

AN ABSTRACT OF THE THESIS OF

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Title Acceleration Waves in Micropolar Elastic Media  
and Formation of Shear bands

Abstract Approved

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The formation of shear bands in isotropic micropolar elastic materials is considered in this thesis. The investigation concerns localized deformation fields due to jumps of second-order gradients of displacement across a standing singular surface. Such a standing surface gives rise to a shear band. The condition for the existence of a shear band is obtained in terms of an appropriate acoustic tensor for the micropolar continuum. The behavior of the inclination angle of the shear band is examined under varied loading conditions. Numerical calculations are presented for a micropolar elastic solid in the two cases, of uniaxial tension, and of tension of a thin plate.

Acceleration Waves in Micropolar Elastic Media  
and Formation of Shear Bands

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# ACCELERATION WAVES IN MICROPOLAR ELASTIC MEDIA AND FORMATION OF SHEAR BANDS

## INTRODUCTION

The mathematical premises of classical continuum mechanics is based on the assumption that the mass density  $\rho$ , defined by

$$\rho(x,t) \equiv \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V}$$

where  $\Delta m$  is the total mass contained in a volume element  $\Delta V$ , is continuous throughout a body and depends only on the position  $x$  of a point in the medium and time  $t$ . However, when the volume  $\Delta V$  is less than a certain critical volume  $\Delta V^*$ , it is known that the ratio  $\Delta m/\Delta V$  begins to show some dependence on  $\Delta V$  and this dependence becomes even greater as  $\Delta V$  approaches zero. Therefore, the ratio  $\Delta m/\Delta V$  does not have a unique limit in reality. Contrary to this fact, the laws of motion and the axioms of constitution are assumed to be valid for every part of the body regardless of its size. Thus, the classical continuum mechanics may be inadequate in the treatment of deformations and motions in the range of  $\Delta V < \Delta V^*$ .

It is also important to note that classical continuum

mechanics loses its accuracy when the length scale of the external physical effect is comparable to the average grain or molecule size contained in the body. In such cases, granular and molecular nature of materials come into play and therefore the intrinsic motions of the constituents must be considered. The wave propagation problem with short wavelength makes this point very clear. For example, in a plate consisting of dumbbell molecules, the rotational motions of these elements give rise to new types of wave not encountered in the classical theory.

The failure of classical continuum mechanics to describe motions of granular media and composite materials have led to the theory of microcontinua. In 1909, E. and F. Cosserat gave a unified theory for bars, surfaces and bodies, in which they introduced a new type of continuum, each point of which is associated with a triad of vectors called directors. This important monograph, however, was buried in the literature until it was revived in various works by H. Grad (1952), W. Gunther (1958), Grioli (1960), Truesdell and Toupin (1960), Aero and Kuvshinskii (1960), Schaefer (1962), Mindlin and Tiersten (1962), Toupin (1962), and Eringen (1962). Later, Eringen and Suhubi (1964) introduced a general nonlinear theory appropriate to microelastic continua. Independently, a microstructure theory of elasticity was published by Mindlin (1964) and a multipolar continuum theory by Green

and Rivlin (1964). The microcontinuum mechanics gives rise to the concept of couple stress, inertial spin and other types of effects which have no counterpart in the classical theory. Among the above theories, the one initiated by Eringen and Suhubi (1964) seems to be a viable theory because of its mathematical elegance and its capability of predicting explicit solutions for quantities such as spin inertia, couple stress and other fields in physical problems. In this theory, the balance laws of classical continuum mechanics are supplemented with additional ones, and the intrinsic motions of microelements contained in a macrovolume  $\Delta V$  are taken into account. Eringen later initiated the micropolar theory, which is a special case of the general microcontinuum mechanics. In this theory, each material point of the body is endowed with an additional degree of freedom, namely an independent rotation, besides translation, and, as a consequence, it is capable of describing a wide variety of motions and deformations of materials including granular media such as rocks and ceramics, fibrous media such as wood and wood composites, oriented media such as liquid crystals, and suspensions such as animal blood, etc. Thus, it should be reasonable to employ this theory for the investigation of shear bands in materials.

The study of shear bands was initiated recently. Thomas (1961) and Hill (1961) considered the case of an

acceleration wave, and in particular, confined their attention to a stationary wave. The latter is a standing singular surface in a material, across which the second-order gradient of displacement suffers a jump while the displacement, the first-order gradient of displacement and the velocity remain continuous, thus leading to the formation of a shear band. The formation of shear bands was further investigated by other workers using different constitutive equations for the materials. Hill and Hutchinson (1975), Tvergaard, Needleman and Lo (1981), Coleman and Hodgdon (1985), and Tokuoka (1986) are among those whose works should be noted.

The purpose of this thesis is to study the formation of a shear band and its properties in a micropolar elastic solid, which has never been investigated before.

# CHAPTER 1

## Singular Surfaces in Classical Continuum Mechanics

### § 1.1 Scope of the Chapter

Since a shear band is a stationary wave as mentioned in the Introduction, it is essential to introduce the concept of waves. A *wave* in continuum mechanics means either a solution to a system of nonlinear hyperbolic partial differential equations or a propagating singular surface, whose speed of propagation depends on the response of the material. It is the latter case, in which we are interested.

In section 1.2, the concept of waves is introduced and the compatibility conditions are obtained. In section 1.3, the basic laws of classical continuum mechanics and the nature of singular surfaces are reviewed. The classification of singular surfaces is given in section 1.4 and, in particular, acceleration waves are discussed in section 1.5. Finally, the introduction of shear bands concludes this chapter.

## § 1.2 Geometrical and Kinematical Conditions of Compatibility

Throughout this thesis, we use a fixed rectangular Cartesian coordinate system. The motion of the material body is described by a one-parameter family of mappings of a material point  $X$  into a spatial position  $x$  at time  $t$ , i.e.,

$$x = x(X,t) \quad \text{or} \quad x_k = x_k(X_K,t) , \quad (1.2.1)$$

where  $x_k$  and  $X_K$  are the rectangular coordinates of the spatial and material points, respectively. Latin subscripts take values 1, 2, 3 and a repeated index in a given term implies summation with respect to that index over the range 1, 2, 3. We assume that

$$\det(x_{k,K}) \neq 0 , \quad (1.2.2)$$

so that the inverse motion

$$X = X(x,t) \quad \text{or} \quad X_K = X_K(x_k,t) \quad (1.2.3)$$

exists in the neighborhood of all points except possibly at some singular surfaces, lines and points which may exist in the body. In (1.2.2), we follow the convention that a comma followed by an index indicates a partial

derivative, for example,

$$x_{k,K} = \frac{\partial x_k}{\partial X_K} .$$

The entire theory of singular surfaces rests upon *Hadamard's lemma* (1903): Let  $s$  be a smooth singular surface which is the common boundary of two regions  $R^+$  and  $R^-$ . Let  $\phi$  be defined and continuously differentiable in the interior of the region  $R^+$  and let  $\phi$  and  $\phi_{,k}$  approach finite limits  $\phi^+$  and  $(\phi_{,k})^+$  as  $s$  is approached upon paths interior to  $R^+$ . Let  $x = x(l)$  be a smooth curve upon  $s$ , and assume that  $\phi^+$  is differentiable along this path. Then

$$\frac{d\phi^+}{dl} = (\phi_{,k})^+ \frac{dx_k}{dl} . \quad (1.2.4)$$

In other words, the theorem of total differentiation holds valid for the limiting values as  $s$  is approached from one side only. If Hadamard's lemma (1.2.4) is applied to the opposite side  $R^-$  of the region, then we obtain

$$\frac{d\phi^-}{dl} = (\phi_{,k})^- \frac{dx_k}{dl} . \quad (1.2.5)$$

The jump of  $\phi$  across  $s$  is defined as

$$[\phi] = \phi^+ - \phi^- . \quad (1.2.6)$$

It follows from (1.2.4) and (1.2.5) that

$$\frac{d}{dt} [\phi] = [\phi_{,k}] \frac{dx_k}{dt}, \quad (1.2.7)$$

which simply means that the jump in the tangential derivative is the tangential derivative of the jump.

If  $\phi$  is continuous across  $s$ , then the result known as *Maxwell's theorem* follows:

$$[\phi_{,k}] = B n_k \quad (1.2.8)$$

where  $n$  is the unit normal to the discontinuity surface, and

$$B = [\phi_{,k} n_k],$$

which expresses the fact that the jump in the gradient of a continuous field across the singular surface is normal to the surface.

A moving surface  $s(t)$  may be represented by the Gaussian forms

$$x = x(p^\alpha, t) \quad \text{or} \quad x_i = x_i(p^\alpha, t), \quad (1.2.9)$$

where  $p^\alpha$  ( $\alpha = 1, 2$ ) are the curvilinear coordinates of the surface. The quantities  $p^\alpha$  in (1.2.9) can be eliminated



to obtain the equation to the surface in the ambient three-dimensional space in the form

$$f(\mathbf{x}, t) = 0 . \quad (1.2.10)$$

The surface velocity  $u(p, t)$  is defined by

$$u(p, t) = \left. \frac{\partial \mathbf{x}}{\partial t} \right|_p \quad (1.2.11)$$

where the notation  $|_p$  means that  $\frac{\partial \mathbf{x}}{\partial t}$  is evaluated holding  $p$  constant. The vector  $u(p, t)$  denotes the velocity of a point of the surface, along its trajectory

$$\mathbf{x} = \mathbf{x}(p, t) \Big|_p . \quad (1.2.12)$$

The speed of displacement is the normal velocity of the surface defined by

$$u_n = \mathbf{u} \cdot \mathbf{n} = - \frac{\frac{\partial f}{\partial t}}{\|\nabla f\|} . \quad (1.2.13)$$

Thus, the speed of displacement  $u_n$  is independent of any choice of the surface coordinates  $p^\alpha$  in (1.2.9), while the velocity  $u$  depends on a particular choice of the surface coordinates.

The surface metric is given by the tensor

$$a_{\alpha\beta} = x_{k\alpha} x_{k\beta} \quad (1.2.14)$$

with

$$x_{k\alpha} \equiv \frac{\partial x_k}{\partial p^\alpha},$$

while the coefficients of the second fundamental form of the surface are:

$$b_{\alpha\beta} = x_{k\alpha;\beta} n_k = \frac{1}{2} \epsilon^{\gamma\delta} e_{klm} x_{k\alpha;\beta} x_{l\gamma} x_{m\delta}, \quad (1.2.15)$$

where  $e_{ijk}$  is the usual permutation symbol for the three dimensional space, and

$$\epsilon^{\alpha\beta} \equiv e^{\alpha\beta}/\sqrt{a}, \quad \epsilon_{\alpha\beta} \equiv e_{\alpha\beta}/\sqrt{a}, \quad a = \det(a_{\alpha\beta})$$

where  $e_{\alpha\beta}$  and  $e^{\alpha\beta}$  are the two-dimensional permutation symbols. In the above expression, a semicolon followed by an index indicates covariant partial differentiation.

When (1.2.7) is applied to a coordinate curve on  $s(t)$ , we have the *geometrical condition of compatibility*,

$$[\phi, k] = B n_k + a^{\alpha\beta} x_{k\alpha} A_{;\beta}, \quad (1.2.16)$$

where

$$A = [ \phi ] .$$

If  $\phi$  is replaced by  $\phi_{,k}$  in (1.2.16), we obtain after some calculation,

$$\begin{aligned} [ \phi_{,kl} ] &= C n_k n_l \\ &+ a^{\alpha\beta} ( B_{;\alpha} + a^{\gamma\delta} b_{\alpha\gamma} A_{;\delta} ) ( n_k x_{l\beta} + n_l x_{k\beta} ) \\ &+ a^{\alpha\beta} a^{\gamma\delta} ( A_{;\alpha\gamma} - B b_{\alpha\gamma} ) x_{k\beta} x_{l\delta} , \end{aligned} \tag{1.2.17}$$

where

$$C \equiv [ \phi_{,kl} n_k n_l ] = [ \phi_{,kl} ] n_k n_l ,$$

which is usually referred to as Thomas's (1957) *iterated geometrical condition of compatibility*.

Thomas also introduced the *delta derivative* or *displacement derivative* of a function  $\phi(x,t)$ , defined on a moving surface, whose speed of displacement is  $u_n$ , as

$$\frac{\delta\phi}{\delta t} = \frac{\partial\phi}{\partial t} + u_n n_k \phi_{,k} . \tag{1.2.18}$$

This derivative is the time rate of change of  $\phi$  as apparent to an observer moving with normal velocity of the surface. By taking the jump in  $\partial\phi/\partial t$  across the surface  $s(t)$ , we can derive the *kinematical condition of compatibility*.

Thus,

$$\left[ \frac{\partial \phi}{\partial t} \right] = - u_n B + \frac{\delta A}{\delta t} \quad (1.2.19)$$

where

$$A \equiv [ \phi ] , \quad B \equiv [ n_k \phi_{,k} ] .$$

We also find after some calculation

$$\left[ \frac{\partial^2 \phi}{\partial x_k \partial t} \right] = \left( - u_n C + \frac{\delta B}{\delta t} + a^{\alpha\beta} A_{;\alpha} u_{n,\beta} \right) n_k + a^{\alpha\beta} \bar{A}_{;\alpha} x_{k\beta} , \quad (1.2.20)$$

$$\left[ \frac{\partial^2 \phi}{\partial t^2} \right] = \left( u_n C - \frac{\delta B}{\delta t} - a^{\alpha\beta} A_{;\alpha} u_{n,k} \right) u_n + \frac{\delta \bar{A}}{\delta t} , \quad (1.2.21)$$

where

$$\bar{A} = - u_n B + \frac{\delta A}{\delta t} .$$

So far our discussion did not involve the motion of the material body. The material description of the surface  $s(t)$  is a surface  $S(t)$  represented by

$$f(x(X,t),t) = F(X,t) = 0 . \quad (1.2.22)$$

The normal velocity of  $S$  is called the speed of propagation, denoted by  $U_N$ , and given by

$$U_N = - \frac{\frac{\partial F}{\partial t}}{\|\nabla F\|} \quad (1.2.23)$$

A singular surface is said to be a propagating surface or *wave* if  $U_N \neq 0$ .

### § 1.3 Basic Laws of Continuum Mechanics

The basic laws of classical continuum mechanics in Cartesian tensor form are [ A. C. Eringen, 1975 ]:

(i) Conservation of Mass

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0, \quad (1.3.1)$$

(ii) Conservation of Momentum

$$t_{kl,k} + \rho (f_k - \dot{v}_k) = 0, \quad (1.3.2)$$

(iii) Conservation of Energy

$$\rho \dot{\epsilon} = t_{kl} v_{l,k} - q_{k,k} + \rho h, \quad (1.3.3)$$

(iv) Entropy Inequality

$$\rho \dot{\eta} + \frac{1}{\theta} q_{k,k} - \frac{\rho h}{\theta} \geq 0, \quad (1.3.4)$$

where  $\rho$ ,  $t_{kl}$ ,  $f_k$ ,  $\epsilon$ ,  $q_k$ ,  $h$ ,  $\eta$  and  $\theta$  are, respectively, the mass density, stress tensor, body force density, internal energy density, heat flux, heat source, entropy density and the absolute temperature. The stress tensor is assumed to be symmetric.

If there exists a singular surface  $s(t)$  in the body  $B$ , then the above equations must be supplemented by the

following jump conditions across the surface:

$$[ \rho U ] = 0 , \quad (1.3.5)$$

$$[ t_{kl} n_k ] + [ \rho U v_l ] = 0 , \quad (1.3.6)$$

$$[ \rho U ( \epsilon + \frac{1}{2} v_k v_k ) ] + [ t_{kl} v_k - q_l ] n_l = 0 , \quad (1.3.7)$$

$$[ \rho U \eta ] + [ \frac{q_k}{\theta} ] n_k \leq 0 , \quad (1.3.8)$$

where  $U$  is the velocity of the moving singular surface.

## § 1.4 Classification of Singular Surfaces

In section 1.2, a singular surface was defined with respect to an arbitrary field  $\phi$  which is discontinuous across a surface. For the present investigation concerning shear bands in elastic solids, we choose for  $\phi$ , the motion given  $\phi = x_k(X,t)$ . The order of a singular surface is defined as the lowest order of the derivatives of  $x_k(X,t)$  that suffers a jump across a surface.

If  $[x] \neq 0$  across a surface, that is, the motion itself is discontinuous, then we have a singular surface of order zero called a *dislocation*.

A singular surface of order one is called a *shock wave*. On such a surface, the deformation gradient and the velocity of the medium may suffer jumps while the motion is continuous.

A singular surface of order two is called an *acceleration wave*. On such a surface, the second order gradient of motion and the acceleration of material particles may suffer jumps while the motion, the deformation gradient and the velocity remain continuous.



### § 1.5 Acceleration Waves

The definition of an acceleration wave was given in the previous section. We assume here that the temperature is continuous across the singular surface. The surface  $s(t)$  is an acceleration wave if the motion, the deformation gradients, and the velocity of material points are continuous, while the second order gradients of motion and the acceleration of material particles are discontinuous across it. Then, from (1.2.17), (1.2.20) and (1.2.21), we have the compatibility conditions across  $S(t)$ , which is the image of  $s(t)$  in the material frame:

$$[x_{k,KL}] = S_k N_K N_L, \quad (1.5.1)$$

$$[\dot{x}_{k,K}] = -U_N S_k N_K, \quad (1.5.2)$$

$$[\ddot{x}_k] = U_N^2 S_k, \quad (1.5.3)$$

where  $N$  is the unit normal to the discontinuity surface, and

$$S_k = [x_{k,KL}] N_K N_L. \quad (1.5.4)$$

Similarly, it follows from (1.3.5) - (1.3.8) that

$$[\rho] = 0, \quad [t_{kl}] = 0, \quad [\epsilon] = 0, \quad [\eta] = 0.$$

We assume that there exists a stress potential or a strain energy function  $\Sigma$  for the elastic medium. Such a material is called *hyperelastic*. Then, the stress constitutive relations for the hyperelastic materials are given by

$$t_{kl} = 2 \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial C_{KL}} x_{k,K} x_{l,L} , \quad (1.5.5)$$

or in alternative form,

$$t_{kl} = \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial E_{KL}} x_{k,K} x_{l,L} , \quad (1.5.6)$$

where  $\Sigma = \Sigma(C, X)$ , and  $C_{KL}$  and  $E_{KL}$  are, respectively, the Green deformation tensor and the Lagrangian strain tensor. In the above expressions,  $\rho_0$  and  $\rho$  denote the mass densities in the undeformed and deformed body, respectively. If the material is homogeneous, then we drop the dependence on  $X$  in  $\Sigma$ .

In the material frame, we have

$$T_{KL} = 2 \frac{\partial \Sigma}{\partial C_{KL}} = \frac{\partial \Sigma}{\partial E_{KL}} . \quad (1.5.7)$$

We also have

$$\begin{aligned} T_{Kk} &= 2 \frac{\partial \Sigma}{\partial C_{KL}} x_{k,L} = \frac{\partial \Sigma}{\partial E_{KL}} x_{k,L} \\ &= \frac{\partial \Sigma}{\partial x_{k,K}} = \frac{\rho_0}{\rho} X_{K,l} t_{lk} , \end{aligned} \quad (1.5.8)$$

which is the well known Piola-Kirchhoff stress tensor. Then, the equation of motion (1.3.2) can be expressed in terms of the Piola-Kirchhoff stress tensor in the form,

$$T_{Kk,K} + \rho_0 f_k = \rho_0 \ddot{x}_k . \quad (1.5.9)$$

Then, we obtain

$$A_{KkLl} x_{l,KL} + \rho_0 f_k = \rho_0 \ddot{x}_k , \quad (1.5.10)$$

where

$$A_{KkLl} = \frac{\partial^2 \Sigma}{\partial x_{k,K} \partial x_{l,L}} .$$

If we assume that the body force density  $f$  is continuous across the surface  $S(t)$ , then (1.5.10) takes the form

$$A_{KkLl} [x_{l,KL}] = \rho_0 [\ddot{x}_k] . \quad (1.5.11)$$

Substitution of (1.5.1) and (1.5.3) into (1.5.11) gives

$$(\mathcal{Q}_{kl} - \rho_0 U_N^2 \delta_{kl}) S_l = 0 \quad (1.5.12)$$

where

$$\mathcal{Q}_{kl} \equiv A_{KkLl} N_K N_L .$$

The spatial form of (1.5.12) is obtained by using the following relations:

$$[x_{k,KL}] = a_k x_{m,K} x_{n,L} n_m n_n ,$$

$$[\dot{x}_k] = U a_k , \quad (1.5.13)$$

where the nonzero vector  $a$  is called the *amplitude vector*, since it is a measure of the strength of the discontinuity. Then, corresponding to (1.5.12), we have in the spatial form:

$$[Q_{kl} - \rho U^2 \delta_{kl}] a_l = 0 , \quad (1.5.14)$$

where

$$Q_{kl} = \frac{\rho}{\rho_0} A_{KkLl} x_{m,K} x_{n,L} n_m n_n$$

is a function of the deformation gradients and the propagation direction  $n$ , and is called the *acoustical tensor*.

Therefore, in (1.5.14), the amplitude vector  $a$  is an eigenvector of the tensor  $Q$  and  $\rho U^2$  is the corresponding eigenvalue. Thus  $\rho U^2$  must satisfy the polynomial equation

$$III_Q - \rho U^2 II_Q + (\rho U^2)^2 I_Q - (\rho U^2)^3 = 0 , \quad (1.5.15)$$

where  $I_Q$  ,  $II_Q$  and  $III_Q$  denote the invariants of the acoustical tensor  $Q$ .

If there exists a shear band, which is a standing singular surface, then the speed of propagation must be zero. Then, (1.5.14) reduces to the form

$$Q \cdot a = 0 , \quad (1.5.16)$$

where  $0$  denotes a zero vector. The equation (1.5.16) is called the *characteristic equation* of the shear band.

## CHAPTER 2

### Review of Micropolar Continuum Mechanics

#### § 2.1 Scope of the Chapter

This chapter is devoted to a brief discussion of micropolar continuum mechanics, the theory initiated by A. C. Eringen [1964]. In section 2.2, the kinematics of micropolar continuum mechanics along with appropriate strain measures is briefly discussed. Section 2.3 presents the balance laws of micropolar continuum mechanics consisting of the balance of mass, linear and angular momenta, conservation of microinertia and energy. The entropy inequality is also presented in this section. Finally, acceleration waves in micropolar continua are investigated in section 2.4.

## § 2.2 Kinematics of Micropolar Continuum Mechanics

A micropolar medium is defined as a classical continuum, to each point of which is assigned another continuum, this latter capable of undergoing only rigid rotations. Thus, each point of a micropolar continuum can translate and rotate independently. The translation is described by the motion

$$\mathbf{x} = \mathbf{x}(X,t) \quad \text{or} \quad x_k = x_k(x,t) , \quad (2.2.1)$$

and the rotation is described by the micromotion

$$\boldsymbol{\xi} = \chi_K \boldsymbol{\Xi}_K \quad \text{or} \quad \xi_k = \chi_{kK} \boldsymbol{\Xi}_K \quad (2.2.2)$$

where  $\boldsymbol{\Xi} = \boldsymbol{\Xi}_K \mathbf{I}_K$  is an arbitrary vector at  $X$  in the material frame, and  $\boldsymbol{\xi} = \xi_k \mathbf{i}_k$  is the vector in the spatial frame to which  $\boldsymbol{\Xi}$  is rotated. We assume that  $\chi$  is an orthogonal tensor, and thus, it completely describes the rotational motion. We also assume that the motion and the micromotion are continuous and possess continuous first-order partial derivatives at all points of the body except possibly at some singular surfaces, lines and points that may exist in the body, which require special attention. We require that

$$\det(x_{k,K}) \neq 0 \quad \text{and} \quad \chi_{Kk}^{-1} = \chi_{kK} , \quad (2.2.3)$$

so that the inverses of both motion and micromotion exist in the regions described above. The angular velocity vector  $\nu$  is given by

$$\nu_k = -\frac{1}{2} \epsilon_{klm} \nu_{lm} , \quad (2.2.4)$$

where  $\nu_{kl}$  is the skew-symmetric *gyration tensor* defined by

$$\nu_{kl} = \dot{\chi}_{kK} \chi_{lK} . \quad (2.2.5)$$

In micropolar continuum mechanics, we define the following deformation tensors :

$$\mathfrak{C}_{KL} \equiv \chi_{k,K} \chi_{kL} , \quad (2.2.6)$$

$$\Gamma_{KL} \equiv \frac{1}{2} \epsilon_{KMN} \chi_{kM:L} \chi_{kN} , \quad (2.2.7)$$

which are called the *Cosserat deformation tensor* and the *wryness tensor*, respectively. The *Cosserat strain tensor* is defined by

$$\mathfrak{E}_{KL} = \mathfrak{C}_{KL} - \delta_{KL} \quad \text{or} \quad \mathfrak{E} = \mathfrak{C} - I . \quad (2.2.8)$$

Naturally,  $\mathfrak{C}$  and  $\Gamma$  are not arbitrary but must satisfy the *compatibility conditions*:

$$\epsilon_{KMN} ( \mathfrak{C}_{ML;N} + \epsilon_{LPQ} \Gamma_{PN} \mathfrak{C}_{MQ} ) = 0 , \quad (2.2.9)$$



$$\epsilon_{KMN} \left( \Gamma_{LN;M} + \frac{1}{2} \epsilon_{LPQ} \Gamma_{PM} \Gamma_{QN} \right) = 0 . \quad (2.2.10)$$

### § 2.3 Balance Laws of Micropolar Continuum Mechanics

As in classical continuum mechanics, to each point  $X$  in an undeformed body  $B$  we assign a mass density  $\rho_0(X)$ , whose dual in the deformed body  $\mathfrak{B}$  is denoted by  $\rho(x,t)$ . Since a material point of a micropolar medium is imagined as a rigid particle, we assign to each material point  $X$ , a positive and symmetric tensor  $J_{KL}$ , the *material tensor of inertia density*.

We often use the tensor  $I_{KL}$ , related to  $J_{KL}$  by

$$J_{KL} = I_{MM} \delta_{KL} - I_{KL} . \quad (2.3.1)$$

The spatial form of  $J_{KL}$  is the spatial tensor of inertia density and is similarly related to tensor  $i_{kl}$  by

$$j_{kl} = i_{mm} \delta_{kl} - i_{kl} , \quad (2.3.2)$$

where

$$i_{kl} = \chi_{kK} \chi_{lL} I_{KL} .$$

Thus, one can easily obtain the relation

$$j_{kl} = \chi_{kK} \chi_{lL} J_{KL} . \quad (2.3.3)$$

In order to generalize the concepts of momentum, moment of

momentum, and energy, we must take into account the effects of rotation of the material. Thus, we postulate the following:

- 1) The momentum density of a micropolar medium is  $\rho v$  where  $v$  is the velocity.
- 2) The moment of momentum density of a micropolar medium is  $r \times \rho v + \rho \sigma$  where  $r$  is the position vector of a point in the deformed body  $\mathfrak{B}$  and  $\sigma$  is the "spin density" defined by  $\sigma_k \equiv j_{kl} \nu_l$ , which is the moment of momentum density due to the intrinsic rotation of the material point.
- 3) The kinetic energy of a micropolar medium is

$$\frac{1}{2} \rho v^2 + \frac{1}{2} j_{kl} \nu_k \nu_l = \frac{1}{2} \rho v^2 + \frac{1}{2} \rho \sigma \cdot \nu,$$

where  $\frac{1}{2} \rho \sigma \cdot \nu$  is the kinetic energy due to the intrinsic rotation of the material particle.

The balance laws for a micropolar medium in local form are

[ A. C. Eringen, 1976 ] :

- (i) Conservation of Mass

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0, \quad (2.3.4)$$

- (ii) Conservation of Microinertia

$$\frac{D j_{kl}}{Dt} = \nu_{km} j_{lm} + \nu_{lm} j_{km}, \quad (2.3.5)$$

- (iii) Balance of Momentum

$$t_{kl,k} + \rho(f_l - \dot{v}_l) = 0 , \quad (2.3.6)$$

(iv) Balance of Moment of Momentum

$$m_{kl,k} + \epsilon_{lmn} t_{mn} + \rho(l_l - \dot{\sigma}_l) = 0 , \quad (2.3.7)$$

(v) Conservation of Energy

$$\rho \dot{\epsilon} - t_{kl} (v_{l,k} + v_{kl}) - m_{kl} + q_{k,k} - \rho h = 0 , \quad (2.3.8)$$

(vi) Entropy Inequality

$$\rho \dot{\eta} + \frac{q_{k,k}}{\theta} - \frac{\rho h}{\theta} \geq 0 , \quad (2.3.9)$$

where  $\frac{D}{Dt}$  is the material derivative defined by

$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\cdot)_{,k} v_k ,$$

and  $m_{kl}$  is the couple stress tensor,  $l_k$  is the body couple density, and the rest of the field quantities are defined in the previous chapter.

If there exists a singular surface in the body, then, accordingly, the above equations must be supplemented by the following jump conditions across the surface:

(i) Conservation of Mass

$$[ \rho U ] = 0 , \quad (2.3.10)$$

(ii) Conservation of Microinertia

$$[ \rho j_{kl} U ] = 0 , \quad (2.3.11)$$

(iii) Balance of Momentum

$$[ \rho U v_l ] + [ t_{kl} n_k ] = 0 , \quad (2.3.12)$$

(iv) Balance of Moment of Momentum

$$[ \rho U \sigma_l ] + [ m_{kl} n_k ] = 0 , \quad (2.3.13)$$

(v) Conservation of Energy

$$\begin{aligned} & [ \rho U ( \epsilon + \frac{1}{2} v_k v_k + \frac{1}{2} \sigma_k v_k ) ] \\ & + [ t_{kl} v_l + m_{kl} v_l - q_k ] n_k = 0 , \end{aligned} \quad (2.3.14)$$

(vi) Entropy Inequality

$$[ \rho U \eta ] + [ \frac{q_k}{\theta} ] n_k \leq 0 . \quad (2.3.15)$$

## § 2.4 Acceleration Waves in Micropolar Elastic Solids

The concept of acceleration waves in classical continuum mechanics was introduced in section 1.5. In this section, we investigate acceleration waves in micropolar elastic media, based on the work of C. B. Kafadar and A. C. Eringen (1971). While the jumps were taken with respect to the second order gradients of motion and the acceleration of material particles for acceleration waves in the case of classical continuum mechanics, we must now consider the discontinuity of the micromotion as well as that of the motion. In micropolar continuum mechanics, an acceleration wave is a surface, across which  $x_{k,KL}$ ,  $\ddot{x}_k$ ,  $\chi_{kK,LM}$  and  $\ddot{\chi}_{kK}$  may suffer a discontinuity while  $x_k$ ,  $\dot{x}_k$ ,  $x_{k,K}$ ,  $\chi_{kK}$ ,  $\dot{\chi}_{kK}$  and  $\chi_{kK,M}$  remain continuous throughout the body. If we apply (1.2.17) and (1.2.21), then we obtain

$$[ x_{k,KL} ] = S_k N_K N_L , \quad (2.4.1)$$

$$[ \ddot{x}_k ] = S_k U_N^2 , \quad (2.4.2)$$

$$[ \chi_{kK,LM} ] = S_{kK} N_L N_M , \quad (2.4.3)$$

$$[ \ddot{\chi}_{kK} ] = S_{kK} U_N^2 , \quad (2.4.4)$$

where

$$S_k = [ x_{k,KL} ] N_K N_L , \quad (2.4.5)$$

$$S_{kK} = [ x_{kK,LM} ] N_L N_M , \quad (2.4.6)$$

and  $U_N$  and  $N_K$  denote, respectively, the speed of propagation and the unit normal to the discontinuity surface  $F(X,t) = 0$ .

The equations of motion could be written as

$$( T_{KL} x_{kL} ),_K + \rho_0 f_k = \rho_0 \dot{v}_k , \quad (2.4.7)$$

$$( M_{KL} x_{kL} ),_K + J \epsilon_{kmn} t_{mn} + \rho_0 l_k = \rho_0 \dot{\sigma}_k . \quad (2.4.8)$$

Before taking the jumps in the above equations, however, we must first find an expression for  $[ \dot{\sigma}_k ]$ . We note that since

$$\sigma_k = j_{kl} \nu_l , \quad (2.4.9)$$

we obtain

$$[ \dot{\sigma}_k ] = j_{kl} [ \dot{\nu}_k ] \quad (2.4.10)$$

as a consequence of  $[ Dj_{kl}/Dt ] = 0$ . We also know from

(2.2.4) and (2.2.5) that

$$\nu_k = -\frac{1}{2} \epsilon_{klm} \dot{\chi}_{lK} \chi_{mK} . \quad (2.4.11)$$

Thus,

$$\begin{aligned} [\dot{\nu}_k] &= -\frac{1}{2} \epsilon_{klm} [\ddot{\chi}_{lK}] \chi_{mK} \\ &= -\frac{1}{2} \epsilon_{klm} S_{kK} \chi_{mK} . \end{aligned} \quad (2.4.12)$$

In order to simplify this expression, we use the fact that

$$(\chi_{kK} \chi_{kL})_{,MN} = 0 . \quad (2.4.13)$$

Taking the jumps of the above, we obtain

$$S_{kK} \chi_{kL} = -S_{kL} \chi_{kK} . \quad (2.4.14)$$

Then, one may introduce, following Kafadar and Eringen [1971], a vector  $s_k$  such that

$$S_{kK} = \epsilon_{klm} s_l \chi_{mK} , \quad (2.4.15)$$

which satisfies (2.4.14). Substituting (2.4.15) into (2.4.12), we obtain

$$[\dot{\nu}_k] = -\frac{1}{2} \epsilon_{klm} \epsilon_{lpq} s_p \chi_{qK} \chi_{mK} . \quad (2.4.16)$$



We use the fact that

$$\epsilon_{klm} \epsilon_{pqm} = \delta_{kp} \delta_{lq} - \delta_{kq} \delta_{lp} . \quad (2.4.17)$$

Then (2.4.16) becomes

$$[ \dot{\nu}_k ] = s_k U_N^2 , \quad (2.4.18)$$

and therefore from (2.4.10), we obtain

$$[ \dot{\sigma}_k ] = j_{kl} s_l U_N^2 . \quad (2.4.19)$$

If we assume that the body force density and the body couple density are continuous across the singular surface, then taking the jumps of (2.4.7) and (2.4.8) , and using (2.4.1) and (2.4.19) for the expressions on the right hand side of these equations, we obtain

$$[ T_{KL,K} ] \chi_{kL} = \rho_0 S_k U_N^2 , \quad (2.4.20)$$

$$[ M_{KL} ] \chi_{kL} = \rho_0 j_{kl} s_l U_N^2 . \quad (2.4.21)$$

It is convenient to define vectors  $a^{(1)}$  and  $a^{(2)}$  such that

$$a^{(1)}_K \equiv \chi_{kK} S_k , \quad (2.4.22)$$

$$a^{(2)}_K \equiv \chi_{kK} j_{kl} s_l = J_{KL} \chi_{lL} s_l , \quad (2.4.23)$$

whose inverses are

$$S_k = \chi_{kK} a^{(1)}_K , \quad (2.4.24)$$

$$s_l = j^{-1}_{lk} \chi_{kK} a^{(2)}_K = \chi_{lL} J^{-1}_{LK} a^{(2)}_K , \quad (2.4.25)$$

where  $j^{-1}_{kl}$  and  $J^{-1}_{KL}$  are the inverses of  $j_{kl}$  and  $J_{KL}$ , respectively. Substituting (2.4.24) and (2.4.25) into (2.4.20) and (2.4.21), we obtain

$$[ T_{KL,K} ] = \rho_0 a^{(1)}_L U_N^2 , \quad (2.4.26)$$

$$[ M_{KL,K} ] = \rho_0 a^{(2)}_L U_N^2 . \quad (2.4.27)$$

Using (2.2.1), (2.2.2), (2.4.1), (2.4.3), (2.4.15) and (2.4.25), we obtain after some calculation

$$[ \mathfrak{C}_{KL,K} ] = a^{(1)}_L N_K N_M , \quad (2.4.28)$$

$$[ \Gamma_{KL,M} ] = J_{KR} a^{(2)}_R N_L N_M . \quad (2.4.29)$$

Then using these expressions in (2.4.26) and (2.4.27), we find

$$\frac{\partial T_{KL}}{\partial \mathfrak{C}_{RS}} N_R N_K a^{(1)}_S + \frac{\partial T_{KL}}{\partial \Gamma_{MR}} J^{-1}_{MS} N_R N_K a^{(2)}_S = \rho_0 U_N^2 a^{(1)}_L , \quad (2.4.30)$$

$$\frac{\partial M_{KL}}{\partial \mathfrak{E}_{RS}} N_R N_K a^{(1)}_S + \frac{\partial M_{KL}}{\partial \Gamma_{MR}} J^{-1}_{MS} N_R N_K a^{(2)}_S = \rho_0 U_N^2 a^{(2)}_L . \quad (2.4.31)$$

Using the relation (2.2.8), we note that

$$\frac{\partial T_{KL}}{\partial \mathfrak{E}_{RS}} = \frac{\partial T_{KL}}{\partial \mathfrak{E}_{RS}} , \quad \frac{\partial M_{KL}}{\partial \mathfrak{E}_{RS}} = \frac{\partial M_{KL}}{\partial \mathfrak{E}_{RS}} . \quad (2.4.32)$$

Then (2.4.30) and (2.4.31) could be written as

$$\frac{\partial T_{KL}}{\partial \mathfrak{E}_{RS}} N_R N_K a^{(1)}_S + \frac{\partial T_{KL}}{\partial \Gamma_{MR}} J^{-1}_{MS} N_R N_K a^{(2)}_S = \rho_0 U_N^2 a^{(1)}_L , \quad (2.4.33)$$

$$\frac{\partial M_{KL}}{\partial \mathfrak{E}_{RS}} N_R N_K a^{(1)}_S + \frac{\partial M_{KL}}{\partial \Gamma_{MR}} J^{-1}_{MS} N_R N_K a^{(2)}_S = \rho_0 U_N^2 a^{(2)}_L . \quad (2.4.34)$$

We define  $Q_{KL}$  and  $a_L$  ( $K, L = 1, \dots, 6$ ) by

$$(Q_{KL}) = \begin{pmatrix} \frac{\partial T_{SK}}{\partial \mathfrak{E}_{RL}} N_R N_S & \frac{\partial T_{SK}}{\partial \Gamma_{MR}} J^{-1}_{M,L-3} N_R N_S \\ \frac{\partial M_{S,K-3}}{\partial \mathfrak{E}_{RL}} N_R N_S & \frac{\partial M_{S,K-3}}{\partial \Gamma_{MR}} J^{-1}_{M,L-3} N_R N_S \end{pmatrix} \quad (2.4.35)$$

and

$$a_K = (a^{(1)}_K, a^{(2)}_{K-3}) , \quad (2.4.36)$$

where each element in the above matrix is itself a  $3 \times 3$  matrix. Then the equations (2.4.30) and (2.4.31) could be expressed as

$$Q_{KL} a_L = \rho_0 U_N^2 a_K . \quad (2.4.37)$$

The tensor  $Q$  in (2.4.35) is the acoustical tensor for micropolar media and  $a$  in (2.4.36) is the amplitude of the discontinuity wave. From (2.4.37), we obtain

$$( Q_{KL} - \rho_0 U_N^2 \delta_{KL} ) a_L = 0 . \quad (2.4.38)$$

Therefore, the amplitude vector is an eigenvector of the tensor  $Q_{KL}$ , while  $\rho_0 U_N^2$  is the corresponding eigenvalue.

## Chapter 3

### Shear Bands in Isotropic Micropolar Elastic Materials

#### § 3.1 Scope of the Chapter

In the present chapter, we turn our attention to the formation of shear bands in isotropic micropolar elastic materials. In section 3.2, the condition for the existence of a shear band, called the *characteristic equation*, is obtained in terms of an appropriate acoustic tensor for the micropolar continuum. Explicit expressions for the characteristic equation are obtained in terms of the stress and couple stress tensors. In section 3.3, the existence of the shear band is investigated for the case of a homogeneous bar subjected to plane stress loadings. We present two cases, one of uniaxial tension, and the other, of tension of a thin plate, in which shear bands are formed. The behavior of shear bands is determined in terms of their angles of inclination, and the applied tension. Numerical calculations are also presented.

### § 3.2 Characteristic Equation of the Shear Band in Micropolar Media

In section 2.4, the propagating condition for an acceleration wave was obtained for micropolar continua in terms of the appropriate acoustical tensor and the amplitude vector. Consider now an acceleration wave traversing through a micropolar elastic medium. We assume that the material is *microisotropic* so that the microinertia tensor is expressed as

$$J_{KL} = J \delta_{KL} , \quad (3.2.1)$$

where  $J$  is a constant.

The propagating condition for an acceleration wave in micropolar media is given by

$$Q \cdot a = \rho_0 U_N^2 a , \quad (3.2.2)$$

where, from (2.4.35) and (3.2.1), the acoustical tensor  $Q$  is expressed

$$(Q_{KL}) = \begin{pmatrix} \frac{\partial T_{SK}}{\partial \epsilon_{RL}} N_R N_S & \frac{1}{J} \frac{\partial T_{SK}}{\partial \Gamma_{L-3,R}} N_R N_S \\ \frac{\partial M_{S,K-3}}{\partial \epsilon_{RL}} N_R N_S & \frac{1}{J} \frac{\partial M_{S,K-3}}{\partial \Gamma_{L-3,R}} N_R N_S \end{pmatrix} \quad (3.2.3)$$

where  $K, L = 1, 2, \dots, 6$  and the amplitude vector is given by

$$a_K = (a_K^{(1)}, a_{K-3}^{(2)}) . \quad (3.2.4)$$

If there exists a shear band in the body, then the speed of propagation must be zero. Thus, (3.2.2) implies that if a shear band exists in the body,

$$Q \cdot a = 0 \quad (3.2.5)$$

for a nonzero vector  $a$ . Thus, the necessary condition for the existence of a shear band in a micropolar material is

$$\det Q = 0 . \quad (3.2.6)$$

We are thus motivated to call (3.2.6) the *characteristic equation* of the shear band in micropolar media. If we define

$$\begin{aligned} Q_{KL}^{(1)} &= \frac{\partial T_{SK}}{\partial \epsilon_{RL}} N_R N_S , & Q_{KL}^{(2)} &= \frac{\partial T_{SK}}{\partial \Gamma_{LR}} N_R N_S , \\ Q_{KL}^{(3)} &= \frac{\partial M_{SK}}{\partial \epsilon_{RL}} N_R N_S , & Q_{KL}^{(4)} &= \frac{\partial M_{SK}}{\partial \Gamma_{LR}} N_R N_S , \end{aligned} \quad (3.2.7)$$

then (3.2.6) can be expressed as a  $6 \times 6$  determinant, which can be put in the following form:

$$\begin{vmatrix} Q^{(1)}_{KL} & Q^{(2)}_{K-3,L} \\ Q^{(3)}_{K,L-3} & Q^{(4)}_{K-3,L-3} \end{vmatrix} = 0, \quad (3.2.8)$$

for  $K, L = 1, \dots, 6$ .

We assume that there exists a stress potential or strain energy function  $\Sigma$  for the micropolar elastic medium. Such a material is called the *hyperelastic micropolar material*. Then, the stress constitutive relations in the material frame are given by

$$T_{KL} = \frac{\partial \Sigma}{\partial \mathfrak{C}_{KL}}, \quad M_{KL} = \frac{\partial \Sigma}{\partial \Gamma_{LK}}, \quad (3.2.9)$$

where  $\Sigma = \Sigma(\mathfrak{C}, \Gamma, X)$ . Furthermore, if we assume that the material is homogeneous, then we can drop the dependence on  $X$  in  $\Sigma$ . For isotropic materials, it has been shown by C.B. Kafadar and A.C. Eringen (1971) that  $\Sigma$  becomes a function of the following 15 joint invariants of  $\mathfrak{C}$  and  $\Gamma$ :

$$I_1 = \text{tr} \mathfrak{C}, \quad I_2 = \frac{1}{2} \text{tr} \mathfrak{C}^2, \quad I_3 = \frac{1}{3} \text{tr} \mathfrak{C}^3,$$

$$I_4 = \frac{1}{2} \text{tr} \mathfrak{C} \mathfrak{C}^t, \quad I_5 = \text{tr} \mathfrak{C}^2 \mathfrak{C}^t, \quad I_6 = \frac{1}{2} \text{tr} \mathfrak{C}^2 (\mathfrak{C}^t)^2,$$

$$I_7 = \text{tr} \mathfrak{C} \Gamma, \quad I_8 = \text{tr} \mathfrak{C} \Gamma^2, \quad I_9 = \text{tr} \mathfrak{C}^2 \Gamma,$$

$$I_{10} = \text{tr} \Gamma, \quad I_{11} = \frac{1}{2} \text{tr} \Gamma^2, \quad I_{12} = \frac{1}{3} \text{tr} \Gamma^3,$$



$$I_{13} = \frac{1}{2} \text{tr} \Gamma \Gamma^t, \quad I_{14} = \text{tr} \Gamma^2 \Gamma, \quad I_{15} = \frac{1}{2} \text{tr} \Gamma^2 (\Gamma^t)^2, \quad (3.2.10)$$

where the notation  $\text{tr}$  is the trace operator, for example,

$$\text{tr} \mathfrak{E} = \mathfrak{E}_{11} + \mathfrak{E}_{22} + \mathfrak{E}_{33} ,$$

and a superscript  $t$  denotes the transpose operator,

$$(\mathfrak{E}_{KL})^t = \mathfrak{E}_{LK} .$$

Henceforth, we assume that the material is homogeneous and isotropic. Then, substituting (3.2.10) into (3.2.9), and using (2.2.8), we obtain the nonlinear constitutive equations for the isotropic micropolar elastic medium up to the second-order terms in terms of the strain tensors. Thus,

$$\begin{aligned} T_{KL} = & \alpha_1 \delta_{KL} + \alpha_2 \mathfrak{E}_{LK} + \alpha_3 \mathfrak{E}_{LM} \mathfrak{E}_{MK} + \alpha_4 \mathfrak{E}_{KL} \\ & + \alpha_5 (\mathfrak{E}_{KM} \mathfrak{E}_{LM} + \mathfrak{E}_{MK} \mathfrak{E}_{ML} + \mathfrak{E}_{KM} \mathfrak{E}_{ML}) \\ & + \alpha_6 \Gamma_{LK} + \alpha_7 \Gamma_{MK} \Gamma_{LM} + \alpha_8 (\Gamma_{MK} \mathfrak{E}_{LM} + \mathfrak{E}_{MK} \Gamma_{LM}) , \end{aligned} \quad (3.2.11)$$

$$\begin{aligned} M_{KL} = & \alpha_6 \mathfrak{E}_{KL} + \alpha_7 (\Gamma_{KM} \mathfrak{E}_{ML} + \mathfrak{E}_{KM} \Gamma_{ML}) + \alpha_8 \mathfrak{E}_{KM} \mathfrak{E}_{ML} \\ & + \alpha_9 \delta_{KL} + \alpha_{10} \Gamma_{KL} + \alpha_{11} \Gamma_{KM} \Gamma_{ML} + \alpha_{12} \Gamma_{LK} \\ & + \alpha_{13} (\Gamma_{KM} \Gamma_{LM} + \Gamma_{MK} \Gamma_{ML} + \Gamma_{MK} \Gamma_{LM}) , \end{aligned} \quad (3.2.12)$$

where  $\alpha$ 's, called the micropolar elastic moduli, are functions of the 15 invariants in (3.2.10), and are generally function of temperature. For isothermal elasticity, it is reasonable to consider these moduli to be constants. Hence, we assume them to be constants. It is known that, in the linear theory, we have

$$\begin{aligned} \alpha_1 &= \lambda I_{\mathfrak{E}} , & \alpha_2 &= \mu , & \alpha_3 &= \mu + \kappa , \\ \alpha_9 &= \alpha I_{\Gamma} , & \alpha_{10} &= \beta , & \alpha_{12} &= \gamma , \end{aligned} \quad (3.2.13)$$

where  $I_{\mathfrak{E}}$  and  $I_{\Gamma}$  are, respectively, the first invariants of  $\mathfrak{E}$  and  $\Gamma$ , and the rest of the  $\alpha$ 's are zero. By substituting (3.2.11) and (3.2.12) into (3.2.7), the components of the acoustical tensor can be computed in terms of the micropolar strain measures:

$$\begin{aligned} Q_{KL}^{(1)} &= \alpha_2 N_K N_L + \alpha_3 (N_K N_R \mathfrak{E}_{RL} + N_R N_L \mathfrak{E}_{KR}) + \alpha_4 \delta_{KL} \\ &+ \alpha_5 (N_K N_R \mathfrak{E}_{RL} + N_R N_L \mathfrak{E}_{RK} + \mathfrak{E}_{KL} \\ &\quad + \mathfrak{E}_{LK} + 2\delta_{KL} N_R N_S \mathfrak{E}_{RS}) \\ &+ \alpha_8 (N_K N_R \Gamma_{LR} + N_R N_L \Gamma_{KR}) , \end{aligned} \quad (3.2.14)$$

$$\begin{aligned} Q_{KL}^{(2)} &= \alpha_6 \delta_{KL} + \alpha_7 (\delta_{KL} N_R N_S \Gamma_{RS} + \Gamma_{KL}) \\ &+ \alpha_8 (\mathfrak{E}_{KL} + \delta_{KL} N_R N_S \mathfrak{E}_{RS}) , \end{aligned} \quad (3.2.15)$$

$$\begin{aligned} Q_{KL}^{(3)} &= \alpha_6 \delta_{KL} + \alpha_7 (\delta_{KL} N_R N_S \Gamma_{RS} + \Gamma_{LK}) \\ &+ \alpha_8 (\mathfrak{E}_{LK} + \delta_{KL} N_R N_S \mathfrak{E}_{RS}) , \end{aligned} \quad (3.2.16)$$

$$\begin{aligned}
Q^{(4)}_{KL} = & \alpha_7 (N_R N_L \mathfrak{E}_{RK} + N_K N_R \mathfrak{E}_{RL}) + \alpha_{10} N_K N_L \\
& + \alpha_{11} (N_R N_L \Gamma_{RK} + N_K N_R \Gamma_{RL}) + \alpha_{12} \delta_{KL} \\
& + \alpha_{13} (N_K N_R \Gamma_{LR} + N_R N_L \Gamma_{KR} + 2\delta_{KL} N_R N_S \Gamma_{RS} \\
& + \Gamma_{KL} + \Gamma_{LK}) . \tag{3.2.17}
\end{aligned}$$

In order to express the acoustical tensor in terms of linear functions of stress and couple stress tensors, we must express the strain tensors in terms of the stress and couple stress tensors. We note that the linear stress-strain relations are given by

$$T_{KL} = \lambda I \mathfrak{E}_{KL} + (\mu + \kappa) \mathfrak{E}_{KL} + \mu \mathfrak{E}_{LK} , \tag{3.2.18}$$

$$M_{KL} = \alpha I_{\Gamma} \delta_{KL} + \beta \Gamma_{KL} + \gamma \Gamma_{LK} , \tag{3.2.19}$$

which are obtained by substituting (3.2.13) into (3.2.11) and (3.2.12). Inverse relations are obtained easily and given by

$$\mathfrak{E}_{KL} = \beta_1 T_{KL} + \beta_2 T_{LK} + \beta_3 I_T \delta_{KL} , \tag{3.2.20}$$

$$\Gamma_{KL} = \beta_4 M_{KL} + \beta_5 M_{LK} + \beta_6 I_M \delta_{KL} , \tag{3.2.21}$$

where

$$\beta_1 = \frac{\mu + \kappa}{\kappa(2\mu + \kappa)} , \quad \beta_2 = \frac{\mu}{\kappa(2\mu + \kappa)} ,$$

$$\beta_3 = \frac{\lambda}{(3\lambda + 2\lambda + \kappa)(2\mu + \kappa)} ,$$

$$\beta_4 = \frac{\beta}{\beta^2 - \gamma^2} , \quad \beta_5 = -\frac{\gamma}{\beta^2 - \gamma^2} ,$$

$$\beta_6 = \frac{\alpha}{(3\alpha + \beta + \gamma)(\beta + \gamma)} , \quad (3.2.22)$$

with  $\beta \neq \gamma$ , as permitted by thermodynamics of micropolar media, and  $I_T$  and  $I_M$  denote the first-invariants of  $T$  and  $M$ , respectively. Substituting (3.2.20) and (3.2.21) into (3.2.14) through (3.2.17), we obtain

$$\begin{aligned} Q^{(1)} = & [\alpha_4 + 3\alpha_5\beta_3 I_T + 2\alpha_5(\beta_1 + \beta_2)N.T.N] I \\ & + [\alpha_2 + 2(\alpha_3 + \alpha_5)\beta_3 I_T + 2\alpha_8\beta_6 I_M] N \otimes N \\ & + (\alpha_3 + \alpha_5)\beta_1 (N \otimes N.T + N.T \otimes N) \\ & + (\alpha_3 + \alpha_5)\beta_2 (N \otimes T.N + T.N \otimes N) \\ & + \alpha_5(\beta_1 + \beta_2)(T + T^t) + \alpha_8\beta_4 (N \otimes M.N + M.N \otimes N) \\ & + \alpha_8\beta_5 (N \otimes N.M + N.M \otimes N) , \end{aligned} \quad (3.2.23)$$

$$\begin{aligned} Q^{(2)} = & \{ \alpha_6 + \alpha_7 [(\beta_4 + \beta_5)N.M.N + 2\beta_6 I_M] \\ & + \alpha_8 [(\beta_1 + \beta_2)N.T.N + 2\beta_3 I_T] \} I \\ & + \alpha_7\beta_4 M + \alpha_7\beta_5 M^t + \alpha_8\beta_1 T + \alpha_8\beta_2 T^t , \end{aligned} \quad (3.2.24)$$

$$\begin{aligned} Q^{(3)} = & \{ \alpha_6 + \alpha_7 [(\beta_4 + \beta_5)N.M.N + 2\beta_6 I_M] \\ & + \alpha_8 [(\beta_1 + \beta_2)N.T.N + 2\beta_3 I_T] \} I \\ & + \alpha_7\beta_5 M + \alpha_7\beta_4 M^t + \alpha_8\beta_2 T + \alpha_8\beta_1 T^t , \end{aligned} \quad (3.2.25)$$

$$Q^{(4)} = [\alpha_{12} + 2\alpha_{13} \{ (\beta_4 + \beta_5)N.M.N + 2\beta_6 I_M \}] I$$

$$\begin{aligned}
& + [2\alpha_7\beta_3 I_T + \alpha_{10} + 2(\alpha_{11} + \alpha_{13})\beta_6 I_M] N \otimes N \\
& + \alpha_7\beta_1 (N \otimes N.T + N.T \otimes N) \\
& + \alpha_7\beta_2 (N \otimes T.N + T.N \otimes N) \\
& + (\alpha_{11}\beta_4 + \alpha_{13}\beta_5) (N \otimes N.M + N.M \otimes N) \\
& + (\alpha_{11}\beta_5 + \alpha_{13}\beta_4) (N \otimes M.N + M.N \otimes N) \\
& + \alpha_{13}(\beta_4 + \beta_5) (M + M^t) , \tag{3.2.26}
\end{aligned}$$

where  $I$  is the identity tensor and the notation  $\otimes$  denotes the tensor product.

Here, we consider two cases for obtaining the characteristic equation.

Case 1. If the effect of the wryness tensor is negligible in comparison with that of the Cosserat strain tensor, then the constitutive equations (3.2.11) and (3.2.12) take the form

$$\begin{aligned}
T_{KL} = & \alpha_1 \delta_{KL} + \alpha_2 \mathfrak{E}_{LK} + \alpha_3 \mathfrak{E}_{LR} \mathfrak{E}_{RK} + \alpha_4 \mathfrak{E}_{KL} \\
& + \alpha_5 (\mathfrak{E}_{KR} \mathfrak{E}_{LR} + \mathfrak{E}_{RK} \mathfrak{E}_{RL} + \mathfrak{E}_{KR} \mathfrak{E}_{RL}) , \tag{3.2.27}
\end{aligned}$$

$$M_{KL} = \alpha_6 \mathfrak{E}_{KL} + \alpha_8 \mathfrak{E}_{KM} \mathfrak{E}_{ML} + \alpha_9 \delta_{KL} . \tag{3.2.28}$$

Then, the corresponding expressions for  $Q^{(1)}$ ,  $Q^{(2)}$ ,  $Q^{(3)}$ , and  $Q^{(4)}$  become

$$\begin{aligned}
Q^{(1)} = & [\alpha_4 + 3\alpha_5\beta_3 I_T + 2\alpha_5(\beta_1 + \beta_2)N.T.N] I \\
& + [\alpha_2 + 2(\alpha_3 + \alpha_5)\beta_3 I_T] N \otimes N
\end{aligned}$$

$$\begin{aligned}
& + (\alpha_3 + \alpha_5) \beta_1 (N \otimes N.T + N.T \otimes N) \\
& + (\alpha_3 + \alpha_5) \beta_2 (N \otimes T.N + T.N \otimes N) \\
& + \alpha_5 (\beta_1 + \beta_2) (T + T^t) ,
\end{aligned} \tag{3.2.29}$$

$$Q^{(2)} = 0 , \tag{3.2.30}$$

$$\begin{aligned}
Q^{(3)} = & \{ \alpha_6 + \alpha_8 [(\beta_1 + \beta_2) N.T.N + 2\beta_3 I_T] \} I \\
& + \alpha_8 \beta_2 T + \alpha_8 \beta_1 T^t ,
\end{aligned} \tag{3.2.31}$$

$$Q^{(4)} = 0 . \tag{3.2.32}$$

Then, in this case, the equations (2.4.33) and (2.4.34) can be written as

$$Q^{(1)}_{KL} a^{(1)}_L = 0 , \tag{3.2.33}$$

and

$$Q^{(3)}_{KL} a^{(1)}_L = 0 . \tag{3.2.34}$$

Thus, the characteristic equations for this case become

$$\det Q^{(1)} = 0 , \quad \det Q^{(3)} = 0 . \tag{3.2.35}$$

By substituting the linear expressions for  $Q^{(1)}$  and  $Q^{(3)}$  from (3.2.29) and (3.2.31) into (3.2.35), and restricting our analysis to an acoustical tensor, that is a linear function of stress, we obtain

$$\det \mathbf{Q}^{(1)} = \alpha_4 (c - a I_T - b N.T.N) , \quad (3.2.36)$$

where

$$\begin{aligned} a &= -[2\alpha_2\alpha_5(\beta_1 + \beta_2 + 3\beta_3) + 2\alpha_3\alpha_4\beta_3 \\ &\quad + \alpha_4\alpha_5(2\beta_1 + 2\beta_2 + 11\beta_3)] , \\ b &= -(2\alpha_2\alpha_5 + 2\alpha_3\alpha_4 + 8\alpha_4\alpha_5)(\beta_1 + \beta_2) , \\ c &= \alpha_4(\alpha_2 + \alpha_4) , \end{aligned} \quad (3.2.37)$$

and

$$\det \mathbf{Q}^{(3)} = \alpha_6^2 (\alpha_6 - d I_T - e N.T.N) , \quad (3.2.38)$$

where

$$\begin{aligned} d &= -\alpha_8(\beta_1 + \beta_2 + 6\beta_3) , \\ e &= -3\alpha_8(\beta_1 + \beta_2) . \end{aligned} \quad (3.2.39)$$

Equating (3.2.36) and (3.2.38) to zero in accordance with (3.2.35), and requiring that the resulting equations to be consistent, we obtain the characteristic equation for this case:

$$a I_T + b N.T.N = c , \quad (3.2.40)$$

with restrictions on the material moduli given by

$$\frac{a}{c} = \frac{d}{\alpha_6} , \quad \frac{b}{c} = \frac{e}{\alpha_6} . \quad (3.2.41)$$

Case 2. If the effect of the Cosserat strain tensor is negligible in comparison with that of the wryness tensor, then the constitutive equations (3.2.11) and (3.2.12) take the form

$$T_{KL} = \alpha_1 \delta_{KL} + \alpha_6 \Gamma_{LK} + \alpha_7 \Gamma_{MK} \Gamma_{LM} , \quad (3.2.42)$$

$$\begin{aligned} M_{KL} = & \alpha_9 \delta_{KL} + \alpha_{10} \Gamma_{KL} + \alpha_{11} \Gamma_{KM} \Gamma_{ML} + \alpha_{12} \Gamma_{LK} \\ & + \alpha_{13} (\Gamma_{KM} \Gamma_{LM} + \Gamma_{MK} \Gamma_{ML} + \Gamma_{MK} \Gamma_{LM}) . \end{aligned} \quad (3.2.43)$$

Then, the corresponding expressions for  $Q^{(1)}$ ,  $Q^{(2)}$ ,  $Q^{(3)}$ , and  $Q^{(4)}$  become

$$Q^{(1)} = 0 , \quad (3.2.44)$$

$$\begin{aligned} Q^{(2)} = & \{ \alpha_6 + \alpha_7 [(\beta_4 + \beta_5) N.M.N + 2\beta_6 I_M] \} I \\ & + \alpha_7 \beta_4 M + \alpha_7 \beta_5 M^t , \end{aligned} \quad (3.2.45)$$

$$Q^{(3)} = 0 , \quad (3.2.46)$$

$$\begin{aligned} Q^{(4)} = & [ \alpha_{12} + 2\alpha_{13} \{ (\beta_4 + \beta_5) N.M.N + 2\beta_6 I_M \} ] I \\ & + [ \alpha_{10} + 2(\alpha_{11} + \alpha_{13}) \beta_6 I_M ] N \otimes N \\ & + (\alpha_{11} \beta_4 + \alpha_{13} \beta_5) (N \otimes N.M + N.M \otimes N) \\ & + (\alpha_{11} \beta_5 + \alpha_{13} \beta_4) (N \otimes M.N + M.N \otimes N) \end{aligned}$$



$$+ \alpha_{13}(\beta_4 + \beta_5)(\mathbf{M} + \mathbf{M}^t) . \quad (3.2.47)$$

Then, in this case, the equations (2.4.33) and (2.4.34) can be written as

$$J^{-1} Q_{KL}^{(2)} a_L^{(2)} = 0 , \quad (3.2.48)$$

and

$$J^{-1} Q_{KL}^{(4)} a_L^{(2)} = 0 . \quad (3.2.49)$$

Thus, the characteristic equations for this case become

$$\det Q^{(2)} = 0 , \quad \det Q^{(4)} = 0 . \quad (3.2.50)$$

By substituting the linear expressions for  $Q^{(2)}$  and  $Q^{(4)}$  (3.2.45) and (3.2.47) into (3.2.50), and restricting our analysis to an acoustical tensor, that is a linear function of couple stress, we obtain

$$\det Q^{(2)} = \alpha_7^2 (\alpha_7 - p I_M - q N.M.N) , \quad (3.2.51)$$

where

$$\begin{aligned} p &= -\alpha_7 (\beta_4 + \beta_5 + 6\beta_6) , \\ q &= -3\alpha_7 (\beta_4 + \beta_5) , \end{aligned} \quad (3.2.52)$$

and

$$\det \mathbf{Q}^{(4)} = \alpha_{12}(r - s\mathbf{I}_M - t\mathbf{N.M.N}) , \quad (3.2.53)$$

where

$$\begin{aligned} r &= \alpha_{12}(\alpha_{10} + \alpha_{12}) , \\ s &= -[2\alpha_{10}\alpha_{13}(\beta_4 + \beta_5 + 4\beta_6) + 2\alpha_{11}\alpha_{12}\beta_6 \\ &\quad + \alpha_{12}\alpha_{13}(2\beta_4 + 2\beta_5 + 4\beta_6)] , \\ t &= -(2\alpha_{10}\alpha_{13} + 2\alpha_{11}\alpha_{12} + 8\alpha_{12}\alpha_{13})(\beta_4 + \beta_5) . \end{aligned} \quad (3.2.54)$$

Equating (3.2.51) and (3.2.53) to zero in accordance with (3.2.50), and requiring that the resulting equations be consistent, we obtain the characteristic equation for this case:

$$s\mathbf{I}_M + t\mathbf{N.M.N} = r , \quad (3.2.55)$$

with restrictions on the material moduli given by

$$\frac{s}{r} = \frac{p}{\alpha_6} , \quad \frac{t}{r} = \frac{q}{\alpha_6} . \quad (3.2.56)$$

### § 3.3 Plane Stress Loadings

In this section, we will investigate the existence of the shear band for the case of a homogeneous bar subjected to plane stress loadings, with no applied couple stress, as shown in figure 1. Thus, we apply case 1 of the previous section. In this case, we can assume that

$$T_1 = \sigma, \quad T_2 = \xi\sigma, \quad T_3 = 0, \quad (3.3.1)$$

where  $T_K$  ( $K = 1, 2, 3$ ) are the principal stresses,  $\sigma$  is the tension applied on the bar, and  $\xi$  is a parameter determined by the loading condition. We assume that  $\sigma > 0$ , and  $0 \leq \xi < 1$ .

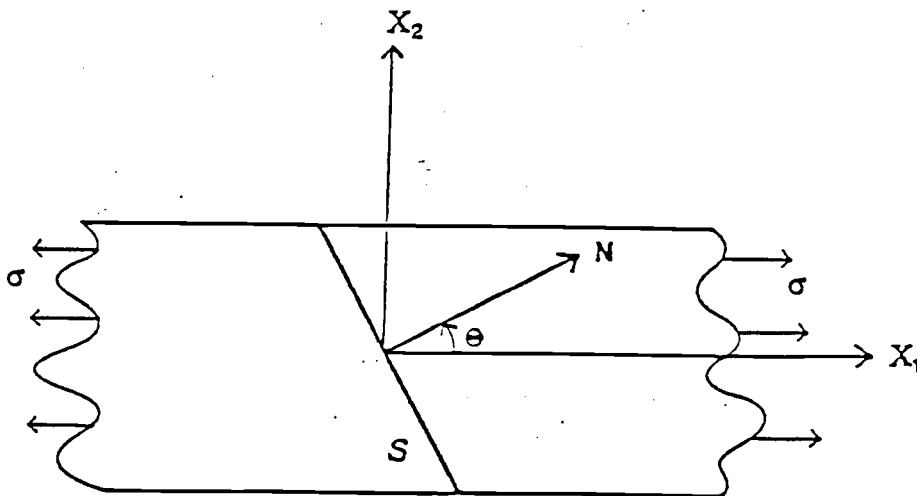


Fig. 1. Plane stress loadings and shear bands

The characteristic equation (3.2.38) can be expressed in this case as

$$f(N) = g(\bar{\sigma}) , \quad (3.3.2)$$

where

$$f(N) = N_1^2 + \xi N_2^2 , \quad (3.3.3)$$

$$g(\bar{\sigma}) = \frac{1}{\bar{\sigma}} - (1 + \xi) \bar{a} , \quad (3.3.4)$$

with

$$\bar{\sigma} = \frac{\sigma}{(c/b)} , \quad \bar{a} = \frac{a}{b} , \quad (3.3.5)$$

where  $\bar{\sigma}$  and  $\bar{a}$  are dimensionless quantities, and  $N_K$  ( $K = 1,2,3$ ) are the components of the unit normal vector  $N$  of the shear band, with the assumption that  $N_3 = 0$ .

We now proceed with our investigation. We note that the function  $f(N)$  in (3.3.2) attains its maximum value 1 when

$$N = (1,0,0) , \quad \theta = 0^\circ , \quad (3.3.6)$$

and its minimum value  $\xi$  when

$$N = (0,\pm 1,0) , \quad \theta = 90^\circ , \quad (3.3.7)$$

since  $N_3 = 0$ .

Let the tension increase from the natural state,  $\bar{\sigma} = 0$ . When the nondimensional tension  $\bar{\sigma}$  takes the value

$$\bar{\sigma}_1 = \frac{1}{(1 + \xi)\bar{a} + 1}, \quad (3.3.8)$$

the function  $f(N)$  takes the maximum value 1 and the state (3.3.6) is realized. That is, the shear band is perpendicular to the direction of the tension.

When the nondimensional tension  $\bar{\sigma}$  takes the value

$$\bar{\sigma}_2 = \frac{1}{(1 + \xi)\bar{a} + \xi}, \quad (3.3.9)$$

the function  $f(N)$  takes its minimum value  $\xi$  and the state (3.3.5) is realized. That is, the shear band is parallel to the direction of the applied tension.

We can express the angle of inclination of the shear band in terms of the applied tension. Thus,

$$\theta = \pm \arcsin \left[ \frac{1 - g(\bar{\sigma})}{1 - \xi} \right]^{\frac{1}{2}}, \quad (3.3.10)$$

where the expression for  $g(\bar{\sigma})$  is given in (3.3.4), and the values of  $\bar{\sigma}$  are restricted to  $\bar{\sigma}_1 \leq \bar{\sigma} \leq \bar{\sigma}_2$ .

Here, we examine the following two cases:

(i) *Uniaxial Tension*. In this case, we have  $\xi = 0$ . Thus,

Table 1. Material Constants

$\mu$	$\lambda$	$\kappa$	$\alpha_3$	$\alpha_5$	$\bar{a}$
1.1907	1.6075	0.1446	29.842	-3.937	1.599

$$\bar{\sigma}_1 = \frac{1}{1 + \bar{a}} , \quad \bar{\sigma}_2 = \frac{1}{\bar{a}} . \quad (3.3.11)$$

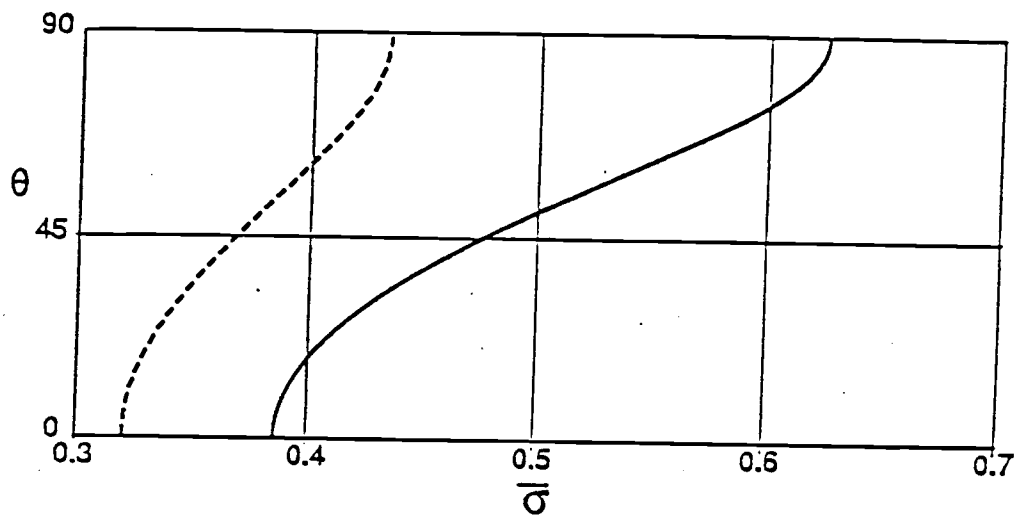
(ii) *Tension of thin plate.* In this case, we can assume that the width of the plate does not change, that is,  $\mathcal{E}_{22} = 0$ .

Then, we obtain

$$\xi = \frac{\lambda}{2\lambda + 2\mu + \kappa} . \quad (3.3.12)$$

The graphs of the angle of inclination  $\theta$  as a function of the nondimensional tension  $\bar{\sigma}$  are obtained for both cases investigated above and the results are presented in figure 2. Table 1 gives the values chosen for the material constants used for obtaining the graphs in figure 2. The experimentally determined values for steel Hecla 37 (0.4%) are used for  $\lambda$  and  $\mu$ , and the choice of the other constants are thermodynamically consistent with the micropolar theory, [Eringen(1976)].

Fig. 2. Inclination angle  $\theta$  versus nondimensional tension  $\bar{\sigma}$  in uniaxial tension (solid line) and in tension of thin plate (dashed line)



## CHAPTER 4

### Scope of Further Work

The present study of the formation of shear bands in isotropic micropolar elastic materials has been limited to isothermal elasticity, that is, the temperature is assumed to be constant throughout the body regardless of the existence of the shear band. In the case of loading of a material under high strain-rates, the localization of deformation occurs along narrow regions of the material. These severe regions of deformation are called adiabatic shear bands because of little time available for the heat generated to diffuse to colder parts of the body. Such shear bands are believed to be precursors to shear fractures, and hence, considerable attention needs to be paid to the study of adiabatic shear bands. Similarly, further work on shear bands should include the investigations of shear bands in plastic materials and composites undergoing deformation. These studies should be given proper attention not only because they are quite interesting research topics by themselves, but also because of their industrial applications. It is important to predict the life of materials with defects experiencing damage under varied loading conditions, especially, in the



presence of shear bands. We believe that micropolar continuum mechanics will be of greater advantage in dealing with such phenomena than the classical continuum mechanics.

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