had learned in the unit.

Monday	Tuesday	Wednesday	Thursday	Friday
Day 1 Application Story "Cereal Boxes"	Day 2 "Monty's Dilemma"	Day 3 Games with Chips	Day 4 "More Chips"	Day 5 "Finding Probabilities"
Day 6 "Two-Stage Trees"	Day 7 "Independent Events"	Day 8 Two-Dice Games	Day 9 "The Hare & the Tortoise Game"	No class
Day 10 "Experimental Probabilities"	Day 11 "Dependent Events"	Day 12 Coin Tossing: Individual Task	No class	Day 13 Carnival Game: Group Task
Day 14 Carnival Game (continued)	Day 15 Carnival Game: Individual Task			

Figure 11. Calendar for Mrs. Talent's probability unit.

A number of the activities will be described as a sample of the unit, including Mrs. Talent's introduction to the unit and the two initial games used as motivation. Also included will be her presentation of the worksheet "Two-Stage Trees," her discussion of "The Hare and the Tortoise Game," and a sampling activity. This section will conclude with a description of the two application tasks used for evaluation.

Introduction to the probability unit. Mrs. Talent began the probability unit by writing the word *probability* on the overhead and asking if any of the students could give her a definition. One student suggested it dealt with "something that you have a chance on that you're not sure about." Another student added, "Like, what will happen if you do something." When the students had nothing further to add, Mrs. Talent put a story entitled "Why I Can't Teach You About Probability Today" on the overhead. Because the print was small, she read the story to the students, asking them to listen for examples of probability.

> I'm really sorry that I can't teach you about probability today, but I haven't had any time to collect good examples. When I left here yesterday I had a few errands to run so I decided to take a chance and go down Main Street. As luck would have it, there was a lot of traffic so it really slowed me down. As I was driving around, I heard the radio announcer say that there was a good probability that it was going to rain, so that made me even more stressed and hurried so I could get home to shut the windows. I stopped to buy a lottery ticket but once again it wasn't a winner–one of these days I'm going to hit it big! One of the places I stopped at was conducting a survey–a taste test where you have to drink some samples of pop and tell which one you liked the best. There was a long line but the sign said they needed a large sampling of people to do it so I stayed. After that I stopped at the grocery store to pick up another box of cereal. They're having a contest and if I get all six different cards I win a prize. Only two more cards to go, I wonder how many boxes I'll have to buy to get them? After that I headed home, only to discover that my insurance company had notified me that my house insurance premium had gone up. Evidently their new data says that we're in a high risk area for earthquakes. By that time I was so stressed that I asked my husband to cook dinner. He said, "Let's flip a coin," and I lost. After dinner I just had to watch the basketball game because all of the experts say that my favorite team has an excellent chance of making it to the championships. So, as I'm sure you realize by now, I just didn't have any time to get good examples of probability for you. Maybe I'll have time to look in the textbook tomorrow.

After reading the story, Mrs. Talent gave the students 2 minutes to jot down the examples of probability they found in the story. The class then took about 5 minutes to list and briefly discuss the examples they had found.

"Cereal Boxes." Following this introduction to probability, Mrs. Talent explained they were going to focus on one of the examples from the story, namely, getting the prizes offered in boxes of cereal. She showed the students a box of Honeycombs that contained one of three possible posters of NBA stars. After explaining they would assume there was the same number of each poster available and they would "mix them all up" before they shipped them to the stores, Mrs. Talent asked the students how many boxes of cereal they thought they would have to buy in order to collect all three posters. The students' responses ranged from "about six" to "probably about nine."

Mrs. Talent suggested they could go buy a case of Honeycombs and do an experiment. However, she pointed out that would be an expensive approach. One student who remembered doing the problem in this same class last year suggested they could use dice to simulate the problem instead. When asked how they could do that, students offered a variety of suggestions. When one student finally proposed assigning two numbers to each poster (e.g., 1 and 2 for the first poster, 3 and 4 for the second poster, and 5 and 6 for the third poster), Mrs. Talent said they would "go ahead and do that." She then had the students set up a chart on their notebook paper as she modeled how they would tally the dice rolls until they had obtained all three posters. As she handed out the dice, Mrs. Talent explained they were to conduct the experiment three different times and find the average number of tries it had taken them. After the students had conducted the experiment and found their average, Mrs. Talent asked for a show of hands to determine the experimental outcome for the class. From those results, she concluded one would have to buy about five boxes to get the three posters. When asked if it would happen for sure, at least one student responded, "No." Mrs. Talent pointed out that "you could buy 20 boxes and never get all three [posters]," but you probably would get them in about five boxes. Mrs. Talent concluded the activity by asking, "How many people had their prediction the same as what they got?"

With about 4 minutes left in the class period, Mrs. Talent moved on to another version of the "Cereal Boxes" problem. In this case, the students were to imagine there were six posters they wanted to get. After thinking about how many boxes were needed to obtain three posters, the students were asked to predict "how many boxes you think you'd have to buy to get all six posters." When asked how they could use dice to simulate this problem, one student suggested they use 1, 2, 3, 4, 5, and 6, or have each side of the die correspond to a different poster. In the few minutes remaining in the period, the students simulated this problem twice, obtaining much more varied results.

After the warm-ups on Day 2, Mrs. Talent returned to the "Cereal Boxes" problem for a 5-minute summary of their results. Again using a show of hands, she had the students report their results. About seven of the students obtained the six posters in less than 10 times on an average, but most needed more than 10, including some who had needed more than 20 times. After concluding "somewhere between 10 and 20 looks like about the most common . . . amount of times," Mrs. Talent revealed which poster was in the box of Honeycombs she had shown them the day before.

"Monty's Dilemma." Mrs. Talent continued on Day 2 by explaining they were going to look at another real-life situation, the game show "Let's Make a Deal." Setting the scene for the students, she reminded them the contestant had a choice from among three doors, one of which contained "a really good prize like a car or a trip or something like that." After the contestant had chosen a door, Monty opened one of the doors containing a gag prize. The contestant was then given an opportunity to stick with the door they had selected or switch to the other door that had not been shown. After setting the scene, Mrs. Talent asked the students if they thought they would be better off sticking or switching or if they thought it would not matter. Of the 27 students in the class, 4 or 5 thought you would be better off sticking with your initial choice. The rest suggested it did not matter.

Mrs. Talent then explained to the students how they were going to simulate the problem. Each pair of students would receive three small paper cups and two pennies. One student would play the part of Monty, hiding one penny under one of the cups. The other student would play the part of the contestant as they acted out the situation 30 times. The first 10 times, the contestant was to stick with their initial choice. They then were to play it 10 times with the contestant switching to the other door. On the last 10 times, the contestant was to flip the second penny to decide whether to stick or switch. After modeling each situation, Mrs. Talent handed out the cups and pennies, which had been counted out ready for distribution.

The students were given about 10 minutes to conduct the simulation. When they were finished, the students were asked to record their results on a transparency Mrs. Talent had prepared. The class' results are shown in Figure 12. Just before class ended, the students were asked to copy down the results so that they could discuss them the following day.

STICK		SWITCH		FLIP	
Won	Lost	Won	Lost	Won	Lost
4	6	5	5	4	6
3	7	6	4	4	6
8	2	5	5	5	5
3	7	4	6	4	6
5	5	4	6	7	3
4	6	7	3	4	6
3	7	6	4	3	7
3	7	6	4	5	5
7	3	10	0	8	2
4	6	2	8	5	5
2	8	9	1	2	8
1	9	4	6	3	7
6	4	5	5	4	6
53	77	73	57	58	72

Figure 12. Display of "Monty's Dilemma" data for Mrs. Talent's class.

After the warm-ups on Day 3, the class returned to their discussion of "Monty's Dilemma." With the class' results shown on the overhead, Mrs. Talent asked the students to total the columns. From their experimental results, Mrs. Talent and the class observed the Switch strategy had the best success rate. One student named Steven began to explain why that was true, noting that "you'll get it right over time if you . . . don't pick one that has [the coin] . . . and then you switch . . . because you'd have more of a chance to not pick one of those coins." Building upon what Steven had observed, Mrs. Talent drew three squares on the overhead, put "P" in one to represent the prize, and asked, "What is the only way that you could win that prize if you stuck with your original guess?" The students recognized they needed "to pick that first" and suggested they had a one-third chance of doing that. Similarly, the students recognized the way to win the prize by using the Switch strategy was to "pick the wrong one," as Steven had said. The students identified this probability as 2/3. Using the Flip strategy, the students agreed the results would be about 50-50. Mrs. Talent concluded the discussion by reminding the students what they had conducted was an experiment and, therefore, "it might not come out exactly the same" if they did it again. However, she pointed out that "over the long haul, that [Switch] strategy is gonna work better than sticking."

"Two-Stage Trees." In analyzing some games with chips on Days 3 and 4, Mrs. Talent introduced tree diagrams as a way of listing "all the different possibilities that could happen in an experiment." On the assignment, the students were asked to draw two multi-stage trees. On Day 5, the students completed a worksheet which involved finding simple probabilities in situations with spinners. In introducing the lesson on Day 6, Mrs. Talent suggested, "What we're going to do today is kind of put the two [tree diagrams and spinners] together."

As introduction to the worksheet "Two-Stage Trees" (Hoffer, 1978), Mrs. Talent provided two examples, one she did with the class and the other, the students worked on independently. In the first example, she had the students draw two spinners (see Figure 13).

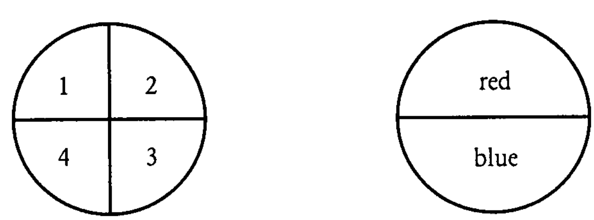


Figure 13. Two spinners used by Mrs. Talent as an example for "Two-Stage Trees."

Using questions intermingled with direct instruction, Mrs. Talent led the students through the process of drawing the tree diagram shown in Figure 14.

T: First of all, the first spinner. How many different ways can you come out on that spinner if you spin?

Ss: One, 2, 3, 4.

T: Okay. So, four different ways. So, what you need to do to represent that spinner, you need to draw the first part of your tree and have it have four branches. . . . And each branch is gonna represent one of the different outcomes, okay. And then, on your . . . assignment, you're gonna have to do this, so . . . on here, what's the probability that anyone of these will come up?

Ss: One fourth.

T: So, what we do is, next to one of the branches, we're just gonna write 1/4 and, really, we should do it all along because each branch has a one-fourth chance of coming out [*writing 1/4 beside each branch*]. . . . Okay, the four different outcomes for the first spinner are . . . what?

Ss: One, 2, 3, 4.

T: Okay, so we're gonna write 1, 2, 3, 4. . . . Okay, the second spinner can come out how?

Ss: Red or blue . . . red or blue.

T: How many different outcomes?

Ss: Two.

T: Two, so what we're gonna do now is off of each one of these, we're gonna make two branches. . . . And what's the probability that it's gonna come out red or blue?

S: One half.

T: One half. And, again, I'm just gonna have you write that next to one of 'em instead of all the way across. . . . What would be the two different possibilities here?

S: Red or blue.

T: Yeh, and you just go through and it's gonna be red or blue, red or blue . . . and so on all the way across. . . . Now, the other thing you're gonna have to do today, you're gonna have to draw the diagram, label the probability of the branches, and then you're gonna have to list all the different outcomes. So, I'll get you started [*as she lists the first two outcomes off to the side of the tree*] and then I'll give you a minute to do this.

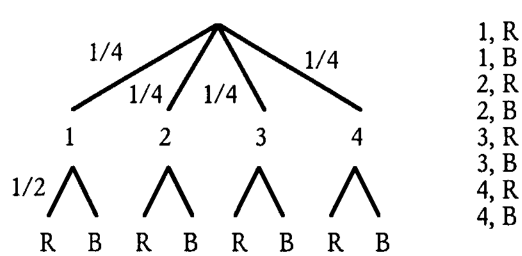


Figure 14. Two-stage tree diagram drawn by Mrs. Talent for spinners example.

After completing the list of outcomes for the two spinners, Mrs. Talent gave the students another example to do completely on their own. In this example, a coin was going to be tossed and then a spinner, which had three equal-sized regions labeled red, white, and blue, would be spun. The students were asked to find the probability of getting a head and white. After a few minutes for students to work, Mrs. Talent asked if someone would tell her how to set up the tree. She then called on Lonnie, who had been absent when they had done trees earlier, to see if he understood. Lonnie explained that you would make three branches "because there's three color things, like red, white, and blue." He went on to have Mrs. Talent draw "two little branches under each one of those" (see Figure 15[a]). When another student suggested he "did it the other way around," Mrs. Talent drew a second tree beside the first (see Figure 15[b]). She then pointed out, "If you want to be really picky, the probability that you're gonna flip a coin and get heads and then spin a spinner and get white is 0 in this instance [on Lonnie's tree] because it's never gonna come up heads first." When students argued that "white and heads is the same as heads and white," Mrs. Talent concluded by saying, "Like I said, if you get picky and you want it in that order [heads and white], you're never going to get it that way. But if you don't care about the order, then you're right. But I care about the order." After explaining to the students what they were to show on the assignment page, Mrs. Talent gave the students the remaining 10 minutes to work on the assignment which involved seven exercises similar to the examples.

(a) Lonnie's tree diagram. (b) Correct tree diagram.

Figure 15. Tree diagrams drawn in Mrs. Talent's class for coin and spinner example.

After finishing the warm-ups on Day 7, the class corrected the worksheet assignment as Mrs. Talent called on people to give their results. Before handing in their papers, Mrs. Talent asked the students to write down on a separate sheet of paper what the probabilities had been beside each of the stages and the final result for each of the items on the worksheet. Then the students were to "look for a pattern or relationship" between the fractions for each item. The students readily recognized "it's just times-ing." Mrs. Talent then concluded, "So, what this means to you is, now, instead of doing a tree diagram to figure it out, if you've got how many different ways the first one could come out and how many different ways the second one could come out, if you take and multiply them, it'll tell what the probability is without having to draw all of it out. After one further example, Mrs.

Talent handed out the assignment for Day 7, a worksheet where the students could apply this property they had discovered.

"The Hare and the Tortoise Game." On Day 8, Mrs. Talent shifted direction in the probability unit away from tree diagrams by having the students play two dice games, one based on dice sums, the other on dice products. After showing tree diagrams would be a cumbersome analysis technique in this case, she provided charts for the students to use instead. The homework assignment, a worksheet entitled "More Dice Games" (Phillips et al., 1986), had presented some difficulties to the students, so on Day 9, Mrs. Talent gave a further explanation of how to do the problems and delayed the date the assignment was to be handed in.

Mrs. Talent then introduced another game the students were going to play with dice. In "The Hare and the Tortoise Game" (Phillips et al., 1986), each player would roll a die three times. Starting with their marker in the center of the game board (see Figure 16), the player would move their marker left if the die landed on an odd number and right if it landed on an even number. The tortoise would receive a point if, at the end of the three moves, the marker was on position P or X. The hare would receive a point if the marker ended up anywhere else. The players were to alternate turns until each had taken 16 turns. After modeling what each turn involved and how to record their results, Mrs. Talent asked the students if they thought the game was going to be fair. About six students suggested it was going to be unfair and the rest were not sure.

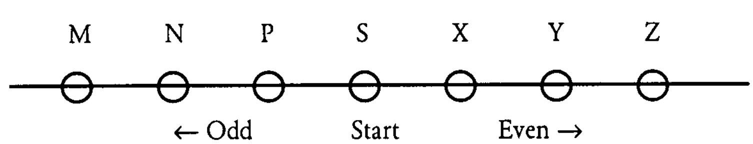


Figure 16. Game board for "The Hare and the Tortoise Game."

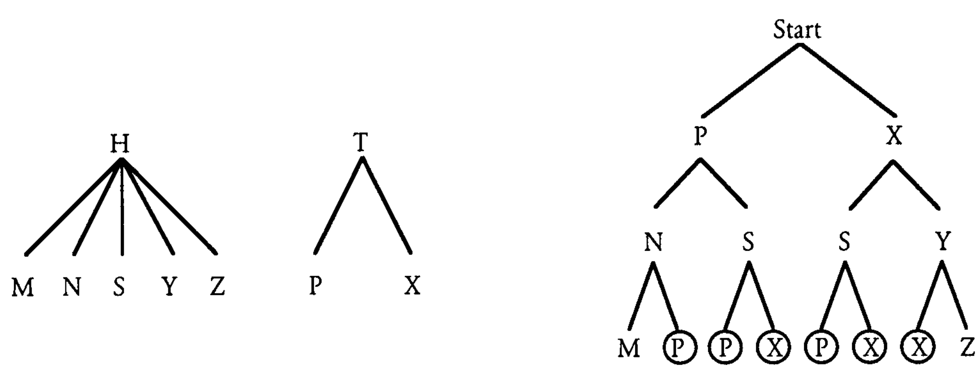
As the students were playing the game, Mrs. Talent circulated among the groups, clarifying the directions and responding to questions. A number of students began observing that it was impossible to end up on some of the positions after three tosses of the die. Mrs. Talent discussed these observations with the individual students.

S: It's impossible to get on an S, isn't it?
T: It's impossible?
S: Yeh.
T: How is it impossible?
S: Because, if . . . when you roll three times . . . let's say you . . . get an odd, then you get an even, then you get another even. Or, if you just said even, even, and then an odd . . . it doesn't work.

T: It'll never go back to S?
S: No.
T: Huh? Could be. See if you ever land there. It looks like you can't. What you said made sense.

Other students also pointed out it was impossible to end up on S as well as on N or Y. They explained that after two moves one ended up in those positions, but "then the third one would have to move you off it." One pair of students took the conjecture one step further, suggesting, "If you use any odd number . . . you can't get it on there [N, S, or Y]."

When the students had completed the game, Mrs. Talent asked them to tally their results and figure out the score. Mrs. Talent then asked if anyone had an idea of how they could use a tree diagram to analyze the game. One student suggested the first stage of the tree would have H and T for hare and tortoise. Then from the H, he suggested drawing branches for M, N, S, Y, and Z (the positions where the hare scored points). Similarly, he wanted to draw branches for P and X from the T (see Figure 17[a]). Another student suggested using odd and even, which Mrs. Talent pointed out "might be a little easier." Explaining that "when you do a tree diagram, you want to look at how many different ways something can come out," she then led the students through the process of drawing the tree diagram shown in Figure 17(b). In this tree, rather than labeling the branches with odd or



(a) Tree diagram suggested by a student.

(b) Tree diagram drawn by Mrs. Talent.

Figure 17. Tree diagrams drawn in Mrs. Talent's class for "The Hare and the Tortoise Game."

even, the outcomes of the die toss, they labeled the branches with the point on the game board where the marker would land. For example, after the first roll of the die, the marker could end up on either P or X. Or if it had been on N after two tosses, the third toss could move it to either M or P. Mrs. Talent pointed out that rather than listing the outcomes by following the branches (e.g., PNM) as they had done before, in this game it was only the last ones that mattered. Circling those results that gave the tortoise points demonstrated there were six "winners for the tortoise" and only two "winners for the hare." Thus, the game favored the tortoise even though, as Mrs. Talent pointed out, it seemed to favor the hare originally. The students' conjecture about impossible outcomes was not discussed, but the tree Mrs. Talent drew clearly supported their claim that some were "even places" and others were "odd places."

Sampling activities. After the usual preliminary activities on Day 10, Mrs. Talent explained students sometimes wonder why she teaches probability because "all you use probability for is when you gamble . . . and you're supposed to be saying that gambling isn't good." Reminding the students they had found "lots of ways that chance is used" from the story on Day 1, Mrs. Talent suggested "what we are going to look at is one of the ways that companies use probability to help them figure out what's going to happen." She first gave an example of how insurance companies use data from accident reports to determine the rates to charge people in different age groups.

As a further example of "how you can take a sample . . . and use that to help predict something," Mrs. Talent conducted a student poll. With the students' assistance, she listed the 18 electives offered at their school. Then she asked the students to imagine the "school fell on hard times." As a result, they were going to have to cut out all electives except the two most popular ones. Mrs. Talent then polled the students to see which two electives they would want to keep. Assuming their class was a "pretty good sample" of the school's student population, they used their data to predict how many students in the school would want to keep the top two choices.

Mrs. Talent concluded the lesson with an example of how results from a poll could be misleading. For instance, she pointed out an advertisement may claim, "9 out of 10 people surveyed prefer Pepsi" without explaining the 10 people asked all work for Pepsi. Mrs. Talent then handed out the students' assignment, a worksheet entitled "Experimental Probabilities" (Phillips et al., 1986). This assignment included four problems dealing with samples or results of polls. In one item, the students were to count the number of times each vowel occurred in a passage from a book and then count all the letters in the same passage. They were to use their results to predict how many *e*'s there would be in a second passage and then to verify their prediction. In the other items, the students were to use the results given from polls to make various predictions.

Application tasks. The last day of instruction, Day 11, followed up on the discovery the students had made one week earlier about multiplying probabilities. Although the term *independent events* had not been used at that time, Mrs. Talent did use the term *dependent events* in an example of drawing two colored cubes from a sack without replacement. In particular, as Mrs. Talent helped the students determine what two probabilities to multiply together, she explained the probability of drawing the second cube depended on what was drawn the first time. This example provided a model for the students to follow on the worksheet, which was their assignment.

On Day 12, Mrs. Talent gave the students the first of two application tasks to be used for evaluation. In introducing the problem, she suggested to the students that if they did a good job, this task would be a good one to include in their portfolios. The Coin Tossing problem, which had appeared on the seventh-grade statewide assessment a couple years before, read as follows: "Three dimes are tossed at the same time. What is the probability that exactly two of the coins will be heads or tails? Explain your answer(s) and your thinking." Suggesting the "question was poorly written," Mrs. Talent clarified with the students what was meant by "exactly two of the coins will be heads or tails." The students had the remaining 15 minutes of the class period to complete the problem. Except for a few students who had questions for Mrs. Talent, the students worked quietly and independently on the problem, many completing their work in about 10 minutes. Although the question had not specified whether an experimental or theoretical probability was expected, all students approached the problem theoretically.

On Day 13 in preparation for the second application task, Mrs. Talent asked the students if they had ever played games at a carnival or at the state fair. When asked if they thought the games were fair, the students provided a number of examples of games which were not set up fairly, which did not offer an equal chance of winning or losing. Mrs. Talent then modeled a sample carnival game. She had three paper bags each containing two green cubes, three yellow cubes, and four white cubes. The player would pay $1 to pick one cube out of each bag and would win $10 if all three cubes selected were green. Mrs. Talent gave the students a couple times to try their luck and then moved on to explain the application task itself.

Next, Mrs. Talent had taken an open-ended question from the current year's statewide assessment. The task, entitled "The Carnival Game," presented the students with the following scenario:

> The Carnival Committee at Greenway Middle School is trying to raise money for the eighth-grade class trip to Washington D.C. Students have submitted ideas for carnival games to the committee for consideration. It is important to make at least $200 on every game they approve.
>
> The committee hopes that about 300 people will play each game. You, as a member of the committee, are given the following plan:
>
> Plan for a Carnival Game:
>
> There are three cans. Each can contains a red, a blue, and a green ball.

> People pay one dollar, then pick out one ball from each can—but they're not allowed to look into the can when they pick.
>
> They win $10 if all three of the balls they choose are the same color.
>
> All the balls are returned to the cans every time a player takes a turn.

Mrs. Talent provided some additional background information, explaining that "last year it was determined people are not willing to risk their money unless the payoff is at least 10 to 1. People thought betting a dollar to win anything less than $10 was not very interesting." The students were given the task to "determine whether or not the carnival game plan outlined on the other paper should be approved or not. If you decide it is not acceptable the way it is, then redesign it so that they will be able to make the profit they want and keep the players happy." On a separate page, Mrs. Talent had outlined the general procedure the students were to follow. In particular, they were to "analyze the game and determine how much profit can be expected," and include a detailed description of their analysis. Then, the students were to show evidence the game will make the money needed or to explain how to change the game so that it will. They were also to "write a final summary to the carnival committee telling them what you found and what you suggest." Finally, on a separate sheet of paper, the students were to describe what each of the group members contributed to the task. It was not stated whether the analysis to be done was to be based on experimental or theoretical results or based on both.

Mrs. Talent allowed the students to form their own groups. The students could work individually, if they chose to do so, or in groups of up to three people. They could use whatever strategies they wanted. Mrs. Talent explained paper bags and cubes were available if the students wanted to test it out that way. The students had the remaining 15 minutes to start working on the problem. Some groups approached the problem theoretically, others experimented with the paper bags and cubes, and still others did not know where to begin.

Mrs. Talent explained to the researcher that her interaction with the students as they worked on such tasks was a bit different from the assistance she usually provided. Her usual approach was to be generous with help, often explaining how to do a problem when students did not know what to do. In this case, however, she would answer questions, but she would not tell them how to do it. Instead she would ask the students questions to see what they understood and help guide the students to identify what they did and did not understand. Mrs. Talent reported this approach frustrated the students, suggesting the students thought, "Just give me a worksheet and show me how to do it. Don't ask me to think."

Because a number of students had been absent on Day 13 for a school-sponsored activity, Mrs. Talent briefly explained the task again on Day 14. For the sake of time, she modified the task, taking off the part about redesigning the game. In the next 20 minutes, the students were to analyze the game and determine how much money it would make. Whatever work they had shown by that time would be what they turned in.

After the students were given the time to work, Mrs. Talent brought the class back together again to share their results. Only one individual and one pair of students thought they would make enough money. When asked how they had found their solution, one student, Jared, explained he had drawn a pie chart (meaning tree diagram) and determined there would have been 3 winners out of 27 possibilities for a probability of winning of 1/9. Dividing 300 by 9, he suggested there would be 34 winners with a net result of losing $40. A second student, Chris, agreed with the 1/9, suggesting there would be 33 winners. Then he subtracted 267 from 330 concluding the game would lose $63. Discussion of these two solutions suggested the difference depended on whether the winners got their dollar back in addition to receiving the $10 prize. Jennifer, who was convinced the game would make money, explained what she had done in her head. Getting matching colors had a probability of $1/3 \times 1/3 \times 1/3$ or 1/27, she suggested. Multiplying by 3, because there were three cans, gave her 1/81 (she multiplied incorrectly). With just one winner of each 81 people who played, she concluded they would make enough money. As the class period ended, Mrs. Talent asked the students to think about these solutions as homework.

As Day 15 began, the class picked up the discussion again. After thinking about it further, Chris had decided he liked Jared's solution of a $40 loss. Mrs. Talent then led the class through a step-by-step analysis of the game. They first identified the probability of drawing any one color from one can as 1/3. Because there were three cans, the probability of any outcome would be $1/3 \times 1/3 \times 1/3$ or 1/27. Drawing a partial tree confirmed there would be 27 outcomes. When the students suggested the 3/27 in Jared's solution was because there were three cans, Mrs. Talent corrected them, stating it was because there were three ways to win by matching red balls, matching blue balls, or matching green balls. The rest of the analysis followed Jared's reasoning.

After giving the students an opportunity to ask questions, Mrs. Talent gave them a follow-up question to the Carnival task as a final individual evaluation task. This problem had the same setting as the Carnival task, 300 people would play and they wanted to make a profit of $200. This time, however, a white ball was added to each can, making a total of four balls in each can. The students were to find out the probability of winning and to calculate the profit or loss, showing and explaining how they solved the problem. After allowing the students time to work on the problem, Mrs. Talent finished off the unit by briefly giving the solution to this final task.

Evaluation of the Probability Unit

Evaluating student learning. Much of Mrs. Talent's daily assessment of the students was done on an informal basis as she walked around the classroom. During these informal times, she was listening to what the students were saying to one another or responding to the questions they asked of her. On two occasions, when many students had not completed their homework assignment, Mrs. Talent sought input from the students whether it was a matter of time or a lack of understanding.

The more formal evaluation of the students included checking their homework assignments, which were generally worksheets. Also included were the application tasks, the Coin Tossing problem and the Carnival task, both of which were "kind of a culminating thing." On one of the tasks, Mrs. Talent explained, "I read them over and . . . realized that they weren't where I wanted them to be." On the other final task, she had evaluated them using a quantified scoring guide. In the end, she suggested if she were going to give the students a grade for the unit, it would probably have been "a combination of all the different things that we did." She explained she would have liked to have had the students do some self-reflection about what they had learned, but they had run out of time.

Evaluating unit effectiveness. Mrs. Talent expressed hope the probability unit had accomplished the goal of raising the students' awareness level about probability. In more specific terms, she thought the students had learned what a tree diagram was, although she was not sure how many of them could draw one without assistance. In addition, she added, "Hopefully, they know that you can conduct experiments to simulate things and, hopefully, they know how to write probabilities . . . and they can calculate simple probabilities like a spinner or something like that. . . . Hopefully they would be able to give some examples of where probability's found in everyday life."

However, in reflecting on the probability unit, Mrs. Talent expressed dissatisfaction, saying, "It just seemed real disconnected to me this year." Although she thought learning had taken place, not as much learning had taken place as she had hoped. Mrs. Talent suggested two factors might have contributed to the difficulties. First, many students had missed a number of days during the unit because of involvement in special end-of-the-year activities. From a teaching perspective, it had been difficult to keep track of who had been present for what parts of the unit. From the students' perspective, it had been difficult to fill in the gaps left by their absences. In particular, many students were absent the day the Carnival task was assigned, giving them far less time to investigate and answer the question.

Second, Mrs. Talent admitted she had not been as thorough in her planning, suggesting, "I thought I knew better what I was doing." Rather than going back and re-reading the background of the activities as she had done in earlier years, she recalled thinking, "Oh, yeh. That was a neat activity. I'll do that." However, because she had not reviewed the instructional background materials, she had not followed through on some of the activities as much as she had in the past.

In the future, Mrs. Talent suggested she would "check the schedule and make sure I'm going to have everybody." She also pointed out she would need to do more preparation herself next year. In addition, she was considering teaching different aspects of probability at different times during the year, allowing her, for example, to do more with sampling than she had been able to do this year.

Mr. English: Influence of Teaching Experience

The final teacher to be introduced is Mr. English. With 27 years of teaching experience and 26 years in the same building, Mr. English considered himself the senior citizen on the staff. Having taught nothing other than mathematics for more than 20 years, Mr. English had become a mathematics leader within his district. Nevertheless, he had not set out originally to be a mathematics teacher.

Teacher Background

School experience. Mr. English grew up in a small farming community in the Midwest and his early learning experiences with mathematics were limited. Prior to high school none of his teachers had backgrounds in mathematics. As was the custom at his small high school of less than 100 students, Mr. English took General Math as a freshman, Algebra I as a sophomore, and Algebra II as a junior. He did not remember if any other mathematics classes were taught, but he recalled no instruction in such areas as geometry, trigonometry, or calculus. Mr. English reported that working with equations in algebra classes came easily to him. However, because of his limited background, anything related to geometry or measurement formulas was "like a foreign language."

Teacher preparation. Because of his success in and enjoyment of English and writing, Mr. English set out to become a language arts teacher. After attending junior college for two years, he transferred to a private liberal arts college in the Northwest to complete his preparation in language arts education. No mathematics was required as part of his program and he had no inclination to take any. As a result, he studied no mathematics in college.

After teaching high school language arts for 1 year, Mr. English returned to college to complete a 6-month program leading to elementary certification at a state university in the Northwest. Although he took whatever mathematics content and methods courses were required by the program, he reported having no recollection of these courses except reading some books on methods of teaching mathematics.

Following completion of the elementary education program, Mr. English began teaching sixth grade, the level at which he thought he would spend the rest of his career. However, 4 years later, when no one applied for a vacant position teaching eighth-grade mathematics, his principal reassigned him to fill that position. Although he had little background in mathematics, Mr. English reported not being overly concerned with this new assignment, recalling he had thought, "Hey, I went through high school and I got a college degree and, granted I didn't take any math, but I have a college degree . . . and I'm only teaching eighth-grade kids. I mean what's . . . so hard . . . about that? And what do eighth-grade kids know anyway that I don't know or that I can't show them?" After teaching eighth-grade mathematics for 5 years, Mr. English began teaching seventh-grade mathematics where he had remained for the next 17 years. At about the time he moved to the seventh-grade level, the state changed certification requirements. To continue teaching mathematics full time as he had been doing, Mr. English was required to obtain a mathematics endorsement.

He then began working on a program at a nearby college of approximately 30 semester hours designed specifically for him. About half of the credits covered mathematics content; the other half focused on teaching mathematics. To fulfill state requirements and supplement the classes and workshops he had been able to take, Mr. English took three independent study classes. One of these classes was an introductory course which surveyed geometry, trigonometry, and calculus as well as probability and statistics. In addition, he recalled taking an independent study course in measurement that included the metric system and one in number theory. Besides presenting content material, these courses also provided games and activities teachers were to implement in their classrooms as part of learning how to teach the content.

Professional experience. Having discovered in the courses he took for certification that there was a great deal he could learn about mathematics and about teaching mathematics, Mr. English began taking advantage of as many opportunities as he could to learn more. He described himself as "a self-directed learner" and learning more about teaching mathematics became his hobby. He subscribed to the NCTM journals and obtained and read all the NCTM yearbooks. For many years, he attended all local and regional NCTM conferences and workshops. He also took additional courses or summer workshops every year or two which, in addition to mathematics education, included courses in assessment, action research, and cooperative learning as well as participation in a writing workshop for teachers. Further, he continued to take advantage of summer programs providing opportunities for teachers to work in the community, linking the mathematics of the classroom and the workplace.

Over the years, Mr. English also became active in the mathematics education community. He provided leadership at the district level as a mathematics consultant, influencing curriculum and textbook decisions and modeling ways of implementing the *Curriculum Standards* (NCTM, 1989). At the state level, he helped write a position paper on teaching mathematics at the middle school level and presented workshops sharing ideas from his classroom. In addition, he had written articles for the state's mathematics education journal and had written and marketed four books of start-up activities for mathematics classrooms, ranging from grade 2 to grade 8. However, despite the course work he had taken, the years he had been teaching, and the professional contributions he had made, Mr. English continued to see himself as a generalist, not a mathematics specialist.

Probability background. The survey course taken as part of the requirements for the mathematics endorsement gave Mr. English a brief introduction to probability and statistics. However, he explained he did not really "get acquainted with probability" until 1988 when he took a training course during the summer in the use of the Middle Grades Mathematics Project materials. As part of that course, they worked through the lessons in *Probability* (Phillips et al., 1986), one of the Middle Grades Mathematics Project units. Experiencing the materials "as a student" had a powerful influence on him, and it was at that time he began teaching probability as part of his curriculum.

The following summer Mr. English participated in an NSF-sponsored workshop in quantitative literacy. This workshop was led by four highly acclaimed teachers who had been trained by the American Statistical Association in the presentation of material from the Quantitative Literacy Project. Although one of the publications in the Quantitative Literacy Series was *Exploring Probability* (Newman, Obremski, & Scheaffer, 1987), the workshop focused primarily on statistics. In addition to some formal teaching, the participating teachers were divided into teams that worked on statistics projects. In gathering, interpreting, and presenting their data, the participants had the opportunity for input and evaluation from local practicing statisticians.

Mr. English reported that these two experiences greatly sparked his interest in probability and statistics. As a result, he sought out workshops on those topics at subsequent regional mathematics conferences. In addition, gathering and presenting statistical data had become a common part of his work in the classroom and the district.

School and Classroom Environment

Mr. English had spent the last 26 years teaching at a middle school located in a rapidly growing suburban community on the outskirts of a large Northwest city. Approximately 425 sixth, seventh, and eighth graders attended the school. Less than 5% were minorities. Economic backgrounds ranged from very poor to very affluent. Mr. English pointed out a great deal of change had occurred in the student body. A survey of 40 sixth graders suggested about half had lived in the community less than 2 years.

The class observed for this study was a seventh-grade pre-algebra class. This class was the top track mathematics class at the seventh-grade level. The 34 students in the class were among the top 20% of the student body, as based on a variety of criteria including placement tests, an algebra readiness test, teacher recommendations, and grades. Mr. English suggested this group of 20 girls and 14 boys were the best thinkers he had taught.

The classroom where Mr. English taught mathematics was packed full. The 34 students in the class filled all available desks, which were arranged in groups of four. The front of the classroom and one side wall were lined with files, bookcases, and stacked-up boxes storing the many resources, games, and manipulatives, which Mr. English had collected and organized. Examples of student work, primarily from the 6-week Enrichment Class Mr. English was teaching to all seventh graders, were on display. The back wall was lined with silhouettes based on estimated measurements and the upper portion of the other side wall was decorated with colorful tessellations. Posters added color to the front bulletin board. The teacher's desk and a desk for the student teacher working with Mr. English were nestled in the front corner, opposite the classroom door. A cart in the center front of the classroom held the overhead projector and boxes for ready-to-use and used transparencies.

Mr. English believed some of the keys to success as a middle school teacher are letting students know you care, modeling a strong work ethic, loving your subject and having a desire to make the subject interesting to students. From the responses of parents and students, it appeared his approach had been successful. He provided examples of appreciative parents who reported their students, previously turned off to mathematics, had become more interested in the subject. And, many students, in a survey conducted as part of a statistics unit, identified mathematics as their favorite subject.

Background of the Probability Unit

Setting educational goals. Mr. English recognized the aims and purposes of education at the middle school level involve more than academic goals. He pointed out, "At the middle school level, we're supposed to be doing things that . . . are fun for the kids, and build kind of a group identity." He suggested accomplishing this aim involves a spirit of cooperation with the elective programs and their special activities and a flexibility in dealing with the absences and multiple interruptions that result. Mr. English admitted, "It's just one of those things . . . you got to keep the total program with the kids in mind and realize, that, at this time, at this level, that's the way it's going to be. Academics, maybe, are not the most important thing at the middle school level." Although finding the right balance between academic and social goals was a struggle, Mr. English set high academic goals for his students. Mathematically, he wanted the students to learn the vocabulary and strategies associated with the different units. But he also wanted them to do more than "just compute with numbers." In particular, he wanted to "extend the students" so that they are able to "solve problems with the different concepts" and to think mathematically and critically.

Mr. English pointed out he started teaching probability because of the *Curriculum Standards* (NCTM, 1989) and as a result of his exposure to the probability materials of the Middle Grades Mathematics Project (Phillips et al., 1986). He also felt there are important reasons for studying the content of probability, explaining, "In this day and age, with so many people wasting their money in lottery situations and gambling, wanting something without having to work for it . . . I think it is important to know what their chances are when they spend that dollar on a lottery ticket or whatever." Further, from a pedagogical perspective, Mr. English suggested getting the students actively involved in experiments using such manipulatives as dice or spinners "just makes the class more interesting."

In addition to these reasons for studying probability, Mr. English outlined a number of goals he had for the students in the probability unit. First, he felt it is important for the students to know the vocabulary and, in particular, to "know what probability means." Second, he wanted the students to participate in a number of different kinds of activities, using dice, coins, cards, and spinners. As a third goal, Mr. English hoped the students will come to "understand the difference between a fair and an unfair game." Fourth, he wanted the students to know a variety of strategies for analyzing probability questions. Finally, Mr. English wanted the students to have "quite a bit of experience with experimental probability" as well as the opportunity to explore the theoretical probability or the "mathematics behind each experiment."

<u>Designing mathematics instruction</u>. From his many years of teaching at the middle school level, Mr. English had gained an understanding of middle school students that influenced what he did in the classroom. In particular, he believed an important aspect of teaching middle school students is to keep them busy, and he generally had something for them to do from the time they walked into the classroom until the final bell rang. For example, he put out a start-up activity for the students to pick up as they entered the

classroom to work on while he took the roll. This page of 8 to 10 problems was selected from the wide variety of activities he had collected. Some of these problems related to the topic being discussed. Others involved using logic or deductive reasoning.

Mr. English also recognized the importance of instruction appropriate for the level of the students. He reported getting a "real strong dose" of Piaget and his levels of learning when he started teaching mathematics, which provided him with "knowledge of how kids learn mathematics." In particular, he suggested it is important to know "what you are able to do with students and move them from a certain level to a different level and not frustrate them too much and discourage them and create negative attitudes."

In his view, an important part of his role as a teacher involved modeling the various procedures or strategies involved in solving problems. Mr. English believed this kind of basic foundation is necessary before expecting students to move on to higher order activities. Mr. English emphasized as well the importance of lots of repetition at the middle school level. "You cannot expect a junior high student to learn it just by doing it once." Instead he believed teachers need to give students lots of practice and opportunities to continually review what they were learning.

In the past 15 years, Mr. English had seen many changes in instructional practice. When he "came on board . . . if you didn't have a quiet classroom you weren't teaching and the kids weren't learning." Now, however, "if you don't have interaction going on, if you don't have things actually happening," then something is wrong with your classroom. Mr. English suggested he had also undergone a transformation in the last few years. According to his own description, for many years he had been a "traditional" teacher or one who followed the textbook exclusively. However, one year he made up his mind not to be traditional any longer. He set a goal for himself to "develop a set of manipulative materials" directly correlated with the lessons he taught. He explained he envisioned transforming his classroom into a math lab. In attempting to reach his goal, Mr. English had collected, tried, modified, and organized boxes full of activities. As a result, his instructional style now included a variety of hands-on activities. Mr. English recognized that, in addition to making his classes more interesting, hands-on activities also help students develop a better understanding of mathematics.

During most of the year, the pre-algebra class had been using the textbook *Transition Mathematics* (Usiskin, 1992). Because Mr. English felt it was better than most textbooks, he had used the textbook extensively for most units, generally following the recommendations given in the teachers' edition for conducting the lesson. After a few minutes spent on the start-up activity, the daily lesson generally began by having students grade the previous day's assignments. One student handed out answer keys. In their groups of four, the students then checked their work and filled out the score sheets, which were collected. Mr. English next told the students what they would be doing for that day. On some days, the next lesson may have involved direct teacher instruction, emphasizing particular concepts. On other days, following the recommendation of the teachers' guide, the teacher had the students work independently without direct teacher instruction. These lessons were designed to be read independently by students. Students then wrote out answers for 20 to 30 questions about what they had read or they completed an exploration or application of the concept. As the students worked, Mr. English walked around the classroom, offering help as needed. He also recognized that students can be a resource to others as they discuss their questions among themselves.

Because the textbook presented the topic of probability in only two or three pages, which Mr. English did not feel was satisfactory, he departed from the textbook and the usual lesson format for the probability unit. He, instead, created an extensive unit utilizing materials from the Middle Grade Mathematics Project (Phillips et al., 1986) and others he had gathered at conferences. The lessons actively involved the students in playing games or doing experiments as well as considering the theoretical analysis of the activities, where appropriate.

Creating the learning environment. The classroom contained tables designed to seat two students each. These tables were arranged to form groups of four students. Within these groups, different configurations of students were used for various classroom activities. Sometimes the students worked in their groups of four. On other occasions, they worked in pairs or individually.

Classroom rules, clearly posted on a bulletin board in the front of the classroom, included:

1. Be in class on time with necessary supplies.
2. Do not leave assigned desk without permission.
3. Keep hands to yourself.
4. Pay attention and follow all instructions.
5. Work quietly. Do not disrupt the class by teasing, putting others down, or saying things to draw attention to yourself!

In addition to the rules, certain routines were followed by the teacher and students in a variety of situations. For example, students were in the habit of picking up the start-up activity from the basket by the door as they came into the classroom. Routine procedures also guided the correction and collection of homework and extra credit assignments. The teacher had prepared score sheets on which the students recorded scores for their assignments and extra credit. These sheets and the assignments were collected in baskets after grading was completed.

During the probability unit, Mr. English changed some of the routines. For example, the start-up activity was part of a packet of materials that also included handouts for the activities and assignments to be done that day. Further, the probability assignments were corrected at various times during the unit, but were not to be handed in until the end of the unit. Students were to write in corrections on their papers so that their assignments could be used for reference as they studied for the unit test. In addition, rather than grading the extra credit daily as was their custom, Mr. English set aside the beginning of class on certain days, generally Fridays, to be what he called "catsup and mustard" days, when they could catch up on grading assignments.

Consistent with his belief in keeping students busy, Mr. English incorporated several different activities in each 48-minute class period. For the first 2 to 5 minutes of each day, the students worked on their warm-ups while Mr. English took attendance and handled other paperwork (passing back papers or getting assignments to students who had been absent). On some of the days, time was then spent either going over the previous day's assignment (4 to 8 minutes) or catching up on grading warm-ups and extra credit assignments (13 to 26 minutes). The remaining 20 to 35 minutes was either spent on one activity or, more likely, divided among two or more activities. On 3 days, there were 5 to 10 minutes left for students to work on an assignment. On the rare occasion when there was free time left at the end of the period, it was no more than 2 minutes.

Planning the probability unit. The probability unit was taught beginning in mid-April. Mr. English indicated that he chose to teach probability in the spring because "it is high interest and a change of pace." Otherwise, when the weather is nice, he suggested it is harder to keep the students involved. Mr. English pointed out that the probability questions on the achievement test taken by the students were generally very basic questions that the students could probably answer correctly even if they had not studied probability. As a result, he felt no pressure to teach probability earlier in the year.

Mr. English believed he was "the only teacher in the district teaching probability." In particular, he was aware of no instruction the students had received in earlier grades. Conversations with the eighth-grade teacher and high school teachers indicated "probability was one thing they weren't doing," although the high school teachers were working to make it a part of their curriculum. Because the students may have no other opportunity to learn probability, Mr. English seemed to feel a responsibility to cover it thoroughly. He observed, however, that they did not "get into applications in careers or insurance or things like that." He hoped they might get some of that at the high school level.

Mr. English suggested that although the students had had no previous instruction in probability, he believed they understood some basic notions about probability. In particular, he felt the students could answer the basic questions on the achievement test they took by intuition. More specifically, he believed the students could identify "your chances" in such situations as flipping a coin, spinning a spinner, or drawing a card from a deck of poker cards. He concluded their knowledge in more complicated situations, however, was limited, pointing out only six students scored 70% or better on the pretest he had given the class.

Mr. English recognized getting students involved in activities using such things as dice, coins, or spinners makes the class more interesting. In planning the unit, however, he wanted to make sure he "wasn't just having the kids play games [without] knowing . . . why I was doing it or where this was leading next." Because one of his goals was for the students to know a variety of strategies, he planned the unit so that it would "move logically" and systematically through different kinds of experiments, beginning with colored cubes, then dice, coins, spinners, and ending with Pascal's triangle. Mr. English explained that many of the games and activities used in the unit were chosen because they presented "a different model" to the students, either dice, coins, cards or other manipulatives or tools.

Transition Mathematics (Usiskin, 1992), the textbook generally being used for the class, covered the topic of probability in approximately three pages. This one section, in a chapter entitled "Patterns Leading to Division," was nestled between sections dealing with the ratio comparison model for division and proportions. The key ideas presented in the section included the range of possible values for probabilities, comparison of probabilities, probability that an event will not happen, and ways of determining probabilities. It ended with a statement of how to determine probability when the situation has equally likely outcomes. This presentation of probability was in contrast to the coverage of probability in *Mathematics Unlimited* (Fennell et al., 1988), the textbook used in the other classes Mr. English taught. This textbook devoted an entire chapter to statistics and probability, including sections presenting the basic notions of probability, tree diagrams and sample spaces, probability and sample spaces, independent events, and dependent events.

In reflecting on the coverage of probability in *Transition Mathematics*, Mr. English observed, "It has one page that is understandable. And then it immediately moves into very abstract, difficult concepts with no . . . nothing in between leading into it and it doesn't have any . . . . It throws multiple concepts at kids on one lesson which violates . . . the principles of teaching for this level." Because he had not found any textbook that he felt gave adequate justice to probability, Mr. English had developed his own unit to teach the subject.

His exposure to the probability materials of the Middle Grades Mathematics Project largely influenced what Mr. English presented in the unit. As part of the unit, Mr. English incorporated materials from 6 of the 10 activities included in *Probability* (Phillips et al., 1986). Other materials from *Probability* were used as homework assignments. Mr. English also incorporated probability activities he had collected at conferences and from a variety of other sources. For example, he used two activities he found in the NCTM Addenda materials, *Dealing with Data and Chance* (Zawojewski, 1991). He also used four activities from the *Problem Solving in Mathematics* series (Lane County Mathematics Project, 1983a, 1983b, 1983c). Other sources included the Mathematics Resource Project (Hoffer, 1978), *Columbus Returns!* (Montana Council of Teachers of Mathematics, 1989), and *Get It Together* (Erickson, 1989). Mr. English also prepared some of his own materials. For example, he had adapted the "Frosted Wheat Yummies" worksheet from a statewide mathematics journal.

In preparing to teach the unit, Mr. English had referred to a wide variety of resources. Even though he taught from the Middle Grades Mathematics Project probability materials (Phillips et al., 1986) every year, reviewing those materials again was part of the preparation process. Mr. English "also took the NCTM yearbook on probability and statistics (Shulte & Smart, 1981) and reread that" as well as reading the newer NCTM Addenda materials (Zawojewski, 1991). In addition to these resources, Mr. English referred to *Mathematics: A Human Endeavor* (Jacobs, 1982) and *What Are My Chances? Book A* (Shulte & Choate, 1977) in developing the vocabulary and basic notions of probability and to the *Smithsonian* article, "Odds Are Against Your Breaking that Law of Averages" (Trefil, 1984) in providing the historical background of probability.

The Probability Unit

The unit lasted for a total of 17 days spread across a 4-week period beginning in mid-April (see Figure 18). Observations were made during the lessons on Days 1 through 12. No observations were made during the final week of the unit, Days 13 through 17, because the teacher and researcher agreed the lessons observed provided a good sample of the unit and because the pressure of being observed was impacting the teacher's other responsibilities.

This section will first give a description of Mr. English's introduction to the unit. Then, as a sample of the probability unit, Mr. English's presentations of six activities will be described. These activities include three dice activities, a spinner activity, a card game, and a simulation of expected value. The section will conclude with a description of the unit test given to the students on Day 17.

Introduction to the probability unit. After a quick overview of what they would be doing that day, Mr. English introduced the probability unit by giving a brief history of probability as reported in a *Smithsonian* article (Trefil, 1984). Mr. English told the story of Antoine deMere, a mathematician in the 1600s, who analyzed a game people were playing in a gambling den. If the player could roll a die four times without once getting a 6, the player would win his bet and receive a payoff. The question asked was, "Is this a fair game?" Mr.

English explained that the probability of rolling a 6 on one roll of a die was 1/6 and the probability of not rolling a 6 would be 5/6, making the odds in favor of the player 5 to 1 on each toss. Mr. English observed, "That sounds kind of fair, doesn't it?" He went on to explain when you roll the die four times, "you take the 5 out of 6 chance and you multiply it four times . . . and what you end up getting is 5 to the 4th power over 6 to the 4th power." After one student calculated that to be 0.48, Mr. English concluded "48% of the time he would win, which means that 52% of the time he would not win." One student began to observe, "So if you play 100 times . . . ," which Mr. English finished by observing "he would win 48 times, but he would lose 52 times. So, in the long run, the . . . gambling house would come out $4 more."

Monday	Tuesday	Wednesday	Thursday	Friday
Day 1 Introductory Activities	Day 2 "Is This Game Fair?" "Doubles in Monopoly"	No class	Day 3 Gum Ratio, Analyzing Dice Sums	Day 4 Lottery, "Fair & Unfair Games"
Day 5 "Which Do You Think Will Be Larger?"	Day 6 "The Top and One Other," "The Hare & the Tortoise Game"	Day 7 Three Coins, "Spinners"	No class	Day 8 "Montana Red Dog"
Day 9 "Montana Red Dog" (reprise), Vowel Frequency	Day 10 "Quiz or No Quiz," "Frosted Wheat Yummies"	Day 11 "Tossing Pennies," "Newspaper Offer"	Day 12 "A Ratty Problem"	No class
Day 13 Area Model	Day 14 Pascal's Triangle	Day 15 Pascal's Triangle	Day 16 Review	Day 17 Unit Test

Figure 18. Calendar for Mr. English's probability unit.

In the remainder of Day 1, Mr. English went on to introduce several vocabulary words associated with probability, conduct a "little experiment" with colored cubes, and check for understanding with an example involving poker cards. The daily packet the students had picked up as they entered class included a worksheet reviewing the vocabulary and an assignment to do as homework.

"Is This Game Fair?" After conducting a review activity and correcting the previous day's homework assignment, Mr. English introduced the first of two activities the students were going to do on Day 2. Using a transparency for "Is This Game Fair?" (Zawojewski, 1991), Mr. English explained the rules and scoring for the game. The two participants, a player and an opponent, would each begin with 10 points. If the player, who was the only one rolling the pair of number cubes, rolled a sum of 7, the opponent transferred three points to the player. If the sum was not 7, the player gave the opponent one point. Mr. English demonstrated what the score would be after one toss that was a 7 and one toss that was not. After explaining they were going to see if it was a fair game, a student asked a question.

> S: What do you mean by fair? Fair to both players? Fair to one player?
> T: Okay. And that's a good question. Fairness means there's an equal chance that either person who's playing the game will win the game. Does that make sense?

As one student handed out the dice, Mr. English instructed the students they were to play the game for 10 rounds. While the students were playing the game, Mr. English circulated among the tables checking on how the activity was progressing. As students began to finish the game, Mr. English asked them to report whether the player or the opponent had won, as he recorded the results on a transparency made from the instructional materials. With final results showing the player had been the winner for 2 of the 12 teams playing, Mr. English asked the students their conclusion. "Thumbs up if it's fair. . . . Thumbs down if it's not fair." With a mixture of responses, Mr. English suggested, "Play it a second time and see." When the second game showed similar results, Mr. English pointed out, "What you're seeing is that the opponent is winning more often."

Then, Mr. English suggested they would analyze the game to see "how you can get those 7s." Using a 6-by-6 addition chart, he filled in the six places where a sum of 7 would occur. Explaining there were 36 squares or sums altogether, Mr. English concluded the probability of rolling a 7 would be 1 out of 6, the probability of not rolling a 7 would be 5 out of 6, and the odds of a 7 would be 1 to 5, having introduced both probability and odds the day before.

In what he later admitted was a spur-of-the-moment idea, Mr. English drew a circle divided into six equal portions. In one section, he wrote the payoff of three points and in the other five sections, he wrote the payoff of one point. Mr. English then observed, "So they're getting five points out of every six rolls, as opposed to the other person only earning three [points]." One student offered a suggestion, "They should get five points there . . . and it would be perfectly even, and then nobody ever would lose their money." A show of hands indicated several students agreed with this correct suggestion.

Another student, apparently not understanding the proportional nature of probability, asked, "What if they roll more times?" Mr. English responded with a reminder, "Remember, we're talking theoretically. That doesn't mean when we conduct the experiment that that's going to happen that way. It's just in theory . . . that's the way it would work."

As a follow-up to this activity, Mr. English had the students do a second activity, "Doubles in Monopoly" (Lane County Mathematics Project, 1983b). This game also involved rolling dice. The two participants, a player and a banker, both began with $10. If the player rolled doubles, the banker paid the player $3. If the player did not roll doubles, the player paid the banker $1. After asking if the rules sounded familiar, Mr. English pointed out "the payoff is the same" as the previous game. In this case, the pairs of students were to roll the dice 20 times, keeping track of their score on the worksheet provided in their packet. After about 3 or 4 minutes to complete the activity, Mr. English began having the students report their results. In this game, the banker had come out ahead 11 times while the player had won only 5 times.

With about 8 minutes remaining in the period, Mr. English asked the students to turn to the summary sheet in their daily packet. On this page, he asked them to "tell me why that last game was unfair . . . and I want you to tell me how to fix it so that it's fair." When one student suggested "it seemed fair," Mr. English referred them to the results. After a few minutes to write down their answers, Mr. English asked if anyone would read what they had written. One student, whose answer had been affirmed by Mr. English as he had been circulating among the tables, volunteered to read her solution, "The dice game seems unfair, because the odds are 1 to 5. If you win, you should get $5 instead of $3 so that you get paid back. The theoretical probability of throwing doubles is 1 in 6. That means five sixths of the times you're going to pay and you'll be paying out $5 but only getting back $3." When she finished reading her summary, Mr. English went on to repeat and expand on what she had said, drawing a circle similar to what he had drawn in the first activity. One other student had time to read her answer before the period ended.

"Which Do You Think Will Be Larger?" Mr. English began Day 3 with the extension of a proportion activity he had done in an earlier period, applying it to probability in this case. The students were asked to count how many pieces of gum were under their table and chair. Based on that sample, the students were asked to predict how many pieces of gum were in their classroom and in the entire school. Also on Day 3, the students completed a series of worksheets on which they (a) determined the theoretical probability of the dice sums, (b) compared the probability to the odds for each possible sum, (c) recorded the experimental results of 36 tosses of the dice, and (d) graphed their experimental results.

After spending the first half of the period on Day 4 (a Friday) catching up on grading the warm-ups and extra credit assignments, Mr. English led the class in analyzing some lottery situations. They first considered the probability of selecting a winning one-, two-, or three-digit number. Then they analyzed a version of the state's Powerball lottery. In the time remaining, Mr. English demonstrated how to analyze a dice game included on the "Fair or Unfair Games" worksheet he had prepared.

As Mr. English had reflected on the first week's lessons over the weekend, he decided he was doing too much talking up front. As Day 5 began, he informed the researcher he planned to get the students more involved in conducting the simulations and doing the analysis. The first activity he had planned for this day was another dice game, "Which Do You Think Will Be Larger?" (Lane County Mathematics Project, 1983a). Mr. English explained to the students they were to play the game for 30 rounds and then decide whether the game was fair or unfair. If they thought the game was unfair, they were to talk about how they could make it fair.

This game involved two players with three regular dice. Player A was to roll two dice and find the product of the two numbers. Player B was to roll one die and multiply the number times itself. The player with the larger product was the winner. Mr. English had the students make a prediction, as asked for on the worksheet, of who they thought would win more often.

After handing out the dice, Mr. English circulated among the tables making sure the students understood the directions. A number of students asked what to do if it was a tie. Mr. English interpreted the question to mean the overall results, suggesting that meant it was fair. When the question was clarified, Mr. English responded that, in case the products were the same, no one received a point. As students began to complete the 30 rolls, Mr. English asked the students, as he moved from table to table, if they thought the game was fair.

When the students had finished and all the results had been recorded, Mr. English asked the students to vote, "If you say it's unfair, based on the fact that B won a whole lot more than A . . . raise your hand if you believe that." When some students responded they thought it "still might be fair," Mr. English asked them to explain their thinking. One student suggested it was fair because he had won. Another student explained that it "depends on what you roll."

Without any further discussion, Mr. English shifted to the process of analyzing the game theoretically, although this process was not part of the assignment on the worksheet. He began by asking the students if they had ideas, saying, "I've asked you to think about it. Maybe you can give me some ideas on how to . . . figure out whether this is fair or not. Anybody have any idea what we might do to figure it out?" Bryan, one of the most successful students in the class, offered an idea which Mr. English sought to clarify.

S: Well, I just finished adding up . . . all the possibilities. If you did 36 dice . . . rolls of the dice for each one . . . the B would get 546, if you add it up. And A would get 461.
T: Did you fill out this chart [*putting up a transparency of the multiplication chart included in their packet*]?
S: No.
T: You did it a different way? Would you . . . could you come up here and draw what you did [*as students are making various derogatory comments about "the brain"*]?
S: I didn't do it on paper.
T: You didn't do it on paper? You just thought it through? So you're conclusively believing that B is going to win?
S: Yes.

Not quite understanding what Bryan had done, Mr. English went on to explain "the way I thought about it." He began by setting up a chart (see Figure 19). Then using a 6-by-6 multiplication table as a guide, he modeled for the students how to complete the chart for the first two possibilities. For example, if the one die landed on a 2, player B's product would be 4. Comparing this product to the multiplication table which would show player A's products, there were five results less than 4, three equal to 4, and 28 results greater than 4. Therefore, B had 5 ways of winning and 28 ways of losing. At this point, Mr. English gave the students a few minutes to complete the chart and find the totals.

There was some disagreement when students began reporting their totals. When three groups agreed on a total of 113 ways to win and 95 ways to lose, Mr. English concluded that was the correct answer. He went on to ask, "With it that close, is it still unfair, or is it fair?" There was some difference of opinion, although most students agreed they'd rather be player B if they had a choice. Then, with about 12 minutes left in the period, Mr. English explained what the students were to do with the other pages in their packet of materials.

# single die	ways to win	ways to lose
1	0	35
2	5	28
3		
4		
5		
6		

Figure 19. Chart Mr. English used for the analysis of "Which Do You Think Will Be Larger?"

"The Hare and the Tortoise Game." During the first part of the period on Day 6, Mr. English and the students analyzed the fairness of a game given in the previous day's packet and then played and analyzed another game using dice. After an interruption in which the band members returned from an outing, Mr. English introduced "The Hare and the Tortoise Game" (Phillips et al., 1986) by reminding the students of the fable about the tortoise and the hare. In this game, a player would begin at the circle marked "Start" in the center of the game board (see Figure 20). The player would roll a die three times, each time moving their marker one place right if the toss was even and one place left if the toss was odd. If after three moves, the marker ended up on P or X, the tortoise received a point. If it ended up anywhere else, the hare earned a point. Because the worksheet was unclear, the students were confused about how to record their results. After trying to clarify the recording procedures, Mr. English just had the students cross out the column headings in the chart and use one column for recording their results.

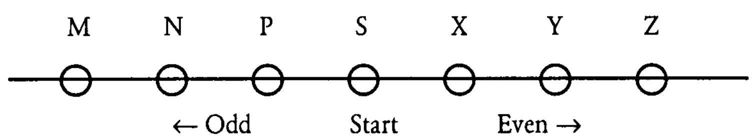


Figure 20. Game board for "The Hare and the Tortoise Game."

One very observant student commented, "But there's some letters you can't do." Mr. English responded, "And that's true. Figure out which ones they are as you play the game." After about 7 minutes of playing the game, and as the rest of the students were finishing up, Mr. English had one student report his results, which Mr. English wrote on a transparency. Using these results as their basis, Mr. English and the class determined the game favored the tortoise. Mr. English concluded, "And, furthermore, did you notice that three of these letters you cannot get? So the probability is zero, it's impossible."

Mr. English then moved on to explain "the way we analyze this one." He began by drawing a tree diagram (see Figure 21), explaining at each branching point one has two choices. With input from the students, Mr. English proceeded to record the outcomes of each branch (e.g., EEO) and the corresponding ending point on the game board. From these results, Mr. English had the class summarize the probability of the hare winning and of the tortoise winning, as well as the odds in favor of each.

With about 3 minutes of class remaining, Mr. English gave the students a follow-up problem to think about at home and come the following day "ready to talk about that." First, the students were to flip three coins 20 times. Then, as Mr. English explained, "I want

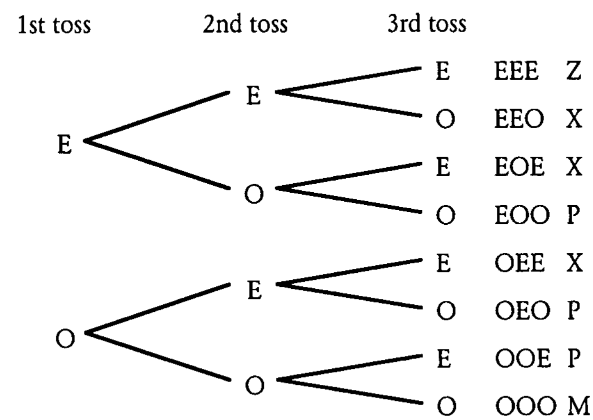


Figure 21. Tree diagram drawn by Mr. English for "The Hare and the Tortoise Game."

you to tell me what the probability is to get a match versus the probability that you get a no match. And I want you to draw me a tree diagram, showing all the outcomes."

"Spinners." After discussing the coin activity as class began on Day 7, Mr. English moved on to an activity called "Spinners" (Phillips et al., 1986). Mr. English first wanted the students to look at the spinners and write down the theoretical probability of each region. From their experience with fractions and percents, the students quite readily came up with correct responses as Mr. English led them through labeling each of the regions on the three spinners given on the page. The next step was to obtain an experimental probability "to see whether those actually agree" with the theoretical predictions. Pointing out the instructions were to spin each spinner 100 times, Mr. English asked if anybody could "suggest a way we might get 100 things but not have to do it." The students offered such suggestions as doing it 10 times and multiplying "what happened by 10" or doing it 20 times and multiplying by 5. Explaining those suggestions were not what he had in mind, Mr. English then told the students he wanted them to spin each spinner 25 times and then to combine their results with three other people for a total of 100 spins on each spinner.

For the next 15 minutes, the students worked on collecting their experimental data. At that point, Mr. English interrupted the activity briefly to give further instructions to the students. On the summary sheet contained in their packets, Mr. English asked the students to make a chart for each of the spinners summarizing the theoretical probability and their experimental probability. The students continued gathering data and making these charts for the remaining 15 minutes of the period. As the period ended, Mr. English commented to the researcher he did not know what else to do with the activity besides just comparing experimental and theoretical results. However, the students had not explicitly been asked to compare the results they had summarized.

"Montana Red Dog." Day 8 was a Friday and, therefore, about 10 minutes was spent at the beginning of class to catch up on grading the warm-up and extra credit assignments. Mr. English then introduced a game with poker cards called "Montana Red Dog," which he had found in the NCTM Addenda materials for grades 5-8 (Zawojewski, 1991). Each group of four students would be dealt four cards. Mr. English's student teacher, who was serving as the dealer, would hold the remaining 20 cards. Each group was to "look at the four cards [they had] and decide as a group what your chance or probability is that you have one card out of the four that will beat the top card that [the dealer] is going to flip over." They were to choose a confidence level ranging from "0—we don't think we can beat it" to "3—we are certain we can beat it." When the dealer's card was revealed, the group received points corresponding to their confidence level if they had a higher card in the same suit. If they could not beat the card, the teacher received the points. For example, after looking at their cards, one group of students may chose a confidence level of 2, being "pretty sure we can beat" the dealer's card. If the dealer selected this group and turned over a 5 of hearts, for instance, the students would receive two points if the group had a heart higher than a 5. Otherwise, the teacher would receive two points.

After clarifying some of the rules, the cards were dealt and the game began. Following each round, the cards from that round were shown and the remaining groups were allowed to reassess their confidence levels. As the game progressed, Mr. English suggested that the later groups should have a higher probability "because they already know what most of the cards are."

The first game ended with a score of 14 to 2, in favor of the teacher. Because the first cards the dealer had turned over had been two aces and a king, the students were convinced the game had been rigged. One student vowed to follow each step the dealer made in the next game. Before beginning the second game, Mr. English handed out a sheet the students could use to keep track of the cards played and to determine what the odds were. The worksheet included a list of cards for each suit, as well as boxes to fill in with the number of cards in each suit that the player could or could not beat (see Figure 22). Totaling these boxes gave the students what their odds were. After a sample demonstrating how to fill in the information, Mr. English suggested "it seems like you should be able to maybe turn this around and the class beat the teacher even though [I] beat you the first time." Playing a little bit more cautiously in the second game, the students remained even through the first few rounds. Nevertheless, they seemed hesitant to wager 0 points even when odds were clearly against them. As the teacher began to move ahead, the students seemed to ignore the odds and wager the point values needed to catch up. They were unsuccessful, however, with the final score of the second game being 11 to 3, in favor of the teacher.

Round ☐

Number of cards you can beat ____	Number of cards you can beat ____	Number of cards you can beat ____	Number of cards you can beat ____	Sum ____	Confidence Level ☐
Number of cards you cannot beat ____	Number of cards you cannot beat ____	Number of cards you cannot beat ____	Number of cards you cannot beat ____	Sum ____	Score ☐

Figure 22. Recording chart created by Mr. English for the "Montana Red Dog" game.

In reflecting on the game of "Montana Red Dog," Mr. English identified some weaknesses or concerns which he pointed out to the researcher. First, after a group had been chosen and played their hand, they had no particular incentive to continue considering the odds or confidence levels. This situation made management more difficult as well as lessening the educational value. Second, Mr. English did not feel the students had been attentive to using odds in the decision-making process, though giving the students an intuitive notion of odds had been one of his goals in using the activity. To address these concerns, Mr. English decided to follow up with a revised version of the game on Day 9.

This time Mr. English intended to show four cards at a time to the entire class so that everyone was involved with each round. He had prepared handouts for the students to keep track of the cards played and to evaluate the odds using boxes as shown earlier in Figure 22. Each student was to calculate the odds and to decide on their own confidence level. Each player would start with $10 and add or subtract amounts corresponding to their confidence level if they won or lost, respectively.

After clarifying some early confusion about how to record the cards played, the teacher and the class played the game for eight rounds. After playing the first few rounds, Mr. English observed, "Now the odds are going to get more and more in your favor, I would think, as the card deck gets smaller. That's my prediction." After finishing eight rounds of play, only two students ended up with less than the $10 with which they had begun. The rest of the class ended up with somewhere between $11 and $18, even though the students had held winning cards only three times in the eight rounds. To conclude the activity, Mr. English pointed out to the class the odds had not swung in their favor when there were fewer cards, as he had predicted. He listed what the odds had been for each round, observing that only two times out of the eight rounds had the odds favored the students.

For the remainder of the class period, the students worked on a sampling activity which involved counting the vowels and letters in two short paragraphs and seeing if there was "any correlation between the probability that occurs in [paragraph] number 1 and what actually happens in number 2." As homework, Mr. English assigned three problems from a worksheet entitled "Experimental Probabilities" (Phillips et al., 1986). Each of the problems gave the students the results from a poll. Using this information, the students were to make certain predictions or find probabilities.

"Frosted Wheat Yummies." After correcting their homework assignments on Day 10, Mr. English announced he had a math quiz with questions on probability. But he was going to do an experiment to see if the class would actually take the quiz or not. He tried the coin tossing experiment three times and the students lost each time. Announcing the students would take the quiz later, he proceeded to introduce the "Frosted Wheat Yummies" activity he had prepared by telling the students a story.

> Several years ago, after studying probability, I decided that I would form a company and I would sell cereal. . . . We decided . . . we would call this Frosted Wheat Yummies. . . . I'm not going to tell you the ingredients of it. I'm just going to tell you that it tastes horrible. But it doesn't matter because I learned when . . . my wife and I raised our three kids, that every time we went to the grocery store it didn't matter what was in the box as long as there was a prize in there. And if there was a prize in the box my kids always insisted that that was their favorite kind. . . . Now, in these cereal boxes I have put together six different colors of fluorescent pens. . . . And every kid 8 years old and under insists that their parents buy that particular brand of cereal when they go to the store.

Mr. English then informed the students they were going to do a simulation to determine how many boxes would have to be bought in order to get all the pens. Referring to the worksheet he had prepared and included in their packet (see Figure 23), Mr. English explained they were going to roll a die. Every time they had a 1 on the die, they would mark a tally for a red pen, and so forth until they had at least one tally in each box. They were to do the simulation five times, corresponding to A, B, C, D, and E. The worksheet then directed the students in the steps of finding the average which Mr. English identified as the expected value. Mr. English went on to observe,

> Now, there's not a way that I think you can set this up theoretically, by looking at the number of outcomes possible in relation to the total outcomes. I don't think you can do that with this. . . . All you can do is conduct an experiment and say, based on the experiment, this is what we would expect to be the number of boxes that parents would have to buy their kids to get all six brands of this prize.

Mr. English concluded by providing some examples of such merchandising approaches in real life.

Mr. English then handed out the dice as the students began working individually on conducting the simulation. When it appeared some students were finished with the worksheet, Mr. English asked them to "find the average for your group." As some of the groups were finished and others were still working, Mr. English asked those who were finished to report their results for the number of boxes and the cost of getting all the pens. Based upon the class' results, Mr. English concluded this way of obtaining the pens would be expensive. But, as he pointed out, it must help sell cereal because the companies had been putting prizes in the boxes "ever since I was a kid."

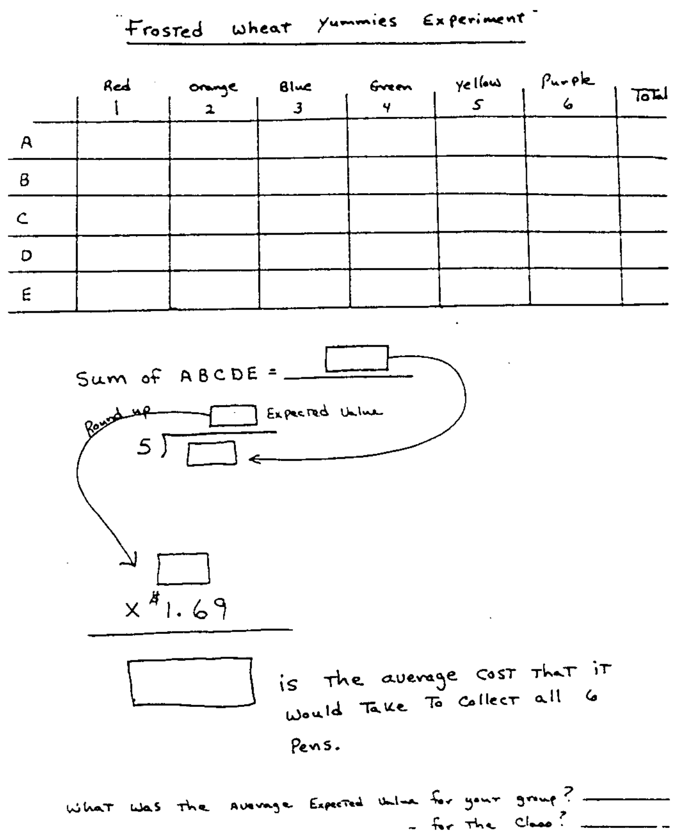


Figure 23. Worksheet created by Mr. English for "Frosted Wheat Yummies."

On Day 11, Mr. English first discussed a "Tossing Pennies" activity (Hoffer, 1978) that had previously been handed out in the students' packets. The students then simulated another expected value situation in "The Newspaper Offer" (Phillips et al., 1986) using poker cards. On Day 12, Mr. English introduced the students to binomial situations with an activity called "A Ratty Problem" (Montana Council of Teachers of Mathematics, 1989). In this activity, the students used a die to simulate the random decision-making process of rats as they ran through a maze and escaped at various openings.

During Days 13 to 17 which were not observed, Mr. English indicated he went on to present the area model and Pascal's triangle, closely following the lessons in the Middle Grades Mathematics Project materials, *Probability* (Phillips et al., 1986). In particular, with the area model, the worksheets had the students analyze a maze using a "matrix with 36 squares." Mr. English reported he also had the students draw and analyze their own mazes. According to Mr. English, 2 days were spent exploring and applying Pascal's triangle. He reported he first reviewed the different binomial situations the students had seen in the unit, including heads or tails with coins and odd or even with dice. He also introduced the nearly binomial situation of the birth of boys and girls to a family. And on one of the days, Mr. English gave the students a 10-question true/false test on which they were to guess without knowing the questions. This test served as an introduction to the activities for Pascal's triangle in the Middle Grades materials.

Unit test. As the final evaluation for the probability unit, Mr. English gave the students a two-part test. For the first part, the students retook the same 18-item test they had taken as a pretest before the unit began. On this part of the test, the items asked the students to fill in a vocabulary word or to find the probability of a given event in situations similar to what they had done in class with dice, coins, cubes, or cards. In a potential application of Pascal's triangle, the students were asked,

> You are taking a true or false test with four questions. What is the probability that you will get a perfect paper?

The second part of the unit test contained an additional 17 items, including further questions about probabilities with dice, coins, and cubes, such as,

> I put 12 colored cubes in a box. Five are red, four are blue, and the remainder of the cubes are green. What is the probability that I will draw out a green cube?
>
> Suppose one die is tossed 36 times. About how many times would you expect a 5 to come up?

Other items had the students judge the fairness of three games, including

> A game is played by two players in which two pennies are flipped. Player A scores a point if the coins do not match. Player B scores two points if heads-heads comes up. Is this game fair? ____ If the game is not fair, then which player should win? ____

Finally, this portion of the unit test also asked the students to identify the odds in one game with dice.

Evaluation of the Probability Unit

Evaluating student learning. In the course of the probability unit, Mr. English suggested he had assessed the students by watching and interacting with them, monitoring their responses. In addition, he had given the students a quiz on Day 10 to check their understanding. The final evaluation of student learning was done with a unit test at the end of the unit. Mr. English explained that, although he used many of the activities from the Middle Grades *Probability* book (Phillips et al., 1986), he did not use their test because "that test, basically, is a very difficult test and, even though you've gone through . . . I used to give it. I had very few kids that would pass that test. And I don't know if it was necessarily designed for kids to get a very high score on it. It is basically to show that they have made progress. But . . . what I've found with that particular test was that it tended to discourage the kids and it discouraged me." Instead, Mr. English prepared his own unit test. But, in preparation for the unit test, Mr. English used the review questions from the Middle Grades Project as something for the class to review and discuss together.

Evaluating unit effectiveness. In evaluating the overall success of the unit, Mr. English pointed to the pretest and posttest results. Although only six students had scored 70% or better on the pretest, Mr. English reported none of the students had fallen below 70% on the posttest and, in fact, "most of them were 100%." In considering specifically what the students had learned, Mr. English listed a number of ideas. First, he suggested "they did learn the vocabulary, . . . how to determine odds, and how to determine probability for different situations." Second, he felt the students had learned "how to diagram" certain probability outcomes. Third, Mr. English felt the students had "gained a good foundation" for considering theoretical and experimental probability. In particular, he observed, "They did learn that just because theoretically you can say it should happen like this, it doesn't mean that it will. In fact, most of the time, it probably doesn't come out exactly that way, but it does come close and the more you do it, the closer it does get to the actual probability." Finally, Mr. English thought that, as a result of some of the simulations they had done, the students may have some second thoughts before getting involved in gambling situations.

Cross-Case Analysis: Teachers' General Pedagogical Knowledge

Now that the four middle school teachers have been introduced and their probability units have been described, this chapter next addresses the research questions. This section will consider the first of those questions: What general pedagogical knowledge do middle school teachers demonstrate in the context of teaching probability? (A detailed analysis of the teachers' general pedagogical knowledge, including examples from the interviews and classroom observations, is presented in Appendix C.)

The picture of middle school teachers' general pedagogical knowledge emerging from the cross-case analysis of the four teachers in this study is one of contrasts. First, the contrast in general pedagogical knowledge between novice and experienced teachers is apparent. Second, the knowledge and beliefs of teachers with a more traditional view toward the learning process stand in contrast to the knowledge and beliefs of teachers with a constructivist approach to learning, the approach encouraged by the reform effort in mathematics education.

Knowledge of Experienced Teachers Versus Novice Teacher

The three experienced teachers, Mrs. Books, Mrs. Talent, and Mr. English, demonstrated a general sense of expertise in their role as middle school mathematics teachers. They expressed confidence in themselves as teachers and displayed mastery of the basic tasks of teaching. Their experience and expertise were reflected in all aspects of their general pedagogical knowledge, including their educational goals, their understanding of the learning process, and their organization and management of the classroom environment.

First, these teachers' general pedagogical knowledge included a recognition of long-term educational goals. These experienced teachers established high academic goals for their students; at the same time, they did not neglect important affective goals. Besides learning basic mathematical skills, these teachers were also striving to meet the goals of the reform movement by involving their students in problem solving, reasoning, and communicating about mathematics as well as helping them see the connections between mathematics and the real world. Their success in reaching these goals varied. Nevertheless, the teachers recognized where mathematics instruction should be heading and what the aims of that instruction should be.

Second, the experience and expertise of Mrs. Books, Mrs. Talent, and Mr. English were reflected in their understanding of the process of mathematics instruction. Although these teachers had different ideas about the role of the teacher and students in the learning process, they nevertheless acted upon a generally consistent set of beliefs. These teachers recognized the need to design instruction appropriate to the nature and background of students, reporting the level of the students influenced their decisions concerning what activities to do in the classroom, how much explanation to provide, and at what pace the class should move. In addition, as a result of their knowledge of students, their instruction included a variety of activities, the use of manipulatives, and opportunities for students to practice what they were learning. Also, despite their differing views about the teacher's and students' roles, these individuals believed teachers play an important and valuable role in the learning process, either as a deliverer of knowledge and/or as a facilitator of learning.

Third, the three experienced teachers demonstrated competence in creating and managing an effective learning environment. There was a degree of flexibility and variety in how they arranged students for instruction. Their classes ran smoothly, generally with an effective use of class time. Classroom routines for dealing with instructional materials were known and followed by the teacher and the students. Expectations for student behavior were posted and consistently enforced. Reminders or warnings were provided when necessary and consequences were meted out when appropriate. Problems were handled expeditiously with a minimum of disruption to the class. Finally, these teachers consistently showed respect for their students.

Mr. Trackman stands in contrast to the overall picture of competence painted for the three more experienced teachers. Although he had been teaching for 3 years and had received tenure in his district, Mr. Trackman continued to deal with problems common to beginning teachers. His difficulties and weaknesses were evident in all aspects of his general pedagogical knowledge.

First, in contrast to the more experienced teachers, Mr. Trackman demonstrated a shallow grasp of either the short-term or long-term goals of mathematics instruction. Specific academic goals were noticeably missing from his list of instructional goals, either for mathematics generally or probability particularly. Although he expressed the goals that students learn how to learn and that they "come up with some sort of understanding of probability," he appeared to have no sense of what that learning should be. Instead, Mr. Trackman's primary focus seemed to be on making mathematics fun, particularly with his use of games during the probability unit. However, his overall goal appeared to be getting through the day and keeping the students busy with something in the process.

Second, Mr. Trackman also lacked a coherent knowledge of learners and the learning process. He attributed success in mathematics to students' efforts or innate ability and emphasized the importance of rote learning. Interestingly enough, he failed to see the teacher's role in the learning process, either as the one in charge or as the one responsible for orchestrating the learning opportunities. In his instructional efforts, his actions frequently were inconsistent with his words. In particular, although he spoke of the ways he dealt with students' wide range of learning abilities and learning styles, there was no evidence in the observed lessons of instructional efforts designed to meet the individual needs of students or to promote the learning of mathematics content. Again, learning did not appear to be the overall goal in Mr. Trackman's classroom.

Third, Mr. Trackman was ineffective in creating and managing an effective learning environment. Several of his instructional routines contributed to an inefficient use of time, including his practice of letting students select their own groups and his pattern of writing on the surface of the overhead projector. Because he often did not plan enough for the students to do, they had too much "free" time to engage in off-task behavior. Further, his classroom management efforts were generally ineffective because he appeared to be more interested in being accepted by the students than assuming his role as classroom leader. In his attempts to be liked by the students, Mr. Trackman had not consistently enforced the posted rules and had participated himself in the off-task behavior and talk. At times, he had also shown a lack of respect for his students.

Many of Mr. Trackman's difficulties are typical of teachers during their first few years of teaching. However, Mr. Trackman did not appear to be interested in growing and learning more about his role as a teacher. In contrast to the more experienced teachers who continued to take advantage of opportunities to learn more about mathematics and about teaching, Mr. Trackman had made no similar effort to participate in ongoing learning opportunities, except for a "couple classroom management classes." He projected a sense of confidence that he had mastered the basic skills of teaching, with the exception of classroom management on which he admitted he was still working. As a result, Mr. Trackman had fallen into a pattern of survival, not realizing he was just going through the motions of teaching. He, in general, was not reflecting on his teaching practices to see the difficulties or to seek solutions. Although he was unhappy about aspects of his situation, he was more likely to blame the problems on the students, the time schedule, or other outside factors, rather than recognizing his responsibility or seeing what he could do to improve the situation.

Knowledge of Teachers with Traditional Views Versus Constructivist Views

The second contrast evident in the teachers' general pedagogical knowledge was the contrast between a more traditional view of the learning process and a constructivist view. On the one hand, Mr. Trackman, Mrs. Talent, and Mr. English held more traditional views about learners and the learning process. To them, learning was viewed as a process in which teachers are the presenters of knowledge and students are the recipients. As a result, the general instructional pattern of these teachers involved teacher-directed activities combined with opportunities for students to practice. To varying degrees, these teachers made adjustments in the pace of instruction, the choice of activities, and the amount of explanation to meet the specific needs of their students. And at least Mrs. Talent and Mr. English recognized the importance of keeping middle school students busy and engaged in concrete hands-on activities.

On the other hand, Mrs. Books held constructivist views of the learning process, largely the result of her own experience during her teacher preparation program. To her, learning mathematics meant a process of constructing meaning and developing understanding. Throughout the process, students were to be active participants as they interacted with mathematical ideas and with other people, including their peers and the teacher. As a result, instruction was student-focused as Mrs. Books interacted with individual students or small groups of students, helping them to identify what they already knew and to construct a new level of understanding. Students were encouraged to share their questions and conjectures with one another and were expected to justify their conclusions with logical and reasonable arguments. Two pedagogical practices distinguished Mrs. Books from the other teachers in this study and enabled her to implement her beliefs about mathematics learning. First, Mrs. Books chose to do fewer activities and spend more time on each. As a result, more time was available to actively involve the students in the learning process as they worked on problems and shared their problem-solving and reasoning efforts. Second, Mrs. Books had made a special effort of establishing a classroom environment where a sense of respect was shown by the teacher and the students for all others in the classroom. In this atmosphere of respect, the students were free to share their ideas, uncertainties, and questions as they dealt with the disequilibrium of the learning process.

The dividing line between these contrasting views was not as clearly drawn as the contrast between novice and experienced teachers. Indeed, although they generally held a more traditional view toward learning, Mr. Trackman, Mrs. Talent, and Mr. English were also aware of the efforts to reform mathematics instruction. To varying degrees, their views and instructional practice had been influenced by those reform efforts. These influences were evident in the goals the teachers stated for mathematics instruction and in the instructional strategies they attempted to implement in their classrooms. In particular, the teachers' goals for mathematics instruction included problem solving, communicating about mathematics, reasoning, and seeing connections within mathematics and between mathematics and the real world, all goals emphasized in the *Curriculum Standards* (NCTM, 1989). Their instructional practice also included efforts to use hands-on manipulatives, problem-solving tasks, cooperative group work, and authentic assessment tasks.

Nevertheless, with the exception of Mrs. Books, the teachers' efforts to implement mathematics instruction as envisioned by the reform efforts generally fell somewhat short of the mark. From a general pedagogical perspective, this result seemed to occur because the teachers lacked a coordinated overall view of the new vision for teaching mathematics. As the teachers attempted to add one or more aspects of the reform to their otherwise traditional framework, the result frequently was inconsistency as the two views were merged. For example, the teachers wanted to involve their students in problem solving and reasoning, but their teacher-directed approach to the activities lowered the cognitive level of the tasks. This decline in the cognitive level of the tasks happened primarily because it was the teacher, not the students, who was doing the problem solving and reasoning. Similarly, Mrs. Talent made use of unfamiliar problems as assessment tasks. However, because most of her instruction had been delivered in a "show and tell" manner, the students had been spectators of the learning process and were not prepared to participate on their own without the teacher's guidance. Thus, although the students apparently were able to follow the teacher's directions and complete the more familiar instructional tasks by mimicking what had been modeled for them by the teacher, they encountered difficulties when given an unfamiliar situation to analyze entirely on their own. Even though they had practiced the various skills involved, they had been given no opportunity to practice applying the skills in unfamiliar settings, until given the assessment task. As a result, the students frequently did not know what to do when given an unfamiliar task. In conclusion, the teachers were aiming at goals consistent with the reform efforts, but, in the end, those goals appeared to more accurately describe the activities done in the classroom (often by the teacher) and not necessarily the learning students took with them from the classroom.

Cross-Case Analysis: Teachers' Subject Matter Knowledge

Next, this chapter addresses the second research question: What is the teachers' subject matter knowledge of probability? Two facets of subject matter knowledge will be reviewed. First, this section will describe the teachers' knowledge of probability content. Second, the teachers' knowledge about the nature of mathematics and probability will be discussed. (A more comprehensive analysis of the teachers' subject matter knowledge is presented in Appendix D and the interview questions referred to are stated in Appendix A: Probability Questions.)

Teachers' Knowledge of Probability Content

A major portion of the pre-observation interview was devoted to exploring the four middle school teachers' knowledge of probability content. Although the interview questions were designed to potentially go beyond what might be familiar material to middle school teachers, it is not unreasonable to expect middle school teachers to have knowledge of probability beyond what they teach, such as what was presented in the interview questions. Nevertheless, the interview responses given by the teachers were correct only 50% to 70% of the time. Mrs. Books was at the upper end of that range, providing more correct answers than the other teachers, although the differences between Mrs. Books, Mrs. Talent, and Mr. English were minimal. Mr. Trackman fell at the lower end of the range, answering only about half of the items correctly. Thus, the teachers interviewed fell within a relatively narrow band even though their subject matter backgrounds were relatively diverse.

The probability content involved in the interviews and observed in the classrooms included the ways of measuring chance occurrences, the basic properties of probability, the strategies for analyzing probabilistic settings, and the applications of probability. In addition to summarizing the teachers' knowledge of this content, this section will review the nature of the teachers' probabilistic intuitions.

Ways of Measuring Chance Occurrences

Both in the interviews and during instruction, the teachers generally expressed the probability of simple events correctly when situations familiar to middle school classrooms were involved. In particular, the teachers were able to find the experimental probability of an event when the results of tossing dice or coins, spinning a spinner, or selecting objects from a bag were known. The teachers also were able to apply the definition of probability (the ratio of the number of favorable outcomes to the number of possible outcomes) to determine the theoretical probability of an event in similar simple situations. However, although the teachers could express probability by interpreting the results of an experiment or by applying the definition of probability, many of the underlying assumptions remained implicit even during instruction. Specifically, the teachers seemed either to be unaware of the assumption underlying the definition of probability that all outcomes are equally likely or to be influenced by the "equiprobability bias" (Lecoutre, 1992), where random events are assumed to be equally likely by nature. Similarly, the teachers seemed to be unaware of the subtle shift made when thinking of experimental probability in terms of relative frequency or when applying the students' knowledge of fractions in place of the definition.

The teachers' knowledge about other ways of expressing the likelihood of uncertain events was more limited. In the interviews, three of the four teachers correctly expressed the odds in favor of an event involving dice. However, during instruction Mrs. Books and Mrs. Talent used the language of probability and the language of odds interchangeably, indicating the distinction between probability and odds may not have been fully understood. Although Mr. English correctly presented the definition of odds to his students, he sometimes used unclear language to describe odds and on one occasion misinterpreted odds as a probability. The fourth teacher, Mr. Trackman, incorrectly believed there was no difference between probability and odds, explaining to his class that one chance out of six (1:6) and one winner compared to five losers (1:5) were "saying the same thing."

Similarly, the teachers lacked a complete understanding of the difference between expected value as a long term average and probability as a long term frequency. Although the teachers provided correct responses to the Newspaper Pay interview item dealing with expected value (probability question #6), none of the teachers acknowledged the idea of expected value or explained why they considered the overall results on an annual basis instead of the probability of the different outcomes on a monthly basis. Mr. English defined expected value as a "long term average" for the two expected value activities he did with his students. He also modeled how to find the expected value, but he appeared to have little more than a procedural knowledge of the concept. In particular, he did not explain why or under what circumstances one might want to consider expected value instead of probability.

Basic Properties of Probability

The teachers' knowledge of the basic properties of probability also seemed to be limited to simple and familiar settings, including their application of the probability of a complement, the possible range of probability values, the sum of the probabilities of all outcomes in the sample space, and the addition and multiplication properties. Although the teachers could correctly apply the basic properties in simple situations, they seemed to lack an explicit understanding of the properties. In particular, their interview responses did not use the corresponding terminology (e.g., complement or impossible event) nor include clear or complete explanations. For the more complex interview questions, the teachers were unsure of when to apply the addition or multiplication properties.

During instruction, the properties occurred primarily on worksheet or textbook assignments. When these assignments were corrected, no discussion identified the properties or checked the students' understanding of them. As a result, important aspects of the properties were not expressed and perhaps were not understood by the teachers. For example, although the instructional materials suggested the students should add the probabilities in order to find the probability of more than one event occurring, neither the materials nor the teachers emphasized that the events must be nonoverlapping.

Strategies for Analyzing Probabilistic Settings

The teachers' effectiveness in theoretically analyzing probabilistic settings also was influenced by the familiarity and complexity of the situation. Prior to teaching their probability units, the teachers used organized lists, tree diagrams, area models, or the multiplication property in response to the interview questions. These strategies along with the definition of probability were used correctly in simple experiments such as selecting marbles from a bag or tossing two coins. However, the teachers' responses to the more complex situations included in the interview questions were, for the most part, incorrect and uncertain. In these cases, the teachers' attempts to solve the more difficult problems often involved misapplications of the multiplication property, a failure to consider the importance of order, or an uncertainty about whether to add or multiply probability values. In addition to the strategies used in the interviews, the teachers also used charts or Pascal's triangle as analysis tools during probability instruction. The knowledge and use of these different strategies, however, varied from teacher to teacher.

The teachers correctly applied these strategies in simple settings and/or reproduced the analysis in situations they had seen modeled. Nevertheless, the teachers seemed to have a procedural rather than a conceptual understanding of the analysis strategies. In particular, although the analysis procedures were modeled for the students, the thinking processes were not made explicit. For example, the teachers did not emphasize the goal of finding all possible outcomes so the definition of probability could be applied. The teachers also did not explain the role of some of the strategies, particularly tree diagrams, in reaching the goal of finding all possible outcomes. Similarly, the teachers did not stress the relationship between the actions in the experiment and the stages of the tree diagram. Nor did they recognize how labeling the stages of the tree diagrams could help the students see that relationship. At times, almost by accident, the teachers did label the tree diagrams in ways that were helpful, but then, at other times, their presentation of the information confused the students. In many cases, these aspects of the analysis process seemed to be known only tacitly by the teachers, if they were known at all. Additionally, these aspects were not emphasized in the course of instruction, although doing so could have been beneficial.

In addition, the teachers' understanding of the analytical strategies seemed to be somewhat tenuous and disjointed. For example, the teachers could analyze a two-stage binomial situation correctly (when tossing two coins), but had difficulty when dealing with a five-stage binomial situation in the Birth item (probability question #4). The teachers could identify probability of different outcomes when spinning one spinner, but were unable to determine the probability corresponding to the outcomes when two spinners were spun. Mrs. Talent included worksheet assignments where students used tree diagrams and the multiplication property to analyze situations with two spinners (with equally likely outcomes), but she had been unable to analyze the Two Spinners item (probability question #5) in the pre-observation interview (in which the spinners had unequally likely outcomes). Each of the teachers applied the multiplication property improperly in the Birth problem (probability question #4), but did not recognize it could be applied in the Two Spinners problem (probability question #5), a problem somewhat less familiar to the teachers. In general, the teachers seemed uncertain about when to consider order, when to add probabilities and when to multiply them, and when or how to set up charts, draw tree diagrams, or apply other strategies for the purpose of analysis.

In some cases, the teachers' understanding of the analysis process seemed to be limited to situations they had seen modeled. Mrs. Books used one-dimensional area models effectively, but was unable to use area models to analyze a two-stage experiment. Mr.

English's understanding of the area model was limited to the square subdivided into 36 smaller squares as modeled in the Middle Grades Mathematics Project materials (Phillips et al., 1986). Mr. English also seemed to have a limited understanding of tree diagrams. In particular, he drew a limited sample of the tree diagrams during instruction, demonstrating only a two-stage binomial tree and a three-stage binomial tree. In one case, he adapted one of these trees to fit a slightly different asymmetrical situation. Even though he found a correct solution in the process, he seemed to see the familiar trees as tools to use or adapt rather than thinking of drawing tree diagrams to fit the situation as a general strategy to be used in the analysis process.

With the exception of Mrs. Books, the teachers did not see conducting simulations as a particular analysis strategy. Although the teachers were able to design simulations in simple straightforward situations, their responses or explanations did not emphasize how their simulation design modeled the characteristics of the problem. As his students were using cards to simulate the "Newspaper Offer," Mr. English recognized the danger of marked cards. But in other instances, several of the teachers did not recognize potential biases in their simulation designs. For example, Mrs. Books was the only teacher to see the impact of the different sized coins on the Newspaper Pay item (probability question #6). Mrs. Books also brought out that the choice of doors the students made as they were simulating "Monty's Dilemma" could be biased, a potential bias overlooked by Mrs. Talent when her class conducted the same simulation.

The teachers' knowledge of the analytical strategies also included some errors and idiosyncrasies. For example, Mr. Trackman did not recall the correct pattern for Pascal's triangle, either when he used it in the post-observation interview or when he assigned the Coin Tossing Exploration which was based on Pascal's triangle. In addition, Mrs. Talent and Mr. English drew their horizontal tree diagrams without a "main trunk," an idiosyncrasy the teachers seemingly had picked up from the Middle Grades Mathematics Project materials (Phillips et al., 1986).

Applications of Probability

The teachers' understanding of the applications of probability also depended on their familiarity with the setting. In particular, the teachers could provide accurate interpretations of probability in application situations familiar to the middle school classroom, such as games involving dice, coins, or cards. However, in less familiar application situations, the teachers' thinking became subjective and intuitive in nature and included errors in interpretation and analysis. For example, the teachers' interpretations of the Weather and Cancer items in the pre-observation interview (probability questions #9 and #10) were inexact and in some cases seemed to be influenced by thinking associated with the outcome approach (Konold, 1991), where the focus is on predicting what will happen next rather than considering the likelihood of the various outcomes. Similarly, Mr. English's interpretation of the gum ratio (number of pieces of gum found underneath the tables:number of tables) as a theoretical probability and his failure to recognize the impact of order during the analysis of the Powerball lottery provide further examples of the difficulties encountered in dealing with the applications of probability.

The teachers provided a number of examples of real-life situations involving uncertainty in the course of the interviews and during probability instruction. However, it was not clear the teachers understood specifically how probability applied in many of those settings. In addition, several of the teachers' examples were informal in nature. As a result, some of the related application activities used as part of probability instruction failed to make explicit connections with probability, often becoming activities dealing with statistics or proportions instead.

Nature of the Teachers' Probabilistic Intuitions

In some of the more complex and less familiar situations encountered by the teachers in the interviews or during instruction, the teachers often resorted to intuitive or subjective responses or guesses. These intuitive responses at times reflected accurate insights into the nature of the problems. For example, Mr. Trackman and Mrs. Talent intuitively found the best strategy for the Two Spinners interview item (probability question #5), even though they could not justify why it was the best. In addition, although the original charts Mr. English created to analyze two dice games may not have been a standard analysis approach, they not only reflected the essence of the probabilistic situations but presented the analysis in a logical step-by-step manner as well.

In other cases, the teachers' intuitive responses were incorrect. Some of these intuitive responses appeared to be influenced by some of the common misconceptions described in chapter II of this research study, including the representativeness heuristic, the gambler's fallacy, a neglect of sample size, the conjunction fallacy, and the inversion of conditional probability. Nevertheless, the teachers' responses to the misconception items included in the pre- and post-observation interviews provided inconsistent evidence. Each of the teachers provided incorrect answers to at least one of the misconception items, but they sometimes gave correct responses to comparable items in different settings. In some cases, incorrect notions, perhaps based on intuition, seemed to exist alongside more correct notions based on mathematical reasoning. In other cases, the inconsistencies may have reflected a misunderstanding or lack of understanding of the problem situation.

Further evidence of misconceptions in the probabilistic reasoning of Mr. Trackman, Mrs. Talent, and Mr. English was observed during probability instruction. This evidence included examples of the gambler's fallacy and a neglect of sample size as well as misconceptions in the teachers' intuitive notions about fairness and random events. Other examples included a confusion between "unusual" events and low-probability events and the expectation that one's odds should improve in the later rounds as the Montana Red Dog card game was played.

Teachers' Knowledge of the Nature of Mathematics/Probability

Knowledge about the nature of mathematics includes what teachers know about how the field is organized, how knowledge grows and is evaluated, and what it means to "do" mathematics. Although no interview questions specifically explored the teachers' knowledge of the nature of mathematics in general or the nature of probability in particular, the teachers' responses to other interview questions and the teachers' classroom practice provided some evidence of the teachers' potential conceptions about the nature of mathematics and probability. In particular, contrasting views about the nature of mathematics were reflected in the responses and instructional processes of the four middle school teachers. These contrasting views were evident in the teachers' conceptions of the sources of authority, the meaning of "doing" mathematics, and the structure of the content.

Elements of a common set of conceptions about the nature of mathematics were reflected in the thinking and instructional processes of Mr. Trackman, Mrs. Talent, and Mr. English. To these three teachers, mathematics was generally portrayed as a fixed collection of separate facts and procedures that teachers were expected to transmit to their students. These procedures were associated with particular types of problems and when applied led to specific and unique answers. The correctness of these answers was determined by the teacher, the textbook, or an answer key. In these classrooms, "doing" mathematics usually meant following the example explained or modeled by the teacher.

However, the conceptions about the nature of mathematics expressed by Mrs. Books were quite different from the conceptions expressed by the other three teachers. To her, mathematics was a world of ideas to be explored, not just a set of facts and procedures to be remembered and applied in particular situations. As such, mathematics was something with which to interact in the process of developing understanding. Because she had this view of the nature of mathematics, students in her classroom were "doing" mathematics in meaningful ways, determining what made sense and was correct on the basis of logical and reasonable arguments.

The teachers saw probability as a branch of mathematics that added something new and different to the middle school mathematics curriculum. On the one hand, they felt a freedom to set aside the textbook, if one was used at all, and include games and activities as part of their probability instruction. But, at the same time, the study of probability was approached in ways similar to the rest of the curriculum, with an emphasis on arithmetic, procedures, and correct answers, at least in the classrooms other than Mrs. Books.'

The teachers' views about the structure of probability, however, were relatively incomplete and naive. In the four classrooms observed, the general presentation of probability was organized around the ideas of experimental and theoretical probability. Mrs. Books also included a subjective or intuitive interpretation of probability along with the experimental and theoretical approaches. Rather than being seen as different interpretations of probability, experimental and subjective probability seemed to be viewed only as stages on the way to the theoretical or "actual" probability value. The teachers did not portray experimental or subjective probability as valid approaches in their own right for estimating the likelihood of a particular event. Nor did the teachers point out that in some situations experimental or subjective probability may be the only approach that is available or accessible.

Two other ways of viewing the structure of probability content were evident in the teachers' thinking. First, Mr. English organized the content of probability around what he saw as the different "models" of probability, namely dice, coins, cards, spinners, and other manipulatives or tools. This structure of the content was evident in how he thought about the subject, in how he organized his probability unit, and in what he said to students. On a smaller scale, Mrs. Talent similarly referred to the different "models" when talking about the different problem settings, although this structure did not seem to influence the overall structure of her probability unit. Mrs. Talent and, to some extent, Mr. Trackman had yet another view of the structure of probability. This view included the topics fair and unfair games, combinations and permutations, and sampling. However, not all of these topics had been included in the probability units, which focused primarily on analyzing games.

In general, the teachers lacked an accurate overall view of the structure of probability. They did not appear to understand how the various strategies were connected, when the strategies could be applied, or how the strategies related to a general approach for solving probability problems. Similarly, they lacked an understanding of which ideas or concepts were foundational to the study of probability and which ideas were important to be learned.

Cross-Case Analysis: Teachers' Pedagogical Content Knowledge

This chapter now addresses the third and final research question, with its three subquestions:

What is the teachers' pedagogical content knowledge concerning the teaching of probability?
(a) What instructional tasks do the teachers use as they teach probability?
(b) What is the nature of classroom discourse during probability instruction?
(c) What is the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability?

This section will be organized around the subquestions. In particular, this section will begin with an investigation of the instructional tasks used by the teachers. Second, the nature of the classroom discourse will be discussed. Third, this section will consider the teachers' knowledge of students' possible conceptions and misconceptions about probability. The section will conclude by reviewing these and other facets of teachers' pedagogical content knowledge.

Selecting the Instructional Tasks

The mathematical tasks teachers use in their classrooms help shape the learning opportunities available to their students. The potential for learning is influenced by the nature of the selected tasks as well as by the manner in which the tasks are implemented in the classroom. (A more extensive analysis of the instructional tasks used by the teachers in their probability units is presented in Appendix E: Selecting the Instructional Tasks.)

Observation of the four middle school teachers in this study revealed that they used a variety of instructional tasks in their probability units. Many of these tasks were hands-on games and activities which allowed the students to actively explore the ideas of probability. Textbook or worksheet assignments provided additional opportunities for the students to practice the skills being learned. The probability units also included simulation and sampling activities, although the teachers generally did not distinguish these tasks from the other activities involving probability. The teachers also used a variety of tasks for the purpose of evaluating student learning, including answering questions over familiar material, solving unfamiliar problems, and writing a letter about their investigation of an activity.

In two classrooms, the textbooks that had otherwise been used for mathematics instruction were either replaced or supplemented by hands-on games and activities during probability instruction. The games and activities used in these classrooms, as well as in the other classrooms, had been collected from a variety of sources. These sources included commercial curricula, supplemental resource books, teacher-developed activities, and textbooks. The two curricula used most extensively were the Middle Grades Mathematics Project's *Probability* (Phillips et al., 1986) and the Math and Mind's Eye materials, *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993). The teachers also had access to a number of supplemental resource books or had become aware of activities from such sources at mathematics conferences or as part of staff development workshops or classes. Mrs. Talent and Mr. English included tasks and instructional materials they had created. The final source for the activities was the students' textbook or the instructional materials accompanying a textbook used previously.

The instructional tasks the teachers selected to use in their probability units held the potential for engaging the students in learning specifically about the content and nature of probability and generally about mathematics. The tasks also potentially involved the students in problem solving, reasoning, communicating about mathematics, and seeing connections with other mathematical ideas and applications to real-world contexts. The extent to which this potential was met depended in large measure on how the tasks were presented and developed.

Conducting experiments, doing the theoretical analysis, and designing simulations presented a number of opportunities for problem solving. However, the problem-solving nature of the instructional tasks had not been emphasized during instruction. In particular, although various problem-solving strategies were modeled for the students, no emphasis was placed on developing a repertoire of strategies, deciding when the strategies applied, or determining what could be concluded as a result. And with the exception of Mrs. Books, the problematic aspects of the tasks became routinized as the teachers and/or instructional materials specified explicit procedures or steps to follow. As a result, the problems were reduced to exercises in following the directions for playing the game or doing the activity, filling in the accompanying handout, and doing the analysis as suggested by the instructional materials or modeled for them by the teacher.

In addition, the instructional tasks theoretically provided a number of opportunities for the students to communicate about mathematics even though the instructional materials accompanying the tasks provided only limited opportunities. In implementing the instructional tasks, the teachers encouraged verbal and written communication in a variety of settings. However, although the students had opportunities to talk with one another and write about what they had learned, the content and nature of their communication may have been more limited than the teachers realized. A more extensive discussion of communication and the nature of the classroom discourse occurs in the next section.

The instructional tasks also provided opportunities for the students to develop their reasoning abilities. These opportunities, however, were also limited in nature. For example, the reasoning expected of the students was primarily limited to simple comprehension or interpretation of experimental or theoretical results, applications of procedures that had been modeled for them, or responses to lower level questions asked by the teacher during the process of analyzing the games or activities. In addition, because the teacher was the one doing most of the reasoning, particularly in those cases involving more complex reasoning, the students were often merely spectators of the reasoning process. When the students were doing some reasoning, they often were not given opportunities to share their thinking with one another. As a result, the potential of the instructional tasks for developing students' reasoning abilities may not have been realized, except perhaps in individual cases.

Finally, in presenting their probability units, the teachers recognized many of the connections between probability and other mathematical ideas, particularly the connections with fractions, decimals, and percents. The teachers also used instructional tasks that represented real-life applications of probability. However, as stated in the section exploring the teachers' subject matter knowledge of the applications of probability, the nature of the connections between the application tasks and probability remained somewhat implicit. In particular, as implemented, the simulations and sampling tasks became applications of statistics and proportions rather than probability. Any specific discussion of probability in real-life settings seemed to be limited to situations involving games.

Orchestrating the Classroom Discourse

Opportunities for learning about the content and nature of mathematics are not only shaped by the instructional tasks in which students are engaged. These learning opportunities are also influenced by the classroom discourse involved as the tasks are being investigated. This section will explore the classroom discourse observed in the four middle school classrooms in response to the question: What is the nature of classroom discourse during probability instruction? (A more comprehensive analysis of the classroom discourse is presented in Appendix E: Orchestrating the Classroom Discourse.)

Two distinctively different pictures of discourse emerge from this investigation. Specifically, the discourse observed in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English differed from the discourse observed in Mrs. Books' classroom. These differences were evident in the roles played by the participants as well as in the patterns and nature of the discourse.

In the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English, the classroom discourse was both teacher-directed and teacher-focused. The teacher's role in these classrooms primarily involved giving or clarifying directions for the tasks, explaining or modeling the analysis of the tasks, and giving and correcting textbook or worksheet assignments. In this role, the teachers were principally interacting with the class as a whole. As part of their role, the teachers were also the primary source of authority regarding both knowledge and discipline.

Consistent with their view of teachers as classroom leaders and presenters of knowledge, the discourse of Mr. Trackman, Mrs. Talent, and Mr. English was generally aimed at delivering information to students and modeling the analysis procedures for them. Teacher-directed dialogues were the dominant pattern of discourse observed in these classrooms, occurring frequently during the reporting of experimental results, the theoretical analysis of the situations, or the presentation of instructional examples. During these dialogues, the teacher was in control of the flow and the eventual outcome of the discourse because the teacher was the one who asked the questions, the one to whom the students addressed their responses, and the one who evaluated the correctness of the students' responses.

For their part, the students in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English played a relatively passive role. Their participation generally involved doing the experiments (following the directions given by the teacher), reporting their experimental results, responding to the teacher's questions during the analysis process, and occasionally asking questions of their own. Only on rare occasions were these students asked to share their observations or explain their thinking about the activities they were doing.

Either believing the students did not have the necessary background knowledge or wanting the lessons to proceed smoothly and efficiently, the teachers chose to closely guide the thinking of the students. The students were led through the analysis process with a series of product questions, some of which suggested the appropriate responses or conclusions to the students. Student conjectures and questions were not encouraged by the teachers and students had limited opportunities to reason things out for themselves. Student responses and contributions were either correct or incorrect. Student errors often were not addressed directly by the teachers who chose instead to demonstrate what the correct answer was.

Although instruction in Mrs. Books' classroom was also teacher-directed, the discourse was less teacher-focused than in the other classrooms. In her interactions with the class as a whole, she set the stage for the investigation of the problems posed and raised issues of concern to that investigation. However, her primary interactions were with individual students or small groups of students as they investigated the problems. During these interactions, Mrs. Books asked questions to guide the students' exploration and probe their thinking. As the students concluded their investigation of the problems, she facilitated the discussion as students shared their results and their conclusions. Throughout the whole-class discussions, Mrs. Books orchestrated the discourse to encourage the participation of all students. In some cases, students were involved in reporting their initial guesses or supporting one conclusion or another. In other cases, students were asked to respond to claims or statements made by other students.

The students in Mrs. Books' classroom played a more active role in the classroom discourse than their counterparts in the other classrooms. They were encouraged to share their predictions and simulation designs with one another. In responding to questions, the students were expected to justify their thinking. Although the students' results and conclusions were subject to the challenge from others in the class, the students felt the freedom to express their conjectures or concerns and were encouraged to explore these ideas. Students' contributions were respected and thinking was subject to revision without embarrassment. With guidance from Mrs. Books, it was the students who made decisions about the reasonableness of ideas or answers on the basis of logical arguments.

In addition to using teacher-directed dialogues, Mrs. Books was attempting to get her students involved in instructional discussions or conversations, where the students were expected to communicate their mathematical ideas with one another as well as with the teacher. Even though Mrs. Books was still involved in guiding the direction of the discussion and monitoring student involvement, the students were free to question and interact with one another in these instructional discussions or conversations. Although occurring only occasionally with the entire class, such conversations occurred more frequently as Mrs. Books interacted with small groups of students.

The resulting discourse in Mrs. Books' classroom involved thinking at a different and deeper level than in the other classrooms. Students were not just following the teacher's directions for doing a simulation. They were considering the characteristics of the problem as they created their own simulation designs. The students were not only reporting their experimental results, but were judging the reasonableness of those data as well. The students were not merely following the teacher as she led them through the analysis of an activity. They were doing their own analysis and discussing their observations and conclusions with their peers.

Addressing Students' Conceptions and Misconceptions About Probability

According to the view of the *Teaching Standards* (NCTM, 1991), learning involves building upon students' prior knowledge and restructuring that prior knowledge to assimilate the new experiences and new ideas encountered. Because of the importance of connecting instruction to the knowledge students already possess, this section will consider the question: What is the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability? (A more specific discussion of the teachers' knowledge of students' conceptions and misconceptions is presented in Appendix E: Addressing Students' Conceptions and Misconceptions About Probability.)

For the most part, the teachers appeared to have a limited perception of what initial knowledge or conceptions their students may have had about probability as they began the probability units. Specifically, the focus of the teachers' perceptions seemed to be limited to the students' knowledge of the chance situations frequently seen in the classroom, such as coins, spinners, dice, or cards. In these situations, the teachers seemed to think the students had notions of probability as an odds or a fraction. The teachers did not seem to consider the students' more general conceptions about chance, such as the proportional nature of chance or the meaning of "happening by chance." Of the teachers, only Mrs. Books seemed to be aware the students may have strong subjective or intuitive notions about probability even before receiving instruction and that these may be incorrect.

Although the teachers could identify how students might respond to some of the misconception items included in the post-observation interview, they did not seem to have an awareness of the common misconceptions of probability discussed in chapter II of this research study. As a result, addressing the misconceptions in the course of their probability instruction was not specifically one of the teachers' objectives. Therefore, the teachers, for the most part, did not select activities designed to address the misconceptions nor create opportunities to address them. They also failed to take advantage of the opportunities to address the misconceptions when questions or comments from students raised the issues. Further, the interviews provided evidence that the teachers themselves held some of the misconceptions, at least in certain settings. As a result, the teachers might have been less attuned to recognizing such errors in the students' thinking when they did occur.

Other than asking the students to define probability or to identify applications of probability, the teachers made little effort to discover what the students actually understood about probability or to connect instruction to those prior conceptions. The exception, again, was Mrs. Books who structured the activities to bring out and address students' subjective notions.

Because of their incomplete knowledge of students' initial conceptions about probability, the teachers' actions during probability instruction were often inconsistent. At times, the teachers concluded the students had no background knowledge of probability because they had received no prior probability instruction. However, at other times, the teachers seemed to base their probability instruction on the assumption that the students had at least a basic understanding of probability concepts.

The efforts of the teachers to engage students' thinking and deal with students' errors and misconceptions during probability instruction also varied somewhat from teacher to teacher. On the one hand, Mrs. Talent and Mr. Trackman did not make any particular effort to explore what conceptions or misconceptions students possessed. In addition, the activities these teachers selected for their probability units were not particularly successful in uncovering or revealing students' conceptual thinking. As a result, these teachers had fewer opportunities to address students' misconceptions. When errors were revealed in students' thinking, Mr. Trackman was generally critical of the errors rather than trying to comprehend the source of the students' misunderstanding.

The activities Mr. English had included in his probability unit had been somewhat more successful in engaging students' thinking. In the process, the students' comments and questions revealed a number of unconventional conceptions or misconceptions. In most cases, however, Mr. English did not recognize the missing piece of knowledge or the misconception in the students' thinking. As a result, he missed many opportunities during the probability instruction to address the students' incomplete or inaccurate conceptions. His efforts to respond to the students' questions or comments by providing an explanation of what was correct also failed to convince the students their thinking was incorrect.

On the other hand, Mrs. Books more actively sought to bring out students' thinking and to deal with potential misconceptions. In particular, Mrs. Book seemed to be somewhat more aware of the potential problems, at least in the case of replacing the beads in the "Cereal Boxes" simulation and of dealing with the issue of sample size in the simulation of "Monty's Dilemma." By asking questions of individuals, groups, or the whole class, Mrs. Books brought up some of the important issues, which were then addressed in a logical manner as the students discussed their thinking. However, even though Mrs. Books was able to bring out students' thinking and to address errors in that thinking in a logical manner, she was not always successful in overcoming the students' powerful subjective beliefs or intuitive thinking, as revealed in the letters written by the students for "Monty's Dilemma" (where some students stood by their initial incorrect predictions despite acknowledging the Switch strategy had the highest probability for selecting the winning door).

Conclusions: Teachers' Pedagogical Content Knowledge

In this exploration of instructional tasks, classroom discourse, and teachers' knowledge of students' conceptions about probability, this section has addressed the final research question: What is the teachers' pedagogical content knowledge concerning the teaching of probability? Pedagogical content knowledge, however, involves more than tasks, discourse, and knowledge of students' conceptions. Among other things, pedagogical content knowledge also includes the teachers' understanding of the purposes for teaching probability and their knowledge of representations of the concepts. (A more detailed analysis of these aspects of the teachers' pedagogical content knowledge is included as part of the presentation in Appendices C and D, respectively.)

One way of conceptualizing pedagogical content knowledge in the case of probability involves teachers' knowledge within the following categories: subject matter for teaching, students' understanding of probability, media (texts and materials) for instruction, and instructional processes (see Figure 1). Marks (1990b), who originally envisioned this view of pedagogical content knowledge, pointed out the "first three categories can be thought of as the main ingredients, the givens, and the last category as the techniques for combining them into a successful stew" (p. 85). The fourth category in this model, namely instructional processes, contained three areas of emphasis. One purely instructional area focused on the presentation of content. The other two areas of emphasis were hybrids combining instructional processes with students and media (texts and materials), respectively. With this structure in mind, this concluding section will summarize the findings of this study for the following categories of pedagogical content knowledge: subject matter for teaching, students' understanding of probability, texts and materials for probability instruction, and instructional processes for the presentation of probability content.

Teachers' Knowledge of Subject Matter for Teaching

Some examples of pedagogical content knowledge focus on the subject matter itself. These examples include the teachers' knowledge of the goals of mathematics and probability instruction, the justification for teaching probability, the important ideas to include when teaching probability, and the ways of representing probability concepts.

The middle school teachers in this study had mathematics goals in line with those envisioned by the NCTM Standards, including problem solving, communication, reasoning, and seeing connections between mathematics and the real world. However, the teachers had a limited understanding of the nature of these goals and the instructional processes for reaching these goals. The teachers, for example, had an incomplete understanding of what the *Curriculum Standards* (NCTM, 1989) meant by problem solving and reasoning. Additionally, in their efforts to develop the students' problem-solving and reasoning abilities, the teachers did not recognize the importance of reflecting on the thinking processes, discussing when particular strategies might be appropriate, or presenting alternative methods.

The teachers also identified a number of goals for probability instruction, many of which paralleled their overall goals for mathematics instruction. For example, the teachers wanted their students to be able to analyze probability questions with a variety of strategies and to see the many applications of probability in the real world. Although these goals were reflected in the instructional activities, the goals more accurately described the activities done in the classroom rather than the learning that occurred. In particular, instruction focused on learning specific strategies to be applied in specific cases rather than on developing the students' problem-solving or analytical skills.

The teachers had a limited understanding of the justification for teaching probability. Although the teachers recognized that probability is important because of its potential impact on students' lives, they were teaching probability primarily because it was required. They did not see probability as a significant application of the mathematical content typically included in the middle school curriculum nor as a foundation for further study in areas such as inferential statistics.

The teachers also had a limited understanding of the important ideas to emphasize when teaching probability. In particular, the teachers lacked an understanding of which ideas or concepts are foundational to the study of probability, concepts such as sample space or equally likely outcomes. The teachers also had difficulty identifying some of the "big ideas," such as the significance of sample size and the role of experimental data, that are important to emphasize when teaching probability. Finally, the teachers did not appear to understand how the important ideas are related, for example, how the various strategies are connected, when the strategies can be applied, how and when decisions about the strategies are to be made, and how the strategies relate to a general approach for solving probability problems.

Similarly, the teachers had an impoverished repertoire of representations for probability concepts. Probability content was generally represented in a straightforward deductive manner, either by providing a definition or example or by modeling the steps of the procedures. When a given explanation was not understood, the teachers had few alternatives other than restating the original explanation or giving a similar example. Some pictorial representations, such as the area model or the circle divided into six sections, related probability to students' understanding of fractions. These representations, however, were not used extensively. No effort was made to connect the more abstract strategies, such as tree diagrams, the multiplication property, or Pascal's triangle, with more familiar strategies such as "make an organized list." Further, the focus of the presentations of these strategies was on the steps to be done instead of when or why the strategy might be applied.

Teachers' Knowledge of Students' Understanding of Probability

Pedagogical content knowledge also includes what teachers know about students' understanding of the subject matter. This knowledge involves an understanding of the possible conceptions and misconceptions students bring to the learning process and the potential difficulties that await them.

The four middle school teachers in this study had a limited understanding of what initial conceptions about probability their students possessed. After talking with teachers in the earlier grades and/or listening to the students themselves, the teachers concluded that their students, for the most part, had received little or no previous probability instruction. The teachers nevertheless believed the students had a basic understanding of certain simple situations involving spinners, dice, or coins, although Mr. English was the only teacher who formally assessed the students' background knowledge. The teachers, however, generally appeared to be unaware of the nature of the students' intuitive beliefs. These beliefs included such issues as what role determinism, luck, or cause and effect play in uncertain events as well as the students' knowledge of the underlying nature of chance and fairness. In addition, the teachers seemed to assume more understanding than appropriate on the part of the students from their use of the basic language of probability.

The teachers also had a limited understanding of what potential misconceptions may be part of students' thinking. Although these teachers could describe how students might respond to items involving the common misconceptions of probability, knowledge of these common misconceptions was not explicitly part of the teachers' knowledge base. In particular, the teachers apparently had not encountered examples or explanations of these common misconceptions in their reading or previous study. As a result, they were not aware of these misconceptions and did not mention them when asked what difficulties students encounter in the study of probability. The interviews conducted with the teachers provided some evidence that the teachers themselves may also have had similar misconceptions in their thinking.

Despite their previous experiences teaching the content, the teachers, for the most part, were not aware of the difficulties students might have in their study of probability. Attention to such difficulties with the mathematical content did not appear to be part of the teachers' thinking. When asked in the post-observation interview to reflect on the difficulties the current students encountered, the teachers could identify some of those difficulties, including the notion of odds and the process of drawing tree diagrams. Other difficulties, such as the importance of order, were either not recognized or not recalled. Nevertheless, even though the teachers had recognized some of the difficulties, this knowledge had not impacted instruction at the time. Other than possibly repeating the already-given explanation, the teachers made little or no effort to reteach the concept causing difficulty for the students.

In addition, the teachers did not appear to have specific knowledge about how students develop an understanding of probability. For example, the teachers made no mention of the importance of proportional thinking as a prerequisite for understanding probability. And although the teachers recognized the role of concrete experiences in mathematics instruction in general, they apparently did not see how experimental results could be used to challenge the faulty intuitive notions students possessed.

Teachers' Knowledge of Texts and Materials for Probability Instruction

Pedagogical content knowledge also includes what Marks (1990b) calls knowledge of media for instruction. In the case of probability, this form of knowledge includes teachers' knowledge of the textbooks, instructional tasks, and/or manipulative materials available for probability instruction.

The teachers did not appear to have any curriculum guidelines to follow concerning what probability activities should be presented and/or what concepts or skills should be mastered at the different grade levels. Without a curriculum to provide articulation and coordination across grade levels, the teachers generally provided a survey of probability topics rather than focusing on mastery of specific objectives or developing mathematical ways of thinking. The resulting probability instruction was therefore broad in terms of topics covered, but shallow in terms of developing potential understanding.

Although the textbook potentially provided some guidance about the structure and development of the probability content, the teachers moved away from using the textbook (if one was being used at all) in favor of utilizing hands-on instructional tasks. In each of the classrooms, hands-on instructional activities provided the focus for probability instruction. These activities had been gathered by the teachers from a variety of sources, including commercial curricula, supplemental resource books, the textbook (in the one classroom where a text was used), and instructional materials accompanying a previously used textbook. Mrs. Talent and Mr. English also included tasks and instructional materials they had created. In addition, Mrs. Books and Mr. English adapted chosen instructional tasks to better meet their goals.

In deciding which instructional tasks to incorporate as part of their probability instruction, the teachers were influenced by a number of factors. First, the teachers chose some of the activities in order to present a specific aspect of probability content (e.g., coins or dice) or to convey something particular about the process of "doing" mathematics (e.g., designing simulations). Second, in making decisions about which activities to use, the teachers also considered how mature their students were and/or what their potential reactions to the activities might be. Third, the teachers chose to include several hands-on activities, believing active involvement on the part of the students was important to the process of learning probability. Although apparently selecting activities based on the content of the task, the nature of students, or principles of learning, these three factors appeared to be considered only superficially. For example, an activity might have been selected because it involved drawing a tree diagram. However, the teachers did not consider what the task might teach students about tree diagrams. Would it be a good task to introduce students to what a tree diagram is? Or would it be a useful part of a series of activities aimed at helping students know when a tree diagram might be helpful or in developing students' ability to draw their own tree diagrams? Similarly, the possible affective reactions of the students were considered, but the teachers did not report any of the instructional tasks were selected because of their potential to connect with students' prior knowledge or to address students' misconceptions.

Along with the hands-on instructional activities, the students were using a variety of manipulative materials, including dice, coins, cards, spinners, and objects drawn from containers. The use of these materials allowed the students to have first-hand experience with chance occurrences. However, in limiting the materials to these familiar embodiments of equally likely outcomes, the students had only a partial experience dealing with situations that involve uncertainty.

The teachers began their probability units with activities that were familiar to the students and/or with tasks that were intended to capture the interest of the students and motivate the study of probability. With the exception of Mr. Trackman, the sequencing of subsequent instructional tasks generally moved from more familiar to less familiar activities and from simpler concepts to more difficult ones. However, it was not clear how much thought was given to the overall sequencing of the instructional tasks, beyond planning one week at a time. In considering the selection and sequencing of the instructional tasks, the teachers appeared to pay more attention to the nature of the tasks (game or simulation) and materials (dice or coins) than to the probability content (tree diagram or expected value). Mr. Trackman's probability unit included a number of problems from the sequencing perspective, primarily because he (a) allowed convenience to outweigh pedagogical concerns, (b) failed to consider the background knowledge of the students, (c) misjudged the difficulty of some of the probability content, and (d) failed to make adjustments for the textbook sections he had omitted.

Teachers' Knowledge of Instructional Processes for the Presentation of Probability Content

Pedagogical content knowledge also includes the instructional processes teachers use in presenting the subject matter. Differences in the instructional processes of the teachers in this study were reflected in the teachers' overall goals and views about learning, their patterns of classroom discourse, their use of questions with students, their expectations about the use of mathematical terminology, and their lesson organization.

First, the teachers' presentation of probability content reflected differences in the teachers' overall goals and views about learning. One goal of Mrs. Books' instruction was to help students develop a conceptual understanding of the mathematics they studied. Her efforts to accomplish this goal were facilitated by her choice to do fewer activities and to explore each activity in greater depth. To develop each students' thinking and understanding to its fullest, Mrs. Books had the students consider important questions individually first. Students then shared their thinking with others in their small group and, finally, the entire class discussed the question. This format provided opportunities for students to form their own ideas based on their prior knowledge and then for those ideas to be shaped by the thinking of others. Mrs. Books also spent considerable time interacting with students within the small-group framework where she could tailor the questions asked to the students involved. In contrast, the focus of the other teachers was more often on the presentation and coverage of content. In these classrooms, more activities were done with less time spent on developing a conceptual understanding of the content covered. Although students worked in small groups as they conducted experiments, most instruction was presented to the class as a whole. Less emphasis was placed on using or developing students' thinking.

Second, differences in the instructional processes used by the teachers in presenting probability content were reflected in the patterns of discourse and interaction between the teachers and their students. Mr. Trackman, Mrs. Talent, and Mr. English used primarily teacher monologues and teacher-directed dialogues as they presented information to the students or guided their thinking in the analysis process. In addition to using teacher-directed dialogues, Mrs. Books encouraged her students to interact with each other in instructional discussions or conversations. These various patterns of discourse were communicating different messages about the nature of learning mathematics, although doing so indirectly. The teacher monologues and teacher-directed dialogues put the teacher in the role of expert, the one who was responsible for delivering information and evaluating the correctness of student responses. On the other hand, in the instructional discussions or conversations, Mrs. Books was trying to create a learning environment where students were interacting with each other as they themselves sought to make sense of the mathematics they were exploring.

Teachers' questions played an important role in the presentation of probability content, included both in the teacher-directed dialogues and the instructional discussions. In their use of subjective questions, the teachers asked students to make intuitive predictions about the results they expected for an activity involving uncertainty or to make judgments based on their knowledge of the likelihood of uncertain events. Product questions, generally seeking a specific factual response, were the questions used most extensively. In particular, Mr. Trackman, Mrs. Talent, and Mr. English typically guided the students through the analysis process by asking series of product questions. On the other hand, Mrs. Books used more open-ended process questions to delve into students' thinking and to guide their exploration and decision making. Metaprocess questions, a fourth category of questions, were used infrequently if at all in the classrooms. However, these questions, which ask students to reflect on their thinking or on the analysis process, could be valuable in helping prepare students to think on their own and to analyze probabilistic situations without the teacher's guidance.

Students' questions also played a role in the classroom presentation and discourse, although not as significant a role as teachers' questions. Most questions asked by students were clarifying either teachers' expectations, activity directions, or probability terminology. Students generally asked few questions dealing with probability content or concepts. In some instances, such questions were inadvertently overlooked or deflected because the teacher did not understand what had been asked nor fully grasp the implications of the question.

Third, the presentation of probability content also reflected differences in the teachers' expectations about the use of formal probability terminology. Two of the teachers, Mrs. Books and Mr. English, expected the students to learn probability vocabulary as one of their instructional goals. Appropriate terminology was introduced and defined as necessary. In addition, at least in Mrs. Books' classroom, the students were expected to use mathematical terminology as they communicated with one another. Mrs. Talent also defined some basic probability terms and used others without defining them. The textbook assignments introduced Mr. Trackman's students to basic probability terminology, although the students were not held accountable for learning the terminology. In each of the classrooms, some of the probability terminology (e.g., event, random, theoretical probability, and simulation) was used incorrectly, either by the teacher and/or the students.

Although, for the most part, the teachers introduced and used the formal language of probability, no consideration was given to understanding what is meant by the informal everyday language used to describe chance events. In particular, no effort was made to relate the fractional values associated with probabilities to expressions such as *likely*, *unlikely*, *possible*, and *probable*. Although such terms are used frequently in everyday life, interpretations of these and other such expressions are quite varied.

Finally, differences in the presentation of probability content were reflected in the organization of the lessons, which varied according to the nature of the instructional task.

The lessons involving games and activities were generally organized around some or all of the following stages: introducing the tasks, making predictions, conducting experiments (or simulations), interpreting the experimental results, and doing the theoretical analysis. On the other hand, the lessons involving textbook or worksheet assignments usually included time for introducing the assignment, working on the assignment, and correcting the assignment. These lesson structures, as well as the nature of the discourse during the different phases, impacted the presentation of probability content in the teachers' classrooms.

The teachers' introductions to the games and activities generally provided an overview of the task by explaining how the investigation of the task would proceed or by relating the task to other tasks included in the probability unit. This introduction, however, usually did not include any discussion of the content, any explanation about learning objectives, or any attempt to tie the activity to students' prior knowledge. The focus of the teachers' introduction instead was on a presentation of the directions and/or rules for the game or activity. The nature of questions asked at this stage of the lessons had the potential of either stimulating students' thinking or prematurely limiting the opportunities for students to reason things out for themselves.

As with the games and activities, the teachers' introductions to the textbook and worksheet assignments provided no specific discussion of any learning objectives. The teachers instead introduced the textbook and/or worksheet assignments with examples designed to prepare the students to complete the items on the assignment. The presentation of these examples, which were chosen to closely match the questions contained in the assignment, generally involved a teacher-directed dialogue with the teachers' questions guiding the students through the solution process. On occasion, the teachers gave students a follow-up example to do on their own, thereby checking for student understanding.

The importance given to the process of making predictions for the games and activities varied. For Mrs. Books, having the students make predictions or subjective guesses was an integral part of the simulation activities they conducted. These predictions were written down and shared with other members of the small groups and/or with the entire class. In addition, the students' rationale for their predictions was also written down and/or shared with their peers. Mrs. Books and her students treated the predictions as data in their own right and displayed or analyzed them accordingly. In "Monty's Dilemma," Mrs. Books asked her students to reflect on their subjective predictions in light of the experimental and theoretical evidence. In the other classrooms, the students were held less accountable for making predictions. When predictions were made, the results were often reported in chorus responses or by a show of hands, without any reasons behind the students' predictions being shared. In this setting, students were able to get by without making any prediction, perhaps fearing they might be "wrong" if they did make one. In addition, Mr. Trackman, Mrs. Talent, and Mr. English less consistently compared or asked the students to compare the predictions that had been made with the experimental or theoretical results obtained later.

Although conducting experiments or simulations was the centerpiece of the probability units, the nature of the interactions between the teachers and their students as the students were conducting the experiments was quite different in the various classrooms. In the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English, many of the interactions focused on the directions given for the game or activity or on the results of the experiments or simulations. In these classrooms, only a limited number of the interactions were aimed at encouraging or probing students' thinking. In contrast, because Mrs. Books involved her students in designing their own simulations and spent more time on the experimental phase of the tasks, she had more opportunities to interact with her students. Although these interactions concerned some of the same topics as in the other classrooms, including the procedures and results of the simulations, the teacher-student interactions often involved thinking at a different level. In particular, rather than just an exchange of factual information such as rules or results, the interactions between Mrs. Books and her students engaged the students in thinking about what they were doing as the simulations were being conducted and encouraged the students to think about the content involved as well.

Two of the teachers, Mrs. Talent and Mr. Trackman, had further opportunities to interact with their students as the students worked on the textbook and/or worksheet assignments. Mrs. Talent reported taking different approaches during these interactions depending on the nature of the task. For the worksheet assignments, Mrs. Talent circulated among the students checking on their progress and offering assistance as needed. In response to students' requests for help, she provided specific answers and whatever guidance the students needed. In addition, as the students worked on their assignments, they frequently looked to Mrs. Talent to confirm their answers. However, when the students were working on the evaluation tasks, Mrs. Talent explained she would ask the students questions to see what they understood and help guide them to identify what they understood and what they did not understand, but she would not tell them what to do. This approach created some frustration, perhaps because the students had not been prepared in previous work to think on their own. Instead they had grown accustomed to being told what to do on the assignments and had come to rely on being told if their answers were correct. This "learned dependency" became a hindrance when they were expected to think independently and apply their knowledge in new situations such as the evaluation tasks. Mr. Trackman was less involved in circulating among the students as they worked, but he did provide assistance when the students sought such assistance. In Mr. Trackman's interactions with various students, there was evidence he treated some students differently from the others. In particular, with some of the weaker students, he opted for telling them the correct answers rather than helping the students understand the question and reason out the solution. In doing so, he also was developing a "learned dependency" on the part of the students.

The teachers had varied approaches to the process of interpreting the experimental results of the games and activities. As the experimental data were shared with the class and analyzed, Mrs. Books encouraged her students to critically evaluate the nature of the data, to assess their reasonableness, and to take into account the possibility of bias. With those issues in mind, the students also considered what information could be obtained from the data and what conclusions could be stated. In the other classrooms, the data were sometimes recorded on the overhead either by the teacher or the students themselves. At other times, the data were reported by a show of hands. At still other times, the data were not reported at all. In these classrooms, no discussion considered the reasonableness of the data and conclusions were usually stated by the teacher, if stated at all.

As in the other phases of the exploration of the probability games and activities, how Mrs. Books dealt with the theoretical analysis stands in contrast to how the theoretical analysis was done in the other three classrooms. As Mrs. Books' students explored "Monty's Dilemma," aspects of the theoretical analysis were intermingled with the experimental phase of the investigation. As she built upon students' ideas, the logical foundations of the problem were revealed, although they were not emphasized as such. In the process of thinking about those ideas, the students determined the theoretical results and the corresponding probabilities on their own. Later, the students were the ones who initiated a theoretical discussion of the outcomes as they were considering the class' experimental results. On the other hand, the theoretical analysis in the other three classrooms was generally a distinct part of the lesson, one which was introduced and directed by the teacher. In presenting the theoretical analysis, Mr. Trackman, Mrs. Talent, and Mr. English generally explained or modeled how the analysis should be done, often making use of teacher-directed dialogues in which the teachers guided the students through the analysis process with a series of questions. Similarly, it was the teacher who stated whatever conclusions were stated for the games and activities. These conclusions focused on the results of the game or activity itself and generally did not involve any reflection on what content or concepts had been involved, how the tasks had been analyzed, or what had been learned. In particular, the teachers provided no discussion of why a specific analysis method had been chosen. Further, the teachers made no effort to review the steps of the analysis process and no attempt to fit the activity into the overall scheme of the unit.

Similarly, the process of correcting the worksheet and textbook assignments focused on correct answers and not on the underlying content or learning objectives. Because there was little or no discussion of the mathematics involved in the assignments, opportunities to check for students' understanding of the basic concepts involved or to explore students' thinking on more challenging problems were generally overlooked.

The organization and structure of the probability lessons provided a number of opportunities to explore students' thinking and to develop their understanding of probabilistic situations. However, with the exception of Mrs. Books, these opportunities were generally overlooked or only partially fulfilled. For example, having the students compare their predictions with the experimental and theoretical results could have been a way to assist students in making the transition from using predictions based on subjective guessing to making decisions based on theoretical considerations. However, with the exception of Mrs. Books, the teachers generally did not ask their students to make such an explicit comparison. Similarly, the other teachers did not ask students to explain the rationale behind their predictions or to draw their own conclusions from the experimental or theoretical data. Further, because of the weaknesses in the introductions and closures provided for the probability lessons, the students were provided little guidance in determining what they were to be learning or in seeing how the lessons being learned fit into the "bigger picture" of probability.

Case Studies: Touching Up the Portraits

Before moving on to a general discussion of the results of this study, this chapter takes one final look at the portraits of the individual teachers. This postscript focuses on the teachers' responses to the various factors influencing their instructional practice.

Mr. Trackman: Influence of Mathematical Background

Mr. Trackman had taken a number of advanced mathematics courses, including one in probability, as part of his major in mathematics education. This experience provided an opportunity for him to establish a strong mathematical foundation for his teaching of mathematics. However, despite his more extensive mathematics background, Mr. Trackman had the most limited knowledge of probability content of the four teachers in this research study. He either had forgotten the basic probability content he had once learned or the basic notions of probability had been lost on Mr. Trackman in the formal abstract nature of the advanced probability course he had taken. In particular, Mr. Trackman saw no relationship between the probability he had studied and what he now was teaching, suggesting, "[The advanced probability class] hasn't had an opportunity to [impact my teaching] because I've been in sixth grade the whole time." By his own choice, Mr. Trackman had not taken the one class offered at his college that included 2 or 3 weeks of basic probability instruction focused on teaching elementary probability. Although, in terms of mathematical background, Mr. Trackman was potentially the best prepared of the teachers, he turned out to be the least equipped in terms of mathematical knowledge.

Mr. Trackman was also the least effective of the teachers pedagogically. His secondary education program, with its greater emphasis on mathematics and lesser emphasis on teaching mathematics, had done little to challenge Mr. Trackman's preconceived ideas about teaching, ideas he had gained from his many years as a student in mathematics classes. The traditional methods by which he had learned mathematics had been good enough for him. If anything, the advanced mathematics classes he had taken may have reinforced his traditional notions about mathematics instruction. However, Mr. Trackman was not particularly effective even in delivering traditional mathematics instruction. His efforts were directed more toward managing the classroom than toward promoting student learning. In many cases, the instruction he planned did not match what occurred and his perceptions of what had happened did not match reality.

Mr. Trackman's secondary education preparation may actually have hindered his growth as a teacher. Because he had taken more mathematics courses than many of his predominantly elementary certified colleagues, he felt he was superior to them in terms of knowledge. As a result, he saw no need to continue learning about mathematics or about teaching mathematics. He would take mathematics courses if they were required for a master's program, but otherwise he felt no need to do so. Although he was a beginning teacher dealing with many of the issues common to beginning teachers, he saw no need for ongoing professional development and growth.

Also because of his perceived expertise, Mr. Trackman was relatively impervious to pressures to reform mathematics teaching. Although he did not fully understand the nature of the reforms, he did not "like the direction that mathematics is going." He instead figured if he waited 5 or 6 years, the pendulum would swing back his direction.

Mrs. Books: Influence of Learning Experiences

Because of her poor experience with mathematics as a high school student, Mrs. Books initially had been discouraged from continuing on to college after graduation. But 10 years later, her attitude toward mathematics would change as she prepared to enter college with the goal of becoming an elementary school teacher. She took a number of community college mathematics courses where her newfound ability to understand mathematics was encouraged. This growing interest in mathematics led her to become involved in a special program for preparing middle school mathematics teachers, a program that further transformed her thinking.

As part of this middle school program, Mrs. Books took seven additional mathematics classes designed specifically for prospective middle school teachers. Not only was she learning more about mathematics in these classes; she was also experiencing a new way of learning mathematics. In the process, the professors served as role models of a new way of teaching mathematics. Mrs. Books credited the middle school program and particularly the modeling of the professors for changing her thinking about mathematics and influencing her approach to teaching mathematics. The probability class in the program had also provided her with instructional tasks and the corresponding background knowledge to implement those tasks in her own classroom.

That influence had profoundly affected Mrs. Books' knowledge and beliefs about learning and teaching mathematics, because in the process Mrs. Books had captured much of the vision of the mathematics education reform effort and was firmly committed to its constructivist philosophy. She was enthusiastic about learning and her instruction captured the essence of the NCTM's vision of the mathematics classroom. Instruction was student-focused as Mrs. Books interacted with individual students or small groups of students, guiding them in their efforts of constructing meaning and developing understanding. Students were active participants in the learning process, sharing their questions and conjectures with one another. Conclusions established within the learning community were based on logical and reasonable mathematical arguments.

Mrs. Books did not necessarily have more mathematical knowledge than the other teachers in this study, particularly in the area of probability, but the knowledge she possessed was of a different quality. Because Mrs. Books saw mathematics as something to be explored and understood, she had a willingness to investigate and a disposition toward making sense of what she found. She was not limited to remembering what she had previously learned. In addition, Mrs. Books was always challenging herself to learn and understand more about mathematics and about teaching mathematics.

Mrs. Talent: Influence of Reform Efforts

Mrs. Talent was a committed and caring teacher, willing to try new ideas and methods for teaching mathematics. As she became aware of the efforts to reform mathematics instruction occurring on a state and national level, she had implemented aspects of the reform and been part of a team sharing the message of the reform with other teachers and parents across the state.

In her efforts to implement the reform, Mrs. Talent had set aside the textbook, choosing instead to focus on meeting the state curriculum guidelines. For instruction, she selected activities and worksheets that would involve the students in exploration and practice of the skills they were learning. In the spirit of the reform, Mrs. Talent used open-ended, unfamiliar problems as evaluation tasks, emphasizing the importance of having the students demonstrate they could apply what they had learned.

In her implementation of the reform, Mrs. Talent recognized changes were called for in her role as the teacher. She acknowledged, "It's real hard to break the habit of 'I am the one up there that's supposed to be doing the teaching.' " However, although Mrs. Talent recognized the tension between the traditional role of the teacher as presenter of knowledge and the new role of the teacher as facilitator of learning, she often continued to operate within the "show and tell" model of teaching. As she modeled how to analyze the probabilistic situations or demonstrated how to do the items on the worksheets, the students were put in the role of spectators. As such, they did not develop the skills and abilities needed to solve problems on their own. This led to difficulties when they were expected to do so on the more open-ended assessment tasks they were given.

Although Mrs. Talent was somewhat disappointed in the students' performance on the assessment tasks, she did not recognize the potential underlying causes. In particular, she had instructional goals and evaluation tasks in line with the vision of the mathematics education reform. However, her traditional "show and tell" mode of instruction did not prepare the students to reach the goals and complete the unfamiliar evaluation tasks. The result was a discontinuity between what the students were able to do and what she had expected them to do. Mrs. Talent had applied specific aspects of the reform effort to the process of mathematics instruction, but she had not seen how the overall process of instruction needed to change in corresponding ways.

Mr. English: Influence of Teaching Experience

Mr. English was an experienced and hard-working teacher, dedicated to helping his students find success in mathematics. From his years of experience, Mr. English had gained knowledge about middle school students and had accumulated instructional activities and methods aimed at teaching mathematics effectively. He had formed beliefs about learning and teaching mathematics that made sense to him; beliefs which were in agreement with widely accepted views. By most standards, Mr. English was an excellent and effective teacher. His students enjoyed mathematics and did well on standardized tests. Their parents appreciated Mr. English's teaching efforts. The teachers and administrators in his district looked up to him as a leader in mathematics education.

When first assigned to teach eighth-grade mathematics, Mr. English did not have a strong mathematical background. His small-town high school had offered only limited opportunities in mathematics. On his way to becoming a language arts teacher he had taken no mathematics in college. In his abbreviated preparation for later becoming an elementary teacher, he had taken the required mathematics methods course but remembered little from the class. Even when assigned to teach mathematics, Mr. English had not sought further education until the state required him to earn a mathematics endorsement. Discovering at this point that there was much he could learn about mathematics and about teaching mathematics, Mr. English set off on a journey to learn all he could. In the years since, he had subscribed to mathematics education journals, had attended mathematics conferences and workshops, and had taken additional mathematics and education classes.

Mr. English reported that his mathematics instruction had changed significantly as a result of a summer workshop where the materials from the Middle Grades Mathematics Project were presented. By his own admission, prior to that workshop, he had been a "traditional" teacher who faithfully followed the textbook. However, the opportunity to experience the instructional materials of the Middle Grades Project as a student had a powerful influence on him and on his instructional practice. Because of that experience, Mr. English set out to develop a set of manipulative materials directly correlated with the lessons he taught. As a result, his instructional style now included a variety of hands-on activities, many of which he had collected at mathematics conferences and workshops. In more recent years, he had begun to include cooperative group and critical thinking activities as well.

Although his instructional practice had undergone somewhat of a transformation, Mr. English's knowledge and beliefs about how mathematics is learned and how it should be taught had not changed significantly. He remained committed to his beliefs in the teacher's roles of delivering knowledge and modeling procedures and the students' roles of listening and practicing. By adding elements of the reform, such as using manipulative materials or cooperative group work, Mr. English believed he had embraced the NCTM's efforts to change mathematics teaching. However, he had not addressed the more fundamental issues about the nature of mathematics (e.g., what it means to think mathematically and what it means to engage in mathematical activity) and the learning and teaching of mathematics, issues that undergird and inspire that reform effort.

CHAPTER V
DISCUSSION AND CONCLUSIONS

Introduction

As a result of the calls for reform in mathematics education and the ever-changing nature of mathematics, today's teachers face the challenge of teaching unfamiliar content in ways that are equally unfamiliar. In view of this challenge, the purpose of this study was to investigate middle school teachers' subject matter and pedagogical content knowledge of probability and its relationship to the teaching of probability. By examining the knowledge and practice of four middle school mathematics teachers, this study has addressed the following questions:

1. What general pedagogical knowledge do middle school teachers demonstrate in the context of teaching probability?
2. What is the teachers' subject matter knowledge of probability?
3. What is the teachers' pedagogical content knowledge concerning the teaching of probability?
   a. What instructional tasks do the teachers use as they teach probability?
   b. What is the nature of classroom discourse during probability instruction?
   c. What is the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability?

This chapter begins with a discussion and summary of the main findings of this research study. This discussion will, in particular, interpret the findings of this study within the context of other mathematics education research efforts. The chapter then concludes with comments regarding the implications of the study for mathematics education, the limitations of the study, and recommendations for future research.

Discussion and Summary of the Main Findings

In addressing the research questions of this study, the goal has been (a) to paint a portrait of the current practice in teaching probability and (b) to provide a picture of the subject matter knowledge and pedagogical content knowledge teachers possess and its relationship to instruction. This section will begin by painting the portrait of probability instruction observed in the middle school classrooms. A discussion regarding the teachers' knowledge and its relationship to probability instruction will follow.

Portrait of Probability Instruction

Throughout this research study, the master portrait against which mathematics instruction has been compared is the vision of mathematics instruction described in the NCTM's *Curriculum Standards* (1989) and *Teaching Standards* (1991). The key components of that vision will be reviewed before the portrait of instruction observed in the middle school classrooms is presented.

Portrait of Instruction Envisioned by the NCTM

The *Curriculum Standards* (NCTM, 1989) and *Teaching Standards* (NCTM, 1991) promote a vision of mathematics classrooms where the goal is to develop the mathematical power of all students. This vision includes mathematics instruction that is aimed at helping students learn how to formulate and solve problems, to reason and communicate mathematically, and to connect the ideas and applications of mathematics. Developing mathematical power also involves helping students make sense of mathematics and helping them rely on themselves to determine whether something is mathematically correct. Further, mathematical power involves the "development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions" (NCTM, 1991, p. 1).

This new vision of mathematics instruction is based on a fundamental rethinking of what "understanding mathematics" means and a new understanding of how students learn mathematics. Students are no longer seen as passive recipients of knowledge but rather as active participants in the learning process as they construct their own understanding of mathematical ideas and concepts. In this context, understanding is viewed as an ongoing process which deepens as conceptions expand and the number and strength of the connections among them develop (Hiebert & Carpenter, 1992; Schifter & Fosnot, 1993).

In mathematics education, this constructivist view of understanding is based on both cognitive and social perspectives of the learning process. From a cognitive perspective, the constructivist approach is based on the following premises: (a) Students have already-existing knowledge as a result of intuition and/or their previous experiences; this prior knowledge influences what they learn; (b) knowledge is not passively received by students; knowledge is actively created or invented by students as they interact with their environment; (c) learning occurs when students integrate the new information and experiences into their existing knowledge structures by modifying their prior knowledge accordingly; and (d) mathematical knowledge is constructed, at least in part, through a process of reflective abstraction, in which students reflect on their physical and mental actions, generalizing from specific cases to form more abstract notions (Davis et al., 1990; Schifter & Fosnot, 1993; Steffe & Kieren, 1994). From a social perspective, Cobb, Yackel, and Wood (1992) argue that learning mathematics is both an individual constructive activity and a collective human activity. As students interact with the teacher and with one another within the classroom community, Cobb et al. point out that learning also becomes a process of acculturation as students are "initiated into the taken-as-shared mathematical practices of wider society" (p. 26).

For this new vision of mathematics instruction to become a reality, the *Teaching Standards* (NCTM, 1991) propose a shift in the environment of mathematics classrooms:

> 1. toward classrooms as mathematical communities–away from classrooms as simply a collection of individuals;
> 2. toward logic and mathematical evidence as verification–away from the teacher as the sole authority for right answers;
> 3. toward mathematical reasoning–away from merely memorizing procedures;
> 4. toward conjecturing, inventing, and problem solving–away from an emphasis on mechanistic answer finding;
> 5 toward connecting mathematics, its ideas, and its applications–away from treating mathematics as a body of isolated concepts and procedures (p. 3).

As this description demonstrates, this new focus of mathematics instruction on reasoning, understanding, and explaining represents a radical departure from the emphasis on memorization and imitation found in conventional mathematics instruction.

In addition to addressing general aspects of mathematics instruction, the NCTM sets specific goals for various content strands, including probability. The *Curriculum Standards* (NCTM, 1989) calls for introducing a number of probability concepts throughout the school curricula. At the K-4 level, the *Curriculum Standards* recommends using experiments and real-world examples to introduce students to initial concepts of chance and to develop their probability sense. At the middle school level, the study of probability is to take on new dimensions, as outlined by the following goals for probability instruction:

> In grades 5-8, the mathematics curriculum should include explorations of probability in real-world situations so that students can–
> - model situations by devising and carrying out experiments or simulations to determine probabilities;
> - model situations by constructing a sample space to determine probabilities;
> - appreciate the power of using a probability model by comparing experimental results with mathematical expectations;
> - make predictions that are based on experimental or theoretical probabilities;
> - develop an appreciation for the pervasive use of probability in the real world. (p. 109)

Thus, the NCTM not only calls for significant changes in mathematics instruction in general, but also calls for changes in probability instruction at the middle school level.

Portrait of Instruction Observed in the Middle School Classrooms

Having described the far-reaching goals set before mathematics teachers, the discussion turns next to the portrait of probability instruction observed in the four middle school classrooms. This portrait will focus on aspects of instruction emphasized by the *Teaching Standards* (NCTM, 1991), namely the important decisions made by classroom teachers in shaping what happens in their mathematics classrooms. These decisions involve the selection of mathematical tasks, the orchestration of classroom discourse, the creation of the learning environment, and the analysis of teaching and learning. This portrait will also focus on the successes and difficulties encountered by these teachers in their efforts to fulfill the vision of the NCTM Standards.

Selection of mathematical tasks. The *Teaching Standards* (NCTM, 1991) recognizes that teachers shape what goes on in mathematics classrooms by the instructional tasks (exercises, questions, problems, applications, projects, activities, and labs) they assign to students. These mathematical tasks not only influence what students may learn about particular mathematical content but also what they come to understand about mathematics in general (Doyle, 1988; Hiebert & Wearne, 1993; Stein, Grover, & Henningsen, 1996). With this in mind, the *Teaching Standards* encourages teachers to select tasks that engage students' interests and intellects and that provide opportunities to deepen their understanding of mathematical ideas and to improve their ability to apply these ideas. An important goal of the reform is for students to be exposed to meaningful and worthwhile mathematical tasks, tasks that offer more than disguised practice of already-demonstrated algorithms. Such tasks provide opportunities for students to make decisions about what to do and how to do it, to judge the reasonableness of their actions and solutions, and to explain and justify their procedures and understandings in written and/or oral form. In the process, these mathematical tasks serve as important vehicles for fostering students' ability to solve problems and to reason and communicate mathematically as well as promoting the development of students' understanding of concepts and procedures.

The middle school teachers observed in this study used a variety of instructional tasks in the course of teaching their probability units. Many of these tasks were hands-on games and activities which allowed the students to actively explore the ideas of probability. In some of the classrooms, textbook or worksheet assignments provided opportunities for students to practice the skills being learned. The probability units also included simulation and sampling activities, although the teachers generally did not distinguish these from the other activities involving probability.

These instructional tasks provided a number of opportunities to address the probability goals outlined in the *Curriculum Standards* (NCTM, 1989). For example, conducting experiments and/or simulations was a central focus of the probability instruction in each of the classrooms. In addition, although the concept of sample space was never formally introduced, several of the strategies modeled for the students involved constructing a sample space. Both experimental and theoretical results were considered for many of the situations investigated. The games and activities also provided opportunities for students to make predictions, although the predictions made were generally based on subjective notions rather than experimental or theoretical probabilities. Finally, many of the tasks portrayed real-life applications of probability.

The extent to which the goals of the *Curriculum Standards* (NCTM, 1989) were met, however, depended in large measure on how the tasks were implemented by the teachers and their students. One teacher was generally more successful in meeting these probability goals. In her classroom, the students were not only involved in conducting simulations; they were also creating their own simulation designs. The reasonableness of experimental results was judged in comparison to the expected theoretical outcomes. Students were making predictions and sharing the rationale supporting their predictions with others in the classroom. The real-life characteristics of the probability applications were subject to discussion. In contrast, students in the other classrooms were told how the experiments and simulations would be conducted, without an opportunity to create their own designs. Experimental and theoretical results were rarely compared explicitly. When predictions were made, the students' rationale behind their predictions were not considered. Finally, with the exception of the games, the nature of probability's role in the real-life applications remained mostly implicit.

In addition to addressing the probability goals of the *Curriculum Standards* (NCTM, 1989), the instructional tasks selected by the teachers also had the potential of involving the students in meaningful ways in problem solving and reasoning. Although the instructional materials themselves did not offer many such opportunities, the problems embedded within the tasks embodied features reformers have associated with promoting higher order learning (Anderson, 1989; Stein et al., 1996). In particular, the instructional tasks involved problems that could have been solved in multiple ways, including listing the sample space, drawing a chart or tree diagram, and applying an area model or Pascal's triangle. Similarly, the tasks potentially involved the use of multiple representations and the requirement that students provide mathematical explanations or justifications.

As with reaching the probability goals, the extent to which the tasks' potential for promoting higher order learning was realized depended on the implementation of the tasks. Again, the teacher who had been more successful in meeting the probability goals of the *Curriculum Standards* (NCTM, 1989) was also more successful in maintaining the cognitive level of the instructional tasks as they were implemented in her classroom. In the other classrooms, however, the potential cognitive level of the tasks tended to decline as the tasks were implemented. A number of factors may account for the differences in these teachers' probability instruction.

First, the potential of the tasks to promote higher order learning was influenced by the problem-solving nature of the tasks as they were implemented and the extent to which the students were directly involved in the problem-solving process. Rather than using the simulation designs suggested by the instructional materials, the students in the one classroom were actively involved in the problem-solving process as they considered how they might simulate the situation and yet avoid as much bias as possible. In addition, the implemented tasks built upon students' prior knowledge and, in the process, stimulated them to extend their mathematical understanding. In the other classrooms, however, the potential cognitive level of the tasks tended to decline as the tasks were implemented because the problem-solving nature of the instructional tasks became routinized as the teachers and/or instructional materials specified explicit procedures or steps to follow. Part of the justification given by the teachers for modeling the strategies to the students was based on the belief the students had no prior knowledge upon which to build. However, in many cases, the tasks were not structured to incorporate what relevant prior knowledge the students may have had because the teacher had a particular solution strategy in mind. For example, rather than introducing tree diagrams by developing the notion of sample space and building upon the students' earlier problem-solving experiences with the "make an organized list" strategy, the teachers explained what a tree diagram was and modeled the steps for drawing one. In this case and in others, the problem-solving process became one of following the steps modeled by the teacher.

Second, the cognitive level of the implemented tasks was impacted by the extent of the teachers' focus on student thinking. Throughout the instructional process in the one classroom, the teacher maintained a focus on mathematical thinking by continually probing the thinking of her students and expecting them to provide explanations and justifications for their responses and solutions. Her success in promoting student thinking also may have been related to the gifted nature of her students, although the impact of this characteristic of the students is unclear. The focus in the other classrooms, however, was generally on the completeness or correctness of the answer instead of on the thinking processes involved. To the students, showing their work meant displaying the procedural steps used to arrive at an answer. In most cases, the students were not expected to provide any explanation or justification for their answer or their solution process. Similarly, at the conclusion of activities or when assignments were being corrected, the focus of the teachers was on getting right answers, not on understanding students' thinking.

Third, the allocation of instructional time influenced the implementation of the tasks. Because the one teacher chose to do fewer activities and spend more time on each one, more time was available to actively involve the students in the learning process as they worked on problems and shared their problem-solving and reasoning efforts. For the most part, the other three teachers had chosen to include a larger number of activities and spend relatively less time on each task. Although this approach served the purpose of the two experienced teachers to keep students busy and engaged in concrete hands-on activities, it provided fewer opportunities to engage the students in sustained thinking and exploration of the mathematical ideas.

These factors, both in maintaining the cognitive level of the tasks and contributing to its decline, are consistent with research findings in other mathematics classrooms (Hiebert & Wearne, 1993; Stein et al., 1996). In particular, Doyle (1988) has noted novel problem-solving tasks involve a certain amount of ambiguity and risk, which stretch the limits of classroom management and intensify the complexity of the teacher's task. Doyle further suggests that, in the interest of maintaining a smooth-flowing and well-managed classroom environment, teachers tend to simplify classroom tasks by focusing on familiar work, where explicit solution procedures are known, or by guiding students quite explicitly through novel or intellectually demanding tasks. However, by providing such extensive guidance, teachers limit their students' opportunities to become autonomous problem solvers and reinforce their dependency on the teacher, as was demonstrated in the classrooms observed.

In addition to these factors already described, this research study suggests one further factor contributing to the decline of the cognitive level of the tasks. In at least three of the classrooms, rather than serving as vehicles for developing students' mathematical understanding and problem-solving abilities, the instructional tasks themselves became the objects of the investigation. For example, the focus was on the number of cereal boxes one might have to buy to obtain a set of prizes instead of on why and when one might want to consider expected value, the mathematical concept embedded in the task. In the process, exploration of the mathematical content became of secondary importance. Consequently, the students did not consider the mathematical content, except at a superficial level, and they had little opportunity to develop a deeper understanding of the concepts or to see connections between the mathematical ideas. As a result, because the students were rarely given an opportunity to step back from the activities and reflect on the mathematical content being learned, the potentially powerful mathematical experiences in many cases became little more than interesting activities.

Orchestration of classroom discourse. The classroom discourse (how the teacher and students interact with the content and with each other) also conveys messages about the nature of mathematics and mathematical activity (Hicks, 1995; Lampert, 1989, 1990). Meaningful classroom discourse, as envisioned by the *Teaching Standards* (NCTM, 1991), is discourse that promotes the exploration of mathematical ideas and fosters students' understanding of mathematics and of mathematical ways of knowing. Teachers are to orchestrate the various forms of communication, including oral, written, and pictorial, in ways that promote this exploration and understanding. In particular, teachers play a role in shaping the classroom discourse by the questions they ask and the efforts they make to probe students' thinking. In addition, by the ways they interact with students, teachers send messages about the ways of thinking and knowing that are valued. For their part, students are also expected to be involved in the exchange of mathematical ideas as they make conjectures and justify the validity of particular claims to themselves and others.

The instructional tasks and the problems embedded within these tasks provided a number of opportunities for the middle school teachers in this study and their students to engage in meaningful classroom discourse. For example, in considering the fairness of the games, the students could have been asked to explain what fair meant to them or how they might change the game to make it fair. In the many experiments the students conducted, they could have been asked to state and justify what they concluded from their results. The theoretical analyses also provided opportunities for the teachers and students to discuss and justify the steps taken in the process of doing the analysis. Thus, through appropriate orchestration of classroom discourse, the potential was there for engaging the students in the type of authentic mathematical activity envisioned by the NCTM Standards. Despite this potential, however, two distinctively different pictures of classroom discourse emerge from this investigation.

One teacher was generally more successful in orchestrating classroom discourse in ways envisioned by the reform, paralleling the success achieved with her use of instructional tasks. Although this teacher continued to play an active role in guiding the direction of instruction, the discourse was primarily student-focused. Throughout the instructional process, students were encouraged to ask questions and share their conjectures with one another. They were expected to explain their results and justify their thinking, all of which was subject to the challenge of others in the class. Nevertheless, all students' contributions were respected and incorrect responses could be revised without embarrassment. With guidance from the teacher, the students were the ones making decisions about the reasonableness of ideas or answers on the basis of logical arguments. Through her efforts to get students involved in instructional discussions, the teacher was trying to create a learning environment where students were interacting with each other as they themselves sought to make sense of the mathematics they were exploring.

As a result of the teacher's goal to explore students' thinking and to promote mathematical understanding, the nature of the discourse in this classroom was distinctively different from the discourse in the other classrooms. During the interactions between the teacher and individuals or small groups of students, the teacher could assess students' thinking and tailor her questions to further understanding or correct misunderstanding. Whether in small groups or with the entire class, process questions were used to probe the underlying thinking of the students and to guide the students' exploration and decision making. In addition, students' thinking played a vital part in the discourse as students were asked to share the supporting rationale for their predictions, to explain their simulation designs, to point out possible sources of bias, and to state and justify their conclusions to the activities, either verbally or in writing. As a result, the discourse in this teacher's classroom involved thinking at a different and deeper level than in the other classrooms. Although students' understanding of probability was not specifically measured in this study, the kind of classroom discourse observed in this classroom has been associated with students' enhanced understanding of other mathematical concepts (Hiebert & Wearne, 1993; Lampert, 1989, 1990; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990).

In contrast, the classroom discourse in the other three classrooms was more teacher-focused as well as teacher-directed. Consistent with their overall view of the teacher as dispenser of knowledge, these three teachers generally interacted with the class as a whole. Within that context, they presented information to students and modeled the analysis procedures for them. Teacher-directed dialogues, using series of product questions (some of which were leading in nature), guided the students through the analysis process. The teachers were also the primary source of authority regarding mathematical knowledge. In particular, the teachers were the ones presenting the material, asking the questions, and evaluating the answers. And either they or the answer key they held determined what was correct or reasonable.

Because the emphasis was generally on correct answers instead of on thinking, the discourse in these three classrooms focused on information and knowledge at a superficial level, on the rules of the game, the results from the experiment, or the steps of the analysis process. Opportunities to get beyond the "how to" and explore the "why" were generally overlooked. The teachers also missed opportunities to explore students' thinking as the students were playing the games and participating in the activities. In particular, the teachers did not probe students' thinking to discover the rationale behind their predictions or their decisions. Nor did the teachers ask the students to draw their own conclusions from the experimental data or to justify the steps of the theoretical analysis. Only on rare occasions were these students asked to share their observations or explain their thinking about the games and activities.

Creation of the learning environment. The nature of the learning environment also influences what students learn about mathematics and mathematical ways of knowing (Cobb, 1986; Nickson, 1992). The *Teaching Standards* (NCTM, 1991) calls upon teachers to create a learning environment in which mathematical thinking is the norm. In particular, teachers are called upon to establish a classroom environment built on mutual respect where students feel free to take risks, to ask questions, and to share their conjectures. This environment also involves an emphasis on sense-making where students are expected to explain their ideas and to justify their solutions. Working together in these ways, the teacher and students become a learning community as they collaborate in their efforts to make sense of mathematical ideas.

For the most part, the teachers observed in this study had established effective and smooth-running classroom environments in the traditional sense. Patterns for interacting with one another had been established. Routines governed the use of instructional materials. Any misbehavior was handled expeditiously with minimal interruption to the ongoing activity of the classroom. However, the extent to which the members of the classroom formed a mathematical community varied from one classroom to the next.

Underlying the one teacher's relative success in implementing instruction as envisioned by the reform was the learning environment she had established in her classroom. She had expended considerable effort in creating and maintaining a learning environment consistent with her social constructivist approach to learning and her view of mathematics as a growing and changing body of knowledge. This teacher had first made a special effort of creating a classroom environment where a sense of respect was shown by the teacher and the students for all others in the classroom. Social norms established appropriate and inappropriate ways of interacting with one another. In this atmosphere of mutual respect, the students could feel free to share their ideas, uncertainties, and questions as they dealt with the disequilibrium of the learning process. Even incorrect notions were valued for what they could contribute.

In addition, the emphasis in this classroom was on making sense of mathematical ideas, whether students were wrestling with a question individually, sharing their thinking in small groups, or discussing ideas with the whole class. Throughout the learning process, students were expected to explain their ideas and justify their mathematical thinking to themselves and to other members of the classroom community. Instead of the teacher being the sole authority for right answers, she wanted her students to assume the responsibility of judging the correctness and reasonableness of their solutions within the context of the classroom and the wider mathematical community.

In contrast, the classroom environment in the other three classrooms reflected more conventional notions about mathematics and mathematics instruction. The teacher was looked to by the students as an authority. From both the teachers' and the students' perspective, answers could always be assessed as correct or incorrect. Mathematics generally was not seen as a subject for exploration or negotiation about the meaning of the content. In two of these classrooms, the teachers consistently showed respect for the students; the third teacher was at times disrespectful. Nevertheless, these teachers made no particular effort to insure students showed respect for each other. As a result of this classroom climate and the prevailing views in the classroom about mathematics, students ran the risk of having their work labeled "incorrect" by the teacher or "stupid" by their peers, as occurred on occasion in these classrooms.

Analysis of teaching and learning. The final aspect of mathematics instruction addressed by the *Teaching Standards* (NCTM, 1991) is teachers' ongoing analysis of teaching and learning. In their efforts to continually improve instruction and promote students' learning, teachers are encouraged to assess what students are learning and to consider how the tasks, discourse, and learning environment impact that learning. Such monitoring of classroom life and students' learning provides potentially valuable information to teachers as they make instructional decisions and adapt their instruction in response to the needs of their students (Thompson & Briars, 1989; Webb, 1992).

In their ongoing analysis of students' learning, the teachers in this study used a variety of assessment methods, including both formal and informal forms of assessment. This variety is evident in the formal tasks used by the teachers to evaluate what students learned from probability instruction. Two teachers used paper-and-pencil tests to evaluate their students' learning. One of these teacher used questions from the textbook materials; the second teacher wrote his own test items. The other two teachers had their students demonstrate what they learned by completing performance tasks. In one of these classrooms, students were expected to apply their knowledge in novel, open-ended problem situations. The other teacher had her students reflect on the activities they had done by completing a written assignment. In one case, the students wrote a letter outlining what they had done to investigate the problem and explaining what they had concluded.

The teachers also gathered information informally. In particular, the three more experienced teachers reported monitoring student learning by observing and interacting with students informally during the probability lessons. Although the fourth teacher occasionally monitored student activity and progress in a similar fashion, he did not report those observations as one of his assessment tools. This informal monitoring of the students, however, appeared to be little more than part of the teachers' routine, at least in three of the classrooms. Seemingly few actions were taken on the basis of the information gathered. For example, little or no effort was made to reteach material, either individually or to the class as a whole, when difficulties were encountered. If any attempt to reteach material was made, it generally involved a restatement of the steps of the assignment. In contrast, the teacher who was having more success in realizing the goals of the NCTM Standards appeared to use her interactions with students more effectively. With the information gained from such interactions as the foundation, the teacher asked further questions to extend or redirect students' thinking. Further, what she learned from the small-group interactions frequently became the starting point for whole-class discussions.

In considering what their students had learned from probability instruction, the teachers in this study based many of their conclusions on the observations they had made during the probability units. In some cases, these judgments seemed to be based on inadequate measures of students' learning. At least one of the teachers equated learning with hearing or having been told the information. In reflecting on what his students had learned, he observed, "They did learn that just because theoretically you can say it should happen like this, it doesn't mean that it will. In fact, most of the time, it probably doesn't come out exactly that way, but it does come close." Although the teacher had made similar statements to the class on a number of occasions, the students had never been given an opportunity to demonstrate that they understood this relationship between experimental and theoretical results, at least not during any of the observed lessons. However, because the teacher had stated the idea, he assumed it had been learned by the students.

At other times, the teachers equated learning concepts with doing activities. For example, because the teachers had demonstrated how to draw tree diagrams and the students had drawn tree diagrams as modeled for them, the teachers seemed to assume the students had learned how to draw tree diagrams. However, because the students only needed to follow the teacher's example as they analyzed similar situations, it is unclear whether the students were actually learning how to do the analysis on their own. Doyle (1988) points to research evidence from task studies that suggests production on tasks and understanding are not necessarily connected. He emphasizes that students may be able to follow procedures for completing assignments without coming to understand the underlying mathematical principles involved. Moreover, when teachers focus only on whether students' answers are complete or correct rather than on student thinking, as three of the teachers in this study did, mistakes and misconceptions in students' understanding may go unnoticed and uncorrected.

In other cases, the teachers did not seem to distinguish between what the students had done and what the teacher had done. In particular, although a number of the teachers' goals were for their students to analyze various situations mathematically, the teacher was generally the one doing the analysis. Specifically, it was the teacher who set up the chart or drew the tree diagram. Although the goal of the mathematics education reform involves the development of mathematical power, which includes the ability to use knowledge flexibly in novel situations, the students had few if any opportunities to engage in such activity during probability instruction. Perhaps as a result, the students who were expected to apply what they had learned in novel situations encountered difficulties when asked to do so on the evaluation tasks. These difficulties seemed to arise because the students, for the most part, had been spectators of the learning process, not necessarily participants. They had observed as the teacher modeled the analysis procedures or they had been explicitly guided through the analysis process by the teacher's questions. However, because the students were given few if any opportunities to engage in such activity on their own, they were generally unprepared to apply their knowledge in novel situations. The students had been taken on a tour of the masterpieces of the Louvre, they had watched the painters at work on Montmartre, and perhaps completed a paint-by-number picture. Then they had been handed an easel, a palette, and a brush and expected to paint a work of art. For the most part, however, this teacher did not recognize the mismatch between the teaching that had occurred and the learning she expected of her students.

A similar mismatch occurred in at least one of the other classrooms. In this case, the misalignment was between the goals stated by the teacher and the instructional strategies and assessment methods he used. Developing his students' problem-solving and reasoning abilities were among the goals listed by the teacher. However, because most of the instruction was delivered in a "show and tell" manner, the students in this classroom had also been primarily spectators rather than participants in the problem-solving and reasoning that went on. Nevertheless, the questions on the unit test were similar to what the students had encountered during probability instruction. In this case, the assessment instrument was closely matched with the instructional strategies, but the student learning measured by the test involved the students' ability to reproduce what had been modeled for them, not their problem-solving and reasoning ability.

The *Teaching Standards* (NCTM, 1991) encourages teachers to monitor "how well the tasks, discourse, and environment foster the development of every student's mathematical literacy and power. Through this process, teachers examine relationships between what they and their students are doing and what students are learning" (p. 20). However, with the exception of the teacher who was generally successful in implementing instruction of the nature envisioned by the reform, the teachers were not attentive to the realignment needed between goals, instructional strategies, and assessment. They had added aspects of the reform to their instructional practice, such as goals and/or authentic assessment tasks, without recognizing corresponding changes were needed in instructional strategies and/or assessment methods. Guskey (1994) reports similar findings from a study investigating the impact of a performance-based assessment system on teachers' instructional practices. The adoption of a statewide performance-based assessment program had resulted in only modest changes in instructional practices. For most teachers, lesson plans, classroom activities, and evaluation of student learning remained unchanged. Guskey concludes teachers, in general, were "ill-prepared to adapt their instructional practices to the new demands of a more authentic, performance-based assessment program. Most teachers had scant knowledge, personal background, experiences, or formal training with the various types of performance-based assessments or ways to use them as instructional tools" (p. 53).

Summary: Portrait of Probability Instruction

Two contrasting portraits of probability instruction emerge from the findings of this research study. In one classroom, the teacher had captured the essence of the reform effort in her probability instruction. The focus of this teacher was on developing the students' conceptual understanding of mathematics and their problem-solving abilities. With these goals in mind, the teacher had made a special effort to create a learning environment built upon mutual respect, where mathematical thinking was the norm and making sense of mathematical ideas was the goal. In the process of exploring real-life problems, the students were actively involved in making predictions, creating their own simulation designs, conducting simulations, judging the reasonableness of their experimental data, and analyzing the situations theoretically. Throughout the instructional process, the teacher maintained a focus on mathematical thinking by continually probing the thinking of her students and expecting them to provide explanations and justifications for their responses and solutions. With guidance from the teacher, the students were the ones making decisions about the validity or correctness of ideas and answers on the basis of logical arguments.

Probability instruction in the other classrooms was consistent with more traditional notions about the nature of mathematics and of mathematics learning and teaching. In these classrooms, instruction was generally teacher-centered rather than student-centered. The teacher was the principal authority and source of knowledge; the students were primarily spectators. For the most part, the focus of instruction was on activities, not on concepts; on correct answers, not on thinking; on procedures, not on understanding; and on doing, not necessarily on learning.

The use of games, simulations, and other hands-on instructional tasks provided opportunities for students to be actively involved in exploring probability content as well as in developing their abilities to solve problems, to reason and communicate mathematically, and to see applications of mathematics. Although the students were actively involved in completing the instructional tasks, their involvement did not include making decisions about what to do or how to do it, stating or justifying their conclusions, or judging the reasonableness of their actions or results. Instead of being engaged in "doing" mathematics, the students were primarily following the directions given to them, reporting the results of the activities, and applying procedures that had been modeled for them. Thus, although the tasks and activities were hands-on, they were not necessarily minds-on. As a result, the cognitive level of the tasks and discourse was limited by the nature of instruction.

Elements of the reform were present in these classrooms, but only in a limited way. Despite the use of hands-on instructional tasks, cooperative group activities, and authentic assessment tasks, the teaching of probability continued to be generally traditional in nature. The teachers had not reexamined their fundamental beliefs about mathematics or about mathematics teaching and learning in light of the constructivist foundations for the reform. The teachers also had not recognized the corresponding realignment between goals, instructional strategies, and assessment methods needed in order to implement the reform.

Relationship of Teachers' Knowledge to Probability Instruction

The various aspects of teachers' knowledge, namely general pedagogical knowledge, subject matter knowledge, and pedagogical content knowledge, have been treated separately for the purposes of analysis. However, in practice, these facets of teachers' knowledge are closely intertwined. In considering how the knowledge of the teachers in this study was related to their probability instruction, this section will explore how the teachers' efforts to implement the reform were either supported or constrained by their (a) knowledge of probability content, (b) conceptions about the nature of mathematics and of mathematics learning and teaching, (c) understanding of the "big ideas" of probability, (d) knowledge of students' possible conceptions and misconceptions about probability, (e) knowledge and skills in orchestrating classroom discourse, and (f) understanding of the nature of the reform.

Teachers' Knowledge of Probability Content

Ball (1988a) identifies several characteristics of subject matter knowledge that are important specifically for teachers. First, Ball suggests teachers' knowledge should be correct or in agreement with the accepted knowledge in the field. Second, Ball argues that teachers' knowledge should be explicit–they should be able to explain. Although tacit knowledge may be valuable in mathematical activity, Ball maintains it is inadequate for teaching, where explanations are important tools of the trade. Third, Ball proposes that teachers need to understand the underlying relationships within mathematics and the interconnections between mathematics and other content areas.

The probability knowledge of the four middle school teachers in this study was correct in most but not all situations commonly seen in the middle school classroom. In particular, they correctly expressed the probability of simple events and correctly applied the basic properties of probability in situations familiar to middle school classrooms. However, in the less familiar and more complex settings, the teachers' knowledge of probability content was uncertain, insecure, and often incorrect. Errors occurred not only in the interviews, but also during instruction, including during presentations of some of the familiar content. In these cases, the teachers' errors often involved misapplications of the multiplication property, a failure to consider the importance of order, or an uncertainty about whether to add or multiply probability values.

The middle school teachers in this study, however, generally lacked an explicit understanding of the probability concepts included in the interviews or presented in the classrooms. Even when they were able to solve the problems correctly, they often could not or did not provide complete explanations of what they had done or, more importantly, why they had done what they had done. This lack of explicit knowledge was demonstrated in how the teachers handled many of the foundational ideas of probability, including the definition of probability, the basic properties, the use of language, and the analysis strategies. At times, the teachers seemed to have an intuitive understanding of the underlying assumptions. However, the teachers often did not recognize what assumptions they had made nor understand the importance of those assumptions. If the teachers in this study had more than an intuitive or tacit understanding of the content, they did not express it either in the interviews or during probability instruction.

Although the teachers in this study did recognize some of the connections between probability and other mathematical topics, particularly fractions, decimals, and percents, they did not see many of the connections within the content of probability itself. Both in the interviews and in the classrooms, the teachers associated specific solution strategies with particular probability problems. Even when they were able to apply different strategies, they often failed to see or explain the connections between the strategies.

In reflecting on the connections teachers made between probability and its real-life applications, evidence suggests the teachers were dealing with two distinct types of probability knowledge. First was their knowledge of "school" probability. This kind of probability involves selecting objects from a bag or playing games with dice, coins, spinners, or cards. Experiments can easily be conducted in these settings. Theoretical probability can be found by applying the basic definition of probability (favorable outcomes/possible outcomes) and/or using various strategies such as an organized list or a tree diagram. As long as the difficulty of the problems did not extend beyond what was familiar in the middle school classroom, the teachers were generally quite comfortable in this world of probability. In these situations, the teachers generally were able to give correct answers and explanations, although much of their understanding remained implicit.

However, a much bigger world of probability exists beyond the boundaries of "school" probability. In this world, one sees the real-life applications of probability, where results may depend on experimental results or on more complex probabilistic models rather than on the definition or simple strategies. The teachers were not as familiar nor as comfortable with this world of probability. Although the teachers could provide numerous examples of real-life situations involving uncertainty, it was not clear they understood specifically how probability applied in those settings. Their examples were often informal and their interpretations, which generally were subjective and intuitive in nature, were frequently incorrect. Further, the related activities used as part of probability instruction failed to make explicit connections with probability, often becoming activities dealing with statistics or proportions instead.

Thus, the probability knowledge of the middle school teachers in this study was limited, both in scope and in nature. In terms of scope, the teachers' probability knowledge was limited to those situations commonly seen in middle school classrooms. Additionally, the teachers' knowledge of probability lacked both the explicitness and connectedness characteristic of a conceptual understanding of the subject matter. These findings regarding the teachers' knowledge of probability are consistent with the conclusions of studies exploring teachers' knowledge of other mathematical content, including place value, whole number operations, and fractions (Ball, 1988a, 1990c, 1991; Khoury & Zazkis, 1994); multiplication and division (Simon, 1993; Tirosh & Graeber, 1989); rational numbers (Post et al., 1991); ratio and proportion (Fisher, 1988); geometry (Swafford, Jones, & Thornton, 1997); area measurement (Baturo & Nason, 1996); elementary number theory (Zazkis & Campbell, 1996); and functions (Even, 1993; Even & Tirosh, 1995; Stein et al., 1990).

The one teacher who had captured the essence of the reform effort in her probability instruction had taken a probability course in the middle school program she completed which focused on both probability content and pedagogy. As part of the class, this teacher had explored the instructional tasks she later implemented in her own classroom. As a result of this experience, this teacher appeared to have developed a relatively rich knowledge associated with the instructional activities. This knowledge helped support her efforts to engage her students in worthwhile mathematical exploration and meaningful classroom discourse. The extent to which this knowledge may have transferred to other settings is unclear. However, although this teacher did not necessarily have more mathematical knowledge than the other teachers in the study, particularly in the area of probability, the knowledge she possessed was of a different quality. Because she saw mathematics as something to be explored and understood, she had a willingness to investigate and a disposition toward making sense of what she found. This attitude was also reflected in her probability instruction.

In contrast, the generally limited probability knowledge of the other teachers was associated with probability instruction that was impoverished, both in terms of the content and processes of instruction. First, the teachers sometimes presented an inaccurate and/or inconsistent picture of the content to students. On a few occasions, the statements the teachers made or the information or results they presented were incorrect. On other occasions, the information presented to students was inconsistent, at times being presented correctly, at other times incorrectly. At the very least, such instruction was potentially confusing to students; at worst, it was misleading.

In addition, the other teachers often presented an incomplete picture of probability concepts. For example, because the underlying assumption of equally likely outcomes was not stated, the students did not realize the definition of probability (number of favorable outcomes/total number of outcomes) applied only in such situations. Further, because no distinction was made between equally likely and nonequally likely outcomes, the students were unaware of those differences. As a result, on at least one occasion, the students applied the definition of probability when outcomes were not equally likely. Thus, because important ideas were absent from probability instruction, the students were potentially left with an inadequate foundation for future study of probability.

Further, the other teachers generally portrayed probability as a disjointed set of procedures. Understanding comprised remembering the procedures to apply in specific cases. Because such understanding leads to knowledge that is compartmentalized, the students may have been constructing knowledge that was not readily usable in other equally relevant contexts.

The generally limited probability knowledge of the other teachers was also reflected in how the teachers presented and/or facilitated probability instruction. Although the teachers selected instructional tasks which had the potential for developing conceptual understanding of the related probability content, the mathematical investigations generally addressed the content only superficially. In addition, because the teachers did not see the assumptions underlying the content nor the connections between probability concepts, the learning opportunities were not structured in ways to develop a conceptual understanding of the content and the activities were not sequenced in ways to make conceptual connections. As a result, learning procedures became the goal, if not by intent, at least by default.

The teachers' generally limited probability knowledge was also reflected in the explanations provided by the teachers. The teachers did not explicitly recognize the underlying assumptions or the important points of the analysis. Perhaps as a result, these assumptions or important points were not shared with the students. In addition, impoverished or inappropriate examples were sometimes provided. Further, even though the teachers often repeated the procedural steps of the analysis, the underlying thinking processes were rarely highlighted. Although understandable to a person who already knows the content, the incomplete and disjointed explanations sometimes led to difficulties for students who were learning the material for perhaps the first time.

Finally, the limited nature of the teachers' probability knowledge inhibited their ability to have meaningful interactions with students. The teachers often did not grasp the significance or the implications of students' questions, and, as a result, they at times were unable to respond to questions appropriately—either by directly answering the question or by reframing the question so that students could figure it out themselves. In addition, the teachers were sometimes unable to understand the difficulties students encountered or to recognize the misconceptions in their thinking. As a result, opportunities to develop students' understanding of probability or to address their misconceptions were missed.

In conclusion, the one teacher, with her relatively rich knowledge associated with the instructional activities, had been able to facilitate instruction in ways that potentially helped students develop their understanding of probability. In contrast, the generally limited probability knowledge of the other teachers was related to probability instruction that (a) provided a structurally weak foundation for further study of probability, (b) placed an overemphasis on procedures arbitrarily applied, and (c) missed opportunities to develop students' understanding of probability. These findings corroborate the conclusions of other studies that have explored the relationship between teachers' mathematical knowledge and their instructional practice (Lehrer & Franke, 1992; Stein et al., 1990).

Teachers' Conceptions About the Nature of Mathematics

Significant differences were evident in the teachers' views about the nature of mathematics, including their conceptions about the sources of authority, the meaning of "doing" mathematics, and the structure of the content. These conceptions were closely intertwined with the teachers' beliefs about the learning and teaching of mathematics. Although the relationship between teachers' conceptions and their instructional practice is more complex than simply cause and effect, teachers' beliefs about mathematics and its teaching have been found to play a significant, albeit subtle, role in shaping teachers' instructional practice (Thompson, 1984, 1985, 1992).

The one teacher most successful in implementing the vision of the reform expressed views about mathematics and the learning and teaching of mathematics that were based on a constructivist philosophy about learning, the same philosophy underlying the NCTM's vision for the mathematics classroom (NCTM, 1989, 1991). To her, mathematics is a dynamic, growing discipline, constantly changing as a result of new discoveries. Rather than an established body of knowledge to be remembered and applied in particular situations, mathematics was seen as a personally constructed, or internal, set of knowledge. As such, mathematics is a world of ideas to explore and with which to interact in the process of developing understanding.

To this teacher, the process of learning mathematics was centered around the students' active involvement in doing mathematics. Through the students' efforts to investigate and solve problems, the students were involved in a process of constructing meaning and developing understanding. According to her view, learning was also a social activity, where each student's developing understanding potentially influenced and was influenced by collaboration with others within the classroom community.

According to this teacher's view, teaching involved being a facilitator of student learning. As a result, her instructional efforts were directed toward creating opportunities for students to actively explore problems and to share their problem-solving and reasoning efforts with one another. Students in her classroom were "doing" mathematics in meaningful ways, determining what made sense and was correct on the basis of logical and reasonable arguments. By choosing to do fewer activities in greater depth, by establishing a classroom environment built upon respect, and by interacting frequently with individual students or small groups of students, the teacher encouraged and guided the efforts of her students as they constructed their own understanding of mathematical ideas. In so doing, she demonstrated an approach to teaching characterized by Thompson (1992) as learner-focused.

To varying degrees, the other three teachers held more traditional views about the nature of mathematics and of mathematics instruction, which can be summarized as follows: (a) School mathematics is assumed to be a fixed body of knowledge, where each type of problem is associated with a particular solution procedure; (b) teaching mathematics involves presenting demonstrations of the procedures and providing opportunities for the students to practice them; (c) learning mathematics involves listening to the teachers' demonstrations, practicing the steps of the procedures, and recalling and applying the procedures when appropriate; and (d) mathematical truth is determined by the teacher and/or the instructional materials. These views are consistent with what has been called a broadcast metaphor (National Research Council [NRC], 1989), the absorption theory of learning (Romberg & Carpenter, 1986), and teaching by telling (Fisher, 1990; Smith, 1996).

This set of beliefs is not unusual. Studies of the beliefs and practices of prospective teachers (Ball, 1988b, 1990a, 1990b, 1990c; Borko et al., 1992; Eisenhart et al., 1993; Wilcox, Schram, Lappan, & Lanier, 1991; Wilson, 1994) and of practicing teachers (Putnam, Heaton, Prawat, & Remillard, 1992; Thompson, 1984; Wood, Cobb, & Yackel, 1991) provide consistent evidence that these beliefs are widespread among teachers and students alike. Observational research on teachers' practices (Schoenfeld, 1988; Stodolsky, 1985) provide additional evidence that such beliefs are reflected in mathematics instruction.

These beliefs were evident in the probability units observed, particularly in the teachers' approach to the analysis process. Believing students did not know how to do the analysis of the probabilistic situations and believing they needed to have that information before being involved in problem solving or reasoning on their own, the teachers focused on delivering and modeling the analysis strategies for the students. In the process, the students became spectators of the analysis instead of participants in it. Even when the teachers used techniques such as hands-on activities or cooperative learning to get the students more actively involved, teaching as telling continued to lurk just beneath the surface, evident in the teachers' interactions with students during the activities.

These strongly held beliefs about mathematics instruction overlook the findings of a growing body of research on how students learn mathematics (Cobb et al., 1992; Davis et al., 1990; Resnick, 1987). Whether or not teachers hold constructivist beliefs, Noddings (1990) and Cobb et al. suggest students are constructing understanding from whatever instruction they receive. What students learn, however, may or may not be what teachers were intending to teach. In particular, students may be constructing inaccurate conceptions about the content and/or nature of mathematics. As Schoenfeld (1988) discovered, even apparently good instruction from a pedagogical perspective may lead to unexpected and undesirable lessons being learned. Among the incorrect messages potentially communicated by the conventional probability instruction were that (a) probability involves procedures to be remembered for particular problems rather than strategies to be applied as part of a general problem-solving process, (b) students are to accept the probability explained to them by the teacher or textbook without the expectation that they can make sense of it for themselves, and (c) the correct answer is what counts, not the thinking behind the answer.

Teachers' Understanding of the "Big Ideas" of Probability

Although the *Curriculum Standards* (NCTM, 1989) provides overall goals and objectives for grades 5 through 8, the middle school teachers in this study did not appear to have any curriculum guidelines to follow concerning appropriate learning objectives or instructional activities for the grade level(s) they were teaching. In particular, the *Curriculum Standards* provided no guidance for these teachers about what probability ideas should be presented and/or what concepts or skills should be mastered at different grade levels. Although textbooks potentially provide some guidance about the structure and development of probability content, the teachers moved away from using the textbook for probability instruction (if one was being used at all) in favor of utilizing hands-on instructional tasks. Therefore, without curriculum guidelines or textbooks to which they could refer, the teachers were left with the responsibility of determining for themselves what content should be covered and how the content should be presented.

For the most part, however, the teachers in this study lacked the knowledge upon which to base such decisions. First, the teachers lacked an understanding of which ideas or concepts are foundational to the study of probability, concepts such as sample space or equally likely outcomes. Second, the teachers had difficulty identifying ideas, such as the significance of sample size or the role of experimental data, that are important to emphasize when teaching probability. Third, the teachers did not appear to understand how the important ideas were related, for example, how the various strategies were connected, when the strategies could be applied, and how the strategies related to a general approach for solving probability problems. Finally, the teachers held relatively incomplete and naive views about the nature and structure of probability.

Because the teachers did not have specific learning objectives or an overall "big picture" of probability in mind and because they did not know the important ideas to emphasize, probability instruction became primarily an investigation of interesting activities. However, there was little continuity from one activity to another and little sense of where an activity fit in the total scheme of things. In addition, without a curriculum to provide articulation and coordination across grade levels, the teachers generally provided a survey of probability topics rather than focusing on mastery of specific objectives or developing mathematical ways of thinking. Therefore, the resulting probability instruction was broad in terms of topics covered, but shallow in terms of opportunities for developing understanding. Thus, although students were exposed to probability ideas and concepts, they were given only an incomplete and superficial picture of the subject and were not expected to gain mastery or competence in dealing with the material in a more general sense.

The teachers identified a number of goals they hoped to accomplish as part of their probability instruction. Many of these goals focused on what students would be doing, including playing games, conducting experiments, and participating in a number of different activities. Other goals appeared to focus on aspects of probability content, including analyzing situations with a variety of strategies, seeing applications of probability in the real world, comparing experimental and theoretical probabilities, and determining whether a game is fair. However, even these goals more accurately described the activities done in the classroom rather than the potential learning that occurred. Prawat (1992b) attributes the tendency to equate activity with learning to a "belief on the part of many teachers that student interest and involvement in the classroom is both a necessary *and* sufficient condition for worthwhile learning" (p. 371). Student interest and involvement had in fact been key aspects considered by the teachers in selecting the instructional tasks for their probability units. The teachers also appeared to pay more attention to the nature of the tasks (game or simulation) and materials (dice or coins) than to the probability content (tree diagram or expected value). Thus, neither the teachers' goals nor the important ideas of the content were of primary concern to the teachers as they planned probability instruction. This is consistent with research cited by Prawat, which suggests that activities rather than ideas are the basic units and starting points for many teachers when they plan lessons.

Prawat (1992b) argues that viewing the curriculum as a network of big ideas is more consistent with constructivist views about teaching and learning. Such an idea-oriented curriculum, however, places heavy demands on teachers' knowledge. Instead of knowing where the teaching and learning process is heading, in terms of one topic following another, Prawat stresses teachers need to develop a "global view, understanding the network of big ideas that helps define a domain of inquiry, and possible relationships among those ideas" (p. 387). In deciding which ideas to emphasize and how to situate those ideas in real-world phenomena, teachers need to understand what is important for students to know from a disciplinary perspective. But teachers also need to consider what students are best equipped to learn, what materials will challenge and stretch the students, and how the students' own search for meaning can be encouraged and accomplished. For the most part, the middle school teachers in this study lacked the knowledge necessary to implement an idea-based curriculum. In particular, the teachers lacked the global view; they lacked an understanding of the network of big ideas. This, however, is not the only knowledge needed in order to implement this interactive view of curriculum. Another essential component is teachers' knowledge of students' understanding of probability, which will be considered next.

Teachers' Knowledge of Students' Understanding of Probability

For the most part, the four middle school teachers in this study demonstrated a limited knowledge of students' understanding of probability. First, the teachers had a limited understanding of what initial conceptions students might possess about probability. In particular, they generally appeared to be unaware of the existence or nature of students' intuitive beliefs. Second, although the common misconceptions have been described in materials directed toward teachers (Hope & Kelly, 1983; Shaughnessy, 1981), the teachers in this study apparently had not encountered examples or explanations of the misconceptions in their previous study of probability. As a result, they had a limited understanding of what potential misconceptions students may have about the subject. Third, despite their previous experiences teaching probability, the teachers were seemingly unaware of the difficulties students might encounter in their study of probability. Finally, the teachers did not appear to have specific knowledge about how students develop an understanding of probability. The research reviewed in chapter II of this research study has provided some information about how an understanding of probability develops. Other research (Fischbein & Schnarch, 1997; Jones, Langrall, Thornton, & Mogill, 1997; Jones, Thornton, Langrall, Johnson, & Tarr, 1997; Lecoutre, 1992; Tarr & Jones, in press; Williams & Amir, 1995) has contributed to a growing understanding of students' probabilistic thinking. The teachers in this study, however, were generally not aware of the findings from these research efforts.

The relationship between the teachers' knowledge of students' understanding and the teachers' probability instruction depended somewhat on the teachers' perspective on mathematics instruction and the extent of the teachers' focus on student thinking and learning. This relationship was evident in the teachers' efforts (a) to connect instruction with students' prior conceptions, (b) to create or take advantage of opportunities to address students' misconceptions, (c) to anticipate and design instruction to overcome difficulties, and (d) to probe students' thinking and develop their understanding of probability.

The three teachers with more traditional views about teaching mathematics made little or no particular effort to connect instruction to or build upon students' previous experiences or prior knowledge. For example, no attempt was made to draw out students' understanding of fairness from their earlier experiences playing games nor to determine whether the students' prior conceptions were accurate or complete. In general, the teachers did not appear to consider students' initial conceptions about probability as they planned probability instruction. Although students' potential affective reactions to the instructional tasks had been considered in selecting the tasks, the teachers had not considered the connection between the tasks and students' prior knowledge.

The manner in which these teachers approached their probability units seemed to be based on multiple, sometimes contradictory, assumptions about the students' initial understanding of probability. On the one hand, because the students had generally had little or no previous probability instruction, the teachers assumed the students had no background knowledge at all. At least no attempt was made to relate instruction to any prior conceptions of probability. Even when the teachers assumed the students understood some very basic notions, no effort was made to verify or build upon those conceptions. On the other hand, although generally believing the students had little or no background knowledge, the same teachers sometimes proceeded with instruction as if the students knew the basic definition or properties of probability.

Because these teachers were generally unaware of the potential misconceptions the students might have or the difficulties they might encounter, the teachers did not plan instruction designed to create opportunities to address potential misconceptions or to assist students through the areas of difficulty. For example, the teachers did not select tasks or ask questions that could have directed the students to consider potential misconceptions in their thinking, such as the impact of order in constructing sample spaces. Nor did the teachers present or sequence instruction in ways that could help students understand how or when to draw tree diagrams.

The teachers also missed opportunities to address the misconceptions or difficulties when such opportunities arose during the course of probability instruction. In particular, the teachers frequently failed to recognize the students' faulty conceptions when they were revealed by students' questions or comments. Even when the teachers realized the students were incorrect, they often did not recognize the nature of the students' misconception. As a result, the teachers either ignored the misconception altogether or attempted to address it with an explanation of the correct answer. They did not attempt to probe the students' thinking to discover the source of their error nor ask questions to redirect the students toward a more correct understanding.

The limited focus on students' thinking and learning in these three classrooms was also evident in the teachers' efforts to probe students' thinking. Rather than using questions to probe and encourage students' thinking, series of questions were used to lead students step by step to the desired conclusions. Students' contributions were usually judged as right or wrong, with little effort directed at understanding the students' reasoning or the source or nature of the students' errors. When errors in students' thinking were revealed, the teachers made little or no effort to ask questions aimed at illuminating the student's faulty line of reasoning, to use other students' explanations to bring out a more correct understanding, to reteach the topic in an alternate way, or to select another activity designed to bring out a more correct understanding.

In exploring how teachers' knowledge of students might impact their responses to students' questions, ideas, or hypotheses, Even and Tirosh (1995) observed that "many of the teachers made no attempt at understanding the sources of students' responses. When asked directly, they found it difficult to explain why students reacted the way they did" (p. 17). This description also characterizes the responses of these three teachers, who held what Thompson (1992) called a content-focused view toward teaching rather than a learner-focused view. This response to students' contributions, particularly their errors, may be explained by one of the premises Thompson states for this view: "It is not necessary to understand the source or reason for student errors; further instruction on the correct way to do things will result in appropriate learning" (p. 136). The teachers in this study had, in fact, responded to student errors by providing an explanation of the correct answer.

To some extent, the teacher with a constructivist view on teaching mathematics stands in contrast to the generally bleak portrait of the teachers' knowledge of and efforts to discover and address students' conceptions and misconceptions of probability. To begin with, this teacher recognized that, even without previous instruction, students have strong intuitive notions as a result of their experiences in a world of uncertainty. In beginning her lessons by having the students make a subjective estimate or guess and give their rationale, she attempted to bring out the students' intuitive notions. Further, she also structured the learning activities so that the possible dissonance between the students' subjective or intuitive notions and the experimental or theoretical evidence might be revealed. Questions were used to probe students' thinking or to raise content-related issues. When errors or misconceptions arose in the students' thinking, pertinent questions redirected that thinking. Possible misconceptions or important issues were brought before the classroom community to be addressed in a logical and reasonable manner. In this context as well, she used the mathematical thinking and arguments of the students themselves to influence and convince their peers as each was allowed to form their own conclusions. Although this teacher did not necessarily have more knowledge of probability or of students' understanding of probability than the other teachers, her instructional efforts were directed more at probing students' thinking and connecting instruction in meaningful ways to the students' prior knowledge. This was largely because the overall focus of her mathematics instruction was aimed at developing students' mathematical understanding and problem-solving ability.

In their investigation of teachers' knowledge of students' cognitions and its impact on instruction, Carpenter et al. (1989) conclude that understanding the knowledge students bring to the topic, the strategies they use in solving problems, and the stages through which they pass in acquiring more advanced strategies allows teachers to structure instruction so that students can connect what they are learning to the knowledge they already possess. However, for the most part, the teachers in this study generally lacked an understanding of these aspects of students' cognitions. Together with the lack of focus on the learner, which characterized the instruction of three of the teachers, probability instruction generally was not connected to students' background knowledge or current conceptions of probability.

Teachers' Knowledge and Skills in Orchestrating Classroom Discourse

Underlying the contrasting portraits of classroom discourse described earlier are significant differences in the teachers' knowledge and perspectives concerning the learning process. In particular, the contrasting nature of the classroom discourse can be traced to the contrast between the one teacher's constructivist views about learning and the more traditional views about learning held by the other teachers. These differences were evident in the teachers' beliefs about the locus of authority for what is accepted as mathematically true or reasonable and in the extent of the teachers' focus on students and their thinking.

Through her efforts to get students involved in instructional discussions, the teacher with constructivist views about learning was trying to create a classroom environment where students were interacting with each other as they themselves sought to make sense of the mathematical ideas they were exploring and where students assumed the responsibility of judging the correctness and reasonableness of their solutions. However, although attempting to involve students in meaningful classroom discourse, the other teachers were having difficulty giving up the teacher's traditional role as sole authority or expert. In particular, rather than allowing students to state what they had concluded from the data obtained in the various activities, in many cases the teachers stated the conclusions themselves.

When the teachers in this study did attempt to open up classroom discussion to include students' strategies and reasoning, they found themselves in unplanned situations, facing the challenge of understanding and responding to the unanticipated questions students asked or the unexpected contributions they made. Although common occurrences in classrooms (Borko et al., 1992; Heaton, 1992; Putnam, 1992), such experiences can be intimidating to teachers who view themselves as the "expert" or "authority" in the classroom. Rather than running the risk of becoming confused and making mathematical mistakes or taking a chance of losing control of the direction of the discussion, the teachers in the more traditional classrooms in this study tended to retreat to their familiar and comfortable routine of presenting material and directing the discourse themselves.

The extent of the teachers' focus on student thinking was related to their decisions about the allocation of instructional time and the use of questions as well as their efforts to encourage the participation of all students, decisions and efforts which influenced the nature of classroom discourse. Because of the one teacher's focus on students and their thinking, she chose to do fewer activities and spend more time on each one. As a result, more time was available to actively involve students in the classroom discourse as they worked on problems and shared their problem-solving and reasoning efforts. However, for the other teachers, who were focused on getting through more activities or covering more content, not enough time was available to involve the students in significant ways, even though the teachers tried. Because it takes less time to tell the students what to do or how to analyze a problem than it does to involve the students in sharing their ideas or making decisions, these teachers generally chose to be more directive in their instructional efforts.

The nature of the questions asked by the teachers was also related to the extent of the teachers' focus on students' thinking. Because of her emphasis on probing students' thinking, the one teacher asked more open-ended process questions to clarify or stimulate students' thinking and to guide their exploration and decision making. In contrast, in their efforts to guide students through the analysis process, the other teachers asked primarily product questions, some of which suggested what the appropriate response should be.

Another challenge facing the teachers was how to get all students engaged in the classroom discourse in meaningful ways. Students in the more traditional classrooms were accustomed to letting the teacher or other students do most of the talking. With the focus on right answers and the potential embarrassment for incorrect answers in these classrooms, other students may have been discouraged from participating. In the one classroom, the teacher made a special effort to encourage the participation of all students. In some cases, students were reporting their initial guesses or supporting one conclusion or another. In other cases, students were asked to respond to claims or statements made by other students.

When a breakdown in communication occurred, the teachers were not the only ones responsible. At times, students were not able to give a clear explanation of what they had done, even when the teacher probed their thinking. Because students were more accustomed to listening and letting the teacher do the talking, they often had no experience preparing them to explain their mathematical thinking or to present a valid mathematical argument.

For the most part, the teachers in this study recognized the importance of involving students in meaningful classroom discourse and were making efforts, albeit generally unsuccessful efforts, to accomplish that goal. The one teacher had been more successful than the other teachers in realizing the goal, but the level and nature of the students' contributions to the discourse had not happened by accident. According to this teacher, much time had been spent at the beginning of the year establishing the expectations and setting the standards for the nature of classroom interactions. By direct instruction and group discussion, she and her students had addressed questions such as: "What should it look like when you have a mathematician sharing? What do the rest of us do? What would we not do? What does it look like if somebody is asking a question in a non-threatening way?" Throughout the year, the students had been given numerous opportunities to participate in the discourse and to practice and improve their ability to communicate. With these opportunities came continual reminders of the expectations. In this way, the teacher had made an effort to provide social scaffolding by establishing the norms for social behavior and expectations regarding classroom discourse (Williams & Baxter, 1996).

However, efforts to provide what Williams and Baxter (1996) call analytic scaffolding, or the structuring of mathematical ideas for the students, were not evident in the discourse of the classrooms in this study. In particular, little or no framework was given to help the students see the interrelationships between the ideas they were studying or to help them see how those ideas fit into the overall picture of probability. Recognizing that there is a fine line between telling students too much and telling them too little, Williams and Baxter suggest teachers may not provide scaffolding of mathematical concepts for their students in their attempts to avoid being too directive. In particular, they concluded the teacher they observed distanced herself from the development of an analytic scaffolding, expecting instead that the analytic scaffolding would arise from the tasks she selected and from the discourse among her students. Such a belief in the ability of students to structure their own learning is called "naive" constructivism by Prawat (1992b). This perspective may describe the teachers in this study, who generally did not appear to be concerned about providing analytic scaffolding for their students. Or, perhaps, the teachers may not have structured the mathematical ideas for their students because such structure may have been a missing element in their own understanding of probability.

Another aspect missing from the classroom discourse in these classrooms was what Cobb, Boufi, McClain, and Whitenack (1997) call reflective discourse or collective reflection. In this form of discourse, the physical and mental actions of the students and teacher become explicit objects of discussion. Cobb et al. distinguish between the psychological process of reflective abstraction, the process by which individual students reorganize their mathematical activity, and the communal activity of collective reflection, which occurs as students participate in reflective discourse, pointing out that "although participation in reflective discourse supports and enables individual reflection on and reorganization of, prior activity, it does not cause it, determine it, or generate it" (p. 266). They, however, conjecture that participation in reflective discourse might support students' mathematical learning by encouraging such individual reflection. Cobb et al. also suggest "one of the primary ways in which teachers can proactively support students' mathematical development is to guide and, as necessary, initiate shifts in the discourse such that what was previously done in action can become an explicit topic of conversation" (p. 269). However, efforts to engage the students in reflective discourse were noticeably absent from the probability instruction observed in this study. Although writing the letters provided an opportunity for students to reflect on their investigation of "Monty's Dilemma," they focused on reporting what they had done rather than reflecting on their actions.

Teachers' Understanding of the Nature of the Reform

Another factor influencing the teachers' efforts to implement the reform was their understanding of the nature of the reform. Based upon their understanding of the mathematics instruction envisioned by the reform, the teachers were making changes in their classrooms. They were moving away from reliance on a textbook, using hands-on activities and manipulatives in its place. As part of probability instruction, the teachers were using instructional tasks designed to provide opportunities for students to solve problems, to reason and communicate mathematically, and to see connections within mathematics. The teachers were also including projects and portfolios (although neither were observed in the probability units) and cooperative learning activities (of which only limited instances of cooperative groups in a formal sense were observed). In addition, one teacher was using authentic assessment tasks for which students were expected to apply what they had been learning to solve unfamiliar problems.

Although among the strategies prescribed by the reform documents (NCTM, 1989, 1991, 1995), these aspects represent only part of the vision of the reform. With the exception of the one teacher who had embraced the constructivist view of learning, the middle school teachers in this study lacked an integrated understanding of the assumptions about learning that research has shown serve as a reasonable foundation for the new vision of mathematics instruction. As a result, the teachers were adding these elements of the reform to their otherwise traditional framework, leaving their traditional pedagogy of telling fundamentally intact. This approach, however, led to discontinuities in the learning process when, for example, the "show and tell" instructional model did not prepare students to solve unfamiliar problems on their own.

The teachers generally seemed to have a somewhat incomplete or inaccurate view of the philosophical foundations of the reform. One teacher was concerned that "sometimes you can't afford the time to spend 2 weeks letting them discover this or that." Because the role of the teacher is no longer to transmit "correct" ways of doing mathematics, some, such as this teacher perhaps, see the constructivist approach as inefficient, free-for-all discovery. However, by selecting appropriate tasks and offering opportunities for meaningful classroom discourse, teachers can facilitate students' efforts to generate powerful ideas.

Another teacher had the impression that "they're wanting everybody to . . . not have their feelings hurt. . . . They want everybody to be right." Rather than the attitude that "anything goes," the *Curriculum Standards* (NCTM, 1989) emphasizes that there is a common core of mathematical understandings that should be a part of every student's experience. This teacher, however, did not understand the teacher's role in helping to correct misconceptions and to shape correct understanding.

In general, several of the teachers also believed students cannot solve problems on their own until after they have specific prerequisites, until after they have some "basic teaching . . . and some hands-on work with it" first. In so believing, the teachers failed to see that students may have prior experiences or knowledge, particularly with probability, that either aid or inhibit their efforts to develop further understanding.

Finally, the teachers had limited views of what is meant by problem solving, reasoning, communicating, and other aspects of the reform effort. In particular, one teacher equated problem solving with the textbook's word problems. The teachers also failed to distinguish between the reasoning the teacher did and the reasoning the students had an opportunity to do.

Knowledge of the reform created some tensions between what teachers realized should be happening and what was actually occurring in their classrooms. For example, the teachers generally realized students should be more involved in classroom discourse. However, as teachers continued to cling to their role as the authority, their goal of greater student involvement was difficult to realize. Nevertheless, these tensions had not been sufficient to cause the teachers to reevaluate their conceptions about learning and teaching mathematics.

Summary: Relationship of Teachers' Knowledge to Probability Instruction

A number of studies conducted in the past decade have described teachers' knowledge, including their subject matter and pedagogical content knowledge, in the areas of place value, whole number operations, and fractions (Ball, 1988a, 1990c, 1991; Khoury & Zazkis, 1994); multiplication and division (Simon, 1993; Tirosh & Graeber, 1989); rational numbers (Post et al., 1991); ratio and proportion (Fisher, 1988); geometry (Swafford et al., 1997); area measurement (Baturo & Nason, 1996); elementary number theory (Zazkis &

Campbell, 1996); and functions (Even, 1993; Even & Tirosh, 1995; Stein et al., 1990). This research study therefore complements the broader research of teachers' knowledge by providing a picture of middle school mathematics teachers' knowledge of probability.

This study also extends this knowledge base by combining interview questions with observations of probability instruction. Previous studies have focused primarily on the knowledge of preservice teachers (Ball, 1988a, 1990c, 1991; Baturo & Nason, 1996; Even, 1993; Even & Tirosh, 1995; Khoury & Zazkis, 1994; Simon, 1993; Swafford et al., 1997; Tirosh & Graeber, 1989; Zazkis & Campbell, 1996) or of practicing teachers (Fisher, 1988; Post et al., 1991) as they participated in tasks set in research contexts (written problem solving and/or individual interviews). Other studies (Putnam, et al., 1992; Thompson & Thompson, 1994, 1996) have observed teachers' instruction but did not explore the teachers' subject matter or pedagogical content knowledge apart from what was observed. Only Lehrer and Franke (1992) and Stein et al. (1990) have combined interviews of teachers' knowledge with observations of mathematics instruction. With the exception of these two studies, the relationship between teachers' knowledge and their instructional practice has not been considered directly. Therefore, in addition to contributing to the expanding picture of teachers' knowledge, this study extends the teacher knowledge research by exploring the relationship between that knowledge and what teachers do in their classrooms.

For the most part, the teachers in this study were competent, caring professionals committed to the task of teaching mathematics to middle school students. They demonstrated a general sense of expertise in their role as middle school mathematics teachers, expressed confidence in themselves as teachers, and displayed mastery of the basic tasks of teaching. The teachers, for the most part, demonstrated competence in creating and managing an effective classroom environment, one perceived by the teachers to be focused on student learning. They also continued to take advantage of opportunities to learn more about mathematics and about teaching mathematics. The teachers observed in this study were familiar with the calls for reform in mathematics education and had assimilated recommended strategies into their mathematics instruction. They had collected a variety of curriculum materials appropriate for teaching probability. Nevertheless, only one of the four teachers taught probability in ways that reflected the general spirit of the reform.

The teachers' knowledge of probability content generally was limited to the situations commonly seen in middle school classrooms. In particular, although one might reasonably expect teachers to have knowledge of probability that extends beyond what they teach, the middle school teachers in this study appeared to have little knowledge beyond what they were expecting students to learn. In addition, the teachers' probability knowledge lacked the explicitness and connectedness which characterize a conceptual understanding of the subject matter. Further, other than the one teacher who understood the dynamic nature of mathematics, the teachers held traditional conceptions about the nature of mathematics and relatively naive views about the nature of probability. In many respects, the limited nature of the teachers' subject matter knowledge of probability extended beyond their interview responses, and was reflected in their efforts to teach probability as well. As a result, the students were potentially left with a generally inadequate and disjointed picture of probability.

The teachers' efforts to teach probability were also related to the teachers' pedagogical content knowledge. Although the teachers selected instructional tasks with the potential to realize the goals of the NCTM Standards, they generally had a limited understanding of the nature of these goals and the instructional processes for reaching the goals. Similarly, the teachers had a limited understanding of the justification for teaching probability and the important ideas to emphasize when teaching probability as well as an impoverished repertoire of representations for probability concepts.

For the most part, the teachers also had a limited knowledge of students' understanding of probability. As a result, the direction and nature of probability instruction in their classrooms were not significantly influenced by what probability knowledge or understanding students already possessed or by what knowledge students demonstrated in the course of probability instruction. The teachers' limited knowledge of students' understanding of probability also appeared to influence the nature and quality of the teachers' responses to students' questions and contributions. In addition, the teachers' ability to recognize and address students' errors and misconceptions was also related to the teachers' subject matter knowledge. Similarly, the extent to which students' understanding of probability was considered in the process of designing probability instruction was influenced by the teachers' conceptions about learners and the learning process.

With the exception of the classroom of the teacher who held more constructivist perspectives toward learning, fairly traditional forms of classroom discourse were evident in the middle school classrooms. The teachers continued to be the authority and source of knowledge, the ones responsible for presenting material and directing the discourse. Rather than probing students' thinking, that thinking was guided by series of questions. Teachers in all four classrooms rarely provided analytic scaffolding for the mathematical ideas being studied. Opportunities to step back and reflect on the mathematics content or on the thinking processes were also rare.

The middle school teachers were making changes in their classrooms in their efforts to implement the reform in mathematics education. However, for the most part, the teachers lacked an understanding of the constructivist foundation upon which the new vision of mathematics instruction is based. Further, because instruction was not necessarily connected to students' prior knowledge and because reflection on the content and thinking processes was not encouraged, learning was disjointed at both ends of the learning process, according to the constructivist view.

The most striking differences in teacher knowledge among the teachers in this study were found in the teachers' conceptions about the nature of mathematics and the teachers' beliefs about learning and teaching mathematics. One teacher envisioned mathematics as a world of ideas to be explored; to her, learning mathematics meant a process of constructing meaning and developing understanding. As a result, she took a learner-focused approach to teaching mathematics, including probability. Her focus on students' thinking and learning was evident in her efforts to connect instruction with students' prior knowledge, to facilitate instruction designed to discover and overcome misunderstandings or difficulties, and to probe students' thinking and develop their understanding of probability. Her focus on students' thinking also impacted the content and nature of the classroom discourse.

Because of her constructivist beliefs about learning and the resulting focus on students' thinking and learning, this teacher had been able to capture the essence of the reform effort in her probability instruction. However, the efforts of the other teachers to implement the reform generally fell short because they (a) lacked an explicit and connected knowledge of probability content, (b) held traditional views about mathematics and the learning and teaching of mathematics, (c) lacked an understanding of the "big ideas" to be emphasized in probability instruction, (d) lacked knowledge of students' possible conceptions and misconceptions, (e) lacked knowledge and skills needed to orchestrate discourse in ways that promoted students' higher level learning, and (f) lacked an integrated understanding of the nature of the reform.

Implications of the Study

This study has explored middle school mathematics teachers' subject matter and pedagogical content knowledge and the relationship between teachers' knowledge and their probability instruction. In general, the findings of this study suggest at least some teachers do not have the knowledge necessary to teach in the ways envisioned by the reform. In order for these teachers to provide more effective probability instruction, the results from this study suggest changes may need to be made and growth may need to occur in general and specific ways. In general respects, teachers first may need to develop useful and personally meaningful theories of mathematics learning which include an understanding of (a) the dynamic nature of mathematics, (b) the constructivist nature of mathematics learning, and

(c) the role of the teacher in facilitating mathematical thinking and understanding. Second, teachers may need to develop the ability to plan and implement instruction of this nature, including (a) how to select and implement appropriate instructional tasks; (b) how to interact effectively with students, including listening, questioning, monitoring, and facilitating classroom discourse in ways that promote the type of learning envisioned; (c) how to establish a learning environment built upon mutual respect; and (d) how to assess students' development of conceptual understanding and mathematical competence (problem solving, reasoning, communicating, and seeing connections). In terms of teaching probability specifically, teachers may need to develop (a) a more extensive, explicit, and connected knowledge of probability; (b) a better understanding of the nature of probability; (c) knowledge of what possible conceptions and misconceptions students may have and how their conceptions of probability develop; and (d) a more complete and integrated knowledge concerning the teaching of probability, including understanding of the goals of probability instruction, the important ideas to teach, instructional strategies to apply, and appropriate representations of probability concepts. These findings have implications for mathematics education reform, preservice teacher preparation, staff development, and curriculum development.

Mathematics Education Reform

The current reform movement in mathematics education puts forward an ambitious agenda for classroom change. Reform documents (NCTM, 1989, 1991, 1995; NRC, 1989) propose significant changes in the content of mathematics that is taught and in the way mathematics is taught and learned. The findings of this study suggest that, as a result, teachers face the tremendous challenge of teaching mathematical content they may not have had an opportunity to learn and content they may not fully understand. In addition, they face the challenge of teaching that content in ways they may not have experienced and in ways that demand rich and flexible teacher knowledge, including general pedagogical knowledge, subject matter knowledge, and pedagogical content knowledge they may not possess or at least know how to apply in an unfamiliar setting.

Although the vision of the reform has been presented in formal documents (NCTM, 1989, 1991, 1995) and discussed in journals and other written materials of national and state organizations, there appears to have been no systematic, in-depth, or comprehensive effort to present the agenda for change personally to practicing teachers. For the teachers in this study, the primary sources of information about the reform in mathematics education had been workshops at mathematics conferences, summer school or staff development classes, or curriculum materials (in many cases obtained in conjunction with the workshops or classes). The result was a disjointed and fragmented knowledge about the new vision for mathematics instruction. The teachers spoke in general terms about the nature of the reform, but their comments and instructional practice revealed an incomplete and sometimes inaccurate understanding of the recommendations of the reform.

Nevertheless, the teachers in this study had responded to the reform efforts by making changes in their mathematics instruction. The teachers' initial response had been to view the reform as a source of content guidelines or of teaching methods. In particular, these teachers were teaching probability, one of the newer content areas recommended for study at the middle school level. They were also including hands-on instructional tasks, cooperative group activities, and authentic assessment tasks. These changes were logical first steps for teachers left to make their own changes and are representative of how other teachers have responded to similar reform efforts (Ball, 1990d; Cohen, 1990; Heaton, 1992; Peterson, 1990; Prawat, 1992a; Remillard, 1992; Schifter & Fosnot, 1993; Wiemers, 1990).

However, the changes the teachers had made appeared to be cosmetic (adding hands-on instructional tasks, cooperative group activities, and authentic assessment tasks) rather than substantive (based on a constructivist perspective on learning). For example, the teachers seemed to treat the new topic of probability as though it was part of traditional school mathematics (with traditional instructional strategies and outcomes). The instructional materials and activities, which were intended to be used to teach mathematics for understanding, were infused with traditional messages about what mathematics is and what it means to understand it. In particular, the materials were used in ways that conveyed a sense of mathematics as a fixed body of right answers rather than a field of inquiry. Classes were conducted in ways that discouraged rather than encouraged exploration of students' understanding and application of that understanding in problem-solving situations.

The resulting probability instruction was a collage of traditional and new or innovative approaches to instruction. However, despite apparent change, only one of the teachers was teaching in ways that captured the essence of the reform effort; that is, with an emphasis on engaging students' thinking and reasoning. The changes called for in mathematics instruction involve more than assimilating new content or new pedagogical methods, as the other teachers had done. The called-for changes involve a fundamental rethinking of teachers' conceptions about the nature of mathematics and a restructuring of teachers' understanding of mathematics teaching and learning. However, the other teachers observed in this study missed the main meaning of the reform as they interpreted that reform through the lenses of their more traditional knowledge, conceptions, and beliefs.

Despite calling on teachers to take a constructivist approach toward mathematics instruction, the designers of the reform have not applied the assumptions of constructivism in their own efforts to bring about change in mathematics instruction. First, the reformers did not consider the impact of teachers' prior conceptions and knowledge. In so doing, they failed to recognize that teachers' conceptions about mathematics and their beliefs about the teaching and learning of mathematics would serve as the lenses through which teachers would interpret the recommendations for change. Second, the reformers acted upon the assumption that changing practice could be accomplished by stating the new vision or enacting new policies. Fundamentally, the policymakers were assuming teachers could be taught a new way to teach by having the new vision presented to them or by enacting new state assessments. Ironically, the reformers were using the same instructional approach they were calling upon teachers to abandon; they viewed policymaking as telling, but wanted teachers to move away from teaching as telling.

Past researchers (Ball, 1988a, 1988b; Simon, 1994) have called for taking a constructivist approach to teacher education, but until recently practice has generally not reflected such a perspective. The findings of this study, however, provide further evidence that the constructivist foundations upon which the new standards are based (NCTM, 1989, 1991) apply equally to teachers in the role of learners. In particular, the teachers' implementation of the new standards was filtered through the teachers' pre-existing practice, knowledge, and beliefs. Their background or prior knowledge had a significant impact on the new understanding they were developing (just as students' prior knowledge impacts their construction of mathematical understanding). In addition, as a result of their experiences, the teachers were constructing their own understanding of mathematics and of mathematics learning and teaching and this understanding was impacted by the nature of the learning opportunities they had experienced, by the degree to which they had reflected on their practice, and by their opportunities to interact with others in a broader learning community.

Teachers are important agents of change in the reform effort currently underway in mathematics education, expected to play a key role in changing what occurs in mathematics classrooms. However, at the same time, teachers are also major obstacles to change because of their adherence to more conventional and outmoded forms of instruction. Teachers, therefore, are not only the agents of change; they are targets of change as well (Cohen & Ball, 1990; Prawat, 1992b; Putnam et al., 1992). To bring about the desired reform, the findings of this study suggest that teacher educators, reformers, and policymakers may need to take a constructivist approach toward teachers' efforts to learn about teaching mathematics in much the same way teachers are expected to take a constructivist approach toward their students' learning of mathematics. This perspective on teacher learning has important implications for teacher preparation and staff development.

Preservice Teacher Preparation

Research has shown that prospective teachers may enter their formal teacher education without a conceptual understanding of the mathematical content they will teach (Ball, 1988a, 1990b, 1990c, 1991; Baturo & Nason, 1996; Even & Tirosh, 1995; Khoury & Zazkis, 1994; Simon, 1993; Tirosh & Graeber, 1989; Zazkis, & Campbell, 1996) and with conceptions and beliefs about the nature of mathematics and the learning and teaching of mathematics that are incompatible with the view of mathematics instruction envisioned by the current reform (Ball, 1988a, 1988b; Holt-Reynolds, 1992; Thompson, 1992). The findings of this study suggest teachers may complete their formal teacher education programs with similar deficiencies in their knowledge base. Therefore, teacher preparation programs may need to do more (a) to challenge the beliefs and conceptions about mathematics teaching that prospective teachers have formed as a result of their "apprenticeship of observation" experiences in traditional classrooms, (b) to provide opportunities for prospective teachers to establish more appropriate conceptions about mathematics and mathematics pedagogy, and (c) to help prospective teachers develop a conceptual understanding of the content they will be expected to teach and the ability to apply knowledge in new situations.

In the past, rather than taking into account what prospective teachers already knew and believed, teachers educators have tended to view prospective teachers as simply lacking particular knowledge and teaching skills. Ball (1988b) suggests "the lack of attention to what prospective teachers bring with them to learning to teach mathematics may help to account for why teacher education is often such a weak intervention–why teachers, in spite of courses and workshops, are most likely to teach math much as they were taught" (p. 3). However, if teacher educators are to take a constructivist approach to preparing teachers, they must consider the beliefs, conceptions, and knowledge of prospective teachers.

Some of these conceptions and beliefs may serve as a useful foundation upon which the preservice teachers can build during their formal study of teaching. Other ideas may be deeply rooted misconceptions about mathematics and the learning and teaching of mathematics. In either case, teacher educators may need to find ways to help preservice teachers bring to the surface and critically examine the knowledge, conceptions, and beliefs they bring with them to teacher education programs. However, for prospective teachers who have been steeped in the instruction of traditional mathematics classrooms, this may not be enough. In order for these prospective teachers to develop classroom practices compatible with the NCTM's vision of good teaching, teacher education programs may need to find ways to challenge the deeply rooted ideas prospective teachers may have developed during their previous school experiences. If not challenged, these beliefs, conceptions, and knowledge will serve as the lenses through which prospective teachers view their preservice experiences and the foundation upon which they will build future practice.

But challenging the firmly embedded conceptions of prospective teachers is not an easy task to accomplish. Conceptual change research in science learning has focused on the process of accommodation, or the change that occurs when old cognitive structures or beliefs must be replaced or reorganized because they are unable to incorporate new knowledge. Posner et al. (1982) suggest that accommodation is not likely to occur until the learner is no longer satisfied with existing conceptions, has a preliminary understanding of a new conception, recognizes that a new conception is plausible, and sees the potential of the new conception to deal with future problems. Thus, teachers need not only the motivation to change their beliefs about teaching and learning mathematics; they need the opportunity to discover there is a better way than what they have previously experienced.

The knowledge and beliefs of one of the teachers in this study had changed as a result of her preservice experience. Her difficulties with mathematics at the middle and high school level may have provided the motivation to change, but the specific source of her motivation is unclear. Several features of the program influenced the transformation in her conceptions about teaching and learning mathematics. First, the prospective teachers took a number of mathematics classes where they experienced a new way of learning mathematics, a way focused on exploration and sense-making. Second, in the process of teaching these classes, the professors were modeling a new way of teaching mathematics, a way consistent with the mathematics instruction envisioned by the NCTM. This future teacher recognized the modeling that was being done and learned from it. It may also be worth noting that these were not individual or isolated courses, but rather a series of classes providing a generally consistent and coherent portrait of mathematics instruction. As a result of her experiences in these classes, this teacher embraced the constructivist perspective on learning mathematics and her instruction reflected this perspective.

The other teachers in this study had not been given a similar opportunity to experience a new way of learning mathematics as part of their preservice programs. Instead, their prior conceptions may have been reinforced rather than challenged by their preservice experiences with mathematics. Each of these teachers either had recognized weaknesses in their school experience or had encountered frustration and difficulty at some point in their prior mathematical experiences. While this realization may have provided some motivation to teach mathematics differently, the teachers were given no opportunity to see or experience any other way.

Providing opportunities for prospective teachers to experience new ways of learning and doing mathematics has been at the core of innovative mathematics courses designed to challenge the traditional beliefs of prospective teachers. In one model program, a sequence of three mathematics courses was designed to establish a learning environment in which preservice teachers could experience mathematics much as their own students might–in a classroom community where students and teacher together engaged in mathematical inquiry and where students were encouraged to make conjectures, to validate their assertions with convincing arguments, and to communicate with others in their attempts to solve problems and make sense of mathematical situations (Ball, 1990a; Schram, Wilcox, Lanier, & Lappan, 1988). In addition to challenging the prospective teachers' traditional beliefs, Wilcox et al. (1991) suggest that creating such a community of learners provides a safe environment in which the preservice teachers can take the mathematical, emotional, and intellectual risks involved in rethinking what they believe. These courses achieved some success in changing preservice teachers' conceptions about mathematics and their perceptions of how mathematics is learned. These changes were most evident in how the preservice teachers thought about themselves as learners of mathematics. Nevertheless, the prospective teachers continued to hold on to their traditional notions about teaching mathematics at the elementary level, as evidenced in the context of student teaching (Wilcox et al., 1991).

A second model program has been based on a framework of cycles of learning, in which the focus on content and pedagogy are separated. Simon (1994) argues that "learning mathematics in a context in which the overarching goal is learning to teach, may not cause much disequilibrium. It may be viewed as non-problematic and even appropriate that in the latter context, the process of learning and teaching is different from what one does to teach mathematics to school children" (p. 90). Therefore, although preservice teachers were asked to reflect on their own learning experiences in the initial mathematics classes, no explicit attention was paid to pedagogy. These new learning experiences, however, served as the basis for a focus on pedagogy in later classes. Evidence concerning the effectiveness of this approach, however, is limited to case studies of two participants in the program (Simon & Brobeck, 1993; Simon & Mazza, 1993). Although the prospective teachers were beginning to change their conceptions about mathematics and mathematics teaching and learning, they generally fell short in their efforts to implement reform-oriented mathematics instruction during their student teaching experience. Simon and his colleagues conclude prospective teachers may need longer, more comprehensive opportunities to learn mathematics in reform-oriented classrooms as well as opportunities for extensive teaching experience in similar classrooms where they can be supervised by educators who have an understanding of the complex issues involved in changing the nature of mathematics instruction.

These findings provide further evidence regarding the complexity of the process of changing prospective teachers' beliefs and conceptions and helping them to establish more appropriate forms of practice. At the very least, the process is one that may require additional time and support. Certainly, the process of challenging teachers' preconceptions may require more than a single mathematics or methods course. Similarly, establishing more appropriate forms of mathematics pedagogy may require support that extends beyond the prospective teachers' relatively brief preservice experience and into their beginning years of teaching.

In addition to challenging the beliefs and conceptions of prospective mathematics teachers and helping them establish more appropriate conceptions about mathematics and more appropriate forms of mathematics pedagogy, teacher preparation programs may need to do more to help prospective teachers develop a better understanding of probability and the teaching of probability. Although Brown and Borko (1992) argue that the acquisition of pedagogical content knowledge should be the primary focus of teacher education programs, the findings of this study suggest teachers first need an improved understanding of probability content. Without a stronger subject matter foundation, meaningful and useful pedagogical content knowledge may be difficult if not impossible to acquire.

According to the findings of this study, prospective teachers need to learn the probability content they will be expected to teach. In reaching this goal, prospective teachers may need learning opportunities that help them (a) develop a conceptual understanding of probability, (b) recognize the underlying assumptions of the content, and (c) understand the role of probability in its applications. Prospective teachers may also benefit from opportunities to see how the topics they will teach are interconnected and how they fit into the "bigger picture" of probability. In addition, because the prospective teachers may have some of the common misconceptions in their own thinking patterns, they may need opportunities to discover and address these incorrect preconceived and/or subjective notions about probability. Because experiences living in a world of uncertainty lead to sometimes faulty intuitive notions about chance occurrences, probability presents a unique challenge to prospective teachers, a challenge they may not encounter in learning algebra or geometry.

Prospective teachers also may need to learn more about teaching probability. In particular, they may benefit from opportunities (a) to learn what representations may communicate probability concepts effectively to students, (b) to learn how students' understanding of probability develops, (c) to learn what conceptions and misconceptions students may have, (d) to learn how to recognize and address these misconceptions, (e) to explore what tasks can be used to develop students' understanding of the concepts, and (f) to see how the probabilistic thinking of students can be developed in the context of probability problems. Research has recently begun to more extensively explore students' probabilistic reasoning and how such reasoning can be assessed and developed (Fischbein & Schnarch, 1997; Jones, Langrall, et al., 1997; Jones, Thornton, et al., 1997; Tarr & Jones, in press; Williams & Amir, 1995). Prospective teachers may potentially benefit from the insights provided by these research efforts.

The preservice opportunities for the teachers in this investigation to study probability had varied from an advanced probability class to perhaps none at all. One of the teachers had taken a formal probability course based upon calculus. Although this course provided the theoretical framework for the basic notions of probability, this teacher saw no relationship between the probability he had studied and what he was now teaching. Whether this failure to see the relationship was a result of the nature of the course, a lack of effort on the part of the prospective teacher, or a combination of these and other factors is unclear. For whatever reason, this class had not proven to be helpful in developing either the subject matter or pedagogical content knowledge of this future teacher.

A second teacher had taken a probability course as part of her mathematics emphasis that combined the study of content and pedagogy. In the process of learning probability content, this prospective teacher experienced probability instruction of the nature envisioned by the NCTM and was exposed to instructional materials she could use to teach probability. Because the class focused both on the content she would teach and on pedagogy for teaching the content, her experiences in the class provided background knowledge she could use and a model she could follow when implementing the same materials with her students. This experience thereby enriched the teacher's subject matter and pedagogical content knowledge, at least in terms of the activities they had done.

The remaining two teachers could recall no study of probability during their preservice teacher preparation (which had occurred several years earlier than the other teachers), although probability may have been included in courses such as Mathematics for Elementary Teachers. Because these teachers had no background knowledge of probability when they were first faced with teaching the subject, they had to seek out opportunities to learn the necessary content, either on their own or through workshops or classes.

The findings of this study suggest that preservice middle school mathematics teachers may need more opportunity to study the content of probability. In particular, they may need more than the 2 or 3 weeks typically spent on probability in the mathematics survey courses taken by many prospective middle school teachers. Higher level, formal study of probability may not necessarily be appropriate or inappropriate for prospective middle school teachers, but prospective teachers who do take such classes may need help in relating the content they are learning to that which they will teach. Rather than more advanced study of probability, the findings of this study provide some evidence for conjecturing that prospective teachers may receive the most benefit from a course focusing simultaneously on probability content and the teaching of probability, particularly if such a course is taught in the same manner in which they will be expected to teach. These opportunities may further be enhanced if the prospective teachers are asked to reflect on these learning experiences.

There are, however, at least two possible difficulties inherent in the vision of teacher preparation outlined in this section. First, there is the risk of challenging the prior conceptions of prospective teachers but falling short of helping them to establish new conceptions. As Ball (1990a) points out, "prospective teachers may come away even less confident than they were before, more worried that they will not be able to teach mathematics so that kids can understand. They may see classrooms and children as daunting, mathematics as a vast sea of things they really do not understand" (p. 15). Thus, although prospective teachers may no longer be satisfied with traditional forms of mathematics instruction, they may have no other recourse but to return to the familiar assumptions and patterns.

A second difficulty arises because it may be impossible and perhaps ill-advised to try as part of teachers' preservice training to teach them everything they might need to know to teach mathematics effectively. Probability is not the only new content area nor the only content area in which prospective teachers lack adequate knowledge to teach mathematics for understanding. Quite simply, it may not be feasible to spend enough time in each of these areas to develop a conceptual understanding of the content and to expand one's corresponding pedagogical content knowledge, including knowledge of students' possible conceptions and misconceptions. This may especially be true considering most middle school teachers come through elementary education programs, in which there is much besides mathematics to learn how to teach. This perhaps suggests the need to require at least a mathematics emphasis, if not special programs and certification, for prospective middle school mathematics teachers.

However, even if such extensive mathematics study were feasible in the preservice context, there is the danger of overwhelming preservice teachers with more information than they can assimilate prior to experiences in the classroom. This may particularly be true of pedagogical content knowledge. If preservice teachers have not made the transition from thinking primarily as students to at least beginning to think as teachers, they may learn what they think they need to learn in order to satisfy the professor without seeing the application of what they are learning in their future classroom. In particular, without the perspective of a teacher, they may not appreciate or value what they are learning. They may also lack the framework for understanding and remembering what they have learned.

Mathematics teachers themselves face a similar dilemma. They cannot possibly teach students all the mathematics they might need to know or to be able to apply in their future lives and occupations. Recognizing this, one of the goals of the *Curriculum Standards* (NCTM, 1989) is to develop students' *mathematical power*, or their "ability to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" (p. 15). Much like mathematics, teaching mathematics is a problem-solving process. Thus, rather than overwhelming prospective teachers with knowledge, they may benefit more from being empowered by their preservice programs with *pedagogical power*. This power may include ways of thinking and acting that are focused on student learning and that view teaching as a continual learning and problem-solving process.

One key aspect of this pedagogical power might involve what Feiman-Nemser and Buchmann (1986) call pedagogical thinking. Recognizing there is a difference between going through the motions of teaching (e.g., giving explanations and checking assignments) and connecting these activities to what students are learning, they describe pedagogical thinking as a focus on students' needs rather than on oneself as the teacher or on the subject matter alone. Prospective teachers may need assistance in shifting their attention from themselves to the students and in focusing their attention on student thinking and on student learning.

Another aspect of pedagogical power might be the recognition that the process of becoming a teacher is a lifelong learning process. Key elements of this lifelong process might include a desire to keep learning about mathematics and about the learning and teaching of mathematics; a disposition to seek ways to make sense of new experiences (mathematical or pedagogical); an intellectual curiosity and inquisitiveness about the surrounding world, including what occurs in classrooms; and a pattern of reflecting on teaching and learning experiences. With a commitment to lifelong learning and a focus on student thinking, even interactions with students potentially become learning opportunities for teachers. If prospective teachers do not already possess such a commitment to lifelong learning, they may need to be encouraged to cultivate such patterns of thinking.

Finally, prospective teachers might be empowered as they learn to take a problem-solving approach to the process of teaching. To do so, they may need assistance in developing the disposition and skills (a) to recognize and seek to understand the various pedagogical problems encountered in mathematics classrooms; (b) to seek, evaluate, and use mathematical and pedagogical information in making decisions and in solving the problems encountered; (c) to analyze, experiment with, and adapt the strategies and alternatives they apply; and (d) to reflect on and examine the effectiveness of their problem-solving efforts. The development of a personal confidence in their ability to solve the problems they may face in the classroom might also be an essential element of this growing sense of self as a teacher of mathematics.

Although pedagogical power would not necessarily replace the need for subject matter or pedagogical content knowledge, it might at least partially make up for and help teachers overcome areas of weakness. For example, teachers may not know what conceptions or misconceptions students might have prior to instruction. But if these teachers view the classroom as a learning environment (for themselves as well as the students) and have a focus on exploring student thinking and making sense of what they experience, they may be able to acquire the knowledge they lack. To do so, however, requires the desire and effort to continue learning.

A focus on student thinking and a desire to continue learning seemed to be important components of the success the one teacher was experiencing in her efforts to implement instruction of the nature envisioned by the NCTM. She did not necessarily have a great deal more mathematical knowledge than the other teachers in this study, particularly in the area of probability, but that did not hold her back. She forged ahead with an attitude that enabled her to learn and grow at the same time she was encouraging her students to learn and grow.

The reform, therefore, not only presents challenges to teachers; it also presents challenges to teacher educators. Developing the pedagogical power of prospective teachers may be a difficult task to accomplish, but cultivating a focus on student learning, a commitment to lifelong learning, and the disposition and confidence to view teaching as a problem-solving process may be a goal worth pursuing. Such pedagogical power, combined with a basic foundation of mathematical knowledge, may at least be a very appropriate starting point for the ongoing process of professional development and growth.

Staff Development

In a continuing effort to improve mathematics instruction, the teachers in this study had participated in a number of staff development opportunities, including workshops, classes, and mathematics conferences. Among other things, these experiences had provided opportunities for the teachers to hear and learn about the reform effort in mathematics education. However, none of these experiences appeared to have caused the teachers to seriously reflect on or reevaluate their basic conceptions about the process of learning or teaching mathematics. In some cases, the teachers felt tension between their traditional teacher-focused approaches and their efforts to increase student involvement. But for the most part, the teachers believed they were teaching in accordance with the reform. The inservice experiences had done little to help the teachers realize their implementation efforts involved a fragmented application of pieces of the reform.

When asked to reflect on their probability instruction, the teachers expressed some dissatisfaction with aspects of their units. In some instances, they were disappointed with what the students appeared to have learned. However, rather than being led to rethink their goals, instructional strategies, or assessment methods, the teachers attributed the difficulties they encountered to the immaturity of the students, a lack of sufficient class time, the many interruptions during the unit, or insufficient planning on their part. Again, the teachers were not inclined to take a serious look at the fundamental conceptions underlying their mathematics instruction. As a result, these teachers lacked the motivation to make the fundamental changes called for in the reform.

The classes, workshops, and mathematics conferences also provided opportunities for the teachers to learn more about mathematics, including probability, and about teaching mathematics. However, Guskey (1986) suggests that teachers "carry with them to staff development programs a very pragmatic orientation. What they hope to gain through staff development programs are specific, concrete, and practical ideas that directly relate to the day-to-day operation of their classrooms" (p. 6). Thus, rather than learning mathematical content, the focus of the teachers at the conferences may have been on finding activities to take back and use in their classrooms. Although the teachers received a number of worthwhile probability activities from these sources, they appeared to learn little more than a procedural knowledge related to the specific activities. In addition, because these workshops or conference sessions were brief and sporadic, the knowledge was obtained in a piecemeal fashion, disconnected from other activities or concepts. As a result, forming an overall integrated understanding of the concepts involved from such experiences had been a difficult if not impossible task for the teachers to accomplish.

The two most experienced teachers, who could not recall studying probability during their preservice preparation, had also taken a class or summer workshop that provided a more extensive investigation of probability content. What the teachers learned from these opportunities varied, no doubt influenced by the teachers' background and personal experiences. For one teacher, using manipulative materials and working through new curriculum materials from the students' perspective influenced him to subsequently use more hands-on activities in his mathematics instruction, but the experience did not significantly change his beliefs about teaching or learning mathematics. The probability knowledge gained in the workshop was limited to a procedural understanding of the particular learning activities they had done. For the second teacher, a recent class that explored the teaching of probability at the middle school level reportedly confirmed the notions she already had about teaching probability, even though these notions were somewhat inadequate. As with the conferences, the teachers' focus in these classes may have been on finding activities to use instead of on developing an understanding of probability.

Prior to taking these classes, these two teachers reported learning much of what they knew about probability from studying on their own. School textbooks, mathematics education journals and yearbooks, supplemental curricula, and resource books had been among the sources used by the teachers. The teachers, however, had developed some unconventional notions about probability, perhaps as a result of their independent study. For example, one teacher began structuring the content of probability around the various materials such as dice, coins, cards, and spinners. And both teachers drew their horizontal tree diagrams without a "main trunk," an idiosyncrasy they perhaps picked up from one of the resource books they used extensively. In addition, as the teachers studied probability on their own, they had not been able to put the pieces together to form a big picture.

Thus, for the most part, the teachers in this study were "Lone Rangers," generally learning about mathematics content and pedagogy on their own or through sporadic and isolated inservice opportunities. The deficiencies in their knowledge base and the difficulties they encountered may be typical of the potential dangers found when the learning process involves little or no interaction with others. In particular, Erlwanger (1973) demonstrated that students learning mathematics in isolation from interaction with others may construct mathematical understandings quite different from the accepted "truth" of the discipline. One might extrapolate that teachers could develop equally inaccurate notions from their independent study of probability.

Teachers also claim to learn subject matter from teaching it. Although Ball and McDiarmid (1990) acknowledge such learning may be fairly common, they point out that "neither teachers themselves nor those who study teaching appear to have written enough about such subject-matter epiphanies to help us understand the conditions that produce them" (p. 445). Given a certain amount of inquisitiveness and a desire to make sense of mathematics, it is conceivable that an unusual outcome, an unexplained result, or a student's question could stimulate further investigation and possibly new insights into the subject matter. In addition, learning more about students' thinking or their difficulties with probability could very possibly occur as teachers interact with students during probability instruction. But it may be less likely that teachers would uncover the concepts and principles underlying the procedures or the connections among them as a result of their teaching experience, although it conceivably could occur as teachers plan probability lessons.

It is unclear what, if anything, the teachers in this study had learned from their previous experiences teaching probability. Reviewing the content in the course of teaching their units reminded the teachers about aspects of the content they had not recalled in the pre-observation interviews. But, for the most part, the teachers did not appear to have been attentive to learning opportunities that may have been available. Although the teachers recognized probability is more involved and more complex than the portion of the content they knew, they seemed to be satisfied that they knew enough to teach probability to middle school students. When the impact of order helped explain the unexpected and incorrect outcomes obtained in different activities, two of the teachers seemed to dismiss further exploration of the topic, even for themselves, suggesting the involved knowledge would be too difficult for middle school students to grasp. The teachers also seemed to be satisfied with what was primarily procedural knowledge. For three of the teachers, at least, making sense of mathematics (conceptually) might not have been consistent with their view of mathematics as a fixed body of facts handed down by experts. In addition, any natural curiosity or inquisitiveness may have been dampened by the competing demands and pressures facing the teachers in the classroom.

The findings of this research study, therefore, suggest the needs of at least some inservice teachers may be surprisingly similar to those of preservice teachers. Despite participation in a number of inservice opportunities, the teachers in this study generally (a) lacked an integrated understanding of the nature of the reform, (b) held traditional views about mathematics and the learning and teaching of mathematics, (c) lacked an explicit and connected knowledge of probability, and (d) possessed an impoverished pedagogical content knowledge about teaching probability. The inservice opportunities in which the teachers participated generally had been unsuccessful in providing the teachers with the knowledge needed to bring about the changes in mathematics instruction envisioned by the NCTM.

To bring about the desired changes in mathematics instruction and to meet the needs of inservice mathematics teachers in the process, staff development efforts may need to take a more connected and comprehensive approach to the professional development of teachers. In particular, staff development efforts may need to do more (a) to guide teachers' development of personally meaningful forms of instructional practice compatible with constructivist learning theory, (b) to help teachers build a stronger foundation of knowledge about mathematics, including probability and the teaching of probability, (c) to provide assistance to teachers as they make fundamental changes in their mathematics instruction, and (d) to encourage collaboration among teachers in learning communities that foster and support ongoing knowledge growth. A number of inservice programs based on the constructivist perspective have been working toward these goals. These programs include the Educational Leaders in Mathematics Project conducted by the SummerMath for Teachers Program at Mount Holyoke College (Schifter & Fosnot, 1993; Simon & Schifter, 1991) and an inservice program developed as part of the Second Grade Classroom Teaching Project (Cobb, Wood, & Yackel, 1990; Cobb et al., 1991).

These programs, first of all, recognized that teachers need assistance in developing a vision of mathematics learning and teaching consistent with the reform effort. As a basis for deciding how to teach mathematics, the teachers were encouraged to reexamine their conceptions about the nature of mathematics and the process of learning mathematics. With respect to these conceptions, teachers currently in the classroom may have more to unlearn than prospective teachers preparing to teach. The preconceived notions gained in their own school experiences may have been further reinforced by their teaching experiences. Although teachers may not be aware of or have seriously considered the implications of these preconceived notions, these notions may have become firmly embedded within their patterns of instruction. Teachers may even have experienced a measure of success in their instructional efforts and be quite satisfied with their approach to teaching mathematics.

Because these firmly embedded conceptions are both the object to be changed and the lens through which new learning takes place, changing teachers' conceptions is a complex and difficult task. As with preservice teachers, current teachers may need motivation to change as well as the opportunity to see and experience a new way of learning mathematics. Somehow they may need to be brought to the point of seeing their current practice as problematic. This may possibly involve a dissatisfaction with instructional outcomes or the recognition of conflict between their beliefs and their actions.

For one teacher participating in a constructivist teaching experiment, the breakthrough came when she conducted interviews with two of her students and discovered they had not learned what she assumed they had from the textbook-based instruction. Only after that discovery did the teacher become motivated to modify her classroom practice and only then did she develop a genuine collaborative relationship with the researchers in an attempt to develop an alternative instructional practice (Cobb et al., 1990). In subsequent inservice efforts, video-recorded interviews of children completing place-value tasks were used during a summer institute as the starting point for discussion about a variety of issues related to classroom instruction. In the course of these discussions, the teachers were caused to reflect on, among other things, student learning in the course of traditional textbook instruction, the separation students typically make between school mathematics and pragmatic everyday mathematical problem solving, and the distinction between correct procedures and conceptual understanding.

Another approach, one forming the core of the summer institute for the Educational Leaders in Mathematics Project, provided opportunities for the teachers to experience the new teaching paradigm. As the teachers reflected on their experiences as learners in that setting, they were in the process constructing new conceptions of learning, teaching, and mathematics content. Subsequent activities focused on student learning and on planning lesson sequences that encouraged students' construction of knowledge. However, rather than telling the teachers specifically how they should teach, the teachers were encouraged to develop their own theories of learning as the basis for their curricular and instructional decisions. In this way, by providing the teachers with the knowledge and tools with which to make their own professional decisions about mathematics instruction, the program was seeking to develop the intellectual autonomy of the teachers.

The process of active self-reflection was a key component of the summer institutes for both inservice programs. In group discussions, the participants analyzed together their experiences as learners. In the process, the teachers gained insights about how knowledge develops and the circumstances that stimulate or inhibit knowledge growth. The teachers continued the process of reflection in daily journal entries. These experiences were intended to encourage self-reflection by the teachers in their own classrooms and to prompt the teachers to view their own classrooms as learning environments for themselves as well as for their students.

Thompson (1984) observes that the extent to which teachers' perceptions are revealed in their classroom practice seems to be directly related to the teachers' tendency to reflect on their actions, beliefs, subject matter, and students. Encouraging teachers to reflect on the effects of their instruction may, therefore, be one way to help teachers identify and resolve whatever conflict and tension there may be between the teachers' beliefs and their practice. To extend this idea one step further, encouraging or requiring action research projects focused on student learning (where the focus of the research is on finding the solution to specific local educational problems and where the results are limited to the setting in which the research was conducted) may be one approach for bringing teachers to the point of critically reflecting on their practice and ultimately bringing about change in mathematics instruction.

In discussing the role of staff development in bringing about teacher change, Guskey (1986) proposes that staff development efforts should initially focus on changes in teachers' classroom practices, which he suggests lead to changes in student learning outcomes. Guskey argues that "significant change in teachers' beliefs and attitudes is likely to take place only *after* changes in student learning outcomes are evidenced" (p. 7). However, the findings of this study suggest that teachers may add aspects of the reform to their current practice without making significant changes either in their instructional practice or in student learning outcomes. In addition, it is perhaps doubtful that teachers will be able to make the necessary changes in practice (e.g., asking questions to probe students' thinking, encouraging students to explain or justify their responses, or assisting students in determining the reasonableness of their answers for themselves) without a fundamental rethinking of the teachers' beliefs and attitudes about learning mathematics. Thus, because of the nature of the changes called for by the reform, the focus of these inservice programs on changing teachers' conceptions about mathematics and the teaching and learning of mathematics may be the appropriate and necessary starting point for teacher change, at least in this case.

Beyond the changes and growth needed in teachers' conceptions about mathematics and mathematics pedagogy, the inservice programs described earlier recognized teachers may also need opportunities to extend their understanding of the mathematical concepts they teach. In some cases, these opportunities were provided as an integral part of the summer institutes. For example, the first 3 days of the Educational Leaders in Mathematics summer institute was devoted to mathematics. By exploring more deeply what, on the surface, seems like familiar territory, the teachers had an opportunity to develop a broader sense of the conceptual issues their students confront (Schifter & Fosnot, 1993; Simon & Schifter, 1991).

In other cases, the idea for a mathematics class or for additional opportunities to learn mathematics content originated with the program participants themselves. As they began implementing instruction based on the constructivist model, the teachers became increasingly aware that their mathematical knowledge was too superficial to allow them to teach as they now wished to teach. Such classes or seminars, which were offered during the school year, also provided opportunities to consider aspects of pedagogical content knowledge, including cognitive models of students' thinking and analysis of student learning and misconceptions.

Understanding that teachers' experiences during the summer institutes were only preliminary steps in the learning process and recognizing that many obstacles face teachers as they try to implement new forms of mathematics instruction in their classrooms, both of the inservice programs were designed to provide ongoing support to teachers during the following school year. This may be one of the missing ingredients of the inservice experiences for the teachers in this study. Although they had participated in a number of inservice classes and workshops that involved studying mathematical content and/or experiencing new ways of teaching mathematics, none of these classes or workshops, appeared to have provided any follow-up support to the teachers or feedback about instruction.

The support offered by the two model inservice programs took different forms. For the teachers in the Educational Leaders in Mathematics Project, the most effective form of support proved to be weekly classroom follow-up by a staff member or resource teacher intern (Schifter & Fosnot, 1993). As teachers returned to their classrooms, they chose from what they found valuable in the summer institute as a starting place for implementation. With the teachers' goals in mind, staff members provided feedback, demonstration teaching, and opportunities for continued reflection. Periodic workshops or seminars during the year provided a second form of support. These workshops included collegial sharing about implementation efforts, hands-on lessons related to common concerns, and small group planning sessions. In addition, teachers were encouraged to meet in small groups at their schools to discuss problems, concerns, and insights. Thus, the follow-up support offered by both inservice programs addressed teachers' pragmatic concerns, helped teachers overcome pressures and resistance to change, and provided guidance as the teachers faced the unexpected issues that arose during their efforts to implement a new form of mathematics instruction.

The support needed by teachers, however, involves more than follow-up efforts of researchers or program staff. Administrators and colleagues may either be another important source of support or they can be obstacles to be overcome as teachers try to make fundamental changes in their mathematics instruction. Brown, Stein, and Forman (1996) propose that the supervisory chain (e.g., principal-teacher-student), with its typical emphasis on assessment, should be replaced by a chain of assistance. According to their model, multiple triads of assisting relationships are established. The primary role of the principal is to assist resource partners (e.g., mathematics educators) to assist teachers by helping to establish an environment that supports such assistance. For their part, resource partners are to assist teachers in their efforts to assist students and to assist teachers to assist each other. Likewise, in addition to directly assisting students, classroom teachers are to assist students to help other students. According to Tharp & Gallimore (1988), the characteristics of effective assistance are mutual respect and trust, intersubjectivity (a common means of communication and shared goals and values), responsiveness (assistance tailored to the needs of the learner), joint productive activity (working together to achieve a clearly defined goal), and reciprocity (both the assistor and the assisted benefit). Although implementing a chain of assistance may involve significant changes in its own right, such forms of assistance may be beneficial to teachers as they seek to implement the broader reform in mathematics education.

Many of the activities of the model inservice programs were built around experiences in learning communities, which provide another form of support. As the responsibility for making important pedagogical decisions shifts from experts and administrators to classroom teachers, collaboration among teachers may become even more essential. In order to make effective instructional decisions, teachers may need opportunities to reflect together on their instructional practice, to help one another plan appropriate lessons, and to explore together the mathematics they teach. Such collaboration may also serve as a force refining and perfecting knowledge as it develops (Wilcox et al., 1991).

After exploring the cases of two teachers who were reinventing their teaching practice in order to teach in more constructivist ways, Peterson and Knapp (1993) conclude that "one way practicing educators can construct a knowledge base for constructivist learning and teaching is through personally participating in diverse communities of researchers, teachers, and learners" (p. 155). They point out these diverse communities may include active participation in professional associations or on teams of teachers within a school. One might also be part of a community of educational scholars and researchers where teachers have access to the current thinking and understanding of the scholars and where the scholars and researchers learn from the experiences of teachers. Peterson and Knapp also emphasize that teachers need to view their own classrooms as learning communities, where they and their students are constantly learning from each other.

Schifter and Fosnot (1993) conclude that "inservice programs will have to recognize that encouraging the development of collaboration among their participants is as integral to their efforts as introducing new models of instruction. As such programs mature and past participants emerge as educational leaders, the latter become the most effective promoters of change, as much in school- and district-wide policy as in their colleagues' classrooms" (p. 18). More importantly, as such leaders become involved in the process of teacher development, the message of change may more effectively be spread and the efforts to accomplish change may be multiplied.

Both the Educational Leaders in Mathematics Project and the inservice program associated with the Second Grade Classroom Teaching Project reported a measure of success in enabling teachers to construct a form of instructional practice consistent with the recent reform movement in mathematics education. Many participants adopted new strategies for teaching mathematics and, perhaps more importantly, a significant number of teachers came to base their instructional decisions on a constructivist view of learning. However, the inservice programs were not universally successful; some teachers were unwilling and/or unable to make significant changes in their mathematics instruction.

In a sense, the results from these inservice programs can be viewed as an existence proof demonstrating that significant changes in mathematics instruction can be brought about through inservice education. However, although such teacher development results are possible, the researchers involved in the programs point out such efforts can be labor, cost, and time intensive. Nevertheless, if the mathematics education community is serious about fulfilling its vision, mathematics educators may need to realize that a commitment of time, labor, and resources might be necessary if teachers are to make the tremendous changes called for by the reform. A commitment to seek effective alternatives may also be necessary in order to bring about the significant and widespread changes required if the NCTM's vision of mathematics instruction is to be fulfilled.

Curriculum Development

Earlier reform efforts have focused on developing "teacher-proof" curricula as a solution to the widespread problems in mathematics education. These efforts have generally been unsuccessful because they failed to realize that, instead of being passive implementers of proposed mathematics curricula, teachers are actively involved in making pedagogical decisions about the content and how to present that content to students. Because these decisions and, thus, teachers' efforts to interpret and implement curricula are influenced by the teachers' knowledge and beliefs (Clark & Peterson, 1986), the "implemented" curriculum may turn out to be quite different from the "intended" curriculum. In other words, because teachers are different and have different backgrounds, their efforts to translate any particular curriculum may result in instruction that differs from the intent of the curriculum developers.

A "teacher-proof" curriculum is not only impossible to truly accomplish; it also is incompatible with the foundations of constructivism, which emphasize the importance of connecting learning to students' interests and prior knowledge. Because the suitability and effectiveness of selected learning activities depend in part on students' prior knowledge, expectations, and interests, and because these factors can be determined only by teachers, a generalized or decontextualized model for instruction cannot serve the needs of students in all situations and at all times. To teach in the ways envisioned by the reform, teachers need to have the freedom to select, adapt, and implement learning activities in ways that support individual students' constructive mathematical activity.

Rather than having too much information and guidance, as in efforts to "teacher-proof" the curriculum, the teachers in this study found themselves in situations where very little guidance was provided to them about the curriculum. For the most part, the teachers had moved away from using textbooks for mathematics instruction, particularly their probability instruction. In the process, the teachers lost the overall sense of coordination and articulation across grade levels that textbooks potentially provide. They also lost potential guidance about the structure and development of probability content. In addition, because the teachers had no general curriculum guidelines to follow for the study of probability, they were left with the responsibility of determining for themselves what content should be covered and how the content should be presented.

In place of textbooks, the teachers were implementing instruction based at least in part on activities, as recommended by the *Curriculum Standards* (NCTM, 1989). These activities generally had been obtained at mathematics conferences or workshops or from supplemental curriculum materials. These sources, however, provided limited background information about the probability content involved in the activities or how that content might fit into the "bigger picture." These sources also provided little if any information about what potential conceptions students might have or what their thinking might be.

Prior to the reform, curriculum developers and textbook designers made many of the important decisions about content, sequence, and teacher actions. But teachers are now responsible for many of these decisions, including allocating time to the various strands of mathematics, establishing the learning goals, selecting and adapting instructional tasks, orchestrating classroom discourse, and assessing student learning. Thus, the reform shifts the responsibility for making key curricular decisions from curriculum developers to classroom teachers. The findings of this study suggest, however, that teachers may lack the knowledge upon which to base such decisions. In particular, the teachers in this study did not have a clear-cut sense of where probability instruction should lead. And although the teachers were using appropriate mathematical activities, they lacked the knowledge necessary to implement the activities in ways that promoted maximum student learning. In addition, the instructional materials themselves provided little information to assist the teachers in implementing the activities effectively in their classrooms.

Although writing "teacher-proof" curricula may be an impossible and undesirable goal, curriculum developers may, nevertheless, be able to contribute to the ongoing process of educating teachers and improving mathematics instruction. In particular, curriculum developers may be able to provide helpful overall guidance to teachers as they plan probability instruction. At one level, in looking across classrooms, coordination and articulation of the curriculum is needed to help teachers answer the following questions: What concepts are appropriate to investigate at each grade level? What should students understand about the concepts at each grade level? What skills should they master? What are the "big ideas" they should learn? What problem-solving opportunities do they need? At the classroom level, teachers may benefit from guidance that helps them determine what goals to set, what activities to use to help students meet those goals, and what factors to consider in choosing learning activities. Teachers may also benefit from information about what difficulties may be encountered as students investigate the mathematical content.

Encouraged by the NCTM *Curriculum Standards* (1989), teachers are looking for activities to incorporate into their mathematics instruction. Curriculum developers, therefore, may be able to help improve probability instruction by providing appropriate instructional activities from which knowledge and skills can be developed. Instructional efforts could potentially benefit from the availability of activities that embody the important mathematical concepts in ways that are relevant to the students and engage them in meaningful exploration of the mathematical content.

However, curriculum developers should not assume teachers will see and understand the content and the conceptual implications involved in the activities. Teachers may need assistance in implementing those activities and connecting them to students' learning of mathematical content. Teachers' efforts to implement these activities may be assisted if important information is provided along with those activities. Among other things, this information might emphasize what concepts are involved in the activity as well as how those concepts fit into the "big picture" of probability in general. This supplemental information might also provide background about students' possible conceptions and related misconceptions, suggestions for probing and developing student thinking, or related questions and avenues for further exploration. In addition to fitting the activity into the overall picture of probability, this information might also highlight the connections to other mathematical topics and concepts.

Rather than a linear and well-defined course to be run, Prawat (1992b) proposes that viewing the curriculum as a network of important ideas to be explored is more consistent with the constructivist views of teaching and learning. In this context, curriculum materials that provide the overall and specific information and guidance described may be beneficial to teachers in that they provide teachers with a sense of direction and the knowledge necessary to make the decisions about how they will explore the conceptual terrain with the students. Teachers would thereby have an overall map of the region as well as information about alternatives for getting from one place to another and additional information about special features of the terrain. Thus, rather than telling teachers what the destination is and how they are to get there, as in curriculum projects of the past, curriculum developers could be empowering teachers to make more informed decisions about instruction for themselves.

Although providing such information may potentially be beneficial, there is a chance that teachers may ignore or perhaps misuse the information provided with the activities. However, the greater risk might be that teachers may avoid such curriculum materials altogether because the amount of supplemental information may be intimidating to them. For the most part, the teachers in this study implemented activities they had experienced at conferences or workshops, not activities they had only read about. Therefore, a more effective approach to the implementation of appropriate curriculum materials may be for curriculum developers or curriculum specialists to work in collaboration with teachers. As teachers have an opportunity to experience the activities and become familiar with the materials, these curriculum specialists can help teachers understand important aspects of the content. As teachers adapt the materials to meet the needs of their classrooms, curriculum specialists can provide assistance and ongoing support. In the process, teachers may learn as they work with curriculum specialists to map out the terrain and curriculum specialists and, ultimately, curriculum developers may learn from the efforts of teachers and students as they explore that terrain. Collaboration may therefore be one of the keys to bringing about significant changes in the mathematics curriculum.

The current reform effort in mathematics education, however, involves more than a new curriculum. The reform involves fundamental changes in content, instructional goals, modes of instruction, and methods of assessing student progress. The reform also involves changes in teacher education and in how all participants in the current system of schooling (students, teachers, administrators, parents, policymakers, teacher educators, and researchers) understand their roles and responsibilities. In past reform efforts, researchers and policymakers have "constructed" knowledge in the form of curriculum or policies and transmitted that knowledge to administrators and teachers who were supposed to "implement" the curriculum or policies in their schools and classrooms. To remain true to the constructivist model, change in mathematics instruction will come about as all participants work together in the ongoing construction of the knowledge base for learning and teaching mathematics and in a collaborative effort to make the reform vision a reality or, perhaps, to revamp the reform agenda.

Limitations of the Study

Various aspects of this study limit the generalizability of the findings to this sample of middle school teachers. Some limitations were inherent in the research design. Other limitations arose during the execution of the study. Finally, unique characteristics of the sample introduced further limitations to the generalizability of the results.

The discussion about common misconceptions of probability has emphasized that one misconception to be avoided is the neglect of sample size. Repeatedly it has been stated that larger samples provide a better representation of the characteristics of a population. Clearly, four teachers is not a large enough sample from which to make inferences about the general population of middle school teachers. However, the case study approach allows in-depth exploration of the variables involved. In so doing, the study of specific and varied cases can enrich one's conceptualization of the general case and serve as a starting point for further investigation. Additional research with a larger sample of middle school teachers is needed to explore the conjectures generated by this study.

The effectiveness of instruction cannot be assessed without student learning being measured. However, student learning was not evaluated as part of this study. Although certain inferences may be suggested from students' questions and comments, it is not known specifically what the students learned from the different probability units. Nor is it known if the instruction in one classroom was any more effective than in others.

This study also was not designed to measure outcomes from the course work or inservice training opportunities in which the teachers had participated. Therefore, it is not known specifically how the different professional opportunities impacted the teachers' knowledge or probability instruction, if at all.

The fact that the researcher would be aware of the teachers' knowledge (as revealed in the pre-observation interviews) during the classroom observations was identified in the research proposal as an unfortunate but unavoidable bias. In the execution of the study, this proved to be less problematic than anticipated. Because the initial analysis of probability knowledge revealed few distinguishable differences in the teachers' knowledge, the researcher began the classroom observations with few if any preconceptions about what differences to expect during probability instruction. In addition, the more extensive analysis of knowledge did not occur until after the observations had been conducted. Nevertheless, the researcher made every effort during the observations to view the teachers as objectively as possible without regard to prior knowledge about the teachers' knowledge of probability.

Additionally, although a variety of data collection techniques were used in this study, the researcher was the primary instrument for data collection and analysis. While the researcher's background and past teaching experiences were helpful in evaluating probability knowledge and interpreting the events in the classrooms, the researcher's background and experiences also had the potential of biasing the data collection and analysis process. The researcher, however, took steps to protect against researcher bias by keeping a journal and reflecting on the thoughts, insights, and decisions made during the process of data collection and analysis. Triangulation provided another protection as multiple sources of data were used to confirm the emerging patterns, themes, and conclusions.

Other limitations arose during the execution of the study, including technical difficulties that occurred during the data collection process. Audio recordings of some observations were not available because the audio equipment failed to work or because the researcher was not present to make the recording. For the few days on which this occurred, the discourse was limited to what was audible on the videotape, which often was only the teacher. Even when audio recordings were made, these sometimes did not pick up students in whole-class discussions or during small-group interactions when the students spoke softly. As a result, some of the discourse record was incomplete. However, the observations for which the discourse record was complete provided sufficient data to reveal patterns. In addition, observational field notes supplemented the recorded discourse and helped to confirm the patterns as well.

One teacher had taught some probability activities earlier in the year. Although these had not been observed, the teacher was still selected to participate in the study because it was thought that her remaining probability instruction would provide adequate data from which to determine patterns and explore variables, which proved to be so.

Participation in the study appeared to influence the probability instruction of at least one of the teachers involved. Based on the teacher's informal comments, this teacher had spent more time in preparation, taught a longer unit, and incorporated more activities than he had previously done or might otherwise have done. The other teachers may have been affected similarly, although no such impact was perceived by the researcher.

Besides the size of the sample, other characteristics of the sample introduced limitations to the generalizability of the results. Specifically, the sample included only teachers who were teaching probability at the end of the school year. These teachers had made a conscious choice to teach probability during the last month or two of the school year; it had not just been pushed off until then. However, other teachers choose to teach probability at the beginning of the year, justifying their decision with some of the same reasons the teachers in this study gave for teaching it later. In particular, these teachers argue that beginning the year with probability activities provides good motivation to the students for studying mathematics as well as a context for introducing and/or reviewing fractions, decimals, and percents.

One teacher was selected as part of the sample because of his secondary mathematics education preparation. However, he may or may not be representative of teachers with such training. Although he received A's throughout high school mathematics, his academic record in college suggests he was not a strong mathematics student. This teacher also had the least amount of teaching experience and was the one having the most difficulties presenting effective mathematics instruction. However, it is not known whether his difficulties in the classroom were related to his secondary preparation, his poor academic record, his limited teaching experience, his attitudes and beliefs about students and teaching, or to other factors.

On the other hand, the teacher having the most success in implementing instruction as envisioned by the *Teaching Standards* (NCTM, 1991) was teaching a class composed of students identified as gifted. Although the teacher claimed this did not change her instructional approach, one might wonder how much of the success she was having in stimulating students' thinking was related to the gifted nature of the students. This question certainly needs further study.

Finally, the teachers involved in this study had originally been chosen because of their apparent strength, either in terms of background or experience. All of the teachers observed had either a secondary mathematics teaching license or an elementary teaching license with an added mathematics endorsement. For the most part, they were considered to be leaders in their schools and/or districts. In addition, because the study involved only teachers who were willing to participate and to allow observation of their probability instruction, the sample was thus limited to teachers with a certain degree of confidence in their teaching. This factor may be important, especially considering probability is relatively new content and viewed as difficult by many. Although these factors may have led to a sample that was potentially above average in terms of teaching ability, their probability instruction did not prove to be exemplary. Further research with a larger sample of teachers who may be more representative of middle school teachers will potentially strengthen the generalizability of this study's results.

Recommendations for Future Research

Although this study has explored the research questions in considerable depth and detail, there are questions left unanswered and additional questions raised. Further research is needed to address these questions. The limitations of this study also suggest further needs for additional research. Finally, the implications of the study's findings propose other potentially fruitful avenues for further investigation.

First, the size and nature of the sample restricted the generalizability of the results. Further research is therefore needed to explore the conjectures generated by this study and to confirm and/or extend its findings. Which portrait of probability instruction observed in this study is most characteristic of middle school classrooms in general? How characteristic and widespread is the teachers' lack of probability knowledge? Or their lack of pedagogical content knowledge concerning the teaching of probability? How widespread are the teachers' traditional conceptions about mathematics and mathematics pedagogy? Do middle school teachers teach more familiar aspects of the mathematics curriculum in similar traditional ways? Or are teachers willing and able to teach in more reform-oriented ways in areas where they have a stronger foundation of knowledge or when their experience has not colored their vision of instruction? Is the discrepancy between goals and outcomes also evident in teaching other mathematics content?

Second, to complete the portrait of probability instruction, future research needs to explore the outcomes of probability instruction in middle school classrooms. What are students learning about the content of probability? What are students learning about the nature of mathematics as a result of probability instruction? What analysis are students able to do on their own? Are they able to analyze only familiar problems (like the ones they see during instruction) or are they able to apply what they learned to analyze unfamiliar or novel problems? What conceptions and misconceptions do students possess before instruction? How are these conceptions and misconceptions impacted by probability instruction? What misconceptions, if any, are created in the students' understanding as a result of the instruction they receive? What misconceptions continue in their thinking after instruction? Does more learning take place and/or different learning take place in classrooms where the instructional approach is compatible with NCTM Standards?

Third, several factors may distinguish the teachers in this study or their classroom settings from other middle school teachers or classrooms. Three of these teachers were teaching probability at the end of the year; the fourth teacher taught probability as a strand during the year. One teacher was teaching a class composed of students who had been identified as gifted. Further research needs to be conducted to explore the influence of these factors. Do teachers who teach probability at the beginning of the year (or during the year) have a different approach to teaching probability from those who teach it at the end of the year? Is the difference in when probability is taught only a matter of the teachers' perspective or are there other differences, perhaps differences in teachers' knowledge? Do differences in student outcomes occur when probability is taught as a strand rather than as a cohesive instructional unit? Was the one teacher more successful in focusing on higher level thinking because of the gifted nature of her students? Is the instruction demonstrated in that classroom feasible with students who are not identified as gifted? If so, how can the same goals in thinking be accomplished?

None of the teachers in this study used technology as part of their probability units, although one can envision uses of computers to simulate random occurrences, for instance. Are other teachers making use of technology in probability instruction? Are special instructional materials available involving technology and probability? What is the nature of these materials? What level of thinking is involved? How effectively are the materials implemented by the teachers? What are the learning outcomes for instruction involving technology?

Another potentially fruitful avenue of research may be an investigation of the impact of teachers' various preservice experiences. At the middle school level, mathematics teachers may have either an elementary or a secondary education background. The teachers in this study had entered the middle school mathematics classroom through a variety of routes. One teacher had added a mathematics endorsement to his elementary preparation;

another had a earned a double major in elementary education and mathematics. A third teacher had completed a special program designed to prepare middle school mathematics teachers where she earned a elementary teaching license with a mathematics emphasis; the fourth teacher had completed a secondary mathematics education program. Further research needs to explore the characteristics of these different forms of preservice preparation and to assess their impact on teachers' knowledge and practice. How much mathematics is required and/or taken in such programs? What is the nature of mathematics instruction in these classes? Are prospective teachers given the opportunity to experience mathematics instruction modeled after the NCTM's vision? What do prospective teachers learn about the content and nature of mathematics in these programs?

One teacher had taken a number of advanced mathematics courses, including one in probability, as part of his major in mathematics education. However, despite his more extensive mathematics background, he was having difficulty presenting effective probability instruction. It is not known whether his lack of success in teaching probability was because the upper division probability course he had taken did not prepare him to teach probability or because he was a poor student and a poor teacher who did not see the pedagogical implications of what he had studied. Further research needs to address these questions. What knowledge of probability do teachers have who have completed a secondary education program? How do they teach probability? Do middle school teachers with secondary education preparation and generally more mathematics background feel a sense of superiority over the majority of their colleagues who have an elementary education background? If so, does this inhibit their ongoing desire to keep learning about mathematics and growing in their ability to teach mathematics?

As a result of taking a probability class that focused both on content and pedagogy, one teacher acquired background knowledge which supported her efforts to implement the same instructional materials in her own classroom. The other teachers had also attended workshops or classes where they became familiar with innovative instructional materials. More needs to be known about the impact of such experiences. Is the knowledge gained from such experiences limited to the specific instructional activities? To what extent does this knowledge transfer to other activities? Do the teachers learn primarily procedural patterns for implementing the activities or do they simultaneously acquire a conceptual knowledge of the content?

Another productive avenue of further research may be an exploration of the impact of teachers' various professional experiences. What types of professional opportunities impact a teacher's probability instruction? What opportunities are effective in helping teachers learn the content of probability? Or assisting them in teaching probability in ways compatible with the NCTM's vision of mathematics instruction?

This study has revealed areas of teachers' knowledge that may influence teachers' ability to implement reform-oriented instruction. Further research needs to explore how such knowledge is acquired. What is the extent and nature of instructional interventions that provide teachers or prospective teachers with a strong conceptual background for teaching probability in the middle school? What preservice or inservice experiences lead teachers to develop appropriate conceptions about the nature of mathematics or a useful personal theory of how students learn mathematics? How might teachers be assisted in identifying the "big ideas" in the probability curriculum? When and how might teachers learn about students' conceptions and misconceptions about probability? What experiences lead teachers to develop the commitment and ability to pursue students' understanding and thinking? How and when might teachers develop the knowledge and skills to orchestrate meaningful classroom discourse? In what ways can the nature of the reform effectively be communicated to teachers?

Finally, in considering the implication of this study's findings for teacher education, various goals were proposed. Research needs to explore how these goals can be realized. What motivates prospective or practicing teachers to reconsider their conceptions about learning and teaching mathematics? How can prospective teachers be assisted to cross the bridge from new conceptions of learning (for themselves) to new conceptions of teaching (for their students)? How can prospective or current teachers be empowered with *pedagogical power*? What experiences help teachers develop a better understanding of probability and of teaching probability? What forms of follow-up support are beneficial to teachers? What is the extent and nature of this support? What forms of collaboration are fruitful for teachers as they attempt to bring about changes in mathematics instruction? How can such collaboration be encouraged among teachers?

Although questions remain to be answered, it is hoped the results of this study, together with the findings of future research, will contribute to the expanding knowledge base of the field, inform teacher education, guide middle school teachers, and, ultimately, improve the teaching of probability in the middle school.

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APPENDICES

APPENDIX A

Pre-Observation Interview Questions

Background Questions:

1. How many years have you been teaching?

2. At what grade are you currently teaching? For how long?

3. The teachers were then asked to give their mathematical autobiography, describing their elementary and secondary school experience as well as their experience in college. As needed, the following prompts were used:

   Describe your experience in elementary (middle, high) school mathematics classes.

   What were your feelings about mathematics during those years?

   Where did you attend college?

   What mathematics courses did you take in college?

   Did you have a specialization or special emphasis in mathematics?

   Do you have an elementary or secondary teaching license?

   Do you have a mathematics endorsement? If so, was it obtained by course work or by testing?

4. Do you recall studying probability in your elementary or secondary school experience? If so, what are your memories of that experience?

   What opportunity to study probability did you have at the college level? For example, was it in a separate course or as part of another mathematics course?

   Have you had any other opportunities to gain knowledge of probability?

5. When do you anticipate teaching your lessons/unit on probability?

   What are your goals for teaching probability?

   What general teaching style will you use and what types of things will you do?

Probability Questions:

1. (One Die problem) One fair die is thrown.

   (a) What is the probability that the die lands
   (i) with a 5 on top?
   (ii) with a 2 or 3 on top?

   (b) What are the odds in favor of landing with a 2 or 3 on top?

2. (Two Coins problem) Jose and Cathy have conducted an experiment tossing two coins. They have recorded the following results:

both heads	5 times
one head, one tail	8 times
both tails	7 times

   (a) According to their results, what is the experimental probability that
   (i) both coins land tails up?
   (ii) one coin lands tails, the other lands heads?

(b) What is the theoretical probability of the events in part (a)?

3. (Marbles problem) A bag contains some red, some white, and some blue marbles. The following probabilities are given:
   - The probability of drawing a red marble is 1/3.
   - The probability of drawing a white marble is 1/2.

(a) What is the probability of *not* drawing a red marble?

(b) What is the probability of drawing a green marble?

(c) What is the probability of drawing a blue marble?

4. (Birth problem) The ratio of boys to girls born is generally about 50:50. A certain family is expecting the birth of their fifth child. The first four children were girls.

(a) What is the probability that the fifth child will be a boy?
(i) less than 50% (ii) about 50% (iii) more than 50%

(b) For families with five children, what is the probability of having four girls and one boy (in any order)?
[After a solution is given] Can you determine the solution in any other ways?

5. (Two Spinners problem) Three students are spinning to get one red and one blue on the given spinners (see Figure A.1).

Mary chooses to spin twice on Spinner A;
John chooses to spin twice on Spinner B; and
Susan chooses to spin first on Spinner A and then on Spinner B.

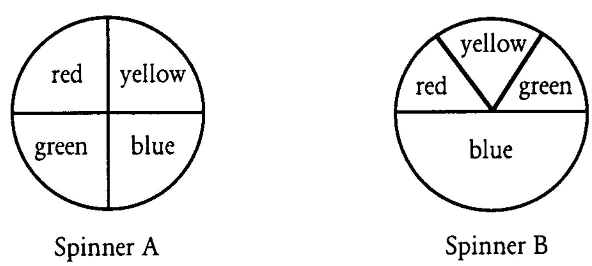


Figure A.1. Spinners A and B for the Two Spinners problem.

(a) Who has the best chance of getting one red and one blue (in any order)?

(b) If you first could spin your choice of the spinners and observe the outcome and THEN decide which spinner to spin second, can you devise a strategy with a greater probability than either Mary, John, or Susan of obtaining one red and one blue?

6. (Newspaper Pay problem) Carey is a carrier for the biweekly small town newspaper. She needs to collect $4 per month from her customers. Instead of paying the $4 each month, one customer makes her the following offer:

As her payment, Carey will draw one bill each month from a bag that will always have the same contents:
three $1 bills,
two $5 bills, and
one $10 bill.

(a) I have here a bag of materials commonly used in probability simulations. How might you simulate this situation? [giving the teacher a bag containing dice, a deck of cards, blank slips of paper, coins, and colored plastic chips]

(b) Perform the simulation for a 12-month period. From your results, would you recommend that Carey accept or reject the offer?

(c) If you were to evaluate this offer theoretically, what would you recommend to Carey?

(d) Compare your answers in parts (b) and (c).

7. (Two Urns problem) Imagine that you are presented with two covered urns. Both of them contain a mixture of red and green beads. The number of beads is different in the two urns: the small one contains 10 beads and the large one contains 100 beads. However, the percentage of red and green beads is the same in both urns. Imagine that you conduct two experiments:

Experiment 1: Without looking, you draw one bead from the *smaller* urn, note its color, and return the bead to the urn. This procedure is repeated until nine (9) beads have been drawn and their colors noted.

Experiment 2: Without looking, you draw one bead from the *larger* urn, note its color, and return the bead to the urn. This procedure is repeated until 15 beads have been drawn and their colors noted.

(a) In which case do you think your chance for guessing the majority color is better? Explain.

(b) If your draws resulted in 3 red and 6 green from the small urn and 9 red and 6 green from the large urn, estimate the percentage of red and green beads contained in the urns. Explain.

8. (a) (Applications problem) How does probability impact your life?

(b) What examples can you give of how it impacts the lives of your middle school students?

9. (a) (Weather problem) What does it mean when a weather forecaster says that tomorrow there is a 70% chance of rain? What does the number, in this case the 70%, tell you? How do forecasters arrive at a specific number?

(b) Suppose a forecaster said that there was a 70% chance of rain tomorrow and, in fact, it did not rain. What would you conclude about the forecaster's statement that there was a 70% chance of rain?

(c) Suppose you want to find out how good a particular forecaster's predicting is. You observe what happens in 10 days for which a 70% chance of rain was predicted. On 4 of those 10 days there was no rain. What would you conclude about the accuracy of this forecaster?

(d) If he or she had been perfectly accurate, what would have happened?

10. (Cancer problem) In a particular population, the frequency of cancer is known to be 1 out of 100. The test screening for the cancer has an overall accuracy rate of 87%. In other words, for patients with cancer it correctly diagnoses the cancer 87% of the time; for patients without cancer it correctly diagnoses them as free of cancer 87% of the time.

A patient has tested positive for cancer.

(a) Estimate the probability that the patient has cancer.

(b) Calculate the probability that the patient has cancer.

APPENDIX B

Post-Observation Interview Questions

The purpose of this second interview is to gather further information about your thoughts concerning the teaching of probability. Some questions are more general; others are a follow-up to the unit you have taught. The goal is understanding the decisions involved in teaching probability.

<u>Pedagogical Questions</u>:

1. What are your purposes for mathematics instruction, in general?

2. As you look back on the year as a whole, could you describe a typical day of mathematics instruction in your classroom (if there is such a thing as a typical day)?

   How does the period begin?

   Is a textbook used?

   What does the teacher do?

   What do the students do?

3. If a parent or administrator were to ask you, what is your justification for teaching a unit on probability at this level?

4. What are your goals in teaching probability to students at this level?

   How effective do you think the unit was in meeting the goals you had for the unit?

5. What were your overall objectives for the unit? (What did you expect the students to be able to do?)

   How did you assess these on a day-by-day basis?

   Which of these objectives were covered on the final assessment (the test or tasks)?

6. What do you think students learned from this unit?

   What are the "big ideas" or the important ideas you want students to remember in the future about probability?

   What probability do the students get in later grades?

7. What aspects of probability present difficulties to students?

   What aspects come easily?

8. Does probability instruction present unique problems from a management perspective because of the focus on student activities?

   If so, how do you deal with these problems?

9. What factors did you consider as you planned the unit (or probability instruction)?

   How did your goals influence the unit?

In what way did your knowledge of the students influence your planning?

10. [Mr. Trackman, Mrs. Talent, & Mr. English] Why did you choose to teach this unit at the end of the year?

    There seemed to be a large number of changes in the schedule (late starts, special activities, etc.). Are these typical of the school year or more typical of the end of the year?

    How did these various changes impact the unit?

11. [Mrs. Books] Rather than teaching a separate unit on probability, you incorporated probability activities throughout the year. What were your reasons for presenting probability in this way?

    How effective do you feel this approach was in meeting your goals for teaching probability?

12. [Mrs. Talent] In the initial interview, you talked about a pattern that you would use in the unit: predict, gather data, model or analyze mathematically, compare. You seemed to use this pattern for the first couple of activities, but less so as the unit went on. Were there reasons for this change in approach?

13. [If a textbook was used at any time during the year] Does the students' textbook present probability?

    If so, describe its presentation of probability.

    [If the textbook was not used for probability instruction] Why did you decide not to use the textbook as part of the unit?

14. Next, let's look at the activities you used in the unit. In each case, why did you choose the particular activity and what was your objective or purpose in using it?

    [This portion of the interview reviewed each of the activities the teacher had included.]

    Were there other tasks/activities that you chose not to use? Why?

15. You did [not] use activities from a variety of sources. Do you have other resources available?

    Have you seen any of the following [if not listed by the teacher]?
    Probability [Middle Grades Mathematics Project]
    Visual Encounters With Chance [Math and the Mind's Eye]
    Exploring Probability [Quantitative Literacy Series]

16. [Mrs. Talent] The coin toss and carnival tasks both came from previous statewide assessments, is that correct?

    You told me when you were presenting the tasks that the students react differently to tasks like these that ask them to think and apply what they've been learning. Can you describe their reactions?

    And you suggested that you responded differently to their questions as they were working on the tasks. In what ways did you respond differently?

17. [Mrs. Talent] In doing Monty's Dilemma, you used a different way of simulating the situations: three cups with a coin hidden under one cup. One student mixed up cups; the second student acted out the choice.

    What were your reasons for simulating the situation in this way?

    Would there be possible biases in the results?

    Would you expect the choices to be random?

18. [Mr. English] You suggested that after I stopped observing you were intending to go on and present the area model and Pascal's triangle. Did you do that?

    If so, can you describe what you did on those days [the activities from the Middle Grades Mathematics Project?]?

    Then, how did you conclude the unit in preparation for the test?

19. The NCTM *Curriculum Standards* emphasize the themes of mathematics as problem solving, mathematics as communication, mathematical reasoning, and mathematical connections.

    In what ways did you incorporate problem solving in the unit?

    What opportunities did the students have to communicate about mathematics?

    In what ways did you encourage students to develop their reasoning abilities?

    What connections were incorporated into the unit?

20. The NCTM *Teaching Standards* envision a classroom environment where "students learn to make conjectures, experiment with alternative approaches to solving problems, and construct and respond to others' mathematical arguments" (p. 56). This vision suggests a shift away from an emphasis on mechanistic answer-finding and away from the teacher as the sole authority for right answers.

    Do you feel the kind of teaching envisioned in the *Standards* is feasible in the middle school classroom? Why or why not?

21. [Mrs. Books] It was clear that one of your goals was to orchestrate a class environment as envisioned by the *Teaching Standards.* How did you go about preparing the students so that they felt free to make conjectures, challenge the conjectures of others, and justify their own conjectures?

    What is involved, from the teacher's perspective, in orchestrating such a classroom environment?

    Would you present these activities in a similar fashion in a regular classroom (as opposed to a classroom with gifted students?

22. An important part of your unit appeared to be an opportunity for the students to do different experiments, as recommended by the NCTM *Curriculum Standards.*

    What do you think students can learn from doing experiments?

    What important ideas about probability could be developed in the experimental setting?

[Examples if not suggested by teachers] Do you expect experimental results to vary? Which do you expect to more accurately reflect the theoretical results, 10 trials or 100 trials?

23. In finding experimental probability, you repeat the experiment for a certain number of trials, and then find the ratio: number of times event occurred/total number of trials.

    What is a general strategy for finding theoretical probability?

    What is the sample space and how is it related to theoretical probability?

    What approaches are there for finding the sample space?
    [Possible answers: organized list, outcome tree, Pascal's triangle, area model]
    [If any not mentioned] Are you familiar with . . . ?

24. [Mr. Trackman] In analyzing the dice sums game that you played with the students you developed the following table (see Figure B.1):

dice sum	2	3	4	5	6	7	8	9	10	11	12
# of ways	1	2	3	4	5	6	5	4	3	2	1

Figure B.1. Summary of number of outcomes leading to different dice sums

    Why didn't you use this same table in analyzing the game from "Is This Game Fair?"

    Are there differences in the games that make order matter in one case and not in the other?

25. How would you describe the student body of your school?

    What is the size of the student body?

    What are the ethnic and economic backgrounds of the students?

    How are students grouped into classes?

26. In general, what is the make-up of the class I was observing (grade level, ability level)?

27. [Mrs. Talent & Mr. English] You were not only teaching probability to the class that I observed, but to your other classes as well, is this correct?

    Did you make any changes in what you presented or how you presented it?

    If so, what factors influenced these changes?

28. What notions do you think the students had about probability before they began the unit?

    What is their understanding, for example, or chance, randomness, and fairness?

    How have these been formed?

    From your knowledge, had the students received any instruction in probability before your class?

29. Research has explored some of the conceptions students and adults have about probability. I am going to show you six questions used in those research studies. For each problem, you are asked to give two responses.

    How would you answer the question yourself?

    How do you think the students [in the research studies] answered the question?

    [The teachers were shown the six misconception questions (see Misconception Questions listed after these Pedagogical Questions). After the teachers had responded to all six questions, the researcher explained what responses had been given by the students in the research studies and explained the nature of the misconceptions.]

30. Would more knowledge about the conceptions and misconceptions of students be useful to you as a teacher?

31. Probability is an area in which teachers have often had little formal instruction.

    [Mr. Trackman] You reported having had a probability class in college, a class you suggested has not been helpful for your teaching. Have you had the opportunity to take any classes or attend any workshops or conferences that have presented information that has been useful for your teaching? Are there things that teacher educators could be doing to assist teachers in teaching such relatively unfamiliar content?

    [Mrs. Books] You reported getting some of your knowledge from classes you have taken and conferences you have attended. Are there things that teacher educators could be doing to assist teachers in teaching such relatively unfamiliar content?

    [Mrs. Talent & Mr. English] You reported gaining much of your knowledge and ideas for teaching from workshops and teaching yourself. Are there things that teacher educators could be doing to assist teachers in teaching such relatively unfamiliar content?

32. Part of the task of teaching is reflecting on what we as teachers do. As you think back and reflect on the unit . . .

    What went well?

    Will you make changes for next year? If so, what are you thinking about changing?

33. [Mrs. Talent] You expressed earlier some dissatisfaction with how things went this year. Have you been able to identify anything specifically?

    Was what you did similar to previous years?

    Has taking the probability class recently made you more aware of areas lacking?

34. [Mr. English] Magdalene Lampert suggests that teachers face contradictory interests in the classroom. In this case, your interest in spending more time on some of the activities was in opposition to your desire to cover a variety of activities and probability situations. Reflect on any tension this struggle caused and your thinking as you dealt with these issues [e.g., the use of the journal pages for reflection].

35. With all teachers I have decided to revisit one of the questions from the first interview:

> In families with five children, what is the probability of having four girls and one boy, in any order?

[For Mr. Trackman] Except this time I have changed the setting:

> If you toss a coin five times, what is the probability of getting four heads and one tail, in any order?

This question is similar to the exploration your students did with flipping coins. If I flip a coin five times, how many different outcomes are possible?

If I get H H H H T, then I end up at the circle marked with red. How many other paths lead to that circle? List each path.

What is true about each of those outcomes?

What is the probability of four heads and one tail?

<u>Misconception Questions</u>:

1. A teacher asked Clare and Susan each to toss a coin a large number of times and to record every time whether the coin landed Heads or Tails. For each 'Heads' a 1 is recorded and for each 'Tails' an 0 is recorded. Here are the two sets of results:

Clare:

010110011001010110110100011100011011010101100100001
010100111001101011001011001011001001011101100111011
010100101100101011000100110101100111011101011000111

Susan:

1001110111101001110010011100100011101111111010101 01
11100000001000101001000001000110001010000000000011001
00000001111100001101010010010011111101001100011000

Now one girl did it properly, by tossing the coin. The other girl cheated and just made it up.
(a) Which girl cheated?
(b) How can you tell?

2. If a fair coin is tossed, the probability it will land tails up is 1/2. In four successive tosses the coin lands tails up each time. What happens when it is tossed a fifth time?

It will most likely land heads up.
It is more likely to land heads up than tails up.
It is more likely to land tails up than heads up.
It is equally likely to land tails up or heads up.

3. The probability of having a baby boy is about 1/2. Which of the following sequences is more likely to occur for having six children?

(a) B G G B G B (b) B B B B G B
(c) about the same chance for each

4. (same assumptions as #3) Which sequence is more likely to occur for having six children?

   (a) B G G B G B (b) B B B G G G
   (c) about the same chance for each

5. Which of the following results is more likely:

   (i) getting 7 or more boys out of the first 10 babies born in a new hospital?
   (ii) getting 70 or more boys out of the first 100 babies born in a new hospital?

   (A) They are equally likely.
   (B) Seven or more out of 10 is more likely.
   (C) Seventy or more out of 100 is more likely.
   (D) No one can say.

6. Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

   Please rank the following statements by their probability, from most probable to least probable:

   (A) Linda is active in the feminist movement.
   (B) Linda is a bank teller.
   (C) Linda is a bank teller and is active in the feminist movement.

APPENDIX C

Cross-Case Analysis: Teachers' General Pedagogical Knowledge

This appendix will consider the general pedagogical knowledge the middle school teachers demonstrated in the context of teaching their probability units. This appendix will first explore the educational goals the teachers stated for the study of mathematics in general and probability in particular. Next, this appendix will investigate how the teachers' efforts in designing mathematics instruction were related to their beliefs about learners and learning and their knowledge of general instructional strategies. Finally, this appendix will report the knowledge and skills demonstrated by the teachers in creating and managing the learning environment.

Setting Educational Goals

One part of general pedagogical knowledge involves the knowledge and beliefs teachers hold about the aims and purposes of education. In the context of teaching probability, this knowledge extends to the teachers' justification for teaching probability and the goals teachers envision accomplishing in the process of teaching probability.

Aims and Purposes of Education

The general goals for mathematics instruction stated by the four teachers ranged from having fun with mathematics to constructing a conceptual understanding of mathematics. (See Table C.1 for a summary of these goals.) Among the affective goals, Mr. Trackman believed getting students to have fun with mathematics was one way to overcome the negative attitudes many students had developed toward mathematics. Similarly, in her lower level classes, Mrs. Talent explained she wanted to "get [her students] turned on" to mathematics by presenting the content in less threatening ways. On the other hand, in her upper level classes, Mrs. Talent wanted to push her students in the process of helping them gain confidence in their abilities to solve problems. In addition, Mr. English explained part of the focus at the middle school level was on building a group identity and individual self-esteem. Even though reaching this goal involved students' participation in special activities, which at times conflicted with academic classes, Mr. English pointed out academic and social goals were both part of the "total program."

The teachers also stated a number of cognitive goals, including goals focused on the products of mathematics. As well as challenging students in areas such as problem solving, Mrs. Talent recognized a need to hone and fine tune students' mathematical skills. An additional goal expressed by Mr. English was to have students learn the vocabulary and strategies associated with the different concepts.

Table C.1
Summary of the Goals Stated by the Teachers for Mathematics Instruction

	Mr. Trackman	Mrs. Books	Mrs. Talent	Mr. English
The students will . . .				
Affective Goals:				
• have fun with mathematics	X			
• get "turned on" to mathematics	X		X	
• gain confidence in problem-solving abilities			X	
• build group identity & individual self-esteem				X
Cognitive Goals: Products				
• hone & fine tune mathematical skills			X	
• learn vocabulary & strategies				X
Cognitive Goals: Processes				
• learn how to learn	X			
• see connections between math & real world	X	X	X	
• be able to solve problems		X	X	X
• develop reasoning, critical-thinking & decision-making skills		X	X	X
• develop written & verbal communication skills		X		
• construct a conceptual understanding		X		

The teachers' understanding of the aims and purposes of education also included goals related to the processes of mathematics. In particular, Mr. Trackman identified teaching students how to learn as one of his educational goals. Mr. Trackman, Mrs. Books, and Mrs. Talent all mentioned the importance of students seeing mathematics as "out there everywhere" in the "bigger world." Further, all the teachers except Mr. Trackman emphasized the importance of problem solving and mentioned reasoning and critical thinking among their goals for students. Finally, Mrs. Books included goals of developing the students' written and verbal communication skills and of having them construct a conceptual understanding of mathematics.

Observations in each classroom revealed efforts to reach these stated goals. For example, Mr. Trackman's inclusion of games as part of his probability instruction provided the students with an opportunity to have fun with mathematics. The warm-ups Mrs. Talent gave her students reviewed their mathematical skills, including their ability to do mental computation. In Mr. English's class, the warm-ups gave the students opportunities to reason logically and deductively. The students in Mrs. Books' class were frequently challenged to solve problems, make decisions, and communicate with each other about mathematics. And each of the teachers tried to help students see the connections between mathematics and the real world. For example, some of the activities included in the probability units, such as "Cereal Boxes," were based on real-life problems.

However, the implemented goals observed during instruction in these classrooms also varied somewhat from the stated goals. For example, Mr. Trackman showed no evidence of trying to help the students learn how to learn. The students were neither given strategies for gaining knowledge nor given assistance in making sense of the content they were studying. In particular, the students were not encouraged to ask questions, make or explore conjectures, or summarize their observations or conclusions, all of which are aspects involved in the process of learning.

In addition, although a majority of the goals stated by the teachers fell into the category of process goals, the cognitive level of the tasks was often lowered in the course of implementing the tasks. For example, several of the activities potentially involved problem solving and reasoning; however, the teacher was the person who was doing the problem solving and reasoning as he or she modeled the analysis of the games and activities. All the students were expected to do was follow along and later reproduce what the teacher had modeled in a similar situation. This approach to the tasks was justified, according to Mr. English, because the students needed to have some "basic teaching of probability," including an introduction of the vocabulary and a demonstration of the strategies, before they could be involved in problem solving and reasoning on their own. As a result, the focus of Mr. English's probability instruction was more on the products (vocabulary and strategies) and less on the processes (problem solving and reasoning). This focus was the case not only in Mr. English's classroom, but also in Mr. Trackman's and Mrs. Talent's classrooms as well, where the expectation for the students was to follow along as the teacher did most of the thinking. Therefore, with the exception of Mrs. Books' class, observations revealed that limited emphasis was put on the process goals during the probability units.

Justification for Teaching Probability

One justification for teaching probability given by three of the teachers in the post-observation interview was that they were required to do so. Mr. Trackman, however, provided no example of who or what stated the requirement. On the other hand, Mrs. Books and Mr. English mentioned the expectations for probability instruction outlined by the *Curriculum Standards* (NCTM, 1989). Mrs. Books also stated that the expectations of her district's tests and outline of essential learnings included probability.

As further justification for teaching probability, Mrs. Books and Mrs. Talent felt an awareness of probability is important because of the many ways it impacts the lives of adults as well as students. Mr. English gave the specific example of people's decisions in lotteries and other gambling situations, a concern shared by each of the teachers over the course of their probability units.

Goals in Teaching Probability

The teachers were also asked about the specific goals they had for their students as they taught the probability unit. Table C.2 summarizes these goals. In many cases, these goals paralleled the goals stated for mathematics instruction in general. For example, Mr. Trackman's goal for the students to play games and have fun paralleled his overall goal that students have fun with mathematics. Knowing the probability vocabulary and strategies fulfilled Mr. English's goal that students learn the vocabulary and strategies corresponding to each unit he taught. Grasping that probability is everywhere is part of seeing the connections between mathematics and the real world, a goal stated by several of the teachers. The numerous goals regarding analysis of the probability problems potentially involved the students in solving problems, reasoning, and communicating about mathematics, which correspond to many of the general goals stated by the teachers for mathematics instruction.

The teachers were asked about their goals for teaching probability in both the pre- and post-observation interviews. In both cases, the teachers' responses seemed to describe what the teachers planned to do or subsequently what they had done in the probability unit. In particular, the teachers talked about goals of playing games, conducting experiments, analyzing situations with a variety of strategies, considering experimental and theoretical probabilities, and determining whether a game was fair. Because these activities were part of the probability units, the teachers' goals appear to be closely matched with the instructional activities they used.

In varying degrees, the goals stated for the probability unit were also reflected in the tasks used to evaluate student learning, although the learning objectives actually evaluated were limited. Mr. Trackman's goal that the students "come up with some sort of understanding of probability" was quite vague. In the end, the "sort of understanding" expected on the unit test was an understanding of the basic probability content presented in the textbook. Mrs. Books talked about goals that involved designing simulations, dealing with bias, using vocabulary and communicating about mathematics. All of these goals were involved in the final task given to the students in "Monty's Dilemma" where they were to write a letter to the researcher explaining how they had simulated the problem and removed what bias they could. One of Mrs. Talent's goals was that the students could apply what they had learned to analyze an unfamiliar task. This goal was the basis of the evaluation tasks given to the students in the Coin Tossing problem and the Carnival task. Mr. English wanted his students to develop a repertoire of strategies for analyzing probability questions. The unit test gave the students opportunities to apply the strategies to the situations they had seen in the probability unit.

Table C.2
Summary of the Goals Stated by the Teachers for Probability Instruction

	Mr. Trackman	Mrs. Books	Mrs. Talent	Mr. English
The students will . . .				
General				
• play games and have fun	X			
• participate in a number of different kinds of activities				X
• "come up with some sort of understanding of probability"	X			
• grasp that probability is everywhere	X	X		X
Knowledge				
• know vocabulary that may be used to express probability		X		X
• know a variety of strategies for analyzing probability questions				X
• become familiar with different measures of central tendency (statistical goal)		X		
Skills				
• learn how to set up a simulation		X		
• be aware of bias (in simulation design)		X		
Analysis				
• be able to conduct an experiment and analyze the results			X	
• have experience with experimental probability & explore the theoretical probability behind each experiment				X
• be able to determine whether a game is fair			X	X
• be able to analyze certain situations mathematically using such strategies as a tree diagram			X	
• apply what they have learned to analyze an unfamiliar task			X	
• be able to make good decisions based on analysis of information			X	

Although the teachers' goals closely match the instructional activities and evaluation tasks, the stated goals more accurately described what the teacher and students did rather than what the students necessarily learned. The contrast between doing and learning is clearly exemplified in one episode that occurred in Mrs. Talent's classroom. The probability unit had begun with simulations of the "Cereal Boxes" problem and "Monty's Dilemma." Next Mrs. Talent had used tree diagrams to analyze some games using chips marked with letters on each side. The students had practiced using tree diagrams and writing probabilities as they completed several worksheets. In correcting one homework assignment, Mrs. Talent had guided the students toward discovering the multiplication property, which they used on a worksheet to find the probabilities of independent events. Then, on Day 8, the class played a game with dice sums and products. To begin analyzing these games, Mrs. Talent had the students complete charts of dice sums and products. Using these charts, she then wrote the probabilities and determined whether or not the games were fair. On the subsequent homework assignment the students were asked to find probabilities of various dice sums or products. Mrs. Talent reported, "[The students] acted like that was the hardest thing I had given them all year," although she could not figure out why it was so difficult. Perhaps one explanation for the difficulty lies in the fact that the students had not been learning what Mrs. Talent thought they had been learning as they had been doing the activities. In particular, much had been left for the students to conclude on their own. For example, the students had been left to discover that probability could be expressed as the number of favorable outcomes divided by the number of possible outcomes. The students had been left to discover this definition of probability because it had never been stated. Similarly, the students had been left to decide when it was appropriate to apply simulations, tree diagrams, the multiplication property, or dice charts because activities where students made those types of decisions were not part of instruction. In addition, because most of the instruction had been delivered in a "show and tell" manner, the students had been spectators of the learning process, not necessarily participants. In this case, the students apparently had been able to follow the teacher's directions and complete the activities by mimicking what had been modeled for them by the teacher, until presented with something just a little bit different. Then the students did not know what to do. They encountered similar difficulties later when presented with unfamiliar tasks as part of the unit evaluation. For those tasks, many students did not know where to begin.

Instances revealing the discrepancy between doing and learning were not as apparent in the other classrooms, although perhaps the same discrepancy existed. A closer look at the tasks the students were asked to analyze might explain, at least in part, the lack of such evidence. The students were rarely given a situation to analyze entirely on their own, but when they were, it almost always was a situation very similar to what the teacher had demonstrated for them previously. For example, as a follow-up to "Is This Game Fair?" which Mr. English had analyzed for the students, he had the students analyze "Doubles in

Monopoly." In both of these games, the player was to receive three points for a given outcome and lose one point for the other possible outcomes. In "Is This Game Fair?" the winning outcome was a dice sum of 7, which has a probability of 1/6. To make the game fair, Mr. English and the class decided the player needed to win five points instead of three points. In the second game, the winning outcome was a roll of doubles on the two dice. This outcome also has a probability of 1/6. Therefore, once the students discovered this similar result, the remainder of the analysis would have been the same as for the first game.

Even the evaluation task given by Mrs. Talent, which began as an unfamiliar group task, ended up as an individual task with which the students were familiar after Mrs. Talent demonstrated the analysis of the initial problem. In the original Carnival task, the carnival planners wanted to make a $200 profit on the game, assuming 300 people would play the game. For the proposed game, three cans each contained a red, a blue, and a green ball. For $1, a player was to pick one ball from each can. If the three balls were all the same color, the player won $10. After the groups had been given time to work and had shared their conclusions about the game, Mrs. Talent demonstrated the theoretical analysis of the game to the students. Then she gave the students a follow-up task they were to analyze individually. This problem was identical to the previous task, except that a white ball had been added to each can. Therefore, only minor changes were needed in the analysis Mrs. Talent had demonstrated to the students.

Mr. Trackman asked his students to analyze one activity that was different from the tasks he had analyzed. In this case, the theoretical analysis of the Coin Tossing Exploration was based on Pascal's triangle. However, because the students had been given no theoretical foundation upon which to develop the patterns in Pascal's triangle, the theoretical analysis of the task was clearly beyond the capability of the students. As a result, many students chose not to hand in the assignment at all, even though they at least could have completed the experimental part of the assignment. In this case, rather than the nature of the task being too simple because it was so similar to previous tasks, the assigned task was too difficult for the students given what they had experienced so far in the probability unit.

A number of the goals shown in Table C.2 focused on the analysis process, which was a centerpiece of each of the probability units. However, as seen in the earlier discussion about the teachers' goals for mathematics instruction, the teacher was generally the one who was doing the analysis. In particular, it was the teacher who set up the chart or drew the tree diagram. Because the students only needed to follow the teacher's example as they analyzed similar situations, the cognitive level of the analysis tasks became something less than analysis, leading to a further discrepancy between the stated goals and what actually happened in the probability units.

Designing Mathematics Instruction

Additional aspects of teachers' general pedagogical knowledge potentially influence how they design mathematics instruction. These aspects include the teachers' knowledge and beliefs about learners and how learning occurs as well as their knowledge of general instructional principles and strategies.

Learners and Learning

In their interview responses and in their instructional practice, the two more experienced teachers, Mrs. Talent and Mr. English, painted similar portraits of learners and the learning process. What they had learned by experience about middle school students influenced what they did in the classroom. For example, both teachers spoke of the importance of keeping middle school students busy. As Mrs. Talent pointed out, "You have to keep them moving and you have to keep a variety or you are going to lose them." Both teachers generally accomplished that goal through a mix of warm-ups, hands-on activities, and direct instruction during each class period.

Both Mrs. Talent and Mr. English also indicated the level of the students influenced their decisions concerning what activities to do in the classroom, how much explanation to provide, and at what pace the class should move. For example, both had made adjustments as they taught probability to classes at a lower level than the classes observed. In Mrs. Talent's case, she reported, "We went slower . . . and I . . . held their hand a little bit more." In particular, she made a special effort to get the students started on worksheet assignments by going through the first few questions with the students. In this way, she hoped to avoid the frustration often encountered in the lower level classes. Mr. English had also made a number of changes as he taught probability to his basic mathematics classes. Specifically, he delayed the presentation of odds and did not cover some of the more advanced concepts such as Pascal's triangle or the "Monte Carlo kind of thing." Instead, the basic mathematics classes had done "quite a few fair and unfair situations using models." Mr. English also pointed out that he and the students kept the notes and dice charts in front of them for reference whenever they did those activities. He explained, "With that group, it's important that . . . you don't . . . expect them to memorize as many things, but you expect them to have the resources and know where to go to get the information."

The instructional practices of Mrs. Talent and Mr. English reflected the view that learning involves a cycle of demonstration by the teacher and practice by the students. For Mr. English, modeling the analysis of the different games and activities was an important part of the probability unit. That modeling was how the students were to learn the various strategies. Similarly, in assigning the worksheets to her students, Mrs. Talent carefully worked through one or two examples for the students to follow. In both classrooms, the teacher's model was followed by similar problems or exercises the students were to do as practice. Both teachers emphasized the importance for middle school students to have opportunities to practice what they are learning and to review what they have learned previously.

Both Mrs. Talent and Mr. English viewed hands-on activities as an important part of the learning process. In the past few years, in particular, Mrs. Talent had moved away from the textbook in favor of hands-on learning. Following Mr. English's experience using manipulatives as part of a workshop exploring the Middle Grades Mathematics Project materials (including Phillips et al., 1986), he had also begun to include hands-on activities in his mathematics lessons. Besides making classes more interesting, Mrs. Talent and Mr. English both recognized that hands-on activities help students develop a better understanding of mathematics.

Mr. Trackman indicated the students in his classes had a wide range of learning abilities. Seeming to attribute success in mathematics to students' effort or innate ability, he suggested mathematics comes easily for some students while others have to work harder at learning mathematics. Mr. Trackman identified still others as lazy. Because of his own experiences with mathematics, Mr. Trackman felt he could identify with the variety of students in his classes. He further suggested this understanding influenced how he responded to student questions. In particular, rather than just repeating an explanation, Mr. Trackman indicated he tried to find out where the student had gotten lost and help the student from that point. In the lessons observed, however, there was no evidence of any efforts to meet individual needs.

Mr. Trackman also suggested that students have different learning styles, although he gave no indication of what he meant. Again there was no evidence of this knowledge impacting his instruction. Other than labeling his students as immature, there was no mention of whether they were concrete or abstract thinkers. There was no suggestion in either his instruction or in the interviews that hands-on activities would help students understand the underlying mathematical ideas. Although some hands-on activities reportedly had been done earlier in the year, Mr. Trackman indicated these were "never really for assignments." The hands-on games and activities in the probability unit had been included because they were fun, not because they facilitated learning. Mr. Trackman's comments seemed to reflect an ill-defined philosophy of learning. On the one hand, he emphasized the rote learning of the "times tables" and other basic mathematics. Yet, he also spoke in a vague way about wanting the students to discover something about probability on their own.

Largely as a result of experiencing a new way of learning mathematics in her teacher preparation program, Mrs. Books had a view of the nature of learners and learning that differed, at least in some respects, from the other teachers. In particular, she felt students are to be active participants in the learning process as they interact with ideas and with people, including their peers and the teacher. Because Mrs. Books believed learning and memory are impacted by doing as well as seeing and hearing, hands-on activities played an important role in the on-going process of learning in her classroom. In addition, Mrs. Books viewed learning as a process of identifying what is already known and combining that understanding with additional pieces of information in coming to a new level of understanding. Therefore, she frequently had students express what they knew in "journal writes" or in discussions with one another. Because Mrs. Books believed new understanding can be formed as one's mind continues to grapple with disequilibrium, she provided time and space in the curriculum for students to revisit in new settings ideas they had seen earlier.

Mrs. Books' views about learners and the learning process were based on a constructivist philosophy about learning, the same philosophy underlying the NCTM's vision for the mathematics classroom. The teachers were asked about this view in the post-observation interview. In particular, they were asked if the vision of a classroom environment where "students learn to make conjectures, experiment with alternative approaches to solving problems, and construct and respond to others' mathematical arguments" (NCTM, 1991, p. 56) was feasible in the middle school classroom. Mrs. Books admitted that was what she was trying to accomplish. Mrs. Talent also believed it was possible, however, she admitted,

> It's real hard to break the habit of, you know, I am the one up there that's supposed to be doing the teaching. . . . I think it's real possible. I also think . . . you have to find a balance. . . . There's sometimes when it fits naturally and sometimes when it doesn't. . . . Sometimes you can't afford the time to spend 2 weeks letting them discover this or that. So you have to kind of make those judgment calls. But, yeh, it's a great way to teach and I'm trying to get better at it.

Mr. English believed such a vision would be feasible at the seventh-grade level where he was teaching "if the kids had had a basic teaching of probability and had some hands-on work with it." However, because the students had not had that prior experience with probability, Mr. English felt it was necessary for him to first present the different models (such as coins, dice, and cards) to the students. Mr. Trackman responded rather emphatically,

> I think it's absurd. Jaime Escalante didn't do it that way, for the most part. . . . I don't like the idea of this . . . I get the impression that they're wanting everybody to . . . not have their feelings hurt . . . with this new style of math and . . . some people are right and some people are wrong. That's the problem. They want everybody to be right and you can't have that and if you say, "Two plus 2 is 5," you're wrong. . . . I don't like the direction that math is going at all and I figure if I just wait it out 5 or 6 years, it will swing back.

These different views about the proposed reform in mathematics education highlight the teachers' different views about learning mathematics and their different approaches to teaching mathematics.

General Instructional Strategies

In describing their mathematics instruction in general, the teachers referred to their use of warm-up activities, textbook assignments, hands-on activities, projects, homework, and cooperative group activities. In some cases, the instructional patterns followed were the same for the probability unit as they were for mathematics instruction in general. In other respects, changes were made in the presentation of the probability units.

Three of the teachers reported using warm-up activities as a regular part of their mathematics instruction. For Mr. English, the warm-ups generally involved some critical thinking or deductive reasoning activities. Mr. Trackman reported his warm-ups, which were "a critical thinking kind of thing," were "geared towards leading into what we were going to do" that day. And, for him, going over the warm-ups often constituted the daily lesson. The warm-ups Mrs. Talent gave her students reviewed the content and skills the students had already studied. During their probability units, only Mr. English and Mrs. Talent continued to start the class with warm-ups.

During the school year, textbooks had been used in two of the classes observed. Mr. Trackman pointed out he used the textbook (Orfan & Vogeli, 1988) extensively because that was "something . . . stable in [the student's] life." For the probability unit, however, the textbook assignments had been supplemented by a number of games and activities. Mr. English pointed out he had taught mathematics exclusively from the textbook for many years. Then, in part because of his exposure to the different units of the Middle Grades Mathematics Project (including Phillips et al., 1986) in a summer school class, he had become aware of the use of hands-on manipulatives for teaching mathematics. As a result, the focus of his teaching shifted from following the textbook to using hands-on activities. However, because Mr. English felt the pre-algebra textbook (Usiskin, 1992) was better than most textbooks he had seen, the textbook was being used in the pre-algebra class observed for this study. The textbook's coverage of probability, however, was neither appropriate nor adequate in Mr. English's view, so he set aside the textbook and planned the probability unit to utilize the many hands-on probability activities he had collected from conferences and other sources.

Instead of following a textbook, Mrs. Talent and Mrs. Books taught mathematics using a variety of hands-on activities and worksheets. Through the use of hands-on activities in Mrs. Talent's class, the students were exploring new content or building upon and applying what they had learned. Worksheets provided further opportunities for the students to practice the skills they were learning. For Mrs. Books, problems set the stage for student exploration. The students initially worked individually on the problems posed to them. As the process continued, the students shared their observations and insights with members of their small group or with the whole class. Additional problems or follow-up activities gave students opportunities to build upon the shared insights and observations. Mrs. Talent and Mrs. Books both continued these instructional patterns in their probability units.

Mr. Trackman pointed out that he and his colleague "try to keep a project going all year." He explained they started out with an NBA project where the students were "keeping track of statistics and figuring out probabilities of NBA players." They had also done another probability project with M&M's. In both cases, these projects may have involved more statistics than probability. However, because the projects had been done earlier in the year, they had not been observed. No project was included as part of the probability unit.

One of the factors Mr. Trackman considered in planning the probability unit was that he did not want the students "to have to take home homework." He explained assigning homework was a "losing battle" because at the end of the year the students "don't do it anyway." Although the other teachers did not express this same sentiment, there generally were few homework assignments made. With the exception of some homework assignments given by Mr. English, time was usually provided in class for students to complete whatever assignments were made. The students had homework only if the assignment had not been finished in class.

Because of the nature of the games and activities used in the probability units, the students were frequently working together in groups. In the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English, the focus of the group activity was generally on gathering experimental data as the students played the game or conducted the experiment or simulation. In Mrs. Books' classroom, in addition to gathering experimental data, the students in their groups were sharing their subjective guesses, discussing their simulation designs, or thinking about issues such as replacement of the beads (in the "Cereal Boxes" simulation activity).

Working in groups had been the specific plan of the teacher in two other cases. Mr. English used some cooperative group activities from *Get It Together* (Erickson, 1989) on two occasions when there was some extra class time. Each member of the group had a card with clues about an unknown spinner. From the information given about the probability of certain outcomes, the students were to figure out what the spinner looked like. In the second case, Mrs. Talent planned for the students to do the Carnival task as a group. As part of the material to be handed in, each student was to explain what each member of the group had done. Because of a shortage of time, Mrs. Talent did not follow through completely with these expectations. Rather than writing down their explanations, the problem itself was discussed with the whole class after the groups had worked for a designated period of time.

In describing what his teaching style would be during the probability unit, Mr. English explained,

> [My] teaching style is a lot of hands-on activities. . . . I basically start out . . . pretty much teacher-directed and I want to kind of introduce them to the vocabulary. . . . I want to involve them in some teacher-directed activities . . . that they play a game, but we do it, [we] analyze the game and I model for them clearly, this is how you analyze it. And, after I've modeled it, then I want them to do problems similar to that where they practice that . . . I give them lots of repetition at this level.

For the most part, this description also applied to the instructional pattern of Mr. Trackman and Mrs. Talent during probability instruction. The teacher first gave the directions for the game or activity. The students then played the game or completed the activity. At that point the teacher directed the class as they considered the experimental data and analyzed the game or activity theoretically. If the assignment was a worksheet, the teacher gave an example or two to demonstrate what was to be done. When the students had completed the activity, the teacher directed the class as they went over and corrected the items. In contrast, the instruction in Mrs. Books' classroom was more student-focused. In exploring the problems, the students generally had input in defining the problem's assumptions and in determining how the simulation would be conducted. Most of Mrs. Books' interaction with the students came as they worked in small groups where she could ask thought-provoking questions appropriate for the individual students.

Creating and Managing the Learning Environment

General pedagogical knowledge also includes the teachers' knowledge and skills in classroom organization and management. This includes how teachers organize the students and how they manage time, materials, and student behavior.

Organization of the Classroom

In the course of the probability units, different groupings of students were used at different times, depending on the nature of the activities. Sometimes, the whole class was involved in instruction or discussion, as in the presentation of the activities or the analysis of experimental or theoretical results. Occasionally, the students worked individually. At other times, particularly when playing the probability games, the students worked in groups of two or more. Mr. English's arrangement of desks in groups of four and Mrs. Books' use of tables seating four students permitted smooth transitions into group activities. Although the desks in Mrs. Talent's class were arranged individually in rows, when she intended to use groups, she rearranged the desks so that students would seat themselves in groups of their choosing as they entered the classroom. These self-selected groups generally worked well together. However, on at least one occasion, when one pair of students was doing more talking than working on the task, Mrs. Talent suggested to these students that they not sit together for future activities.

The desks in Mr. Trackman's class were arranged in rows, but grouped two or three abreast. However, Mr. Trackman did not utilize these groupings when arranging students for activities. When groups were called for in an activity, students in his class were allowed to move around and form their own groups. Besides using up class time, this means of forming groups led to problems when the students left over did not want to work together. On at least two occasions, it was necessary for boys and girls to work together in the same group. Working with students of the opposite gender was met with resistance from the students and a lack of cooperation when Mr. Trackman forced them to work together.

Management of Time

Because Mrs. Books had a self-contained classroom, her time schedule was more flexible than the other teachers' schedules. In general, she planned about 45 minutes for mathematics each day, although the actual time ranged from 15 minutes (one day when there was an assembly) to nearly 60 minutes. On the other hand, Mr. Trackman and Mrs. Talent had 43-minute periods and Mr. English had 48 minutes of class time. These periods also were shortened at times to accommodate special school events.

Although Mr. Trackman suggested in the pre-observation interview that "we cram as much as we can into [the 43 minutes] as possible," observations revealed a different story. Generally between 10 and 27 minutes were spent discussing the results from the previous day's activity and introducing the activity for the current day. For the remainder of the class time, from 16 to 33 minutes, students were expected to be working on the assignment.

Although Mr. Trackman attempted to expand the activity by giving the students multiple ways of playing the games, the students frequently finished their work before the period ended. Often, whether they finished the activity or not, the attention of the students drifted to off-task conversations and behavior.

On the other hand, in a period that was only 5 minutes longer than Mr. Trackman's, Mr. English's students often completed at least two different activities. Rather than introducing one activity and giving the students the rest of the period to complete it, Mr. English more closely monitored the time as he led the students through the process of gathering experimental data and then considering the theoretical analysis of the problem. Because the students were kept actively involved in tasks throughout the class period, the students had little opportunity for off-task behavior.

The students in Mrs. Talent's class generally completed a single activity in each period. But a considerable amount of time, sometimes as much as 10 to 15 minutes, was spent on completing and correcting the five warm-up problems. The remainder of the time was generally occupied with the teacher explaining and the students carrying out the current day's activity or assignment. Because of the time spent on the warm-ups, the probability activities sometimes had been cut short. However, judging from the amount of off-task discussion occurring during the warm-ups, more time than necessary was allotted to the warm-up activities. Therefore, Mrs. Talent could have made more time available for the probability lessons while also continuing her practice of starting the class with warm-ups. Otherwise, there was only a limited amount of off-task conversation and behavior.

In contrast to these teachers who did one or more activities each lesson, the students in Mrs. Books' class spent several days on each problem they investigated. This amount of time was necessary because Mrs. Books wanted to get the students involved in the process of designing their own simulations. It also gave them time to gather their data, share their results with the class, and discuss the implications of their findings. Because of the involvement of the students in the activities, not much off-task behavior was observed.

The nature of the probability activities presented two challenges to the teachers in terms of managing time. First, the activities often involved carrying out an experiment or a simulation a repeated number of times and reporting the results to the class. In such situations, some students invariably worked at a faster pace than other students. The teachers were then faced with the challenge of keeping on-task those students who had completed their trials, while waiting for the other students to finish. The teachers dealt with this challenge in different ways. Mr. English often began collecting the class data as soon as some of the students were finished conducting their trials. At other times he handed out some cooperative group problems to those who had finished. On one occasion, Mrs. Books suggested to students that they could find the mean, median, and mode for their data as they waited for the rest to complete their trials. For one activity involving dice sums, rather than repeating the experiment a fixed number of times, Mrs. Talent had the students roll their dice as many times as they could in 4 minutes. The benefit was the students were all actively involved in the activity throughout the entire time period. And, even though the totals varied from group to group, by converting their results into percents, the students could still compare their findings.

The second challenge to time management, somewhat unique to probability, was the process of collecting class data. Again, the teachers had different ways of handling this challenge. Mr. Trackman and Mr. English generally wrote down the results as students reported them orally. However, it was difficult at times to hear the results being reported when this was done at the same time some students were finishing up the activity and others were taking advantage of the opportunity to talk. As a consequence, more time was spent repeating the already-reported results. For several of their activities, Mrs. Books and Mrs. Talent had the students themselves record their results on a transparency the teacher had prepared. Because this approach involved students getting out of their seats and moving around, it too was potentially time-consuming, especially in regaining control and getting the students back in their seats. In one activity, when Mrs. Talent was not concerned about collecting the specific data, she had the students report their data by a show of hands as she called out the range of possible outcomes. This approach worked quite efficiently as a way of at least allowing her to get a general sense of the results. However, it did not necessarily give the students any sense of the results.

Management of Materials

Because the probability units, as a whole, involved hands-on activities and instructional materials other than a textbook, there was a considerable amount of materials to be dealt with in the course of the lessons. In addition, there was the usual flow of homework assignments, late papers, and assignments for students who had been absent.

Each of the teachers used an overhead projector almost exclusively for large group teaching situations. Just beneath the overhead, Mr. English had easy access to separate boxes for clean transparencies and ones that had been used. He, as well as Mr. Trackman and Mrs. Talent, also had a sink available in the classroom to wash off transparencies, when needed. On many occasions, the teachers used ready-made transparencies corresponding to the activity they were doing. Although Mr. Trackman prepared and used some transparencies, he generally wrote right on the overhead glass. This practice involved time, particularly in setting up the grid of circles for the Coin Tossing Exploration, for example. It also took time to wash off the overhead each time that was necessary, which Mr. Trackman did with a water bottle and a towel. But it also meant that what had been written could not be referred to again, once it had been washed off.

Each teacher used a variety of ways to keep track of the flow of papers. Because all six periods Mr. Trackman taught were the same, he did not have to keep track of multiple sets of different materials as the other teachers did. In general, the teachers had routines associated with correcting homework assignments and designated baskets or places for students to hand in their assignments. In Mr. Trackman's class, assignments for students who had been absent were placed in large envelopes on one of the bulletin boards. Mr. English put together packets for the students to pick up each day as they came into the classroom. Because these packets included the warm-up activities, handouts for class activities, worksheets for whatever assignment there was, and a summary page, it saved the time that otherwise would have been spent in handing out papers. Further, Mrs. Talent and Mr. English both took advantage of the time when students were working on their warm-ups to return assignments to students.

The probability lessons also involved the use of manipulatives and other materials, which the teachers generally stored in containers on shelves or in cabinets available in their classrooms. As a result, the teachers had easy access to the materials and could distribute them to the students as needed. At times, the teachers handed out the materials. At other times, they asked students to hand the materials out. Or alternatively, in Mrs. Books' case, a variety of materials were made available and the students chose what they wanted to use for their simulations. On at least one occasion, Mrs. Talent expedited the distribution process by having the paper cups and chips already counted out and combined, ready for distribution. The teachers were also careful to allow time at the end of class so that materials could be picked up and put away.

For the most part, the classroom routines were known and followed by the students, with the result that class was not disrupted unnecessarily in the process of dealing with papers or materials. At times, some of Mr. Trackman's students asked what they were supposed to do with their assignment, perhaps because it was something other than their familiar textbook assignments. Except where noted otherwise, the classes ran smoothly, from an organizational perspective.

Management of Student Behavior

The students in the classes observed were typical middle school students, full of energy, activity, and conversation. They took advantage of any opportunity to talk with one another, whether at the beginning of class, in periods of transition, while they were working, or when they had finished their work. However, with the exception of Mr. Trackman, the teachers were generally in control of the class and able to keep the students focused on the assigned tasks.

Mrs. Talent and Mr. English involved the students in warm-ups as soon as the class began. These two teachers and Mrs. Books had activities planned that generally kept the students occupied throughout the class period. Expectations for student behavior were posted and consistently enforced. Most examples of misbehavior involved minor lapses in paying attention or in talking when they were supposed to be listening. These individual cases of discipline were handled expeditiously and with a minimum of disruption to the flow of the class, generally with a comment or warning directed to the student involved.

As necessary, these three teachers provided reminders of behavior expectations. For example, during the times of sharing, Mrs. Books reminded the students that they were to talk to the class, not to her, and she reminded the others of their role as listeners. At other times, the teachers provided warnings, when necessary. For instance, when the activity and noise level of the class as a whole became disruptive, Mr. English had a series of caricatures he put on the overhead, which progressed from a request to "be quieter" to a warning, and finally to one that assigned penalties. In this case and in others, Mrs. Books, Mrs. Talent, and Mr. English generally followed through with consequences when it was necessary.

In contrast to the other three teachers, Mr. Trackman had more difficulties in maintaining control of his class and keeping them focused on the tasks assigned. Although he had established a management system with rules and consequences clearly posted, it did not appear the rules had been consistently followed and enforced. In particular, the consequences or penalties had not been applied regularly or severely enough to change students' behavior. The result was frequent interruptions by Mr. Trackman with admonitions to individuals to sit down, to be quiet, or to throw out their gum. For example, although it was nearly the end of the year, students were caught almost daily with gum. Because the only penalty seemed to be having to throw it out, the students tried day after day to see if they could get away with it, and both the students and the teacher seemed to be aware of the "game" they were playing.

During the time when students in Mr. Trackman's class were to be working on their assignments, there was a great deal of off-task behavior and conversation. This situation resulted in part because Mr. Trackman had not planned enough work to keep the students busy. But Mr. Trackman also encouraged, rather than discouraged, such off-task behavior by participating in it. For example, on one occasion, Mr. Trackman took a meter stick away from one student who had been waving it around as a bat and a sword. But, then, Mr.

Trackman used it as a putter and discussed golf with the student. On another occasion, Mr. Trackman joined in an off-task discussion the students were having in Pig Latin.

Mrs. Books put a great deal of emphasis in her classroom on showing respect for one another. In fact, "respect each other" was first on the list of posted behavioral expectations. This respect was a key component in creating an environment where students felt free to share their conjectures and their questions with one another. First of all, in all her dealings with students, Mrs. Books' showed respect for the students, even when it was necessary to correct or discipline them. And with frequent reminders, Mrs. Books encouraged the students to respect one another. For example, when students were sharing their ideas, Mrs. Books reminded the other students that they were to listen to one another, to carefully consider the implications of what the person had to say, and to challenge the person in a non-threatening way when they did not agree with what the person had shared.

Mrs. Talent and Mr. English also showed respect for their students. However, the students in these classrooms did not always treat each other with respect. On one occasion when many of Mr. English's students were gone on a trip with the band, several students remaining in class referred to the absent students as "band nerds." On other occasions, a few students made derogatory comments about "the brain" in the presence of one bright member of the class. In these cases, Mrs. English made no comment about the disrespectful remarks, although he was very concerned that he himself treated the students with respect.

On the other hand, some of Mr. Trackman's comments showed a lack of respect for the students in his class, particularly those who did not speak English as their first language. On a number of occasions, for example, Mr. Trackman made statements mimicking an Asian accent or repeating errors students had made. But other students were also treated with disrespect. In one of the review questions on Day 10, Mr. Trackman labeled the categories of the circle graph as "boys that are smart, . . . boys that are not smart, . . . girls that are smart, and girls that . . . are blond. Oh, I'm sorry. That are not smart." And those students who made the honor roll had been released the day before the unit began for what Mr. Trackman called a "smarty" party. As with the teacher's remarks, some of the comments made by the students also showed disrespect for others in the classroom. For example, when one student asked why 1 was not included on the list of dice sums, another student responded, " 'Cuz you can't roll a 1, stupid." In some cases, Mr. Trackman objected to what the students said, but in other cases, their comments were ignored.

APPENDIX D

Cross-Case Analysis: Teachers' Subject Matter Knowledge

This appendix provides a comprehensive analysis of the teachers' subject matter knowledge. Two facets of subject matter knowledge will be explored. First, this appendix will investigate the teachers' knowledge of probability content. Second, the teachers' knowledge about the nature of mathematics and probability will be considered.

Knowledge of Probability Content

A major portion of the pre-observation interview was devoted to exploring the four middle school teachers' knowledge of probability content. This knowledge included the teachers' knowledge of the definition of probability, the distinction between probability and odds, experimental and theoretical probability, the basic properties of probability, strategies for analyzing probability situations, simulations and expected value, and applications of probability. In addition, other questions in both the pre- and post-observation interviews considered whether the teachers' understanding of probability included any of the common misconceptions discussed in chapter II of this research study. (See Appendix A: Probability Questions and Appendix B: Misconception Questions for a listing of the interview questions.)

For each question, the background of the question and possible appropriate responses will first be discussed. Then, the responses of the four middle school teachers to the interview questions will be explored. Finally, this section will investigate what knowledge of probability content was exhibited during probability instruction and how the teachers' knowledge of probability content was related to their probability instruction. In particular, this investigation will focus on how the teachers represented the corresponding concepts of probability. Although teachers' knowledge of the ways of representing concepts is considered to be part of pedagogical content knowledge, those representations will be discussed in this section because of the direct relationship that became evident between the teachers' knowledge of probability content and the ways they presented and represented that content.

Definition of Probability

Situations where observations or measurements of chance occurrences can be made, such as tossing a coin or throwing a die, are called *experiments.* An *outcome* is any one of the possible occurrences in an experiment. For example, if the experiment is throwing a standard six-sided die and recording the number of dots showing on the top face, then there are six possible outcomes, namely 1, 2, 3, 4, 5, or 6. The set of all possible outcomes for an experiment is called the *sample space*. Therefore, in this case, the sample space is the set {1, 2, 3, 4, 5, 6}. Any subset of this sample space is called an *event*. In this experiment, an event might be that the face that lands on top has four dots, or that the die lands with an odd number of dots on top. As in this example, an event may be a single outcome or may include several outcomes.

In some situations, each of the possible outcomes in the sample space has the same chance of occurring or, in other words, the outcomes are *equally likely*. If all outcomes are equally likely, the probability of a given event is defined as the ratio comparing the number of favorable outcomes to the number of possible outcomes. For example, when rolling a fair die, the assumption the die is fair suggests each of the six faces has the same chance of occurring. Therefore, the probability of rolling a 4 is 1 out of 6 or 1/6 because there is one outcome where that occurs out of the six possible equally likely outcomes. Similarly, the probability of rolling an odd number is 3/6 or 1/2 because there are three favorable outcomes, namely 1, 3, and 5, out of the six possible outcomes. Symbolically, the probability of these events would be expressed as P(4) = 1/6 and P(odd) = 1/2, respectively.

In applying this definition to find the probability of an event, it is important to establish the outcomes as equally likely. For example, even though the red portion is one of three regions on the spinner pictured in Figure D.1(a), the probability of the spinner landing on the red portion is not 1 out of 3. However, by dividing the spinner into four equally likely portions, as in Figure D.1(b), one can then apply the definition of probability to determine the probability of the spinner landing on red, which would be 1 out of 4. This section will explore the teachers' knowledge of this basic definition of probability as revealed in the interviews and as reflected in their classrooms.

Figure D.1. Sample spinners with (a) unequally likely and (b) equally likely outcomes.

<u>As revealed in the interviews</u>. The first question in the probability knowledge interview presented a situation where the definition of probability could be applied. The teachers were asked the following two-part question in the One Die problem (probability question #1):

One fair die is thrown. What is the probability that the die lands
(a) with a 5 on top?
(b) with a 2 or 3 on top?

In the first part, the probability is 1 out of 6, or 1/6, because, out of the six equally likely ways the die could possibly land, there is one way of the die landing on 5. In the second case, there would be two favorable outcomes out of the six equally likely outcomes. Therefore, the probability would be 2 out of 6, 2/6, or 1/3.

Each of the teachers correctly identified these two probabilities. However, none of the teachers in their explanations explicitly mentioned the idea of equally likely outcomes. This assumption, however, was implicit in their responses. For example, Mr. Trackman explained, "There are six sides to the dice, so there are six numbers on the dice, and they are all different numbers, so there are six possibilities and one of them coming up would be the 5, unless it's weighted." In a similar fashion, the other teachers referred to the fact there were six sides to the die or six different numbers on the die, but never stated the sides or numbers were equally likely to occur.

Table D.1 summarizes the responses of the teachers to the One Die problem. The second part of the item, the distinction between probability and odds, will be considered in the section following the discussion of how the teachers' knowledge of the definition of probability was reflected in their teaching.

Table D.1
Summary of the Teachers' Responses to the One Die Problem (Probability Question #1)

	Number of teachers responding correctly	Number of teachers responding incorrectly
Definition of probability	4	0
Distinction between probability and odds	3	1

As reflected in the classroom. Based on Jacobs (1982), Mr. English defined probability as "a measure of chance." The vocabulary worksheet he gave to students on Day 1 further defined probability as "the number of favorable ways divided by the total number of ways that an event can happen." No mention was made by Mr. English that this definition applies only when outcomes are equally likely, as suggested by Jacobs, but ignored by Phillips et al. (1986), the other source used by Mr. English in defining the vocabulary. Similarly, the textbook pages assigned by Mr. Trackman on Day 4 defined the "probability of an outcome" as the ratio: "number of favorable outcomes/number of possible outcomes" (Orfan & Vogeli, 1988, p. 384). The example given on the same page was of an experiment with equally likely outcomes, but again it was not emphasized that the definition applied only in such cases. Mrs. Talent overlooked defining probability altogether, but she clearly was using this same definition with its assumption of equally likely outcomes throughout her probability unit. If Mrs. Books defined probability, she had done so during activities earlier in the year. In the lessons observed, the underlying understanding of probability seemed to be based on equally likely outcomes.

Nearly all of the situations seen by the students in all four classrooms were those with equally likely outcomes. For example, the prizes in the boxes of cereal were equally likely to occur, marbles or cubes in a sack were equally likely to be selected, or the sections of the spinners were all of equal size. Those situations that did not have equally likely outcomes were converted into ones with equally likely outcomes, by accident or design, without any discussion distinguishing between outcomes that were equally likely and those that were not. For example, in dealing with spinners that had sections of different sizes, Mr. English and his students thought in terms of fractions and decimals. If the definition of probability as stated on Day 1 by Mr. English (without the condition of outcomes being equally likely) was applied to the spinner previously shown in Figure D.1(a), one might think the probability of red was 1 out of 3 because there is one section favorable to red out of three possible sections. However, rather than applying the definition of probability in this case, Mr. English and his students would have identified red as 1/4 or 0.25 of the circle based upon their understanding of fractions and decimals. Because fractions of a whole are based on equal-sized pieces and decimals are based on 100 equal-sized pieces, thinking in terms of fractions or decimals converted the problem into one with equally likely outcomes. However, there was no discussion of the subtle shift made in moving from applying the definition to applying their knowledge of fractions. It may be Mr. English and the students did not even realize the shift they had made. Similarly, rather than saying a dice sum of 3 is 1 out of 11 possible sums, each of the teachers used the addition chart based on the 36 equally likely ways the two dice could land and from that derived the probability of the dice sums. Thus, nearly all of the situations the students encountered either had equally likely outcomes or could at least be thought of in terms of equally likely outcomes.

No discussion in any of the observed classrooms contrasted equally likely outcomes with the outcomes that were not, such as results from dropping thumbtacks or predicting the weather. As a result, the students seemed to operate under the false assumption that outcomes were equally likely in all cases. For example, in an activity called "Quiz or No Quiz," Mr. English suggested he was going to do an experiment to see whether or not the class would take the quiz he had prepared. As he put up a triangular array (see Figure D.2), he explained he was going to toss a coin five times. Starting from the top, he would move the chip he was using as a marker left if the coin landed heads and move the chip right if the coin landed tails. If, after five tosses, the chip landed at either end, in position 1 or position 6, then the class did not have to take the quiz. If the chip ended up in the middle, then they would take the quiz. When asked what they thought the probability was of the chip landing in the outside positions, a chorus of students responded, "One third." In their minds, out of the six possible ending positions for the chip, the outer two positions were favorable, leading thus to a probability of 2 out of 6 or 1/3. The six positions, however, are not equally likely because more paths lead to the middle four positions than the outer two.

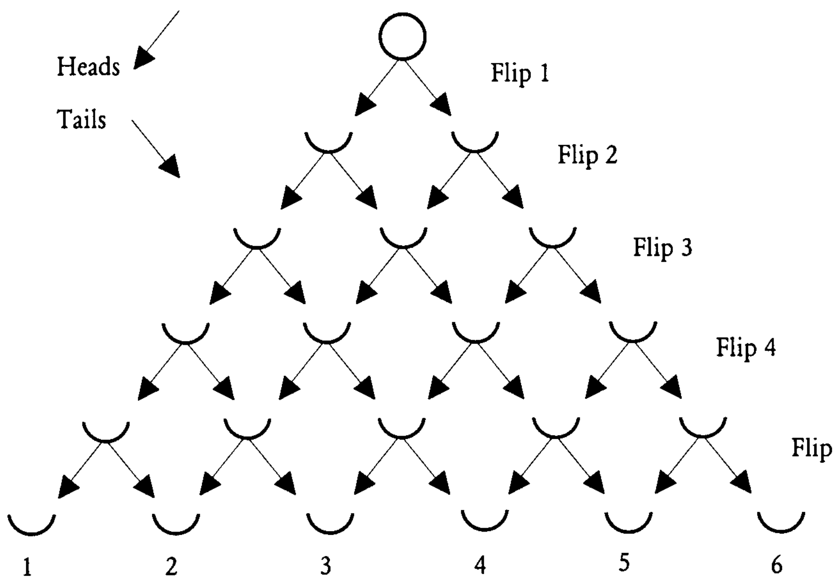


Figure D.2. Triangular array for "Quiz or No Quiz."

Thus, the teachers correctly applied the definition of probability, at least as far as they went. It is unclear whether the teachers were unaware of the equally likely assumption underlying the definition of probability; whether they were correctly assuming equally likely outcomes, but doing so intuitively; or whether their thinking was influenced by the "equiprobability bias" (Lecoutre, 1992), where random events are assumed to be equally likely by nature. However, what the teachers omitted led to a potential misconception about probability. In the case of "Quiz or No Quiz," Mr. English seemingly did not recognize the incorrect assumption the students were making and, therefore, did not address the misconception.

Distinction Between Probability and Odds

Another way of expressing the likelihood of an event occurring is to express the odds in favor of the event. The concept of odds, however, is frequently confused with probability. In settings where all outcomes are equally likely, odds in favor of an event are defined as the ratio comparing the number of favorable outcomes to the number of unfavorable outcomes. This definition contrasts with the definition of probability where the number of favorable outcomes is compared with the total number of possible outcomes. Probability, thus, is a part-to-whole ratio and odds compare part to part. For example, the probability of rolling a 4 on a die would be 1 out of 6, but the odds in favor of rolling a 4 would be 1 to 5, usually written as 1:5, because one outcome is favorable and five outcomes are not.

This concept is widely misunderstood, even in its common occurrences. For example, one state lottery prints the odds of winning the jackpot prize as 1:3,529,526 while announcing the "chances of winning the jackpot prize are 1 in 3,529,526." The first uses the symbols of odds, the second uses the language of probability. And, in fact, this representation is reporting the probability of winning. In this case, the odds of winning would be 1 to 3,529,525. This section next considers the teachers' knowledge about odds.

As revealed in the interviews. To see if the teachers in this study understood the distinction between probability and odds, the One Die problem provided a follow-up to the earlier questions. In the previous part of the problem, the teachers had been asked to identify the probability of rolling either a 2 or a 3 on one die. As a follow-up, they were asked to give the odds in favor of that event occurring. If the teachers seemed to be confused by the question or hesitant to respond, they were asked if the odds were the same as the probability. In this case, the probability of a 2 or a 3 would be 2 out of 6 or 1/3. However, the odds in favor of a 2 or 3 would be 2:4 or 1:2 because there are two favorable outcomes on the die compared to the four unfavorable outcomes.

Of the teachers interviewed all but one recognized the difference between odds and probability. Most explained that odds were the comparison of the "ways to get it" to the "ways not to get it" or of "something happening versus not happening." Although Mrs. Talent was somewhat hesitant in her response, she knew there was a difference and correctly expressed the odds in this situation. On the other hand, Mr. Trackman responded the probability and the odds "would be the same." A summary of the teachers' responses to this item was previously shown in Table D.1.

As reflected in the classroom. Mr. English was the only teacher who formally introduced the concept of odds to his students. On the vocabulary sheet handed out on Day 1, he defined the odds in favor of an event as "the number of favorable ways divided by the number of unfavorable ways." Because ratios can be expressed as quotients, the use of "divided by" could be considered correct in this case. However, it might be preferable to use "compared to" instead because two parts of the whole are involved. In any event, he was correct in stating that the odds ratio involves the number of favorable ways and the number of unfavorable ways. In reviewing the vocabulary on Day 2, Mr. English put four red trapezoids and two blue rhombi on the overhead projector and asked what the odds were in favor of selecting a trapezoid. By arranging the pieces in two groups, each with two trapezoids and one rhombus, Mr. English helped the students visualize the nature of the odds ratio.

On a number of occasions during the unit, Mr. English had the students express their theoretical results both as probabilities and as odds. Using both simultaneously initially seemed to confuse some of the students. For example, in analyzing one game with odd and even dice products on Day 4, one group of students had written 9/27 without knowing if this result was the probability or the odds (9:27 are the odds in favor of rolling an odd product with two dice). But the Montana Red Dog card game played on Days 8 and 9 provided an excellent setting for clarifying the concept of odds. Using a chart of the four suits, students determined the number of ways they could win compared to the number of ways they might lose. This activity clearly presented odds as a comparison between parts.

Using odds to express the theoretical results for games proved to be helpful in the analysis of the fairness of the games. For example, on Day 2, Mr. English used odds in the analysis of "Is This Game Fair?" In this game, the player received three points if the roll of two dice yielded a sum of 7; otherwise, the opponent received one point. After calculating the odds in favor of rolling a sum of 7 were 1 to 5, Mr. English drew a circle divided into six equal-sized pieces. He explained one region corresponded to the player winning and the remaining five corresponded to the opponent. The students readily observed that to make the game fair, the player with one way to win needed to get five points each time in order to balance out the five points the opponent would get from winning one point five times as often. In essence, getting from the 1:5 ratio of possible outcomes to a 5:5 ratio of resulting scores involved finding a common multiple of the two elements of the ratio of outcomes.

Although using the circle to represent odds effectively communicated to the students, Mr. English did not use odds in the analysis of any other game, even though it would have been very appropriate for a series of worksheets he gave the students entitled "Fair or Unfair Games." On each of these worksheets, Mr. English described the rules for a game, assigned particular payoff values, and asked the students to determine if the games were fair or unfair. In one such game, two dice were rolled and the numbers showing on top were added. Player A received five points if the sum was 7, player B received two points if the sum was greater than 7, and player C received two points if the sum was less than 7. In modeling the analysis of the game, Mr. English chose to work with the probabilities, which were 2/12, 5/12, and 5/12 for player A, B, and C, respectively. He then multiplied the numerators by the point values to determine the game was, in fact, fair. Using the earlier approach and a generalized form of odds (because there are three players), one could consider the ratio: ways of A winning:ways of B winning:ways of C winning, which, in this case, would be 2:5:5 (exactly the numerators with which Mr. English was working). Multiplying by the point values again would verify the game is fair. If it were not fair, this representation would be a good starting point toward finding a common multiple and determining what points to award each player.

Although Mr. English knew the difference between odds and probability and correctly reported the odds in many situations, he encountered some difficulties in presenting the concept of odds to his students. As Mr. English introduced both probability and odds on the first day of his probability unit, he admitted, "It bothered me a little bit at first, because I . . . I had a hard time thinking through the difference between what is probability and what's an odds." That uncertainty revealed itself in some inconsistencies in his conceptualization, representation, and interpretation of odds, particularly during the first few days of the unit and, perhaps, contributed to some of the confusion observed among the students.

On the vocabulary sheet, odds in favor of an event were correctly defined as "the number of favorable ways divided by the number of unfavorable ways." Usually, Mr. English described odds as the comparison of ways of winning to ways of losing, although on one occasion, he referred to odds as the ratio of winning compared to the chances for the other person to win. In most games between two people, these would most likely be the same. However, in one game in which a tie could occur, Mr. English suggested the probability of player A winning was 4 out of 6, but the odds were 4 to 1, even though he recognized elsewhere the two numbers "have to add up to the total ways."

There was also some confusion in how Mr. English represented odds. One problem was the language used in expressing odds. For example, in summarizing one example on the first day, Mr. English stated, "Odds is different than the probability itself, because the probability of a 6 [on a die], again, was 1 out of 6, whereas the odds of a 6 is a 1 out of 5." Expressions such as "1 out of 5" were frequently interchanged with "1 to 5" or "1 over 5." A second problem arose by expressing odds as percents. As part of an activity analyzing two-dice games on Day 3, Mr. English had the students express the odds as fractions and percents. However, on the "Fair or Unfair Games" worksheet for Day 4, he reversed himself, correctly suggesting the students "cross off the part about percent, because I don't think we need to look at percents for odds." Problems arise because both of these expressions, 1 out of 5 or 20%, imply a part out of a whole rather than a part-to-part comparison. For example, if there are 5 ways out of 36 ways of getting a dice sum of 6, giving odds as 5 out of 31 or 16% does not accurately communicate what the true odds are.

It was just such a situation that led to an error in the interpretation of odds. In the same activity on Day 3, the odds in favor of a dice sum of 7 were calculated as 1 to 5 or 20%. In summarizing the results of the activity, Mr. English concluded, "If you look at the odds of winning when you're using two dice, you can see that the highest percent chance of winning or your greatest probability of winning is 20% on the 7 or the doubles. That's the highest chance you have. . . . That means 20 times out of 100 times you play the game you're going to win." In this case, Mr. English had taken the odds of 20% and mistakenly interpreted it as a probability. He was correct in stating the greatest probability occurs on the 7 or doubles, but the probability of that happening is 1 out of 6 or approximately 17%.

Although the other teachers did not introduce the concept of odds, their failure to make a distinction between probability and odds led to potential and real difficulties in the course of teaching their probability units. In the case of Mrs. Books and Mrs. Talent, the language of probability and the language of odds were used interchangeably without any distinction being made. For example, after Mrs. Talent's students had suggested the probability of a chip landing on *A* (if it had *A* on one side and *B* on the other side) was 50%, she went on to explain "you have a 50-50 chance or a 1 out of 2 chance of getting an *A*." Similarly, Mrs. Books used both 50-50 chance and 1/2 chance in reporting what she heard students talking about. In these cases, the students seemed to understand the commonly used language, but not making the distinction may reinforce the incorrect notion that probability and odds are the same thing. That such an incorrect notion is not only possible but existed was demonstrated by what one student in Mrs. Books' class wrote in her letter concerning "Monty's Dilemma." In that letter, the student stated the following theoretical conclusions: "The Stick method has odds of 1 out of 3 or 1/3. . . . The Flip method has odds of 50-50. . . . The Switch method has odds of 2 out of 3 or 2/3."

As revealed in the pre-observation interview, Mr. Trackman believed odds and probability are the same thing. In particular, he did not understand the distinction between probability as a part-to-whole ratio and odds as a part-to-part comparison. His belief that odds and probability are the same not only misrepresented the concepts but also created some confusion for his class. However, Mr. Trackman was not the only one who was unclear about the distinction between odds and probability. While the students were working on an activity on the first day of the unit, the other sixth-grade mathematics teacher came to Mr. Trackman with a question. He also had begun teaching probability and did not understand why the book defined probability as a ratio. In his mind, ratios referred only to comparisons of one part to another part, such as the ratio of girls to boys in a classroom. Mr. Trackman explained to him that ratios could also be used in part-to-whole comparisons such as the book's definition of probability.

As a result of this conversation with his colleague, Mr. Trackman was left with the impression that the book provided a confusing explanation of probability, which it did not. Because the students would be doing the textbook assignment on Day 4 when he would be gone, Mr. Trackman decided to try to straighten out the confusion in advance. On Day 2, he asked the students what the probability was for rolling a 4 on one die. The students readily gave the probability as 1 out of 6, which Mr. Trackman wrote as 1:6 and identified as a ratio of part to whole. Mr. Trackman then continued, "Now, there's another way to write this, and it becomes confusing. [The other sixth-grade teacher] and I were talking about this yesterday, and I wanted to cover it ahead of time, so that you understood. There's another way of writing this, and it's like this [*as he writes 1:5*]. And the book goes over it this way . . . ." After the students suggested it had been "rounded," Mr. Trackman explained, "What they did is they compared part . . . to other parts." He provided a second example of buying six tickets, one of which would be a winner. Then he explained, "This is like saying, if one of them is a winner, five of them are gonna be losers. Or, for every one winner, there's five losers. And that's the way the book explained it. And the book will list both of them. So you have to be aware of what you're really looking for." Mr. Trackman concluded his presentation with a general illustration.

> What it's like . . . you've got a guy standing here [point *A* in Figure D.3] and he needs to get over here [point *B*]. And two people give him sets of directions. One guy says, "Go straight [east]; turn left." The other guy says, "Go straight [north]; turn right." . . . Two different ways to get to the same point. . . . These [1:6 and 1:5] are kinda saying the same thing, they're just saying it a different way.

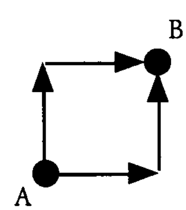


Figure D.3. Example used by Mr. Trackman of "two . . . ways to get to the same point."

Mr. Trackman returned to this issue in reflecting on what difficulties students encountered with the probability unit. He pointed out the students could not "differentiate between odds, between probability, between the different outcomes, the way they're saying it . . . whether 1 out of 5 possibility or a ratio of 1 to 4. They don't realize that's necessarily the same thing." He went on to admit, "it's very confusing for adults that know what they're talking about." Because Mr. Trackman himself did not understand the distinction between probability and odds, it is perhaps no surprise that his students were confused.

Experimental and Theoretical Probability

Besides the definition of probability in situations with equally likely outcomes, a second way of determining the likelihood of an event is to repeat an experiment a number of times and find the relative frequency the event occurred. This is called the *experimental probability*. In some cases, such as when rolling weighted dice, experimental probability may be the only way of determining the likelihood of an event. Such empirical approaches would also be used to determine the likelihood of an 18-year-old male driver being involved in an automobile accident in a given period of time.

In other cases, a *theoretical probability* can be assigned to an event based on ideal occurrences. In order to determine theoretical probability, one must be able to identify the possible outcomes and to determine what the likelihood of the outcomes would be if the "perfect" experiment were conducted. Various strategies can be applied to determine this likelihood. In contrast to experimental probability which may vary from trial to trial, theoretical probability does not. This section next explores the teachers' understanding of experimental and theoretical probability in the case of simple experiments.

As revealed in the interviews. The second probability knowledge question presented the teachers with the following scenario in the Two Coins problem (probability question #2):

> Jose and Cathy have conducted an experiment tossing two coins. They have recorded the following results:
>
> | | |
> |---|---|
> | both heads | 5 times |
> | one head, one tail | 8 times |
> | both tails | 7 times |
>
> According to their results, what is the experimental probability that
> (a) both coins land tails up?
> (b) one coin lands tails, the other lands heads?

By adding together the times each outcome had occurred, all of the teachers found the experiment had been conducted 20 times. From there, each correctly determined the experimental probabilities asked for were 7/20 and 8/20, respectively. A common concern among the teachers was expressing the fractions in simplest form. Mr. English went one step further, by correctly expressing the experimental probabilities as 35% and 40%, respectively. Rather than explicitly applying the idea of relative frequency, the teachers seemed to adapt the definition of probability to fit the experimental situation, considering the number of favorable trials compared to the total number of trials. The end result is a relative frequency, but it was not clear if the teachers were aware of the subtle shift they had made.

After providing the experimental probabilities in the Two Coins problem, the teachers were asked to determine the theoretical probability for the same two events. In particular, the teachers were asked to determine the probability that both coins land tails up and the probability that one coin lands tails and the other lands heads. In this case, there are four possible outcomes: both coins land heads (HH), the first coin lands heads and the second coin lands tails (HT), the first coin lands tails and the second lands heads (TH), and both coins land tails (TT). Assuming the coins are fair, each of these four outcomes (HH, HT, TH, and TT) would be equally likely to occur. Therefore, applying the definition of probability in equally likely situations, the probability both coins land tails up would be 1 out of 4 or 1/4. Because two outcomes result with one head and one tail, the probability of that event would be 2 out of 4 or 1/2.

In responding to this question, Mr. Trackman initially suggested, "There's three possibilities: two heads, two tails, and one head and one tail." However, he quickly corrected himself as he realized there were actually four outcomes which he listed as HH, TT, HT, and TH. Mrs. Talent drew a tree diagram as in Figure D.4 to determine the four outcomes of HH, HT, TH, and TT. Mr. English suggested one could draw a tree diagram, but did not do so. However, he and Mrs. Books provided a list of outcomes similar to Mrs. Talent's. Although no one specifically mentioned these outcomes would be equally likely, they all made that assumption as they identified the probability of two tails as 1 out of 4 and the probability of one head and one tail as 2 out of 4 or 1 out of 2. Table D.2 summarizes the teachers' responses to the Two Coins item.

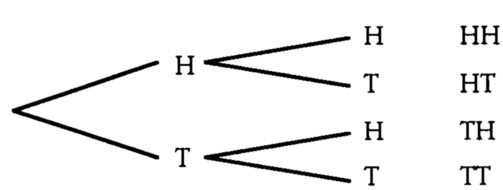


Figure D.4. Tree diagram drawn by Mrs. Talent for the Two Coins problem.

Table D.2
Summary of the Teachers' Responses to the Two Coins Problem (Probability Question #2)

	Number of teachers responding correctly	Number of teachers responding incorrectly
Experimental probability	4	0
Theoretical probability	4	0

As reflected in the classroom. Experimental and theoretical probability provided the general structure for the probability units and finding both experimental and theoretical probability in a variety of settings was the principal activity involved in the probability units. This approach to probability was true for all four teachers, although their understanding and use of the vocabulary varied somewhat.

Although much of the activity in all four classrooms centered around finding experimental and theoretical probabilities, only two of the teachers used the terms to any significant degree. Experimental and theoretical probability were two of the vocabulary terms Mr. English defined for the students at the start of the probability unit. Similarly, as Mrs. Books began the "Cereal Boxes" activity, she reminded her students of the process they had applied in an earlier activity. In particular, this process involved giving a subjective estimate, doing an experiment, and then considering the "theoretical piece." On one day, Mrs. Talent identified the results the students had found as the experimental probability. But, other than that, the terms experimental or theoretical probability were rarely, if ever, heard in her classroom or in Mr. Trackman's classroom.

Mr. English's use of the term *theoretical probability* in class activities generally demonstrated he understood the term in situations involving chance occurrences. However, references to theoretical probability in situations involving sampling revealed some confusion. For example, in one sampling activity, Mr. English had his students count the number of pieces of gum underneath their table and set up a ratio of the number of pieces of gum to the number of tables. He gave the example of a 5 to 1 ratio and referred to this as the theoretical probability. In a second sampling activity, Mr. English had the students count the number of vowels and the total number of letters in a short written passage. He then also referred to the resulting ratio of vowels to total letters as a theoretical probability. In these cases, his use of theoretical probability to describe a gum ratio or the frequency of vowels was not clear. Admittedly, interpreting theoretical probability and even experimental probability in such sampling situations is not a clear-cut matter. Certainly a ratio of five pieces of gum to one table does not fall within the normal range from 0 to 1 of values for any probability. One could perhaps argue that the actual number of pieces of gum in the room might be the theoretical value and what the students observed at their own tables is an experimental result, but neither would be a probability. In the case of the frequency of vowels, it depends somewhat on the purpose of the count. If one were going to randomly select a vowel from the passage, then the students' count could be used as the theoretical basis for computing probabilities. However, if one wants to use the count to predict the frequency of letters in another passage, which to some extent was the goal of this activity, then this sample seems to play the role of an experimental sample. One might consider the theoretical value to be the actual frequency of vowels in the target passage. Or, perhaps, it is the incalculable frequency of vowels in all that has ever been written. As in these examples, applying the terms theoretical probability and experimental probability to situations involving sampling is difficult to do.

In the instances when Mrs. Books used the terms experimental probability and theoretical probability, she did so correctly. Mrs. Talent's one use of experimental probability was also appropriate. However, Mrs. Talent and Mr. Trackman did not take advantage of a number of other opportunities when it would have been appropriate to use the vocabulary in distinguishing between experimental and theoretical results. Instead, in comparison to the experimental results, Mrs. Talent and Mr. Trackman referred to theoretical probability as "how it comes out mathematically" or as "one way of looking at it . . . apart from the numbers [the experimental data] that you got."

Basic Properties of Probability

A number of basic properties can be applied to help identify the probability of various events. These properties deal with the probability of a complement, the possible range of probability values, and the sum of the probabilities of all outcomes in the sample space. First, the event that something happens and the event that the same thing does not happen are called *complements* of each other. The probabilities of complementary events always add up to 1, which leads to the following property: P(complement of event E) = 1 - P(event E). An additional property states the probability of an event must be in the range between 0 and 1, inclusive. At one end of this range are events that cannot occur, or *impossible* events, whose probability is 0. At the other end are events that are *certain* to happen which have a probability equal to 1. Finally, a property similar to the result for a complement states that the sum of the probabilities of all outcomes in the sample space equals 1. This property can be used, for example, to find the probability of one outcome if all the other probabilities are known.

As revealed in the interviews. To explore the teachers' understanding of the basic properties of probability, they were given the following situation in the Marbles problem (probability question #3):

> A bag contains some red, some white, and some blue marbles. The following probabilities are given:
> - The probability of drawing a red marble is 1/3.
> - The probability of drawing a white marble is 1/2.
>
> (a) What is the probability of *not* drawing a red marble?
> (b) What is the probability of drawing a green marble?
> (c) What is the probability of drawing a blue marble?

The basic properties described can help identify the probability of these events. For example, the first question asked of the teachers deals with a complement. Therefore, the probability of "not drawing a red marble" can be found by subtracting the probability of drawing a red marble, namely 1/3, from 1, giving a result of 2/3. Although none of the teachers used the term complement, they correctly identified the probability of not drawing a red marble as 2/3. As Mr. English pointed out, "If you look at all the events [and] add them all together, it's got to equal one whole. And there's three thirds in a whole, of which one third is red. That means that two thirds are not red."

In the scenario presented to the teachers, there were no green marbles in the bag. As a result, because it was impossible to draw a green marble from the bag, the probability of that event would be 0. Some of the teachers suspected they were being tricked when asked to find the probability of drawing a green marble, thinking they had not been given all the information. When it was clarified the bag contained only red, white, and blue marbles, all of the teachers recognized there would be "zero chances" of drawing a green marble because "there aren't any." In this case as well, the teachers did not use the term impossible event.

The property dealing with the sum of probabilities can be used to find the probability of drawing a blue marble from the bag. There are three possible outcomes in this experiment; therefore, P(red) + P(white) + P(blue) = 1. Because the probabilities of drawing a red marble and of drawing a white marble are given as 1/3 and 1/2, respectively, the probability of drawing a blue marble can be calculated to be 1/6. The teachers all used this idea in finding the probability of selecting a blue marble from the bag. As Mrs. Books read the problem initially, she pictured the situation using an area model as shown in Figure D.5.

R	W	W
R	W	B

Figure D.5. Area model drawn by Mrs. Books for the Marbles problem.

As Mrs. Books was drawing the picture, she explained, "Just looking at the area model, and knowing that 1/3 and 1/2 would both fit in nicely into 6 . . . and we have red marbles are 1/3 of what is in our bag. Two sixths is also equal to 1/3. Um, the white marbles in there are 1/2. And half of my diagram would be 3/6. And that would leave us that our blue marbles are 1/6." The other teachers used similar reasoning, adding the 1/3 and 1/2 to get 5/6 and recognizing the blue would be the 1/6 that was left. A summary of the teachers' responses to the Marbles problem is given in Table D.3.

Table D.3
Summary of the Teachers' Responses to the Marbles Problem (Probability Question #3)

	Number of teachers responding correctly	Number of teachers responding incorrectly
Complement	4	0
Impossible event	4	0
Sum of probabilities	4	0

As reflected in the classroom. Applications of the basic properties of probability were seen in the lessons and/or assignments in three of the classrooms observed. No evidence of the properties was observed in the context of the simulations conducted in Mrs. Books' classroom, although the properties may have been introduced in earlier activities. However, although present in the lessons and/or assignments, the concept of complement, the range of probability values, and the sum of probabilities generally were neither stated explicitly nor emphasized in the course of instruction with a few exceptions. In general, most examples or applications of the basic properties of probability occurred on the assignments the students were given rather than in class discussions. However, on most occasions, when the assignments were corrected, only the answers were read. No discussion identified the properties or checked the students' understanding of them.

The occurrences of the idea of complement were generally limited to items on textbook or worksheet assignments. For example, three exercises on one of the textbook pages assigned by Mr. Trackman asked the students to find the probability of the complement of an event or, in other words, the probability something would not happen. These exercises seemed to rely on the students' understanding of the common sense meaning of *not* without ever leading the students to any understanding of it from a mathematical perspective. Similarly, some of the worksheets assigned by Mrs. Talent included items involving the complement. One that proved particularly troublesome to the students was finding the probability of spinning "4 or not 4" on a spinner with sections numbered 1 to 8.

Mr. English used the idea of complement initially in the opening historical example. In this case, Mr. English described a game people had been playing in a gambling den of the 1600s. If the player could roll a die four times without once getting a 6, the player would win his bet and receive a payoff. In finding the probability of not getting a 6, Mr. English explained, "If you have six numbers . . . on a die, one of them is a 6, so you have a 1 out of 6 chance to lose [by getting a 6]. . . . On the other hand, you have 5 out of 6 chance that you will not get that 6 on the die. Right? Because there's five other numbers on the die." Mr.

English relied on the students' intuition and the relative simplicity of the situation in finding the probability, as did the other teachers. His students also encountered the idea of complement in questions appearing on homework assignments, where they were asked to find the probability of not getting such and such. The worksheet completed after the introduction of the vocabulary on the first day asked the students to fill in the blank: "The probability that something will happen plus the probability that it will not happen always equals ___." Ironically, the students were asked to draw the conclusion before they had seen any examples or there had been any discussion other than the historical example quoted.

In contrast to the other basic properties, the one explicitly stated was that the values of probabilities range from 0 to 1 or from 0% to 100%. This property was explicitly stated and repeated by both Mrs. Talent and Mr. English. In Mr. English's class and in a textbook assignment given by Mr. Trackman, the terms *impossible event* and *event certain to happen* were used to describe the events whose probabilities were 0 and 1, respectively. In addition, in Mr. English's class, the idea of certain and impossible events kept recurring throughout the unit in various formal and informal situations. For example, on several occasions, Mr. English reminded the students at the end of class that "there is a probability of 0 that you will leave if anybody is out of his seat. And there is a probability of 1 that you will leave on time if everybody is in their seat."

In assignments made by Mr. Trackman and Mr. English, students were asked in one exercise to add up the probabilities of all the outcomes for a given experiment. Presumably they would find the sum to be 1, but it was not made explicit that these were all possible outcomes and that they were nonoverlapping outcomes. In both cases, a follow-up question had the students find an unknown probability if the probabilities of the other outcomes were given. For example, the students were asked to find the "probability of choosing the third color" if the probability of choosing a green sock was 1/10, the probability of choosing a red sock was 3/10, and there were only three colors of socks. In one other case, when presenting the Coin Tossing Exploration, Mr. Trackman again observed, without further explanation, the sum of the probabilities would be 1.

A related property was explicitly stated in the portions of the textbook assigned by Mr. Trackman. To find the "probability of two events that cannot occur at the same time," students were told to "find the sum of the probabilities of the outcomes." In addition, a worksheet assigned by Mrs. Talent suggested the students should find the P(2 or 8) by adding P(2) and P(8), which referred to outcomes on a spinner. Unfortunately, examples of situations where adding probabilities would not apply were not used to help students understand when it was appropriate to add probabilities and when it was not. For example, one exercise on a worksheet assigned by Mrs. Talent and Mr. English asked the students to use the results from a poll to find the probability a student chosen at random "rides a bike or opposes the rule." In this case, the two outcomes joined by the *or* were not mutually exclusive or, in other words, they could have occurred at the same time. As a result, it was not appropriate to add the probability the student rides a bike and the probability the student opposes the rule. However, when the assignment was corrected there was no discussion of how students solved the problem or even if they had solved it correctly. Further, the nature of many of these exercises made it possible for students to add together the favorable outcomes rather than adding together the probabilities. For example, if there are six red pens and three blue pens in a jar containing 18 pens, then the probability of selecting either a red or a blue pen is 9/18 because there are 9 ways of selecting either a red or a blue pen out of the 18 possibilities. This procedure would be a common sense approach, but no connection was made between this strategy and adding the probabilities.

Strategies for Analyzing Probability Situations

The situations or experiments considered so far, including rolling dice, tossing coins, or drawing marbles from a bag, are typical of the simple experiments seen in middle school textbooks and classrooms. Other experiments included in the interview were more complex, going beyond what may commonly be seen at the middle school level. In particular, these involved situations with multiple stages and unequally likely outcomes or binomial outcomes. Although somewhat more complicated than middle school problems, the strategies applied to solve them are among those introduced at the middle school level, including tree diagrams, the area model, and Pascal's triangle. This section will explore the teachers' knowledge and use of these and other strategies.

As revealed in the interviews. The spinner shown in Figure D.6 illustrates an experiment where the outcomes are not equally likely. To find the probability, for example,

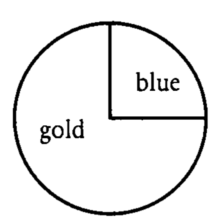


Figure D.6. Sample blue/gold spinner.

of spinning the spinner twice and getting matching colors on both spins, one could apply the strategies used in the simple experiments. In particular, the spinner could be subdivided into equally likely regions and a tree diagram drawn to determine the outcomes (see Figure D.7).

From the diagram, it can be seen that, out of the 16 equally likely outcomes, one results in matching blues and nine result in matching golds. Therefore, the probability of spinning matching colors would be 10 out of 16 or 5/8. However, as can be imagined, this process could get complicated very quickly, as the number of outcomes and/or stages of the experiment increase.

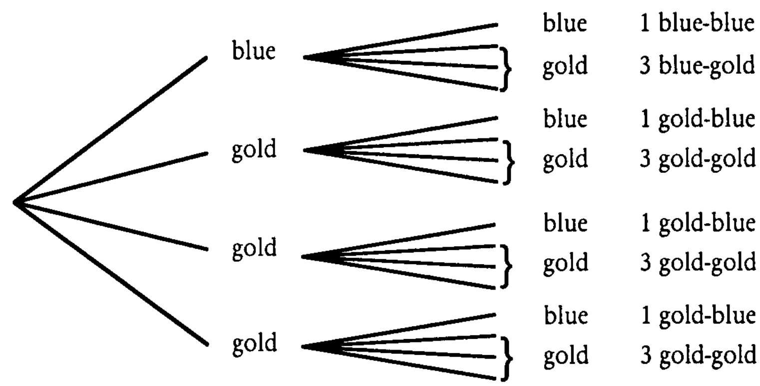


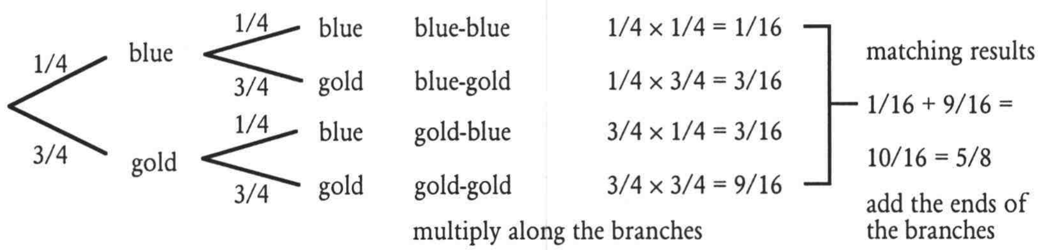
Figure D.7. Outcome tree diagram for the blue/gold spinner example.

In situations like this example, two properties can be applied to simplify the process of determining probabilities. First, "to find the probability of several things happening in succession, [one can] multiply the probabilities of the individual happenings" (Jacobs, 1982, p. 482). In the example of the spinner shown previously in Figure D.6, the probability of getting a blue on the first spin and a blue on the second spin would be $1/4 \times 1/4$ or 1/16. One could justify this result by considering the theoretical outcome of spinning the spinner 16 times. Theoretically, one fourth of the 16 spins, or 4 spins, should result in blue on the first spin. On one fourth of these 4 spins, or on 1 spin, the blue on the first spin will be followed by a blue on the second spin. Thus, blue on both spins would occur 1 out of 16 times, confirming the result of multiplying. Similarly, one could find the probability that both spins would be gold by multiplying $3/4 \times 3/4$ which equals 9/16.

A second property states that to find the probability of two or more events that cannot occur at the same time, one adds the probabilities of the individual events. For example, because two blue spins cannot occur at the same time as two gold spins, the probability of spinning matching blues or spinning matching golds would be the sum P(2 blues) + P(2 golds) or $1/16 + 9/16$ which equals 10/16 or 5/8, which agrees with the results from the outcome tree (see Figure D.7).

Two strategies that could be applied to solve problems such as these would be a probability tree diagram or the area model as shown in Figure D.8. Both strategies apply the properties of multiplying and adding as indicated in the figure. The probability tree in Figure D.8 simplifies the outcome tree shown previously in Figure D.7 by combining branches where the outcomes are the same and weighting the branches according to the probabilities.

Probability Tree Diagram



Area Model

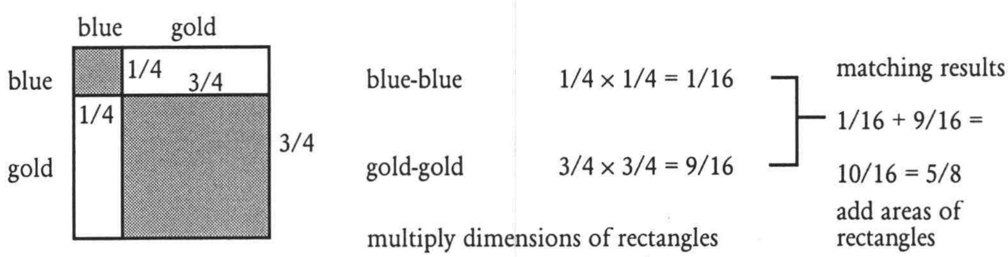
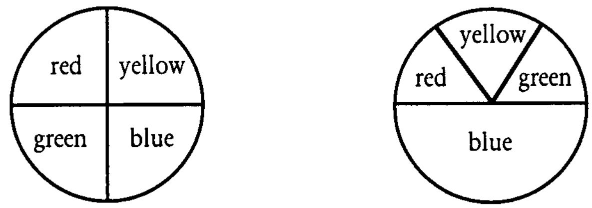


Figure D.8. Two strategies for solving the blue/gold spinner example.

In the probability knowledge interview, the teachers were presented with a similar, although somewhat more complicated, situation. The Two Spinners problem gave the teachers the following scenario (probability question #5):

> Three students are spinning to get one red and one blue on the given spinners (see Figure D.9).
>
> > Mary chooses to spin twice on Spinner A;
> > John chooses to spin twice on Spinner B; and
> > Susan chooses to spin first on Spinner A and then on Spinner B.
>
> Who has the best chance of getting one red and one blue (in any order)? (Lappan & Even, 1989)

The solution to this problem could be illustrated either with a probability tree or an area model. The solution using a probability tree diagram is shown in Figure D.10. From that solution, one sees John and Susan have an equal chance of getting a red and a blue in any order and both have a better chance than Mary.



Spinner A Spinner B

Figure D.9. Spinners A and B for the Two Spinners problem.

Each of the teachers could correctly identify the probability of spinning a red or the probability of spinning a blue when considering the individual spinners separately. In all cases, their initial approach to the problem was to write down the probability of a red and the probability of a blue on the spinners each person was spinning. However, two major difficulties arose as they attempted to analyze the problem further.

Mary (Twice on Spinner A)

$$\left(\frac{1}{4}\times\frac{1}{4}\right)+\left(\frac{1}{4}\times\frac{1}{4}\right)=\frac{1}{8}$$

John (Twice on Spinner B)

$$\left(\frac{1}{6}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{6}\right)=\frac{1}{6}$$

Susan (Spinner A, then B)

$$\left(\frac{1}{4}\times\frac{1}{2}\right)+\left(\frac{1}{4}\times\frac{1}{6}\right)=\frac{1}{6}$$

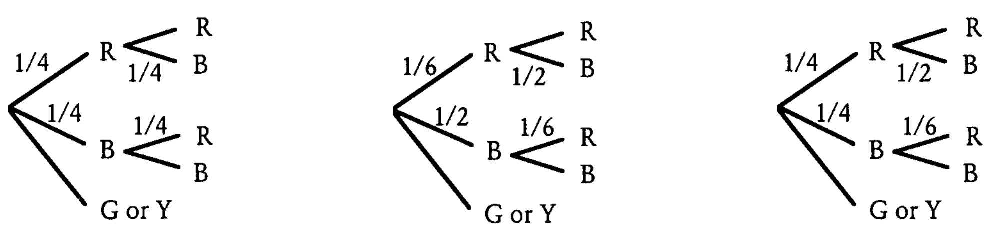


Figure D.10. Probability tree diagram solution for the Two Spinners problem.

First, the teachers did not adequately consider how order influenced the probabilities. In particular, the question had asked about "getting one red and one blue (in any order)." In considering the strategies of the first two students, all teachers wrote down 1/4 and 1/4 for Mary and 1/6 and 1/2 for John. These values account for the probability of getting a red followed by a blue, but ignore the possibility of getting a blue first and then a red. When they came to Susan, three of the teachers realized there were two possibilities (see Figure D.11). As Mr. Trackman observed, "That one really messes things up, depending on which order and which one she gets." In writing down the probabilities for Susan, Mr. Trackman and Mrs. Books first wrote down 1/2, the probability of either a red or blue on the first spin. They then branched from the 1/2 to show probabilities on the second spin of 1/2 for blue or 1/6 for red. Although recognizing the two possibilities, lumping together the red and blue on the first spin would lead them to an incorrect answer. On the other hand, Mrs. Talent correctly wrote down two separate outcomes, considering 1/4 (for red) and 1/2 (for blue) first and then 1/4 (for blue) and 1/6 (for red). Mr. English, remaining consistent in thinking only of red followed by blue, wrote down 1/4 and 1/2 for Susan.

$$\frac{1}{2} \left< \begin{matrix} \frac{1}{2} \\[2ex] \frac{1}{6} \end{matrix} \right. \qquad\qquad \begin{matrix} \frac{1}{4} & \frac{1}{2} \\[2ex] \frac{1}{4} & \frac{1}{6} \end{matrix} \qquad\qquad \begin{matrix} \frac{1}{4} & \qquad \frac{1}{2} \end{matrix}$$

Mr. Trackman & Mrs. Books — Mrs. Talent — Mr. English

Figure D.11. Teachers' strategies for Susan's spins on the Two Spinners problem.

The second difficulty, once the separate probabilities were written down, was what to do with the probabilities. As Mr. English pointed out, "I'm trying to think. Do I add or do I multiply?" He eventually concluded incorrectly that "it doesn't matter whether I multiply or add . . . it's that third one. She's got the best chance." Similarly, Mr. Trackman seemed to compare just the fractions he had written down, also incorrectly suggesting as a "guesstimate . . . based on my chicken scratch" that Susan had the best chance. Mrs. Talent incorrectly focused on adding the fractions, with her results also favoring Susan's strategy.

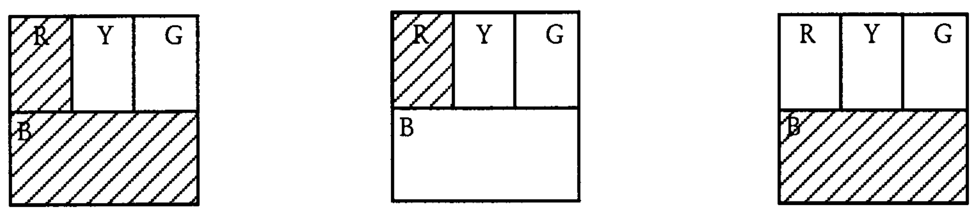
Mrs. Books became quite involved in trying to solve the problem, reporting, "I love these complex problems that we can think about." Although she had initially ignored order, after further consideration, she recognized Mary could get a red and blue in either order. She then wrote down the probabilities as in Figure D.12, concluding, "Mary's chances of getting what she wants in either direction is going to be 1/16." She went on to question, "There's a part of me that's looking at . . . . You basically have two spinners. They are independent. So I'm wondering if it's not feasible to look at it that there's 1/16 of [red-blue] happening. There's 1/16 of [blue-red] happening so the chances that she is going to succeed would be 1/8." Although somewhat hesitant, Mrs. Books was correct in thinking she should add the two probabilities. She then pointed out another way of looking at the problem, namely that

red	$\frac{1}{4}$	$\frac{1}{4}$	=	$\frac{1}{16}$
blue	$\frac{1}{4}$	$\frac{1}{4}$		

Figure D.12. Mrs. Books' work on the Two Spinners problem.

"she has half a chance on the first spin of getting either red or blue. Her second spin, though, she's limited to just 1/4 possibility of what she wants." Using similar reasoning, Mrs. Books observed John had 4/6 chance of getting either a blue or red on his first spin. However, she became stumped in dealing with the second spin because the colors she would need to get were not equally likely. She then began to draw area models corresponding to the spinners. For example, in considering John's strategy, she drew the picture in Figure D.13.

$\frac{4}{6}$ | $\frac{1}{6}$ OR $\frac{1}{2}$



[Blue or red on first spin of Spinner B] | [Red on second spin of Spinner B] OR [Blue on second spin of Spinner B]

$$\frac{4}{6} \times \frac{1}{6} = \frac{4}{36} = \frac{1}{9}$$

[blue-red]

$$\frac{4}{6} \times \frac{1}{2} = \frac{4}{12} = \frac{1}{3}$$

[red-blue]

Figure D.13. Mrs. Books' area models for the Two Spinners problem.

She then multiplied 4/6 × 1/6 getting 4/36 or 1/9 for spinning blue and red and 4/6 × 1/2 getting 4/12 or 1/3 for spinning red and blue. After drawing a similar picture for Susan and determining she had a 1/12 chance of getting blue and red and 1/4 chance of getting red and blue, she concluded John had a better chance of getting one red and one blue because "he has a 1/3 chance of getting it and then 1/9 of a way, which is bigger than 1/4 and 1/12 combined, and definitely bigger than 1/8." Although she was on the right track with her analysis of Mary and had a sense of when to multiply and when to add, she confused the issue by incorrectly combining the probabilities corresponding to the first spin. As a result, the probability she calculated mistakenly included the red-red and the blue-blue outcomes. Further, she did not make the transfer between the area model for a one-stage experiment where the region is the whole to the area model representation of a two-stage experiment where the sides of the square represent the whole for each stage, ranging from 0 to 1. Figure D.14(b) demonstrates the correct area model solution corresponding to John's strategy.

(a) Mary (Twice on Spinner A) (b) John (Twice on Spinner B) (c) Susan (Spinner A, then B)

$$\left(\frac{1}{4}\times\frac{1}{4}\right)+\left(\frac{1}{4}\times\frac{1}{4}\right)=\frac{1}{8} \qquad \left(\frac{1}{6}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{6}\right)=\frac{1}{6} \qquad \left(\frac{1}{4}\times\frac{1}{2}\right)+\left(\frac{1}{4}\times\frac{1}{6}\right)=\frac{1}{6}$$

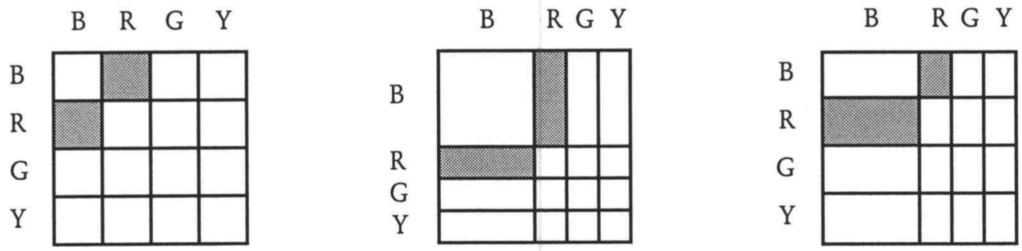


Figure D.14. Area model solution for the Two Spinners problem.

To explore the teachers' thinking and reasoning skills in probabilistic settings, a follow-up question was asked involving the two spinners.

> If you first could spin your choice of the spinners and observe the outcome and THEN decide which spinner to spin second, can you devise a strategy with a greater probability than either Mary, John, or Susan of obtaining one red and one blue?

In talking about this problem the teachers made a number of observations. Mr. English pointed out "your highest potential for red is 1/4 [on Spinner A] and the highest potential for blue is 1/2 [on Spinner B]." As a result, Mr. English suggested, based on his intuition, that he would do a combination of the spinners. Mrs. Books indicated she would spin Spinner B first because "we have 4 out of 6 ways of getting [red or blue]." She then suggested she would switch to Spinner A to take advantage of the 1 out of 4 ways of getting the second color. However, in calculating the probability for this strategy, she was surprised it did not turn out better than John's strategy (which she had calculated incorrectly as 1/3 for red-blue and 1/9 for blue-red). Similarly, Mrs. Talent also suggested she would spin on Spinner B first because the 2/3 chance to "get the colors you want" would be greater than the 50-50 chance on Spinner A. After indicating, "I might just insist that I want to spin on B twice," Mrs. Talent began to see the potential for choosing after she knew the result of the first spin. Changing her mind, she then suggested if she got a blue on the first spin, she would spin on Spinner A because the chances for a red were better. If the first spin was red, then she definitely wanted to spin Spinner B again. Using similar intuitive arguments, Mr. Trackman explained his strategy.

> I would spin Spinner B first, because the probability of getting the blue, you could assure yourself of the blue. . . . Your percentages of getting what you want is 2/3 as opposed to 1/2 by spinning Spinner A first. If you get the red, you stay on Spinner B and spin for your blue. So I would say . . . well, that's

> what I would say. I would spin B first and if you got the red stay with B, and if you got the blue go to Spinner A.

A probability tree diagram (see Figure D.15) illustrates that, using this strategy suggested by Mrs. Talent and Mr. Trackman, the probability of getting one red and one blue in any order would be 5/24, which is slightly better than John's and Susan's 1/6 chance. With the exception of Mrs. Books, the teachers had not arrived at probability values for the three students' strategies and did not calculate the probability of their strategy as justification that it was better. Mrs. Books, having calculated the students' probability (although incorrectly for John and Susan), seemed to be caught up in the calculations rather than taking a more general look at the problem. Table D.4 summarizes the teachers' responses to the Two Spinners questions.

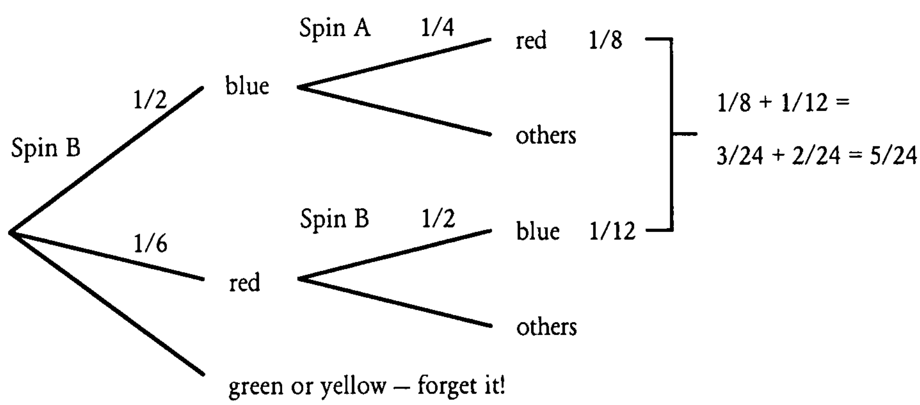


Figure D.15. Best strategy for the follow-up to the Two Spinners problem.

Table D.4
Summary of the Teachers' Responses to the Two Spinners Problem (Probability Question #5)

	Number of teachers responding correctly	Number of teachers responding incorrectly
Identified probabilities on individual spinners	4	0
Considered possible order of outcomes for all three students	1	3
Multiplied to find probabilities	1	3
Obtained correct probabilities		
Mary	1	3
John	0	4
Susan	0	4
Devised strategy to maximize chances	2 (intuitively)	2

Experiments which consist of a sequence of smaller identical experiments each having two possible outcomes are called *binomial.* Tossing coins is an example of a binomial experiment because there are only two outcomes, heads or tails, on each toss. Another example of a binomial experiment was presented to the teachers in the Birth problem (probability question #4).

> The ratio of boys to girls born is generally about 50:50. For families with five children, what is the probability of having four girls and one boy (in any order)?

There are a number of ways of finding this probability. Because there are two possible outcomes for the first child and, for each of these outcomes there are two possible outcomes for the second child, there would be 2×2 or 4 different outcomes for two children (comparable to the four outcomes for tossing two coins). Extending this pattern suggests there would be $2 \times 2 \times 2 \times 2 \times 2$ or 32 different outcomes for the birth order of five children. Because the question involves one boy, one might realize the boy could be born in any one of the five positions: first, second, third, fourth, or fifth. Therefore, there would be 5 outcomes with four girls and one boy out of the 32 possible outcomes for a probability of 5/32. An organized list of outcomes or a tree diagram (see Figure D.16) would be other ways of determining the favorable and possible outcomes. Pascal's triangle (see Figure D.17) is another strategy which applies in cases of binomial probability and

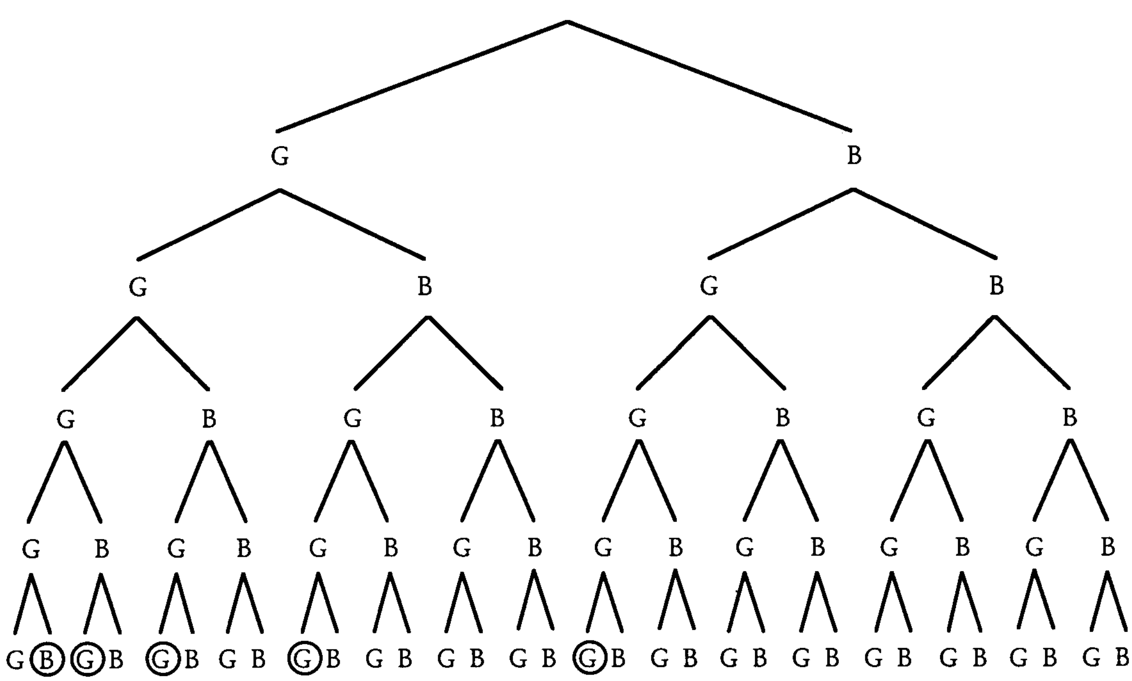


Figure D.16. Outcome tree for the Birth problem.

summarizes the outcomes from the tree diagram. In this case, with five children in the family, the corresponding row of the triangle is 1 5 10 10 5 1. The numbers in this row indicate there is 1 way of getting all girls, 5 ways of getting four girls and one boy, 10 ways of getting three girls and two boys, and so on. Finding the sum of the numbers in the row provides the number of possible outcomes. The results from the triangle confirm the probability of 5/32 found earlier.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Figure D.17. Pascal's triangle for the Birth problem.

In the pre-observation interview, the teachers were first asked to solve the problem. Then, to explore their understanding of the connections or relationships between the various strategies, the teachers were asked if they could determine the solution in any other way. Additionally, to investigate what impact teaching the unit may have had on the teachers' knowledge, this question was revisited in the post-observation interview.

When initially asked the question, Mrs. Talent suggested she could draw a tree diagram and Mrs. Books began to draw a tree diagram. However, realizing the tree diagram was going to be complicated, both teachers resorted to the multiplication property as an alternative. The multiplication property was also the first strategy applied, in one form or another, by Mr. Trackman and Mr. English. Using the multiplication property, Mrs. Books, Mrs. Talent, and Mr. English found a probability of 1/32, with reasoning something like, "I would say the probability of having a girl the first time is 1/2. The probability of having a girl second time is 1/2. . . . Third time is still 1/2, fourth is 1/2. And then a boy would be 1/2, so you have $(1/2)^5$ which is 1/32." This result correctly finds the probability of one particular birth sequence, namely GGGGB. However, it ignores the different orders in which the four girls and one boy may occur. Mr. Trackman had a slightly different approach to the problem, although still using the multiplication property. In what he identified as an uneducated guess, Mr. Trackman responded,

> [It] looks like the probability would be cut in half each time. So . . . you have a boy and a girl, I would . . . I would have to say that they cancel each other out [*crossing off one boy and one girl*]. To have the girl, 50-50 chance. To have

> the next girl I would say 25% chance. Then to have the next one would be 12.5%. So to have that kind of order, I'd go with 12.5%, or 1 out of 8 . . . uh, probability, off the top of my head.

He incorrectly crossed off the boy and girl and, as a result, found only the probability of having three girls in a row.

When asked how else they might solve the problem, Mr. Trackman and Mrs. Books did not offer any further strategies in the first interview. When prompted to consider other strategies, Mrs. Talent and Mr. English both realized the problem said "in any order." Wondering if that might make a difference, Mrs. Talent began drawing a tree diagram, pointing out, "I never used to use these tree diagrams very much, but the more I use them with kids, the more it helps me . . . keep myself organized." As Mrs. Talent began drawing the branches for the fifth child, she stopped and began considering which sequence of branches would yield four girls and one boy. After finding four such sequences (and missing one), she concluded the probability was 4 out of 32 or 1/8. Trying to reconcile this result with her earlier answer of 1/32, Mrs. Talent observed, "because the girls could be in any order, that's where you get the 4 out of 32." As Mr. English began considering order, he recognized it was "the same thing as heads-heads-heads-heads and then tails. It's the same thing as flipping the coins. And if the order is not important . . . I could make a list." He began to make a list, putting the boy in the last position and then moving the boy to the fourth position. At this point he recognized there would be five outcomes, so that the result would be 5 out of 32. When asked about other approaches, Mr. English began thinking of other representations besides coins that could be used in presenting the problem to students. He concluded you could use "anything that has two sides. You could do dice, odd, even. You could do cards, red or black, 'cause half the cards in the deck are red, if you take the jokers out. The other half are black. . . . Anything that is represented by a binomial situation." Thus, Mr. English not only recognized various strategies could be applied to this problem, his response also revealed connections he saw between this situation and other structurally similar problems.

In the course of teaching their probability units, both Mr. Trackman and Mr. English had done activities based on Pascal's triangle. As a result, this strategy was the one they chose to use when the question was revisited in the post-observation interview. Writing down the first few rows of Pascal's triangle, Mr. English provided the following explanation: "Okay. This is . . . five children, so I need to go down to row 5 . . . 10, 10 [*as he writes down that row*] . . . . What's the probability of having four girls and one boy, in any order? Okay, this would be five girls [*referring to the first 1 in the row*]. This is . . . there's 32. And this would be four girls and one boy [*referring to the 5 appearing next*]. And that's what you're asking, so it's 5 out of 32. If I used the right row . . . 'cause I don't have my . . . is that right?" Similarly, when this question was revisited in the second interview with Mr. Trackman, in the context of tossing coins, he recognized he could use the page he had assigned to the students for the Coin Tossing Exploration. Turning to the Pascal's triangle-like arrangement of circles, he incorrectly labeled the fifth row 1, 2, 3, 3, 2, 1. After identifying which circle corresponded to four heads and one tail, he then concluded the probability of that occurring would be 2/12 or 1/6. At the researcher's encouragement, Mr. Trackman wrote down the five different paths that led to the destination circle he had identified. With some hesitation, he recognized there would be 32 different outcomes possible for tossing five coins. He then concluded the probability he was looking for was 5 out of 32. It did not seem to trouble him that this result was different from what he had given earlier. He concluded he had been "pretty close . . . 1 out of 6 was pretty close. That would be 5 out of 30."

In the post-observation interview, Mrs. Books and Mrs. Talent both returned to the strategies they had used in the first interview. Mrs. Books again suggested finding $(1/2)^5$. Then she wondered what she would find if she drew a tree diagram. When she completed the tree, she discovered there were, in fact, 32 branches. Inspecting the tree she found 5 ways out of 32 to get four girls and one boy. Referring back to the 1/32, she observed, "I think if we were just to kind of change the . . . order that this happened in, we would also get 5 out of 32." As Mrs. Talent was reviewing the partially-completed tree diagram she had drawn in the first interview, she began arranging the favorable outcomes in an organized list. She quickly realized there would be five possible positions for the boy, giving a probability of 5/32. After completing some additional branches of the tree, she found the tree also led to 5 favorable outcomes out of the 32 possible outcomes. Table D.5 summarizes the teachers' responses to the Birth problem in both interviews.

As reflected in the classroom. The teachers presented a number of different strategies for analyzing probability situations to their students. These strategies included organized lists, charts, tree diagrams, and the multiplication property, as well as more advanced strategies.

Because "make an organized list" is one of the problem-solving strategies commonly taught in elementary school, this strategy would be a logical starting point for listing possible outcomes in situations involving probability. It was the strategy most often used by Mr. Trackman. However, the other teachers generally used different strategies to find the possible outcomes, without relating those strategies to making an organized list. In one situation, Mr. English did make an organized list–when trying to determine the possible ways of choosing two cards from a hand with one face card and five cards that were not face cards. Following his model, making an organized list was then the strategy used by the students for similar items on their homework. But that was the only situation in which an organized list was utilized in any of the classrooms other than Mr. Trackman's.

Table D.5
Summary of the Teachers' Responses to the Birth Problem (Probability Question #4)

	Pre-observation interview	Post-observation interview
<u>Strategies used</u>		
Mr. Trackman	"uneducated guess," multiplication property	Pascal's triangle, organized list
Mrs. Books	tree diagram, multiplication property	multiplication property, tree diagram
Mrs. Talent	multiplication property, tree diagram	organized list, tree diagram
Mr. English	multiplication property, organized list	Pascal's triangle
<u>Correct result obtained</u> (with strategy)		
Mr. Trackman	no	yes (organized list)
Mrs. Books	no	yes (tree diagram)
Mrs. Talent	no	yes (list & tree diagram)
Mr. English	yes (organized list)	yes (Pascal's triangle)

Mr. Trackman used an organized list to determine the possible outcomes for three of the activities he included in the probability unit, "Paper, Scissors, or Rocks," "Is This Game Fair?" and the Dice Sums game. One question on the "Paper, Scissors, or Rocks" worksheet asked the students to "make a list of the ways three players could show the signs." In listing the possible outcomes of tossing two coins for "Is This Game Fair?" and for the Dice Sums game, Mr. Trackman used an organized list rather than the 6-by-6 addition table usually used to analyze the possible dice sums.

However, in making an organized list in these situations, Mr. Trackman did not consider the underlying importance of order. This error led to the misrepresentation of the probability of various outcomes. The impact of this error was observed in the directions given to the students in the "Paper, Scissors, or Rocks" game and in the theoretical results presented for that game as well as for "Is This Game Fair?" For example, in explaining the assignment to be done for the "Paper, Scissors, or Rocks" game, Mr. Trackman had suggested "rock, paper, rock is the same thing as rock, rock, paper because you still have two rocks and a paper. So you don't need to duplicate that." In analyzing the game the following day based on this assumption, Mr. Trackman listed 10 possible outcomes (see Figure D.18). In three of these outcomes, all players were showing the same sign, giving points to player A.

In six of the outcomes, player B would have received points with only two players showing the same sign. Mr. Trackman identified getting all different signs as "the least likely to happen," giving player C a point only 1 time in 10.

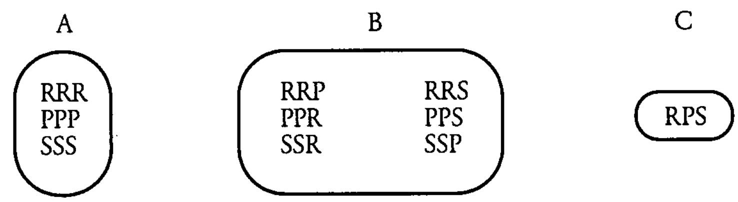


Figure D.18. Possible outcomes suggested by Mr. Trackman for "Paper, Scissors, or Rocks."

However, if three players, A, B, and C, are playing the game, the outcome of rock, paper, rock (where player B is the one showing paper) is different from rock, rock, paper (where player C is the one showing paper). When order is properly considered, there are 27 different equally likely outcomes. Player B is still the most likely to win with 18 of the 27 outcomes having only two similar signs. Player C is the second most likely winner, rather than the least likely one according to Mr. Trackman's analysis, because there are six possible ways that the players could each show a different sign. Finally, player A is the least likely to win, with the same three possible ways of all showing the same sign as Mr. Trackman had listed. Consideration of order thereby affected the probabilities of the results and would also impact the ways of making the game fair.

The issue of order also influenced the analysis Mr. Trackman did for "Is This Game Fair?" As they began to discuss the game based on dice sums, the students had suggested two ways of getting a sum of 3. However, for the sake of simplicity and saving time, Mr. Trackman incorrectly responded they were going to try it with just one possibility, 1 and 2, instead of considering both that and the combination of 2 and 1.

Just over a week later, Mr. Trackman played another game based on dice sums with the students. After playing several rounds of that game, in which the students were expected to discover the pattern, Mr. Trackman presented the theoretical outcomes to the students, this time correctly. "There's 1 real possibility for [a sum of] 2, there are 2 for [a sum of] 3, there are 3 for [a sum of] 4, . . . and there's 4, 5, 6, 5, 4, 3, 2, 1 possibilities" (as he wrote down the number of outcomes for the other possible sums). These two ways of analyzing the dice sums give significantly different results as illustrated in Figure D.19.

In the post-observation interview, Mr. Trackman was asked if the table used in the Dice Sums game could be applied to "Is This Game Fair?" Mr. Trackman agreed it could. Although he observed the students had been able to understand the table later on, he suggested, "I didn't want them to get confused on that initially, knowing the maturity level." When asked if he would consider all 36 possibilities if he did "Is This Game Fair?" in the future, Mr. Trackman responded, "Probably not. . . . Because it worked out pretty well, because they understood it pretty well, when we did the game on the board . . . the *X*s and they understood it there, but I didn't think they . . . that was later on and they had already done a lot of probability and so they were thinking in [a] probability mindset. That was only the second day we had done it and they weren't . . . I didn't think that they were ready." In summarizing his decisions, Mr. Trackman suggested that he was applying "the KISS theory, Keep It Simple Stupid." Thus, even when the difference was pointed out to him, he did not recognize the importance of order. Nor did he see the potential confusion this inconsistency could have caused with some of his students.

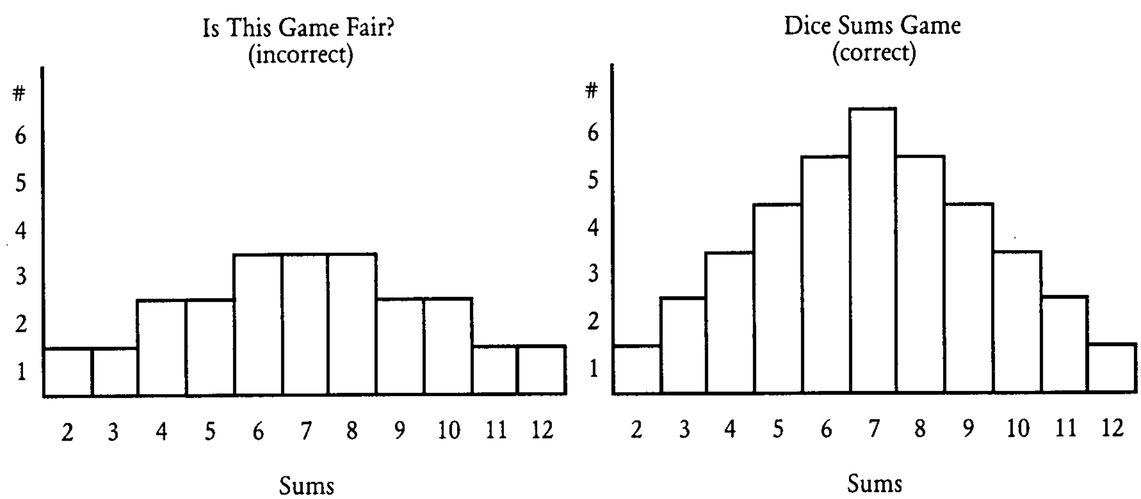


Figure D.19. Comparison of Mr. Trackman's analysis of dice sums.

Charts were the representation used most widely by the teachers. In fact, Mr. English suggested the students would probably call him "Chart Man," because he used them so often. However, Mr. English felt he was typical of middle school teachers because "we use charts quite frequently." He suggested he liked "to do it that way" because charts were an easy way to set up a problem and were meaningful to students at the middle school level.

In all four classrooms observed, charts were used most frequently to record individual experimental results and to collect class data. The instructional materials generally included some form of chart on which the students could record their experimental results. In collecting class data, Mr. Trackman usually just made a list of the data on a blank transparency or on the overhead glass itself, but the other teachers generally used charts set up for the purpose. Blackline masters for some of these charts were provided with the instructional materials. In other cases, the teachers had drawn up simple charts for recording the class data.

Besides being used to collect and record experimental data, charts were also used as a tool in the analysis process, especially in situations involving dice. In particular, using charts with dice seemed to be one of the points Mr. English felt was important for students to learn. For example, he explained to the students, "Whenever you analyze different situations, you have to set them up a little bit differently. . . . This is how you set up an analysis of a dice game. You have to make a chart. If we're adding the dice, you make an addition table." Later in the unit, he reminded the students, "Anytime you use dice, you've got to analyze the chart."

The 6-by-6 addition table for finding dice sums was the most commonly seen chart. The students in Mrs. Talent's class made use of it to analyze one game and complete one assignment. Mr. English referred students to the chart in analyzing dice sums on at least 4 different days. Instead of using the 6-by-6 addition table when he analyzed the dice sums, Mr. Trackman made a partial array of the dice outcomes and summarized the sums as shown in Figure D.20. A 6-by-6 table of dice products was also used by Mrs. Talent and Mr. English in the analysis of a number of class activities and homework assignments.

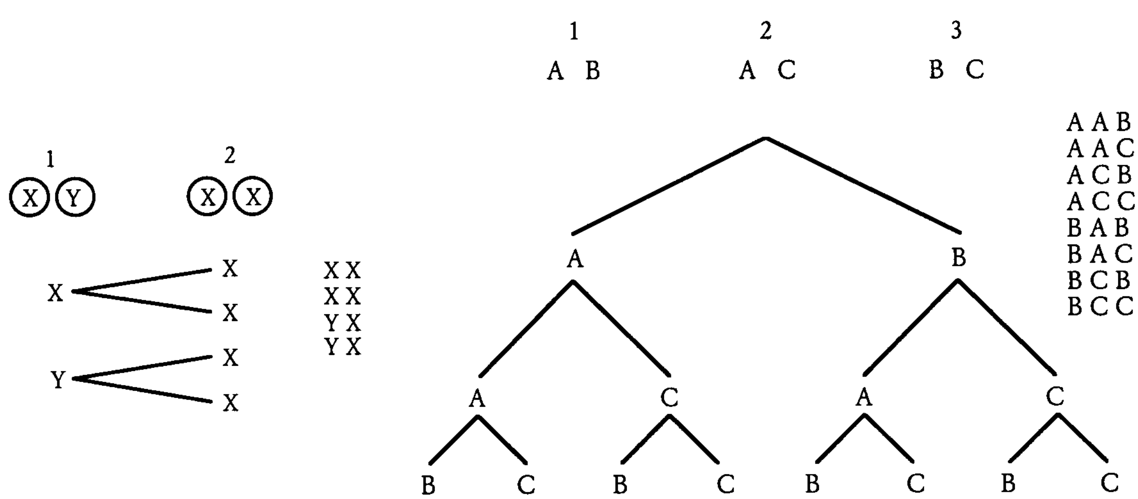
dice sum	2	3	4	5	6	7	8	9	10	11	12
# of ways	1	2	3	4	5	6	5	4	3	2	1

Figure D.20. Summary of number of outcomes leading to different dice sums.

In addition to the charts of dice sums and products, Mr. English created some original charts in order to analyze the games in "Which Do You Think Will Be Greater?" and "The Top and One Other." The worksheets themselves focused on experimental results and the instructional materials did not offer any suggestions for doing a theoretical analysis of the games. Therefore, Mr. English needed to create his own charts for analyzing the problems. In particular, the game played in "Which Do You Think Will Be Greater?" presented a unique problem to analyze because to determine the winner a comparison between two outcomes was involved. One player rolled two dice and multiplied the two numbers; the other player rolled one die and squared the number obtained. In analyzing the problem, Mr. English considered the outcomes from the perspective of the player with the one die. From this perspective, there were six outcomes to consider and, in each case, the ways of winning, losing, or tying were evident from a 6-by-6 multiplication chart. His chart shown in the case study (see Figure 19 in chapter IV) and his reasoning were quite clear, easy to follow, and captured the essence of the probabilistic nature of the situation. Likewise, the chart created for "The Top and One Other" presented the analysis in a logical step-by-step manner. However, there was no discussion about how to set up charts such as those used in these two games. As a result, the students were given little or no opportunity to learn how to do such analysis on their own.

Drawing a tree diagram was another strategy commonly used by the teachers for analyzing situations involving probability. At this level, the focus was on the tree diagram as a way of listing the possible outcomes. Mrs. Talent and Mr. English introduced tree diagrams to their students and made extensive use of them as an analysis tool. The textbook Mr. Trackman used in his class included a section on tree diagrams which he skipped because he did not like the section and did not "feel comfortable in teaching it."

Mrs. Talent introduced tree diagrams on Days 3 and 4 as a strategy for analyzing two games in which chips were being flipped. In the first game, one chip had the letter x on both sides and the other had an x on one side and a y on the other. The tree Mrs. Talent drew is shown in Figure D.21(a). In the second game, three chips were flipped, one with an A side and a B side, one with an A side and a C side, and one with a B side and a C side. Mrs. Talent first displayed the information about the chips across the top and then began demonstrating how to draw the tree diagram. To illustrate that trees could be drawn either horizontally or vertically, she drew the second tree vertically (see Figure D.21[b]). When this tree "grew out" of the display for the second chip, some students were confused, thinking the labels should be A and C instead of A and B. The corresponding homework assignment asked students to draw tree diagrams in two similar settings involving three chips and four chips.



(a) Tree diagram for "Chips" game 1. (b) Tree diagram for "Chips" game 2.

Figure D.21. Tree diagrams drawn by Mrs. Talent for "Chips."

On Day 6, Mrs. Talent gave the students one more assignment dealing with tree diagrams. The assigned worksheet focused on drawing two-stage trees. In completing an example involving two spinners, Mrs. Talent led the students through the drawing process with a series of questions and actions:

T: First of all, the first spinner. How many different ways can you come out on that spinner if you spin?
Ss: One, 2, 3, 4.
T: Okay. So, four different ways. So, what you need to do to represent that spinner, you need to draw the first part of your tree and have it have four branches. . . . And each branch is gonna represent one of the different outcomes. And then, on your . . . assignment, you're gonna have to do this, so . . . on here, what's the probability that any one of these will come up?
Ss: One fourth.
T: One fourth. So, what we do is, next to one of the branches, we're just gonna write 1/4. . . . Okay, the four different outcomes for the first spinner are . . . what? What could it come out?
Ss: One, 2, 3, 4.
T: Okay. So we're gonna write 1, 2, 3, 4. . . . Okay, the second spinner can come out how?
Ss: Red or blue.
T: How many different outcomes?
S: Two.
T: So, what we're gonna do now is off of each one of these, we're gonna make two branches. Okay? And what's the probability that it's gonna come out red or blue?
S: One half.
T: One half. And again, I'm just gonna have you write that next to one of 'em instead of all the way across, okay. And then, what would be the two different possibilities here?
S: Red and blue.
T: Yes. And you just go through and it's gonna be red or blue, red or blue . . . and so on all the way across.

Mrs. Talent then listed the final outcomes off to the side of the tree and wrote down the probability of each outcome. After a second example, the students completed a worksheet of seven similar problems. Tree diagrams were seen again on Day 9 when a tree was used to analyze "The Hare and the Tortoise Game."

Mr. English introduced tree diagrams as part of the vocabulary on the first day by giving the example of tossing two coins (see Figure D.22). The next use of a tree diagram was on Day 6, when one was drawn in the analysis of "The Hare and the Tortoise Game." In this setting, a die was rolled three times with results being even or odd. As practice following this game, Mr. English asked the students to draw a tree diagram representing the outcomes for tossing three coins. That diagram was essentially the same, except that heads and tails replaced the odd and even outcomes. In the coin toss activity "Quiz or No Quiz," Mr.

English began the analysis by drawing the tree through the second coin toss. After using multiplication to generalize the situation, he concluded by drawing only the outside branches of the tree, H-H-H-H-H and T-T-T-T-T. Tree diagrams were also used in analyzing other coin situations from the homework assignments on Days 11 and 12.

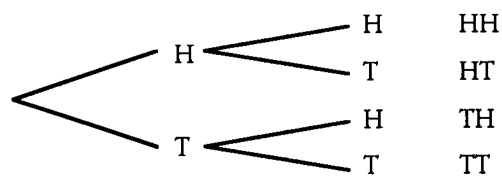
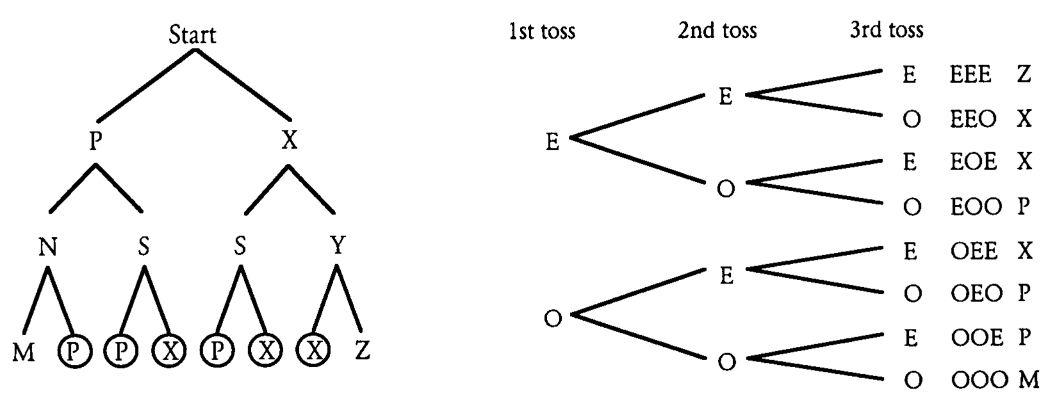


Figure D.22. Tree diagram for tossing two coins.

Although Mrs. Talent and Mr. English both drew tree diagrams in their analysis of "The Hare and the Tortoise Game," the diagrams they drew were somewhat different (see Figure D.23). In particular, the tree diagrams had the same structure, but were labeled differently. In hers, Mrs. Talent used the positions on the game board where the marker ended up as the labels; Mr. English used the odd/even outcomes of the dice roll. Using the specific outcomes of the action in the experiment, in this case the dice roll, is perhaps more typical of how tree diagrams are usually drawn. Nevertheless, the version Mrs. Talent drew led to a correct analysis as well. Besides the labeling of the outcomes, her version also differed from the usual tree diagram in that the final outcome was at the end of the last branch (e.g., Z) and not the sequence of outcomes along the branches (e.g., EEE). Although different from the more typical tree diagram, the one Mrs. Talent drew provided stronger



(a) Mrs. Talent's tree diagram.

(b) Mr. English's tree diagram.

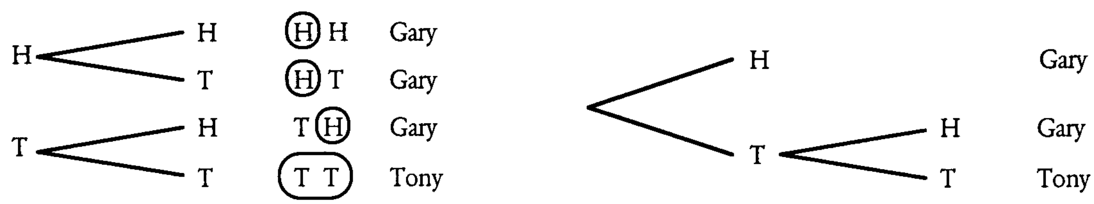
Figure D.23. Tree diagrams drawn for "The Hare and the Tortoise Game."

evidence in support of the students' conjectures that some positions on the game board were not possible outcomes, but this point was not discussed.

A number of features are worth noting concerning Mrs. Talent's and Mr. English's presentation of tree diagrams. First, although both modeled the process of drawing tree diagrams, the thinking processes involved were never made explicit. Even the series of questions Mrs. Talent used were not emphasized other than the one time they were asked. In particular, one key question seemed to be omitted: What sequence of actions will occur in the course of the game or the experiment? It is these actions that determine the stages of the tree and how the outcomes are labeled. It was at this point students had difficulties when trying to draw their own tree diagrams. For example, when Mrs. Talent asked the students how they would draw the tree diagram for "The Hare and the Tortoise Game," one student suggested starting with one branch for the hare and another for the tortoise. However, in this case, the actions involved rolling a die three times and observing whether it landed on an odd number or an even number. Similarly, when the students were trying to represent the Carnival task in which colored balls were to be selected from the different cans, some of the students began by drawing three branches corresponding to the three cans. The actions, however, involved selecting balls first from can 1, then from can 2, and finally from can 3. In the follow-up question to "The Hare and the Tortoise Game" in Mr. English's classroom, students were asked to draw a tree representing tossing a coin three times. In this case, some of the students drew a tree with the appropriate structure, but because they were not focusing specifically on the actions, they did not label the outcomes correctly. Labeling the stages as Mr. English did in Figure D.23(b) or as Mrs. Talent did in Figure D.21(a) could have helped students better understand the process of drawing a tree diagram if the link between the actions in the experiment and the stages of the tree diagram was made explicit. Further, to be more helpful, the display of the chips in Figure D.21(b) should have been written vertically corresponding to the stages.

Second, there was little discussion about when the use of a tree diagram would be appropriate or not appropriate. Mr. English generally associated the use of tree diagrams with and recommended their use for situations involving coins. For both teachers, the use of tree diagrams was limited to binomial situations like tossing coins and to simple two-stage trees. At some point in each of their probability units, as they were beginning to analyze the outcomes for tossing two dice, Mrs. Talent and Mr. English began to draw a tree diagram to represent the possibilities. After drawing the first set of six branches at the second stage, they stopped, pointing out to the students that the tree diagram in this particular situation would be "too complicated." Other than that, no specific discussion about when it was appropriate to use a tree diagram and when it would be less appropriate or inappropriate took place in these teachers' classrooms.

Third, although Mr. English used tree diagrams on several different occasions, only two versions of tree diagrams were demonstrated, the two-coin tree and the three-coin tree. Further, these two diagrams were used in analyzing problems, even when they did not quite fit the situation. For example, the situation in the "Tossing Pennies" activity was somewhat unique. Two boys, Gary and Tony, were going to play a "friendly game" tossing pennies. If the first toss of the penny landed heads, Gary won a point and the penny was not tossed again. If the first toss was tails, they tossed again. Heads on this toss gave Gary one point, and tails gave Tony a point. In presenting an analysis of the problem, Mr. English used the standard tree diagram for tossing two coins (see Figure D.24[a]). Thus, rather than drawing a tree to fit the situation of the problem, he started with the two-coin tree diagram and adapted it to fit the problem. In doing so, he correctly concluded that Gary would win three fourths of the time and Tony would win only one fourth of the time. In this case, rather than thinking of drawing tree diagrams to fit the situation as a strategy to use in the process of analysis, it appeared Mr. English saw the familiar trees as analysis tools; the task was to decide which tool could be used or adapted to fit the situation. No students raised any question about the tree diagram used, but it potentially could have been confusing to have drawn something that did not occur. In the process, students may not be learning to be independent problem solvers, learning instead to be dependent on the teacher's analysis.



(a) Mr. English's tree diagram. (b) Tree diagram matching the action.

Figure D.24. Tree diagrams for "Tossing Pennies."

In addition, by using this tree diagram, an opportunity was missed to address a possible misconception. The assignment page included a conversation between Gary and Tony: Gary explained the rules of the "friendly game," after which Tony observed, "That's not fair! You'll win twice as often." To make it fair, Gary offered to let Tony get two points when he won, as opposed to his receiving one point when he won. If a tree diagram were drawn to fit the situation (see Figure D.24[b]), some might argue it was now a fair game.

Gary would win one point on two branches and Tony would win two points on one branch. What may easily be overlooked, however, is that these outcomes are not equally likely.

Finally, with the exception of the sample tree diagram drawn on Day 1, the tree diagrams that Mr. English drew did not have a "main trunk" (compare Figures D.22 and D.24[a]). His tree diagrams began by listing the outcomes of the first stage of the experiment. This characteristic was also true of the one horizontal tree Mrs. Talent drew. This error seemed to be one both teachers may have picked up from the Middle Grades Mathematics Project materials (Phillips et al., 1986), where their trees developed from an organized list of the outcomes in a chart. Although it may present no difficulty at this level, it would be a problem in a probability tree diagram for there would be no branches to label with the corresponding probability. For example, this difficulty would apply to the asymmetrical probability tree that represents "Tossing Pennies" (see Figure D.25).

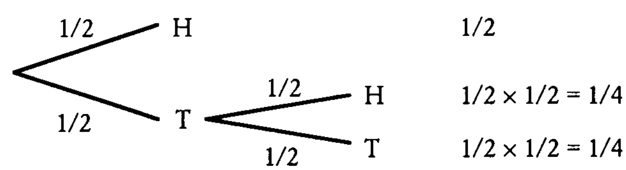


Figure D.25. Probability tree diagram for "Tossing Pennies."

The multiplication property was another strategy used by both Mrs. Talent and Mr. English in analyzing a number of probability situations. This property states that, when a particular outcome can be represented as a sequence of simpler outcomes, none of which affect the other outcomes, the probability of the overall outcome is the product of the probabilities of the simpler outcomes. For example, if a coin is tossed and then a die is rolled, the probability of getting a tail and a 3 is found by multiplying 1/2 (the probability of getting a tail on the coin) times 1/6 (the probability of rolling a 3 on the die). In addition, this property would justify multiplying the probabilities along the branches of the probability tree diagram for "Tossing Pennies," as shown in Figure D.25.

After working with tree diagrams for 3 days, Mrs. Talent had the students discover this multiplication pattern. On their homework from Day 6, the students had drawn several two-stage tree diagrams. At each stage they had been instructed to write the probabilities beside the corresponding branches. After they had corrected the assignment, Mrs. Talent had the students write down the probabilities corresponding to the two stages of each tree and write down the probability of the final outcome. When asked to find a pattern between these fractions, the students recognized the two fractions corresponding to the stages had been multiplied to give the final probability.

In summarizing the students' observation, however, Mrs. Talent confused two related multiplication properties when she stated, "So, what that means to you is, now, instead of doing a tree diagram, to figure out, if you've got how many different ways the first one could come out and how many different ways the second one could come out, if you take and multiply them, it'll tell what the probability is without having to draw all of it out." One property, called the fundamental counting property, deals with the number of outcomes when a sequence of occurrences are possible. According to this property, "If an event A can occur in r ways, and for each of these r ways, an event B can occur in s ways, then events A and B can occur, in succession, in $r \times s$ ways" (Musser & Burger, 1997, p. 474). For example, an experiment that involves flipping a coin and tossing a die has 2×6 or 12 possible outcomes. Rather than dealing with the number of outcomes, the multiplication property stated earlier involves finding the probability of a sequence of simple outcomes by multiplying the probabilities of the simpler outcomes. In the example given, the probability of heads and a 6 would be $1/2 \times 1/6 = 1/12$. Thus, in her summary, Mrs. Talent had begun as if she was stating the fundamental counting property, but ended up talking about multiplying probabilities. Though the two ideas are related, Mrs. Talent's statement could have been confusing to the students.

Mrs. Talent followed this discovery with a practice worksheet the students were to complete by multiplying probabilities rather than drawing trees. Some students encountered difficulties on the worksheet, apparently because of the limited nature of the examples the students had seen. For the first set of exercises on the worksheet, a spinner with equal-sized sections numbered 1 to 5 would be spun and a coin would be tossed. The students had no difficulty finding P(3, heads) and P(5, tails) because these were like the examples that had been given. However, at least one student became confused when trying to find P(odd number, tails), as the following conversation indicates:

S: Is that right?
T: For number two . . . how did you get that?
S: Oh. That was supposed to be number three.
T: What's the probability that you'll get an odd number if you spin that . . . spinner?
S: Three.
T: Out of how many total?
S: Five.
T: Okay.
S: So is it one third? 'Cause the other one I was thinking . . . it was out of 10.
T: Well, 5 out of 10 is 1/2, but 3 out of 5 is 3/5. You can't reduce that.
S: Three fifths?

T: Huh? That's what you just told me. You said you can get 3 out of 5. Well, that's 3/5.
S: But . . .
T: For that.
S: How do you . . . how do you find out what that number is?
T: Well, how many odd numbers are there . . . on the spinner?
S: Three.
T: And how many numbers total?
S: Oh! I get it!

However, it did not appear the student had gotten it, for he encountered further difficulties on the next item, which asked the students to find P(not 1, heads).

S: Would it be 1/4?
T: How many are not 1s?
S: Four.
T: Okay. Out of . . . ?
S: Five . . . a fourth of a circle?
T: Well, is 4 out of 5 a 1/4?
S: Huh? Maybe . . .
T: Well, if it's asking . . . what's the chance that you're going to spin that and not get a 1, it is 4 out of 5 total, right? . . . That doesn't have to be . . . 1/4.
S: Oh! So, how do you do it . . . Would it be 4 though?
T: It's not 1 out of 4! You just told me it was 4 out of 5! . . . How do you write 4 out of 5?
S: Four on top of 5.
T: Four on top of 5, right.
S: Oh, man.

In this case, two reasons might help explain why the student was having trouble. First, Mrs. Talent had not defined probability as the ratio of the number of favorable outcomes to the number of possible outcomes. That definition seems to be one piece the student is missing. But the student also keeps trying to express the probability as a unitary fraction (1 over something). For example, there were three ways of getting an odd number on the spinner, so the student thought the probability should be 1/3. Or with four ways of getting "not 1," he wanted to express the probability as 1/4. The student may have formed this incorrect notion because the examples used in discovering the property and the three additional examples given by Mrs. Talent had all involved probabilities expressed as unitary fractions. For example, the probability of getting a 3 when rolling a die and a tails on a coin flip is $1/6 \times 1/2 = 1/12$. Not having seen any other fractions in the examples and not knowing the definition of probability apparently combined to leave the student confused.

In a later lesson, Mrs. Talent also demonstrated to the students how the multiplication property could be used in situations where the initial outcomes did affect the other outcomes. For example, she showed the students a bag containing three yellow cubes, three green cubes, four white cubes, and one black cube from which she was going to select two cubes without replacement. Mrs. Talent asked what the probability would be of selecting a green cube and then a white cube. The students readily identified the probability of drawing out a green cube initially as 3/11. Then, Mrs. Talent explained they were going to assume they "did pull a green cube out the first time." After then identifying the probability of selecting a white cube from the remaining cubes as 4/10 or 2/5, the class multiplied the two fractions to find the probability of pulling a green and then a white. The students then completed a practice worksheet with problems similar to this situation.

Once the students in Mrs. Talent's class had discovered the multiplication property, it seemed to replace tree diagrams as the primary analysis tool. In particular, the multiplication property was used by the students in their analysis of the open-ended Carnival task. However, the students appeared to use the multiplication property without fully understanding what it represented. For example, in the Carnival task, the students multiplied $1/3 \times 1/3 \times 1/3$ to find the probability of drawing matching balls from three cans each containing a red, a blue, and a green ball. The students then multiplied by 3 because, as they reported, there were three cans. By multiplying rather than drawing a tree diagram, the students lost track of what the outcomes looked like and that there were three ways the balls could match. They were, therefore, coming to the correct result, but for the wrong reason.

Mr. English also used the multiplication property in analyzing some of the activities he presented to his students. For example, in his introduction to the probability unit, Mr. English used the multiplication property in finding the solution to an historical gambling problem. In this case, to find the probability of not getting a 6 in four tosses of the die, Mr. English explained, "You take the 5 out of 6 chance [of not getting a 6 on one toss] and you multiply it four times." He used a similar approach in analyzing lottery situations, in extending the results of the tree diagram in "Quiz or No Quiz," and in finding the probabilities of each opening in "A Ratty Problem."

However, Mr. English also encountered a problem when he used the multiplication property in analyzing the lottery situations on Day 4. In preparation for analyzing the state's Powerball lottery, Mr. English led the students through an analysis of some simpler "lottery" games. In considering their chances of correctly choosing the winning one-, two-, or three-digit numbers, Mr. English and the students concluded the chance of picking any digit correctly was 1/10. To find the probability of matching multiple digits they concluded one needed to multiply the 1/10s together. Following this discussion, the class considered a version of the state's Powerball lottery in which five balls were to be selected from balls numbered 1 to 40 and the powerball was to be selected from balls numbered 1 to 10. The students readily caught on that, because the balls were not replaced after each selection, the number of possible balls decreased each time. And following the pattern of the earlier examples, they concluded the probability of winning the jackpot by matching all six numbers was $1/40 \times 1/39 \times 1/38 \times 1/37 \times 1/36 \times 1/10$. Although the result when calculated did not seem reasonable to Mr. English, he was not aware of where they had made any error. Unlike matching two- or three-digit numbers, where position is important, this jackpot could be won by selecting the correct numbers in any order, except for the powerball. Therefore, there would have been more than one way to have been a winner out of the $40 \times 39 \times 38 \times 37 \times 36 \times 10$ ways the six balls could have been selected. Thus, in applying the multiplication property in this case, Mr. English had unknowingly fallen into one of the traps hidden within the content of probability–the importance of order in the difference between permutations (where order matters) and combinations (where order does not matter).

In both classrooms, the students seemed able to follow the multiplication pattern without much difficulty. However, it was not clear whether the students understood why one multiplied or if they would know when such a strategy was appropriate and when it was not. In the simple lottery situations, the justification might have been more clearly seen. Of the 100 possible two-digit numbers between 00 and 99, the probability of selecting the winning one is 1 out of 100, which is equivalent to $1/10 \times 1/10$. But in the other situations, the connection between multiplication and the actual probability may not have been as evident. No attempt to develop the rationale for multiplying was made. While the multiplication property is a useful analysis tool in some situations, it is more abstract than a list of outcomes or a tree diagram. And as these examples demonstrate, dangers are involved in the use of the multiplication property, particularly when order is pertinent but not considered.

In addition to the strategies already discussed, other representations were also presented to students. For example, during the final week of the probability unit, Mr. English presented an area model and Pascal's triangle as additional ways of analyzing probability situations. These lessons, however, were not observed. When asked about these strategies later, Mr. English's understanding of them seemed to be limited to the types of situations he had seen in the Middle Grades Mathematics Project materials, *Probability* (Phillips et al., 1986). For example, the activities and assignments involving the area model all used a square divided into 36 smaller squares. When asked in the follow-up interview if the area model could be applied to solve the Two Spinners problem (probability question #5) from the pre-observation interview, Mr. English concluded it probably could because "at least the model we used had 36 squares, and those spinners could be designed so that you could mark in the . . . they would correlate well with 36." He went on to wonder aloud if the area model always used 36 squares, admitting that was the only application he had seen.

Although Mr. Trackman did not introduce Pascal's triangle formally, it did provide the theoretical foundation for the Coin Tossing Exploration he assigned on Day 3. Mr. Trackman, however, did not utilize the correct pattern for the triangle and, in fact, used different patterns on different occasions. On Day 3, prior to assigning the Coin Tossing Exploration to the students, Mr. Trackman showed the researcher his values for the theoretical probabilities in the triangle (see Figure D.26[a]). These values were based on at least two false assumptions. First, he believed an even split of heads and tails (e.g., 1H, 1T; or 2H, 2T; or 3H, 3T; etc.) always had the probability of 1/2. Second, he assumed some values were repeated in the rows that followed as indicated by the arrows. Mr. Trackman correctly identified the results on the ends of each vertical row. Then, using the symmetry of each vertical row and applying the fact that the sum of each vertical row is 1, Mr. Trackman had determined the other values in each row. However, because of his false assumptions, some of the probabilities differed from the actual values in Pascal's triangle in rows beyond row 3, as shown in Figure D.26(b).

Row		1	2	3	4	5	6
							1/64
						1/32	
					1/16		3/64
				1/8		3/32	
			1/4		3/16		→ 3/16
		1/2		3/8		→ 3/8	
	1		1/2		→ 1/2		→ 1/2
		1/2		3/8		→ 3/8	
			1/4		3/16		→ 3/16
				1/8		3/32	
					1/16		3/64
						1/32	
							1/64

(a) Mr. Trackman's version on Day 3

4	5	6
		1/64
	1/32	
1/16		6/64
	5/32	
4/16		15/64
	10/32	
6/16		20/64
	10/32	
4/16		15/64
	5/32	
1/16		6/64
	1/32	
		1/64

(b) Correct version

Figure D.26. Two versions of Pascal's triangle.

In giving the assignment to the students, Mr. Trackman added to what was asked for on the worksheet. In addition to recording their experimental results and finding the probabilities based on those results, Mr. Trackman asked the students to figure out the theoretical probability of each stop. Because he did not understand the complexity of the pattern himself, he provided very little information for the students to go by in finding or verifying their results. After identifying the probability of the "Start" circle as 1 out of 1 and the probability of each circle in the next row as 1 out of 2, Mr. Trackman suggested each vertical row should add up to 1 and, as the students continued, they should begin to notice a pattern. One student in the back of the classroom had already suggested a pattern, either not heard or ignored by Mr. Trackman. This student had proposed the pattern would continue with "1 in 3 . . . 1 in 4." Because the numbers had not been connected in any meaningful way to the coin tossing outcomes, this student and others had no way of verifying whatever patterns they discovered. In fact, if the student interpreted the outcomes of tossing a coin twice as zero, one, or two heads, then his value of 1 out of 3 could perhaps be justified in his mind.

In the post-observation interview, a follow-up question was asked about the probability of having four heads and one tail, in any order, if a coin were tossed five times. Mr. Trackman referred to the arrangement of circles used in the Coin Tossing Exploration. At that time, he labeled the corresponding row 1, 2, 3, 3, 2, 1, and suggested the probability would be 2/12. Extending that pattern would give the triangle shown in Figure D.27. As with his earlier version of Pascal's triangle, this is an interesting pattern, but one that is meaningless in the context of tossing coins.

						1
					1	
				1		2
			1		2	
		1		2		3
	1		2		3	
1		2		3		4
	1		2		3	
		1		2		3
			1		2	
				1		2
					1	
						1

Figure D.27. Another version of Pascal's triangle, according to Mr. Trackman.

In the analysis of the "Cereal Boxes" problem, Mrs. Books introduced her students to a variety of statistical ways of representing and analyzing their experimental data. At the first stage, the class found the range and mode for their subjective guesses. Then, after each student or pair of students had conducted the simulation 10 times, the class pooled their data by making a line-plot. While waiting for others to finish their trials, the students were encouraged to find the mean, median, mode, and range for their own data.

Once all of the class data were collected and displayed on the line-plot, Mrs. Books' introduced the students to a second way of displaying data, a box-plot or box-and-whisker. As described in the case study, Mrs. Books made a number of misstatements in the process of modeling how to construct a box-plot, even though she initially modeled the process correctly and clearly knew the correct steps to follow. Her errors may have been the result of stress (problems with the computers had changed her plans) or evidence that her own knowledge of this relatively new technique was still under construction.

As examples for the students, Mrs. Books drew three different box-plots for a given set of data. In the first, she drew the box encompassing 90% of the data. The other examples presented 80% and 50% box-plots. In each case, Mrs. Books placed the lower end of the box at the lower end of the data. The 50% box-plot thus encompassed the lower 50% of the data. The problem in this case involved buying cereal boxes to get a full set of prizes, so one might be justified in choosing to consider the fewest number of boxes possible. However, the more common box-plot encompassing the middle 50% of the data from the lower quartile to the upper quartile was not mentioned. And there was no discussion about when it might be preferable to consider the lower 50%, the middle 50%, or the upper 50%. As a result, the students received a limited view of the use of box-plots.

<u>Simulations and Expected Value</u>

Expected value or mathematical expectation is an important concept related to probability, one which is used to determine values such as insurance premiums or admission to games with payoffs. The expected value of an experiment is the "long range average" or the average value of the outcomes over many repetitions. For example, one might play a game where a \$5 prize will be paid for tossing two coins and having them both land heads; otherwise, the player will lose \$1. Theoretically, if the player were to play the game four times, he or she would expect to win \$5 one time (HH) and lose \$1 the other three times (HT, TH, TT). Therefore, the player's average winnings for the four games would be [\$5 × (1 time) + (-\$1) × (3 times)]/4 or $5 \times (1/4) + (-1) \times (3/4) = 5/4 - 3/4 = 2/4$ or \$.50. In a formal sense, when there are several possible outcomes (\$5 and -\$1, for instance) and different probabilities of obtaining these outcomes (1/4 and 3/4, respectively), the expected value is calculated by multiplying the different probabilities times the respective outcomes and adding these products. In this case, that would be equivalent to the $5 \times (1/4) + (-1) \times (3/4) = 2/4$ in the previous calculation.

A simulation is one approach used to find probability or expected value in situations where finding the experimental probability may be too expensive, time-consuming, or otherwise impractical and determining a theoretical probability may be too difficult or impossible. This approach involves modeling the mathematical characteristics of the problem with the use of random devices such as dice, coins, spinners, or a random-number generator.

The next interview question involved both expected value and a simulation. As a result, the discussion of the teachers' knowledge of these probability concepts will be combined in this section. Also considered in this context will be a comparison between experimental and theoretical probability results.

As revealed in the interviews. The teachers were given a situation to simulate in the following Newspaper Pay problem (probability question #6):

> Carey is a carrier for the biweekly small town newspaper. She needs to collect \$4 per month from her customers. Instead of paying the \$4 each month, one customer makes her the following offer:
>
> > As her payment, Carey will draw one bill each month from a bag that will always have the same contents:
> > three \$1 bills,
> > two \$5 bills, and
> > one \$10 bill.
>
> (variation of Phillips et al., 1986)

Although it would be possible to do this experiment repeatedly with actual money in a bag, a simulation might be more practical, particularly in a classroom setting. With this situation as the setting, the teachers were asked to design a simulation; to conduct a simulation and interpret the results; to analyze the situation theoretically; and to compare their experimental and theoretical results.

As the first part of the problem, the teachers were given a bag containing materials commonly used in probability simulations, including dice, coins, colored chips, a deck of cards, and blank slips of paper. They were asked how they might simulate the situation described. In the customer's proposal, Carey will be drawing one bill at random from a bag containing six different bills. Therefore, to simulate the situation, one needs to have a way of modeling six equally likely outcomes, where three can be assigned an outcome of \$1, two can be assigned an outcome of \$5, and one assigned as \$10. Many of the materials in the bag could be used to model the characteristics of this problem. The slips of paper could be labeled with corresponding dollar amounts and drawn from a bag. The chips or cards could be chosen to include three of one color or value, two of another, and one of a third. They then could be drawn from the bag or shuffled and chosen at random. Similarly, one could roll a die with three faces of the die assigned to correspond to the \$1 bills, two to the \$5 bills, and one to the \$10 bill. Although the bag contained pennies, nickels, and dimes, which correspond to the value of the bills, using the coins presents the possibility of bias because the different sizes of the coins could influence the outcome.

In general, the teachers recognized the many possible ways the problem could be simulated, giving examples of each of the ways described. Mrs. Books, however, was the only one who identified the difficulty in using the coins, pointing out the "different sized coins . . . would be a problem because that would be easy to tell which was which." Although Mr.

Trackman did not suggest one could not use them, he concluded he "wouldn't mess with the dice or cards."

After describing how the problem could be simulated the teachers were asked a further question.

> Perform the simulation for a 12-month period. From your results, would you recommend that Carey accept or reject the offer?

In conducting the simulation Mr. Trackman and Mrs. Talent used the colored chips; Mrs. Books used a die. Mr. English chose the deck of cards, explaining that was what he used in teaching a similar problem from the Middle School Mathematics Project materials (Phillips et al., 1986). After conducting the simulation for 12 trials, the teachers added up the total amount they would have earned, with results varying from \$41 to \$58. They then compared their totals to the \$48 Carey would normally have earned in a year at the rate of \$4 per month.

Most of the teachers made a preliminary recommendation to Carey based on their experimental results. However, Mrs. Talent, whose experimental result had been \$47, suggested, "I would conclude that you would need to do it more times to see if it really was supposed to be that close or if it were just a fluke. I don't think we have enough information to make the decision."

Because it is possible to analyze the problem theoretically in this case, the teachers were next asked, "If you were to evaluate this offer theoretically, what would you recommend to Carey?" This problem is dealing with mathematical expectation or expected value. In any given month, Carey has 3 out of 6 chances of receiving less than the usual \$4 if she accepts the offer and 3 out of 6 chances of receiving more than \$4. The expected value, on the other hand, can be used to predict the average in the long run, if the experiment were repeated many times. In this case, the probability of receiving \$1 is 1/2, of receiving \$5 is 1/3, and of receiving \$10 is 1/6. Therefore, according to the definition of expected value presented earlier, Carey's expected monthly earnings would be $1 \times (1/2) + 5 \times (1/3) + 10 \times (1/6)$ which is 23/6 or approximately \$3.83 on the average, compared with her usual monthly charge of \$4. From this result, it looks like Carey will be shortchanged in the long run if she accepts the customer's offer. In particular, she could expect to earn $12 \times 23/6$ or only \$46 in a year, on the average, compared to \$48 at the rate she usually charges.

Perhaps because the teachers had been asked to perform the simulation for a 12-month period, they considered the theoretical results on an annual basis as well. Thus, rather than considering the probability of what would happen in a given month, the teachers were considering how it would turn out in the long run over a year's period. Although they were not finding an expected value as such, their approach led to a comparable conclusion. None of the teachers gave any justification for considering the annual results; it seemingly was a convenient way to compare their experimental and theoretical results.

Each of the teachers applied the proportional nature of probability to determine how many months out of the year Carey would get the various amounts. For example, Mrs. Talent observed, "So that means out of every six times, she can expect to get three 1s, two 5s, and one 10. Okay, so if she did it six times, she'd get 3, 2, and 1. So if she did it 12 times, she'd get 3, 2, and 1 again, which would mean six 1s, four 5s, and two 10s, which would give her . . . $46." In a similar fashion, Mrs. Books used the probability values to arrive at a result, explaining, "Half of our months would be 6 months. We could expect her theoretically to get $6 for those 6 months. One third of our months is four of our months. And for those 4 months, we would expect to get $5, for a total of 20. And 2 months would be 1/6 of our year and we would expect to get $20, 2 times the 10. So we would have a total . . . of $46." Based upon the theoretical result, Mr. English suggested he "would not take the offer" and the other teachers made similar recommendations. However, Mr. Trackman seemed to be tempted on the basis of his experimental results, concluding, "I'd have to say that I recommend against it, but if you have the Midas touch, then you . . . I would go with it. I'd go with it. I'd take the gamble. Because $2 . . . Actually, $2, if it was me, $2 is not a big deal with the chance of getting the extra $10 [*which he had done experimentally*]. Although there's a chance of losing and only coming out with $12."

As a final part of the Newspaper Pay problem, the teachers were asked to compare the results of their simulation with the theoretical results they had calculated. If the two had been close to the same, did the teacher expect the experimental results to always be a good estimate? If the results had been quite different, what would the teachers expect if they repeated the simulation?

Mr. Trackman, whose experimental result had been $58, suggested he "would expect to do worse," so he "wouldn't do it again." Mrs. Talent pointed out Carey "could do the experiment 10 to 12 times and she could come out ahead every time, but she may fall way short another." In general, the teachers agreed with Mrs. Books' observation that "there may be some fluctuation that could come up . . . but if we were to do it over time, we would approach closer and closer to 46." Or, as Mr. English suggested, "if you took a computer and did it a million times . . . you'd probably get [close] to the theoretical probability." Table D.6 summarizes the teachers' responses to the Newspaper Pay simulation item.

As reflected in the classroom. Mrs. Books, Mrs. Talent, and Mr. English each included simulations in their probability units. Each of these three classes conducted a simulation of the "Cereal Boxes" problem in one form or another. In this problem, the students were to determine how many boxes of cereal they would have to buy in order to obtain a full set of the prizes hidden in the cereal boxes. In addition, Mrs. Books and Mrs. Talent both had their students simulate "Monty's Dilemma," a decision-making problem arising from a television game show. In this case, after choosing one of the three doors and having one of the other doors opened to reveal a gag prize, the contestant is given the option of sticking with the door they initially selected or switching to the other remaining door. Mr. English's students also conducted simulations of the "Newspaper Offer," where they evaluated an offer similar to the one the teachers evaluated in the pre-observation interview, and of "A Ratty Problem," where they modeled rats running a maze.

Table D.6
Summary of the Teachers' Responses to the Newspaper Pay Problem (Probability Question #6)

	Number of teachers responding correctly	Number of teachers responding incorrectly
Designing the simulation		
Identified how to use		
blank slips of paper	4	0
colored chips	4	0
cards	3	1
dice	3	1
Identified bias in using coins	1	3
Interpreting simulation results		
Calculated expected value	4	0
Made reasonable recommendation	4	0
Analyzing simulation theoretically		
Calculated expected value	4	0
Made reasonable recommendation	4	0
Comparing experimental & theoretical results		
Recognized variability of experimental results	4	0
Recognized value of more trials	4	0

In conducting the simulations, Mrs. Talent and Mr. English did not focus on the idea of simulation as such. Instead, the simulation activities seemed to be presented as just additional activities involving experimentation. In particular, Mrs. Talent and Mr. English chose the method the students would use to carry out the simulation. On the other hand, one of the goals Mrs. Books stated for her probability instruction was that students would learn how to set up a simulation. Therefore, the nature of the problem was discussed and the students were expected to decide for themselves how they would model the mathematical characteristics of the problem. In the course of individual interactions with the students as well as class discussions, the issues of randomness, replacement, and bias were explored. As a result, the students discovered the advantages and disadvantages of a number of simulation designs. Interestingly enough, the simulation design used by Mrs. Talent for "Monty's Dilemma" had been rejected by Mrs. Books' students. These students decided that there might be some bias in having people act out the problem by choosing between three paper cups. In particular, they felt one might be able to tell where the prize was hidden or might not be completely random in the choices made.

Mr. English was the only teacher who introduced the concept of expected value to his students. The use of expected value occurred in two activities, "Frosted Wheat Yummies" and "Newspaper Offer." The first activity was Mr. English's version of the "Cereal Boxes" problem. In this case, six different fluorescent pens were prizes in boxes of cereal. To find out how many boxes one would have to buy to obtain all six pens, Mr. English had the students do a simulation by rolling dice. After giving the directions to the students, Mr. English explained,

> What we are trying to find is called expected value. Now there's not a way that I think you can set this up theoretically, by looking at the number of outcomes possible in relation to the total outcomes. I don't think you can do this. I think . . . all you can do is conduct an experiment and say, based on the experiment, this is what we would expect to be the number of boxes that parents would have to buy their kids to get all six brands of this prize.

It is not clear whether Mr. English meant this statement as an explanation of expected value or of why they were doing a simulation. It does provide a rationale for doing the simulation, but does not really explain what expected value is or why they would be finding it. Later, as the students were conducting their simulations, Mr. English wrote a definition on the overhead: "Expected value is the mean average number of boxes you would expect to buy in order to get all six colored pens." Although this statement provided a definition of expected value in this situation, it still gave no rationale for finding expected value. For example, there was no discussion that some could get all the pens in 6 boxes, others in 10, and still others might have to buy 20 or more boxes before they had all the pens. One way to express these results is to determine what would happen on the average or in the long run if all the results were put together. Thus, expected value is used as the long range average.

Mr. English had prepared a worksheet on which the students could tally their results for the five trials they were asked to do. The worksheet then provided very specific directions indicating which numbers to add and which to divide in the process of finding "the average cost that it would take to collect all six pens." Because the worksheet provided such specific directions, the activity became more an exercise in following directions than one aimed at developing an understanding of expected value. In the process, at least some of the students lost track of the meaning of the problem. This difficulty was evident when one of the stronger students in the class reported a final result of 12.3 boxes, having ignored even the instructions to "round up" the quotient, which was identified as the expected value.

On the following day, Mr. English did a second simulation involving expected value, one very similar to the question asked of the teachers in the pre-observation interview. In the "Newspaper Offer" (Phillips et al., 1986),

> A girl . . . is delivering papers and she charges $5.00 a week. . . . One of her customers [who] is a mathematician . . . makes a proposition to her and he says, "Rather than paying $5.00 a week, why don't I put one $10 bill and five $1 bills into . . . a paper bag. . . . You can just reach in and draw out, without looking, two bills and, whatever you get, that will be how much I have to pay you."

To determine if this was a fair offer, the students were going to conduct a simulation and find the "long term average" of their results. As he introduced the activity, Mr. English explained, "Today's activity is a little bit different than some we've been doing [because] today's activity uses cards." Thus, instead of seeing this activity as a second one with expected value, Mr. English focused on the materials involved in doing the simulation, namely, the poker cards. The worksheet again guided the students through the steps of finding the long term average, which Mr. English identified as expected value. The students were thus introduced to the idea of expected value, but it was not clear what understanding they gained of the concept. For example, the homework assigned after the students' simulation of the "Newspaper Offer" began with the item: "The customer will place a $5 bill and three $1 bills in a bag. Sue will draw out two bills." This item led to the following discussion among three students:

> S1: Ryan, what'd you say for the first one?
> S2: Unfair.
> S3: It is not.
> S2: It is.
> S3: Un uh.
> S2: It is too.
> S3: It's three and three. How can it be unfair?
> S2: Because she only . . . she has an average of [$4] . . . so that's unfair. She'd be losing money.
> S3: Un uh, 'cause it's three and three.
> S2: She . . . makes $5 [*referring to usual pay rate*] . . .
> S3: So what? She still gets money. Isn't that what we're supposed to figure out . . . if she gets money?
> S2: Obviously she is going to get money, friend.

In this case, there are six possible outcomes. For three of these outcomes, Sue receives $6; for the other three she receives $2. It was these outcomes the third student was considering, and because the two results were equally likely to occur, he felt the offer was fair. However, he failed to realize that Sue loses $3 the weeks she draws out $2 but only gains $1 the weeks she draws out \$6. As a result, although the results were equally likely to occur for a given month, in the long run, Sue would lose money. This student apparently had missed the rationale for considering expected value.

Although Mrs. Talent and Mrs. Books did similar "Cereal Boxes" activities, neither introduced the idea of expected value. To make it more convenient for reporting results, Mrs. Talent asked the students to find the average of their three trials, but no significance was given to this average. And rather than expected value, Mrs. Books focused on a variety of other ways the data could be expressed, including both a line-plot and a box-plot.

An obvious but, perhaps, often overlooked idea is that results of experiments and simulations will not necessarily turn out as the theoretical analysis predicts and, in fact, will vary from trial to trial. However, this fact may be important in trying to understand the role of probability in our world where things do not turn out theoretically. Mr. Trackman only indirectly hinted at that idea, but the other teachers made more direct comments, such as, "Just to remind you again, this was an experiment so . . . if we did it again, it might not come out exactly the same. It probably wouldn't come out exactly the same." In fact, when some of the experimental results came out to exactly match the theoretical prediction for one of the strategies in "Monty's Dilemma," some of Mrs. Books' students challenged the results, suggesting they were "very unlikely."

With the exception of Mrs. Books' class, the students were not involved in directly comparing the experimental and theoretical results. It was more like, "Here are the experimental results and here are the theoretical results," without any further discussion. The teachers may have observed they were different, but the students were never asked to state that conclusion themselves. Nor was there any attempt to see if the students had learned that important idea. In one case Mr. English came close to doing so. He and the class had written down the theoretical probabilities for the different sections of three different spinners. Then the students had collected data for 100 trials on each of the spinners. On the summary sheet in their packet, Mr. English asked the students to make a table for each of the spinners, listing the theoretical probabilities and the experimental probabilities. However, the students were not asked to do anything further with the data. In particular, they were not asked to write down any conclusions from the tables. Nor were they asked, for instance, if the two results were the same and, if not, why not. Mr. English probably assumed they would notice the differences, but it is not certain that they did. In addition, when asked in the post-observation interview what "big ideas" the teachers hoped the students had learned, none of the teachers mentioned understanding the differences between experimental and theoretical probability.

Applications of Probability

Although the study of probability had its origin in the study of games, probability has many other applications in today's world. Both the interview questions and the classroom observations provided opportunities to explore the teachers' knowledge of the various applications of probability.

As revealed in the interviews. The pre-observation interview included questions exploring what applications of probability the teachers could identify outside their classrooms, in their lives as well as the lives of their students. In addition, the teachers' understanding of two real-life applications of probability was also investigated.

The Applications item (probability question #8) first asked the teachers, "How does probability impact your life?" In response to this question, the teachers provided a number of examples. Both Mr. English and Mrs. Talent indicated they do not participate in the lottery or other gambling situations because they "know what the odds are." Mrs. Talent also explained probability shows up in the fine print of contests in the marketplace and sweepstakes in the mail. Insurance rates were another example given by Mr. English. Mrs. Books pointed out probability provides information useful in making decisions as a consumer. In particular, she cited how advertisers and businesses use probability in trying to sway her decisions. Mr. English also indicated information based on probabilities influences decisions about his health and lifestyle. Mr. Trackman provided a further example of how probability impacted his life every day, suggesting, "When I drive 45 minutes . . . to and from work, I've got to figure out what is the quickest way back. And there's about seven or eight different ways that I will take in the course of the month because I hear that there's an accident here and so what's the quickest route around that?" By trial and error, it appeared he had determined what would most likely be the quickest route under the given conditions. Similarly, Mrs. Talent saw combinatorics, or finding "how many combinations of things are possible," as part of probability. Other examples given by the teachers were more informal and subjective in nature, related to the uncertainties of life. These examples included the probability of getting lost on an unfamiliar road, the probability of remaining in their current home, or how the probability of success is influenced by educational opportunities.

The Applications item also asked the teachers, "What examples can you give of how [probability] impacts the lives of your middle school students?" The teachers were less specific in response to this question. The examples given focused primarily on the role of probability in the games the students play. Mrs. Books pointed out the students are also consumers, and are therefore influenced by some of the same marketing strategies as adults. Mrs. Books also provided an example from her classroom. In particular, after she had begun picking up homework on a random basis as the school year progressed, she realized students were making some decisions and taking some risks based on the likelihood the papers were going to be collected. Mrs. Talent's response focused more on the positive impact that learning about probability would have on students. In particular, she suggested raising the students' awareness level might help them realize the importance of reading the fine print, analyzing their choices, and asking intelligent questions when making decisions involving chance occurrences (such as contests that come in the mail). A summary of the teachers' responses to the Applications item is given in Table D.7.

Table D.7
Summary of the Teachers' Responses to the Applications Item (Probability Question #8)

	Number of examples given for teacher's life	Number of examples given for students' lives
Mr. Trackman	2	2
Mrs. Books	3	2
Mrs. Talent	4	3
Mr. English	6	2

Two questions in the pre-observation interview asked the teachers to interpret information of a probabilistic nature in real-life settings. The first of these items was the Weather problem (probability question #9) in which the teachers were asked the following series of questions:

(a) What does it mean when a weather forecaster says that tomorrow there is a 70% chance of rain? What does the number, in this case the 70%, tell you? How do forecasters arrive at a specific number?
(b) Suppose a forecaster said that there was a 70% chance of rain tomorrow and, in fact, it did not rain. What would you conclude about the forecaster's statement that there was a 70% chance of rain?
(c) Suppose you want to find out how good a particular forecaster's predicting is. You observe what happens in 10 days for which a 70% chance of rain was predicted. On 4 of those 10 days there was no rain. What would you conclude about the accuracy of this forecaster? If he or she had been perfectly accurate, what would have happened? (variation of Shaughnessy, 1985)

According to a local meteorologist, "a 70% chance of rain" means "given similar atmospherics, 70% of the time there is measurable rain" (M. Zaffino, local weather forecaster, personal communication, July 20, 1995). In other words, out of 100 days when the atmospheric conditions are similar, 70 days will have measurable rain and 30 days will not. Having no rain on such a day does not necessarily mean the forecaster's prediction was wrong. Instead, it just ended up being one of the 30 days out of the 100 in which there was no rain. Given a prediction of 70% rain, one could expect 7 out of 10 days to have measurable rain. Therefore, having 6 days with rain and 4 days without rain would mean the prediction was fairly accurate, missing the expected results on only one of the days.

In general, the teachers had no knowledge of how the prediction was arrived at or of specifically what it meant. They interpreted the prediction in a more subjective way and their "degree of belief" varied somewhat. For example, Mr. Trackman observed, "70% chance of rain means it's . . . you're going to get rained on." Mrs. Talent suggested, "if they say there's a 70% chance of rain, it's going to rain . . . or you can be pretty sure it's going to rain." Mrs. Books was perhaps more accurate when she observed, "the chance that it will rain is 70%; the chance it won't rain is 30%, so it's more likely that it will rain. It's more than 50% chance of rain."

If there was no rain, the teachers generally attributed that outcome to an error by the forecaster or to a change in the weather conditions, although Mr. Trackman pointed out, "they've reserved the 30% right to say, 'I said 70%, I didn't say 100% chance of rain.' " And even though Mrs. Books suggested "we just happened to hit the 30%," she believed the forecaster "was probably a little over-confident." However, in general, the teachers concluded "something happened that they [forecasters] thought wasn't or didn't happen that they thought was."

Given the difficulty in predicting such things as weather, the teachers seemed to give the forecaster the benefit of the doubt, suggesting the forecaster was "fairly accurate" when there was rain on 6 out of 10 days in which the forecast had been 70% chance of rain. Mr. Trackman observed, "I'd say [the forecaster] was pretty accurate," even though he identified the forecaster's accuracy rate as 60% when only 6 out of 10 days had rain. But as Mrs. Talent observed, "I've heard that anybody who can predict something and get it right even 50% of the time would make a mint on . . . the stock market, so I guess he's a pretty good predictor."

The teachers' responses to the Weather problem contain evidence that at least some of their thinking was based on what Konold (1991) calls the outcome approach. Konold describes this non-probabilistic form of reasoning as follows:

> According to this alternative interpretation, which I will refer to as the "outcome approach," the primary goal in situations involving uncertainty is not to arrive at a probability of occurrence but to successfully *predict* the outcome of a *single* trial. Given this objective, a question that explicitly asks for the probability of an outcome is interpreted as asking whether the outcome will, in fact, occur on the next trial. For example, asked to explain a weather forecaster's prediction of 70% chance of rain, many students respond that they take that to mean that it *will* rain. Asked what they would conclude if it did not rain, these same students hold that the forecaster's prediction would then have been wrong. They also will argue that a forecaster

> is performing sub-optimally when it rains on 70% of the days for which 70% chances were given. Probability values are evaluated in the outcome approach in terms of their proximity to the anchor values of 100%, 0%, and 50%, which have the respective meanings of "yes," "no," and "I don't know." Thus students reasoning according to the outcome approach will argue that 70% is sufficiently close to 100% to warrant the assertion, "It will rain tomorrow." (p. 146)

As suggested by Konold, Mr. Trackman had interpreted the 70% chance of rain by saying, "You're going to get rained on." Similarly, Mrs. Talent suggested, "It's going to rain," before she qualified her response by adding, "You can be pretty sure it's going to rain." Further, as indicated by Konold, several of the teachers believed the forecaster had made an error when there was no rain.

A second setting involving the interpretation of a real-life application of probability was presented to the teachers in the Cancer problem (probability question #10).

> In a particular population, the frequency of cancer is known to be 1 out of 100. The test screening for the cancer has an overall accuracy rate of 87%. In other words, for patients with cancer it correctly diagnoses the cancer 87% of the time; for patients without cancer it correctly diagnoses them as free of cancer 87% of the time.
>
> A patient has tested positive for cancer.
>
> (a) Estimate the probability that the patient has cancer.
> (b) Calculate the probability that the patient has cancer.
> (variation of Eddy, 1982)

All the teachers provided responses based on their intuition, putting their confidence in the results of the screening test. As a result, they concluded there was an 87% chance the person has cancer. Mr. Trackman expressed an even stronger belief, concluding, "I would say the probability is . . . that the patient definitely has cancer." None of the teachers could provide any strategy for actually calculating the probability.

This problem was based on an example reviewed in chapter II of this research study. The contingency table shown in Figure 3 in chapter IV presents a solution. In particular, if 10,000 people are tested, 100 would have cancer and 9,900 would not. Of the 100 who have cancer, 87% or 87 would have a positive test result and, of the 9,900 who do not have cancer, 13% or 1,287 would have a "false" positive test result. Thus, of the 1,374 who have positive test results, 87 or only 6% actually have cancer.

When the solution to this problem was shown to the teachers, they were surprised at how different the result was from their intuitive response. The difference comes from a common misunderstanding of conditional probability. In particular, the probability of a positive test given the person has the disease, which the accuracy rate expresses, is confused with the probability of having the disease given a positive test result. Table D.8 summarizes the teachers' interpretations of these real-life situations.

Table D.8
Summary of the Teachers' Interpretations of Real-Life Situations (Probability Questions #9 and #10)

	Number of teachers responding correctly	Number of teachers responding incorrectly
Weather		
Meaning of 70% chance of rain	3 (subjectively)	1
Interpretation when no rain	2 (subjectively)	2
Description of accuracy rate	2 (subjectively)	2
Cancer		
Probability patient has cancer	0	4 (subjectively)

As reflected in the classroom. Each of the teachers had the desire to help students see how probability impacts their lives. To accomplish that goal, the teachers incorporated a variety of examples and activities related to or based on applications of probability. Two of the teachers, Mr. Trackman and Mrs. Talent, even began their probability units by addressing the question of how probability relates to the everyday life of the students.

Mr. Trackman began his probability unit by asking students where probability occurred in the world around them. Although it was not clear the students understood what probability was, they offered suggestions such as in accounting or on sports cards. After pointing out "sports has probability throughout," Mr. Trackman proceeded to give a number of examples from the world of sports, including the Preakness Stakes, the Super Bowl, and a baseball manager's decision to make a pitching change late in a game. No further discussion of the applications of probability occurred during the unit, except for the textbook's problem-solving assignment, which demonstrated how probability may be involved in scheduling events that depend on the weather.

The two activities observed in Mrs. Books' class were both based on real-life situations. The "Cereal Boxes" activity investigated how marketing strategies can be related to probability. As the students determined the basic assumptions in the problem, the discussion focused on many real-life questions such as packaging, restocking, and distribution. However, as the activity progressed, many of the students realized the assumptions they had made did not reflect the reality of the marketplace. In the process of doing the simulation, one student observed, "Mrs. Books, this isn't real life, 'cause if they were actually going to do this, they wouldn't put the same amount of them all in there. They'd have one that they'd only put in one in a hundred boxes, so that way you'd have to keep buying them." The second activity, "Monty's Dilemma," was also based on a real-life situation, a television game show. Thus, in these examples, Mrs. Books demonstrated to the students that probability problems arise in the world around them.

The imaginary story with which Mrs. Talent began her probability unit contained a myriad of examples of the applications of probability, ranging from the times someone takes a chance on something based on a hunch to the more formal applications of probability in establishing insurance rates. Other examples included weather predictions, lottery games, taste tests, and sports predictions. In conducting the same two activities as Mrs. Books ("Cereal Boxes" and "Monty's Dilemma"), Mrs. Talent was further connecting probability to real life. Later in the unit, in responding to a hypothetical argument that probability was only related to gambling, Mrs. Talent brought up an example to which the students would soon relate personally, the reason why automobile insurance rates are so high for teenagers. In one final example, Mrs. Talent conducted a poll of the students, asking them about their favorite school electives. In this example, she demonstrated how the results of the sample could be used to predict the responses of the entire school population. She also showed the students how such samples can be biased and the results misused.

In the midst of an experiment Mr. English was doing with colored cubes on the first day of the unit, one student asked the question, "What's the probability of this affecting us in the next 100 years?" Perceiving the student was more interested in being a distraction than actually seeking an answer, Mr. English responded, "What's the probability of my answering that question?" as he moved on with the activity. In this particular case, Mr. English did not return to address the question, although what the activity was modeling has widespread application. In the experiment, Mr. English was selecting several samples of three colored cubes from a box containing 10 colored cubes. From these samples, the students were to predict the colors of the 10 cubes in the box. In similar fashion, samples are taken to predict the outcome of political elections or the success of a new product.

In other instances throughout the unit, Mr. English did include examples of how probability applies in real-life situations. After a second sampling activity, one sampling the frequency of vowels in a selected paragraph, Mr. English gave the students an example of when such information had been applied.

> During wartime . . . when governments encode secret messages that they send to . . . the armies and different people. . . . And one of the jobs of some of the people in the Pentagon is to try to intercept those messages and figure out what the code is. And I think they do that by the frequency in which certain symbols occur. Based on the fact that the letter *e* occurs so much more than other letters in a language, then they can kind of translate the percentages into letters.

Further, in the "Frosted Wheat Yummies" activity, Mr. English demonstrated how certain merchandising techniques based on chance occurrences can be used to sell products.

One particular area or application Mr. English included in the probability unit was how probability impacts the games people play, particularly when money is involved. In addition to considering the fairness of a number of games, Mr. English also provided some historical perspective as well as exploring one of the current state lottery games. Mr. English began the unit explaining how probability had been "born in the study of games of chance– in a gambling den" (Trefil, 1984, p. 67). In this introduction to probability, Mr. English explained how one particular simple dice game favored the house. Mr. English also introduced "Montana Red Dog" as "a game that evolved out of the Old West, years and years ago." But in the analysis of the state's Powerball lottery, Mr. English demonstrated people continue to play games of chance, although he hoped the students would now be more aware of their chances in such games.

Although the teachers incorporated a variety of examples and activities related to or based on applications of probability, the nature of the connection between the examples or activities and probability usually remained implicit. For example, the teachers did not explain specifically how probability was involved in conducting taste tests or in predicting weather or outcomes of sporting events. The only real connection made was the fact that probability is related to events that involve some uncertainty.

Similarly, the connection between probability and the simulation and sampling activities generally remained implicit. For example, in the "Cereal Boxes" simulation, each prize had a specific probability of being in a given box, depending on the number of different prizes and the assumptions of the problem. This probability value would have been an important mathematical characteristic to keep in mind when designing how to conduct the simulation. This characteristic, however, was not specifically discussed as the students in Mrs. Books' class designed their simulation. In the other classrooms, where the teacher decided how the simulation would be conducted, this connection to probability was lost altogether. As a result, the activity became an application of statistics instead as the students were gathering data, finding an average, and interpreting the results. Even in the one class where expected value was introduced, the focus was on the averaging process not on the reasons for considering expected value. No question, for example, asked, "If you buy eight boxes of cereal, what is the probability you will have a full set of prizes?" In a similar fashion, the sampling activities were not much more than problems involving proportions, with the links to probability remaining obscure. Thus, any specific discussion of probability in real-life settings seemed to be limited to situations involving games, such as "Monty's Dilemma" or the state lottery.

Common Misconceptions of Probability

Both the pre- and post-observation interviews included items exploring whether the teachers' understanding of probability involved any of the common misconceptions of probability described in chapter II of this research study. This section will discuss the teachers' responses to the interview questions involving the misconceptions related to representativeness, the gambler's fallacy, a neglect of sample size, the conjunction fallacy, and an inversion of conditional probability. This section will also provide examples of misconceptions of probability revealed in the teachers' probability units.

As revealed in the interviews. As described in chapter II, the representativeness heuristic suggests the likelihood of an event or a sample is determined by the degree to which that event is similar to the major characteristics of its parent population or reflects the random process by which it is generated. Three items in the post-observation interview addressed the issue of representativeness.

In the first, the Random Digits problem (misconception question #1), the teachers were given the following item (Green, 1983a):

> A teacher asked Clare and Susan each to toss a coin a large number of times and to record every time whether the coin landed Heads or Tails. For each 'Heads' a 1 is recorded and for each 'Tails' an 0 is recorded. Here are the two sets of results:
>
> Clare:
> 01011001100101011011010001110001101101010110010001
> 01010011100110101100101100101100100101110110011011
> 01010010110010101100010011010110011101110101100011
>
> Susan:
> 10011101111010011100100111001000111011111101010101
> 11100000001000101001000001000110001010000000011001
> 00000001111100001101010010010011111101001100011000
>
> Now one girl did it properly, by tossing the coin. The other girl cheated and just made it up.
> (a) Which girl cheated?
> (b) How can you tell?

These sequences are two of the many such sequences that have an equal likelihood of occurring. However, the question is not which sequence is more likely, but which girl cheated. Therefore, the reasons provided in support of a person's choice are of particular interest. In this case, Clare has a fairly even distribution of 0s and 1s without any long strings of either. On the other hand, Susan's results contain several long strings of 0s and 1s. Reasoning according to the representativeness heuristic, one might think Clare's results are more likely to have actually occurred because they remain close to the 50-50 distribution expected of coin tosses. However, in a truly random experiment, strings of outcomes do occur. Because Clare's results did not have any such strings that are truly more representative of random occurrences, Green identified her as the one who most likely cheated. Alternatively, because one might expect the student to reason according to representativeness and create data close to the expected 50-50 distribution, Clare would again appear to be the culprit.

In responding to the Random Digits item, Mr. English and Mrs. Talent both suggested Susan had cheated, providing reasons reflecting the representativeness heuristic. Mr. English observed Susan seemed to be following a pattern, whereas Clare's results looked a little more random. Mrs. Talent agreed, suggesting, "If I were gonna cheat, I would probably want to make it look random [and] Clare's looks more random. . . . Susan's got a whole string of tails . . . nine in a row, and . . . it's supposed to be 50-50 chance heads or tails, and you'd think it wouldn't go nine times, but it could." On the other hand, Mrs. Books saw a pattern in Clare's results, suggesting to her Clare made up her results. Mr. Trackman also thought Clare cheated because her results had such a "nice even break up."

The representativeness heuristic was also considered in a pair of Birth Sequence questions (Shaughnessy, 1977) in misconception questions #3 and #4.

R1: The probability of having a baby boy is about 1/2. Which of the following sequences is more likely to occur for having six children?

(a) B G G B G B (b) B B B B G B
(c) about the same chance for each

R2: (same assumptions as R1) Which sequence is more likely to occur for having six children?

(a) B G G B G B (b) B B B G G G
(c) about the same chance for each

Reasoning according to the representativeness heuristic would lead one to chose answer (a) in response to both questions. In the first question, the three boys and three girls in answer (a) appear to be more representative of the 50:50 ratio of boys to girls than the five boys and one girl in answer (b). In the second question, answer (a) looks like a more random sequence of births than answer (b). However, these particular sequences are just 3 of the 64 possible sequences, all of which are equally likely to occur, given the assumptions of the problem.

In the first item, Mr. English chose response (a), reasoning that in Pascal's triangle there were more outcomes with three girls and three boys than with five boys and one girl. In his thinking, he mistakenly was not considering the order in the particular sequences given. In addition, Mrs. Books initially suggested there were more combinations with three of each, but when realizing order was being considered she observed, "You're going to have the same chance." Mrs. Talent said she would look at each birth as a separate event, so the given sequences would have about the same chance. In comparing the likelihood of BGGBGB and BBBBGB, Mr. Trackman reflected on his own experience.

> In my family, I would probably say [BBBBGB] because we had four boys. We didn't have five and six. The next one was probably going to be a girl. . . . I think realistically it just doesn't seem like I've ever seen a family of six that has had . . . an even distribution, so I would lean more towards the more boys or more girls because . . . I think that's because of prior knowledge of biology and knowing that it truly isn't one half.

Although his response does not necessarily reveal thinking based on representativeness, it does provide glimpses of thinking based on the gambler's fallacy ("the next one was probably going to be a girl") and the availability heuristic ("it just doesn't seem like I've ever seen a family . . .").

In responding to the second item, all but Mr. Trackman correctly recognized the two sequences would have "about the same chance." However, Mr. English indicated "intuitively I want to say (a), because . . . that's a little more spread out than (b) is." Mr. Trackman responded, "I would never answer (b) with the three boys and then three girls. It just seems like there would be more opportunity to have them come intermittent."

One of the errors leading from the idea of representativeness is called the negative recency effect or the gambler's fallacy. Two questions presented the teachers with scenarios where one might use reasoning based on the gambler's fallacy. As part of the Birth problem (probability question #5) in the pre-observation interview, the teachers were asked the following question:

> The ratio of boys to girls born is generally about 50:50. A certain family is expecting the birth of their fifth child. The first four children were girls. What is the probability that the fifth child will be a boy?
>
> (i) Less than 50% (ii) About 50% (iii) More than 50%

If the teachers began considering genetic factors, they were asked a parallel question with coins. In the post-observation interview, the teachers were given a similar scenario in the Coin Toss item (misconception question #2) taken from the fourth NAEP (Brown & Silver, 1989).

> If a fair coin is tossed, the probability it will land tails up is 1/2. In four successive tosses the coin lands tails up each time. What happens when it is tossed a fifth time?
>
> It will most likely land heads up.
> It is more likely to land heads up than tails up.
> It is more likely to land tails up than heads up.
> It is equally likely to land tails up or heads up.

Reasoning according to the gambler's fallacy would suggest a boy or a heads is more likely to occur the next time because "it's about time." However, because the birth of a baby and the toss of a coin are independent events, the probability is still about 50% in both cases. In response to the Coin Toss item, the teachers all agreed it's "equally likely to land tails up or heads up, because the fifth toss is not dependent at all on the preceding four tosses."

In response to the Birth question, Mr. English, Mrs. Talent, and Mrs. Books agreed the probability of a boy was still about 50% "because each birth is a separate event." On the other hand, Mr. Trackman gave the following response.

> I would have to say that the probability of the fifth child being a boy would be *much* more than 50% . . . because of the ratio being 50-50. Now, let's see . . . there's all the experimental error in there. You could ask, "Well, is this [the] ratio of boys to girls generally born to the same family or to the country?" Um, you'd have to deal with . . . family traits. But I would have to say, from this, if you've got four girls, then your next four should be boys . . . so that the probability would be much more than 50%.

However, when asked, "If this were a situation of tossing coins and you had four heads in a row," he responded the probability of a tail "would still be at 50%." He went on to explain genetics was the difference between the two situations, although he seemed confused about the role genetics would play. As he explained, "If you've got a family that . . . the mother has daughters, daughters, daughters, daughters . . . and all she has are sisters, all her mother had were sisters, then I would question whether she would continue to have daughters." If anything, genetics would suggest in this particular family there might be a tendency to have girls. In that case, the likelihood of a boy might be less than 50%, not more than 50% as Mr. Trackman suggested. In addition, his response not only demonstrated a lack of understanding of probability, but also about biology and the determining role of the father in such situations.

Another possible misconception resulting from the expectation of representativeness is a neglect of sample size. This misconception leads to the conclusion that any sample, no matter its size, will have the same characteristics as the original population. However, even though larger samples generally resemble the original population, smaller samples may include a great deal of variability. Two questions in the interviews explored the teachers' understanding of the importance of sample size.

In the pre-observation interview, the teachers were given the following scenario in the Two Urns problem (probability question #7):

> Imagine that you are presented with two covered urns. Both of them contain a mixture of red and green beads. The number of beads is different in the two urns: the small one contains 10 beads and the large one contains 100

> beads. However, the percentage of red and green beads is the same in both urns. Imagine that you conduct two experiments:
>
> Experiment 1: Without looking, you draw one bead from the *smaller* urn, note its color, and return the bead to the urn. This procedure is repeated until nine (9) beads have been drawn and their colors noted.
>
> Experiment 2: Without looking, you draw one bead from the *larger* urn, note its color, and return the bead to the urn. This procedure is repeated until 15 beads have been drawn and their colors noted.
>
> In which case do you think your chance for guessing the majority color is better? Explain. (Bar-Hillel, 1982)

In this case, because the beads are replaced each time, the population size has no impact on the sample. Therefore, one would get more information from the larger sample with 15 draws. The teachers, in general, did not recognize the impact of replacing the beads. As a result, although they preferred a larger sample, they were thinking of relative sample size. As Mrs. Talent pointed out, "Nine draws out of 10 things is a bigger sample than 15 out of 100 things." Using similar reasoning all of the teachers incorrectly chose to draw from the smaller urn.

As a follow-up question, the teachers were given: "If your draws resulted in 3 red and 6 green from the small urn and 9 red and 6 green from the large urn, estimate the percentage of red and green beads contained in the urns. Explain." A number of responses might be appropriate to this item. If one puts more weight in the larger sample, then one might lean the direction of 60% red (9 out of 15) and 40% green (6 out of 15). If one also considers the results from the smaller urn, the 60% red might be reduced. Or, because the samples were taken from similar populations, the samples might be combined leading to a 50-50 split of the red and green. Consistent with their responses in the first part of the problem, the teachers believed green was the majority color, again mistakenly basing their conclusions on the results from the smaller urn.

In the post-observation interview, the teachers were asked a second question related to sample size in the Hospital problem (Schrage, 1983) in misconception question #5.

> Which of the following results is more likely:
>
> (i) getting 7 or more boys out of the first 10 babies born in a new hospital?
> (ii) getting 70 or more boys out of the first 100 babies born in a new hospital?
>
> (A) They are equally likely.
> (B) Seven or more out of 10 is more likely.
> (C) Seventy or more out of 100 is more likely.
> (D) No one can say.

Those reasoning according to representativeness would conclude the two results are equally likely because the results are proportional. However, such a variation from the overall 50:50 ratio of boys to girls would be more likely to occur in smaller samples. Therefore, "getting 7 or more boys out of the first 10 babies born in a new hospital" would be more likely than "70 or more boys out of the first 100 babies."

The teachers each identified "getting 7 or more boys out of the first 10 babies" as the more likely event. As Mrs. Books observed, "You would expect some variability in a small group that would even itself out in a large group."

Another misconception sometimes related to representativeness is what Tversky and Kahneman (1983) call the conjunction fallacy. As explained in chapter II, the probability of a conjunction, A and B, cannot exceed the probabilities of its constituents. However, this conjunction rule is sometimes ignored when the conjunction seems more representative than one of the parts. An example exploring this potential misconception was given to the teachers in the Bank Teller item (misconception question #6) of the post-observation interview (variation of Tversky & Kahneman, 1982b).

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.
>
> Please rank the following statements by their probability, from most probable to least probable:
>
> (A) Linda is active in the feminist movement.
> (B) Linda is a bank teller.
> (C) Linda is a bank teller and is active in the feminist movement.

In this case, the third statement is the conjunction of the other two statements. Because persons who are bank tellers *and* active in the feminist movement would be a subset of the larger two sets as shown in Figure D.28, the conjunction would be no more likely than either of the parts of the conjunction. However, because Linda is more representative of one who is active in the feminist movement, the tendency is to rank the conjunction as more likely than being a bank teller. All the teachers were in agreement in ranking statement A as the most likely. All but Mr. Trackman agreed statement C would be ranked second, thus reasoning according to the conjunction fallacy. As reasons, Mrs. Books suggested "the third would be that she's a bank teller because I don't think a philosopher would have much fun being a bank teller." Mrs. Talent also ranked "just a bank teller last," despite the fact that she recognized "in order to be a bank teller and active in the feminist movement, she'd have to be the bank teller first." And even after the conjunction fallacy was explained, Mr.

English remained emphatic, "She's not going to be a bank teller, I'll tell you that right from the beginning, a person like that. Not from the bank tellers I know."

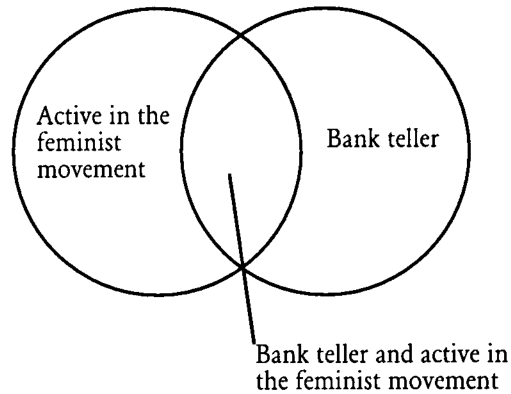


Figure D.28. Venn diagram representing the conjunction in the Bank Teller problem.

A final potential misconception investigated in the interviews was the inversion of conditional probability. This error was explored in the Cancer problem stated earlier (see probability question #10). As seen in the discussion of the Cancer problem, the teachers believed the probability the patient had cancer corresponded to the 87% accuracy rate of the test. The accuracy rate of the test is actually a measure of correctly reporting what is or is not there. Given a person has cancer, it will show a positive test 87% of the time and, given a person does not have cancer, the test will show a negative result 87% of the time. Thus, the accuracy of the screening test expresses the probability the test will be positive, *given the person has cancer.* However, what the question was asking is the inverse: What is the probability the person has cancer *given the test result is positive?* This result is quite different, as was shown earlier. Table D.9 summarizes the teachers' responses to the items exploring the common misconceptions of probability.

As revealed in the classroom. Further examples of errors in reasoning on the part of the teachers were revealed in the course of teaching their probability units. Such errors in reasoning were observed particularly during the probability lessons of Mr. Trackman, Mrs. Talent, and Mr. English. No evidence of misconceptions was seen during the observed lessons of Mrs. Books. Some of these errors corresponded to the common misconceptions identified in chapter II of this research study. However, the teachers' thinking also revealed misconceptions or reasoning errors in addition to those described earlier.

The most common misconception reflected in Mr. Trackman's teaching was the gambler's fallacy. For example, in the "Paper, Scissors, or Rocks" game, when one group reported that player A had 21 points, player B had 10 points, but player C had not won a single round, Mr. Trackman suggested, "This person's probably gonna get one pretty soon . . . the odds are there." In another situation later in the unit, when Mr. Trackman was playing the Dice Sums game with the class, the students had figured out that a sum of 7 was quite likely to occur. When in one game they had rolled the dice 23 times without any 7s, Mr. Trackman suggested "the dam is ready to break loose."

Table D.9
Summary of the Teachers' Responses to the Misconception Items

Misconception/question (#1 = Interview #1, etc.)	Number of teachers responding correctly	Number of teachers responding incorrectly
Representativeness		
Random digits (#2)	2	2
Birth sequence 1 (#2)	2	2
Birth sequence 2 (#2)	3	1
Gambler's fallacy		
Birth (#1)	3	1
Coin toss (#2)	4	0
Neglect of sample size		
Two urns (#1)	0	4
Hospital (#2)	4	0
Conjunction fallacy		
Bank teller (#2)	1	3
Inversion of conditional		
Cancer (#1)	0	4

In analyzing the Dice Sums game, Mr. Trackman's response to a student's question reflected another common misconception. Hope and Kelly (1983) suggest that "people have a tendency to confuse the categories of 'unusual' events with those of low-probability events" (p. 567). For example, because a royal flush in spades is an "unusual" highly-valued poker hand, people tend to think that it has a lower probability than other five-card hands. However, the probability of obtaining any particular set of five cards is the same. In the probability unit, after Mr. Trackman and the class had developed the theoretical results for the dice sums in 36 tosses of the dice, one student asked, "If you rolled the dice 36 times, would there be that many [tallies] by each [sum]?" Mr. Trackman responded, "You'd probably be more lucky to get it to land like that if you rolled it 36 times than you would be any other thing."

In addition, rather than recognizing that individual experimental results may vary greatly and that small samples are less reliable than larger samples, Mr. Trackman seemed to give significance to the individual results the students obtained. For example, although the worksheet for the "Paper, Scissors, or Rocks" game asked the students about theoretically rescoring the game after they had listed the possible outcomes, Mr. Trackman asked the students to figure out a way to rescore the game to make their experimental results more even. This question led to a whole host of rescoring schemes, but there was no discussion of how there could be so many different ways of making the game fair. In particular, the issue of sample size and unreliability of smaller samples was never explicitly addressed.

Finally, Mr. Trackman also demonstrated some unconventional thinking about the concept of fairness. For example, he used "rock, paper, scissors" as an example of how to resolve the issue when two students both "call shotgun at the same time" as they are going out to the car. In particular, he explained, "That's one way that you'll decide things . . . because, you know, there's a chance that you can win and there's a chance [you can lose] . . . and it kinda makes it fair." In his statement, Mr. Trackman seems to be suggesting it does not matter what chance one has, but that it is fair as long as one has a chance to win. Later, during the activity, Mr. Trackman focused on ways of evening out the scores in order to make the game fair. In this case, it appears that incorrect notions, perhaps based on intuition, can exist side by side with more correct notions based on mathematical reasoning.

As Mrs. Talent was summarizing the results for "Monty's Dilemma," she made a statement which revealed a questionable notion about random events.

> When you look at it, it's actually 2 out of 3 chance you'll pick the wrong one, and switching would make you win [and] 1 out of 3 chance you'll pick the right one and sticking would make you win. So switching is the better option. Okay? And the reason we did this experiment is because lots of times there are things out there that look completely random. Like, when I first saw this problem, I thought, "Well, this is stupid. You know, you got two doors, and what's the difference? It's a 50-50 chance." But when you look at it, there is a big difference. . . . And so, lots of things that you look at everyday, and you think, [it] doesn't make a difference, but when you analyze it mathematically, it does.

In this statement, Mrs. Talent seems to be saying she initially thought the results would be 50:50 or "completely random," but when she did the mathematical analysis, it turned out otherwise." Whether she meant to or not, Mrs. Talent seems to be suggesting the results are no longer "completely random" once she found mathematically that the probabilities for the Stick and Switch strategies are 1/3 and 2/3, respectively. Nevertheless, because the placement of the prize and the initial choice of the contestant are made randomly, the outcome of the game will still be random, unless the contestant knows something he or she should not know. The mathematical analysis just says contestants will have a better chance of winning the prize in the long run if they use the Switch strategy. Associating the concept of randomness with the idea of equally probable, which Mrs. Talent appeared to be doing in this instance, is a misconception Lecoutre (1992) calls the "equiprobability bias."

An error appeared in Mr. English's intuitive thinking as they were playing the "Montana Red Dog" game. In playing this card game, eight groups of students would each be dealt four cards. As it was each group's turn to play, their four cards and the one card turned over by the dealer would be revealed to the rest of the players. Before the game began, the following brief discussion took place:

T: Now if there's eight groups and he [the dealer] goes to the first group and he ends up at the eighth group, who has the highest chances of winning?
S: The last group.
T: Why?
S: Because they know what most all the cards are.
T: Because they already know what most of the cards are but . . . however, there will still be some cards in his hand [the dealer's] that you won't know. But their probability of winning will be higher.

Thus, both Mr. English and the students seemed to think that knowing more information would result in better odds as the game progressed. That is, they thought they would have a better chance of winning the later rounds because fewer cards would be left in the deck.

The following day Mr. English played a variation of the game with his students. In this case, four cards were shown to the students and the students individually decided on their own confidence level. After the dealer's card was revealed, the game continued with four more cards being shown to the students. As they began to play the variation, Mr. English again suggested, "Now, the odds are going to get more and more in your favor, I would think, as the card deck gets smaller. That's my prediction." Much to his surprise, what he predicted did not occur. As Mr. English brought the activity to a close, he listed on the overhead what the odds in favor of the students had been for each of the rounds (see Figure D.29). Then he observed, "My prediction was as we got fewer and fewer cards the odds would swing into our favor. That's kind of what I thought would happen. They really didn't. Only two times out of eight were the odds in your favor."

Round	Odds
1	9 to 39
2	12 to 31
3	31 to 7
4	9 to 24
5	7 to 21
6	16 to 7
7	8 to 10
8	5 to 8

Figure D.29. Results for variation of "Montana Red Dog" game in Mr. English's class.

The two versions of the game present slightly different scenarios. In the original game, each group of students held the same four cards for each round. Rather than the odds in favor of a given hand improving in this case, the odds would tend to remain fairly steady. For example, a group might hold a strong hand where they initially could beat three times as many cards as they could not beat (e.g., with odds in favor of 36 to 12). If the cards played are truly random, the number of cards revealed from the "could beat" and "could not beat" categories would also have the same ratio, as would the cards remaining. In four rounds, for example, the cards shown might have included 15 cards that could have been beaten and 5 cards that could not have been beaten. The resulting odds would be 21 to 7, still with the same 3:1 ratio. Therefore, it would make no difference if this group was selected early or late in the game. Similarly, the other groups would have odds in later rounds comparable to the odds with which they began. Therefore, the odds of the groups in the later rounds would not necessarily improve as cards are known.

In the second version of the game, each round of the game was played with a newly dealt hand. Assuming these were chosen from a well-shuffled deck, they would be a random selection of the remaining cards. Some hands may have favorable odds; many will not, as shown in the results from the class. But, most importantly, the size of the remaining deck does not influence the resulting odds.

In this example, Mr. English and his students seemed to have the same intuitive notion: If I keep playing, my odds will improve. Although repeated several times, Mr. English expressed the notion as his prediction or as a conjecture. In the end, when the results did not support his conjecture, he pointed this fact out to the students. Overall this discussion provided a positive example of making and evaluating conjectures, although that process was not explicitly discussed.

This section concludes the investigation of the four middle school teachers' knowledge of probability content as revealed in the interviews and reflected in their classrooms. Attention now shifts to the second facet of subject matter knowledge, the teachers' knowledge of the nature of mathematics and the nature of probability.

Knowledge of the Nature of Mathematics/Probability

No interview questions specifically explored the teachers' knowledge of the nature of mathematics in general or the nature of probability in particular. However, the teachers' responses to other interview questions and the teachers' classroom practice provide some evidence of the teachers' views about the nature of mathematics and probability.

Knowledge of the Nature of Mathematics

Knowledge of the nature of mathematics includes what teachers know about how the field is organized, how knowledge grows and is evaluated, and what it means to "do" mathematics. This section explores what knowledge of the nature of mathematics the middle school teachers demonstrated in the context of teaching probability. In particular, what these teachers believed about the structure of the content, the sources of authority, and the meaning of "doing" mathematics will be considered.

Structure of the content. In his interview responses, Mr. Trackman explained he liked mathematics because of its logical step-by-step nature and the security he found in knowing there was one right answer. For him, the content of school mathematics was the set of topics defined by the textbook or determined by the standardized tests taken by his students. And although he included some activities other than the textbook in the probability unit, it was still the textbook that determined what was important, because the material from the textbook was the only part of the unit evaluated.

Although Mr. English had generally moved away from using the textbook in teaching mathematics, he had chosen to use the textbook in the class observed except for the probability unit. As a result, his organization of the content of mathematics was still influenced by what the textbook contained. Although Mr. English expressed a desire for students to be able to "think mathematically . . . [and] critically," to some extent he still seemed to view mathematics as a set of facts and procedures associated with the different areas of the content. For example, among his stated goals, Mr. English wanted the students "to know vocabulary and . . . strategies that are associated with different units" and "to solve problems with the different concepts." This kind of limited thinking, with its focus on specific strategies for particular problems, was evident in the probability unit where making a chart was associated with dice activities and drawing a tree diagram was the strategy used to analyze coin problems.

Mrs. Talent had also discontinued using a specific textbook in her mathematics classes. Instead of being guided by a textbook, she focused on meeting the curriculum guidelines established by the state's Department of Education. At least some of Mrs. Talent's mathematics instruction focused on learning procedures or strategies to apply in particular situations. Examples of this focus on specific strategies for particular situations were seen in her lessons on percent problems observed just prior to the probability unit. This focus was not as evident during the probability unit where Mrs. Talent expected her students to become familiar with a variety of strategies. However, only one way of doing each problem was demonstrated. And even though Mrs. Talent wanted her students to recognize doing an experiment was an alternative strategy, she did not emphasize it as such during instruction.

In contrast to the other three teachers, Mrs. Books was following a curriculum that focused more on developing a conceptual understanding of mathematics and less on learning a given set of facts or procedures. She viewed mathematics as a multifaceted and "fascinating world of its own," but one that also has "many connections to other worlds." The multifaceted nature of mathematics influenced Mrs. Books to think of mathematics as multiple areas similar to the various mediums in which an artist might work. In at least two instances within the post-observation interview, Mrs. Books referred to mathematics in a plural sense. In one case, she explained her reasons for incorporating probability and other mathematical content throughout the school year by saying, "I think it presents more of a well-rounded picture of what mathematics are." In the second case, she expressed her desire that students see "how mathematics are to be used in probability." Mrs. Books also viewed mathematics as something with which people can interact as they build an understanding of the subject. Rather than focusing on remembering specific ways to solve problems, she emphasized exploration as an important step in the process of developing understanding. This attitude was demonstrated in how Mrs. Books herself approached problems. As she explored different ways of solving the Two Spinners problem in the pre-observation interview, she commented, "I love these complex problems that we can think about."

Within the context of the middle school curriculum, probability was generally seen as something relatively new. In particular, the teachers only recently had begun teaching the subject of probability. In some respects, probability was treated as something different within a curriculum whose focus was on the arithmetic of natural and rational numbers. For example, two of the teachers, Mr. Trackman and Mr. English, who otherwise were following a textbook in the classes observed, did not do so for the probability unit, choosing instead to actively involve the students in games and activities.

The teachers recognized the connections between probability and other mathematics topics including fractions, decimals, and percents; ratio and proportions; and statistics. However, in making the connections with these topics, the emphasis of probability instruction was often placed on procedures and correct answers. For example, the focus was more on simplifying fractions than on ascribing meaning to what the fractions represented. Thus, even though this relatively new part of the mathematics curriculum provided opportunities to actively involve students in learning about mathematics in new ways, probability generally was treated as just another part of a curriculum that focused on learning procedures and finding correct answers.

Sources of authority. Different sources of authority were evident at different times within the classrooms observed. At times, the teachers saw themselves as the primary source of authority. In some cases, the textbook or instructional materials provided the correct answers. At other times, students were given the opportunity to determine the correct result or to justify their responses. The balance between these different sources shifted as one moved from one classroom to the next.

In Mr. Trackman's classroom, establishing mathematical truth was not necessarily based on logical arguments. Instead the correctness of ideas or responses was most often determined by some authority. For example, when analyzing the probability of the different dice sums, Mr. Trackman arbitrarily decided when order would be considered. Because they were running short of time on one particular day, he decided they would consider 1 and 2 as the only possibility leading to a sum of 3. However, at a later time, the possibility of 2 and 1 was also considered. In a further example, when students were asked why they thought an answer on the Owl and Oyster Riddle worksheet was correct, they suggested it was because the teacher had done it. Thus, the students looked to the teacher for verification of their answers and accepted what he did as truth. However, the teacher himself also looked to outside sources for justification, when available. When questioned by the researcher about claims he made in the analysis of "Paper, Scissors, or Rocks," he referred to the Lane County materials and suggested that he had followed their model (which he had not).

When dealing with probability content, Mr. Trackman seemed to have contradictory views about establishing what was correct. On the one hand, finding the one correct answer still appeared to be the primary goal. For example, when one group obtained results on "Paper, Scissors, or Rocks" that were nearly even, Mr. Trackman asked them "to play it again another 25 times and then see if it's still fair . . . 'cuz it's not supposed to be fair." Thus, rather than realizing that their results were just one of many possible outcomes, the students were led to believe they had done something wrong. However, on the other hand, Mr. Trackman portrayed the process of analyzing problems as an open-ended activity. In particular, on Day 3 he suggested that "part of doing experiments . . . involves figuring out ways to analyze the data . . . and sometimes you've got to come up with [a way] on your own." For example, Mr. Trackman led the class in considering the results for "Is This Game Fair?" from several different perspectives. First, they considered how many players had won and how many opponents had won after 10 rolls of the dice. Second, some of the students, who had also played the game until one person had all 20 points, reported how long some of those games had been, and considered how many had been won by the opponent and by the player. Mr. Trackman concluded the analysis by finding the average length of these games based on the data from the class. However, no attempt was made to come to any overall conclusions about the fairness of the game based on these experimental results, so in this case there apparently was no definitive answer.

In some instances, Mr. English expected the students to determine what was correct. This approach was particularly true when dealing with computational or experimental results. For example, after the students had completed the table in analyzing the game "Which Do You Think Will Be Larger?" Mr. English had asked the students to add up the results in the two columns of the table. When several different results were given, Mr. English suggested they needed "one other group to verify" which result was correct. In other words, the totals obtained by the most students were judged to be correct. In another case, when two different sets of odds had been given by students as they played "Montana Red Dog," Mr. English asked for a show of hands to determine the correct set. Thus, rather than going back and verifying the figures and the totals mathematically, the correct answer was established by majority rule. While one would hope the correct answer would be the result found by a majority of the students, using this approach as a frequent way of finding the correct solution may send an incorrect message to the students about how mathematical truth is established.

In instances other than computational or experimental results, Mr. English saw himself as the primary source of authority regarding the content as well as correct and incorrect answers. This belief largely was because he did not feel the students had the prior experience or background knowledge needed to make such decisions on their own. However, rather than using students' responses or conjectures as opportunities to develop their ability to determine mathematical truth, Mr. English was quick to affirm correct responses or to point out errors. For example, when one student reported that he felt a game they had considered on a homework assignment was unfair, Mr. English responded, "Now Timothy says it's unfair, and he is correct." In this case, Timothy was not given an opportunity to explain why he felt it was unfair, nor were the other students given the chance to evaluate Timothy's reasoning.

In addition, Mr. English seemed to view answers as either right or wrong, with no allowance for revising conjectures in the process of developing what was correct. For example, when one student's tree diagram for tossing three coins was shown, Mr. English labeled it as an "incorrect diagram," even though the structure was correct. The student had not understood the language used in stating the problem and as a result had not used correct labels for the diagram. But rather than seeing this student's solution as a diagram "under construction" or as an opportunity to help students better understand the process of drawing tree diagrams, the student's tree diagram was labeled incorrect and a correct diagram was drawn by another student.

Mrs. Talent involved the students to a larger degree in the process of reporting answers, at least for the warm-ups and homework assignments. Rather than reading the correct answers herself, as Mr. Trackman and Mr. English did, Mrs. Talent called upon students to report their answers. From a pragmatic perspective, this process took more time, but perhaps communicated that students can be involved in the process of determining answers. However, when a student reported an incorrect answer, it was overruled by the teacher or answer key as the final authority.

On at least one occasion, Mrs. Talent presented a misleading picture of how mathematical truth is established. The students had been asked to draw a tree diagram representing the toss of a coin and the spin of a spinner and to find the probability of getting a heads and a white. When one student drew a tree diagram with the spinner first and then the coin toss, Mrs. Talent pointed out it was impossible to get heads and white on that tree. When students argued white and heads were the same thing, Mrs. Talent explained, "Like I said, if you get picky and you want it in that order, you're never going to get it that way. But if you don't care about the order, then you're right. But I care about the order." Thus, rather than order being important for mathematical reasons, order made a difference only because the teacher said so, or so it could seem to the students.

On a few occasions, both with individuals and the class, Mrs. Talent asked students to justify their thinking. For example, when some students observed as they were playing "The Hare and the Tortoise Game" that it was impossible to land on some of the places, Mrs. Talent asked the students why they thought that was the case. In her interactions with the individual students, she encouraged and affirmed their thinking with such questions or comments as, "How is it impossible?" or "What you said made sense." Later, in the class discussion of the Carnival task, Mrs. Talent had students explain their solutions and wanted the class to decide which was the correct solution. Therefore, opportunities to develop mathematical truth based on the reasoning of the students were provided. However, other opportunities such as these could have been developed. For example, she could have asked the students to state and justify their conclusions to the simulations the class conducted.

In contrast to the other three classrooms, Mrs. Books had worked hard to establish a sense of community among her students where they could make decisions about the correctness of ideas or answers on the basis of logical and reasonable arguments. In this context the students had a number of opportunities to share their observations and conclusions as well as their thinking in support of their responses. Their conjectures and conclusions were subject to challenge by other members of the class. And their thinking, if incorrect, could always be revised as they gained a better understanding of what they were dealing with. For example, this sense of determining mathematical truth was demonstrated in how Mrs. Books dealt with the issue of replacement of the beads in the "Cereal Boxes" simulation. Rather than telling the group of students they were in error when she observed they were not replacing the beads, she raised the issue with a question. By encouraging the students to share their reasons for replacing or not replacing the beads, she guided the students to determine the correct approach for themselves. The discussion was concluded when those students who thought the beads did not need to be replaced changed their minds, being convinced by the reasons given by their fellow students. When the same issue was later raised with the entire class, the final decision was similarly based on the exchange of logical and reasonable arguments.

Mrs. Books' time as the authority figure was limited. When necessary and appropriate, as in the presentation of the box-plots, she contributed vocabulary or modeled different strategies. However, for the most part, Mrs. Books acted as a facilitator raising questions for individuals or groups of students to consider and guiding the students' exploration of the mathematical problems.

Process of "doing" mathematics. Mrs. Books, perhaps more than any of the other teachers, grasped the significance of mathematics as a process as well as a dynamic body of knowledge. As a result, her students were involved in doing mathematics in meaningful ways and were coming to understand both the content of mathematics and the process of doing mathematics. In particular, the students in Mrs. Books' class were trying to make sense of and solve problems. They were encouraged to make conjectures and to explore their own ideas. They were challenged to reason about mathematics and to communicate with one another. In this setting, the students were determining mathematical truth based on logical justification. This process of determining truth occurred through interaction with one another under the guidance of Mrs. Books, as happened when the issue about the replacement of the beads was raised.

For the most part, however, the activities students engaged in as they were doing mathematics in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English differed from those the students were doing in Mrs. Books' class. Although it varied somewhat from classroom to classroom, doing mathematics in these classrooms generally meant following the directions of the teacher or the worksheet, applying strategies modeled by the teacher in the same or very similar situations, or practicing skills demonstrated to them. Students in Mrs. Talent's and Mr. English's classes were given some opportunities to communicate their reasoning and to solve problems on their own, but these opportunities were not as pervasive as in Mrs. Books' classroom. And even though students were making conjectures, such activity was not specifically encouraged and the students were rarely given opportunities to explore or justify their conjectures.

Knowledge of the Nature of Probability

The discussion concerning the nature of probability in chapter II of this research study outlined four schools of thought regarding the nature of probability. These views included theoretical, experimental, subjective, and formal interpretations of probability. Three of these views were represented in the classrooms observed. In presenting the content of probability to their students, each of the teachers distinguished between experimental and theoretical probability. For the most part, the classroom games and activities were first investigated from the experimental perspective. Then, where possible the teacher modeled the theoretical analysis for the students. However, rather than both experimental and theoretical probability being presented as equally significant approaches, experimental probability was generally just something done on the way to the theoretical analysis. In particular, there was no discussion about when each approach is applicable and why each is important.

In addition to being a step on the way to the "actual" or theoretical probability, doing experiments was seen as a way of giving the students something to do. Mrs. Talent and Mr. English also emphasized the importance of experimental probability as a concrete foundation for more abstract ideas. Mrs. Talent suggested, "Hopefully, if they get a broad enough base with experimental probability, then you can take that to the next level the next time you hit it." Similarly, Mr. English reported, "When it comes to, sort of an abstract thing, where you're talking about it in an abstract way, you have a . . . concrete foundation or you have something . . . to put the labels on or to relate back to. And that's what these [experiments], that's what they did."

Mrs. Books was the only teacher who seemed to be aware of the subjective or intuitive interpretation of probability. Although the other teachers on occasion had their students make predictions, Mrs. Books was the only one to use the term *subjective probability* and incorporate it in a formal way into her instruction. She also recognized the difficulty involved in overcoming the subjective notions both she and her students held, suggesting, "[The students] really struggle with what they believe in their mind versus what the experimental and theoretical evidence shows." Again, rather than a distinct interpretation of probability, subjective probability seemed to be viewed only as a stage on the journey to the theoretical or "true" probability value.

In addition to distinguishing between experimental, theoretical, and subjective probability, the teachers' thinking revealed two other ways of viewing the structure of probability content. First, Mr. English organized the content of probability around what he saw as the different "models" of probability, namely dice, coins, cards, spinners, and other manipulatives or tools. This organization of the content was evident in a number of ways.

For example, in reflecting on the notions that students might have about probability before instruction, Mr. English pointed out "they have a notion of flipping a coin, what your chances are there. . . . They can take a simple spinner . . . because they've studied percents, they can take a spinner and talk about your chances there. . . . [With] poker cards . . . they could probably tell you what your chance of getting an ace [is]."

This view of the structure of probability also influenced how Mr. English organized the probability unit. In discussing the factors considered in planning the unit, he reported, "I wanted it to move . . . systematically through different kinds of simulations. . . . I think I started out with colored cubes, then dice, then coins, then spinners, and then ending up with Pascal's triangle." This thinking was also evident in his reasons for choosing many of the particular activities. After presenting the basic concepts using colored cubes, Mr. English introduced several dice activities because "dice are something that the kids have had more experience with than maybe any other kind of model." In identifying the reasons for choosing some of the other activities, Mr. English repeatedly emphasized that the activities were chosen because they represented a different model, whether dice, coins, cards, or spinners.

This view of the structure of the content was also reflected in comments Mr. English made to the students. For example, on the first day when explaining what the unit would involve, he suggested to the students that "we'll do several kinds of experiments . . . we'll do them with dice, and coins, with cards, spinners . . . ." A few days later, he explained to the students, "I'm trying to focus in on dice for a couple days, because it's just a logical place and I have the dice to do it with." And as he approached the end of the unit, Mr. English was reviewing with the students what they should have learned when he said, "You should have right now a pretty good understanding of how the dice probabilities go [with dice you use the chart]. . . . You should have somewhat of an understanding about coins [you use a tree diagram]."

On one or two occasions, Mrs. Talent also referred to the different "models" when talking about the different problem settings. For example, in handing out the worksheet assignment for practice in writing probabilities on Day 5, she explained "it's going to give you several different models to look at. . . . One of the models that they use is a deck of cards." Later she pointed out that another model included on the worksheet was drawing cubes or marbles from a bag. Although this structure influenced the examples she used, it did not seem to influence the overall structure of the probability unit.

For probability as a whole, Mrs. Talent seemed to have a second structure in mind. A few days before the probability unit began, she had explained to the researcher one of the questions she reconsiders each year as she thinks about teaching probability. In particular, she struggles with how to fit together fair and unfair games, combinations and permutations, and sampling. In the probability unit that subsequently unfolded, Mrs. Talent focused primarily on the chance occurrences associated with fair and unfair games. She included one day of sampling activities and expressed a wish that there would have been more time to do more with that aspect of probability.

Although Mr. Trackman spoke of probability as a "new wing of mathematics," something other than arithmetic, little evidence suggests how Mr. Trackman viewed the structure of probability. Games were clearly the focus of his probability instruction, but he also recognized some of probability's applications in actuarial-type problems. From some of the examples of applications he gave, it was clear that, for him, counting techniques such as permutations and combinations were part of probability. In particular, he had given the example of a sound technician trying to determine which combination of all the different knobs on the sound board might produce the best results for a singer's voice. From these examples, Mr. Trackman's view of probability may have included at least some of the categories identified by Mrs. Talent.

APPENDIX E

Cross-Case Analysis: Teachers' Pedagogical Content Knowledge

This appendix provides a comprehensive analysis of the teachers' pedagogical content knowledge. In particular, this appendix will begin with an investigation of the instructional tasks used by the teachers. Second, the nature of the classroom discourse will be analyzed and discussed. Third, this appendix will consider the teachers' knowledge of students' possible conceptions and misconceptions regarding probability.

Two aspects of pedagogical content knowledge were considered in earlier sections. These include the teachers' understanding of the purposes for teaching probability, (considered along with the teachers' knowledge of educational goals in general in Appendix C), and the teachers' knowledge of representations of the concepts (explored in conjunction with the teachers' knowledge of the same concepts in Appendix D).

Selecting the Instructional Tasks

The mathematical tasks teachers use in their classrooms help shape the learning opportunities available to their students. The potential for learning is influenced by the nature of the selected tasks as well as by the manner in which the tasks are implemented in the classroom. This section will first describe the instructional tasks used by the teachers in their probability units. Second, the sources of the instructional tasks will be identified. Third, this section will consider the factors influencing the selection of the tasks. Fourth, the sequencing of the instructional tasks will be investigated. Finally, the section will discuss the nature of the instructional tasks used by the teachers during probability instruction.

Description of the Instructional Tasks

The teachers used a variety of different tasks and activities in the course of teaching their units on probability. These ranged from hands-on games and activities to textbook and worksheet assignments, from tasks actively exploring the ideas of probability to assignments practicing the skills the students were being taught. The students were also involved in doing simulations and sampling activities, although the teachers generally did not distinguish these from the other activities involving probability. Table E.1 provides a list of the different tasks the teachers incorporated in their probability units.

Games. The teachers used a large number of tasks that would be classified as games because they involve competition between individuals or teams, the distribution of points, and the designation of winners and losers. Because the games are based on chance

Table E.1

Instructional Tasks Included in Probability Units

	Mr. Trackman	Mrs. Books	Mrs. Talent	Mr. English
Number of days	10 days	7 days (observed)	15 days	12 days (observed)
Games	Paper, Scissors, or Rocks Is This Game Fair? Dice Sums Pig	River Crossing (not observed)	Chips Two-Dice Games The Hare & the Tortoise Game	Is This Game Fair? Doubles in Monopoly Lottery Fair & Unfair Games Which Do You Think Will Be Larger? The Top & One Other The Hare & the Tortoise Game Montana Red Dog Tossing Pennies
Chance Activities	Coin Tossing Exploration			Analyzing Dice Sums Three Coins Spinners Quiz or No Quiz The Maze (not observed) Pascal's Triangle (not observed)
Simulation Activities		Cereal Boxes Monty's Dilemma	Cereal Boxes Monty's Dilemma	Frosted Wheat Yummies Newspaper Offer A Ratty Problem
Sampling Activities		Sampling, Confidence & Probability (not observed)	School Electives	Colored Cubes Gum Ratio Vowel Frequency
Textbook Assignments	Experiments & Outcomes Probabilities of Events Problem Solving			
Worksheet Assignments	Owl & Oyster Riddle		More Chips Finding Probabilities Two-Stage Trees Independent Events More Dice Games Experimental Probabilities Dependent Events	Vocabulary Blocks & Marbles More Dice Games Experimental Probabilities Newspaper Pay
Evaluation	Unit Test (textbook material only)	Letter about Monty's Dilemma	Coin Tossing Task Carnival Task	Unit Test

occurrences such as rolling dice or flipping coins, they provided an opportunity to introduce the ideas of probability. Besides this natural connection to probability, the teachers cited the familiarity of students with games and the potential for having fun as reasons for including games as part of probability instruction. Because the games were a significant part of the probability units, many of them have already been described in the individual case studies.

Because of the competitive nature of the games, they generally were successful in getting the students involved in the learning activity. At the same time, the games potentially engaged the students with the mathematical content as well. As Mr. Trackman suggested in introducing his probability unit, "games and making games fair" was one of the issues addressed. Some of the games used by Mr. Trackman and the other teachers, such as "Is This Game Fair?" and "Chips," asked the students to judge the fairness of the game based on the students' experience in playing it. Other games, including "Paper, Scissors, or Rocks" and "The Top and One Other," asked students how they could make the game fair. The games also provided opportunities for students to develop and explore ways of analyzing chance occurrences. For example, "Paper, Scissors, or Rocks" asked students to list the possible outcomes, "Chips" introduced students to tree diagrams, and "Two-Dice Games" presented the dice sums and products tables as analysis tools. Yet other games were settings where students could apply what they were learning about probability as they played the games. In the River Crossing game used earlier in the year by Mrs. Books and the Dice Sums game used by Mr. Trackman, the students were predicting the likelihood of dice sums based on the discoveries they were making as they played the game. Similarly, "Montana Red Dog" presented opportunities for the students in Mr. English's class to make decisions based on their understanding of odds.

<u>Chance activities</u>. Some of the tasks, while not competitive in nature, still actively involved the students in the exploration of probability content. These tasks included the Coin Tossing Exploration Mr. Trackman's students did as well as a number of activities Mr. English included in his probability unit. Each of these chance activities involved doing an experiment of some kind, whether spinning a spinner, rolling dice, or tossing coins. The activities themselves usually focused only on experimental probabilities. However, in each case, the teachers extended the activity to include a theoretical analysis as well.

As in the case of the games, the probability activities also introduced the students to the outcomes of chance occurrences. In the Coin Tossing Exploration and "Quiz or No Quiz," students began to discover the nature of Pascal's triangle. Analyzing Dice Sums, the Three Coins problem, and "Spinners," provided opportunities for the students to compare experimental and theoretical probability. Further, during the last week of Mr. English's unit, in activities that were not observed, Mr. English modeled the use of an area model and Pascal's triangle as tools for finding probabilities.

Simulation activities. Mrs. Books, Mrs. Talent, and Mr. English each included activities in their probability units that were simulations. In these activities, actually conducting an experiment would have been impractical. To simulate an experiment instead, the students modeled the mathematical characteristics of the problem with dice, coins, cards, or other random devices. One of these simulation activities was the "Cereal Boxes" problem, which each of these three teachers did in one form or another. In this problem, the students were to determine how many boxes of cereal they would have to buy in order to obtain a full set of prizes hidden in the cereal boxes. In addition, Mrs. Books and Mrs. Talent both had their classes simulate "Monty's Dilemma," a decision-making problem arising from a television game show. In this case, after choosing one of three doors and having one of the other doors opened to reveal a gag prize, the contestant is given the option of sticking with the door they initially selected or switching to the other remaining door. Mr. English included two other simulations in his probability unit. In the "Newspaper Offer," he had his students simulate an offer one customer made as an alternative to paying the customary monthly charge. In "A Ratty Problem," students simulated rats running a maze by flipping a coin at each point of decision where the rat had to choose whether to turn right or left.

For Mrs. Books, the simulation activities provided opportunities to have the students create their own simulation designs for the situations. In the process, the issues of randomness, bias, and replacement were discussed. In addition, the simulations, "Cereal Boxes" and "Newspaper Offer," both potentially introduced the idea of the "long term average" or expected value. This was the specific goal of the "Newspaper Offer" and was the approach Mr. English used in his version of the "Cereal Boxes" activity. In their versions of "Cereal Boxes," Mrs. Talent had the class summarize the data with a show of hands and Mrs. Books focused on the various measures of central tendency and on representations of the data such as line-plots and box-plots. Further, Mr. English used "A Ratty Problem" as an introduction to binomial probability.

Sampling activities. In addition to the tasks dealing with chance occurrences, Mrs. Books, Mrs. Talent, and Mr. English included activities in their probability units that focused on sampling as an application of probability. In an activity done earlier in the year by Mrs. Books and not observed, the students were asked to predict the contents of a "hidden sack" on the basis of samples that had been drawn from the sack. As part of the introductory activities on the first day of Mr. English's probability unit, he conducted a similar experiment with students using colored cubes. After drawing out 10 different samples of three cubes each from a box containing 10 cubes, the students were to predict what the colors of the 10 cubes were.

In a second sampling activity, included by Mr. English on Day 3, the students used the results from sampling the amount of gum under their table to predict the amount of gum that would be found in their classroom or the entire school. This activity had been motivated by a proportion activity Mr. English had done in his first period class. Because he had been surprised and concerned by the results, he decided to extend the activity into his other classes, in this case relating it to probability. As he took attendance, he asked each pair of students to count the number of pieces of gum underneath their table and chairs. Mr. English then had them write a ratio of the number of pieces of gum to the number of tables, which he described as a theoretical probability (an incorrect use of the terminology). Then, by setting up and solving a proportion, Mr. English went on to explain they would be able to predict how many pieces of gum were in the entire room underneath the 17 tables. After reporting the actual number was 58 pieces of gum, Mr. English and the students multiplied that result by 20 to determine how much gum might be in the 20 classrooms of the school.

A third sampling activity, conducted by Mr. English on Day 9, involved vowel frequency in two short paragraphs he had given the students to analyze. In these paragraphs, the students were to count the number of vowels and the total number of letters. Mr. English then suggested, "You are going to set up a probability just based on counting. And that will be a theoretical probability and you want to turn that into a percent." After doing the same thing for the second paragraph, the students were to "see if there is any correlation between the probability that occurs in [paragraph] number 1 and what actually happens in [paragraph] number 2." As a homework assignment following the activity with vowel frequency, Mr. English assigned three problems from a worksheet entitled "Experimental Probabilities" (Phillips et al., 1986). Each of these problems gave the students the results from a poll. Using this information the students were to make certain predictions or find probabilities.

Before assigning the same "Experimental Probabilities" worksheet, Mrs. Talent also gave her students an example involving sampling. After having the students help her list the school's electives, Mrs. Talent sampled the class members to determine what the two most popular electives were and, based on that, had the students predict how many students from the entire school population might select those electives.

The purpose of these sampling activities was different from the purpose of the games and activities involving chance occurrences. Where chance occurrences were involved, the goal usually was to find probabilities of the different events. On the other hand, in the sampling activities, the general purpose was to predict something about the characteristics of a larger population, the color of the cubes or number of vowels, for example. However, at least in Mrs. Talent's and Mr. English's classrooms, no mention was made of this distinction between the chance activities and the sampling activities in the course of the unit.

Textbook assignments. Mr. Trackman was the only teacher observed who made assignments from the students' textbook. He suggested he had made the textbook assignments primarily because he was going to be absent and a substitute teacher would be teaching on those days. Two of the textbook assignments, sections entitled "Experiments and Outcomes" and "Probabilities of Events," helped establish a basic foundation of probability knowledge by introducing some of the vocabulary and stating some of the properties. The exercises did not actively engage the students in any experimentation or exploration, but rather asked students to respond to questions dealing with fairly traditional situations, such as drawing letters out of a hat or selecting pens from a jar. The textbook assignments included two pages identified as problem solving. On "The Class Picnic," one of the problem-solving pages, students were to answer questions using information provided to them about the likelihood of various weather conditions.

Worksheet assignments. Mr. Trackman assigned one worksheet during the probability unit he taught and Mrs. Talent and Mr. English each included a number of worksheets in their probability units. For Mr. Trackman and Mrs. Talent, these worksheets often served as the learning activity for the day. On the other hand, the worksheets Mr. English assigned were generally for homework as follow-up to a hands-on activity the students had done in class. These worksheet assignments were distinguished from the other tasks and activities because they involved no experimentation or active exploration. They instead provided practice in skills that had been modeled for the students.

The worksheet Mr. Trackman assigned served as follow-up to one of the textbook assignments, giving students further practice in finding the probabilities of events joined by *and* and *or*. The worksheet assignments given by Mrs. Talent provided further practice writing probabilities, finding simple probabilities, drawing tree diagrams, and applying the multiplication property. The worksheets used by Mr. English included practice using the basic vocabulary and properties of probability as well as some of the analysis strategies.

Evaluation tasks. The teachers used a variety of tasks for the purpose of evaluating student learning. Mr. Trackman gave his students a unit test, using questions covering the material from the textbook assignments only. Mr. English also used a unit test at the end of his probability unit. His test covered the strategies and concepts covered in the unit, with the exception of expected value. Questions were similar to what students had encountered in the activities and assignments they had done. Using a different tactic, Mrs. Talent gave her students two problems to solve which had previously been used in statewide assessments to see if the students could apply what they were learning in unfamiliar settings. To assess her students' learning, Mrs. Books asked her students to write a letter about how they had approached and investigated "Monty's Dilemma."

Sources of the Instructional Tasks

The tasks used by the teachers in their units on probability, or in some cases the ideas for the tasks, had been gathered from a variety of sources. These sources included commercial curricula, supplemental resource books, teacher-developed activities, and textbooks. Commercial curricula designed to be used for probability instruction accounted for 42% of the tasks used by the teachers. In particular, the two curricula used were the Middle Grades Mathematics Project's *Probability* (Phillips et al., 1986) and the Math and the Mind's Eye materials, *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993). The teachers did not follow either curricula activity by activity, but rather selected from among the activities provided. In some cases, the teachers took an idea for a task and implemented their own version, as Mrs. Talent did when she used the box of Honeycombs to introduce her version of the "Cereal Boxes" problem.

Supplemental resource books were the source for 27% of the tasks used. These tasks included five activities from the *Problem Solving in Mathematics* series (Lane County Mathematics Project, 1983a, 1983b, 1983c); three activities from the NCTM Addenda materials, *Dealing with Data and Chance* (Zawojewski, 1991); three activities from the Mathematics Resource Project materials, *Statistics and Information Organization* (Hoffer, 1978); and three activities from other sources. In some cases, the teachers had direct access to these materials. In other cases, the teachers had become aware of the activities at staff development workshops or classes or at mathematics conferences.

Teacher-developed tasks, which accounted for 19% of the tasks, were also included in two of the probability units. First, Mrs. Talent took a poll of her students' favorite school electives as an example of how sampling can be used to make predictions. This activity served as an introduction to additional sampling questions on the worksheet, "Experimental Probabilities," from the Middle Grades Mathematics Project curriculum. Second, Mr. English also included several original activities in his probability unit. These included the analysis of the state lottery and the sampling activity based on the number of pieces of gum the students found beneath their desks. In addition, Mr. English used some instructional materials he had created himself. These included (a) a worksheet reviewing probability vocabulary; (b) a series of worksheets exploring dice sums, including worksheets for recording experimental results, calculating theoretical probability, comparing experimental and theoretical probability, and comparing theoretical probability and odds; (c) a worksheet, "Frosted Wheat Yummies," presenting his version of the "Cereal Boxes" problem; and (d) a series of five worksheets analyzing "Fair and Unfair Games."

The source for the remaining 12% of the tasks was the textbook that was being used or had been used in the class. Mr. Trackman assigned three of the five textbook sections covering probability to his students. He also adapted a chapter resource page as the basis of the Coin Tossing Exploration. In addition, although Mrs. Talent was not using the textbook in her advanced math classes, she used two practice worksheets, "Independent Events" and "Dependent Events," from the instructional materials accompanying the textbook they had used previously.

Each of the teachers was influenced by a primary source, but drew on other resources as well. For example, Mr. Trackman relied primarily on the textbook, but also was aware of the NCTM Addenda materials, *Dealing with Data and Chance* (Zawojewski, 1991). Mr. Trackman also included games that he or his colleague had found from other sources. Mrs. Books used materials exclusively from *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993), but was influenced by other sources including the *Curriculum Standards* (NCTM, 1989). Mrs. Talent and Mr. English both used *Probability* (Phillips et al., 1986) from the Middle Grades Mathematics Project as their primary source, but each drew upon a large collection of materials gathered from conferences and other sources.

Selection of the Instructional Tasks

In selecting which tasks to incorporate in their probability units and in deciding what adaptations of the tasks were necessary or appropriate, the teachers were influenced by three particular factors. The factors referred to by the teachers included the mathematical content of the task, the nature of their students, and the ways in which students learn mathematics. One factor not mentioned by the teachers was the existence of any overall curriculum or scope and sequence that guided their organization and coverage of probability content.

Mathematical content. In providing reasons for why the teachers selected a particular task, they often referred to the mathematical content involved in completing the task. In a general sense, Mrs. Talent and Mr. English both expressed a concern of knowing why they were doing an activity and where it was leading in terms of content. More specifically, they and the other teachers indicated they chose certain activities or tasks because they introduced the concept of fairness or the use of a tree diagram, for example. Other tasks were chosen because they provided opportunities for the students to practice what they were learning. Mrs. Talent and Mr. English also had certain models in mind that they wanted the students to see. For instance, Mrs. Talent explained she had included one task because she wanted the students "to see some games with dice." Similarly, Mr. English chose many of his activities so that the students would have experience in situations using dice, coins, spinners, and cards.

While the others focused on the content in a general sense, Mrs. Books also was concerned about what the tasks conveyed about the process of "doing" mathematics. As a result, she adapted the tasks to involve the students in the process of designing the simulations they were going to conduct. She was also interested that the tasks gave the students opportunities to strengthen and build upon what they had learned in earlier probability activities.

Students. What the teachers knew about their students was a major consideration in selecting the tasks they would include in their probability units. Because of the relative immaturity of his sixth-grade students, Mr. Trackman and his colleague had decided to "go more with things that they're used to as opposed to teaching them new games." This, in particular, was the reason given for starting the unit with "Paper, Scissors, or Rocks" and including games and activities with dice and coins. When deciding which activities from *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993) to include and which to leave out, Mrs. Books considered, in part, whether the activities would be of interest to the students. Similarly, Mrs. Talent was concerned about what the reactions of the students might be to the various activities she considered including in the probability lessons. For example, she did not include some activities because she believed "they'll think it's stupid or they won't buy into it." Mr. English expressed the importance of choosing activities that would be at the right level for students, helping them move from one level to the next without creating discouragement, frustration, or negative attitudes.

Learning principles. Mr. English also seemed to take into consideration how students learn mathematics. In choosing to develop his probability unit from the activities he had gathered at conferences rather than continuing in the textbook, Mr. English pointed out, "[The textbook] has one page that is understandable. And then it immediately moves into very abstract, difficult concepts with . . . nothing in between leading into it and . . . it throws multiple concepts at kids on one lesson which violates . . . the principles of teaching for this level." As a result, he decided to focus on hands-on activities, which he felt were more appropriate for teaching the concepts of probability. Mrs. Books and Mrs. Talent used hands-on activities for mathematics instruction in general. In so doing, they also recognized the importance of concrete activities in the learning process for middle school students.

Mr. English also altered the procedures for "Montana Red Dog" in order to keep students more actively involved in making decisions throughout the game, believing their involvement was an important part of the learning process. Similarly, Mrs. Books did not follow the instructions given for "Cereal Boxes" and "Monty's Dilemma," choosing instead to get the students involved in the process of designing their own simulations.

Curriculum guidelines. No curriculum guidelines appeared to be available, either at the school, district, or state level, to which the teachers could refer for guidance concerning appropriate learning objectives or instructional activities at different grade levels. Although the *Curriculum Standards* (NCTM, 1989) provides overall goals and objectives for grades 5 through 8, no guidance is provided for what probability ideas should be presented and/or mastered at the different grade levels. Because no such articulation of the curriculum was available, the teachers generally approached their probability units with the goal of giving the students a survey of various probability topics rather than focusing on mastery of certain specified objectives.

In addition, because no curriculum provided articulation across grade levels, the teachers did not know what had been done in prior years or might be done in later years, unless the teachers themselves talked with teachers at the other grade levels. Mrs. Talent, for example, had called the sixth-grade teachers to find out what "kinds of things" her students had done the year before, so that she would not duplicate activities. From her calls, she learned some sixth-grade classes had done no probability and some had done "a little bit." From conversations with the seventh-grade teachers, Mrs. Books had learned her students might have the opportunity to do the "Checkers" activities from *Visual Encounters with Chance* (Shaughnessy & Arcidiacono, 1993) as seventh graders, so she chose other activities for them to do as sixth graders. Similarly, Mr. Trackman had been asked by the seventh-grade teachers to "get [the students] excited about probability" in preparation for a special project with actuarial data the students would do as seventh graders. From talking with other teachers in his district, Mr. English had concluded he was "the only teacher in the district [who] is teaching probability." However, Mr. English emphasized the problem of finding out what was being done at the different grade levels.

> The problem with it, though, is that . . . I'm just going to speak for our school system and I have to believe it's probably typical of other school systems. You would think that from grade level to grade level, there's a lot of communication, wouldn't you? Logically, there should be. . . . There isn't anything. There's none. And, unless, unless I seek it out, unless I go to the high school teachers and say, "What are you doing with the kids that I laid the basis for?" . . . or unless I go the eighth-grade teacher, "What are you doing?" or "Are you interested in knowing what I did?" or if I go to the sixth-grade teacher and I'm saying, "I'm interested in doing this. What have you done prior?" . . . We don't do that. And the reason we don't is because after-school time, a lot of your teachers are coaching or involved in things. . . . Again, it's that idea of . . . there is too much expected of us. We are spread too thin. We don't have enough time . . . or even if we did, because of

> different responsibilities that people have, to get together, it's a very complex thing. And, as a result, the coordination and the continuum is at the sacrificial altar. It's just not there.

Without such coordination across grade levels, teachers were left to decide for themselves what probability activities to include in their units and which probability ideas to cover.

Sequencing of the Instructional Tasks

In general, it appeared the teachers focused on planning one week at a time. As a result, it is difficult to determine how much thought was given to the overall sequencing of activities and tasks. Further, because the teachers themselves did not seem to have an overall plan for the unit, they did not provide the students with any sense of where the unit was going or how the activities all fit together. On one or two occasions, Mrs. Talent and Mr. English looked back with students to think about what they had covered and what the students should be able to do as a result, but these were the only times the students or researcher were given any sense of an overall picture.

Although planning on a weekly basis, Mr. English seemed to have some sense of how he wanted the probability unit to develop. As foundation for the experiments, which he saw as an important part of the unit, Mr. English introduced the vocabulary and basic concepts of probability on the first day. In reflecting on the rest of the unit, he explained he had wanted the probability unit "to move . . . systematically through different kinds of simulations. . . . I think I started out with colored cubes, then dice, then coins, then spinners, and then ending up with Pascal's triangle." Although occasional changes in this overall plan had occurred in order to include a particular activity such as the analysis of the Powerball lottery or to deal with the unexpected absence of a large number of students on "Take Your Daughter to Work Day," this pattern had generally been followed in the unit.

In addition, Mr. English's decisions concerning the sequencing of activities had also been influenced by familiarity of the students with the various materials. For example, Mr. English suggested he had included several dice activities early in the probability unit because "dice is something that the kids have had more experience with than maybe any other kind of model." In the end, the sequence of activities in Mr. English's probability unit not only moved students from more familiar activities to less familiar ones, it also presented the content of probability in a somewhat logical fashion, moving from simpler concepts to more difficult ones, from finding probabilities with simple charts to the more complex Pascal's triangle.

Mrs. Talent began her probability unit with two activities intended to capture the interest of the students and motivate the study of probability. She finished the unit with two evaluation tasks designed to have the students apply what they had learned. However, in between the beginning and ending, the unit seemed to lack a sense of direction. Although no major problems with sequencing the series of games and worksheets were evident, a better arrangement of activities may have been possible in at least two cases. First, after the opening activities, Mrs. Talent provided an introduction to tree diagrams by using them to analyze games with chips. Because these games involved three and four chips and the rather unusual situation of a chip with *x* on both sides, the resulting trees were fairly complicated. On the other hand, the worksheet assigned 2 days later modeled how to draw two-stage trees. Of the two assignments, the second seemed like a more appropriate introduction to tree diagrams. In the second case, Mrs. Talent observed a connection between the two-stage trees and the multiplication property and, as a result, moved the worksheet practicing multiplication with independent events forward in her plans. However, she left the corresponding worksheet with dependent events where it had been scheduled. As a result, the connection between independent events and dependent events was interrupted by several activities scheduled in the interim.

Mrs. Books was, in general, following the Math and the Mind's Eye materials, *Visual Encounters with Chance*. In so doing, she had introduced the students to the basic notions of probability in the activities done earlier in the year. To some extent, the two activities observed, "Cereal Boxes" and "Monty's Dilemma," built on the earlier activities. However, at the same time, the activities could stand on their own. Therefore, sequencing issues were not as critical in her classroom as in the others.

In contrast, Mr. Trackman's decisions concerning the sequencing of instruction did not appear to be based on mathematical or pedagogical considerations, but rather on pragmatic or personal concerns. A number of problems in the sequencing of activities occurred as a result. First, a concern for convenience outweighed pedagogical concerns. It appeared that one of Mr. Trackman's main concerns was planning around his absence on Days 4 and 5. Because he felt the textbook pages would be easier assignments for the substitute teacher to give, he chose to delay the textbook assignments until the days he was going to be absent. As a result, the students were not introduced to the basic definitions or properties of probability until Day 4, after they had been assigned more difficult situations.

A second problem arose because Mr. Trackman failed to consider what background knowledge students had or needed. He chose to begin the unit with activities familiar to the students, such as rolling dice and tossing pennies. Although it would have been possible and very appropriate to have introduced the basic notions of probability with these activities, Mr. Trackman did not do so. However, even though he believed the students had not had any prior instruction in probability, he operated under the assumption that the students knew how to find and express probabilities in these familiar settings.

Further sequencing problems arose because Mr. Trackman misjudged the difficulty of the content involved in the tasks he assigned. Although he appeared to give consideration to the difficulty of the tasks, he was judging the difficulty of the activity by the nature of the materials used or the number of rules in the game, not by the difficulty of the content. Rolling dice and tossing pennies in such activities as "Is This Game Fair?" and the Coin Tossing Exploration are simple activities at an experimental level. However, finding theoretical probabilities in situations involving dice and coins is not always simple. Had Mr. Trackman left the activities as explorations of experimental probabilities, as they were designed, there would have been no problems. Mr. Trackman's problems came when he tried to extend the activities to include a theoretical analysis as well. On the one hand, in what Mr. Trackman reported as an effort to simplify the analysis of "Is This Game Fair?" he had not considered all the outcomes possible when rolling two dice. But, on the other hand, he completely misjudged the difficulty of the next activity, the Coin Tossing Exploration, when he assigned the task of finding the theoretical probabilities associated with Pascal's triangle. And ironically, the students were assigned the introductory sections from the textbook after this very difficult activity.

Finally, there were additional sequencing problems because Mr. Trackman did not consider the impact of doing some of the textbook sections and omitting others. For example, because they had not covered the section introducing the idea of probability as a relative frequency, the students did not know what the term meant when it appeared in the assignment from the "Problem Solving" section. In addition, the last three questions on the unit test came from the sections that had been omitted. In particular, two of the items on the test applied the idea of relative frequencies. And in the section "Tree Diagrams and Compound Events," tree diagrams had been used to find all possible outcomes in situations similar to the bonus breakfast problem (Find the probability that Larry will have eggs and juice for breakfast if he could choose pancakes, eggs, cereal, or toast, and milk or juice.), but this section had been skipped as well.

Nature of the Instructional Tasks

The instructional tasks were primarily designed and selected to provide opportunities for the students to learn various aspects of probability content. However, the tasks also held the potential for involving the students in problem solving, reasoning, communicating about mathematics, and seeing connections with other mathematical ideas and applications to real-world contexts. In addition to being goals of the *Curriculum Standards* (NCTM, 1989), problem solving, communication, reasoning, and connections were among the goals stated by the teachers for their mathematics instruction. This section, therefore, will explore what opportunities the instructional tasks made available for reaching these goals. In particular, the discussion will consider what the instructional materials themselves and the teachers' implementation of the tasks contributed toward realizing the potential of the instructional tasks. This section will also include the teachers' perceptions of how these goals were addressed in their probability units.

Problem solving. As envisioned by the *Curriculum Standards* (NCTM, 1989), problem solving involves tackling questions where no strategy for finding a solution is evident. The extent to which the instructional tasks promoted problem solving in this sense varied. At one end of the spectrum was the "Problem Solving" assignment Mr. Trackman made from the textbook. Although set in a real-life situation, the questions about the probability of rain on the day of the school picnic involved a fairly simple and straightforward application of the skills and knowledge learned in the earlier sections of the chapter. Therefore, the problem-solving nature of the assignment was limited. At the other end of the spectrum, however, were a number of instructional tasks that provided opportunities for students to grapple with actual problems. For example, many of the games asked the students to determine if the game was fair and, if not fair, to figure out how to make the game fair. Each of the simulations was based on an initial problem the students were asked to solve. And although generally containing exercises (in which the solution involved straightforward application of familiar algorithms) rather than problems (in which the solution involved more flexible or higher levels of thinking), the worksheet assignments occasionally included a nontypical item such as: "A bag contains two yellow marbles, four blue marbles, and six red marbles. . . . How many marbles must be added to the bag to make the probability of drawing a blue marble equal to 1/2" (Phillips et al., 1986, p. 17)?

Many of the problems posed could potentially be solved by a variety of problem-solving approaches, including such strategies as doing an experiment, making an organized list, or drawing a tree diagram. As designed, the handout accompanying many of the tasks directed the students to do an experiment and record the results. However, although the students conducted a variety of experiments, they did not come to understand the role of doing an experiment as a problem-solving strategy. For example, when Mrs. Talent's students were trying to decide where to begin in solving the unfamiliar problems given to them as evaluation tasks, they did not consider doing an experiment until Mrs. Talent suggested they could if they wished. And even when some students began conducting an experiment, they did not seem to see how that helped them solve the problem. Thus, when the tasks involving experimentation were implemented, not enough emphasis was placed on when doing an experiment was appropriate and on what can be learned by doing one.

In all possible cases, the teacher raised the level of the task by also considering the theoretical probability. In a few cases, the instructional materials also looked at the task from a theoretical perspective. However, in the cases when the instructional materials did consider the theoretical results, they generally specified a particular strategy. For example, "Paper, Scissors, or Rocks" asked the students to "make a list of the ways three players could show the signs" (Lane County Mathematics Project, 1983b, p. 205). Similarly, the handout for "Chips" asked the students to draw a tree diagram. When the teachers considered the theoretical probability for the different tasks, they, too, generally demonstrated for the students the particular strategy to be used rather than having the students choose from their developing repertoire of strategies.

Therefore, instead of the students becoming familiar with a variety of problem-solving strategies they could use to solve probability problems, they came to rely on being told which strategy to apply in each situation. This became apparent in the response of some of Mr. English's students to the "Newspaper Pay" worksheet assignment. In the "Newspaper Offer" task done just prior to the worksheet assignment, the students had simulated drawing bills from a paper bag by selecting poker cards. In analyzing the situation from a theoretical perspective, Mr. English used an organized list to find the possible outcomes and the theoretical probability. The students applied this strategy to the first two similar items on the worksheet without difficulty. However, at least some of the students had difficulty in deciding how to solve the remaining items which involved tossing coins and rolling a pair of dice. Although these were situations the students had seen before, they had become accustomed to following the strategy modeled for them without thinking about which strategy might be appropriate and why. When faced with something that did not fit the specific pattern modeled for them, the students did not know what to do.

Mrs. Books' approach to the simulations for "Cereal Boxes" and "Monty's Dilemma" stands in contrast to the routine nature of problem solving in the other classrooms. Rather than suggesting, "Here's how we are going to find the answer to the problem," Mrs. Books asked the students what information they needed to know about the problem, what they expected the answer might be, and how they would simulate the situation. In particular, the simulation activities provided an opportunity for students to learn about designing and conducting simulations. However, of the three teachers who used simulation activities, Mrs. Books was the only one who took advantage of this opportunity. Because the students were actively involved in designing their own simulations and thinking about the problem-solving process, Mrs. Books believed they might question teachers in the future who gave them a specific way to do a simulation. On the other hand, Mrs. Talent and Mr. English determined what materials the students would use and how the students would conduct the simulations done in their classrooms. As a result, these activities became little more than additional chance activities to be conducted, from both the teachers' and the students' perspective.

In presenting many of the instructional tasks, the teachers followed a general pattern that included making a prediction, gathering experimental data, and doing the theoretical analysis. The use of this pattern provided a number of opportunities for the students to be engaged in solving problems. Making a prediction potentially tapped into the students' intuitive or subjective understanding. Gathering experimental data and doing the theoretical analysis subsequently provided opportunities for the students to address shortcomings in their subjective understanding and to develop a more accurate understanding of the probabilistic situations.

One such activity that followed this pattern was the Dice Sums game Mr. Trackman played with his class. To begin the game the students made predictions about the most likely dice sums. As they played the game, they were able to revise their predictions based on what they were observing experimentally. Mr. Trackman began the game giving no strategy to the students, rightly expecting the students would soon discover a pattern on their own as the game was played. As they continued to play the game, Mr. Trackman gave the students some clues by playing the fourth and fifth rounds himself on the board. However, Mr. Trackman never gave the students an opportunity to report their observations or to explain the patterns they were finding. Instead, before the sixth and final round, he presented the theoretical results to the students without explanation of how the theoretical results were obtained. When he proceeded to develop and justify the theoretical pattern with the students the following day, the students did not seem to be particularly interested. After all, they already knew the results. Mr. Trackman commented he liked the Dice Sums game because "it got them . . . you could just see the wheels turning, you could smell the smoke" as the students were thinking about their results. However, even though the activity had successfully involved the students in problem solving, Mr. Trackman failed to take advantage of the opportunity to have the students report their own conclusions. In this case as in others, the problem solving opportunities presented by the tasks were not recognized and/or were overlooked by the teachers.

In reflecting on how problem solving had been involved in their probability instruction, the teachers provided a number of examples reflecting different views about the nature of problem solving. In Mr. Trackman's case, he pointed out, "We did the story problems. We also did the 'Is This Game Fair?' We got them into creative, finding out that it was wrong and then creating one that was fair, based upon their results as well as based on theory . . . the probability that it should happen." Mrs. Books explained the students had been involved in problem solving as they designed their simulations and tried to remove bias from their designs. Mrs. Talent believed the "thinking part . . . in the tasks and in different situations" had involved the students in problem solving. In his response, Mr. English thought about the different problem-solving strategies, "When I think of problem solving, I think of guessing and checking, drawing diagrams, which we did. . . . We used charts, your addition charts and tables are an example of that. We never worked backwards on anything. I tend to think . . . what I've been trained in. I think we probably used about half of them in the approaches we used." Thus, Mr. English concluded he had modeled a number of the problem-solving strategies for his students in the course of the probability unit.

Communication. In theory, the instructional tasks used by the teachers in the probability units provided a number of opportunities for the students to communicate with each other and with the teacher, both orally and in writing, about mathematics and about probability. For example, in considering the fairness of the games, students could be asked to explain what fair means to them or how they might change the game to make it fair. In the many experiments the students conducted, they could be asked to state and justify what they concluded from the results. The theoretical analyses also provided opportunities for the students to explain and justify conjectures they made in the process of doing the analysis.

However, in practice, the instructional materials gave the students few opportunities to communicate about probability. For the most part, the textbook and worksheet assignments asked the students to supply only an answer without asking for any explanation. The handouts accompanying the games and activities had the students focus on tallying experimental results and perhaps answering some related questions. Only two of these handouts asked the students for any explanation. For example, the handout for "Is This Game Fair?" which both Mr. Trackman and Mr. English included in their probability units, asked the students to explain before they played the game why they thought the game might be fair or unfair. In addition, the handout accompanying "Tossing Pennies," which Mr. English gave as a homework assignment, asked students to "give reasons" to support their response to the item, "Based on your data in the table, is the game fair" (Hoffer, 1978, p. 619)? However, when these games were played, these opportunities for communication were largely ignored. In particular, neither Mr. Trackman nor Mr. English had the students make a prediction before playing "Is This Game Fair?" Further, when correcting "Tossing Pennies," Mr. English presented an analysis of the problem without asking students to state or justify their reasoning or conclusions.

Even though the instructional materials did not emphasize communication, the opportunities still existed for the teachers to get their students involved in communicating about mathematics. The teachers did take advantage of several opportunities, although only to a limited extent. For example, after the lessons on Days 1 and 2, Mr. English had the students demonstrate what they had learned on a written summary page. At other times, Mr. English asked students to share their ideas for analyzing "Which Do You Think Will Be Larger?" or their tree diagrams for representing the Three Coins problem. As part of the assignment for the evaluation tasks, Mrs. Talent asked her students to explain their thinking or describe their analysis process. At other times, she also asked her students to support conjectures they made, such as the observation that some results were impossible in "The Hare and the Tortoise Game." Further, on the Coin Tossing Exploration, Mr. Trackman asked his students to explain why the experimental and theoretical results were or were not the same. However, it was again Mrs. Books who most extensively involved her students in mathematical communication. She expected her students to explain their thinking at various times throughout their simulations of "Cereal Boxes" and "Monty's Dilemma." For example, the students described how the problem situation could be simulated, explained why the beads should be replaced, and justified why they thought certain experimental results were biased.

In reflecting on the role of communication in the probability units, the teachers identified many of their efforts to encourage verbal and written communication. Mrs. Books pointed out the students were communicating "in their small groups, with their partners, through their letters, through sharing at the overhead, so it was large group, small group, [and] written communication." Mr. Trackman also explained his students had been "working in groups, in teams . . . communicating different ways of making the games fair." In addition to the summary pages, Mr. English suggested he "encouraged [the students] to discuss [the tasks] among themselves . . . I heard a lot of good interaction among the kids using the terminology. That convinced me that they were communicating." Mrs. Talent also pointed to the various ways her students were involved in communication, "Anytime I asked them to justify. . . . They'll explain it to me as I'm walking around, and I'll say, 'Well, how'd you get that?' They can communicate verbally and, then, on the [Carnival] task, they had to communicate in writing and on the dime thing they had to communicate in writing."

<u>Reasoning</u>. The probability units also provided opportunities for the students to be involved in reasoning and thinking about mathematical ideas. The instructional tasks used as part of the probability instruction entailed thinking and reasoning at many levels, from simple recall to the more complex thinking strategies that would be typical of "doing" mathematics (e.g., conjecturing, justifying, etc.). As with problem solving and

communication, the extent to which reasoning was involved in the learning tasks depended on how the teacher developed and implemented the tasks.

At one level, much of the thinking and reasoning in the probability units involved simple comprehension and interpretation of experimental or theoretical results. For example, after playing one of the games and tallying the results, the accompanying handout frequently asked the students to indicate if the game was fair. At other times, the students were asked to report factual answers to questions such as, "How many times did the player win?" or "How many ways can you roll doubles with two dice?"

At times, the students were asked to apply procedures that had been explained or modeled for them. This involved an understanding of the definition of probability in response to questions such as, "Given 36 possible outcomes and 6 ways of getting a sum of 7, what is the probability of rolling a sum of 7?" The students also applied various strategies such as drawing a tree diagram or making an organized list. In each case, the steps to follow had been specified either by the teacher or the instructional materials. For example, to complete the worksheet, "More Chips," Mrs. Talent had specified the students were to draw the tree diagram, list the possible outcomes, indicate how many ways each player could win, give the probabilities for each player, and state whether or not the game was fair. Similarly, "Two-Stage Trees," a second worksheet Mrs. Talent assigned, asked the students to draw the two-stage tree diagrams for a number of spinner situations and then "(a) write the probability on each branch of the tree; (b) write the outcome below the branches; and (c) write the probability for each outcome."

In some cases, the meaning of the concept seemed to get lost in the process of following the specified procedures. For example, handouts for both the "Newspaper Offer" from *Probability* (Phillips et al., 1986) and "Frosted Wheat Yummies," which Mr. English had created, led the students through the steps of calculating the expected value. However, in the process of filling the numbers into the appropriate blanks on the handout, the concept of expected value as an average of the results was easily lost, even though the final result was identified as the "long term average." For example, in a small-group discussion about one offer made on the "Newspaper Pay" worksheet, one student argued it was a fair deal because the news carrier had three chances of receiving more than the usual rate of $5 and three chances of receiving less, ignoring the long term effects which expected value represented. Thus, in some cases, the calculation process became the end in itself without attention being paid to understanding the related concepts. In addition, because of the lack of closure provided by the teachers to the activities, the students did not have an opportunity to reflect on the concepts they were to have learned.

The theoretical analysis of the games and activities provided opportunities for students to be engaged in more complex reasoning. However, the students' participation in the analysis process was generally limited to responding to the teacher's questions. For example, after playing a few rounds of the Dice Sums game, Mr. Trackman could have asked his students to report their observations about the likelihood of the various dice sums. It also would have been an excellent opportunity to see if the students could justify why sums of 6, 7, and 8 are more likely than the other sums. However, when the analysis was done the following day, the students were just expected to respond to the following series of questions: "How many possible . . . combinations could you come up with from rolling the two dice?" "Now, how many possibilities from the list that we would make if we continued the list out, how many different combinations would add up to 2? . . . to 3? . . . [etc.]," "What is the probability of getting the 2? . . . of rolling a 3? . . . [and so on]."

On the other hand, Mrs. Books wanted her students to participate in thinking and reasoning about mathematics much like a mathematician would do. For example, when the question about replacement came up as the students were doing their simulations of "Cereal Boxes," Mrs. Books asked the students to think about the following question:

> So as an individual, I want you to think right now. Given the fact that you have a container of beads, there are an equal number of each of the colors of beads in there. Is it going to matter while you're doing your experiment, if you're taking the beads out and not replacing them, or if you're taking the bead, drawing it, and putting it back in? Would you make a decision for yourself, without talking to anybody else, on which of those two styles is going to give you the most accurate information based on the conditions that we put on our experiment yesterday.

After the individuals had made their decisions, the class discussed the question with students offering their reasons why the beads should or should not be replaced. The question was answered when the class became convinced by the reasons supporting the need to replace the bead. In Mrs. Books' classroom, the students had these opportunities to be involved in reasoning, in part, because the students were given time to explore problems such as "Cereal Boxes." In particular, 3 days were spent on the investigation of "Cereal Boxes" in Mrs. Books' class, where both Mrs. Talent and Mr. English did the comparable activity in a single day's lesson.

In reflecting on their probability units, the teachers recognized many of the opportunities made available by the instructional materials to promote their students' reasoning abilities. Mrs. Books spoke in a general way of how she hoped to promote reasoning by asking "thought-provoking questions . . . that were not leading but would allow [the students] to bring out some new understanding." Mr. Trackman pointed to a particular example, observing, "I know the game where they had to mark the *X*s, they had to decide, based on their results . . . they had to make decisions based upon . . . their observations." In addition to the warm-ups that included "logic kinds of things . . . that [the students] had to reason out," Mr. English suggested reasoning was involved in "the analysis. Anytime you analyze fair or unfair, that's a reasoning kind of situation, so that's incorporated all through it." Mrs. Talent emphasized,

> I'm always working on reasoning. But, I think throughout all of it I tried to talk about the reasoning behind . . . the analyzing mathematically, and why do we do that? Why do we use a tree? . . . Why do we do the experimental part? When they went from tree diagrams to learning that if you just multiply the two separate events together, it'll come out, they kind of reasoned that one out themselves. And so, I guess I was trying to, throughout, trying to get them to make some connections through reasoning.

However, the perceptions of the teachers about the reasoning students were doing did not always appear to match what occurred in their classrooms. For example, although Mr. Trackman's students may have been doing some reasoning as they played the Dice Sums game, they were given no opportunity to share their observations or the reasoning behind their strategies. Similarly, because Mr. English generally modeled the analysis process, he was doing most of the reasoning, not the students. Even though Mrs. Talent believed she had talked about the reasoning behind the analysis procedures, little evidence suggests that she did so, at least in a consistent way.

Connections. The instructional tasks also provided opportunities for students to see the connections between probability and other mathematical ideas as well as see the connections between probability and real-world contexts. The connection receiving the most attention in the classrooms was the one between probability and fractions, decimals, and percents. On several occasions, the teachers took advantage of this connection to review the students' skills in simplifying fractions and converting fractions into decimals and percents. For some activities such as the Analyzing Dice Sums worksheets, Mr. English specifically asked the students to express their results as fractions and as percents. At other times, the students used both fractions and percents to express probability without being asked to do so, as happened in the other classrooms as well.

Ratios and proportions are two additional concepts closely linked with probability. This connection was utilized in three of the classrooms. Mr. English presented odds as the ratio of the number of favorable outcomes to the number of unfavorable outcomes. In preparation for a sampling activity, he also had the students find the ratio comparing the number of pieces of gum to the number of tables. In addition, although Mr. Trackman did not introduce the term odds, he presented the part-to-part ratio as one way of expressing the likelihood of chance occurrences. Further, Mrs. Talent and Mr. English both made use of proportions in demonstrating how to predict characteristics of a population from the results of a sample.

The teachers also made use of the connections between probability and statistics. In particular, various statistical methods were used for displaying and interpreting the data from the probability experiments and simulations. For example, Mrs. Books presented line-plots and box-plots as ways of displaying the data from the "Cereal Boxes" simulation. She also had the students find the mean, median, mode, and range for their data. In addition, Mr. English asked his students to make a bar graph of their experimental results in Analyzing Dice Sums. Further, Mr. Trackman found the average of the results reported to him by the students for "Is This Game Fair?" In an effort to simplify the process of reporting data, Mrs. Talent asked her students to find the average of their three trials in the "Cereal Boxes" activity. Although these averages represented the experimental expected value, neither Mr. Trackman nor Mrs. Talent introduced the concept of expected value to their students.

Some of the tasks and instructional materials used by the teachers incorporated other mathematical topics not directly related to probability. In particular, some of the examples and/or worksheets used by Mr. Trackman, Mrs. Talent, and Mr. English included questions dealing with the ideas of multiples or primes. For example, students were asked to find the probability of rolling "a [dice] sum which is a multiple of 3." Mr. English also reviewed geometry vocabulary by putting several shapes on the overhead and asking the students to find the probability of selecting a trapezoid, for example. However, in some of these cases, the probability ideas took second place to a discussion about what a multiple or a trapezoid was.

The teachers did not take advantage of other opportunities to make connections between mathematical topics. Specifically, the schemes for making a game fair by evening out the points were based on finding common multiples, although the teachers did not seem to recognize this connection or make it explicit. In one case, as Mr. Trackman was explaining the assignment for "Paper, Scissors, or Rocks," he gave an example of how the scores might turn out, "Let's say that . . . they had . . . 16 for player A, 1 for player B, and 8 for player C." After deciding the game was not fair, he went on to explain how they could make the game fair by giving 1 point to Player A, 16 points to player B, and 2 points to player C for favorable outcomes. Because each player would then earn 16 points, Mr. Trackman suggested the game would be fair. Similarly, after Mr. English and his class had determined the odds favoring the player in "Is This Game Fair?" were 1:5, they decided the player needed to receive five points each time a sum of 7 appeared in order to make the game fair. Each of these examples used the idea of common multiple, but that aspect of the problem was not discussed.

In addition to the connections within mathematics, some of the real-life applications of probability were also presented. In particular, approximately 25% of the instructional tasks used in the probability units portrayed a real-life situation or were set in a real-world context. Some of these involved applications of probability in the students' everyday world. This included the version of the "Cereal Boxes" activity done by Mrs. Books, Mrs. Talent, and Mr. English. In this case, students began to understand the impact that offering prizes based on chance has on merchandising. Further, the questions about scheduling the class picnic from the textbook "Problem Solving" assignment Mr. Trackman gave his students illustrated how the uncertainty of weather impacts decisions.

Additionally, several of the instructional tasks applied probability to actual game situations. The problem in "Monty's Dilemma," which Mrs. Books and Mrs. Talent had their students simulate, was based on a decision-making situation on the TV program, "Let's Make a Deal." In addition, "Montana Red Dog" was given as an historical example of an actual card game. Further, in Mr. English's exploration of the state's Powerball lottery, he was providing a current example. Finally, the Carnival task was placed in a setting familiar to the students, a school carnival.

The sampling tasks also provided a number of opportunities for the students to see connections between probability and real life. Whether it was the preference stated by a student in response to a poll or the number of pieces of gum the student counted under his or her desk, the students had the chance to see how predictions can be based on polls or samples.

Other connections to real-life situations, however, were somewhat artificial. For example, "The Newspaper Offer" presented a newspaper carrier with an alternative to receiving the usual monthly payment. This scheme and similar offers on the worksheet "Newspaper Pay," were based on chance occurrences such as drawing money from a paper bag, rolling dice, or tossing coins. Such situations would rarely arise, but the problem presented an interesting setting nonetheless.

The teachers had different perceptions of the connections made as part of their probability instruction. Mr. Trackman focused on the connections between the probability unit and projects done earlier and between probability and other mathematics topics, observing, "We had done the M&M's project earlier in the year . . . and also we had done stuff with NBA . . . calculating the percentages. . . . [We] tried to relate it to fractions. We tried to relate the probability to decimals." Mrs. Books emphasized the connections with "being a consumer," pointing out how the students had discussed whether the assumptions made in the "Cereal Boxes" simulation were true in the marketplace. Mr. English outlined the connections to real life as well as within mathematics, explaining, "Well, connecting up to real life, the lottery and gambling situations. We certainly did that. Connecting up to game situations that the kids had already played like Monopoly. . . . And then all the way through we were always connecting up to decimals and percents and fractions." In addition to connections to real life and other mathematical topics, Mrs. Talent referred to desired connections within the probability concepts, "Hopefully a connection to real life. The connections to other types of math because they were using fractions. . . . So they pulled in some things that they knew there. They didn't get this like I wanted them to get it, but I wanted them to make a connection between the experimental and the theoretical. That if there's a situation, you can do it both ways."

Orchestrating the Classroom Discourse

Opportunities for learning about the content and nature of mathematics are not only shaped by the instructional tasks in which students are engaged. These learning opportunities are also influenced by the classroom discourse involved as the tasks are being investigated. This section will explore the classroom discourse observed in the four middle school classrooms. This section will first discuss the forms, patterns, and components of the classroom discourse. Then various lesson segments will provide frames for viewing the nature of classroom discourse during probability instruction.

Forms of the Classroom Discourse

The classroom discourse took the form of both oral and written communication. The most common form was oral communication. The teachers were verbally explaining the rules for the activities or asking questions as part of the analysis. The students were responding to questions, participating in a discussion, or reporting their results. Some of the verbal communication took place within small groups as students were doing the activities. At other times, the discourse involved the entire class.

The teachers also created a number of opportunities for students to communicate in writing about the probability ideas they were exploring. Although many of these involved completing a written worksheet or homework assignment, other opportunities were also provided. For example, after the students in Mrs. Books' class had explored "Monty's Dilemma," they were asked to write a letter to the researcher discussing their initial prediction, the experiment they conducted, the results they obtained, and their conclusion about the best strategy to use.

To encourage his students to think about and record what they had learned each day, Mr. English included a summary page in the students' daily packet of materials. At the end of the lesson on Day 1, he asked the students "to summarize what you think the lesson was all about." On Day 2, the students were asked to write up their analysis of a follow-up activity they had done. Although Mr. English began the probability unit with plans for the students to complete the summary page each day, he discontinued the practice after the first 2 days. When a student asked later why they were not completing the summary pages, Mr. English replied, "We did the first day and I decided . . . for your class I just wanted to talk about it rather than having you write it. . . . If this class was a little bit longer, I would have you doing those summary sheets every day, but it's kind of packed as it is." Mr. English did have the students use the summary page on one other day during the probability unit. In this case, the students were asked on Day 7 to make charts of their experimental data and theoretical predictions for the outcomes of three spinners. Although this provided an opportunity to compare the experimental and theoretical outcomes, doing such a comparison was not part of the assignment as explicitly given.

Mrs. Talent also provided opportunities for her students to communicate in written form. For example, she had the students write up the solution to the Coin Tossing problem she had taken from a statewide assessment. In addition, the groups working together on the Carnival task were asked to "write a final summary to the carnival committee telling them what you found and what you suggest." However, because of a number of absences and time constraints, this final assignment was abbreviated. As a result, the discussion of results was done orally as students were asked to share their solutions.

Patterns of the Classroom Discourse

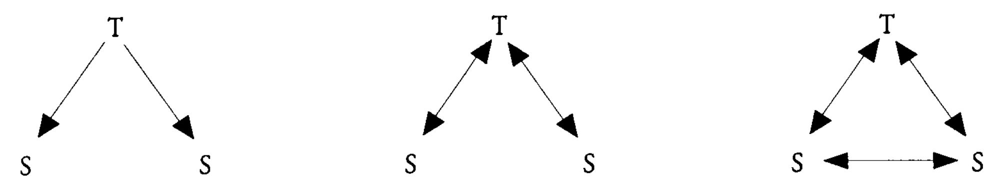
A review of the discourse occurring in the four classrooms reveals primarily three particular patterns of discourse involving the teachers and their students. These patterns include teacher monologues, teacher-directed dialogues, and instructional discussions or conversations. In addition to these patterns of interaction between the teachers and their students, interaction also occurred between students.

Teacher monologues. At times the teachers engaged in brief monologues. These occurred primarily when the teachers were introducing activities or explaining the rules for a game. For example, as introduction to "Montana Red Dog," Mr. English told the students,

> All right, you're going to play a game that evolved out of the Old West, years and years ago. And it was written up in the National Council of Teachers of Mathematics . . . what is called the *Standards* document with a unit on probability. And it's a game that involves a poker deck. [The student teacher] is my dealer today. He is going to deal each group four cards out of the

> standard poker deck. Now the standard poker deck has four suits, 13 cards to a suit. We have taken the jokers out. Ace is high, two is low. You are going to take the four cards that he gives you in your group and only your group members should look at them. Don't show them to any other groups or any other kids and don't try to look at anybody else's. Okay. Now, for this activity, your goal as a class is to try to beat me as the teacher. You are trying to get more points than I get.

Mr. English then went on to read and explain the rules for the game, which were displayed on the overhead projector. As in this case, the students sometimes had a copy of the rules or the handout for the activity to refer to during the teacher's explanation. At other times, the students were given nothing to refer to as they listened. During these monologues, the direction of the discourse flowed from the teacher to the students (see Figure E.1[a]).



(a) Teacher monologues (b) Teacher-directed dialogues or recitations (c) Instructional discussions or conversations

Figure E.1. Patterns of classroom discourse.

The teacher's purpose during these monologues was to deliver information about the games or activities to the students. The role of the students was to listen, to receive the information, to respond to questions if any were asked, and to ask questions when given an opportunity if they did not understand the rules or what they were to do. Because the rules for the games or directions for the activities were generally given on the handout the students received, the students could have been given an opportunity to read the directions and figure out what they were to do on their own. However, the teachers had decided it would be more effective and efficient to present this information to the students. In addition, this presentation often included a demonstration of the actions, a sample of results, or an example of keeping score, which helped clarify what the students were to do.

Teacher-directed dialogues. The second and clearly the predominant pattern of discourse observed in the four classrooms was teacher-directed dialogues or recitations. These dialogues were conversations between two parties with the teacher as one party and the students, either individually or as a group, as the other party (see Figure E.1[b]). This pattern of discourse occurred frequently, particularly during the reporting of experimental results, the theoretical analysis of the situations, or the presentation of instructional examples. For example, as Mrs. Talent introduced one of the worksheet assignments, the following dialogue took place:

T: You're going to get a worksheet where the whole worksheet is practicing writing probabilities. And it's going to give you several different models to look at. . . . One of the models that they use is a deck of cards. And if you don't use cards a lot, a lot of people aren't familiar with them. And in a deck of cards, if you don't include the jokers, how many cards are there? Does anybody know?
Ss: Fifty-two . . . 54.
T: There's 54 with the jokers.
S: Fifty-two without.
T: Fifty-two without. Okay, raise your hand if you can tell me . . . what cards are in one suit, if you know what I'm talking about. One suit, do you know what the suits are?
Ss: Spades, clubs . . .
T: Okay, these are clubs [*holding up a card*]. And . . .
Ss: Spades, diamonds . . .
T: Spades, okay. . . . Hearts, okay. And the last one is . . . diamonds. Who can list off, if I pick all the cards that have diamonds on them, what are they?
S: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king.
T: Okay, there's all these cards in the clubs, and there's all of them in spades, and all of them in hearts, and all of them in diamonds. . . . How many cards would this be [*pointing to the list of cards for each suit*]?
Ss: Thirteen.
T: . . . There's 13 cards. Okay. Sometimes, instead of calling the ace the 1, what do they do?
Ss: It's the 14.
T: Yeh, it's like the highest card. And it depends on what you're doing with the cards, but sometimes the ace represents the low card, sometimes it represents the high card.
S: Some games you can use it for either.
T: And some games you can use it for either. So it just depends. . . . So there's 13 cards in each suit and then there's four suits. That's where the 52 comes from: . . . 13 times 4 is 52. The reason you need to know that is they may ask you a question on the worksheet, "What is the probability . . . ?" Let me just ask you one . . . if you shuffle this up and you pulled a card out, what would be the probability that it would come out a king of diamonds?
Ss: One out of 52 . . . 4 out of 52 . . .
T: A king of diamonds?
Ss: One in 52.
T: Okay. . . . That's 1 out of 52.

After a second probability question involving cards, Mrs. Talent proceeded to give similar examples involving a spinner and drawing marbles from a paper bag before handing out the students' assignment.

As in this sample dialogue, the exchanges generally were initiated with a question or a request by the teacher. For example, Mrs. Talent asked, "How many cards are there?" For their part in the dialogue, the students provided responses to the teacher's question or request. This response was often, but not always, followed by a reaction from the teacher. This reaction sometimes involved feedback or evaluation of the response ("Yeh, it's like the highest card."). At other times, the teacher repeated and/or expanded on the student's response ("Okay, there's all these cards in the clubs, and there's all of them in spades, and . . . ."). At still other times, particularly when an incorrect answer had been given, the teacher repeated the question ("A king of diamonds?").

In these dialogues, the teacher was clearly in control of the flow and content of the discourse. It was the teacher who was asking the questions. (The nature of the questions asked will be discussed in a later section.) And it was the teacher who was the focal point of the dialogue, the one to whom the students addressed their responses. Further, it generally was the teacher who evaluated the correctness of the students' responses.

Instructional discussions. The third pattern of discourse, an instructional discussion or conversation, occurred far less frequently than the teacher monologues or the teacher-directed dialogues. Ideally, in these instructional discussions or conversations, the teacher is just one of several participants involved in the exchange of mathematical ideas. Students are not only interacting with the teacher but communicating with one another as well (see Figure E.1[c]). This was the pattern of discourse Mrs. Books was attempting to implement in her classroom. For example, on one occasion, Mrs. Books called upon a student to direct the discussion. In this case, the student, Audrey, had expressed a concern to Mrs. Books about the number of 5s reported as data in the "Cereal Boxes" simulation. The object of the activity was to see how many cereal boxes one might have to buy in order to get a set of five prizes being given away inside the boxes. The value 5 was turning out to be the mode of the class' data. However, Audrey did not think it was reasonable to buy only five boxes and obtain all the prizes as frequently as it had occurred.

T: Audrey, do you want to share your feelings about that 5? Because we talked a little bit. Come on up and tell us what you were talking about. And if you're not listening to Audrey or facing Audrey, remember that that is an issue of respect.

A: [*as Audrey (A) comes to the front of the classroom to lead the discussion*] Well, I just looked up there and I saw how many 5s were there. And I thought that probably wasn't exactly right because with what I was doing, rolling dice, it was really hard to get all the numbers in a row. And I just couldn't understand why it was so many 5s. It's, it's really hard to get 5s.

T: Okay, any comments on Audrey's idea? Do you want to call on some people and get some . . .

S1: It might be easier with the . . . [*inaudible*]

S2: I agree with her . . . it was really hard. I kept getting the same numbers over and over. And it was really hard to get . . . the number I was missing.

S3: I also agree with Audrey because I . . . [*further student comments were too far from the microphone*].
A: [*in response to student's observation about using beads*] How many beads did you have altogether?
S4: Five.
A: Oh, well, hmmm. I guess that's a little easier then, probably . . . but . . . still . . .
S5: Deborah and I, we were using beads. And we had 25 and it's really hard to get 5. We only got one 5, and all the others were trying to get, to see how many . . . it's hard to get the 5. Five times, when there was just one, you always either . . . or you draw the same color twice.
T: How many are in agreement with Audrey that that 5 just feels kind of funny? [*about 15 students raise their hands*]

Although this student-led discussion was a unique occurrence, the freedom of the students to question and interact with one another was reflected at other times in the classroom discourse. The resulting pattern of discourse was perhaps a blend of teacher-directed dialogue and instructional discussion. For example, the following sample is part of a dialogue/discussion that took place with the entire class after the students had been given time in their small groups to design a way of simulating the "Cereal Boxes" problem:

T: Okay. What I'd like to do is have you share . . . have one person share what you have thought of. And, as a listener, what I would like for you to do is to listen to your fellow classmates . . . [and] think about, mathematically, if their design is going to work. How might the information that you glean from them cause you to want to maybe make some changes in how you would run your own simulation? Okay, Eric, would you stand up and share what your table has decided.
S1: Instead of using cereal cartons, we were going to use the cups and instead of using the stickers, we're going to use beads. Then we're going to take . . . 25 cups and put the different beads in them. And then we're going to have a person at our table that hasn't been looking draw. And if we need to replace some as needed . . . and see how many . . . and then we'll do it a couple times and then take the average.
T: Okay. Thank you. Another table that's ready?
S2: Ours is exactly the same . . . except we had five cups.
T: Okay. What do you mean five cups?
S2: Eric said that his table had 25 cups and we just have five.
T: So . . . listening to them, did you want to change to 25 cups?
S2: No. . . . It's just a different way.
S3: How many beads did they have? One in each carton?
S1: I'd like to argue with this, 'cause if you . . .
T: Okay. I'm going to let you argue later, Eric. . . . Okay. Let's come over to . . . I heard this table came up with something about dice.
S4: We were going to take five dice and then cover up the 6 dot and then shake . . . roll them and see what happens.
T: So you kind of roll five at a time?
S5: What would happen if you . . . if it landed on the part that . . . the 6 dot was covered? Would you roll it again?
T: Would you count that as a . . . trial?
S4: No. If you rolled it again.

S5: *You need it to be one fifth of a chance, so you'd have to make your own dice. We could make our own dice and make it five-sided.*
T: So, James, you're saying that all they'd have to do is roll that again if they ended up on a 6?
S4: Yeh.
T: Aaron, would that still give them 1/5?
Ss: Yes. . . . No, you'd still have one sixth of a chance of rolling it on . . .
T: Okay, we'll let them kind of think about that. Let's go on to another group.

Although the teacher was still involved in guiding the direction of the discussion and monitoring student involvement, the students were beginning to interact with each other, as noted by the italicized student comments in the preceding example. The students, in particular, were asking questions of their fellow students such as how many beads the group intended to have in each cup. In orchestrating the discourse, Mrs. Books determined which comments to pursue ("You're saying that all they'd have to do is roll that again . . . .") and which to delay ("I'm going to let you argue later, Eric."). Mrs. Books also referred some of the issues back to the students for their evaluation ("Would that still give them 1/5?"). As in this case, the pattern was not one to which the students were accustomed; therefore, they needed reminders of whom they were to address and what they were to do as listeners.

Although a somewhat unusual pattern in the whole-class setting, instructional discussions occurred more frequently as Mrs. Books interacted with small groups of students. For example, during the class' simulation of the "Cereal Boxes" problem, Mrs. Books had the following conversation with one small group:

T: How's it coming? How come this bead's laying here?
S1: Would you put them back in?
T: Do you have to put them back in? Or do you leave them out when you draw . . . ?
S2: You open the box.
S1: Oh yeh. It should be gone. So . . .
S3: But they restock.
T: They do restock.
S4: Okay, so you should put them back in.
S2: You don't open the cereal and then find the toy and then say, "Okay," then give the cereal back [saying], "I don't want it."
T: Okay, let me ask you a question. I always like questions. [*as she draws out one bead*] . . . Now, according to Jacob, when we draw one out . . . .
S1: You open the cereal.
T: Okay, we got a box with an orange. Now we would . . . draw one out, right?
S1: Another box.
T: Okay. And it's taken us . . .
S3: . . . restock . . . so shouldn't we put them back in there?
S2: They don't put the same toys back in there.
S4: Yeh, but if you didn't . . .
T: So which do you think is going to give you more accurate results?

Ss: Put it back in . . . put it back in.
S3: You need the same odds every time.
T: So, do you need those same odds every time?
S2: Yeh, okay, I see.
T: Do you guys all agree with that? Kristin, does that make sense?
S2: It has to be the same odds.

In this case, Mrs. Books raised the initial question and guided the overall direction of the conversation, but the students were contributing their thoughts as they interacted with one another and with Mrs. Books.

At the end of her probability unit, Mrs. Talent tried to get the students involved in a discussion about their conclusions to the Carnival task. After the students were given time to work on the problem in groups, Mrs. Talent brought the class back together again to share their results. Three students explained what they had done, although clarifying their explanations required numerous questions from the teacher. Then Mrs. Talent attempted to get the students interacting with each other to determine which solution was correct. For example, because Chris provided a solution different from Jared's, Mrs. Talent asked the students, "Convince them that he is wrong, or Chris, convince us that Jared's wrong or . . . something like that." When no other students joined in the discussion, Mrs. Talent commented, "Okay, I've got two people talking to me, really, and the rest of you are sitting there. . . . Someone else? . . . Besides you two." However, her efforts to involve other students in the discussion were not successful. Because the time working on the task had been limited, many students evidently had made little progress on the task. As a result, they had little to contribute when the class as a whole discussed it.

Student interaction. In addition to the teacher-student interactions, students were also interacting with other students at various times during the probability units. Because the teachers were the focus of the data collection processes for this study, only limited observations can be stated about the nature of this interaction between students.

Although some of this interaction was invariably of an off-task nature, the students were also interacting with one another concerning the assigned tasks. In the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English, this student interaction usually took place in the context of playing the games and doing the activities. This involved such things as determining the different roles of the group members, recording results, and keeping score. In some cases, it also involved sharing ideas about making the game fair. However, when results were shared and discussed as a class, the interaction was primarily between teacher and students as the students reported their results to the teacher and responded to the teacher's questions.

In one case, Mr. English tried to get students to interact with each other about the content. During the experimental phase of "Which Do You Think Will Be Larger?" Mr. English encouraged two students to discuss their differing conclusions with each other.

S1: Now what are we supposed to do?
T: Okay, now you're . . . Well, who won, first of all?
S1: B had one more, not counting the tie.
T: Was it fairly close?
S1: Yeh.
T: Okay. Do you think the game is fair or unfair?
S1: I think it's unfair.
T: Why?
S1: Because B always is going to have, like a square number where it's . . .
T: Did your results conclusively prove that the game was unfair or was it really close?
S1: I think it was close.
S2: Well, it was kind of fair, but . . .
T: Okay, now you're saying it's fair. She's saying it was unfair. What I want you to do is kind of debate back and forth about . . . you tell her why you think it's fair; you tell her why you think it's unfair.
S1: It depends what you roll.
T: Hmm.
S1: It depends what you roll, Mr. English.
T: Okay. See if you can figure the probabilities behind this.

In this case, it appears the students had neither the knowledge tools nor the communication skills necessary to carry on the debate. Further, the complicated nature of the game made it a difficult task on which to initiate student communication.

In Mrs. Books' classroom, more opportunities were provided for the students to interact with one another, both in small-group settings and in large-group discussions. In their small groups, the students were sharing their subjective guesses as well as their potential simulation designs. As in the other classrooms, the students were interacting with each other as they conducted their simulations and recorded the results. But the students continued to interact with one another when the results were discussed within the larger group as well. In fact, Mrs. Books often reminded the students who were sharing that they were to talk to their peers not just to her. In this whole-class context, the students were reporting their results and explaining their reasoning to one another. Those results or reasons were subject to challenge by their peers. As a result, the interaction between students focused more on the mathematics than the interaction in the other classrooms did.

Components of the Classroom Discourse

The classroom discourse during the probability instruction was made up of a variety of different components. This section will explore the use of the language of probability and the use of questions by both the teachers and the students. The section will also consider the teachers' responses to various contributions made by students.

The language of probability. The extent to which the language of probability was used varied from classroom to classroom. At one end of the spectrum, Mr. Trackman made minimal use of the vocabulary associated with probability. At the other end of the spectrum, Mrs. Books and Mr. English both included familiarity with the language of probability as one of their instructional goals. In between was Mrs. Talent who made limited use of probability vocabulary.

The only terminology Mr. Trackman used was the word *probability* itself and that term was often used interchangeably with chance, odds, or percentage, sometimes correctly, sometimes incorrectly. For example, in referring to his earlier experiences with probability, Mr. Trackman suggested, "I always had a running percentage of what my free-throw percentage was . . . during games. I was always trying to [get] that probability up." During the probability unit, as Mr. Trackman was introducing the dice game, "Pig," the following exchange took place:

T: Now, what are the chances of [the dice] landing both 1s?
S: One in 36.
T: One in 36. So the odds . . . it's very likely that in 36 rolls you're gonna get to 100, so the chances are slim that you're gonna get double 1s.

Although Mr. Trackman used the term *odds* in his statement, what was really being discussed was the probability of the dice landing on double 1s. But as has been seen earlier, probability and odds had the same meaning to Mr. Trackman, thereby explaining why he freely interchanged those two terms.

Although Mr. Trackman did not make use of the language of probability, the sections of the textbook he assigned to students introduced some basic probability vocabulary. This vocabulary included the ideas of *experiment*, *outcome*, *equally likely*, *impossible* event, and "an event that is *certain* to occur," which were presented by example rather than definition. The term *event* was defined as "a collection of one or more outcomes." The *probability of an outcome* was defined as the ratio of the number of favorable outcomes to the number of possible outcomes, although it was not explicitly stated that this definition applied only where outcomes were equally likely. Although this vocabulary was presented by the text, the students were not held accountable for knowing it. In particular, the only term the students encountered on the unit test was the word *probability*.

Mrs. Talent made limited use of probability terminology in the course of her probability unit. In particular, she used but did not define *simulate*, *experiment*, *impossible*, *certain*, *tree diagram*, and *experimental probability*. She defined *fairness* only after the students had played two games dealing with fairness. Toward the end of the probability unit, Mrs. Talent explained the meaning of the term *dependent events* used as a title on a worksheet assignment. She, however, had not defined the term *independent events* used several days earlier on a similar worksheet. As in Mr. Trackman's classroom, the students were not held accountable for knowing or using the language of probability. As a result, probability vocabulary was generally not used and, if required, the appropriate term was sometimes not known. For example, in describing how he had solved the Carnival task, one student explained he had used a *pie chart* when he meant a *tree diagram.*

Believing exposure to "some of the language that they're going to hear" was important for students, Mrs. Books used mathematical terminology when it was appropriate and introduced specific terms when needed. Some of the language of probability, such as the terms *subjective*, *experimental*, and *theoretical*, had evidently been introduced in the activities done earlier in the year. Mrs. Books also used the words *simulation*, *random*, *biased*, and *skewed* with the assumption that they were understood by the students. As she began the discussion about *bias*, however, one student asked what she meant by the term. Mrs. Books responded by explaining "bias means that it's not going to be truly random. That there's something that's going into the factor that is going to shift the results away from what we would see if it was truly without bias and truly random." In addition, during consideration of the "Cereal Boxes" data, Mrs. Books reminded the students of the meaning of the statistical terms *mean*, *median*, *mode*, and *range* and introduced the students to *line-plots*, *box-plots*, and *outliers*. As they were drawing their box-plot of sample data, Mrs. Books referred to the length of the box as the range of the data. Otherwise, vocabulary was used appropriately.

Besides using mathematical terminology herself, Mrs. Books expected her students to use mathematical language where appropriate as well. For example, after the initial discussion about "Monty's Dilemma," Mrs. Books asked her students, "Would you write down which strategy [you would use] and a statement as to why you think that that strategy is the one that's going to get you that whopping big prize. If you can use some mathematical terms in there to help communicate to somebody else why you believe that, that's going to be helpful." Later, when the students were writing their letters to the researcher as the final assignment for "Monty's Dilemma," they were again encouraged to "use mathematical language to communicate." Mrs. Books, however, wanted to insure the mathematical language used by the students was understood by others. For example, before the term *outlier* had been introduced in class, one student used it during a discussion Mrs. Books was having with a small group about one of the outcomes in the "Cereal Boxes" simulation.

T: So, do you think [that outcome of 28 boxes is] a helpful piece of data or do you think that . . .

S1: I think it's an outlier.
T: An outlier? What's an outlier? [*She asks a second student in the group.*]
S2: I don't know. Uh . . .
S3: Outlier?
T: So, [student 1], if you're going to use words over here, is it important that you maybe explain what you're talking about?
S1: It's out there.

Although Mrs. Books expected her students to define the terms they used, she was not as careful in her own use of the language of probability. In particular, she did not define the term *random*, assuming instead that its meaning was understood by the students. However, from what the students wrote in their letters about "Monty's Dilemma," there is evidence that not all students had a correct understanding of the term. For example, one student summarized the theoretical results in the chart shown in Figure E.2. Theoretically, a person using the Flip strategy would choose the winning door 50% of the time, which is what the student's experimental results had shown. Thus, rather than understanding random to mean "occurring by chance" or "lacking a pattern," the student associated random with equally likely outcomes, a misconception Lecoutre (1992) calls the "equiprobability bias." A second student seemed to be thinking along similar lines when she summarized the results of the Switch strategy, "I also switched to switch because it is easier to win when you use the bag way [*confusing simulation design and game strategy*]. This is because if there are two gag prizes and one real prize, there is more of a chance of choosing one of the gag prizes. But that is good because if you draw a gag prize then [when] you switch you will win instantly. So there is a little bit of bies [sic] in that." In this case, the student correctly recognized the Switch strategy would more likely than not lead to winning the prize. However, as a result, she concluded there was bias because the outcomes would not be equally likely. This led to a misunderstanding of *bias* as well.

methods	Stick	Flip	Switch
percentage of wins	33.3%	random	66.6%

Figure E.2. One student's summary of the theoretical results for "Monty's Dilemma."

Mrs. Books and her students freely used both the language of probability and the terminology of odds, often in the same discussion. In the simple situations with which the students were dealing, they used the terminology appropriately and seemed to understand what they meant by expressions such as "1 out of 3 chance" or "50-50 chance." As pointed out earlier in the cross-case analysis of the teachers' subject matter knowledge, Mrs. Books did not introduce the concept of odds and made no distinction between those expressions representing odds and those representing probabilities. The letters written for "Monty's Dilemma," however, revealed that not all students used the corresponding language appropriately. For example, one student concluded, "The stick method has odds of 1 out of 3 or 1/3. . . . The flip method has odds of 50-50. . . . The switch method has odds of 2 out of 3 or 2/3." In this case, the results reported by the student for the Stick and Switch strategies are probabilities, not odds.

Because Mr. English believed having a foundation of common terminology was essential to their study of probability, he introduced the vocabulary to be used in the probability unit as part of the lesson on Day 1. Mr. English used *Mathematics: A Human Endeavor* (Jacobs, 1982) and *Probability* (Phillips et al., 1986) from the Middle Grades Mathematics Project as his models for presenting the vocabulary. In particular, Mr. English defined the terms *theoretical probability*, *experimental probability*, *event*, *outcome*, *odds*, *tree diagram*, and *binomial*. The worksheet the students completed following the presentation of vocabulary also introduced the terms *probability*, *impossible*, and *certain*. At later points in the unit, *fairness*, *expected value*, and *correlation* were defined, the latter after Mr. English had informally used the word while describing an activity. Though not defined formally, Mr. English also used the terms *experiment* and *simulation* as part of the unit. Although most of the vocabulary was defined and used correctly in the unit, the use of some terms seemed to reflect an inaccurate or incomplete understanding of the concepts. These included the definition or use of the terms *event*, *theoretical probability*, and *simulation*.

In introducing the term *event* on the first day of the unit, Mr. English suggested it "is simply, in this case, the dice roll. That's an event. If I'm using coins, the coins would be the event. It's a description of whatever it is we're doing at the time." This definition is not consistent with how the term is used in the field of probability and, in fact, it would be difficult to find the "probability of an event" in that sense. Generally, an event is defined as "any subset of the sample space" (Musser & Burger, 1997, p. 459) or any set of possible outcomes. For example, in the case of rolling a die, the event might be getting an outcome of a 6 or getting an even number.

As part of the presentation of the vocabulary on the first day, Mr. English suggested *theoretical probability* "is where you look at something without actually doing an experiment." The vocabulary worksheet, however, defined theoretical probability as "analyzing how something will happen before you conduct an experiment." The question is not when the analysis happens, but whether it is based on experimental results or on "ideal occurrences" (Musser & Burger, 1997, p. 462).

In most sources, a *simulation* is an approach used to find a probability or expected value when a mathematical analysis of the problem may be too difficult or impossible and when experimentation may be too expensive, time-consuming, or otherwise impractical. This approach involves modeling the mathematical characteristics of the problem with the use of random devices such as dice, coins, spinners, or a random-number generator. For example, instead of buying boxes of cereal to see how long it takes to obtain the entire set of six different prizes, one can toss a die until each of the six faces has been obtained. On the other hand, if a game involves tossing coins or rolling dice, doing that activity involves experimentation, but not simulation. Mr. English, however, seemed to use the terms *simulation* and *experiment* interchangeably. He defined a simulation as "an activity where you actually are rolling the dice or flipping coins or doing something." In describing to the students what the unit would involve, Mr. English suggested that "we'll do several kinds of experiments . . . they're called simulations. We'll do them with dice, and coins, with cards, [and] spinners." Further, in introducing an activity with colored cubes on the first day of the unit, Mr. English suggested, "We're going to do a little experiment. . . . I have 10 cubes. . . . I'm just going to shake this box, hold it over my head, and I'm going to draw out of the box 10 times. . . . And this will kind of be a simulation to start us out." The activities he included in the probability unit did involve a number of simulations. For example, dice were used to simulate "Frosted Wheat Yummies," his version of the "Cereal Boxes" simulation activity; cards were used to simulate drawing money from a paper bag in the "Newspaper Offer" activity; and tossing dice simulated the random decisions of rats as they ran a maze in "A Ratty Problem." However, no distinction was made between these situations and such problems as drawing cubes from a box or playing games involving the tossing of pennies.

Knowing the probability vocabulary was one of the goals Mr. English had for his students in the probability unit. As a result, the students were held accountable for knowing the vocabulary on the unit test. In particular, the students were asked to fill in the blank on the following item: "__________ is a word that means 'the chance that something will happen.' " The items on the unit test also used the terms *probability*, *odds*, *outcomes*, *certain*, and *fair*. In addition, the students were expected to understand the use of the notation *P(yellow)*. However, beyond knowing the vocabulary for the unit test, the students did not make extensive use of the probability terminology.

Questions. A second component of the classroom discourse during probability instruction involved the questions asked by the teachers and by the students. The teachers' questions played an important role in the classroom discourse, included in both the teacher-directed dialogues and the instructional discussions or conversations. Four categories of questions, covering most of the academic questions asked, are of interest to this study: subjective questions, product questions, process questions, and metaprocess questions. Other types of questions were less frequently asked but are also discussed.

Subjective questions are somewhat unique to the study of probability, in part because of the subjective nature of some of the decisions involved. With some of these subjective questions, the teachers were asking the students to make an intuitive prediction or judgment about the results they expected for an activity involving uncertainty. For example, before the students in Mrs. Talent's class simulated the "Cereal Boxes" problem, she asked them, "If I wanted to collect all three posters, how many boxes of cereal would you think I would need to buy?" Or after explaining the situation in "Monty's Dilemma, Mrs. Books instructed her students, "On your paper . . . would you write down what *you* would be most inclined to do, and *why* you would be most inclined to do that [*italics represent teacher's emphasis*]."

Other subjective questions were intended to explore students' judgments based on their knowledge of the likelihood of uncertain events. For example, after calculating the odds for each round of the "Montana Red Dog" game, Mr. English repeatedly asked his students how confident they were that they would beat the card the dealer would turn over. They had a choice of the following levels of confidence: "0—We don't think we can beat it; 1—We think we might beat it; 2—We are pretty sure we can beat it; and 3—We are certain we can beat it." In this case, rather than asking the students the same set of questions in each case, the order in which the questions were asked was related to the odds reported. For example, in one round when the odds in favor of the students were 12:31, Mr. English asked the questions in the following order: "Just out of curiosity, how many are . . . going with 0 again this time? . . . How many are going with a 1? . . . Anybody going with a 2? . . . And I assume nobody's . . . you're going with a 3?" However, in a later round when the odds in favor of the students were 31:7, Mr. English responded as follows: "Okay, odds are in your favor now. How many are betting a 3? . . . How many are betting a 2? . . . And a 1? . . . Anybody still going with 0?" Thus, by implication at least, Mr. English was indicating which direction the "bets" should go.

Product questions formed a second category of questions. These questions generally involved thinking at the knowledge or comprehension level of Bloom's taxonomy. In response to these questions, the students were expected to provide a factual response such as a word or number. These product questions were generally closed in nature because either the teacher had a particular answer in mind or a particular answer would be correct in the given setting. Examples of product questions are given in Table E.2.

Product questions were the ones used most extensively, particularly in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English. These questions appeared during all phases of the probability lessons. Specifically, series of product questions generally served as the basis for the analysis of the tasks. For example, after handing out a 6-by-6 chart

Table E.2
Examples of Product Questions Appearing in the Probability Lessons

Response expected	Examples
Recall factual information (from earlier work)	When we talk about fractions, what do we talk about the part being, compared to this part? When we started the River Crossing and you had to . . . estimate or decide where to put your boats, how did you do it the very first time?
Report factual information (e.g., experimental results)	How long was your game to get to 20? Who won?
Identify factual information	How many different ways can you come out on that spinner? Who wins on even, even, even?
Provide factual information based on comprehension	What's the theoretical probability of rolling doubles? What are the odds the tortoise will win [*after outcomes have been shown*]?

for analyzing one of the "Two-Dice Games," Mrs. Talent had the following dialogue with her students:

> T: Where do you think they got the six numbers [along the top of the chart] from?
> S: The dice.
> T: Okay. That's the six numbers on the die and then, down the other side, is the other die. So there are the two dice. And, now, if you're doing sums, if you rolled a 1 and a 1, what would the sum be?
> S: One.
> T: The sum?
> S: Two.
> T: Two! Okay. So, you're going to fill in a 2. And it's kind of like those multiplication charts you have in the lower grades. . . . So, like a 2 and a 1 would give us a sum of . . . ?
> S: Three.
> T: And a 3 and a 1, and so on. Would you take about 30 seconds and fill this chart in [*allows time for the students to complete the chart*]. . . . If you have a 6-by-6 grid, how many squares are in here?
> S: Twelve?
> T: Six-by-six?
> S: Thirty-six!
> T: Thirty-six. So the total number of sums that are in here, we can go ahead [and] write in 36. Now, somebody . . . out of 36, count up how many are even.
> S: Eighteen. . . .
> T: So, if 18 out of 36 are even, how many does that leave to be odd?
> S: Eighteen. . . .
> T: So, should this be a fair game?
> S: Yes!

T: It should be. . . . The chance that you'll get an even sum is 1/2 and the chance you'll get an odd sum is 1/2. So, mathematically, this is a fair game.

Not only were product questions used to guide students' thinking processes in general, they sometimes were used very specifically to lead students to a particular conclusion. In some cases this happened more directly than others. For example, to check the students' understanding of the definition of probability, Mr. English displayed a transparency of nine playing cards and asked the students what the probability would be of selecting a card that was a prime number. When several different answers were given, Mr. English went card by card, asking whether or not it was a prime. He then gave the students a second example.

T: Okay, what's the probability of drawing a multiple of 5? The probability of drawing a multiple of 5 out of those?
Ss: [*various answers given*]
T: Okay. Is 10 a multiple of 5?
Ss: Yes.
T: And is 5 a multiple of 5?
Ss: Yeh.
T: So, it's two out of how many?
Ss: Nine.
T: Two out of 9.
Ss: Yes.

In this case, Mr. English only asked about the favorable outcomes, presumably to expedite the discussion so that he could move on to giving the students their assignment.

Process questions were a third category of teachers' questions. These questions, which required thinking beyond the comprehension level, were generally more open-ended and designed to delve into students' thinking. In particular, they were not necessarily seeking a single specific answer. Examples of these questions are shown in Table E.3.

Mrs. Books used process questions quite extensively throughout her probability unit for a variety of purposes. In particular, she used such process questions to stimulate students to think ("Can you think of something that would help you make an informed guess that you would like to know?"); to clarify what students had said ("So you would have more than one sticker in a box?"); and to clarify students' simulation designs ("So what are you going to do with the sixth side of the die?"). Mrs. Books also used process questions in guiding students' exploration and decision making. For example, as she interacted with the different groups as they were doing the "Cereal Boxes" simulation, she asked one pair of boys, "Are you getting a random . . . [result]?" She asked another group, "How come this bead's laying here?"

Table E.3
Examples of Process Questions Appearing in the Probability Lessons

Response expected	Examples
Interpretation of experimental or theoretical data	Is this a fair game? [*after students had played "The Hare and the Tortoise Game"*]
Explanation	Why [can't you end up on S]?
Analysis	Can you figure out what the probability is that B is going to win as opposed to A going to win?
Evaluation	So which do you think is going to give you more accurate results [replacing the bead or not]?

Process questions were used less frequently and less effectively by Mr. Trackman, Mrs. Talent, and Mr. English. After giving a sample of the possible experimental results for "Paper, Scissors, or Rocks," Mr. Trackman asked his students, "How would we . . . make that game more fair?" After one student suggested playing the game again, Mr. Trackman asked if there was a different way of scoring the game. In this case, Mr. Trackman seemed to have a particular answer in mind; when the students did not provide that answer he directed them toward it. Similarly, after Mr. English's students had played "The Hare and the Tortoise Game," they were asked "How can you make this game fair?" When there was no response, Mr. English asked, "How many [points] are you going to give the tortoise?" Thus, he also guided his students in the direction he wanted them to think.

Although process questions held the potential for encouraging and exploring student thinking, Mr. Trackman, Mrs. Talent, and Mr. English often did not effectively take advantage of that potential. After her students had finished tallying up their experimental results for "The Hare and the Tortoise Game," Mrs. Talent asked, "Is this a fair game?" The students responded with a chorus of "Yes" and "No," but Mrs. Talent did not follow up, other than to have the students finish filling out the activity handout. In a similar setting, after Mr. English's students had played "Is This Game Fair?" he questioned them, "What's your judgment about it? Is this fair or not fair?" When he also received a mixed chorus of "Yes" and "No" responses, he encouraged the students to play the game again. However, neither teacher pursued the questions they asked to uncover students' thinking or the reasoning behind their responses.

In two isolated instances, Mrs. Talent and Mr. English asked process questions that seemed to connect with students and elicit more thoughtful responses. When Mrs. Talent was monitoring her students as they played "The Hare and the Tortoise Game," some individuals commented to her that it was impossible to end up on certain points on the game board. In response to the students' observations, Mrs. Talent asked, "Why?" or "How is it impossible?" In these cases, the questions encouraged thinking because they related directly to what the students were doing and thinking as they played the game. Similarly, as Mr. English's class prepared to play "Montana Red Dog," he asked, "Now if there's eight groups and he [the dealer] goes to the first group and he ends up at the eighth group, who has the highest chance of winning? . . . Why?" This question also seemed to engage the students' thinking, although in this case that thinking was incorrect, as Mr. English and the class later discovered. In contrast to these limited examples of the effective use of process questions, Mrs. Books was more effective in engaging students' thinking with her process questions, perhaps because she more frequently interacted with students in a small-group setting where she could assess the students' thinking and ask questions that would "bring out some new understandings."

A fourth category of teachers' questions would be what one might call metaprocess questions, or questions that ask students to reflect on their thinking or on the analysis process. Examples of metaprocess questions might include the following: What were the benefits of using a chart in this situation? Why did you choose to apply the multiplication property here and not use a tree diagram? What can we learn in the comparison between experimental and theoretical outcomes? Are there other strategies that would be appropriate in this situation? This type of question, it seems, would be valuable in helping prepare students to think on their own and to analyze probabilistic situations independent of the teacher's guidance. However, this type of question was noticeable by its absence from the probability units, with the exception of one situation in Mrs. Books' classroom. During the discussion of the results for the "Cereal Boxes" simulation activity, Mrs. Books asked the students, "What can we glean from looking at a line-plot as far as information?" Later, after presenting box-plots and having the students draw their own box-plots, Mrs. Books had the students compare the benefits of the two representations, asking, "What kind of data can we gather from the box-plots as opposed to the line-chart [*meaning line-plot*] that we had earlier? What can you tell very easily from this [box-plot]? What do you lose from the line-chart to the box-plot? We're making some judgments right now in valuing each of these different types of plots." As the discussion continued, Mrs. Books gave the students the following scenario to consider:

> What if you had a simulation that had 10,000 pieces of data, and you had a choice to look at a line-chart with all 10,000 pieces of data or you had a choice of looking at a box-plot. Think about which one you would like to see that data displayed in. It's going to be presented to you at a board meeting. How many would like to see the line-chart with all 10,000 pieces of data? [*Two students raise their hands.*] How many want to see a box-plot that represents the 10,000 pieces of data? [*About 12 students raise their hands.*] Now,

> wait a minute. Why don't you want to see all 10,000 pieces of data there at your board meeting so that you can get a really good picture of what's going on with your company?

Finally, the activity sheet the students completed at the end of the "Cereal Boxes" activity asked the students to reflect further on the comparison between the two representations with the question, "We have used both line-plots and box-plots to visualize the data from the cereal box simulation experiment. What are some advantages of each of these types of plots? What are some disadvantages? Explain." In this series of questions, the students had been asked to reflect on the methods they had used to represent data, but this period of reflection was a unique occurrence within the probability units.

The teachers asked a number of other questions that do not fit into these four categories. These included questions teachers asked to check that students understood what they were to do to complete an assignment. For example, after giving directions for one of the games, Mr. English asked, "Any questions before we start this one?" In a similar setting, Mrs. Talent asked, "Does everybody understand?" The teachers also used questions to monitor student progress as they were working on an activity. As the students were playing one game, Mr. English inquired, "Okay, how many have finished up one game so far?" As Mrs. Books circulated among the groups as they worked on an activity, she often asked, "How are you guys doing?" Although these questions did not stimulate significant responses from the students, they served the limited purpose for which they were asked.

Besides falling into the categories described, the questions asked by the teachers during probability instruction had other interesting characteristics as well. One of these characteristics was that the teachers sometimes did not just state a single question but asked multiple questions at the same time. For example, as Mrs. Books was demonstrating how to draw a box-plot for a sample with 19 pieces of data, she asked her students, "How would we find out from the 19 pieces of data where 90% of the data would fall? What could we do? . . . How would you go about finding out where 90% of that data is? You're a mathematician. Your job is to find out and report back to your company where that 90% confidence would be. How might you do that? Would you talk about it at your table what your strategy would be." As in this case, the follow-up questions either restated or rephrased the original question. In other cases, the teachers followed up with different but related questions. These series of questions perhaps had the potential of either stimulating students' thinking or confusing them. On the one hand, asking the questions in different forms might have communicated with more students, where each could hear it in a form that might be more meaningful. However, on the other hand, it could also be confusing if the students are left wondering just which question it was they were to answer. Other students may not be able to see the relationship between the questions or follow the logic and may give up instead of trying to answer the question(s).

A number of questions Mr. Trackman asked during his probability unit were quite vague. Because of this, the intended question-and-answer sessions became more of a guessing game, as in the following excerpt where they were talking about the representation of probability as a fraction:

> T: When we talk about fractions, what do we talk about this part being, compared to this part?
> S: Denominator and numerator.
> T: Well, we have the numerator and denominator, but what do we call this in relationship to the other one?
> S: Uh, uh . . .
> T: Okay. The first part we call . . . the top one we call the part. This bottom was . . .
> S: Parts per million.
> T: No, not parts per million. You're still at Outdoor School.
> S: Oh, the whole.
> T: Yeh, the whole! This is the whole thing [*pointing to the denominator*] and this is the part of it [*pointing to the numerator*].

The teacher was not the only one asking questions in the classrooms during probability instruction. The students were also asking questions, although not as often as the teacher. Some of the students' questions were seeking to clarify the teacher's expectations. For example, as students were completing one assignment, they asked, "Do you want it in fractions or in percents?" Or as students were conducting an experiment, a student asked, "Are we supposed to write this down?" At other times, students sought to clarify the directions given for the tasks with questions such as "What do you do if you tie?" or "What if you don't have a spade?"

Some of the students' questions in each of the classrooms dealt with probability terminology. For example, when students in Mr. Trackman's class encountered unfamiliar terminology on the "Problem Solving" assignment, they asked, "What is relative frequency?" After Mrs. Books explained that she wanted the class "to talk a little bit about types of bias," one student asked, "What do you mean by bias?" As the students in Mrs. Talent's class were drawing tree diagrams to complete a worksheet, one asked, "Why do they call it a two-stage tree?" During a discussion about the fairness of a game they had played, one of Mr. English's students asked, "What do you mean by fair? Fair to both players? Fair to one player?"

Although most of the questions asked by students were clarifying either teachers' expectations, activity directions, or probability terminology, the students also asked a few questions dealing with the probability outcomes. For example, as the students in Mrs.

Talent's class were playing "The Hare and the Tortoise Game," a student observed, "It's impossible to get on an S, isn't it?" After playing the same game, one student in Mr. English's class asked, "How come it landed on P and X so much?" After Mr. Trackman had summarized the dice sums outcomes, a student asked, "If you rolled the dice 36 times, would there be that many [tallies] by each [sum]?" The occurrence of these questions, however, was quite limited, except in Mrs. Books' classroom.

Mrs. Books encouraged her students to ask questions throughout the instructional process. For example, as their investigation of the "Cereal Boxes" activity began, she asked the students, "Can you think of something that would help you make an informed guess that you would like to know?" In response, the students asked a number of questions, including, "What are the odds of getting each of the stickers? Did one carton of boxes . . . cereal boxes all have the same sticker put inside? Is there any indication of what's inside? Are all the stickers in one store? How many boxes are left in the store you are at? Are there the same amount of stickers 1 and stickers 2 . . . ?" Later, as the students shared their ideas for conducting their simulations, Mrs. Books encouraged the students to ask questions of each other. During this discussion, one student asked, "How many beads did they have? One in each carton?" Another asked, "What would happen if you . . . if [the die] landed on the part that . . . the six dot that was covered? Would you roll it again?" The students were also encouraged to raise questions about the results. As the experimental data were being recorded, one student questioned the number of 5s being reported. In this case, Mrs. Books allowed the student to share her concern with the entire class. Although rare in the other classrooms, questions of this nature were an ongoing part of the classroom discourse in Mrs. Books' classroom.

<u>Responses to students' conjectures</u>. In some of the questions students were asking, they were actually expressing conjectures or insights they had about the activities. However, with the exception of Mrs. Books' class, the students generally were not encouraged to make or share any conjectures or observations. On the rare occasions when such conjectures or observations were made, the response from the teachers varied.

In analyzing "Is This Game Fair?" Mr. Trackman had (incorrectly) determined there were 3 ways the player could win with a dice sum of 7 and 18 ways the player could lose with a sum other than 7. Because the player received three points for a win and lost one point otherwise, Mr. Trackman concluded the opponent would generally win. One student spoke up asking, "Yeh, but what if you times it by 6?" With 3 ways of winning and 18 ways of losing, awarding six points for a winning toss would even up the scores, theoretically making the game fair. In response to the student's observation, Mr. Trackman replied,

> What if we times it by 6? That would be something we could investigate if we had more time. But we don't. And that, you know, our little theory through this . . . that's part of the trial and error . . . that probability involves. Sometimes you just kinda, you're not sure about things, so you just kinda do it. You go with the theory and then you try it out. You see if it really works . . . if the theory works. Okay. In this little theory, it looks like they should be even. Okay? Any questions about what we've done or where we've gone?

Thus, in Mr. Trackman's classroom, there was no time for pursuing student conjectures. Even if there had been time, it is not clear Mr. Trackman knew how to deal with such conjectures. In some cases, he did not understand what the students had said. In other cases, he did not know appropriate ways to explore or respond to their conjectures.

In the other classrooms, students' conjectures were treated more positively. Yet, even in these cases, the responses varied. One example of this variety is evident in the different responses given by Mr. English and Mrs. Talent to similar observations made by their students during "The Hare and the Tortoise Game." In this game, students were moving a marker along a game board marked with the letters M, N, P, S, X, Y, and Z. Starting at point S, the students moved the marker left if the roll of a die was odd and moved the marker right if the roll of the die was even. Each turn involved three rolls of the die. As the students were playing the game in Mr. English's and Mrs. Talent's classrooms, they observed it was impossible to end up on some of the letters after the third roll of the die. Mr. English responded to one student's observation by saying, "All right. And that's true. Figure out which ones they are as you play the game." Later, after he and the class had analyzed the game, Mr. English asked, "And furthermore, did you notice that three of these letters you cannot get? So the probability is 0; it's impossible." Although his response encouraged the students to figure out which ones were impossible, the students were given no opportunity to report what they had found. And by his quick affirmation of the observation, he provided no opportunity for the students to explain or justify their thinking.

When Mrs. Talent's students made the same observation, she responded with, "Why not?" or "How is it impossible?" This response encouraged the students to try to explain their reasoning. Without totally affirming their explanations, Mrs. Talent encouraged them to think and investigate further with comments such as, "Okay. See if that's true. If you never land on it, you might be right." Or she suggested, "Could be. . . . It looks like you can't. What you said made sense." As the students thought further about the situation, some began to observe there were "odd places" and "even places." Another pair of students observed one would get similar results "if you use any odd number" of tosses of the die. Thus, the students had been stimulated to think about the problem. However, this issue was not discussed as the class shared their experimental results or analyzed the problem theoretically, even though the tree diagram drawn for purposes of analysis offered clear supporting evidence some letters were "even places" and others were "odd places." Thus, pursuing conjectures was valued on an individual basis but not seen as part of the overall goal for the activity or for the class as a whole.

Sharing of students' ideas and conjectures was an integral part of the classroom culture and discourse for Mrs. Books and her students. In one particular case, a student became concerned when the data being recorded for the "Cereal Boxes" activity showed a large number of 5s. Because they were trying to obtain five prizes from the boxes of cereal, this student did not feel it was realistic to accomplish the goal as often as it appeared when buying only five boxes of cereal. Mrs. Books responded to the student's concern by asking, "So which table . . . do you need to go investigate tables and find out where those 5s are coming from and talk to them?" After the student discovered the source of many of the 5s, she asked the boys at one table, "I was just wondering. How did you get . . . 5s that often?" And, later, as the entire class was considering the data, Mrs. Books gave the student an opportunity to share her concern with her classmates. As a result, other students concurred that it was hard to obtain the five prizes in only five tries.

<u>Responses to students' errors</u>. The teachers' responses to incorrect answers or ideas were largely influenced by the teachers' views about the nature of mathematical knowledge and of the learning process. Consistent with her constructivist views of learning, Mrs. Books treated students' solutions or mathematical conjectures as reasonable hypotheses, whether they were "right" or "wrong." She accepted the students' contributions and used them as the starting point from which she tried to guide the students to a correct understanding based on sound mathematical reasoning. This approach was evident in matters as basic as computation as well as in more abstract situations. For example, when an incorrect calculation was proposed, Mrs. Books directed the students' attention to reasoning out what was correct, as demonstrated in the following excerpt:

T: Well, how would we find 90%. Given this data, where would we find out where 90% would be? . . .
S: [There are] 19 total and . . . you figure 9/10. . . . So you divide by 0.9.
T: Nineteen divided by what?
S: [By] 0.9. . . . That's about 21.
T: Let's think about this, 90% of 10 is how many?
Ss: Nine.
T: Nine [and] 90% of 9 is what?
Ss: About 8.
T: About 8, so if we add those two together?
S: Then it's about 17.
T: So, is it 19 divided by 0.9?
S: It's, I think it is . . . No, times . . . maybe 19 multiplied by 0.9?
T: [*Using a calculator*] So, if we have 19 times 0.9, that gets us to 17.1. . . . Does that seem reasonable?

Because Mrs. Books viewed the students' understanding as a work in progress, she made allowances for revision to occur. In some cases, she seemed to orchestrate the discourse in ways that provided opportunities for the students to discover their own errors and make correct decisions. For example, Mrs. Books had discovered some students were not replacing the beads they were selecting from containers in the "Cereal Boxes" simulation. After presenting the class with the choice between replacing the beads and not replacing the beads, the question was discussed:

T: How many of you feel that it is not necessary to replace the beads . . . that you can draw them out and put them to the side? . . . [*one or two hands raised*] Okay, we've got a couple that are thinking that way. How many think it's needed that you put the bead back into the container? [*most students raise their hands*] . . . Okay, can I have one person that feels real adamant that you've got to put those beads back in, and if you don't put those beads back in, you're going to have major trouble . . . Can I have you address us on that issue? [*A brief discussion follows with several students giving reasons why they think the beads should be replaced.*]

T: Christie, you felt that it didn't matter [*thinking that the beads did not have to be replaced*].

S: I changed my mind.

T: And what was the deciding piece for you? What helped you?

S: Well, I was just thinking and, then, when the people said that . . . since you were always restocking, the boxes are being restocked, and then it just kind of clicked.

T: And sometimes that happens.

In this case, by first calling on students to present the reasons supporting replacement of the beads (the correct answer), the other students had an opportunity to reconsider their response in light of the reasons given in favor of replacement. As happened in this case, this was enough to change the minds of the students who had been in error.

Because students' ideas or thinking were always subject to revision, Mrs. Books had created a classroom environment where student input was valued, even though sometimes it was admittedly incorrect. As the class considered the simulation data reported for the Stick strategy of "Monty's Dilemma," students questioned the 100 and 0 reported by one student. The following discussion occurred in response:

T: Would the person that got this 100 and 0 . . .

S: David.

T: David? Do you want to . . . Do you feel that that's unlikely?

D: [*David(D)*] Yeh, I do.

T: Do you want to tell us what you did and what you would do differently next time? Come on up front, so . . . Let's listen.

D: [*as he comes to the front of the classroom*] With the dice . . . I had, um, 1 and 2 were grand prize and 3, 4, 5, and 6 were gag prizes and, like, when 1 and 2 . . .

S: So you rolled a 1 and 2, 100 times? That is . . .

D: I just rolled it and whenever I got a 1 or a 2, I just . . .

T: Did you make a tally mark when you got a 3, 4, 5, or 6?
D: Yes. Well, no.
T: No? So, is it important that, maybe, he tally every roll?
Ss: Yes.
T: And, David, if you tallied every roll, do you think you would have come . . . What do you think you would have gotten, about?
D: Probably around 30.
T: Okay, is this what you were trying to tell me real quickly before we started? Do you wish I would have had time to look into it then?
D: Yes.
T: But, obviously, that gave us something to look at to see if we thought there was some bias there. Thanks for coming up and sharing that one.

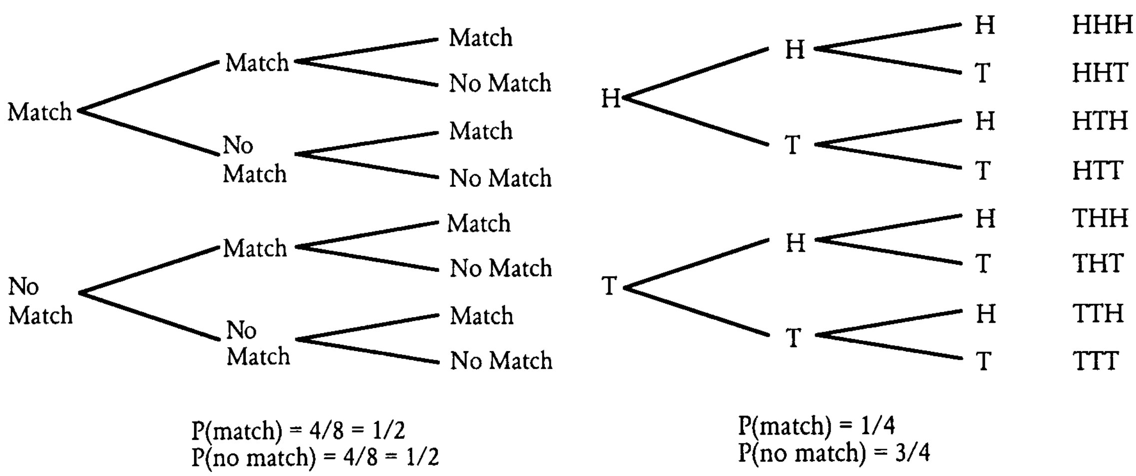
Although admitting the error he had made potentially could have been very embarrassing for the student, he evidently knew that addressing errors was an expected and accepted part of the classroom environment. And, to some extent, Mrs. Books accepted part of the burden of error because there had not been time to address his question before the lesson had begun.

In contrast to Mrs. Books' constructivist approach to student errors, Mr. English seemed to hold a more traditional view about the correctness of mathematical ideas, as illustrated in the following presentation of students' work for the Three Coins problem, which had been assigned as homework:

T: Remember I said we had three coins. We are going to flip the coins and we're going to see how often the coins match and how often they don't match. And we want to draw a diagram showing that. And it's called a tree diagram. So here's . . . one person's thinking about it. She took the first coin and she said it will either match or it won't match. [*Mr. English is demonstrating the tree diagram drawn by a student (see Figure E.3[a]).*]
S: What's it have to match though?
T: And I don't know that you were necessarily thinking of the first coin, but you just thought that's the way to set this [diagram up]. [*Mr. English continues to draw the tree and, then, he analyzes the results based on the final outcomes listed.*] . . .
S: What do you mean by "match"? Don't all the three coins have to match . . . to be a match?
T: Now, this is an incorrect . . . incorrect diagram. And the reason it's incorrect is you start out incorrectly because that first coin can't match [*indicating the first part of the tree*]. If you flip it, what's it to match with? . . . So you can't have your outcomes as "match" and "no match." If you flip that one coin, what are the outcomes?
Ss: Heads or tails.
T: Heads or tails. Let's go to . . . [*to student from table by the door*] Do you want to come up and draw the way you did it? [*The second student draws a correct tree diagram (see Figure E.3[b]).*]
T: Okay, the difference between that one and the first one [*showing the first tree diagram again*] . . . the diagram basically sets up correctly, because you've got one thing here and one thing here [*stage 1*] and you have two branches coming off that one thing [*stage 2*] and you have two branches

> coming off each thing [*stage 3*]. So you end up with eight outcomes. That is correct. . . . Actually the only difference is these are not the outcomes for coins [*pointing to the "match" and "no match" labels*]. The outcomes for coins are heads or tails, so you have to list them like that. You only look at the match or no match when you are all done.

In this case, the tree diagrams were either correct or incorrect, at least initially. There appeared to be no room in his thinking for partially-correct ideas or ideas "under construction."



(a) First student's tree diagram. (b) Second student's tree diagram.

Figure E.3. Tree diagrams drawn by Mr. English's students for the Three Coins problem.

The first student had come to Mr. English with a question before class began because she had not understood what he meant by match and no match. Being uncertain of her answer, she had rather reluctantly given Mr. English permission to use it as an example. Mr. English expressed to the researcher concerns he had about presenting a student's answer that is wrong. However, in this case, he had felt it would be okay because the student was generally a strong student. In addition, Mr. English pointed out that by using the incorrect tree diagram as an example, he discovered five other students had drawn a similar diagram. And, as he explained to the class, "Sometimes it's good to analyze how we think about something and contrast it with . . . the right way to do it and learn from it." Nevertheless, when students know their contributions will be judged to be right or wrong, as they were in this case, the risk involved in putting forth ideas or answers can discourage even the strong students from participating in the discourse.

Mrs. Talent's approach to dealing with students' errors seemed to be based on a concern for students' feelings and a desire not to label students' work as incorrect, at least in situations other than correcting homework. On two different assignments, Mrs. Talent asked how students would draw a tree diagram to represent the problem or example being explored. In both cases she initially received incorrect suggestions from students, which she dealt with in slightly different ways. In one case, Mrs. Talent asked how students would analyze "The Hare and the Tortoise Game," in which a die had been rolled three times and movement of the player on the game board had been determined by the odd or even outcome on the die.

T: How could we use a tree diagram? Does anybody have any idea? . . .
S: Just put the hare and the tortoise at the top of the tree . . . Then put . . . M, N, S, Y, or Z for the hare [*the winning points for the hare*] . . .
T: Like branches?
S: Yeh . . . and put a P in that place for the tortoise . . . .
T: Is there any way we could do this with less . . . branches?
S: [Use] like odd or even . . .
T: That might be a little easier. . . . What you want to do when you do a tree diagram, you want to look at how many different ways can something come out. And, when you roll the die . . . the first time, how many different ways . . . what's going to happen?

In this case, the initial suggestion was incorrect because the student's tree diagram did not represent the actions of the problem, namely rolling the die. But rather than addressing the student's error, either at the time or in later discussion, Mrs. Talent continued to take suggestions until she heard an idea that led to a correct analysis. Although Mrs. Talent's comment, "What you want to do when you do a tree diagram, you want to look at how many different ways can something come out," might have been intended to address the student's error, it did not do so directly. The student could have been using this thinking when he drew his tree diagram. In particular, he could have been thinking there are two ways the game can come out, the hare can win or the tortoise can win. And there are five ways the hare can win and two ways the tortoise can win. If he was thinking in this way, the point he was missing is that the tree diagram needs to correspond to the actions that take place in the experiment. This potential misunderstanding, however, was not addressed. In addition, not only was the second student's suggestion to use odd and even "easier," it was also more appropriate. Specifically, it was correct because it did represent the actions involved in the game.

During an earlier lesson, Mrs. Talent had demonstrated a two-stage tree diagram representing an experiment with two spinners. To check the students' understanding before assigning a worksheet, Mrs. Talent gave the students a second example.

T: Okay . . . I'm gonna have you do this one completely on your own. . . . I want to know what . . . the probability [is] of flipping the coin and getting heads and then spinning this spinner and . . . you'll get white.

Would you draw out the tree diagram that shows for these two, and then . . . list the outcomes and then figure out what the probability of heads and white are. [*after time working independently*] . . . Can somebody . . . tell me how to set the tree up?

S: You make three little branches . . . because there's three color things, like red, white, and blue. . . . Then you make two branches under each one of those . . . .

T: By the way, what was the probability that you'd come out with [heads and white]?

S: One in 6?

T: Actually, if you did it this way, is there any of these combinations that come out heads and then white?

S: No.

T: So you have to be a little bit careful, because this is asking for, if you flip a coin and then spin the spinner. . . . And if you want to be really picky, the probability that you're going to flip a coin and get heads, and then spin a spinner and get white is 0, in this instance, because it's never gonna come up heads first.

S: But that's really picky.

T: It is really picky, but you know how I am. I get that way.

In this case, the student had reversed the actions of the problem, spinning the spinner before flipping the coin. Mrs. Talent pointed this error out to the students. However, rather than basing the judgment on mathematical reasons, Mrs. Talent suggested it was wrong only because she happened to be picky. As a result, the students may fail to realize the important role order plays mathematically in certain situations.

Rather than taking students' input as reasonable hypotheses or as a starting point from which to guide students to a correct solution, it seems Mrs. Talent tried to dispatch incorrect suggestions as expeditiously as possible, without specifically labeling them as incorrect. In these two cases, rather than stating the students' tree diagrams were incorrect, Mrs. Talent indicated only that another diagram was preferable because it was "easier" or because she was "picky." This seems to be an overcompensation for the concern about labeling students' work as incorrect or wrong. Further, because she did not always address the error that had been made, opportunities for developing students' understanding were overlooked.

In Mr. Trackman's responses to students' errors, he did not seem to be particularly concerned either about labeling errors and how students would feel as a result or about guiding students to correct their misunderstanding. His focus, instead, appeared to be on getting the correct answers so they could get through the lesson. Mr. Trackman had reported to the researcher that one of the things he liked about mathematics was "the security in knowing that there is a right answer." In his focus on correct answers, Mr. Trackman did not seem to give consideration to student errors or to the thinking behind the errors. For example, as he was beginning the analysis of "Is This Game Fair?" which was based on dice sums, the following interaction took place:

T: We're rolling . . . two dice. How many different numbers, totals of the two dice, added, sums, how many different totals are there?
Ss: Twelve . . . a lot . . . 36 . . . 24 [*various answers suggested*].
T: No. Not 12.
S: Thirty-six. [*various answers given*]
T: What's the lowest number you can get?
Ss: One . . . 0 . . . 2.
T: What's the highest?
Ss? Twenty-four . . . 24 [*laughter*] . . . 12 . . . 99 [*various other answers*].
T: How many possible numbers are there?
S: Eleven.
T: Eleven. Thank you. There's 11 possible. You guys, you don't have 1. You've got 2 through 12. That's right. Wonderful. Anyway . . .
S: [*student makes a comment or asks a question*]
T: I've been asking the same questions all day. You guys, are just not here.

Because the focus was on answers and not on thinking, it almost seemed like the students' expectations were, "If we give an answer, any answer, that's okay. He'll pick out the correct one from the choices we give him." For his part, Mr. Trackman seemed to treat the incorrect answers as an annoyance. There was a hint of exasperation in his comment, "Wonderful," as if he was saying, "You finally got it right . . . about time."

As the lesson continued and he was having the students list the possible dice outcomes leading to the different sums, one student suggested an incorrect pair.

T: [What about 8?]
S: [*mumble . . . 4 and 4*]
T: Um, 4 and 4.
S: Two and 6.
T: Two and 6.
S: Seven and 1.
T: Seven and 1?
S: Three and 5.
T: How can you get 7 and 1?
S: Oh, no.
T: [*laughs*] I've gotten somebody that's . . . every period.
S: One and 7 [*thinking the order was wrong when he suggested 7 and 1*]
T: One and 7? When was the last time you rolled one dice and got a 7?
S: Me.
T: A six-sided and got a 7? Huh?
S: I've got one that has 7 on it.
T: I'm sure you have. One of those . . . weird funky ones.
S: Ha, ha! Those 26 letters one.
T: Yeh, whatever.

In this case, the student had made a common and understandable error. As he was focusing on the mathematics of the problem, he had forgotten the constraints of the situation being discussed. Mr. Trackman reminded the students they were dealing with dice, but his comment was more a put-down of the student than an attempt to correct his misunderstanding.

Mr. Trackman seemed to express a similar exasperation and impatience when one student questioned Mr. Trackman's notation when he was listing the dice pairs leading to the various sums.

S: Doesn't the dot mean "times"?
T: No, I'm just. . . [listing the numbers and happened to put a dot between them].
S: There like, um . . .
T: Don't worry about . . . [it]. You're right. You're right. Absolutely right. It does mean "times."

To him, he was just listing pairs of numbers, and the punctuation he used did not make a difference. He seemed frustrated when the student did not see it similarly and questioned what he had done.

Nature of the Classroom Discourse

The learning opportunities for the games and activities were structured around some or all of the following stages: introducing the tasks, making predictions, conducting experiments, interpreting the experimental results, and doing the theoretical analysis. In addition, the lessons involving textbook or worksheet assignments usually included time for introducing the assignment, working on the assignment, and correcting the assignment. Each of these different segments of the probability lessons provides a frame for viewing the nature of classroom discourse. This section will explore the picture of classroom discourse as it is seen through these frames. This section will also consider what the classroom discourse contributed toward structuring the learning opportunities and portraying the overall picture of probability as taught in these classrooms.

Introducing the games and activities. One primary focus of the teachers' discourse was the introduction of the games and activities. The approaches used by the teachers included asking questions, recalling fables, telling stories, and relating real-life examples. For example, Mr. Trackman asked his students, "When you're . . . going out to the car and two of you call shotgun at the same time, how do you solve it?" He then reminded the students they sometimes used "rock, paper, scissors" to decide. From there he introduced a game called "Paper, Scissors, or Rocks." Similarly, Mr. English introduced "Doubles in Monopoly" by asking, "Just out of curiosity, whenever any of you play Monopoly and you land in jail, how many of you pay the $50 to get out [instead of trying to roll doubles]?" Alternatively, in presenting "The Hare and the Tortoise Game," Mr. English recalled the fable of the tortoise and the hare, although no connection was made between the fable and the game (as in the fable, the game appeared to heavily favor the hare when, in fact, the tortoise turned out to be the winner). Further, Mr. English introduced the "Frosted Wheat Yummies" simulation activity by telling the students an imaginary story about his strategy for marketing cereal by putting prizes in the boxes. In addition, Mrs. Talent introduced the similar "Cereal Boxes" problem using a real-life example of a Honeycombs box offering free posters. Mrs. Books introduced a similar problem by stating the question and explaining it was a question she had explored in one of her college classes. However, although the teachers sometimes used questions, stories, or examples to introduce the games and activities, they more commonly introduced the tasks with comments such as, "Today we're going to play a game," or "Here's what we're going to do today."

In some cases, the teachers provided an overview of the task by explaining how the investigation of the game or activity would proceed. For example, in his introduction to "Is This Game Fair?" Mr. English explained, "After we [have played the game for 10 rolls of the dice], I'm going to ask you to report to me the results . . . of how it came out. . . . And then we'll analyze this and see if it's a fair game . . . and try to determine, if it's not, how we could make it fair." At other times, the teachers provided an overview by relating the game or activity to the other tasks included in the probability units. In presenting "Monty's Dilemma," Mrs. Talent pointed out, "We're gonna take a look at another situation that's a real-life situation. We're going to simulate it by conducting an experiment like we did yesterday. And then . . . first you're gonna decide before the experiment which way you think it's going to go, and then after the experiment we'll see if the experiment changes your mind a little bit."

Although comments of this nature were fairly common, the introduction of the games and activities usually did not include any discussion of the content or of any learning objective for the tasks. For example, as Mr. English was introducing the "Newspaper Offer" simulation activity, he explained, "Today's activity is a little bit different than some that we've been doing. . . . Today's activity uses cards . . . poker cards again [rather than dice or coins as many of the activities have]." Here, Mr. English focused on the differences between this and earlier activities, ignoring the fact that the "Newspaper Offer" activity and the "Frosted Wheat Yummies" simulation immediately preceding it both dealt with expected value. Similarly, although Mrs. Talent frequently began her lessons with an explanation of "what we're going to do," this explanation focused on the specific task, not on the content involved. For example, as introduction to the "Chips" games, Mrs. Talent said, "What we're gonna do, we're gonna play two games and you're gonna decide, before you play, whether you think the games are fair or unfair. And then you're gonna play. Then, after we play, we're gonna see how it turns out by an experiment, and then, if we have time today, I'll show you how to analyze 'em mathematically." From an instructional perspective, Mrs. Talent's goal was to present the tree diagram as a "way that you can list out all of the different possibilities that could happen in an experiment." However, this goal of introducing a new analysis tool was not mentioned in the overview of the task. Likewise, as Mrs. Books introduced the "Cereal Boxes" activity, she did not speak of the box-plot she intended to introduce as a way of representing the data from their simulation, although that was one of her goals for the activity.

A presentation of the directions and/or rules for the game or activity was generally the focus of the teachers' introduction. The teachers often read the rules or directions from the page handed out to students, adding comments of their own at times. In some cases, an explanation was given of how the students would be arranged for the task. For the Dice Sums game, for instance, Mr. Trackman chose to have teams with the boys playing against the girls. In addition to explaining the rules for the games and how the students would be arranged, the teachers usually gave the students an example of how to play the game, explained how to record the results, and, in some cases, provided a sample of the final results. For example, in presenting "Paper, Scissors, or Rocks," Mr. Trackman provided the following example: "Let's say that . . . this group here, they had 16 for player A, 1 for player B, and 8 for player C. Is that a fair game?" After establishing this was not a fair situation, Mr. Trackman went on to introduce the idea of rescoring the game by asking,

> Have you ever played against your brother or sister in basketball or something and they spot you a few points? . . . They spot you about 10 points, and then they go to 11? [*chuckle*] Well, that's making the game more fair, 'cuz you still have a chance to win then. Okay? Well, one thing they suggested in a couple of other periods was, okay, the person, the team that got 16, theirs would only count as 1, the team that got 1, theirs . . . each time they got one would count as 16 points, and the team, the person that got 8, theirs would count for 2 points each time. Then if you multiply that out, they get 16, 16, and 16 and then it would seem to be fair.

In this case, Mr. Trackman was assuming the students could determine what was important from his brief example.

The presentation of the simulation activities varied somewhat from teacher to teacher, particularly in the extent to which the procedures were determined by the teacher. For example, after setting the stage for the "Cereal Boxes" problem with a box of Honeycombs she had purchased, Mrs. Talent stated the following assumptions about the problem.

> Here are the three posters and assuming that they put, like they make out . . . I don't know, of how many, probably millions of boxes of cereal a day,

> but assuming that they put the same number, like there's just as many of the Shawn Kemp's as there are the Patrick Ewing's. They're all equal. And they mix them up and they put them in cereal boxes and they ship them out to the stores. If I want to collect all three posters, how many boxes of cereal would you think I would need to buy?

Mrs. Talent then went on to explain how the students would simulate the situation using dice. In his version of the "Cereal Boxes" problem, Mr. English provided a similar explanation of how the students would conduct their simulation. On the other hand, after Mrs. Books had briefly stated a similar problem, she suggested the students might have "some questions that would need to be answered" before they made their initial subjective guesses. In this case, the assumptions of the problem were revealed in response to the students' questions, as the following excerpt indicates:

> S: What are the odds of getting each of the stickers?
> T: Okay. We're going to assume . . . because I don't have any information to the contrary that there is an even mix of each of the five stickers.
> S: Did one carton of boxes . . . cereal boxes all have the same sticker put inside?
> T: No, we're going to assume that it's a random mix within any carton, any truckload . . . any trainload that would come out. . . .
> S: Is there any indication of what's inside?
> T: . . . No, there is no indication when you get the box . . . which sticker is inside. . . .
> S: How many boxes are left in the store you are at?
> T: We're going to assume that they continue to restock those shelves.

Although Mrs. Books was the one stating the assumptions, as the other teachers had done, it was the students who were determining what needed to be known. With the assumptions agreed upon, the students proceeded to make and discuss their initial responses. Then, with the assumptions still in mind, Mrs. Books had the students decide how they could simulate the problem. As a result, the students in Mrs. Books' class were more actively involved in thinking about the problem than the students in other classes had been.

Teacher monologues were the predominant pattern of discourse as the teachers introduced the games and activities and explained the rules or directions. These generally brief monologues were interrupted occasionally by students' comments or questions seeking clarification. For example, as Mrs. Talent was explaining the rules for two games using chips marked with letters on both sides, one student questioned her about the materials they would need, "Do we need three cups?" Later, another student asked, "How many times do we do this?"

At other times, a dialogue was going on between the teacher and students, initiated by questions from the teacher. Some of these questions potentially stimulated the students to think about the situation. For example, after reading the directions for the "Pig" game, which indicated a player's turn would end and all the points earned in that turn would be lost if the player rolled a 1 on the die, Mr. Trackman asked, "What is the probability of throwing a 1?" Other questions, however, seemed to prematurely limit the opportunities available for students to reason things out for themselves. For instance, after explaining the goal of "The Top and One Other" was to obtain the highest total score, Mr. English asked his students, "Now, since you're trying to get the larger number . . . the larger total, when player A takes the top number [on the die], which side number is he going to take [as the 'other' number to add to the top]?" In this case, rather than letting the students discover for themselves that they should choose the largest side number, the teacher pointed out the strategy to them, reducing the game to one of rolling the dice and adding the numbers. In one case, the typical pattern of the teacher-directed dialogue was reversed. When Mrs. Books asked her students to think about what they needed to know about the "Cereal Boxes" situation, it was the students who were asking the questions and the teacher who was providing the responses in the question-and-answer session that followed, as seen in the excerpt quoted earlier.

Making predictions. For a number of the games and activities, making a prediction about the final outcome was an important part of the investigation. However, the importance given to the process of making predictions varied from class to class. For Mrs. Books, having her students make a prediction was an integral part of the simulations her class did. In particular, this prediction, or subjective estimate as she called it, along with the experimental and theoretical parts of each activity provided the structure for the exploration of the problems. A similar pattern of prediction, experimentation, and theoretical analysis was the pattern Mrs. Talent followed for the three games and two simulation activities included in her probability unit. Having students make predictions was also involved in four of the games and activities Mr. English used during his probability instruction. In Mr. Trackman's classroom, the Coin Tossing Exploration was the only activity where the students were asked to make a prediction. In response to the question, "Which stop has the highest probability?" the students were to indicate the stop they *thought* had "the most chance." However, because they were to answer this question *after* conducting the experiment, their response was less a prediction and more an interpretation of the results.

How the predictions were made and/or recorded influenced the extent to which the students were involved in the process of making a prediction. For example, after the "Cereal Boxes" problem had been explained and the assumptions established, Mrs. Books asked the students, "Right now, on your piece of paper, what I would like for you to do is write down what your best guess is and also what your rationale behind that guess is. Don't show it to anybody else at your table. Write down why you are making that guess. Could be a couple sentences that would explain your guess. But please don't share it with anyone else right now." As the students were writing down their predictions, Mrs. Books circulated among the students, checking that each student was making a prediction and interacting with some students about the prediction they had made. She next had the students briefly share their predictions with the other students at their table. Finally, she had the students reveal their predictions to the whole class. For example, in the "Cereal Boxes" simulation activity, the students reported their predictions in a round robin fashion as Mrs. Books wrote them on the overhead. In this way, all students were encouraged to make a prediction and were held accountable for their participation.

Mr. English and Mrs. Talent each had their students write down a prediction for one of the activities included in their probability units. In particular, the handout accompanying "Which Do You Think Will Be Larger?" provided a space for students to enter a prediction, which Mr. English had his students do. Mrs. Talent had her students write down and circle their prediction for the "Cereal Boxes" simulation activity. However, in neither case were the predictions shared with others in the class.

For the other activities involving predictions, the students in Mr. English's and Mrs. Talent's classes reported their predictions as part of a chorus response or by a show of hands. For example, in response to Mr. English's question, "Anybody want to guess what the probability is of you landing on the outside [two circles of 'Quiz or No Quiz']?" a chorus of students responded, "One third." After giving the students about 15 seconds to decide which strategy in "Monty's Dilemma" would give them the best chance of winning the prize, Mrs. Talent took a vote by a show of hands. Although these approaches provided opportunities for students to make predictions, it did not hold them as accountable as writing down that prediction or sharing it with other students. As a result, it appeared many did not respond or raise their hands at all, opting for the "I don't care" category suggested by one of Mrs. Talent's students or the "I don't know" response.

In some cases, it seemed the students potentially saw their predictions as answers to a mathematical question. From their previous experience they had come to expect answers to mathematical questions to be either right or wrong. The fear of being wrong perhaps explains the hesitancy on the part of some students to make a prediction. Mrs. Talent may have sensed the same tendency to view answers as right or wrong when she encouraged the students to make a guess on the "Cereal Boxes" simulation activity by saying, "And just so you make a little prediction here, would you write down on your paper and circle it, how many times you think it's going to take you to get all three [posters]. Just make a little prediction, and it's okay if it ends up being wrong." On the other hand, her comment may have reinforced this notion by suggesting their predictions could be wrong, leaving the students with an unfortunate impression.

Besides writing down their predictions, Mrs. Books also had her students write down a reason or rationale for their subjective guesses. These supporting reasons were part of what the students were to share with the other students at their table when the predictions were discussed. In the case of Monty's Dilemma, Mrs. Books then used some of the things she had heard from the students as a starting point for the class' discussion about the Stick and Switch strategies. In particular, she explained, "As I wandered around, these came up: 50-50, equal chance, 33 and a little bit. What are those things? How come those are things that you're talking about? How might they be helping you make your decision?" After a brief discussion of the students' thinking, Mrs. Books introduced the Flip strategy and asked the students to consider what had been discussed as they reflected on the three strategies, made their choice, and wrote a statement explaining their choice.

In the other classrooms, however, the students' thinking behind their predictions was not considered. As a result, some excellent opportunities to explore students' understanding of and reasoning about probability were overlooked. For example, in Mr. English's classroom, the students were asked to consider the following offer made by one customer to a girl who had a paper route: "Rather than paying \$5 a week [the usual rate], why don't I put one \$10 bill and five \$1 bills into a paper bag. You will shake the bag and mix up the bills and you can just reach in and, without looking, draw out two bills. Whatever you get, that will be how much I have to pay you." After explaining the situation, Mr. English indicated, "The question is . . . would you do that? Would that be . . . a fair kind of payoff? How many think it would be unfair? You would not do it. You wouldn't take a chance [*several students raised their hands*]." By his questions Mr. English was leading the students to suspect the fairness of the offer, but it would have been interesting to have explored students' thinking. In this case, the paper girl would get either \$2 or \$11. Because one outcome is below the standard charge of \$5 and one is above, some students might think the offer would be fair. Or because the paper girl stands to gain \$6 when selecting the \$11 outcome and only losing \$3 when selecting the \$2 outcome, students might think the offer favors the paper girl. If students were given the time and were able to calculate the probabilities of the outcomes, they might feel the offer is not fair because the \$2 outcome (with a probability of 2/3) is more likely than the \$11 outcome (with probability of 1/3). Even if they could not calculate the probabilities, they might suspect that result because the \$1 bills outnumber the \$10 bill. Or, finally, one might wonder if the students would take into consideration how this might turn out, not for one particular month but in the long range. When expected value is calculated and the gains and losses are averaged out over a longer period of time, this offer turns out to be fair. In particular, the expected value is the same as the standard monthly charge. However, such opportunities to explore students' thinking, in this classroom as well as in the classrooms of Mr. Trackman and Mrs. Talent, were not fulfilled because a chorus response or a show of hands would not reveal what students are thinking.

The predictions the students made were dealt with in a variety of ways. In the "Cereal Boxes" simulation activity, Mrs. Books treated the predictions as data in their own right. After recording the students' predictions, she and the students discussed what the range and the mode of the predictions were. They also compared their predictions to those the students in the other class had made. The students' predictions, however, were not compared to the results obtained from the simulation.

In the case of "Monty's Dilemma," the students' predictions were revisited after the simulation had been conducted. After discussing the simulation results, Mrs. Books asked the students to recall the initial prediction they had written down and asked how many would make a "modification based on your trials." Further, in the letter the students were to write, they were to report their initial prediction and its rationale and to indicate the strategy they now would use to play the game. Thus, in this case at least, the students were asked to consider their subjective predictions in light of the experimental and theoretical evidence.

Mrs. Talent briefly referred back to the students' predictions in two of the five activities where predictions had been made. After the results from the "Cereal Boxes" simulation had been reported, Mrs. Talent asked, almost as an afterthought, "Oh, how many people had their predictions the same as what they got? You predicted right?" The response to this question was not clear. Some students raised their hands; others responded verbally. In reviewing the "Chips" game the day after the game had been played, Mrs. Talent reminded the students about the predictions they had made, "First I had you predict whether you thought it was gonna be fair or unfair. Most of you thought it was gonna be unfair. And how did it turn out?" Although students responded that the game turned out to be unfair, further analysis found it to be fair. Agreeing with a student's observation that "it doesn't look like it [is fair]", Mrs. Talent asked, "Why do people think it's unfair?" One student suggested it looked unfair because "there's three *X*s and one *Y*." In conclusion, Mrs. Talent pointed out the *X-X* chip was not even needed; the results were determined by how the *X-Y* chip landed. In this case, Mrs. Talent was not only recalling the students' predictions, but also uncovering part of the subjective reasoning on which the predictions were based.

Mrs. Talent provided no follow-up to the other three activities involving predictions. This was also the pattern followed by Mr. English. And although one of the questions Mr. Trackman's students were to answer for the Coin Tossing Exploration was, "Are these [experimental] results the same as the [predicted] probabilities?" there was no follow-up to the activity or to the questions.

Some of the games included in the probability units did not explicitly involve making a prediction, but did involve making decisions based on predictions or subjective guesses as the games were played. For example, in playing the Dice Sums game, the students in Mr. Trackman's class were trying to predict the most likely sums when two dice were rolled. Choosing numbers for the lottery, selecting confidence levels for "Montana Red Dog," and setting traps for the rats in "A Ratty Problem" were all instances where Mr. English's students were taking actions based on predictions or subjective guesses. And by establishing what the theoretical outcomes for the spinners were before doing an experiment, Mr. English and his students were in essence making a prediction about what the experimental results might be.

These instances of making implicit predictions also provided excellent opportunities to explore students' understanding and thinking, if the students had been asked to provide a rationale for their decisions or compared their results and their expectations. Having the students compare their predictions with their results also would have been a way to assist students in making the transition from using predictions based on subjective guessing to making decisions based on theoretical considerations. But the students were not asked to reveal their thinking. As in the other cases, the teachers did not take advantage of these opportunities and perhaps did not even recognize the opportunities the activities provided.

Conducting experiments. Following the presentation of the games and activities by the teachers and the predictions made by the students (if any were made), conducting experiments or simulations was generally the next stage involved in the exploration of the games and activities. This was the stage of the investigation in which the students were actively involved, either tossing pennies, rolling dice, or simulating a TV game show. As the experiments or simulations were being conducted, the teacher and students had a number of opportunities to interact with each other. The nature of these interactions and of the discourse during this time was quite different in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English in comparison to Mrs. Books' classroom.

In the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English, many of the interactions focused on the directions given for the game or activity. These interactions involved clarification of the rules or procedures ("How many times do we play it?"); the arrangement of students ("I don't have a partner."); or the distribution of materials ("If you're done with your dice, could I have them back?"). Where the presentation of the rules and directions had been made to the class as a whole, the clarification of the rules and procedures generally occurred with small groups or individuals. These interactions were initiated both by the teacher ("Do you guys know what you're doing?") and by the students ("Am I doing this right?").

Other interactions in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English focused on the results of the experiments or simulations. Although in some cases these results would later be reported for the entire class, these interactions involved small groups or individuals reporting what their results had been. Sometimes it was the teacher asking about the results ("What happened when you had to stick?") or questioning an incorrect outcome ("Where did you get [a dice sum of] 14?"). At other times, students were volunteering their results with statements such as, "I got exactly 75 and 25," or "Evens won that one, odds the first one."

These interactions involved a limited number of instances aimed at encouraging or probing students' thinking. For example, Mr. Trackman asked one group, "How would you change the rules [to make the game fair]?" On several occasions, Mr. English asked individuals or small groups questions such as, "So what confidence level are you going to have?" or "Why is it unfair?" In the following dialogue, Mrs. Talent asked one student about her results for a game involving dice products:

T: Do you think this is going to be fair?
S: No!
T: Who is it going to favor?
S: Even!
T: Is that because you got those first?
S: No! Because . . . No, because most of them are that . . . most of the numbers . . . once you take and multiply it come up even.

However, discourse such as this, which explored students' thinking, was limited in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English.

In contrast, Mrs. Books' approach to the experimental phase of the activities differed from the other teachers in two significant ways. First, she involved the students in the process of designing how to simulate the problem. Second, in part because of this, more time was spent on this phase of the investigation. As a result, there were more opportunities for interaction between the teacher and students. Many of these interactions occurred between the teacher and individuals or small groups. However, some of the issues involved in designing the simulations were considered by the class as a whole after they first had been considered individually and in small groups. This included the discussions concerning possible bias in the simulation designs for "Monty's Dilemma" and whether or not the beads should be replaced after they had been selected from the paper sack in the "Cereal Boxes" simulation. Thus, not only were there more opportunities for the teacher and students to interact with each other, but these interactions often involved thinking at a different level in

Mrs. Books' classroom. Rather than just an exchange of factual information such as rules or results, the interactions between Mrs. Books and her students engaged the students in thinking about what they were doing as the simulations were being conducted and encouraged the students to think about the content involved as well.

Although the interactions concerned some of the same topics as in the other classrooms, including the procedures and results of the experiments, the interactions between Mrs. Books and her students often involved thinking at a different level. In particular, because the students had decided for themselves how to do their simulations, part of Mrs. Books' interest was in learning how the students were going to simulate the situation. Therefore, as she went from group to group, she asked, "What are you guys going to do?" She also guided the students to consider aspects of their design with such questions as, "Are you sure . . . are you getting a random . . . [result]?" or "So what are you going to do with that sixth side of your die [when there are only five prizes to be won]?" Thus, rather than discourse that just clarified rules established by the teacher or activity handout, the discourse in these cases encouraged the students to think about and to justify what they were doing.

Mrs. Books also guided the students to consider issues she saw as important. For example, as the students were conducting their simulations for the "Cereal Boxes" activity, Mrs. Books asked one group, "How come this bead's laying here?" On the following day, she raised the issue of replacement with the entire class.

> Some groups . . . had the beads in a container and they were drawing them out and setting them on the table. They weren't putting them back in. Other groups were drawing and replacing. So there's two different styles that started to go on. . . . So as an individual, I want you to think right now. Given the fact that you have a container of beads, there are an equal number of each of the colors of beads in there. Is it going to matter, while you're doing your experiment, if you're taking the beads out and not replacing them, or if you're taking the bead, drawing it, and putting it back in? . . . Which of those two styles is going to give you the most accurate information based on the conditions that we put on our experiment yesterday?

Mrs. Books explained she hoped to ask students "thought-provoking questions . . . that were not leading but would allow them to bring out some new understanding." In so doing, she tried to encourage her students to think about critical issues without specifically telling them what they should conclude.

Similarly, as Mrs. Books was asking the students about their results, she was also encouraging them to think about those results. For example, she asked one group, "Is there any pieces of unexpected data coming up?" When another group reported being surprised by having to buy 28 boxes of cereal before getting all the prizes, Mrs. Books asked, "So, do you think [that's] a helpful piece of data or do you think . . . he [would] be better off just to forget that piece of information?" To follow up on that question, she asked the group how that number would impact the different averages they were to calculate. Thus, rather than just an exchange of factual information such as rules or results, the interactions between Mrs. Books and her students engaged the students in thinking about what they were doing as the simulations were being conducted and encouraged the students to think about the content involved as well.

Interpreting experimental results. For at least one of the tasks, the "Cereal Boxes" simulation activity, the experimental stage was the final one considered. In this case, the theoretical analysis would have been beyond the grasp of middle school students. Three of the teachers included some version of the "Cereal Boxes" activity in their probability units. How these teachers dealt with the experimental data in this situation provides a contrasting picture of the ways the experimental results were treated. This picture also presents the varied approaches used in interpreting experimental results in general.

At the start of the second day of the "Cereal Boxes" simulation activity in Mrs. Books' classroom, she had her students reflect on the preliminary data that had been reported. These data were first summarized in response to the following questions: "The data that we have so far shows a range from what number to what [number]? . . . Do you expect that we will get any data [above the 17, the highest so far]?" The students were further stimulated to think about the results by the following series of questions asked rhetorically: "Do you think, by looking at this data, that our sample simulations are pretty accurate? Do you think . . . we're dealing without bias in here? . . . Do we have anything that looks like it doesn't belong or things that do belong?"

As the second day came to a close and the trials had been completed, Mrs. Books again directed the students' attention to their data with the question, "What's our mode?" After the class determined their mode was 5, Mrs. Books asked, "Does that seem reasonable?" Because they were trying to obtain five prizes randomly hidden in the cereal boxes, some of the students had earlier expressed to Mrs. Books their doubts that the result of 5 could have occurred as often as reported. After a recount showed no change in the number of 5s, the students were given an opportunity to discuss their concern. When several students agreed a mode of 5 did not seem reasonable, Mrs. Books asked, "What would you have expected the mode to be?"

As the activity continued on the third day, the class returned to the issue concerning the number of 5s, agreeing the other class' results were more along the lines of what they would have expected, with modes of 7 and 11. After discussing what type of information could be "gleaned" from the data displayed on the line-plot, Mrs. Books went on to demonstrate a box-plot as "another style of recording information." Using the box-plots, the class considered the question, "How many boxes of cereal would you tell somebody that they would need to buy in order to be 90% certain?"

In this example, Mrs. Books and her students spent the good part of two different days interpreting the experimental data from several different perspectives. Throughout the process, the students were encouraged to critically evaluate the nature of the data, to assess their reasonableness, and to take into account the possibility of bias. With those issues in mind, they also considered what information could be obtained from the data and what conclusions could be stated.

This approach to the experimental data was also how Mrs. Books dealt with the data from "Monty's Dilemma." In that case, after the data had been recorded on a transparency, Mrs. Books asked her students "to examine the results . . . to see if you see anything on these that is suspect." After inspecting the data, students questioned results such as 100-0 for the Stick strategy (in which there had been a procedural error) and 100-100 for the Flip strategy (which was exactly the theoretical expectation). Students were also asked to consider what range of experimental data would be reasonable once the theoretical outcomes had been suggested. Finally, the students were asked to reconsider what strategy they would choose, based on the experimental data and the class discussion.

After Mr. English's students had completed their version of the "Cereal Boxes" simulation activity, they reported their results to Mr. English who was recording the data on the overhead projector. These results included the average number of cereal boxes bought and the total cost. After the class data had been reported, Mr. English observed, "All right. Just look at the numbers up here. Can you see why . . . do you understand why the companies put prizes in the boxes? . . . You could probably buy this set of [six] pens for under $8, my guess is. And yet, if you do it this way and they really want to get the prize, you're spending between two and three times as much to get those pens." With the simulation activity thus completed, Mr. English moved on with other class activities.

This example was representative of Mr. English's approach to interpreting experimental results. Time was spent recording the experimental data so that they were available for all to see. However, there generally was no discussion about the reasonableness of the data. In particular, the reported data were questioned or challenged in only one activity. In this case, after simulating the "Newspaper Offer" with playing cards, the results reported by two students were questioned. Because the results seemed unreasonable to Mr. English, he wondered if the cards had been marked. One student admitted he had been able to identify the face card, and his data were ignored as a result.

In addition to recording the experimental data, conclusions were usually stated from the data. However, as in the "Cereal Boxes" activity, it was the teacher, not the students, who generally stated these conclusions. In a few instances, the students were somewhat more involved. For example, after recording some of the outcomes for "Which Do You Think Will Be Larger?" Mr. English asked his students, "Are any of you ready to conclusively say this is unfair?" In this case, however, rather than being asked, "What do these data show us?" the students in essence were being asked to agree or disagree with what the teacher had already concluded.

Mr. English seemed to assume that stating the conclusions was part of his role as the teacher. When not stating the conclusions himself, Mr. English made leading comments or asked leading questions, suggesting to the students what their response should be or what decision they should make. For example, after the students had played "Doubles in Monopoly," Mr. English asked them to explain on the summary page in their packet "why that last game was unfair." However, not all the students were convinced the game was unfair. When one student suggested, "It seemed fair to me," Mr. English referred her to the summary of the experimental results. Clearly, in his request, Mr. English stated the main conclusion about the game, leaving it to the students only to supply the reason. On one occasion, Mr. English even supplied the reason. In this case, after the results for "Which Do You Think Will Be Larger" had been reported, Mr. English asked, "If you say it's unfair, based on the fact that B won a whole lot more than A, it's unfair . . . raise your hand if you believe that."

Mrs. Talent was the third teacher who conducted the "Cereal Boxes" simulation activity. After completing a simpler version, in which the students were trying to obtain the three posters advertised as prizes in a box of Honeycombs, Mrs. Talent had the students report their results in response to a series of questions. She first asked about an unusual result with the questions: "Anybody get three for three . . . they rolled three times and got the three posters? . . . Did anybody have it happen all three times they tried? . . . Anybody have it happen more than once? . . . Twice you got 3 out of 3?" Then, by a show of hands, the students reported the average of their three trials in response to the questions, "Did anybody get an average number of tries that was larger than 5? . . . How many people got then 5 or lower as your average? . . . Anybody get around 4 . . . 4 or 5, in there?" With this picture of the results, Mrs. Talent proceeded to conclude, "So, okay, that means by the experiment we just did, that means that if you buy probably four, five, or six boxes, somewhere in there, about five boxes, you should be able to get all three [posters]." Before moving on to a second version of the problem, Mrs. Talent added the following caveat: "Will it happen for sure that way? [*to which the students replied, "No."*] . . . You could buy 20 boxes and never get all three, but probably . . . ." The second version of the problem involved obtaining six prizes. The interpretation of the experimental data for that case was handled in a similar manner.

This example of Mrs. Talent's treatment of experimental data is representative of her general approach. In one case, "Monty's Dilemma," she had the students record their data on an overhead transparency so that the entire class could see the results. Otherwise, the results were reported by a show of hands, as she had done with "Cereal Boxes," or the results were not reported publicly at all. In particular, rather than having the students report their experimental data for the three games included in the probability unit, Mrs. Talent's focus in these situations seemed to be on guiding the students to complete the accompanying handout which generally asked students to record experimental results and calculate experimental probabilities.

Stating conclusions based on the data also seemed to get only cursory coverage as well. In some cases, Mrs. Talent stated a conclusion, as she did in the "Cereal Boxes" simulation. At other times, the students were questioned about a conclusion, but none was specifically stated. For example, after playing a game with chips, Mrs. Talent asked the students, "Do you think this game is fair?" Although the students responded with both, "Yes," and, "No," Mrs. Talent only replied, "You do if you're player 1 [*who was favored theoretically by 3:1 odds*]," without making any connection to the theoretical results. With no further comment, Mrs. Talent directed the students to stack up their materials and turn to the reverse side of the handout.

Although Mr. Trackman did not have a version of the "Cereal Boxes" simulation as an activity in his probability unit, he did include four games and one chance activity involving experimentation. In two of the games, the experimental data were reported by the students and discussed. In these, Mr. Trackman did not have a specific approach in mind. As he suggested to students, "part of doing experiments . . . involves figuring out ways to analyze the data." Later in the same activity, he added, "You come up with stuff, you start writing down numbers. When you get it all written down, you start to notice a trend . . . and stuff like that."

Because Mr. Trackman had asked the students to play "Is This Game Fair?" in two different ways, the experimental results could be considered from a number of different perspectives. He first asked the students to report whether the player or opponent had won when the game was played 10 rounds as directed on the handout. After it was reported four players and eight opponents had won, Mr. Trackman observed, "Okay, that's about the way it should have turned out. That's the way it's been turning out [in the earlier periods]. Well, that's one way of analyzing the data." He then moved on to the results from the second version of the game, in which the students had continued playing until the score was 20 to 0. After having the students report how many rounds they had played and who had won these games, the following brief discussion took place:

T: Okay. . . . We've got some . . . . There's nothing necessarily we can notice from all that. One thing we can . . . [*to student*] Yes?
S: One thing we can notice is that the opponent . . .
T: Yes! The opponent. It look's like he's gonna win. So does that mean this game is fair?
Ss: No! [*plus other answers*]
T: Probably not, probably not. We'll look over that in a little bit.

Mr. Trackman then proceeded to find the average length of the games reported by that particular period and to report what the averages had been in the other classes. He then concluded, "The length of the games would take probably around 18 to 20, somewhere in there."

As in this example, quite a bit of class time was devoted to reporting and considering experimental data, for at least the two games. However, it is not clear what the students could learn from the process. Mr. Trackman looked at the data for the games from multiple perspectives, but he generally stopped short of stating definitive conclusions. He did not give the students any opportunities to state their own conclusions, seemingly assuming that stating conclusions was part of the teacher's role. When the one student began to point out an observation, Mr. Trackman jumped in to finish his statement.

<u>Doing the theoretical analysis</u>. As in the other phases of the exploration of the probability games and activities, how Mrs. Books dealt with the theoretical analysis stands in contrast to how the theoretical analysis was done in the other three classrooms. As the class began the "Cereal Boxes" activity, Mrs. Books reminded the students that making a subjective prediction, doing an experiment, and considering the theoretical analysis had been the pattern they followed in earlier activities. Of the two activities observed in Mrs. Books' classroom, only "Monty's Dilemma" could be considered from a theoretical perspective (at least at the middle school level). But rather than the theoretical analysis being a distinct part of the overall lesson, aspects of the theoretical analysis were intermingled with the experimental phase of the investigation. For example, as part of the discussion about removing bias from their simulation designs, one student suggested putting slips of paper in a paper sack. Building upon that idea, Mrs. Books considered with the class how they could conduct their simulations using plastic tiles in a similar way. As she talked the students through the process of how the trials could be conducted for each of the strategies, Mrs. Books revealed the logical foundations of the problem, without emphasizing them as such, in the following excerpts:

T: Okay, let's think about this. If the red [tile] is always the prize . . . and, if you pull it out, you can tell. The Stick is real easy, right?
S: How?
T: 'Cuz you can see, reaching in and pulling it out, and know whether you have the prize or not . . .

T: Now, if you had the one where it says Flip. . . . If you drew the red one, you have won. But you flip the coin that says, "Switch," and if you have to switch to the other one, you've automatically . . .
S: Lost.
T: If you flip the coin and it says, "Stay," then you have won. . . . But [what about] the other bad prize that's in there?
S: It's eliminated.
T: It's eliminated because they have shown you that one. . . .

T: Okay, now let's look at . . . the third one is that you automatically switch. So, if you pulled a gag one out of the sack . . .
S: You have to switch.
T: Have you won or lost?
Ss: Lost.
T: If you have to automatically switch?
S: No, you've won.
T: If you pull the losing one out of the bag and you have to switch . . .
S: You've won.
T: You've won. If you pull the winning one out of the bag and you have to switch, [then] you've lost.

These ideas were not identified as the theoretical analysis, but presented as information for students to think about as they conducted their simulations. In particular, this discussion had not completed the theoretical analysis by assigning probabilities to the different outcomes, but as the students began conducting their simulations, they began to determine the probabilities on their own. The following day, two boys described to Mrs. Books what they had concluded.

S1: [Originally] I thought it was a 50-50 chance for any of them, but I figured out later it was . . . 33 and 1/3 chance for the Stick, 50-50 for the Flip, and then . . . 66 and 2/3 for the [Switch].
T: So when did . . . you first decide that your original impression that it was 50-50 for all of them . . . when did that change?
S2: Well, as soon as Jeremy and I started to do the Stick, we realized there's 33 . . . . It was simple. And then the Switch, we realized that . . .
T: So, how many trials had you done when all of a sudden you thought . . .
S1: Five or 10 of each . . . we realized . . .
S2: Yeh, I mean, I mean, usually in the beginning there's like 5 on the win and 15 on the lose, and this time it was the other way around. It was like 10 on the win and 20 on the . . . [*possibly meaning the reverse?*]
S1: Yeh. And so then we started thinking about it, and we realized the, um, percentage. If you got the, um, uh, if you got the prize the first time, the real prize, you'd lose. That was the only way you could lose [on the Switch strategy].

S2: And so, it's a one-third chance that you get that prize, so two-thirds [chance] . . . you don't . . . .

Later, as the whole class was considering the simulation results shown on the overhead, other students also brought up the theoretical expectations. For example, after considering what data might be "suspect" for the Stick strategy, such as a reported result of 100-0, one student made an observation.

S: Well, um, there's supposed to be one-third chance of winning on the Stick, theoretically.
T: Can you clarify what you're talking about for me?
S: Well, uh, there's, uh, two chances that you can . . . if you're sticking, uh, there's two chances you could lose, and there's one chance that you can win . . . out of the three.
T: So, if I were to draw prize doors up here . . . [*she draws on the overhead*]
S: Yeh, so, so, uh, usually they'd probably get about 33 or so . . . yeh, because, like, um, it's . . .
T: So, you're saying that . . .
S: It doesn't really matter if . . . that they, um, afterwards, that they unveil one, that they take, like, one of the nonprize ones away because they're going to be sticking with [the first] one, so it's still a one-third chance.
T: How many would agree with Eric that on that Stick, that it's a 1 in 3 chance that you would win?
S: Theoretically, you would get . . .
T: What do you mean theoretically?
S: Well, if you were not, like, predicted, you'd probably get about in the range of 33 wins and 66 losses and then, the other way around, vice versa for, uh, switched . . .
T: Okay, well, let's, let's stay with just the Stick. So you're saying theoretically, mathematically, that we have 1 out of 3. Does our data support this, other than this one over here [*the 100-0 result dismissed earlier*]? Are they close enough to 1/3 that we would accept them, or is there something else in here that we're thinking . . .

In this case, the theoretical result suggested by the student became the standard for judging the reasonableness of the experimental results. As the discussion of the results for the Stick strategy continued, the class considered how much the results could vary from the expected 33 wins and 66 losses and still be judged reasonable.

Although the theoretical approach to the Flip strategy was not explicitly stated, the consideration of the experimental data was based on an underlying assumption of an equal 50:50 ratio between wins and losses. Similarly, Eric's suggestion that the Switch strategy was the complement of the Stick strategy guided the discussion of the experimental results in that case. Following the discussion of the experimental results, Mrs. Books asked the students what strategy they would now choose. No further theoretical analysis took place.

As a conclusion to the activity, Mrs. Books asked the students to write a letter "discussing your initial prediction and your reason. Then discuss how you conducted your experiment, any bias, and how you might have had bias initially, and how you changed it, your results, [and] how you would play the game. And you want to use mathematical language to communicate, but you want to make sure that everything is thorough." Although not specifically asked to do so, some of the students included their theoretical conclusions in their letters. For example, one student wrote, "There is a two-thirds chance of winning [on the Switch strategy] because there are two gag prizes. When you [initially pick] a gag, you win because you switch to the prize."

Therefore, as the students had been conducting their investigation of "Monty's Dilemma," Mrs. Books had given them the theoretical ideas with which to work and the opportunity to form their own conclusions. In the end, it was the students themselves who were making the connections and doing the theoretical analysis. And in this case, the discussion of the theoretical outcomes was initiated by the students, not the teacher.

In contrast, the theoretical analysis in the other three classrooms was generally a distinct part of the lesson, one which was introduced and directed by the teacher. For example, after considering the experimental results for "Is This Game Fair?" Mr. Trackman introduced the theoretical analysis as "one way of looking at it . . . apart from the numbers you got." For the same activity, Mr. English suggested, "Let's analyze this game in terms of how you can get those 7s." For "The Hare and the Tortoise Game," Mrs. Talent indicated, "Let's see how it comes out mathematically."

In presenting the theoretical analysis, Mr. Trackman, Mrs. Talent, and Mr. English generally explained or modeled how the analysis should be done. This presentation of the theoretical analysis most often involved teacher-directed dialogues in which the teacher guided the students through the analysis process with a series of questions. These questions were usually asked of the whole class, although occasionally an individual student was singled out to respond. For example, after the experimental results for "Monty's Dilemma" had been considered in Mrs. Talent's class, she directed the students' attention to the theoretical approach in the following dialogue:

T: Okay . . . let's take a look at the game. . . . If you have three doors or curtains or whatever you want to call them, how many of them had a real good prize behind them?

S: Only one.

T: Only one. . . . What is the only way that you could win that prize if you stick with your original guess?

S: . . . to pick that first.

T: Okay, does that make sense? . . . The only way you would win the prize, if you were using the Stick strategy, is if you would have picked it to begin with and you stuck with it. Then you would win, right? What is the chance that you will pick that one?

S: Thirty-three percent.

T: Thirty-three percent because it's 1/3. Okay. . . . So, you have a 1 out of 3 because there's only one prize door and three doors. So what they say in probability is you have a 1 out of 3 chance that you'll pick that door. And if you stick, you would keep that. Okay? What's the only way that you can win the prize by using the Switch strategy?
S: Pick the wrong one . . .
T: Okay. . . . Pick the wrong one to begin with. So, the only way you can win, if you're always gonna switch is if you picked the wrong one to start with 'cuz you would switch to the right one. What's the chance you're gonna win, I mean, pick the wrong door?
S: One out of 2.
T: How many doors are there?
S: Three.
T: How many are loser doors?
Ss: Two.
T: Two. So the chance that you're gonna pick one of these is actually 2 out of 3. Okay. So, when you look at this mathematically, the reason it came out that you win more often by switching is because you're more likely to pick a losing door. . . . You're more likely to pick one of these two to start with and then, if you switched, you would switch to the right door.

Although a teacher-directed dialogue was the most common discourse pattern used for presenting the theoretical analysis, Mr. English delivered the analysis in the form of a monologue on two occasions. For example, after putting up a transparency on which he had written out the solution (see Figure E.4), Mr. English presented the following analysis of the "Newspaper Offer," where the paper carrier was given the choice of $5 per week or selecting two bills from a bag containing one $10 bill and five $1 bills:

> Let's talk about the theoretical now. . . . I went ahead and made the chart ahead of time because I wasn't sure how much time we'd have [*referring to the transparency on which he had written out the steps of the analysis*]. What we are going to look at are possible combinations of two cards. You could either . . . if you let the face card [which simulates the $10 bill] represent the letter A as a symbol, and the number cards [which simulate the $1 bills] represent 1, 2, 3, 4, 5, then you can make a list of the ways that they might come up. . . . You're going to have . . . this A is the $10 bill and there's five ways to pair that up. So you get A-1, A-2, A-3, A-4, and A-5. . . . Then if you just get the numbered pairs, you might draw a 1 and a 2, or a 1 and 3, 1 and 4 [*pointing to the different cases as he mentions them*], . . . we've done this kind of thing where we made lists before. You can kind of see how that works out. There's 15 outcomes when you get done. You just do that by making a organized list. . . . All right now . . . all of these ways right here are worth $11, so there's five ways to get $11. And I'm going to take that 5 times $11 and that gives $55. Five ways to get $11, so the payoff there is $55 [*pointing to the transparency*]. There are 10 ways here in which you end up getting $2. And 10 times 2 is 20, so out of those 15 ways or 15 trials there . . . you're going to get $75. You take the $75 and then divide it by the 15 outcomes. That comes out to be a $5 average or an expected value of $5, which is the same as what she charges the customer anyway. So that's a fair situation. It doesn't matter which one she would do. In the long run, with enough customers, or

> with enough . . . if she would do it for 30 weeks . . . she probably would earn the same or maybe more, maybe less. Anyway, she'd come out real close.

Possible Combinations

Let Face Card = A
Let numbered cards = 1, 2, 3, 4, 5

A-1	1-2	2-3	3-4	4-5
A-2	1-3	2-4	3-5	
A-3	1-4	2-5		
A-4	1-5			
A-5				

There are 15 outcomes.

5 ways to get \$11	$5 \times 11 =$ \$55
10 ways to get \$2	$10 \times 2 =$ \$20
	\$75

\$75 ÷ 15 outcomes = \$5.00 average

Figure E.4. Mr. English's analysis of the "Newspaper Offer."

Thus, in this case Mr. English explained how he had analyzed the problem using an organized list. However, because he had written the solution out in advance, he did not utilize any student input in the presentation of the result.

In some cases, before presenting the theoretical analysis or in place of presenting the theoretical analysis themselves, Mrs. Talent and Mr. English asked how the students thought the theoretical analysis should be done. For example, as Mrs. Talent began the analysis of "The Hare and the Tortoise Game," she asked, "How could we use a tree diagram to do this? Does anybody have any idea?" After one incorrect way was suggested, Mrs. Talent asked, "Is there any way we could do [it] with less . . . branches?" One student suggested using "odd or even," to which Mrs. Talent responded, "That might be a little easier." She then proceeded to lead the students through the process of drawing a tree diagram, although she used the game board positions as labels for the tree instead of odd or even.

Later in the unit, Mrs. Talent also involved the students in the discussion of the Carnival task. After determining by a show of hands that many of the groups thought the game would not make the amount of money needed, Mrs. Talent asked, "Can somebody . . . tell me what their thinking was? Could you . . . prove to me that you wouldn't make enough? . . . Anybody want to try?" Two students volunteered to share their results. By drawing a tree diagram, Jared had found that 3 out of every 27 people would be a winner or 1 out of every 9 people. Dividing 300 by 9, he had found 33.333... or 34 people would win. Jared concluded, "Because every time they won, they'd win \$10 . . . I said those people won

\$340. . . . And because they [the ones running the game] won a dollar from every person in the line, they won \$300 . . . if everyone played. . . . [That meant the game lost] 40 bucks." The second student, Chris, suggested that because "you'd paid off . . . the 33 people that won," he had taken 300 minus 33 to get \$267 received. In comparison to the \$330 paid out in prize money, Chris concluded the carnival would lose \$63 on the game. Mrs. Talent then encouraged the class, "Convince them that he is wrong or, Chris, convince us that Jared's wrong or . . . something like that." The consensus of those involved in the resulting brief discussion seemed to be that "no matter what the game was, they got \$300," but Chris kept arguing that 33 winners "are getting their money back." Mrs. Talent then asked the class,

> Now, let's see, Chris, you say they come up with a loss of \$63. . . . And Jared comes out with a loss of . . . \$40 . . . if you go with 34 people. Okay. If you had to vote . . . which way you thought was the more correct . . . since no one else really wants to come up and show me how they did it . . . . How many people think that you'd go with the idea that they'd probably lose \$40? . . . And how many people would go with . . . the \$63?

Mrs. Talent then called on Jennifer, who had indicated the game would make money, to explain her reasoning. She indicated that, by doing it in her head, she had found there would be 1 winner in every 81 people. As the class period ended, Mrs. Talent asked the students to "think about this [as homework] and see if you can figure out who was right."

Other than helping the students who shared their solutions to clarify their explanations, Mrs. Talent had not participated in the analysis. In particular, she neither presented the theoretical analysis nor indicated which, if any, of the students' solutions were correct. After thinking about the problem further, Chris indicated the following day that he had decided he like Jared's solution. At this point, in preparation for a very similar problem Mrs. Talent was going to assign as an individual evaluation task, Mrs. Talent led the class through a step-by-step analysis of the game, closely following Jared's reasoning.

Mr. English asked for student input in the process of analyzing four of the games and activities he included in the probability unit, although he used that input to varying degrees. After interpreting the experimental results for the "Newspaper Offer" simulation, Mr. English asked, "Does anybody have any idea how we might analyze that?" One student responded, "Make a chart," as if he thought this was the standard answer to give. When the student could not explain how he would set up a chart, Mr. English proceeded to explain how he had analyzed the situation using an organized list.

In two of the games and activities, the students previously had been shown how to analyze a situation similar to what they were asked to analyze. For example, after Mr. English had demonstrated how to analyze "Is This Game Fair?" he asked the students to analyze "Doubles in Monopoly." Both of these games involved rolling two dice with the player receiving three points for a specific outcome and losing one point otherwise. These winning outcomes were rolling a sum of 7 or rolling doubles, both of which have a probability of 1/6. After playing "Doubles in Monopoly" and sharing the experimental results, Mr. English asked the students, "On your summary sheet . . . I want you to tell me why that last game was unfair. . . . Tell me how to fix it so that it's fair. . . . Look at the results, look at the rules, and analyze it and tell me why it's unfair." After allowing some time for the students to complete the assignment, Mr. English explained, "One of the things I'd like to do . . . is to have two or three of you just read what you said." He called on one girl to read what she had written, knowing the student had a correct answer from what she had shown him as he circulated around the class as the students worked. The student read her response, patterned after the analysis done for the earlier game, "The dice game seems unfair, because the odds are 1 to 5. If you win, you should get $5 instead of $3 so that you get paid back. The theoretical probability of throwing doubles is 1 in 6. That means five sixths of the times you're going to pay and you'll be paying out $5 but only getting back $3." After she finished reading her solution, Mr. English added the following:

> Okay. Now, translated, what she basically said was that she figured out . . . and she told me this, that there were six ways to get doubles: double 1, double 2, double 3, double 4, double 5, double 6. There are 36 sums, so that's 6 ways out of 36. It's exactly the same as the 7 situation. So one out of the six times, she's going to get $3. And she drew this [circle divided into six sections as was drawn for "Is This Game Fair?"] . . . on her summary page. She said she was going to get the $3 one time, but five times . . . the other person's going to get $1. So out of six times, the other person comes out $2 ahead.

After this explanation, there was time for one other student to read what she had written on her summary page.

Similarly, Mr. English used students' solutions as part of the analysis for the Three Coins problem, which had been assigned as a follow-up to the tree diagram drawn for "The Hare and the Tortoise Game." In this case, the tree diagram for the Three Coins problem would be exactly the same as for "The Hare and the Tortoise Game," except for the labels. "The Hare and the Tortoise Game" considered the odd and even outcomes for dice; the Three Coins problem dealt with coins. Mr. English first presented a tree diagram one student had labeled incorrectly with "match" and "no match." Mr. English had chosen to use the incorrect tree diagram, in this case, because as he pointed out to the class, "Sometimes it's good to analyze how we think about something and contrast it with . . . the right way to do it and learn from it." After a second student had demonstrated the correct tree diagram, Mr. English concluded, "The outcomes for coins are heads and tails, so you have to list them like that. You only look at the match or no match when you are all done."

Mr. English asked for student input in the analysis of one other game, "Which Do You Think Will Be Larger?" In this case, the game had not been preceded by a similar task that already modeled the analysis procedures for the students. And, in fact, this was perhaps one of the more difficult tasks included in the unit, from the analysis perspective. In this game, player A rolls two dice and multiplies the two numbers. Player B rolls one die and multiplies the number times itself. The winner of the round is the player with the larger product. Mr. English and the students had previously analyzed some dice games where, for example, two dice were rolled and the numbers were multiplied. In this case, one player received two points if the product was odd and the other player received one point if the product was even. Because both outcomes were based on the product of the two dice, the definition of probability as the ratio of favorable to possible outcomes could be applied. However, in "Which Do You Think Will Be Larger?" two different actions were taken by the players and the winner was determined by comparing the outcomes obtained. The different actions and the comparison complicate the analysis process.

Perhaps not recognizing the complexity of the analysis in this case, Mr. English explained to the researcher that he wanted to get the students more involved in the analysis process. Therefore, after Mr. English and the students had considered the experimental results for "Which Do You Think Will Be Larger?" he asked the students, "Now, I've asked you to think about it. Maybe you can give me some ideas on how to . . . figure out whether this is fair or not. Anybody have any idea what we might do to figure it out?" One student offered a suggestion,

S: Well, I just finished adding up . . . all the possibilities. If you did 36 dice . . . rolls of the dice for each one . . . the B would get 546, if you add it up. And A would get 461.
T: Did you fill out this chart? [*putting up a transparency of the multiplication chart included in their packet*]?
S: No.
T: You did it a different way? Would you . . . could you come up here and draw what you did?
S: I didn't do it on paper.
T: You didn't do it on paper? You just thought it through? So you're conclusively believing that B is going to win?
S: Yes.

In analyzing the game, the student had imagined what outcomes each player would get for 36 rolls of the dice and then he had found the sums of those 36 outcomes. To find the sum for player A, the student had in effect added up the 36 numbers in the 6-by-6 multiplication chart Mr. English had mentioned. However, the student had made an error, for the total should have been 441 instead of 461. To find the total for player B, the student had found the sum of the squares, assuming each single digit had occurred six times (to have 36 outcomes corresponding to the 36 outcomes for player A). In this case, his result of 546 was correct. The student then compared the total sums. Although this reasoning led the student to a correct conclusion, his analysis was not completely correct because it failed to account for the comparison of individual outcomes for each round. Nevertheless, the student demonstrated some significant insights into what was a fairly difficult problem to analyze. However, Mr. English could not follow the student's explanation and, therefore, did not make use of the student's input. When efforts to clarify what the student had done were unsuccessful, Mr. English proceeded to show the students how he had analyzed the problem with a chart.

In addition to being the ones who generally did the analysis in the classrooms of Mr. Trackman, Mrs. Talent, and Mr. English, the teachers were also the ones who generally stated whatever conclusions were given, but the nature of these conclusions varied. In some cases, little or no conclusion was given at all; the analysis just ended and the teacher went on to the next activity. For example, after Mr. Trackman had presented the outcomes for the Dice Sums game, he asked the students a series of questions identifying the probabilities of the dice sums from 2 through 12. After completing that list, the discussion of the game ended as follows:

T: Now, if you added all those fractions ups, what should it add up to? Pete?
S: Six thirty-sixths.
S: What?!
T: No. Ding.
S: Thirty-six.
T: Thanks for playing. Yes, 36 . . . 36ths or . . . ?
S: One whole.
T: One whole. . . . Okay. What I need to have happen right now . . . is I need a volunteer [to hand out the dice for the next game].

At other times, the teachers stated conclusions to the game or activity. However, in most cases, the focus of the conclusion was on the game or activity itself, not on the methods used in the theoretical analysis. For example, Mr. Trackman concluded the analysis of "Is This Game Fair?" by observing, "So, the opponent should win." As Mrs. Talent concluded the analysis of "The Hare and the Tortoise Game," she pointed out, "If I said, 'Hey, I'll play a game and the only way you can win is if you land here [*pointing to two spaces*] and I get to win if it lands everywhere else [*five spaces*].' You'd say, 'I'm not going to play it. It's not fair. You're going to win more often.' But, the tortoise [with only two spaces] actually wins more." For "Which Do You Think Will Be Larger?" Mr. English concluded, "The B guy has a little bit better chance of winning than the A guy, 'cause he's got a few more ways to win as opposed to losing." In particular, in these cases, there was no discussion of why a specific analysis method had or had not been used. For example, Mrs. Talent did not review the steps she had used in drawing the tree diagram for the analysis of "The Hare and the Tortoise Game." Nor did Mr. English explain how the students could set up a chart as he had for analyzing games like "Which Do You Think Will Be Larger?"

In a limited number of instances, the teachers did reflect on some of the broader issues. For example, after playing and analyzing "Is This Game Fair?" and "Doubles in Monopoly" on Day 2, Mr. English reflected on what the students should have learned, observing, "Okay, what you should have learned a little bit about today is the concept of fairness and unfairness, how to make something fair, how to analyze dice activities when you're rolling two dice, and the concept of what do we mean by odds. Those things I think we talked about." After doing the analysis of the "Chips" games, Mrs. Talent provided a rationale for learning to analyze similar situations, telling the students, "What you can use this for is, once you know how to analyze something like this, what it will tell you is, if you play a game like this, if you play 100 times, how many times would you expect it to match, out of 100? . . . About 75 times and a "No Match" only 25 times. So you can make predictions with this kind of stuff after you know how to analyze a little bit."

At other times, Mrs. Talent or Mr. English concluded the theoretical analysis by comparing the theoretical results with what had happened experimentally. After concluding the Switch strategy provides a better chance of winning the prize in "Monty's Dilemma," Mrs. Talent cautioned, "Sometimes it doesn't work that way. I've had classes that have done this where, like, there's probably somebody here, they lost eight times and won only twice by switching. So, it doesn't always mean it's gonna work, but over the long haul, that strategy is gonna work better than sticking." Similarly, after considering how "The Top and One Other" would come out theoretically, Mr. English concluded, "If we go back and look at the experimental results . . . a lot of times your experiment will not always verify or be exactly the same thing as the theoretical probability. But we did the experiment and we looked at the experimental probability and it came real close to that."

At yet other times, the teachers concluded the theoretical analysis of the games and activities by checking for students' understanding of the analysis or, at least, going through the motions of checking for understanding. For example, as Mr. Trackman completed his analysis of "Is This Game Fair?" he asked, "Any questions about what we've done and where we've gone? Okay?" When there was no response, he had the students put their names on their papers, hand the papers in, and get ready for the next activity. Similarly, Mrs. Talent concluded the analysis of the "Chips" games by asking, "Now I want to know if you have any questions about how I drew the tree or how I got any of these or how you got them. You guys understand this now?" After Mr. English had explained the theoretical solution to the

"Newspaper Offer," he checked the students' understanding by inquiring, "Do any of you have a question on anything here? Is there anything confusing to you about this? Do you understand how I paired these things up? How I did it?" There generally was no response to such questions, perhaps giving the teacher the false impression that all the students understood what had been done.

Introducing textbook and worksheet assignments. The nature of the discourse involved in the presentation of the textbook and worksheet assignments differed from the discourse involved in presenting the games and activities. In particular, because these did not involve hands-on activities, no rules or directions needed to be explained.

Mr. Trackman was the only teacher to make textbook assignments. In doing so, he provided no introduction or explanation, at least for the one textbook assignment where the presentation was observed (the other two textbook assignments were made by a substitute teacher). The assignment was simply written on the overhead and the students were expected to get to work on it.

A more extensive introduction was generally provided for the worksheet assignments included in the probability units. For example, Mr. Trackman introduced the one worksheet he assigned by asking the riddle printed on the page: "What do you get when you cross an owl with an oyster?" To discover the letters to fill in for the solution to the riddle, the students had to find probabilities for various outcomes, including several joined by *and* or *or*. In one situation, a spinner was divided into 12 equal-sized pieces numbered 1 to 12. The students were asked to find, for example, the probability that the spinner would stop on a number that is a "multiple of 2 *and* a multiple of 3." In another question, they were to find the probability that the spinner would land on a number that is a "multiple of 2 *or* a multiple of 3."

To help the students understand the directions, Mr. Trackman worked the first item on the worksheet with them. Then, he presented an example to help the students distinguish between the conjunctions *and* and *or*. For his explanation, Mr. Trackman drew three ovals and labeled them A, B, and AB, meaning, he said, A times B (see Figure E.5). He then proceeded to ask the students the following sequence of questions: "What's the probability of getting a multiple of A? . . . What's the probability of getting a multiple of B? . . . What's the probability of having a multiple of A and B?" When the students gave a variety of responses to the last question, Mr. Trackman pointed to each of the ovals as he then asked them, "Is this a multiple of A and B?" The students responded, "No," for each of the first two ovals. Mr. Trackman then pointed to the AB oval as he concluded, "This is a multiple of A and B 'cause . . . it's got a factor of A and it's got a factor of B . . . *and* . . . both of them. Okay. So this would be 1/3."

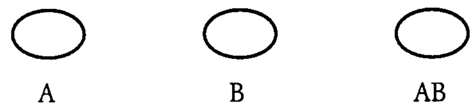


Figure E.5. Mr. Trackman's example for distinguishing between *and* and *or.*

When he moved on to the question of the "probability of getting one with a multiple of A *or* a multiple of B," he again received a variety of responses from the students. As he pointed to oval A with the question, "Is this a multiple of A or B?" he continued to get both yes and no responses. Mr. Trackman replied, "Yes, no, yes, no. If you say 'yes, no' enough times you're going to get it right. Okay, it's yes? That is a multiple of what?" When at least one student responded, "A," Mr. Trackman went on to conclude, "So it is a multiple of A or B." The students displayed similar confusion with oval B before Mr. Trackman pointed out that it was a multiple of A or B "because it's a multiple of B. It's one of the two." After the students agreed AB was a multiple of A or B, Mr. Trackman concluded, "It's 3 out of 3. See this is talking about either one of them. Either one of them. Either/or. And this is both of them. It has to be both of them combined."

In the end, the example Mr. Trackman provided, with letters in place of numbers, did not appear to have helped the students understand the distinction between *and* and *or* because it was more abstract than the items on the worksheet. Or rather than being confused by the conjunctions, the students may not have understood Mr. Trackman's use of the term *multiple* in this abstract setting.

Mrs. Talent included seven worksheet assignments in her probability unit. Three of these worksheets, "More Chips," "More Dice Games," and "Experimental Probabilities," were assigned to follow up games or activities done in class. As a result, these worksheets needed little introduction. Nevertheless, Mrs. Talent generally provided a brief explanation of what the students were to do. For example, before handing out "More Chips," Mrs. Talent explained,

> What you're going to have to do for your assignment is . . . you have to draw two tree diagrams like this. And one of the tree diagrams has four chips, so you'd have to branch out one more time. And another one has three chips like this one [we just drew]. And then the third thing is . . . you're gonna have to invent a game like this and make it fair. Like this one is not fair because . . . [*unfinished thought*] . . . you have to invent a game with chips or coins that would be fair and you have to explain your rules. Okay, let me give you the paper and make sure you know what to do.

After handing out the worksheets, Mrs. Talent provided further explanation about the first item and then the students began working on the assignment.

The remaining worksheets were not directly related to other assignments, but were daily lessons by themselves. These Mrs. Talent usually introduced by giving the students two or three examples focusing primarily on how to complete the items on the worksheet. For instance, on an assignment chosen by Mrs. Talent to give the students practice in "writing probabilities," she gave the students samples of the "different models" they would encounter on the worksheet. In particular, she made sure the students knew what a deck of cards contained and how to write the corresponding probabilities. She also gave them examples of writing probabilities corresponding to spinners and to the selection of a cube from a sack of cubes. These examples involved a series of questions asked by the teacher. The following dialogue is the portion related to probabilities on spinners:

T: On this spinner, if you just put this down on a flat surface and you spun, you know, just normal, what's the probability that you would land on yellow?
Ss: One out of 4.
T: Okay, does that make sense? It's 1 out of 4? Okay, what's the probability that you would not land on yellow?
Ss: Three fourths.
T: Okay. And then, what's the probability you'd land on either yellow or red?
Ss: Two fourths, 1/2.
T: One half, 50%. Okay, so the spinner works kind of the same way [as the cards in the earlier example]. What you're gonna see on the worksheet is . . . you're gonna see a spinner with numbers. For instance, I think it divides it into eight sections, and it just has the numbers 1 through 8. And they're gonna ask you, "What's the probability that you'll spin and get a 6?" So, what would it be?
Ss: One out of 8 . . . 1 out of 6 . . .
T: That you'll spin and land on just the 6? Just this section . . . 1 out of . . . ?
Ss: Eight.
T: The eight choices. So, we would say 1 out of 8. Now, where it throws people off sometimes is they'll ask you questions like, "What's the probability that it'll be greater . . . than 3?"
Ss: Five eighths . . . 3/8 . . .
T: What does greater than 3 mean?
S: Five eighths.
T: What is greater than 3?
S: Four or higher.
T: Yeh, 4, 5, 6, 7, 8. So, you've got five chances out of . . .
Ss: Eight.
T: Eight. So, you would say 5/8. Does that, is this making sense?

As in this case, the examples given were designed to prepare the students to answer the questions contained on the worksheet. In particular, Mrs. Talent gave examples using *not, or,* and *greater than* because the worksheet included such items. Other than stating the worksheet was for practice "writing probabilities," there was no discussion of learning objectives or content. In particular, the definition of probability as the ratio of the number of favorable outcomes to the number of possible outcomes was not stated, although it was clearly assumed in these examples.

In this situation, Mrs. Talent provided several examples. In introducing the other three worksheets she presented two examples in each case. The first example provided a model for the students to follow. As with the excerpt quoted previously, this first example generally involved a teacher-directed dialogue with the teacher's questions guiding the students through the process. Mrs. Talent then gave the students an example to do on their own, thereby checking for student understanding. For example, after demonstrating a two-stage tree diagram representing an experiment with two spinners, she asked the students to draw a two-stage tree representing a coin toss and a spinner. On another assignment, after summarizing the multiplication property, Mrs. Talent applied it to a situation involving a die and a coin. She then asked the students to apply it to a situation involving a die, a coin, and a spinner. Finally, for a later assignment, Mrs. Talent explained how to apply the multiplication property in a dependent situation such as selecting colored cubes from a paper bag without replacement. After finding the probability of selecting a green cube and then a white cube as an example, she checked the students' understanding by asking them to find the probability of selecting a black cube and then a yellow cube.

The worksheets Mr. English assigned to his students were primarily given as homework or follow-up to activities done in class. For example, after introducing the vocabulary and basic properties of probability on Day 1, Mr. English assigned "Blocks and Marbles" and a vocabulary worksheet as homework. Because the worksheets were related to activities done in class, the procedures or analysis techniques involved had already been modeled for the students. As a result, Mr. English provided only a limited introduction, reminding the students of what had been done in class. For example, following the "Newspaper Offer" simulation activity, Mr. English provided the following explanation for the "Newspaper Pay" assignment.

> All right, now let's look at the problems you're going to do [*picking up a copy of the packet and holding it up*]. Turn the page. It's called "Newspaper Pay." You've got six problems. . . . Read the directions with me as I read them to you. "In each of the following situations, the customer should pay $5.00 per week for newspaper." And that's exactly what we did in this situation here. "Sue, the paper carrier, has to decide which of the schemes of chance would give her a fair deal over the long run." In each case, you've got to decide what Sue should do. Should she accept or reject the proposal? In number 1, "the customer will place a $5 bill and three $1 bills in a bag." And Sue is going to draw out two bills. You've got to analyze that exactly like I analyzed this [*referring to the list outcomes on the overhead*], except, in this case, we had a $10 bill and five $1 [bills]. In the case of the problem here, you've only got a $5 bill and three $1 bills. So make a list of all the ways that could happen and

> then [*pointing to what they did for the problem done in class*] multiply to get the expected value and then divide that amount by the number of outcomes that are possible and you'll see what the expected value would be theoretically. And then, on that basis, then decide whether she should accept of reject. Okay. So you tell me whether you accept or reject and you tell be why . . . you show me how you arrived at that.

Thus, for these three teachers who gave worksheet assignments, the focus of their presentation was on how to complete the worksheets. And, as with the games and activities, there was no specific discussion of any learning objectives involved.

Working on assignments. In two of the classrooms, students were given class time to work on textbook and/or worksheet assignments. This provided further opportunities for Mrs. Talent and Mr. Trackman to interact with their students on a small group or individual basis. Mrs. Talent also allowed time in class for the students to work on the evaluation tasks she assigned, time during which she interacted with the students as well.

As Mrs. Talent's students were working on the worksheet assignments, she circulated among the students, checking on their progress and offering assistance as needed. At times, she monitored students' progress with questions such as, "Do you know how to do this?" or "You guys doing okay?" At other times, she responded to students with their hands raised for help by asking, "Okay, how can I help?" The students' questions generally focused on particular items from the assignment. For example, on the "More Chips" worksheet, one student asked, "Which ones are red or blue?" On the "Finding Probabilities" worksheet involving spinners, another student asked, "What's '4 or not 4'? What does that mean?" In response to the students' questions, Mrs. Talent provided specific answers and whatever guidance the students needed, as in the following excerpt:

T: What can I do for you?
S: I'm trying to figure out . . . player 1, player 2 . . . [the] win thing.
T: Okay, how does player 1 win?
S: If both the red chips are showing . . .
T: Okay, you've got three chips here, right? One came up A, the other came up A, and the other one . . . . Which ones do you want to be the two red chips and which one do you want to be the blue chip? It doesn't matter, just . . .
S: These two.
T: Okay, first two are red. So it says that player 1 wins if both red chips show A, if the blue chip shows A or all three chips show A, so you need to find all the different times that that happens.
S: . . . move these around?
T: Right.
S: This one?
T: Well, does that fit? Does that fit? Do both red chips show A? But the blue chip does, so that's a winner.
S: One, 2, 3, 4, [5 out of 8].
T: Okay, so is this a fair game?

S: No.
T: Okay.

As the students worked on the assignments, they frequently looked to Mrs. Talent to confirm their answers, often asking, "Is that right?" Mrs. Talent sometimes responded briefly with, "Looks good." At other times she responded more extensively with, "Um, let's see what you did. This must represent spinner 1, spinner 2, and did you get six different outcomes? Looks good. And then, don't forget to write in the [probabilities]." In instances when the students were incorrect, Mrs. Talent asked questions such as, "How did you get that?" as she sought to find out where the students had made their error.

However, in contrast to the guidance she provided on the worksheet assignments, Mrs. Talent reported taking a different approach as she interacted with students as they worked on the evaluation tasks. Although she would try to answer the students' questions, she suggested she would not tell them what to do. Mrs. Talent explained she instead would ask the students questions to see what they understood and help guide them to identify what they understood and what they did not understand. For example, when one student asked for help as he worked on the Carnival task, he received the following response:

T: What part don't you understand?
S: None of it!
T: Okay, did you read it over again?
S: Yeh.
T: Okay, so . . .
S: Is it just like . . . doing it for three things?
T: You could try that. Basically, you just need to tell me, are they going to be able to make . . . $200 with this game. . . . So, you've got to figure out a way . . . to analyze it a little bit to . . . determine if they can do that or not . . . . We've done enough stuff where you should be able to do that. . . . You have enough background to be able to figure this out. . . . You're going to have to . . . think it through and draw on all your resources to . . . see what you can come up with.

Mrs. Talent reported this approach frustrated the students, suggesting the students thought, "Just give me a worksheet and show me how to do it. Don't ask me to think." Part of the students' frustration may be that they had not been prepared in previous work to think on their own. In particular, the students had grown accustomed to being told what to do on the worksheets and had come to rely on being told if their answers were correct. However, this "learned dependency" became a hindrance when they were expected to think independently and apply their knowledge in new situations such as the evaluation tasks Mrs. Talent provided at the end of the unit.

The students in Mr. Trackman's class had time in class to work on their assignment on two of the days observed. Mr. Trackman seemed to see this as a time to work on his own tasks. On one of the days, he told the students, "I'm going to be at the back working at my desk. . . . If you have questions, don't hesitate to come back." Some students did come to his desk with their questions. At other times, he circulated among the students for brief periods. The few questions Mr. Trackman asked focused on getting the students to work on their assignments. For example, after making one textbook assignment, he asked different students, "Did you get it done yet?" or "Did you work on it yet?" For students to get assistance it was necessary for them to ask for help. This they did with questions such as, "What is relative frequency [*a term that had not been defined*]?" or "How would I reduce this [fraction]?" As the students worked on the worksheet with the Owl and Oyster Riddle, many of their questions focused on getting the correct letters in the appropriate blanks at the bottom of the page. For example, the questions they asked included, "Is this J?" or "Does TSAT spell anything?" The answer to the riddle, rather than the mathematics, also seemed to be Mr. Trackman's focus, as in the following dialogue with one student:

S: I don't get this one . . . a multiple of 3 or . . .
T: What's this word . . . this letter [*referring to the first blank which was a word by itself*]?
S: *U.*
T: *U? U*'s a word?
S: Well, that's . . .
T: What are the only one-letter words you can have?
S: *I.*
T: Or . . . ?
S: *A.*
T: *A,* and that's it.
S: Oh, I messed up.
T: And in the first part, there's no *I,* so it has to be . . .
S: *A.*
T: *A.*

Although in this case, Mr. Trackman helped the student reason out what the correct answer would be, he addressed neither the student's question nor the probability question being asked.

In Mr. Trackman's interactions with various students, there was evidence he treated some students differently from the others. For example, one part of the Owl and Oyster Riddle worksheet presented the students with 15 different numbered shapes including triangles, squares, circles, pentagons, and hexagons. One of the items asked the students what the probability was that a shape picked at random "is a multiple of 3 or a square." When two boys asked for help on that item, Mr. Trackman began pointing to each shape with the question, "Is it a 3 or a square?" When the boys failed to identify shape 6 as a multiple of 3, Mr. Trackman corrected them. In this way, Mr. Trackman guided the boys with his questions to the correct result of 7 out of 15. Later, one girl came to Mr. Trackman with a question on the same item. In this case, Mr. Trackman responded as follows:

T: Which one? Which one are you on?
S: [Number] 10.
T: Ten is 7.
S: How?
T: Okay, there's a multiple of 3 [*underlining shape 3*], there's a multiple of 3 [*shape 6*], there's a multiple of 3 [*shape 9*], there's a multiple of 3 [*shape 12*], there's a multiple of 3 [*shape 15*]. Right?
S: Yeh.
T: Now it also could be a square, so it would be that or a square. There's a square [*shape 3*], there's a square [*shape 7*], there's a square [*shape 9*] . . .
S: And there's a square [*shape 13*].
T: And there's a square [*shape 13*]. So all the ones that are underlined is the answer for 10.

Mr. Trackman may not have been conscious of this differential treatment. Rather, he may have grown tired of answering this question throughout the day and turned to this approach as a quick way of dealing with it. Or because this particular student frequently asked for help, with reducing fractions for instance, Mr. Trackman may have opted for telling her the answer, thinking she might have trouble reasoning through the problem. Whether or not it was a conscious pattern of differential treatment, this girl at least seemed to have discovered Mr. Trackman would give her the correct answers if she asked. A similar request for help in the Coin Tossing Exploration had yielded similar results when Mr. Trackman began to fill in the probabilities for her.

Rather than specifically a case of treating boys differently than girls, however, it may instead reflect a pattern of how Mr. Trackman treated stronger students in contrast to weaker students. In particular, one of the boys in the class seemed to have discovered he could also get Mr. Trackman to give him the answer. As the student was working on the unit test, he raised his hand for assistance on the questions dealing with the initial line graph, with the following results:

T: First one is the table . . . the average. What's the average? Yeh, you got 8, you got 10 there, you got 14 there, and you got 4. What's the average number?
S: Oh, you add them up.
T: And then . . . add them up and then . . .
S: Divide by 4?
T: Divide by 4.

A few minutes later, the same student went up to Mr. Trackman with a further question on the same item.

S: Is that right?
T: Which one? Okay, that's total, if you added 'em up okay. There's four, right? So, each one has, added together they each have, how many?
S: [*answer not heard*]
T: Well, now what . . . the average is saying, remember, is that it's about the same for this, this, this, and this. There's four different . . . seasons, so this is asking for the average per season. You gave me the total for the whole year. Okay? So break it up into fourths.
S: Yeh.

Interestingly enough, Mr. Trackman was assisting the student in finding the average per season when that was not one of the questions asked. Mr. Trackman seemed to have jumped to that conclusion based on the title given for the line graph, "Average Seasonal Rainfall in Tallahassee." He had, however, provided the student with at least one of the correct answers when he had read the graph results as he said, "You got 8, you got 10 there . . . ." Nevertheless, this example provides further evidence that at least some of the students received more productive help than others from Mr. Trackman.

In addition, a considerable amount of the discourse between Mr. Trackman and his students did not focus on mathematics or the task at hand. In much of this informal interaction with students, Mr. Trackman was trying to relate to the students or create a "fun" classroom environment. For example, the off-task topics of discussion included the movies, musical groups, or sports that were of interest to the students. Mr. Trackman also interjected his sense of humor in response to students' comments. For example, a frequent response when students said, "Oh, I see," was " 'Oh, I see,' said the blind man to his deaf dog." Although he was trying to create a relaxed and "fun" classroom with his comments, his sense of humor sometimes served as a distraction when students tried to figure out what it was he had said.

Correcting assignments. Because the games and activities involved experimentation, which produces diverse results, the results from those tasks were not graded. The only assignments needing to be corrected as such were the textbook and worksheet assignments. For these, Mr. Trackman, Mrs. Talent, and Mr. English used a variety of different approaches. In particular, the students were involved to varying degrees in the process of correcting the assignments.

To correct the textbook assignments, Mr. Trackman listed the correct answers on the overhead for the students to check. In the two lessons observed where textbook assignments were corrected, Mr. Trackman did not ask if there were any questions or difficulties on the assignment. In particular, there was no checking for understanding with regard to the content of the assignment. The only discussion came in response to a student's question, as seen in the following excerpt:

S: Were we supposed to reduce these?
T: You don't have to reduce it. Reducing's not important . . . not on this one.
S: How do we know if we got it right?
T: Well, you have to determine whether or not it's an equivalent fraction.

After giving the students time to check their answers against the correct answers posted, Mr. Trackman showed the students the grading scale and proceeded to have the students report their scores. Thus, for the textbook assignments, there was no substantive discourse at all, either in assigning or in correcting the assignments.

As they were correcting the Owl and Oyster Riddle worksheet, however, Mr. Trackman asked, "Were there any problems anybody encountered?" After asking a couple times what problems had been encountered, Mr. Trackman reminded the students of one problem.

T: I know some of you had, I think on the second word or on the third word they had TSAT.
S: TSAT?
T: Okay. You knew the *T* was right. Why?
S: Because you gave it to us.
T: Because I gave it to you. Okay. So there's part of the word. So now it comes down to, basically, the *A* or the *S* being incorrect. How do you do that? How do you determine which is right?
Ss: You do one of 'em. . . . You do one over.
T: Okay . . . you go to where the problems could be. . . . You do the problem over. And the second time, you . . . figure out where your mistake was. . . . And then you figured out that it was THAT.

Mr. Trackman then proceeded to show the students the answer to the riddle and explain how scores would be assigned. As with the textbook assignments, there was no discussion of the mathematics involved with the assignment.

Mr. English used several different means for providing the correct answers to the worksheet assignments. On the Vocabulary worksheet and the first item of the "Newspaper Pay" assignment, Mr. English called on students to supply the answers. For the "Blocks and Marbles," "More Dice Games," Vowel Frequency, and "Experimental Probabilities" worksheets, Mr. English read the answers to the students from the answer key. For the remaining five items on the "Newspaper Pay" assignment, Mr. English presented the analysis and final solution to the students.

With the exception of the "Newspaper Pay" assignment, there generally was little or no discussion of the mathematics involved in the assignment. After reading the answers to the "Experimental Probabilities" worksheet, Mr. English asked, "Any questions on that?" Because he had used some of the same worksheets in previous years, he was aware of some items that presented difficulties to students. After reading the correct answers to these items, Mr. English sometimes asked, "How many got that one right?" indicating, for example, that "when they put in the *or's* and the *and's* and the *nor's*, that . . . tends to bother people."

Mr. English, however, did not adequately foresee difficulties with one item on the "Blocks and Marbles" worksheet. Given a bag containing "two yellow marbles, four blue marbles, and six red marbles," the students were asked, "How many marbles must be added to the bag to make the probability of drawing a blue marble equal to 1/2?" Because the goal seemed to be getting through the correcting process as quickly as possible, Mr. English did not take time to explore this item, even though students suggested a variety of answers besides the answer of "four blue marbles" given on the answer key. Rather than exploring the other answers suggested by the students, Mr. English indicated they should "go ahead and count it correct," noting other answers were possible (as indicated on the answer key). However, some of the responses given by the students were incorrect. In particular, some students gave an answer of two blue marbles. This represents a common error. Adding two blue marbles would bring the number of blue marbles to six which is one half of the original amount, but adding two blue marbles would also change the total number of marbles. In this case, 6 out of 14 is not equal to 1/2. This item provided an opportunity to explore students' thinking, but that opportunity was overlooked in the process of correcting the assignment.

Mr. English took a different approach to correcting the "Newspaper Pay" worksheet, which he identified as "probably one of the harder pages of the unit." As they began to correct the assignment, Mr. English asked, "Is there anyone that would like to come up and model [number 1] on the overhead?" One student volunteered and wrote out the analysis for the situation, which involved choosing bills from a paper bag, in the same format as modeled by Mr. English the previous day. However, when no one volunteered a solution to a similar situation for the second item, Mr. English reported, "You should reject number 2." He then went on to demonstrate the analysis. When he had no further volunteers, Mr. English went ahead and provided the solutions and/or analysis to the remaining four items, which involved situations with coins and dice. On items 3 and 5, he reminded the students about the analysis strategies to apply in those situations, namely tree diagrams and charts, respectively.

Some of the worksheets Mrs. Talent assigned were handed in without being corrected or discussed by the class. Other worksheet assignments were corrected by the class. In these instances, Mrs. Talent called on students to provide the answers. Correct responses were reinforced with comments such as, "You're right," or, "Good." Incorrect responses led to a more extensive discussion, as in the following example:

T: And then what did you get for the second part on that one?
S: One fourth.
T: What did you get?
S: One fourth?
T: One fourth? You guys agree with that?
Ss: Yes . . . No . . . It's 1/2.
T: Yeh, that one. Notice there is a little cloud there that says, "Careful." The reason is, if you list out all outcomes, it's heads, heads . . . head, tails . . . tails, heads . . . tails, tails. And so, it says, "What's the probability of getting one head and one tail?" It could be a head and a tail or a tail and a head, and there is two of them.
S: Oh.
T: So it is a tricky one. So it is 2 out of 4 or 1/2.

Although the focus was on the answers as the assignments were being corrected, the students needed to have written down more than the answers in order to receive credit. For example, Mrs. Talent reminded the students on one assignment,

> Check your paper right now and make sure you've got all these things. You've got a tree diagram for every problem, and off to the side, remember I said you had to put the . . . probability, like 1/4 or 1/2, for each branch. And then, also, you had to list out the outcomes. If you don't have the outcomes listed, you'll get it back for a redo. Then you need to have the answer in the little blank there. Okay?

One of the worksheet assignments, after it was corrected, served as a springboard for the next lesson. In particular, before the students handed in the "Two-Stage Trees" worksheet, Mrs. Talent asked them, "What I'd like you to do is go through [each of the items on the worksheet] . . . and write the probability for the first one, probability for the second one, and then what you got for your answer. And then see if there is a relationship there that's always the same . . . that'll always work." With this request, Mrs. Talent had the students discover the pattern of the multiplication property for finding probabilities when a sequence of outcomes in involved.

Structuring the learning opportunities. In addition to impacting the content and nature of instruction, the classroom discourse also potentially contributes to learning by elucidating the structure of instruction in at least two respects. First, the classroom discourse potentially shapes and/or reveals the structure of the lessons themselves. Specifically, the introduction and closure provided for a learning task are of particular importance to the learning process as they provide answers to questions such as: What is the motivation for doing the activity and/or learning the related content? What learning objectives are expected to be met in the process of doing the task? Or, as part of closure, what expected learning outcomes have been achieved?

As seen in the earlier portions of this section, these questions were rarely addressed. While the introduction of the games and activities frequently included an overview of the task, that overview focused on the steps the students would follow in completing the activity, such as making a prediction, conducting an experiment, and/or doing the theoretical analysis. Similarly, the introduction of the textbook and worksheet assignments provided examples of the steps to be followed in completing the assignment. However, the overview of the tasks generally did not include any extensive discussion of the content involved or express expectations for what students were to learn or be able to do, much less how it fit in with the unit in general.

Likewise, whatever closure was provided to the tasks did not focus on learning. When conclusions to the games and activities were stated, if they were stated at all, these generally focused on the game or activity itself. Thus, the conclusion stated how many boxes of cereal one would have to buy to get all five prizes or whether the game was fair or not. The conclusions, however, did not involve any reflection on what content or concepts had been involved, how the tasks had been analyzed, or what had been learned. In particular, the teachers provided no discussion of why a specific analysis method had been chosen. Further, the teachers made no effort to review the steps of the analysis process and no attempt to fit the activity into the overall scheme of the unit. Similarly, the process of correcting the worksheet and textbook assignments focused on correct answers and not on the underlying learning objectives.

As seen in these descriptions and in the earlier sections, the probability lessons were lacking introductions and closures focused on learning. As a result, because these key elements of the lesson structure were weak or nonexistent, the students were provided little guidance in determining what they were to be learning, whether concepts or skills.

Second, the classroom discourse not only provides potential structure to lessons themselves, but it also may elucidate the structure of the unit as a whole. Of importance in the context of this study are the answers to questions such as: How was the subject of probability introduced and/or motivated? What general overview of the probability unit were students given? What connections were made between daily lessons? What overall indications were students given about expected learning outcomes? The four middle school teachers answered these questions in varying degrees, as the following review of their probability instruction will indicate.

Mr. Trackman began his probability unit with a reminder of an earlier project and a question about the applications of probability. In particular, he introduced their study of probability with, "Today what we're going to be doing is we're going to be working on . . . probability. We're starting probability, and we've done some of it with the NBA project. Probability and statistics, they're . . . interwoven together. We're gonna be doing stuff with probability. Probability . . . does anybody know where probability is used in our lives . . . and where it's . . . and where it happens?" After he and the students had given some examples, Mr. Trackman went on to explain, "Games and making games fair is kinda what we're gonna investigate this week." However, rather than focusing on what the students might learn from the games, Mr. Trackman seemed more concerned about how much commotion would be created by the games, when he pointed out,

> We have about 6 or 7 assignments in the book that we could do, and I've got about 10 assignments that we can do over the next few weeks that are games or activities or something of that sort. Obviously the 10 that are activities, games, are a little bit more fun, not that they're so much fun that you're gonna [say]. . . "Oh, I can't . . . I just don't want to leave math class." No, but they're a little more fun than . . . than book assignments, wouldn't you agree? . . . Anyway, if you're good and quiet, like we are, then we'll continue to do activities. . . . The louder you get, the more book assignments that you're gonna get.

Although similar threats were repeated at times during the unit, they appeared, in the end, to be idle threats. Nevertheless, Mr. Trackman seemed more concerned about the noise level of the classroom than about the learning of the students. In particular, Mr. Trackman provided no explanation of anticipated learning objectives in his introduction to the unit or to the individual lessons. At no time during the probability unit did he pause and reflect on the unit or point out to students what he thought they should be learning. Nor was there any effort to make connections between the games, activities, worksheets, or textbook assignments included in the unit.

Mrs. Books had initially introduced probability in activities done earlier in the year. Because these had not been observed as part of this research study, it is not known what introduction or overview she provided at that time. One approach Mrs. Books used in motivating the tasks that were observed was portraying the problems as interesting questions to be explored. This she did, primarily, by providing examples of others who were interested in the problem or had also explored it. For example, as she introduced "Cereal Boxes," Mrs. Books explained, "This lesson that we're going to do today . . . is one that I have done in college, that one of my professors just recently did with his college class. It is one that one of the seventh- and eighth-grade teachers is getting ready to start on Friday." Similarly, she described "Monty's Dilemma" as follows:

> This problem has been written up in several magazines and several of the magazines have gone out to universities. And it is a problem that generates a lot of conversation. There are still people that are convinced that different ways are right. When I was in my Internet class and was just kind of surfing around, looking at different things, this problem came up on the Internet. They had changed it just a little bit, but it was essentially the same problem. So, you can see that it is a problem that people are curious about.

In addition to being interesting questions to explore just for the sake of exploring them, Mrs. Books also pointed out how each "mirrors something that does happen in real life."

Although Mrs. Books did not provide a single cohesive probability unit, but rather presented the probability tasks at different times during the year, she did make connections between the activities. In particular, as she introduced the "Cereal Boxes" simulation activity, she reminded the students of the structure of previous tasks. She explained that this structure, including a subjective estimate, a simulation, and the theoretical analysis, would be the pattern they would follow as they explored the new problems. In addition, as their investigation of the problems moved from phase to phase and from day to day, Mrs. Books also provided guidance and reminders. For example, after the students had made and shared their subjective estimates for the "Cereal Boxes" problem, Mrs. Books directed their attention to the next step, "The next task that you're going to have at your table is to design a simulation or experiment that has some criteria. One is that it meets our discussion that we had before . . . that you based your guesses on. . . . And the other thing is that it has to be as fair of a simulation as you can [have]." Similarly, on the second day of "Cereal Boxes," she began the lesson with a reminder, "Yesterday we started collecting some data for our experiment [and] a question came up."

Mrs. Books generally did not provide any indication to the students of her learning objectives for the activities. This, in part, may be a result of her view of learning as an individual process. Thus, although she wanted her students to learn about designing simulations, removing bias, and representing measures of central tendency, she believed students might be at different points in the process. Similarly, she did not take time at any point to look back and reiterate what learning had been expected. However, to some extent, the concluding assignment to both of the activities had the intent of having the students do that themselves. For example, the final activity sheet for "Cereal Boxes" asked the students to respond to three questions, the last of which was assigned as extra credit:

1. Now that you have completed the cereal box activity, how many boxes would you need to buy to collect all five stickers? Explain your reasoning.

2. We have used both line-plots and box-plots to visualize the data from the cereal box simulation experiment. What are some advantages of each of these types of plots? What are some disadvantages? Explain.

3. Suppose there are six stickers in the cereal boxes. How many boxes would you need to buy to collect all six? ________(your guess)

   Devise an experiment to test the number of boxes you would have to buy to collect six stickers. Collect some data based on your experiment. Show the results of your data below. Make a 90% box-plot for your data. How did your results compare to your guess above?

For "Monty's Dilemma," Mrs. Books asked the students to write a letter to the researcher "discussing your initial prediction and your reason. Then discuss how you conducted your experiment, any bias, and how you might have had bias initially, and how you changed it, your results, [and] how you would play the game."

It is not known what feedback Mrs. Books provided to the students regarding either of these final assignments, but the probability activities seemed to lack an overall sense of closure. In particular, the letters written by the students for "Monty's Dilemma" revealed some incorrect notions about expressing probability and about bias. Although Mrs. Books may have addressed these in her feedback to the letters, it still seems the class as a whole could have benefited from a brief discussion focused on what could be concluded about the activity, about their approach to exploring the question, and about their study of probability in general. Specifically, in the case of "Monty's Dilemma," this conclusion could have included a summary of appropriate ways of expressing probability (e.g., giving probabilities versus giving odds).

Mrs. Talent began her probability unit by writing the word *probability* on the overhead and asking, "Does everybody have a piece of notebook paper out? Please take out some notebook paper and I'd like you to write this word at the top. It's a word, hopefully, you have seen before and it won't be a brand new one. . . . Can anybody give me a good definition or what you remember about it? What you do with it? Anything?" After a couple students had responded to her question, Mrs. Talent next presented a story with several examples of how probability impacts people's everyday lives, which set the stage for the first simulation activity the class was going to do. Thus, in her introduction to the unit, Mrs. Talent was motivating their study of probability by its applications. However, her introduction to the unit provided no overview or explanation of what the probability unit would involve. Throughout the unit, the focus was generally on just the task for each particular day, with two exceptions. On Day 6, Mrs. Talent reviewed what had been done the two previous days as introduction to the "Two-Stage Trees" worksheet by saying, "Okay, last week, on Friday, and then the day before that, you worked with tree diagrams a little bit. Some of you still are kind of . . . not sure you know how to do those. And then on Friday, you worked with some spinners. And what you're gonna do today is kind of put the two together. And draw out what this would look like in a tree diagram." In introducing the "Two-Dice Games" on Day 8, Mrs. Talent again reflected on earlier activities,

> Yesterday . . . you took a look at . . . how you can take two or three things that are happening and figure out the probability that, like you're going to spin and get something and then get something else. . . . You figured out to do these yesterday. . . . Today, what we're going to do . . . is . . . I'm going to have you play another game, and at first, we're going to do an experiment

> like, remember when with the XY chips and the ABC chips and . . . "Let's Make a Deal" thing. We experimented first and then we took a look at the math part to see how it should have turned out. That's what we're going to do today.

With the exception of these comments, Mrs. Talent provided no overall view of the structure of the probability unit and made no connections between the various tasks included in the unit.

However, at one point, Mrs. Talent reflected back to consider what the students should have learned.

> [I] need your attention . . . so I can explain what we're doing. And kinda re-cap where you should be right now . . . Right now, here's what you should be able to do with probability . . . if I gave you a test right now. I . . . I should be able to give you a situation and say, "What's the probability that if you have a bag of marbles, you're going to pull out a certain color. You should be able to just . . . be able to write that. You should be able to, if I say "You're going to spin a spinner, and then flip a coin," you should be able to figure out the chance of getting like a four and a heads. Something like that. You should be able to take something and draw a tree diagram for it. And then, yesterday, you learned that you could also use a chart. Like when we did . . . the six. One, 2, 3, 4, 5, 6 and 1, 2, 3, 4, 5, 6. And, you multiplied or added. So you learned that you could draw a chart. So, basically, all we've worked with, so far, has been looking at different situations, and then how do we write out all the different outcomes possible. Then, how do we write the probability . . . .

Later, as Mrs. Talent introduced the Carnival task she was using as an evaluation task, she explained, "The purpose for doing this is you've got to be able to take the probability that you've learned . . . . We've been working with probability for 3 weeks, believe it or not! And, now it's time to see if you can apply it. Can you take a situation that's real and use what you've learned to figure out if it's good or bad? So that's what I'm after."

As Mr. English began his probability unit, his focus seemed to be on some format changes he was going to make for the unit. In the process of explaining those changes, he also gave the students somewhat of an overview of the probability unit, although it is not clear if that had been his intent.

> We're going to change the format of class just a little bit. One of the things we will do is that you will not be turning in anything that has to do with probability until we are all done. And, at the time you take the unit test, everything that you do in this unit will be stapled to that and turned in. A second thing that we'll be doing a little bit differently than you've been used to is at, at the end of each lesson, starting today, the last part of the class, you will have a summary sheet and basically what you are to do with that is to summarize what you think the lesson was all about. Include any vocabulary words that might have been talked about. If you do an experiment . . . in probability, we'll do several kinds of experiments . . . they're called simulations. We'll do them with dice, and coins, with cards, spinners, various things that we'll use and you will do an experiment and

> you'll study the probability of how it came out in the experiment, as compared to what we call the theoretical probability. And then we'll make some comparisons in writing. We'll turn these into percents, so that one thing shows such and such a percent and the other thing shows such and such a percent. Then, you can write a statement of comparison about it. Either they agree or they're pretty close or they didn't agree at all.

The probability unit did involve a number of experiments followed by theoretical analyses, but, perhaps because the use of the summary pages was discontinued after the first few days, the students did not do any "comparisons in writing" between the experimental and theoretical results. In the end, a major goal of the unit seemed to be the presentation of a variety of analysis tools such as charts, tree diagrams, and Pascal's triangle, but this emphasis was not mentioned in the overview Mr. English provided.

Mr. English faithfully presented an overview at the beginning of each day's lesson, which often involved more than one learning activity. This overview of the lesson generally included an outline of the tasks the students would be doing and/or an explanation of the assignments in their daily packets. For example, after taking some time to correct assignments at the end of the first week of the probability unit, Mr. English outlined the tasks for the remainder of the period.

> We're going to do three things today, in the remaining time or as long as we have time. First thing we're going to do is conduct a lottery and see whether anybody wins. Second thing we're going to do is analyze the two games that I gave you yesterday on the last worksheet in that packet. And I'm going to explain one of them and help you, just kind of talk you through it. And, then, you're going to talk through the other one in the groups that you're in and see if you can figure out how to analyze it and make it a fair game. And then, if there's time, I have a quiz that has 10 questions. I want to see whether you have learned the things that I have tried to teach you this week.

As in this example, the overview usually referred only to the tasks that would be done and did not explain what the students were expected to learn from doing the tasks. On one occasion, as Mr. English was explaining a series of worksheets the students would complete in analyzing dice sums, he reminded the students about the meaning of experimental and theoretical probability. Other than that, the daily overviews did not refer to probability content.

Mr. English also presented an occasional summary at the end of the probability lessons, although these did not occur as frequently as lesson overviews. For example, at the end of Day 2, Mr. English concluded, "What you should have learned a little bit about today is the concept of fairness and unfairness, how to make something fair, how to analyze dice activities when you're rolling two dice, and the concept of what do we mean by odds. Those things I think we talked about." Later in the unit, as the class completed the "Montana Red

Dog" game, Mr. English suggested, "All right, you've learned a little bit about odds. You've learned a little bit about maybe how you can use odds to help you win something." Although these statements did not indicate specifically what the students should have learned about fairness or odds, the students were at least reminded of the content covered in the lessons. At other times, Mr. English was somewhat more specific. For example, in making a transition from the analysis of the state's lottery to consideration of fair and unfair dice games, Mr. English provided the following summary:

> When you . . . whenever you analyze different situations, you have to set them up a little bit differently. We set up the lottery different than when we set up a dice game. This is how you set up an analysis of a dice game. You have to make a chart. If we're adding the dice, you make an addition table. Then you can look at the 36 probabilities, 36 is going to be your bottom number because there's 36 outcomes on that table. And then you can set up the probabilities for the sum of each one occurring.

Later in the unit, Mr. English summarized the analysis of a coin tossing situation by observing, "With coins, what you want to do is to make a little [tree] diagram, look at the outcomes, look at how they scored, see who wins, and figure out what the final score is going to be." And as the unit test approached, Mr. English reminded the students of what they should have been learning:

> We're getting down to the final . . . 2 days of this unit. We are getting real close to a unit test. Today's activity is a little bit different than some that we've been doing. You should have right now a pretty good understanding of how the dice probabilities go, the analyzing games with dice. You should have somewhat of an understanding about coins. Today's activity uses cards . . . poker cards again. We've done that a little bit. . . .

At none of these times of reflecting on learning did Mr. English turn things around and ask the students, "What have you learned about odds?" or "How would you analyze a game with dice?" This kind of personal reflection had been Mr. English's original intent with the summary pages he had included in the students' daily packets. However, the use of these summary pages had been discontinued because Mr. English did not feel there was time to complete them.

Painting the "big picture." In addition to possibly providing structure to the probability lessons, the classroom discourse potentially places the lessons being learned within the larger structure of the content of probability in particular and mathematics in general. In doing this, the discourse contributes to the "bigger picture" students come to understand by answering questions such as: How do the concepts fit together? How does one idea relate to the rest? What are the big overriding ideas that should be remembered?

A review of the classroom discourse in the four classrooms reveals very little discourse focused on answering these questions. As seen in the previous section, Mrs. Talent and Mr. English provided occasional overviews of what the students should be learning. In these overviews, Mrs. Talent and Mr. English listed the different types of problems the students should be able to solve. For example, Mrs. Talent suggested the students should be able to find probabilities in situations with marbles or spinners. Likewise, Mr. English reminded his students they should have a "pretty good understanding of how the dice probabilities go [and] . . . somewhat of an understanding about coins." Both teachers also reminded their students of two strategies they had used to find probabilities, namely, charts and tree diagrams. But rather than these being part of an overall approach to solving probability problems, the teachers portrayed them as specific strategies for specific problems. For example, Mr. English observed, "With coins, what you want to do is to make a little [tree] diagram, look at the outcomes, look at how they are scored, see who wins, and figure out what the final score is going to be."

One general approach to solving probability problems involves applying the definition of probability in cases of equally likely outcomes, where the probability of an event is defined as the ratio of the number of favorable outcomes to the number of possible outcomes. An important goal in this process is finding the sample space or the list of all possible outcomes. Several of the strategies introduced in the probability units, including making an organized list, constructing a chart, or drawing a tree diagram, are useful means of accomplishing this goal. Mrs. Talent introduced tree diagrams as "a way to organize how many different combinations are possible." At another point, Mrs. Talent made reference to a general approach, saying, "All we've worked with, so far, has been looking at different situations and then how do we write out all the different outcomes possible. Then, how do we write the probability . . . ." However, because she had not stated the definition of probability, the connection between writing out the outcomes and writing the probability may not have been recognized by the students. In general, the teachers failed to portray any overall approach for finding probability, focusing instead on applying specific strategies in specific situations.

Using the various strategies as part of a general approach to solving probability problems would be one way of the viewing the content of the probability units presented by the teachers. Experimental and theoretical probability also played prominent roles in the study of probability in the four classrooms observed. In general, experimentation was viewed as a way of getting the students actively involved in the lessons or experimentation was portrayed as a step on the way to the "actual probability." The teachers did not portray experimentation as a valid approach in its own right for estimating the likelihood of a particular event. Nor did the teachers point out that in some situations experimentation is the only approach available or accessible. With the exception of Mrs. Books, the teachers also failed to address important issues related to experimentation, such as sample size and potential sources of bias.

Although assigning some validity to each stage of the process, Mrs. Books portrayed the transition from subjective through experimental to theoretical as a process through which understanding of a problem evolves. After the students had made their subjective guesses for "Monty's Dilemma," Mrs. Books suggested, "We're going to collect some data to see if we're headed in that right direction." In reviewing the general steps of a previous activity, Mrs. Books stated, "Then the third part . . . was the theoretical piece . . . where you took your results and looked at it through arithmetic and decided what that was going to look like."

At one point, Mr. Trackman seemed to portray experimentation as one way of confirming theoretical results. In responding to a students' suggestion for making a dice game fair, Mr. Trackman explained,

> What if we times it by 6? That would be something we could investigate if we had more time. But we don't. And that . . . our little theory through this . . . that's part of the trial and error . . . that probability involves. Sometimes you just kinda, you're not sure about things, so you just kinda do it. You go with the theory and then you try it out. You see if it really works. . . . In this little theory, it looks like they should be even.

Although experimental results may provide evidence supporting a theoretical conclusion, such results do not prove the validity of the conclusion.

Mrs. Talent expressed disappointment that her students had not understood the value of doing an experiment. In reflecting on the unit, she commented, "They didn't get this like I wanted them to get it, but I wanted them to make a connection between the experimental and the theoretical. That if there's a situation, you can do it both ways." Perhaps one reason the students did not see the importance of the experimental approach was because many of Mrs. Talent's comments emphasized the unreliability of the experimental results the students had obtained in the activities the class had done. For example, after considering the class' experimental results for "Monty's Dilemma," Mrs. Talent observed, "Okay, now just to remind you again, this was an experiment so we don't know if, if we did it again, it might not come out exactly the same. It probably wouldn't come out exactly the same." At other times, she seemed to portray almost a mistrust in experimental results. In comparing the experimental and theoretical results for a dice game, she concluded, "So, mathematically, this is a fair game. And if you didn't get it to come out fair, that's part of probability. You never know. It might come out or might not." Such variability is a factor in doing experiments, but such comments outnumbered the times the students were asked to draw conclusions from their experimental data. And ultimately, experimental probability was "not the actual probability." In reflecting on the experimental results for one activity, Mrs. Talent summarized what the students had done,

> What you guys are doing on this page . . . is you conducted an experiment, and they asked you to take your experimental results, and write the probability. Not the actual probability. We're going to do that right now. And so, this is what they call experimental probability. Okay? Now, we could probably go around the room and everybody's answers here would be different, and that's because it's an experiment, and when you do an experiment, you don't know how it's going to turn out.

Therefore, because of the way experimental probability was portrayed, it is perhaps no surprise that the students did not recognize the importance of doing experiments.

The differences that occur between experimental results and the theoretical probability was an oft-repeated theme in what the teachers observed as part of the their conclusions for the activities. For example, in summarizing the results for "The Top and One Other," Mr. English observed, "Which a lot of times, your experiment will not always verify or be exactly the same thing as the theoretical probability. But we did the experiment and we looked at the experimental probability and it came real close to that." In reflecting on what the students had learned from the probability unit, Mr. English suggested, "They did learn that just because theoretically you can say it should happen like this, it doesn't mean that it will. In fact, most of the time, it probably doesn't come out exactly that way, but it does come close and the more you do it, the closer it does get to the actual probability." It is unclear whether or not the students had learned these ideas. Although Mr. English often noted the difference between experimental and theoretical probability, the students were never given an opportunity to make such an observation themselves. And none of the activities explored the impact of doing more trials.

The law of large numbers, to which Mr. English was referring, suggests that as the number of trials gets very large, the probability is high that the proportion of times an event occurs (the experimental probability) is close to the theoretical probability. When thinking about doing an experiment, the number of trials to be done is an important consideration. This important idea was not addressed by the teachers, with the exception of Mrs. Books. Mr. English had an opportunity to compare the results from few and many trials when his students suggested spinning the spinners 10 or 20 times and multiplying those results by 10 or 5 instead of spinning 100 times. However, he did not recognize the students' conceptual error and overlooked the opportunity. Mr. Trackman and Mrs. Talent frequently ignored the importance of doing many repetitions when they considered the experimental outcomes from three or fewer trials.

Implicitly recognizing the variability of experimental data, Mrs. Books probed her students to determine what they thought were reasonable experimental results. For example, after deciding the theoretical outcome for the Stick strategy should be 33 or 34 wins out of 100 trials, she asked her students if they thought the results were reasonable at 34, 33, 32, . . . 25. In the same context, she also explored the students' thinking about the impact of increasing the number of trials, asking them what would be reasonable if they considered their combined data from 2,100 trials. Finally, she asked, "Would we expect it to be closer to that 34% with 5,000 trials?" Mrs. Books, however, stopped short of explicitly stating any conclusion or having the students do so. Nevertheless, she at least had introduced the issue as an important idea to think about.

In the post-observation interview the teachers were asked, "What are the big ideas or the important ideas you hope the students will remember in the future about probability?" The teachers' responses reflected a wide variation in their thinking. Mr. Trackman focused on the notion of uncertainty, stating, "That it is only probable. . . . That's one thing I harp on. It just . . . means that this is probably going to happen. This is the likelihood of it happening, not necessarily to guarantee that if you roll the dice six times, you're going to roll a 1 one of those times."

In response to the interview question, Mrs. Books focused on her goals, including what the students had learned about simulations, bias, and measures of central tendency as well as the connections of probability with the area model and fractions. She also pointed out, "A big idea that I don't think is there but . . . I would like for them to continue to work on is . . . confidence. How confident are you . . . of your answer?"

Mrs. Talent outlined a number of important ideas in her response, stating

> I think, number 1, that it is everywhere and they're going to run into that. And the other important idea is when they see a situation that they feel involves chance, that maybe they'll stop and think, "Well, wait a minute; if I think about it logically, let me figure this out." That they'll know that they can do that. They might not remember how, but they'll know that they can do it. And just maybe apply some of the simple things that we learned how to do, like be able to say, "Well, your probability of doing that is 1 in whatever." When they see a probability, like on the back of a lottery ticket, they know what that means . . . because hopefully they'll get it again next year . . . .

The issue that immediately came to Mr. English's mind was gambling. He suggested, "I survey my kids to see how many of their parents participate in gambling. The situation is about 50% and my guess is that probably about that many of those kids are going to go and get into gambling situations. And I think, based on some of the Monte Carlo kinds of simulations we did with dice and cards, they are going to remember that and . . . at least they're going to maybe have a little better basis for doing that."

As these responses indicate, the teachers generally lacked knowledge of specifically what important ideas could or should be communicated as part of the study of probability. They similarly failed to see or communicate the connections between the concepts included in the probability units. In some cases, the possible connections between concepts were lost because the teachers' focus was on the activities themselves rather than on the concepts the activities explored. For example, as Mr. English was introducing the "Newspaper Offer" activity, he explained, "Today's activity is a little bit different than some we've been doing [because] today's activity uses cards." In focusing on the use of cards, he failed to see this activity as a second application of the idea of expected value which he had introduced the day before.

The teachers also may have failed to make the connections between concepts because they did not have a good understanding of those concepts. Mr. English's introduction of the expected value activity, for example, suggests he may not have had a clear understanding of expected value. After giving the directions for the activity, Mr. English observed,

> What we are trying to find is called expected value. Now there's not a way that I think you can set this up theoretically, by looking at the number of outcomes possible in relation to the total outcomes. I don't think you can do this. I think . . . all you can do is conduct an experiment and say, based on the experiment, this is what we would expect to be the number of boxes that parents would have to buy their kids to get all six brands of this prize.

Rather than an explanation of expected value, this appears to be a rationale for conducting a simulation. In the process, he did not make the connection between expected value as a long-term average and probability as a long-term frequency.

Mr. English had been the only teacher to make connections between probability and odds as different ways of expressing the likelihood of an event occurring. Mr. Trackman had suggested there were two different ways of expressing probability, but he incorrectly said they "were the same thing." Mrs. Talent had used an assignment with two-stage tree diagrams as the springboard for developing the multiplication property, but she had not really made many connections between concepts otherwise. Because Mrs. Books spent more time on fewer activities, more opportunities were available to develop the relationship between the involved concepts. This included the issues of bias and sample size that were addressed in the context of designing and evaluating the results of the simulations.

For the most part, the teachers made few connections between the concepts involved in their probability units, either because they did not recognize those connections or because they were focusing more on doing activities than on learning concepts. Similarly, the discourse during the probability instruction did little to portray any overall picture of probability.

Addressing Students' Conceptions and Misconceptions About Probability

According to the view of the *Teaching Standards* (NCTM, 1991), learning involves building upon students' prior knowledge and restructuring that prior knowledge to assimilate the new experiences and new ideas encountered. Because of the importance of connecting instruction to the knowledge students already possess, this section will consider the teachers' knowledge of the possible conceptions and misconceptions middle school students may have about probability. The section will first explore the teachers' perceptions of middle school students' knowledge of probability. Second, the teachers' efforts to discover and connect instruction to the students' initial conceptions about probability will be described. Third, this section will investigate the teachers' efforts to address the errors and misconceptions encountered during probability instruction.

Teachers' Perceptions of Students' Knowledge of Probability

The teachers reported that most of their students had received no prior instruction in probability. For example, when Mr. Trackman was asked if students previously had received any instruction, he responded, "Not that I know of." Although Mrs. Books believed the students should have received some instruction, she felt that only "about half of them did . . . based on what [the students] were talking about." In checking with the sixth-grade teachers, Mrs. Talent learned that only one of them had taught probability in the previous year. Therefore, Mrs. Talent concluded that "for most of [the students] it was their first major exposure to probability." Based on his knowledge of what other teachers were doing and on the results of the pretest he had given the students, Mr. English also concluded the students "have had no instruction."

Although they believed the students had not had any prior instruction in probability, each of the teachers believed the students had some intuitive understanding of simple probability situations. For example, Mrs. Talent reported, "I think most kids, by the time I get them, intuitively, if you say, 'What's your chance when you spin this spinner that it's going to come out red?' they can tell you 1 out of 4." Mrs. Books similarly felt the students "could work with . . . a spinner." In addition, Mr. Trackman suggested, "Flipping coins . . . they realize that that's 1 out of 2." Mr. English provided a more extensive summary of what he thought students knew prior to instruction.

> They have, as far as a notion of chance, they have a notion of flipping a coin, what your chances are there. They do . . . with two coins, they get it wrong, even the advanced kids do. They can take a simple spinner . . . because they've studied percents, they can take a spinner and talk about your chances there. We have . . . poker cards . . . they could probably tell you that your chance of getting an ace . . . . They know there's 52 cards in a deck and they know there's four aces, so they can probably tell you what the chances are there. Although most of them would not . . . They'd more likely say 1 out of 52 than they would 4 out of 52, for whatever reason there is. . . . But the average kids . . . I really think they really don't have a notion at all other than those very very basic notions.

In addition to these simple probability situations, the teachers felt the students had some awareness of the language of probability. Mrs. Talent suggested, "If I say, 'a 1 out of 3 chance,' they kind of know what that means." Mrs. Books also felt the students were familiar with chance expressed as 50-50 or 1/2, although she did not clarify if the students knew the difference between expressions of odds (50-50) and expressions of probability (1/2). However, Mr. English did not think the students understood the meaning of probability itself, suggesting, "They don't know how to define it. They don't know that it means chance."

Further, Mrs. Talent indicated, "Most students have an awareness of the games because they've played games." However, Mr. Trackman suggested, "Their idea of fairness is, if they lose, it's not fair." He added, "They're still pretty . . . ethnocentric [*meaning egocentric*]. . . . They all assume it's going to happen to them. They haven't developed a defeatist attitude yet, or they haven't developed an attitude of . . . realizing that it's not their fault, that it just happens."

Finally, Mrs. Books pointed out that much of the students' understanding of chance is based on intuitive or subjective notions. However, these intuitive or subjective notions often present difficulties, according to Mrs. Books. In particular, she explained, "There's so much of that subjective that continues to sway our decisions even though that theoretical is lying right there."

In an attempt to assess the teachers' knowledge of what the possible misconceptions of the students might be, the teachers were asked how they thought students might respond to some of the same misconception items asked of the teachers in the post-observation interview. The chosen items were those misconception questions that dealt with the representativeness heuristic (where the likelihood of an event is determined by how representative the event is of the expected distribution or random generating process) and two of the misconceptions related to representativeness, namely, the gambler's fallacy and the neglect of sample size. The answers the teachers thought students would give are compared with responses students have given (as reported in the research literature). The correct responses to these items were discussed earlier, during the analysis of the teachers' knowledge of probability content.

The Random Digits problem (misconception question #1) presented the following scenario (Green, 1983a):

> A teacher asked Clare and Susan each to toss a coin a large number of times and to record every time whether the coin landed Heads or Tails. For each 'Heads' a 1 is recorded and for each 'Tails' an 0 is recorded. Here are the two sets of results:
>
> Clare:
>
> 01011001100101011011010001110001101101010110010001
> 01010011100110101100101100101100100101110110011011
> 01010010110010101100010011010110011101110101100011
>
> Susan:
>
> 10011101111010011100100111001000111011111101010101
> 1110000000100010100100000100011000101000000000011001
> 00000001111100001101010010010011111101001100011000
>
> Now one girl did it properly, by tossing the coin. The other girl cheated and just made it up.
> (a) Which girl cheated?
> (b) How can you tell?

Reasoning according to the representativeness heuristic, one might think Clare's results are more likely to have actually occurred because they remain close to the 50-50 distribution expected of coin tosses. On the other hand, Susan's results seem to deviate from the expected 50-50 distribution because of the long strings of 0s and 1s contained in her sequence. Mr. Trackman, Mrs. Books, and Mrs. Talent each thought the students would think Susan was the one who cheated, specifically referring to the long strings of 0s and 1s. Mr. English responded, "I don't think they would know. I think they would make a guess." In this case, Mrs. Talent and Mr. English had themselves chosen Susan as the cheater, also citing reasons based on representativeness.

When Green (1983a) had asked the Random Digits item of students aged 11 to 16, 53% of the students concluded Susan had cheated because her results varied too far from the 50-50 proportion and because her sequence contained runs that were too long. The teachers thus recognized the students might expect a fairly even distribution of heads and tails within the sequences, consistent with Green's results.

The Birth Sequence items (misconception questions #3 and #4) also dealt with the representativeness heuristic (Shaughnessy, 1977):

> R1: The probability of having a baby boy is about 1/2. Which of the following sequences is more likely to occur for having six children?
>
> (a) B G G B G B (b) B B B B G B
> (c) about the same chance for each
>
> R2: (same assumptions as R1) Which sequence is more likely to occur for having six children?
>
> (a) B G G B G B (b) B B B G G G
> (c) about the same chance for each

With the exception of Mr. Trackman, the teachers themselves had correctly recognized the sequences would be equally likely to occur. When asked to predict students' responses, Mr. Trackman, Mrs. Books, and Mrs. Talent thought the students would say BGGBGB was more likely than BBBBGB because "they would tune in on the 50:50." Mrs. Books and Mr. English suggested the students similarly would choose BGGBGB over BBBGGG because "they're going to expect more randomness in that pattern."

The Birth Sequence items had originally been asked of college students, where responses indicated a reliance upon the representativeness heuristic. In particular, Shaughnessy (1977) reported 50 out of 80 subjects chose BGGBGB as more likely than BBBBGB because the first sequence fit more closely with the 50:50 expected ratio of boys to girls. Similarly, 28 out of 80 selected BGGBGB as more likely than BBBGGG because the first sequence appeared more random. An additional 23 subjects had indicated the two sequences had "about the same chance," but gave incorrect reasons based on representativeness. Consistent with these findings, the teachers believed the thinking of their middle school students would be influenced by the representativeness heuristic.

The Coin Toss item (misconception question #2) addressed the gambler's fallacy, a misconception associated with the representativeness heuristic (Brown & Silver, 1989):

> If a fair coin is tossed, the probability it will land tails up is 1/2. In four successive tosses the coin lands tails up each time. What happens when it is tossed a fifth time?
>
> It will most likely land heads up.
> It is more likely to land heads up than tails up.
> It is more likely to land tails up than heads up.
> It is equally likely to land tails up or heads up.

When asked how they expected students might respond, the teachers provided two responses. First, each of the four teachers thought the students would expect it to land heads up. As Mrs. Books explained, "I think developmentally that's where they're at." However, with the exception of Mrs. Books, the teachers also suggested some students might think it was going to land tails up, believing the game was rigged or the coin was weighted. None of the teachers suggested the students would recognize the coin tosses as independent events, meaning heads up or tails up were equally likely on the next toss, although each of the teachers had correctly given that as their response.

The Coin Toss item had been among the NAEP results reported by Brown and Silver (1989). Approximately half of the 7th graders and 11th graders had correctly responded that the coin was equally likely to land tails up or heads up, even after four consecutive tails. However, 38% of the 7th graders and 33% of the 11th graders believed a heads was "most likely" or "more likely" on the next toss. The teachers therefore thought at least some of their students would be among this group whose responses reflected thinking in accordance with the gambler's fallacy.

The Hospital problem (misconception question #5) addressed the issue of sample size with the following question (Schrage, 1983):

Which of the following results is more likely:

(i) getting 7 or more boys out of the first 10 babies born in a new hospital?
(ii) getting 70 or more boys out of the first 100 babies born in a new hospital?

(A) They are equally likely.
(B) Seven or more out of 10 is more likely.
(C) Seventy or more out of 100 is more likely.
(D) No one can say.

In response to this item, Mr. Trackman and Mrs. Books suggested the "students would probably say that they're equally likely because they are the same percentage." Mrs. Talent and Mr. English were uncertain how the students might respond. Each of the teachers had themselves correctly understood smaller samples would have more variability.

The Hospital item had been given to two different groups of subjects. Of the students aged 11 to 16 in Green's (1983a) study, 25% reported the two events were equally likely while 61% thought no one could say. When Schrage (1983) asked the same question of 153 of his education students at the University of Dortmund, Germany, except with the "no one can say" answer omitted, 60% believed the two events were equally likely to occur. As in the other cases, the teachers' responses were consistent with these findings.

Thus, the teachers were quite accurate in their perceptions of how students might respond to the items involving potential misconceptions. Of the items the teachers answered, they provided responses consistent with the research findings. However, the teachers had not given responses to all of the items. Although Mrs. Books had responded to all five items, Mr. Trackman had responded to only four, Mrs. Talent to three, and Mr. English to two. When no response was chosen, the teachers usually suggested the students would not know how to respond or the teacher did not know how the students would answer. Mr. English, in particular, seemed to believe the students frequently would not know how to respond, apparently equating no prior instruction with no knowledge or intuitive beliefs in such situations.

Nevertheless, although the teachers in many cases could identify how students might respond, knowledge of the common misconceptions was not explicitly part of the teachers' knowledge base. In particular, the examples and explanations of the misconceptions provided in the post-observation interviews presented new information to the teachers.

Students' Initial Conceptions About Probability

As the teachers began their probability units, the attempts they made to discover what background knowledge of probability their students possessed varied from teacher to teacher. Similarly, the teachers' efforts to connect their probability instruction with students' initial conceptions about probability also differed.

Mr. English had given his students a pretest before he began probability instruction. This pretest focused on what knowledge the students had of the situations that would be included in the probability unit. For example, the students were asked about the probability of guessing correctly on a true-false question, the number of outcomes possible when two coins are flipped, or the probability of rolling a sum of 11 on two dice. When 79% of the students scored below 70% on the pretest, Mr. English concluded they had no background knowledge of probability. As Mr. English pointed out, "Judging from the pretest . . . a lot of them didn't know even what probability was."

Based on his conclusion that the students had no background knowledge of probability, Mr. English planned to provide a foundation for the students on Day 1 by introducing the basic vocabulary and presenting some simple examples. However, seemingly ignoring the students' supposed lack of background knowledge, the historical example presented before any of the introductory material involved several more advanced ideas, including complement, odds, and the multiplication property. In this example, Mr. English told the story of Antoine deMere, a mathematician in the 1600s, who analyzed a game people were playing in a gambling den. If the player could roll a die four times without once getting a 6, the player would win his bet and receive a payoff. In analyzing the game, Mr. English explained that the probability of rolling a 6 on one roll of a die was 1/6 and the probability of not rolling a 6 would be 5/6, making the odds in favor of the player 5 to 1 on each toss. He went on to explain when you roll the die four times, "you take the 5 out of 6 chance and you multiply it four times . . . and what you end up getting is 5 to the 4th power over 6 to the 4th power." Thus, the analysis of the game in this example involved probabilistic thinking that likely went beyond the understanding of the students, particularly if they had no background knowledge of probability, as Mr. English assumed.

During his probability unit, Mr. English related instruction to the students' background knowledge in one situation involving spinners. In this case, he called upon the students' knowledge of fractions and percents to identify the theoretical likelihood of the spinner landing on the different sections of the spinner. In other situations, Mr. English also relied on the students' previous experience with fractions, decimals, and percents. However, no specific attempt was made with any of the activities to relate instruction to the students' prior conceptions about probability, particularly to what their intuitive notions might be.

Whether it was their intent or not, Mr. Trackman and Mrs. Talent began their probability units in ways that could have been used to discover students' basic understanding of probability. For example, Mr. Trackman began his probability unit with the question, "Does anybody know where probability is used in our lives . . . and where it happens?" The students may not have taken the question seriously when they initially responded, "math class" or "jobs." But when the students suggested such things as accounting or sports cards, it was not clear that they understood what probability is. Mr. Trackman, however, did not take advantage of the opportunity to follow through and clarify the meaning of probability, choosing instead only to provide further examples.

Similarly, Mrs. Talent began her probability unit with a question about what probability is and how it applies to everyday life. After writing the word *probability* on the overhead, Mrs. Talent asked,

T: So . . . what is this? Can anybody give me a good definition or what you remember about it? What you do with it? Anything? Give me some background.
S1: Uh, something that you have a chance on that you're not sure about.
T: Okay, he said, "Something you have a chance on that you're not sure about." What else?
S2: Like, what will happen if you do something.
T: Okay, what will happen. Anything else you can think of when you think of this word? . . . Nothing? That was quick. Okay.
S3: Have we ever done this?

In this class, at least some of the students knew probability had something to do with chance occurrences. At the same time, probability seemed to be a new area, at least to the third student. Mrs. Talent then read a story to her students that included a number of situations involving chance or probability. The students were able to identify such applications as taking a chance on something, making a weather forecast, buying a lottery ticket, putting prizes in cereal boxes, flipping a coin to make a decision, setting insurance rates, making sports predictions, and conducting a taste test. Thus, even though the students had some difficulty defining probability, they were able to recognize its role in events involving uncertainty.

Although many of the students apparently had not had previous probability instruction and had only a limited understanding of what probability actually is, evidence in these two classrooms supported the teachers' assumption that students had some understanding of simple probability situations, either intuitively or from previous instruction. For example, without any introduction to what probability is or how it is expressed, Mr. Trackman's students responded, "One out of 6," when asked what the probability of getting a 4 on one die was. Similarly, without any introduction of terminology, the students in Mrs. Talent's class were using 50-50 or 33% to express the chances of various simple outcomes. However, Mrs. Talent may have assumed more than she should have from the students' use of the terminology. In particular, she apparently assumed the students knew probability could be expressed as the ratio of the number of favorable outcomes to the number of possible outcomes. As a result, she never presented that definition of probability, a fact that may have contributed to the difficulties some students encountered later in the probability unit. Other than their introductory questions, neither Mr. Trackman nor Mrs. Talent made any apparent effort to connect instruction to students' initial conceptions or background knowledge of probability.

The initial probability activities in Mrs. Books' class had been done earlier in the year and, as a result, had not been observed. Therefore, it is not known how Mrs. Books brought out the students' prior conceptions of probability initially. However, based on the later probability activities, Mrs. Books appeared to be the only teacher who considered students' conceptions of probability in designing instruction. In particular, she recognized the difficulties often created by the intuitive or subjective notions. She pointed out, "The students really struggle with what they believe in their mind versus what evidence, when there is experimental and theoretical shown. They know that this is theory, this is what should happen, yet it's that gut feeling." To address these subjective notions, Mrs. Books began each activity by having the students state their subjective estimate or prediction and give their supporting rationale. In this way, Mrs. Books structured the activity so that the students could then compare their subjective predictions with the experimental and/or theoretical results as the simulation was conducted and the problem was analyzed (when this was possible). In addition, Mrs. Books used what she heard students discussing in their groups as a starting point for the class discussion. As a result, she was bringing the students' thinking before the classroom community for consideration and evaluation.

Nevertheless, even these efforts by Mrs. Books were not always successful in overcoming the students' strong intuitive beliefs. The letters written by the students at the end of "Monty's Dilemma" provide evidence of just how powerful and persistent some of the students' subjective notions were. For example, one student reported the following experimental results: 47 wins and 53 losses with Stick, 54 wins and 46 losses with Flip, and 74 wins and 26 losses with Switch. Then she concluded, "That's why I decided to use the Switch strategy. Since I came out winning so much with [the] experiment, then maybe I would switch my prediction to switch. But then again, all of that was based on experimental data, while my first prediction [to stick] was based on instincts, something I have had good luck with. And anyway, I usually have [better] luck with my first choices than my second ones." In this case, the student seems to return to her original instinct to stick, despite the strong experimental evidence favoring the Switch strategy, which the student acknowledged. A second student first stated her theoretical observations regarding the three strategies, "For the 'Flip' . . . I found that the chances of winning are 50-50 because you have an equal chance of switching or staying. For the 'Switch' . . . there is a two-thirds chance of winning this way because there are two gag prizes. When you pull out a gag, you win because you switch to the prize. The 'Stick' strategy is the least likely to win because you pick one of the three doors and stay, so there is a one-third chance of winning." Then she stated her final conclusion about which strategy she would use, "I, in the end, stayed with 'Stick' because backing out of my original whim seems uncomfortable, wrong to me." Thus, even a correct theoretical understanding of the situation did not convince this student to give up her original subjective conviction.

Students' Errors and Potential Misconceptions About Probability

The students' understanding of probability concepts was not formally assessed as part of this study, either before or after instruction. Nevertheless, the comments made and questions asked by the students during the probability lessons revealed some of their conceptions about probability, including some errors and potential misconceptions. However, it is not known how many of these conceptions were due to intuition and how many were due to prior or current instruction. This section will consider what students' comments and questions during the probability lessons revealed concerning their conceptions about chance and fairness and their errors in analysis. Examples of the common misconceptions (as described in chapter II of this research study) that occurred during probability instruction will also be included. As part of the analysis, this section will explore the teachers' responses to these conceptions and misconceptions.

Conceptions about chance and fairness. A number of the students' comments and questions revealed some lack of understanding or some unconventional conceptions about chance and fairness, two fundamental ideas in the study of probability. For example, after Mr. English had summarized the results for "Is This Game Fair?" on the second day of the unit, one student asked a question:

T: They're going to win 1 out of 6 times; they're going to lose 5 out of 6 times. So if you pay them the $5 when they win, they're going . . . it's going to come out to be fair. Does that make sense?
S: . . . What if they roll more times?
T: Remember . . . remember, we're talking theoretically. That doesn't mean when we conduct the experiment that that's going to happen that way. It's just in theory . . . that's the way it would work.

This particular student did not seem to understand the proportional nature of probability. Mr. English, however, did not recognize the missing piece in the student's thinking and, therefore, did not address the student's lack of understanding in his response.

One or more students in Mr. English's class seemed to think luck could play a role in the outcome of an uncertain event. For example, as Mr. English and his class were analyzing the results for "Which Do You Think Will Be Larger?" one student suggested, "You could get lucky," as if luck could override the theoretical conclusions. At a different time, after determining one's chances of winning a game based on dice sums was only 20%, Mr. English asked the following question:

T: Why is it then a good idea not to play that game?
S: You're going to lose your money.
T: Because it's an unfair game.
S: What if you're lucky?
T: Okay. Okay. And some people are lucky. That happens. But some people are also addicted to gambling and they're not so lucky.

Again, the student seemed to think that a person who was lucky could overcome the unfavorable odds. Or perhaps the student was thinking if one is lucky and wins, then the game may be fair. In this case, Mr. English did not clarify that a person might be considered "lucky" if they were part of the 20% who would be expected to win, but that being a winner would not make the game fair.

The students also displayed some questionable ideas about fairness. For example, after playing "Is This Game Fair?" as an introductory activity, Mr. English suggested they were going to see how the scores came out and then "we'll analyze this and see if it's a fair game." At this point, one student asked, "What do you mean by fair? Fair to both players?

Fair to one player?" Mr. English responded by explaining that "fairness means there is an equal chance that either person who's playing the game will win the game." Thus, rather than using this as an opportunity to explore what this and other students understood about fairness, Mr. English provided a definition and moved on with the activity.

Mrs. Talent and Mr. Trackman also had fairness as one of the themes in their probability units. Neither teacher, however, took time to explore what the students believed about fairness. In Mrs. Talent's case, she did not consider what the students believed about fairness or even what fairness meant until after the students had already completed a couple activities dealing with fairness. Then, realizing she had not defined the term, Mrs. Talent explained what she meant, "What I'm talking about is if you and I sit down to play this game, we both have an equal chance of winning. Okay? Now if I always win . . . you know how little kids are, they always have to win. If I always have to win or I'm not happy, to me that's fair . . . . But, really, it's not fair. Fair means you've got either person that could win." Although Mr. Trackman similarly believed some students equated fairness with their winning the game, he did not address this misconception during his probability lessons. Instead, without explaining what fairness meant, he directed the students to consider how they could even out the scores of the game players.

On two different occasions, after experimental data were gathered for two other games, Mr. English asked students whether the games were fair or not. On both occasions, a student (perhaps the same in both cases) responded, "It depends on what you roll." This student or these students seemed to recognize the short-term unpredictability of chance occurrences such as a dice roll, but did not realize that there is a long-term regularity which can be quantified and analyzed theoretically. In response to the same question about the fairness of one of the games, another student suggested the game "still might be fair" because he had won. Therefore, although the experimental data revealed that player B had won many more times than player A, this student believed the game was fair because he, as player A, had won. In other words, because either player A or player B *could* win, the game was judged to be fair in the student's mind. Rather than responding to either of these students directly, Mr. English rephrased the question and asked, "If we played another 30 rounds, how many would predict that B would win?" In this case, however, there was no follow-up to verify the students' predictions.

<u>Errors in analysis.</u> The students' comments and questions during the probability units also revealed a number of other errors or misconceptions in their thinking. Among these errors were neglecting the context of the problem, ignoring the importance of order, and overlooking the need for replacement in certain settings.

Mr. Trackman did not appear to have given thought to what might be easy or hard for the students or what errors they might make. For example, in the Dice Sums game, one student wondered why they did not have 1 on the list of possible sums. And other students suggested such pairs as 7 and 1 in listing possible outcomes. In similar settings, Fischbein, Nello, and Marino (1991) had found that students sometimes forget the limitations of the situation when focusing on the mathematics of the problem. However, rather than treating these as possible errors to be expected, Mr. Trackman and even the students were critical of the students who made the errors.

One student in Mr. English's class revealed another common error in analysis. As Mr. English was introducing tree diagrams as part of the vocabulary on Day 1, he drew the tree diagram representing two coins being tossed as an example. As he was listing the four possible outcomes (e.g., HH, HT, TH, and TT), one student commented that HT and TH represented the same thing. In response, Mr. English explained, "It's the same except when you're talking about the order that it occurs, it's not the same. Okay. And that's another thing that's a little confusing. If I said, 'Match or No Match,' that'd be simple. Match would be heads-heads or tails-tails; no match would be if it wasn't that." Later, on Day 12, as the teacher was drawing a tree diagram to represent tossing three coins in a homework item, the same student said, "No, you only have to do one of those [referring to HHT and HTH] 'cause it doesn't matter what order you get it in." This time the comment was not heard by the teacher, so there was no response. However, it appears the explanation given on the first day was not effective in changing the students' misconception about order, nor did any of the activities in the 3 weeks observed prove effective in dispelling this misconception.

Students in Mrs. Talent's classroom also wrestled with the issue of order. During a lesson on drawing two-stage tree diagrams, the students were asked to draw a tree diagram representing the toss of a coin and the spin of a spinner. Then they were to find the probability of getting a heads on the coin and a white on the spinner. One student who was called upon to show his tree diagram had drawn the spinner first and then the coin toss. Mrs. Talent pointed out it was impossible to get heads and white on this student's tree diagram "because it's never gonna come up heads first." Some of the students, however, argued that "white and heads is the same as heads and white." Mrs. Talent concluded by saying, "If you get picky and you want it in that order [heads and white], you're never going to get it that way. But if you don't care about order, then you're right. But I care about the order." Thus, although the issue was raised, the importance of order was not fully addressed.

Another potential misconception was addressed in Mrs. Books' class as the students were doing their simulations of the "Cereal Boxes" activity. As Mrs. Books circulated among the tables, she observed one group had drawn a bead from their sack and set the bead aside.

Mrs. Books then raised the issue of replacement by asking "How come this bead's laying here?" As the group considered the issue, one student suggested you would not return the bead because once a box had been opened, it was not returned. But another student believed the beads should be replaced because the shelves were being restocked. Another student pointed out you needed the same odds every time. In the end, these arguments seemed to convince the group members that they needed to replace the bead. Mrs. Books brought up the issue of replacement as class began the following day. She first asked each student to think about the question. Then by a show of hands, she discovered who would replace the bead and who would not. Beginning with a student who felt the bead should be replaced, Mrs. Books asked for an explanation of the student's thinking. After some further discussion, she called on one student who would not have replaced the bead. This student reported that she had changed her mind after hearing the other person's explanation. In this way, Mrs. Books had guided the class to consider the need to replace the bead in a logical and convincing manner by having the class community discuss their thinking.

Common misconceptions. Of the potential misconceptions discussed in chapter II of this research study, the only one observed in these teachers' classrooms was a neglect of sample size. This potential misconception showed up in various forms in the different classrooms. On the "Spinners" activity, Mr. English and the class first determined the theoretical probability for each of the regions on the three spinners. Suggesting the students probably did not want to spin each spinner 100 times, as the instructions asked them to do, Mr. English asked, "Can anybody suggest a way we might get a hundred things but not have to do it?" The students responded with suggestions to "do it 10 . . . and multiply what happens by 10" or do it "20 and then multiply by 5." Whether they realized it or not, the students were assuming that a sample of 10 or 20 would give results in the same proportion as a sample of 100. Mr. English seemingly did not recognize this as a misconception. He went on to explain what he had in mind, namely that each student was to spin each spinner 25 times and combine those results with the results of three other students for a sample of 100 spins. Although Mr. English did not realize it, how he structured the activity would have provided an opportunity to explore the misconception by comparing the theoretical results the class had predicted with the results from 25 spins, 100 spins, and the class total. In the end, Mr. English had the students make a chart for each spinner, recording their theoretical predictions and their experimental results. However, he did not ask them to state any conclusions based on the results recorded in the charts.

One pair of students in Mrs. Books' class decided to use a similar shortcut in gathering data for their simulation of "Monty's Dilemma." Soon after the groups had begun conducting their simulations, one pair exclaimed that they were finished as Mrs. Books walked by them. Mrs. Books then discussed with them what they had done.

T: So, how did you do this without the tallies?
S: All you have to do is do it five times, and then . . . multiply it.
S2: Wouldn't that be much easier, and then you have 100 times?
T: Why do we do things more than five times and multiply it?
S: 'Cuz you get more randomness?
T: Aah. If we had stopped after five simulations on Cereal Box, would we have had good data?
S: No.
T: Why not?
S: Next time you could have gotten a 28 or something.
T: Hmm. So, does doing it five times and multiplying it, how does that impact your data?
S: I think we should just do it 20 times, then multiply it.
T: Is that going to be random?
S: Yeh.
T: When you report to your audience to tell them which strategy is best, . . . are you able to tell them what 20 trials is like or what 100 trials is like?
S: Twenty.
T: 'Cuz does multiplying it by 5, does that really give you what 100 trials would be like?
S: I think so.
T: You think so?
S2: Let's keep doing it.
T: That would be interesting.

In this case, by asking questions, Mrs. Books helped the students realize the importance of doing the 100 trials. During the discussion about bias on the following day, these students reported their error as one example of possible bias.

The next day, as the class was considering their experimental results for "Monty's Dilemma," Mrs. Books asked some general questions related to sample size. In particular, the class had determined that theoretically a person using the Stick strategy should win about one third of the time or 33 times out of 100 trials. To see how much variation the students might expect from that amount, Mrs. Books asked them if they thought the results were reasonable at 34, 33, 32, . . . 25. By the time she reached 25, only one student still had his hand raised. She repeated the question, asking which results would be reasonable if the results for the class were combined for 2,100 trials. Finally, Mrs. Books asked, "Would you expect it to be closer to that 34% with 5,000 trials?" Although she was trying to emphasize that results are generally closer to the theoretical value when sample size is larger, the response from the students was unclear. At least some students seemed to get the point, emphasizing that it would get closer to 33 1/3% to be exact. However, without any specific conclusion being stated, the class moved on to consider the results for the Flip strategy.

The issue of sample size was not explicitly addressed by Mrs. Talent or Mr. Trackman during their probability units. However, by their actions, both teachers implicitly demonstrated a lack of concern for sample size. Because time was limited, Mrs. Talent had her students conduct only a small number of trials for the simulation activities they did. For example, the students conducted three trials for the first version of the "Cereal Boxes" simulation (with three prizes available) and only two trials of the second version (with six prizes). Similarly, the students did 10 trials for each strategy of "Monty's Dilemma," compared to the 100 trials Mrs. Books' students had done. At no point did Mrs. Talent discuss with the class whether the small number of trials was sufficient evidence upon which to base their conclusions. Nor was there any discussion of whether more trials would be more accurate or convincing.

In a similar fashion, Mr. Trackman put considerable emphasis on individual experimental results. For example, after the groups had played "Paper, Scissors, or Rocks" for 25 rounds each, Mr. Trackman asked the students how they would rescore the game to make it fair. In this case, each of the groups had results that were quite different, leading to a wide variety of rescoring schemes. However, there was no discussion about these differences between one group's results and another's or about the importance of gathering sufficient data to see a consistent pattern before drawing one's conclusions.

Of the common misconceptions described in chapter II of this research study, the neglect of sample size was the only one observed in these classrooms. This may have been because of the lack of emphasis placed on revealing students' thinking during the probability units. It may also reflect the fact that the activities included as part of instruction were not of the sort to bring out any other misconceptions. Nonetheless, although the specific activities may not have addressed potential misconceptions directly, the general settings of the activities provided opportunities to explore students' thinking related to the misconceptions. For example, the general representativeness heuristic or the gambler's fallacy could have been brought up in the context of the coin-tossing and dice-throwing activities included in many of the probability units. For example, a string of heads or tails would have provided an opportunity to discover what the students might expect to happen next. The teachers, however, did not take advantage of such opportunities.

When asked about addressing the common misconceptions at the middle school level, Mr. English responded, "Well, yeh, if you had . . . a clear focus, with that being your focus . . . you could do situations like that . . . just look at them and talk about the . . . I think you'd have to talk about them." In his response, Mr. English appeared to be thinking in terms of teaching the students how to respond to specific questions such as the "boy/girl in the family situation" rather than addressing the misconceptions in more general terms.