AN ABSTRACT OF THE THESIS OF

Arabinda Kumar Nandi for the Doctor of Philosophy
(Name) (Degree)

in Electrical Engineering presented on 3/9/71
(Major) (Date)

Title: Characterization and Optimization of Output-Controllable Systems

Redacted for Privacy

Abstract approved:
James H. Herzog

This study is concerned with some aspects of the output-controllability problem associated with the vector differential system

$$(1) \quad \dot{\underline{x}} = \underline{A}(t)\underline{x} + \underline{B}(t)\underline{u}$$

$$(2) \quad \underline{y} = \underline{C}(t)\underline{x}$$

A new characterization of output-controllability for a linear, constant, multivariable system has been obtained based on its canonical structure in the sense of Kalman. Particular attention has been given to the optimization of an output-controllable system by maximizing the determinant of the output-controllability matrix subject to certain constraints. For time-invariant systems, differential equations have been derived to find the optimal observation matrix $\underline{C}$ when the control interval changes. Sensitivity analysis of the optimized system and mean-squared error analysis due to parameter uncertainty have been carried out. Finally, the effect of state feedback on

output-controllability has been briefly considered for a particular class of linear multivariable systems.

CHARACTERIZATION AND OPTIMIZATION OF OUTPUT-CONTROLLABLE SYSTEMS

by

Arabinda Kumar Nandi

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1971

APPROVED:

Redacted for Privacy

Associate Professor of Electrical and
Electronics Engineering

Redacted for Privacy

Head of Department of Electrical and
Electronics Engineering

Redacted for Privacy

Dean of Graduate School

Date thesis is presented 3/9/71

Typed by Erma McClanathan for Arabinda Kumar Nandi

ACKNOWLEDGMENTS

I wish to express my deep appreciation to Dr. J. H. Herzog for his help and encouragement during the course of this investigation. My obligations are many and any attempt at listing them is bound to be futile.

I thank Dr. D. Guthrie, Jr. for many helpful discussions over a period of two years.

It is time to thank all my teachers, most notable of them being Professor L. A. Zadeh of the University of California, Berkeley, who introduced me to mathematical system theory in inimitable style. I also express my gratitude to Professor P. P. Varaiya of the University of California, Berkeley for acting as an (external) adviser when this research was in progress.

My wife, Bina, has made non-technical but essential contribution to this work. Her encouraging words have enabled me to tide over many difficult situations. My parents have been very thoughtful and generous always, and I wish to express my appreciation to them.

Finally, my thanks go to Professors J. Mingle, F. Oberhettinger, J. Saugen and S. Stone for serving on my graduate advisory committee, and Mrs. E. McClanathan for doing a fine typing job.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\underline{a}, \underline{b}, \underline{x}, \underline{y}$	denote column vectors
$\underline{A}, \underline{B}, \underline{N}, \underline{W}$	denote matrices
$\underline{I}$	denotes the identity matrix
$\underline{a}', \underline{W}'$	denote transpose of the vector $\underline{a}$ or matrix $\underline{W}$
$\triangleq$	equals by definition, denotes
$\forall$	for all
$\in$	belongs to
$\{x\}$	set with elements x
$\{x/P\}$	set with elements x possessing property P
$\supset$	contains
$\subset$	is contained in
$< \underline{x} = \underline{x}'$	denotes a row vector
$\underline{x} > = \underline{x}$	denotes a column vector
$< \underline{x}, \underline{y} >$	denotes inner product of vectors $\underline{x}$ and $\underline{y}$
$\Vert \underline{x} \Vert$	(Euclidean) norm of $\underline{x}$
det. $\underline{A} = \lvert \underline{A} \rvert$	determinant of the square matrix $\underline{A}$
(t_1, t_2)	open interval $t_1 < t < t_2$
$[t_1, t_2]$	closed interval $t_1 \leq t \leq t_2$
E^n	space of ordered n-tuples of real numbers with inner product (Euclidean n-space)
$\dot{\underline{x}}$	time derivative of $\underline{x}(t)$
$\underline{W}^{-1}$	denotes inverse of a square matrix $\underline{W}$
f	denotes a function
f(t)	denotes the value of f at time t. This convention is not adhered to strictly
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	implies and is implied by, if and only if
$\equiv$	identically equal over some time interval
E(x)	expectation of a random variable x
$E(x^2)$	variance of a zero-mean random variable x
a.e.	almost everywhere
$\doteq$	approximately equal

CHARACTERIZATION AND OPTIMIZATION OF OUTPUT-CONTROLLABLE SYSTEMS

I. INTRODUCTION

In the theory and application of automatic control, one is often interested in controlling the outputs of a dynamical system in a definite fashion by using suitable inputs. The natural question is whether this can always be done. As will be shown shortly, there are instances where this is not possible. On the other hand, one can have situations where outputs can be manipulated with inputs but the control is rather inefficient. In other words, one may need excessive control effort or control energy to perform the task. The latter situation leads automatically to the question of optimization.

In the present study, attention has been confined to the responses of systems at certain instants of time and not over time intervals. In case of dynamical systems it is usually not possible to change a physical variable instantaneously. Such systems follow the cause and effect rule of Newton and are often mathematically described by a vector differential equation relating the well known physical vector called "state" to the inputs. On solving the differential equation subject to given initial conditions, one gets the state evolution with time. Embedded herein is the Principle of Causality which says in essence that all future behavior of a real physical system is determined by its present state and future inputs. Any information about the past behavior is unnecessary.

The outputs or measurable variables are usually related to the state vector in some known fashion. Assuming this knowledge, the question of output-controllability is intimately connected with that of state-controllability. Part of this work is concerned with an exploitation of this basic idea.

1.1. General Considerations

The systems to be considered are all special cases of those which can be represented by a pair of equations of the form

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t) \tag{1.1a}$$

$$\underline{y}(t) = \underline{g}(\underline{x}(t), \underline{u}(t), t) \tag{1.1b}$$

In these equations, $\underline{x}(t)$ stands for the state vector at time t, $\underline{u}(t)$ stands for the input vector and $\underline{y}(t)$ stands for the output vector. Because of physical considerations and mathematical tractability, some restrictions are imposed on the transformations $\underline{f}$ and $\underline{g}$. These so-called smoothness conditions on $\underline{f}$ and $\underline{g}$ are discussed in detail in Zadeh and Desoer (1) and Desoer (2). In this study it is assumed that $\underline{f}$ and $\underline{g}$ are linear in $\underline{x}$ and $\underline{u}$, so that

$$\underline{\dot{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) \tag{1.1c}$$

$$\underline{y}(t) = \underline{C}(t)\underline{x}(t) + \underline{D}(t)\underline{u}(t) \tag{1.1d}$$

Furthermore, the vector-valued functions of time $\underline{x}$, $\underline{u}$ and $\underline{y}$ are all assumed to be real. Because of this, in (1.1c) and (1.1d), $\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$ and $\underline{D}(t)$ represent real piecewise continuous matrices of appropriate dimensions. In what follows, $\underline{x}$ is n-dimensional, $\underline{u}$ is p-dimensional and $\underline{y}$ is q-dimensional.

With (1.1c) and (1.1d) in mind, the following basic question can be asked: Given any initial state $\underline{x}(t_o) = \underline{x}_o$, and the characterizing matrices $\underline{A}(t)$, $\underline{B}(t)$ and $\underline{C}(t)$ ($\underline{D}(t) = \underline{0}$ in the main body of this research) can any desired output $\underline{y}$ be obtained at some (possibly specified) future time $t_1 > t_o$ by using a suitable control $\underline{u}$ over the time interval $[t_o, t_1]$. By suitable control one generally means a bounded, piecewise continuous vector-valued function defined over $[t_o, t_1]$. By and large, an answer to this question exists in the control literature. Without going into great detail, an example is cited to justify the previous statement that such a control is not always possible. The simple one input-two output system illustrates the point

$$\underline{\dot{x}} = \begin{bmatrix} \alpha & o \\ o & \alpha \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{1.2a}$$

$$\underline{y} = \underline{x} \tag{1.2b}$$

where α is any real scalar and u(t) is an arbitrary real function. If one starts with $\underline{x}_o = \underline{0}$, then for any choice of u(t), the two outputs are such that $y_1 = y_2$ for each t. Thus, an arbitrary $\underline{y}$ cannot be attained.

In the remainder of this chapter, the concepts of controllability, observability and canonical decomposition are discussed. These ideas form essential background material for Chapter II which is concerned with a study of output-controllability and its characterization. The question of quality of output-controllability and its optimization is considered in Chapter III. The optimization procedure applies to both time-invariant and time-varying systems. In Chapter IV, a sensitivity analysis of the optimized system and a mean-squared error analysis due to parameter uncertainty is done. In Chapter V, the effect of state feedback on output-controllability is discussed. Finally, the thesis is concluded with a summary of principal results obtained and suggestions for future research in Chapter VI.

1.2. State-Controllability

Since the early part of 1960, the concept of controllability (and its dual observability) has had profound influence on linear system theory. Even earlier, the mathematical implications of controllability were utilized by the L. S. Pontriagin school of mathematicians in Russia in their celebrated work on optimal control (3).

Similarly, in the U.S.A., J. P. LaSalle used this notion in his study of the time optimal control problem (4). Beginning in 1960, R. E. Kalman and his associates have imparted great physical meaning and mathematical structure to this notion through a series of excellent papers (5, 6, 7, 8). Other definitions of controllability have been proposed by various authors in connection with their study of optimal control problems. For example, Roxin (9) uses the term "attainable" in his work on optimal control of finite dimensional systems not necessarily linear. However, in the engineering world, the definitions and characterizations due to Kalman are well known and are used in this study whenever necessary. One starts out with the state equation (1.1c) that describes the system S.

Definition 1: The system S defined by (1.1c) is completely (state) controllable at time t_o if and only if any state $\underline{x}_o$ at t_o can be transferred to the origin at some finite $t_1 > t_o$ using a suitable control $\underline{u}_{[t_o, t_1]}$. In general, $\underline{u}(t)$ and t_1 may depend on both $\underline{x}_o$ and t_o.

Remarks:

(1) The system S may be completely controllable at t_o and yet may not be completely controllable at $T > t_o$. For time-invariant systems, this difficulty does not arise.

(2) The above definition is concerned with transferring all possible initial states to the origin in a finite length of time. As remarked by Kalman (10), this

definition is most appropriate if one has the linear regulator in mind. The following is an obvious extension of Definition 1 and can be found in Zadeh and Desoer (1), and Kreindler and Sarachik (11).

Definition 2: The system S defined by (1.1c) is completely (state) controllable at time t_o if and only if any state $\underline{x}_o$ at t_o can be transferred to any state $\underline{x}_1$ at some finite $t_1 > t_o$ using a suitable control $\underline{u}_{[t_o,t_1]}$. In general, $\underline{u}(t)$ and t_1 may depend on $\underline{x}_o$, $\underline{x}_1$ and t_o.

As is obvious from Definitions 1 and 2, one always uses the phrase "controllable at time t_o" in connection with general time-varying systems. In some cases, controllability does not depend on the initial time t_o and this leads to the following:

Definition 3: The system S defined by (1.1c) is completely (state) controllable if and only if any state $\underline{x}_o$ at t_o can be transferred to any state $\underline{x}_1$ at some finite $t_1 > t_o$ using a suitable control $\underline{u}_{[t_o,t_1]}$. In this definition t_o does not play a particularly important role.

It is well known (12) that (1.1c) has the solution

$$\underline{x}(t_1) = \underline{\Phi}(t_1,t_o)\underline{x}_o + \int_{t_o}^{t_1} \underline{\Phi}(t_1,t)\underline{B}(t)\underline{u}(t)dt \qquad (1.3a)$$

The (nxn) matrix $\underline{\Phi}(t_1,t_o)$ occurring in (1.3a) is nonsingular for all t_1 and t_o and is usually called the state transition matrix. Furthermore, it satisfies the matrix differential equation

$$\frac{d\,\underline{\Phi}(t_1,t_o)}{dt_1} = \underline{A}(t_1)\,\underline{\Phi}(t_1,t_o),\ \underline{\Phi}(t_o,t_o) = \underline{I},\ \text{the identity matrix}$$

Setting $\underline{x}(t_1) = \underline{x}_1$ and using some standard properties of the state transition matrix (1), one gets

$$-\left[\underline{x}_o - \underline{\Phi}(t_o,t_1)\underline{x}_1\right] = \int_{t_o}^{t_1} \underline{\Phi}(t_o,t)\underline{B}(t)\underline{u}(t)dt \qquad (1.3b)$$

An input $\underline{u}_{[t_o,t_1]}$ that transfers $(\underline{x}_o - \underline{\Phi}(t_o,t_1)\underline{x}_1)$ at t_o to $\underline{0}$ at t_1 also transfers $\underline{x}_o$ at t_o to $\underline{x}_1$ at t_1. In view of this statement, Definitions 1 and 2 are equivalent (1). If t_o and t_1 are fixed, in order that (1.3b) is true for arbitrary $\underline{x}_o$ and $\underline{x}_1 \epsilon E^n$ (the n-dimensional Euclidean space) it is necessary and sufficient that Range L = E^n, where L is the integral map on the right hand side of (1.3b) (2).

The following characterization of (state) controllability due to Kalman (6) and Kalman, Ho and Narendra (8) is both beautiful and powerful.

Assertion 1: The system S is completely controllable at time t_o if and only if the (nxn) symmetric matrix

$$\underline{W}(t_o,t_1) = \int_{t_o}^{t_1} \underline{\Phi}(t_o,t)\underline{B}(t)\underline{B}'(t)\,\underline{\Phi}'(t_o,t)dt \qquad (1.4)$$

is positive definite for some $t_1 > t_o$.

Proof. Let t_1 be the smallest t such that $\underline{W}$ has maximal rank.

$\Leftarrow$ Since $\underline{W}(t_o,t_1)$ is assumed positive definite, the inverse exists. One has to find a control function $\underline{u}(t)$, $t \in [t_o, t_1]$ such that the expression (1.3b) is valid. By choosing in (1.3b)

$$\underline{u}(t) = -\underline{B}'(t)\, \underline{\Phi}'(t_o,t)\underline{W}^{-1}(t_o,t_1)\left[\underline{x}_o - \underline{\Phi}(t_o,t_1)\underline{x}_1\right] \tag{1.5}$$

and using (1.4), the equality is at once established.

$\Longrightarrow$ Let $\underline{W}(t_o,t)$ be positive semidefinite for all $t > t_o$. There is no loss of generality in setting $\underline{x}_1 = \underline{0}$ in (1.3b). Also, let $t = t_2 > t_o$. Let $\underline{x}_2 \neq \underline{0}$ be such that

$$\underline{x}_2' \, \underline{W}(t_o,t_2)\underline{x}_2 = 0 \tag{1.6}$$

Define

$$\underline{u}_2(t) = -\underline{B}'(t)\, \underline{\Phi}'(t_o,t)\underline{x}_2 \tag{1.7}$$

Then

$$\int_{t_o}^{t_2} \| \underline{u}_2(t) \|^2 dt = \int_{t_o}^{t_2} \underline{x}_2' \, \underline{\Phi}(t_o,t)\underline{B}(t)\underline{B}'(t)\, \underline{\Phi}'(t_o,t)\underline{x}_2 dt$$

$$= \underline{x}_2' \, \underline{W}(t_o,t_2)\underline{x}_2$$

$$= 0, \text{ because of } (1.6) .$$

Since $\underline{u}_2(t)$ is piecewise continuous and bounded on $[t_o, t_2]$, it is, therefore, zero almost everywhere on $[t_o, t_2]$. If the plant is completely controllable at t_o, then from Definition 1 there exists an $\underline{u}_1(t)$ such that

$$\underline{x}_2 = -\int_{t_o}^{t_2} \underline{\Phi}(t_o,t)\underline{B}(t)\underline{u}_1(t)dt$$

And,

$$\|\underline{x}_2\|^2 = \int_{t_o}^{t_2} \underline{u}_1'(t)\underline{B}'(t)\ \underline{\Phi}'(t_o,t)\underline{x}_2 dt$$

$$= - \int_{t_o}^{t_2} <\underline{u}_1(t),\underline{u}_2(t)>\ dt, \text{ because of (1.7)}$$

$$= 0 \ (\text{since } \underline{u}_2(t) = \underline{0} \text{ almost everywhere}) \qquad (1.8)$$

Because of (1.8) $\underline{x}_2 = \underline{0}$, which is a contradiction. This contradiction proves the fact that if $\underline{W}(t_o,t)$ is positive semidefinite for all $t > t_o$, the system cannot be completely controllable.

For time-invariant systems, a simpler result is available (6, 8).

Assertion 2: A time-invariant system is completely controllable if and only if

$$\text{Rank}\left[\ \underline{B},\ \underline{A}\ \underline{B},\ \ldots\ldots\ldots,\ \underline{A}^{n-1}\underline{B}\right] = n \qquad (1.9)$$

The brackets in (1.9) denote a composite matrix of n rows and np columns.

Proof. Because of stationarity, controllability does not depend on t_o. One lets $t_o = o$ and t_1 any positive number.

$\Leftarrow$ Let rank of (1.9) be n and yet the system not completely controllable. Then as in the second part of Assertion 1, there exists a $\underline{x}_1 \neq \underline{0}$ such that

$$\underline{x}_1' \, \underline{W}(o,t_1)\underline{x}_1 = 0 \tag{1.10a}$$

or

$$\int_o^{t_1} \underline{x}_1' e^{-\underline{A}t} \, \underline{B} \, \underline{B}' e^{-\underline{A}'t} \, \underline{x}_1 dt = 0 \tag{1.10b}$$

or

$$\int_o^{t_1} \| \underline{B}' e^{-\underline{A}'t} \, \underline{x}_1 \|^2 dt = 0 \tag{1.10c}$$

Since the integrand in (1.10c) is non-negative and continuous, it follows

$$\underline{B}' e^{-\underline{A}'t} \, \underline{x}_1 \equiv \underline{0} \, , \quad 0 \leq t \leq t_1 \tag{1.11}$$

Differentiating (1.11) n-1 times and setting $t = 0$, one gets

$$\underline{B}' (\underline{A}')^k \underline{x}_1 = \underline{0}, \quad k = 0,1,2,\ldots\ldots,n-1 \tag{1.12}$$

Relation (1.12) implies that $\underline{x}_1 \neq \underline{0}$ is orthogonal to all the columns of the matrix in (1.9). The rank of the composite matrix cannot be n as such which is a contradiction.

$\Longrightarrow$ Let the system be completely controllable and yet rank of the composite matrix in (1.9) be less than n. There exists a non-zero vector $\underline{x}_1$ satisfying (1.12). By the Cayley-Hamilton theorem it follows that

$$\underline{B}' \underline{e}^{-\underline{A}'t} \, \underline{x}_1 = \underline{0} \, , \text{ for all t.} \tag{1.13}$$

This implies that

$$\int_o^{t_1} \underline{x}_1' \, e^{-\underline{A}t} \, \underline{B} \, \underline{B}' e^{-\underline{A}'t} \, \underline{x}_1 dt = 0 \tag{1.14a}$$

or

$$\underline{x}_1' \, \underline{W}(o,t_1)\underline{x}_1 = 0 \tag{1.14b}$$

Therefore, $\underline{W}(o,t_1)$ is positive semidefinite. Since $t_1 > 0$ is arbitrary, this contradicts complete controllability.

The notion of controllability as discussed above is important in many ways. Controllability is needed to prove existence of optimal control for linear systems with quadratic performance criterion, as shown by Kalman (5). A stronger form of controllability called "total controllability" was found to be necessary and sufficient condition for the uniqueness of the solution to certain optimal control problems by Kreindler (13). Again, controllability and its dual observability (to be introduced in the next section) is needed to study stability. One also requires some kind of uniformity conditions (Silverman and Anderson, 14). Finally, Roxin (9) and Marcus and Lee (15) have used the idea of controllability in the context of non-linear systems.

1.3. Observability

The concept of observability was introduced as a dual to that of state-controllability by Kalman (5). Other interpretations can be found in Zadeh and Desoer (1) and Kreindler and Sarachik (11).

Again, one starts with the equations (1.1c) and (1.1d) and sets $\underline{u}(t) = \underline{0}$ and $\underline{D}(t) = \underline{0}$.

Definition 4: The free system S is completely observable at time t_o if and only if any state $\underline{x}_o$ at t_o

can be determined uniquely from a knowledge of $\underline{y}(t)$ on $[t_o, t_1]$ for some finite $t_1 > t_o$.

In Definition 4, the phrase "observable at time t_o" has been used. In some cases, observability does not depend on the initial time t_o and this leads to the following:

Definition 5: The free system S is completely observable if and only if any state $\underline{x}_o$ at t_o can be determined uniquely from a knowledge of $\underline{y}(t)$ on $[t_o, t_1]$ for some finite $t_1 > t_o$.

Remarks:

(1) From Definition 4 it can be interpreted that complete observability of S at t_o does not imply complete observability at $T > t_o$.

(2) In the later works of Weiss and Kalman (7), this kind of observability has been called anti-causal observability.

(3) In Definition 5, t_o does not play any significant role.

The following characterization of observability occurs in (7, 11).

Assertion 3: The system S is completely observable at time t_o if and only if the (nxn) symmetric matrix

$$\underline{N}(t_o, t_1) = \int_{t_o}^{t_1} \underline{\Phi}'(t, t_o)\underline{C}'(t)\underline{C}(t)\,\underline{\Phi}(t, t_o)dt \qquad (1.15)$$

is positive definite for some $t_1 > t_o$.

Proof. Given in (11) and is omitted.

For time-invariant systems there exists the following test of observability.

Assertion 4: A constant system is completely observable if and only if

$$\mathrm{Rank}\left[\underline{C}', \underline{A}'\underline{C}', \ldots\ldots (\underline{A}')^{n-1}\underline{C}'\right] = n \tag{1.16}$$

The brackets in (1.16) denote a matrix with n rows and nq columns.

Proof. Given in (1, 11) and is omitted.

Though this discussion of the concept of observability is short, its importance cannot be underestimated. In fact, it is the key notion in problems connected with state determination.

1.4. Canonical Decomposition Theorem

The canonical decomposition theorem or the canonical structure theorem due to Gilbert (16) and Kalman (17) is a powerful tool in linear system theory. Basically it says that at any fixed instant of time, the components of the state vector of a linear dynamical system with respect to a suitable (possibly time-varying) basis or coordinate system can be arranged in four mutually exclusive parts as below:

Part (1): Completely controllable but unobservable

Part (2): Completely controllable and observable

Part (3): Uncontrollable and unobservable

Part (4): Uncontrollable but completely observable

The following version of the canonical structure theorem is due to Kalman (17).

Theorem: Consider the linear dynamical system (1.1c-1.1d).

(i) At every fixed instant t of time, there is a coordinate system in the state space relative to which the state vector can be decomposed into four mutually exclusive parts

$$\underline{x} = (\underline{x}^1, \underline{x}^2, \underline{x}^3, \underline{x}^4)$$

which correspond to the scheme outlined above.

(ii) This decomposition can be achieved in many ways, but the number of state variables $n_1(t), \ldots\ldots n_4(t)$ in each part is the same for any such decomposition.

(iii) Relative to such a choice of coordinates, the system matrices have the canonical form

$$\underline{A}(t) = \begin{bmatrix} A^{11}(t) & A^{12}(t) & A^{13}(t) & A^{14}(t) \\ 0 & A^{22}(t) & 0 & A^{24}(t) \\ 0 & 0 & A^{33}(t) & A^{34}(t) \\ 0 & 0 & 0 & A^{44}(t) \end{bmatrix}; \quad \underline{B}(t) = \begin{bmatrix} B^1(t) \\ B^2(t) \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{C}(t) = \begin{bmatrix} 0 & C^2(t) & 0 & C^4(t) \end{bmatrix}$$

Remarks:

(1) The coordinate system necessary to display the canonical form of $\underline{A}(t)$, $\underline{B}(t)$ and $\underline{C}(t)$ will not be continuous in time unless $n_1(t), \ldots\ldots. n_4(t)$ are constants. For time-invariant systems this difficulty does not arise.

(2) For periodic or analytic systems the dimension numbers $n_1(t), \ldots\ldots. n_4(t)$ are constants and the canonical decomposition is continuous with respect to t.

(3) $n_1(t) + n_2(t) + n_3(t) + n_4(t) = n, \forall t.$

In the next chapter, the time-invariant form of this theorem will be used to establish a characterization of output-controllability for constant systems.

II. OUTPUT-CONTROLLABILITY AND ITS CHARACTERIZATION

The notion of output-controllability is believed to be due to Bertram and Sarachik (18). Later, Kreindler and Sarachik (11) studied the problem fairly thoroughly and obtained many new results. Some different aspects of the output-control problem have been considered by Weiss (19), Sivan (20), Brockett (21), Brockett and Mesarovic (22) and others.

The main result of this chapter is a characterization of output-controllability based on the canonical structure theorem of section 1.4.

2.1. Preliminaries

Recall the equations (1.1c) and (1.1d) defining the system and set $\underline{D}(t) = \underline{0}$. Such plants are often called purely dynamic. The following definitions and characterizations are along the lines of Kreindler and Sarachik (11).

Definition 6: The system S defined by (1.1) is said to be output-controllable on $[t_o, t_1]$ if and only if any final output $\overline{\underline{y}}$ can be attained at time t_1 while starting with arbitrary initial conditions $\underline{x}_o$ at t_o using a suitable control $\underline{u}_{[t_o, t_1]}$.

Assertion 5: The system S is output-controllable on

$[t_o, t_1]$ if and only if the symmetric (qxq) matrix[1]

$$\underline{W}_y(t_o,t_1) = \underline{C}(t_1)\int_{t_o}^{t_1} \underline{\Phi}(t_1,t)\underline{B}(t)\underline{B}'(t)\,\underline{\Phi}'(t_1,t)dt\,\underline{C}'(t_1) \tag{2.1}$$

is positive definite.

Proof. Given in (11) and is omitted.

Remarks: From the form of (2.1) it is obvious that starting at any initial state $\underline{x}_o$ at t_o, the fact that any final output $\bar{\underline{y}}$ can be attained at $t_1 > t_o$ does not imply that $\bar{\underline{y}}$ can be attained at $t_2 > t_1 > t_o$. For time-invariant systems, a simpler check of output-controllability is available (11, 21).

Assertion 6: A time-invariant system is completely output-controllable if and only if

$$\text{Rank}\left[\underline{C}\ \underline{B},\ \underline{C}\ \underline{A}\ \underline{B},\ldots\ldots\ \underline{C}(\underline{A})^{n-1}\underline{B}\right] = q \tag{2.2}$$

The brackets in (2.2) denote a composite matrix of q rows and np columns.

Proof. Given in (11) and is omitted.

2.2. Output-Controllability and Canonical Structure

In this section an attempt has been made to identify the output-controllability problem of a linear, time-invariant system with its canonical structure. Intuitively, one feels that the completely (state) controllable and observable subsystem would have a key role to play here.

---

[1] $\underline{W}_y(t_o,t_1)$ is called the output-controllability matrix.

Consider the time-invariant system representation[2]

$$S: \quad \begin{aligned} \dot{\underline{x}} &= \underline{A}\ \underline{x} + \underline{B}\ \underline{u} && (2.3a) \\ \underline{y} &= \underline{C}\ \underline{x} && (2.3b) \end{aligned}$$

Case 1. Let $t_o = 0$ and $\underline{x}(0) = \underline{0}$. This is usually called the reachability problem. Define

$$\underline{M}_y(0,t_1) = \underline{C}^2 \int_o^{t_1} e^{\underline{A}^{22}(t_1-t)} \underline{B}^2 \underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)} \underline{C}'^2 dt \tag{2.4}$$

Assertion 7: The system S is output-controllable if and only if the (qxq) symmetric matrix $\underline{M}_y(0,t_1)$ in (2.4) is positive definite for $t_1 > 0$.

Proof. One follows the pattern of Kalman (6).

$\Leftarrow$ Let $\underline{M}_y(0,t_1)$ be positive definite at $t_1 > 0$. Let $\underline{y}(t_1) = \bar{\underline{y}}$ be the desired output. Then by choosing

$$\underline{u}(t) = \underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)} \underline{C}'^2 \underline{M}_y^{-1}(0,t_1)\bar{\underline{y}} \tag{2.5}$$

and recalling the fact that $\underline{x}(0) = \underline{0}$ and $\underline{x}^4$ is uncontrollable, one gets

$$\underline{y}(t_1) = \underline{C}^2 \int_o^{t_1} e^{\underline{A}^{22}(t_1-t)} \underline{B}^2 \underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)} \underline{C}'^2 dt\, \underline{M}_y^{-1}(0,t_1)\bar{\underline{y}} \tag{2.6a}$$

$$= \bar{\underline{y}} \tag{2.6b}$$

The last result (2.6) is a consequence of (1.1), (1.3a), (2.4) and section 1.4.

---

[2] In (2.3), $\underline{A}$, $\underline{B}$, $\underline{C}$ are assumed to have the canonical forms.

$\Longrightarrow$ Let $\underline{M}_y(0,t_1)$ be positive semidefinite for all $t_1 > 0$ and yet the system S be completely output-controllable. Then for t_1 (arbitrary) > 0, there exists some $\underline{y}_2 \neq \underline{0}$ (a q-vector) such that

$$\underline{y}_2'\underline{M}_y(0,t_1)\underline{y}_2 = 0 \tag{2.7}$$

Defining,

$$\underline{u}_1(t) = \underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)}\underline{C}'^2\underline{y}_2 \tag{2.8}$$

one gets

$$\int_o^{t_1}\|\underline{u}_1(t)\|^2 dt = \underline{y}_2'\int_o^{t_1}\underline{C}^2 e^{\underline{A}^{22}(t_1-t)}\underline{B}^2\underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)}\underline{C}'^2\underline{y}_2 dt$$

$$= \underline{y}_2'\underline{M}_y(0,t_1)\underline{y}_2 \quad , \text{ by } (2.4)$$

$$= 0 \qquad\qquad , \text{ by } (2.7)$$

Since $\underline{u}_1(t)$ is piecewise continuous and bounded on $[0,t_1]$, it is zero a.e. on $[0,t_1]$. On the other hand, if the plant is completely output-controllable, a control function $\underline{u}_2$ exists such that

$$\underline{y}_2 = \int_o^{t_1}\underline{C}^2 e^{\underline{A}^{22}(t_1-t)}\underline{B}^2\underline{u}_2(t)dt \tag{2.9}$$

and, therefore

$$\|\underline{y}_2\|^2 = \int_o^{t_1}\underline{u}'_2(t)\underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)}\underline{C}'^2\underline{y}_2 dt$$

$$= \int_o^{t_1}\underline{u}'_2(t)\underline{u}_1(t)dt = 0 \tag{2.10}$$

This implies $\underline{y}_2 = 0$, which is a contradiction.

Since this discussion is limited to time-invariant systems, one has as a consequence of Assertion 7 the following simpler test of output-controllability.

Assertion 8: A constant system is completely output-controllable if and only if

$$\text{Rank}\left[\underline{C}^2\underline{B}^2,\ \underline{C}^2\underline{A}^{22}\underline{B}^2,\ldots\ldots\ \underline{C}^2(\underline{A}^{22})^{n_2-1}\underline{B}^2\right] = q \qquad (2.11)$$

The brackets in (2.11) represent a composite matrix of q rows and n_2p columns.

Proof. Because of stationarity, set $t_o = 0$. Let t_1 be any positive number.

$\Leftarrow$ Let the rank of (2.11) be q and yet the system not completely output-controllable. Then, as in the second part of Assertion 7, there exists a $\underline{y}_2 \neq \underline{0}$ such that

$$\underline{y}_2' \underline{M}_y(0,t_1)\underline{y}_2 = 0 \qquad (2.12a)$$

or

$$\underline{y}_2' \int_o^{t_1} \underline{C}^2 e^{\underline{A}^{22}(t_1-t)}\underline{B}^2\underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)}\underline{C}'^2 dt\,\underline{y}_2 = 0 \qquad (2.12b)$$

or

$$\int_o^{t_1} \|\underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)}\underline{C}'^2\underline{y}_2\|^2 dt = 0 \qquad (2.12c)$$

Since the integrand in (2.12c) is non-negative and continuous, it follows

$$\underline{B}'^2 e^{\underline{A}'^{22}(t_1-t)}\underline{C}'^2\underline{y}_2 \equiv \underline{0}\ ,\quad 0 \leq t \leq t_1 \qquad (2.13)$$

Differentiating (2.13) n_2-1 times and letting $t \to t_1$ from the left, one obtains

$$\underline{B}'^2(\underline{A}'^{22})^k\underline{C}'^2\underline{y}_2 = \underline{0},\quad k = 0,1,2,\ldots\ldots n_2-1 \qquad (2.14)$$

From (2.14), it is concluded that $\underline{y}_2 \neq \underline{0}$ is orthogonal to all the columns of the composite matrix on the left hand side of (2.11). Therefore, the rank of the composite matrix cannot be q which is a contradiction.

$\Longrightarrow$ Let the system be completely output-controllable and yet rank of the composite matrix in (2.11) be less than q. There exists a non-zero vector $\underline{y}_2$ satisfying (2.14). By the Cayley-Hamilton theorem it follows that

$$\underline{B}'^{2} e^{\underline{A}'^{22}(t_1-t)} \underline{C}'^{2} \underline{y}_2 \equiv \underline{0} \,, \quad 0 \leq t \leq t_1 \tag{2.15}$$

This implies that

$$\underline{y}_2' \int_0^{t_1} \underline{C}^2 e^{\underline{A}^{22}(t_1-t)} \underline{B}^2 \underline{B}'^{2} e^{\underline{A}'^{22}(t_1-t)} \underline{C}'^{2} \underline{y}_2 dt = 0 \tag{2.16a}$$

or

$$\underline{y}_2' \underline{M}_y(0,t_1) \underline{y}_2 = 0 \tag{2.16b}$$

Therefore, $\underline{M}_y(0,t_1)$ is a positive semidefinite matrix. Since $t_1 > 0$ is arbitrary, this contradicts complete output-controllability.

Case 2. Let $t_o = 0$ and $\underline{x}(0) \neq \underline{0}$. Because of (2.3) and section 1.4, one has

$$\underline{y} = \underline{C}^2 \underline{x}^2 + \underline{C}^4 \underline{x}^4 \tag{2.17}$$

Setting $\underline{u} = \underline{0}$,

$$\underline{x}^4(t) = e^{\underline{A}^{44} t} \underline{x}^4(0) \tag{2.18}$$

and

$$\underline{\dot{x}}^2(t) = \underline{A}^{22}\underline{x}^2 + \underline{A}^{24}\underline{x}^4$$

$$= \underline{A}^{22}\underline{x}^2 + \underline{A}^{24}e^{\underline{A}^{44}t}\underline{x}^4(0) \tag{2.19}$$

Therefore

$$\underline{x}^2(t) = e^{\underline{A}^{22}t}\underline{x}^2(0) + \int_o^t e^{\underline{A}^{22}(t-\sigma)}\underline{A}^{24}e^{\underline{A}^{44}\sigma}d\sigma\underline{x}^4(0) \tag{2.20}$$

Finally, from (2.17)

$$\underline{y}(t) = \underline{C}^2\underline{x}^2 + \underline{C}^4\underline{x}^4$$

$$= \underline{y}_f(t) = \text{free response at time } t$$

It is possible to calculate $\underline{y}_f(t)$ because of (2.18) and (2.20). Thus, in order to move $\underline{y}(0) = \underline{y}_o$ to $\bar{\underline{y}}$ at a subsequent time $t_1 > 0$, one only has to synthesize an input that takes $\underline{y} = \underline{0}$ at time $t = 0$ to $\bar{\underline{y}} - \underline{y}_f(t_1)$ at time $t = t_1$. This problem has already been treated under Case 1.

2.3. Example 1

Consider the canonical form (Example 2 in Kalman (17) modified).

$$\underline{\dot{x}} = \underline{A}\ \underline{x} + \underline{B}\ \underline{u} \tag{2.21a}$$

$$\underline{y} = \underline{C}\ \underline{x} \tag{2.21b}$$

where

$$\underline{A} = \begin{bmatrix} 2 & 4 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Writing $\underline{x}' = [x_1, x_2, x_3, x_4]$, in terms of the notation of section 1.4, one has

$$\text{Part (2)} \begin{cases} x_1 = \text{controllable and observable} \\ x_2 = \qquad \text{"} \end{cases}$$

$$\text{Part (4)} \begin{cases} x_3 = \text{uncontrollable but observable} \\ x_4 = \qquad \text{"} \end{cases}$$

The following partitioning of the $\underline{A}$, $\underline{B}$, $\underline{C}$ matrices is almost immediate

$$\underline{A}^{22} = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \qquad \underline{A}^{24} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \underline{B}^{2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\underline{A}^{42} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underline{A}^{44} = \begin{bmatrix} -3 & -2 \\ 0 & 1 \end{bmatrix} \qquad \underline{B}^{4} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{C}^{2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{C}^{4} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The controllable and observable subspace $\underline{X}^2$ is of dimension two. One forms

$$\underline{R} = \left[\underline{C}^2 \underline{B}^2 ,\ \underline{C}^2 \underline{A}^{22} \underline{B}^2 \right]$$

$$= \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix} \tag{2.22}$$

Since $\underline{R}$ has rank 2, the system defined by (2.21) is not output-controllable.

To develop these ideas further and for the sake of comparison, the standard test of output-controllability (2.2) is applied to the non-canonical representation

$$\dot{\bar{\underline{x}}} = \bar{\underline{A}}\ \bar{\underline{x}} + \bar{\underline{B}}\ \underline{u} \tag{2.23a}$$

$$\underline{y} = \bar{\underline{C}}\ \bar{\underline{x}} \tag{2.23b}$$

Then

$$\underline{A} = \underline{T}\ \bar{\underline{A}}\ \underline{T}^{-1} \tag{2.24a}$$

$$\underline{B} = \underline{T}\ \bar{\underline{B}} \tag{2.24b}$$

$$\underline{C} = \bar{\underline{C}}\ \underline{T}^{-1} \tag{2.24c}$$

In (2.24), $\underline{T}$ is a constant, nonsingular matrix of appropriate dimension that takes (2.23) to (2.21). Details about $\underline{T}$ can be found in Kalman (17). Write

$$\bar{\underline{R}} = \left[\bar{\underline{C}}\ \bar{\underline{B}},\ \bar{\underline{C}}\ \bar{\underline{A}}\ \bar{\underline{B}}, \ldots\ldots.\ \bar{\underline{C}}(\bar{\underline{A}})^{n-1}\bar{\underline{B}} \right]$$

$$= \left[\underline{C}\ \underline{B},\ \underline{C}\ \underline{A}\ \underline{B}, \ldots\ldots.\ \underline{C}(\underline{A})^{n-1}\underline{B} \right], \text{ by (2.24)}$$

In this example, n=4 and using $\underline{A}$, $\underline{B}$, $\underline{C}$ from (2.21), one has

$$\underline{\bar{R}} = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 2 & 3 & 5 & 5 & 9 & 11 & 19 \\ 0 & 1 & 4 & 6 & 4 & 8 & 12 & 20 \end{bmatrix} \qquad (2.25)$$

In (2.25), the rank of $\underline{\bar{R}}$ is 2 which confirms the earlier conclusion that (2.21) or, equivalently, (2.23) is not output-controllable. The first four columns of $\underline{\bar{R}}$ are the same as that of $\underline{R}$ above. The last four columns yield no additional information. Indeed, the last four columns of $\underline{\bar{R}}$ can be generated by forming the matrix

$$\underline{R}_1 = [\,\underline{C}^2(\underline{A}^{22})^2\underline{B}^2,\ \underline{C}^2(\underline{A}^{22})^3\underline{B}^2\,] \qquad (2.26)$$

Furthermore, since $\underline{A}^{22}$ is a (2x2) matrix it follows that

$$(\underline{A}^{22})^2 = 2\,\underline{I} + \underline{A}^{22} \qquad (2.27)$$

$$(\underline{A}^{22})^3 = 2\,\underline{I} + 3\,\underline{A}^{22} \qquad (2.28)$$

Because of (2.26), (2.27) and (2.28), it is concluded that $\underline{\bar{R}}$ contains the same information as $\underline{R}$.

2.4. Remarks

(1) The criterion proposed in the form of Assertion 8 is not intended as a substitute for the standard test of output-controllability (2.2). In order to use (2.11) one needs the canonical form. If the canonical form is available, (2.11) would be simpler than (2.2) in many cases.

(2) The main object of this discussion is to explain the role played by subsystem 2 in the context of the

output-controllability problem. This point of view is not emphasized in (11, 22).

(3) As a consequence of section 1.4 and Assertion 8, it follows that S defined by (2.3) cannot be output-controllable if $n_2 < q (\leq n)$, where q is the number of (independent) rows of $\underline{C}$. This fact is verified in Example 1 above. In the special case of single-input systems, p = 1 and the composite matrix in (2.11) cannot have rank = $q > n_2$.

III. OPTIMIZATION OF AN OUTPUT-CONTROLLABLE SYSTEM OPERATING OVER FINITE INTERVALS

As stated in the introduction, one is often interested not only in control but also in the quality of control. In this chapter, which forms a major part of this work, the above question is pursued in a definite manner to be outlined shortly. To be more specific, a system is obtained that is not only output-controllable on a given interval but also optimal with respect to a certain performance criterion. The performance criterion chosen for this study is the volume of the region (in the output space E^q) that can be reached from the origin by using a fixed amount of control energy. The developments apply to stationary as well as to nonstationary systems albeit with greater difficulty. The last section of this chapter is concerned with time-invariant systems where the relationship between the optimal observation matrix $\underline{C}^*$ and the control interval has been investigated.

3.1. Problem Formulation

For convenience, the state equations for a purely dynamic plant are repeated here

$$\dot{\underline{x}} = \underline{A}(t)\underline{x} + \underline{B}(t)\underline{u} \tag{3.1a}$$

$$\underline{y} = \underline{C}(t)\underline{x} \tag{3.1b}$$

Consider the question of attaining an output $\bar{\underline{y}}$ (an arbitrary q-vector with real components) at time t_1 when the plant is in some initial state $\underline{x}_o$ at time $t_o < t_1$. By Assertion 5 in Chapter II, the real and symmetric matrix $\underline{W}_y(t_o,t_1)$ must be positive definite if this kind of output control is desired. It is to be noted that t_o and t_1 are specified in the problem itself.

Use Equations (1.3a) and (3.1b) to get

$$\bar{\underline{y}} = \underline{y}(t_1) = \underline{C}(t_1)\left[\underline{\Phi}(t_1,t_o)\underline{x}_o + \int_{t_o}^{t_1} \underline{\Phi}(t_1,t)\underline{B}(t)\underline{u}(t)dt\right] \tag{3.2}$$

so that

$$\bar{\underline{y}} - \underline{C}(t_1)\underline{\Phi}(t_1,t_o)\underline{x}_o = \int_{t_o}^{t_1}\underline{C}(t_1)\underline{\Phi}(t_1,t)\underline{B}(t)\underline{u}(t)dt \tag{3.3}$$

Let $\underline{y}_1$ be the output at t_1 due to free motion. Then

$$\underline{y}_1 = \underline{C}(t_1)\underline{\Phi}(t_1,t_o)\underline{x}_o \tag{3.4}$$

Now, by choosing

$$\underline{u}(t) = \underline{B}'(t)\underline{\Phi}'(t_1,t)\underline{C}'(t_1)\underline{W}_y^{-1}(t_o,t_1)(\bar{\underline{y}}-\underline{y}_1) \tag{3.5}$$

and substituting $\underline{u}$ in the right hand side of (3.3), the identity in (3.3) is at once established because of (2.1). Let

$$\begin{aligned}\underline{C}(t_o)\underline{\Phi}(t_o,t_o)\underline{x}_o &= \underline{C}(t_o)\underline{x}_o\\ &= \underline{y}(t_o)\\ &= \underline{y}_o\end{aligned} \tag{3.6}$$

It will be shown that the minimal control energy required for the transfer $\underline{y}_o$ (at t_o) to $\bar{\underline{y}}$ (at t_1) is given by

$$E_{min} = (\bar{\underline{y}}-\underline{y}_1)'\underline{W}_y^{-1}(t_o,t_1)(\bar{\underline{y}}-\underline{y}_1) \tag{3.7}$$

Also, (3.7) is realized by using the control function $\underline{u}(t)$ in (3.5).

Thus, to prove (3.7) one computes

$$\int_{t_o}^{t_1} \|\underline{u}(t)\|^2 \, dt = \int_{t_o}^{t_1} \langle \underline{u}(t), \underline{u}(t) \rangle \, dt$$

$$= (\bar{\underline{y}}-\underline{y}_1)'\,\underline{W}_y^{-1}(t_o,t_1)\underline{W}_y(t_o,t_1)\,\underline{W}_y^{-1}(t_o,t_1)(\bar{\underline{y}}-\underline{y}_1)$$

$$= (\bar{\underline{y}}-\underline{y}_1)'\underline{W}_y^{-1}(t_o,t_1)(\bar{\underline{y}}-\underline{y}_1) = E_{min} \tag{3.8}$$

The expression (3.8) is usually called control energy (Kalman, 6). The fact that (3.7) or (3.8) represents the minimal control energy for the transfer $\underline{y}_o$ to $\bar{\underline{y}}$ is proved below along the lines of Kalman, Ho and Narendra (8).

Let $\underline{u}(t)$ and $\underline{v}(t)$ be two control functions that carry out the desired transfer $\underline{y}_o$ to $\bar{\underline{y}}$. In addition, it is assumed that $\underline{u}(t)$ is defined by (3.5) and $\underline{u}(t) - \underline{v}(t) \neq \underline{0}$ on $[t_o,t_1]$. Write

$$\bar{\underline{y}} - \underline{C}(t_1)\,\underline{\Phi}\,(t_1,t_o)\underline{x}_o = \bar{\underline{y}} - \underline{y}_1$$

$$= \underline{z}_1$$

From (3.3) one gets

$$\underline{z}_1 = \int_{t_o}^{t_1} \underline{C}(t_1)\,\underline{\Phi}\,(t_1,t)\underline{B}(t)\underline{u}(t)dt$$

$$= \int_{t_o}^{t_1} \underline{C}(t_1)\,\underline{\Phi}\,(t_1,t)\underline{B}(t)\underline{v}(t)dt$$

Hence

$$\underline{0} = \int_{t_o}^{t_1} \underline{C}(t_1)\ \underline{\Phi}\ (t_1,t)\underline{B}(t)\left[\underline{v}(t)-\underline{u}(t)\right]dt$$

or

$$0 = \int_{t_o}^{t_1} <\underline{C}(t_1)\ \underline{\Phi}\ (t_1,t)\underline{B}(t)\left[\underline{v}(t)-\underline{u}(t)\right],\ \underline{W}_y^{-1}(t_o,t_1)\underline{z}_1>\ dt$$

$$= \int_{t_o}^{t_1} <\left[\underline{v}(t)-\underline{u}(t)\right],\ \underline{B}'(t)\ \underline{\Phi}'(t_1,t)\underline{C}'(t_1)\underline{W}_y^{-1}(t_o,t_1)\underline{z}_1>\ dt$$

$$= \int_{t_o}^{t_1} <\left[\underline{v}(t)-\underline{u}(t)\right],\ \underline{u}(t)>dt,\ \text{by (3.5)}$$

Define

$$\underline{\sigma}(t) = \underline{u}(t)-\underline{v}(t),\ t_o \leq t \leq t_1$$

Then

$$\int_{t_o}^{t_1} <\underline{v}(t),\underline{v}(t)>\ dt = \int_{t_o}^{t_1} <\underline{u}(t)-\underline{\sigma}(t),\ \underline{u}(t)-\underline{\sigma}(t)>\ dt$$

$$= \int_{t_o}^{t_1} <\underline{u}(t),\underline{u}(t)>\ dt + \int_{t_o}^{t_1} <\underline{\sigma}(t),\underline{\sigma}(t)>\ dt$$

$$> E_{min}$$

3.2. Problem Statement and Solution

For output-controllable systems, expressions have been derived for the minimal control energy required to transfer

$\underline{y}_o$ (at t_o) to $\bar{\underline{y}}$ (at t_1). In the light of the work done by Kalman, Ho and Narendra (8), one can introduce some (scalar) figures of merit ω as a measure of the quality of output-controllability. The natural candidates for ω are (1) the determinant of $\underline{W}_y^{-1}(t_o,t_1)$ and (2) the trace of $\underline{W}_y^{-1}(t_o,t_1)$. In what follows, the case where ω = det.$\underline{W}_y^{-1}(t_o,t_1)$ is considered.

The following question can now be asked: For specified $\underline{A}(t)$, $\underline{B}(t)$, t_o and t_1, what choice of the (qxn) matrix $\underline{C}(t_1)$ would minimize ω? Furthermore, to make the problem meaningful and interesting, the additional restriction is imposed on $\underline{C}(t_1)$ that its rows are independently norm-constrained by unity. Now

$$\begin{aligned} \omega &= \det.\underline{W}_y^{-1}(t_o,t_1) \\ &= 1/\det.\underline{W}_y(t_o,t_1) \end{aligned} \tag{3.9}$$

Therefore, the above minimization problem involving ω is equivalent to the maximization of det.$\underline{W}_y(t_o,t_1)$. To understand the physical implications of minimizing ω, one goes back to (3.7). Let $\underline{x}_o = \underline{0}$. Then for an arbitrary but fixed $\bar{\underline{y}}$

$$E_{min} = \bar{\underline{y}}'\underline{W}_y^{-1}(t_o,t_1)\bar{\underline{y}} \tag{3.10}$$

where

$$\underline{W}_y(t_o,t_1) = \underline{C}(t_1)\underline{W}_x(t_o,t_1)\underline{C}'(t_1) \tag{3.11}$$

and, from (2.1)

$$\underline{W}_x(t_o,t_1) = \int_{t_o}^{t_1} \underline{\Phi}\,(t_1,t)\underline{B}(t)\underline{B}'(t)\,\underline{\Phi}'(t_1,t)dt \qquad (3.12)$$

It is obvious from (3.12) that the (nxn) matrix $\underline{W}_x(t_o,t_1)$ is real, symmetric and, at least, positive semidefinite.

Since (3.10) represents a positive definite quadratic form (23), the ideal solution to the optimization problem lies in the minimization of each (positive) eigenvalue of $\underline{W}_y^{-1}(t_o,t_1)$ by choosing an appropriate $\underline{C}(t_1)$ that satisfies the constraint conditions mentioned before. However, by minimizing ω, one does not minimize the eigenvalues (of $\underline{W}_y^{-1}$) individually but their product. This approach is attractive because of the mathematical simplifications that result and leads to satisfactory optimization in most cases. The principal mathematical tools used in this section are the theory of quadratic forms and the spectral properties of real, symmetric, positive definite (and semidefinite) matrices (Friedman, 23; Bellman, 24; Gantmacher, 25; Courant and Hilbert, 26).

Assuming $\lambda_1,\lambda_2,\ldots\ldots,\lambda_q$ to be the (not necessarily distinct) eigenvalues and $\underline{u}_1,\underline{u}_2,\ldots\ldots,\underline{u}_q$ the orthonormal eigenvectors of $\underline{W}_y^{-1}(t_o,t_1)$, one writes

$$\bar{\underline{y}} = \alpha_1\underline{u}_1 + \ldots\ldots + \alpha_q\underline{u}_q \qquad (3.13)$$

Then, from (3.10)

$$\bar{\underline{y}}'\underline{W}_y^{-1}\bar{\underline{y}} = \alpha_1^2\lambda_1 + \ldots\ldots + \alpha_q^2\lambda_q \qquad (3.14)$$

The boundary of the hyperellipsoid that can be reached using unit control energy is obtained by solving

$$\alpha_1^2\lambda_1 + \,\ldots\ldots + \alpha_q^2\lambda_q = 1 \tag{3.15}$$

The square of the volume V of the q-dimensional ellipsoid is given by

$$V^2 = K / \prod_{r=1}^{q} \lambda_r, \text{ where K is a constant} \tag{3.16}$$

Using the fact

$$\det.\underline{W}_y^{-1}(t_o,t_1) = \prod_{r=1}^{q} \lambda_r \tag{3.17}$$

and, (3.9) gives

$$V^2 = K \det.\underline{W}_y(t_o,t_1) \tag{3.18}$$

The idea behind the proposed optimization is clear from (3.18).

Also, there exists an orthogonal matrix $\underline{R}$ (may depend on t_o and t_1) such that

$$\underline{R}'\underline{W}_y(t_o,t_1)\underline{R} = \underline{\Lambda} \tag{3.19}$$

where $\underline{\Lambda}$ is a diagonal matrix with strictly positive elements. Furthermore,

$$\begin{aligned} \det.\underline{W}_y(t_o,t_1) &= \det.\underline{\Lambda} \\ &= 1 / \prod_{r=1}^{q} \lambda_r \end{aligned} \tag{3.20}$$

Since $\underline{W}_x(t_o,t_1)$ in (3.12) is real, symmetric and has rank $m \geq q$ (otherwise the system would not be output-controllable on $[t_o,t_1]$), there exists a nonsingular, orthogonal matrix $\underline{T}$ (may depend on t_o and t_1) such that

$$\underline{T}'\underline{W}_x\underline{T}=\underline{D} = \begin{bmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_m & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} ; \quad \begin{array}{l} m \geq q \\ d_i \geq d_{i+1} \end{array} \tag{3.21}$$

and

$$\underline{W}_x(t_o,t_1) = \underline{T}\ \underline{D}\ \underline{T}' \tag{3.22}$$

Because of (3.11) and (3.22)

$$\underline{W}_y(t_o,t_1) = \underline{C}(t_1)\underline{T}\ \underline{D}\ \underline{T}'\underline{C}'(t_1) \tag{3.23}$$

Using the notation

$$\underline{C}(t_1) = \begin{bmatrix} <\underline{c}_1 \\ \vdots \\ <\underline{c}_2 \\ \vdots \\ <\underline{c}_q \end{bmatrix} , \quad \underline{T} = \begin{bmatrix} \underline{t}_1> & \underline{t}_2> & \cdots & \underline{t}_n> \\ & & & \\ & & & \end{bmatrix} \tag{3.24}$$

one gets

$$\underline{C}(t_1)\underline{T} = \begin{bmatrix} <\underline{c}_1,\underline{t}_1> & \cdots & <\underline{c}_1,\underline{t}_n> \\ <\underline{c}_2,\underline{t}_1> & \cdots & <\underline{c}_2,\underline{t}_n> \\ \vdots & & \vdots \\ <\underline{c}_q,\underline{t}_1> & \cdots & <\underline{c}_q,\underline{t}_n> \end{bmatrix} \tag{3.25}$$

Therefore

$$\underline{W}_y(t_o,t_1) = \begin{bmatrix} d_1 <\underline{c}_1,\underline{t}_1> \ldots d_m<\underline{c}_1,\underline{t}_m> \\ d_1 <\underline{c}_2,\underline{t}_1> \ldots d_m<\underline{c}_2,\underline{t}_m> \\ \vdots \qquad\qquad \vdots \\ d_1 <\underline{c}_q,\underline{t}_1> \ldots d_m<\underline{c}_q,\underline{t}_m> \end{bmatrix} \begin{bmatrix} <\underline{c}_1,\underline{t}_1> \ldots <\underline{c}_q,\underline{t}_1> \\ <\underline{c}_1,\underline{t}_2> \ldots <\underline{c}_q,\underline{t}_2> \\ \vdots \qquad\qquad \vdots \\ <\underline{c}_1,\underline{t}_n> \ldots <\underline{c}_q,\underline{t}_n> \end{bmatrix} \quad (3.26)$$

It is to be noted that the first matrix on the right hand side of (3.26) has its last (n - m) columns made of zeros which are not shown explicitly. From (3.26)

$$\det.\underline{W}_y(t_o,t_1) = \begin{vmatrix} \sum_{j=1}^{m} d_j <\underline{c}_1,\underline{t}_j>^2 \ldots & \sum_{j=1}^{m} d_j<\underline{c}_1,\underline{t}_j> <\underline{c}_q,\underline{t}_j> \\ & \\ \sum_{j=1}^{m} d_j <\underline{c}_q,\underline{t}_j><\underline{c}_1,\underline{t}_j> \ldots & \sum_{j=1}^{m} d_j<\underline{c}_q,\underline{t}_j>^2 \end{vmatrix}$$

(3.27)

It is well known that the columns of $\underline{T}$ in (3.24) span E^n and are orthogonal. In this application they are assumed to be orthonormal.

Let the rows of $\underline{C}(t_1)$ be norm-constrained as below

$$\|\underline{c}_1\|^2 \leq 1, \|\underline{c}_2\|^2 \leq 1, \ldots\ldots, \|\underline{c}_q\|^2 \leq 1 \quad (3.28)$$

Then the problem is to maximize (3.27) subject to (3.28). It will be shown shortly that the maximizing vectors $\underline{c}_1$, $\underline{c}_2$, ..., $\underline{c}_q$ must all lie on the boundary of the n-dimensional unit hypersphere.

Solution:

(i) Consider the scalar case $q = 1$

Since $\underline{C}(t_1) = \langle\underline{c}_1$ = a row vector, one gets

$$\underline{W}_y(t_o,t_1) = \langle\underline{c}_1,\underline{W}_x(t_o,t_1)\underline{c}_1\rangle \tag{3.29a}$$

$$= \text{a positive scalar}$$

and

$$\underline{W}_y^{-1}(t_o,t_1) = 1/\langle\underline{c}_1,\underline{W}_x(t_o,t_1)\underline{c}_1\rangle \tag{3.29b}$$

In this case whether one optimizes on the basis of det. $\underline{W}_y^{-1}(t_o,t_1)$ or trace $\underline{W}_y^{-1}(t_o,t_1)$, it makes no difference. Because of (3.27) and (3.28), one arrives at the relatively simple problem of maximizing

$$f(\underline{c}_1) = \sum_{j=1}^{m} d_j \langle\underline{c}_1,\underline{t}_j\rangle^2 \tag{3.30}$$

subject to

$$\langle\underline{c}_1,\underline{c}_1\rangle \leq 1 \tag{3.31}$$

In this situation, it is easy to check that the maximizing vector $\underline{c}_1$ does lie on the boundary of the constraint hypersphere.

By way of illustration, let $q = 1$, $m = 3$ and $d_1 \geq d_2 \geq d_3 > 0$. Then the optimal solution vector is $\underline{c}_1^* = \underline{t}_1$.

(ii) Consider the vector output case $q > 1$.

Let $\mathcal{E}$ be the set of all (qxn) matrices $\underline{C}(t_1)$ that satisfy the constraint conditions in (3.28). Then

$$\mathcal{E} = \{\underline{C}(t_1) \mid \|\underline{c}_1\|^2 \leq 1,\ldots\ldots,\|\underline{c}_q\|^2 \leq 1\} \tag{3.32}$$

Also, let

$$\hat{\mathcal{E}} = \{ \underline{C}(t_1) \in \mathcal{E} \mid \underline{W}_y(t_o,t_1) \text{ is nonsingular} \} \tag{3.33}$$

Obviously, $\mathcal{E} \supset \hat{\mathcal{E}}$. It follows that $\hat{\mathcal{E}}$ is nonempty because of the assumption on $\underline{W}_x(t_o,t_1)$ that its rank is at least q. One needs to find a $\underline{C}^*(t_1) \in \hat{\mathcal{E}}$ such that det.$\underline{W}_y(t_o,t_1)$ is a maximum. In connection with the diagonalization shown in (3.21), it is important to recall the conditions

$$m \geq q \tag{3.34a}$$

$$d_1 \geq d_2 \geq \ldots\ldots d_q \geq d_{q+1} \ldots\ldots d_m > 0 \tag{3.34b}$$

Also, if one chooses the vectors $\underline{c}_1, \underline{c}_2, \ldots\ldots, \underline{c}_q$ in the subspace spanned by $\underline{t}_1, \underline{t}_2, \ldots\ldots, \underline{t}_q$, the following orthogonality condition holds

$$\langle \underline{c}_j, \underline{t}_r \rangle = 0, \; j = 1,2,\ldots\ldots,q \text{ and} \qquad r = q+1, q+2, \ldots\ldots m \tag{3.35}$$

Because of (3.35), the summation index m in (3.27) can be replaced by q. After elementary calculations

$$\text{det.}\underline{W}_y(t_o,t_1) = \prod_{k=1}^{q} d_k \begin{vmatrix} \langle \underline{c}_1,\underline{t}_1 \rangle & \ldots\ldots & \langle \underline{c}_1,\underline{t}_q \rangle \\ \langle \underline{c}_2,\underline{t}_1 \rangle & \ldots\ldots & \langle \underline{c}_2,\underline{t}_q \rangle \\ \vdots & & \\ \langle \underline{c}_q,\underline{t}_1 \rangle & \ldots\ldots & \langle \underline{c}_q,\underline{t}_q \rangle \end{vmatrix}^2 \tag{3.36}$$

While the matrix $\underline{C}^*(t_1)$ has not been found yet, it is obvious from the form of (3.36) that the maximizing vectors $\underline{c}_1^*, \underline{c}_2^*, \ldots\ldots, \underline{c}_q^*$ must lie on the boundary of the unit sphere in E^n, i.e.,

$$\| \underline{c}_k^* \| = 1, \; k = 1,2,\ldots\ldots,q \tag{3.37}$$

Now, the elements of the first row in the determinant on the right hand side of (3.36) are the coordinates of the vector $\underline{c}_1$ with respect to an orthonormal basis $\{\underline{t}_j\}_{j=1}^{q}$. Therefore

$$\sum_{j=1}^{q} <\underline{c}_1, \underline{t}_j>^2 = 1 \tag{3.38}$$

and, this is true for $\underline{c}_2, \underline{c}_3, \ldots\ldots, \underline{c}_q$. Using the Hadamard Inequality for determinants (Courant and Hilbert, 26; Bodewig, 27) one concludes

$$\max \; (\det.\underline{W}_y) = \prod_{k=1}^{q} d_k \tag{3.39}$$

From (3.9)

$$\omega_{\min} = \frac{1}{\max(\det.\underline{W}_y)} = \frac{1}{\pi d_k} \tag{3.40}$$

Furthermore, the maximum in (3.39) and the minimum in (3.40) are achieved when the different rows of the determinant in (3.36) are orthogonal. Since $\underline{t}_1, \underline{t}_2, \ldots\ldots, \underline{t}_q$ are fixed, the simplest choice is

$$\begin{array}{c} \underline{c}_1^* = \underline{t}_1 \\ \vdots \quad\quad \vdots \\ \underline{c}_q^* = \underline{t}_q \end{array} \tag{3.41}$$

Remarks:

(1) The choice of the $\underline{c}_k^*$ vectors in (3.41) is not unique.

(2) Linear independence between the rows of the matrix $\overset{*}{\underline{C}}(t_1)$ is accomplished automatically because of the choice given by (3.41).

(3) In case $q = n$, a trivial situation develops. Since $\det.\underline{W}_y = (\det.\underline{C})^2 \det.\underline{W}_x$, for any control interval, $\det.\underline{W}_y$ can be maximized by choosing the rows of the $\underline{C}$ matrix to be (constant) orthonormal vectors. Optimization, in this case, is independent of control interval.

3.3 Example 2

The following example shows how $\omega = \det.\underline{W}_y^{-1}(t_o, t_1)$ can be minimized according to the ideas developed in sections 3.1 and 3.2.

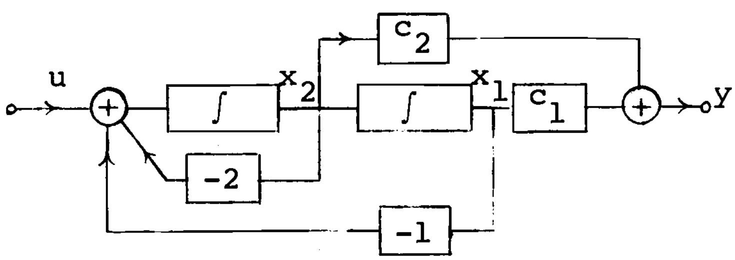


Figure 1.

The state equations read

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.42a}$$

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \underline{x} = \underline{c}'\underline{x} \tag{3.42b}$$

Therefore, by setting $t_o = 0$, one obtains

$$e^{\underline{A}t} = \begin{bmatrix} e^{-t}(1+t) & te^{-t} \\ -te^{-t} & e^{-t}(1-t) \end{bmatrix} \tag{3.43}$$

Since the plant is stationary, it is sufficient to work with $\underline{W}_x(0,1)$. By definition,

$$\underline{W}_x(0,1) = \int_o^1 e^{\underline{A}(1-t)}\underline{b}\,\underline{b}'e^{\underline{A}'(1-t)}dt \tag{3.44}$$

Using $e^{\underline{A}t}$ from (3.43) and $\underline{b}$ from (3.42a), one gets

$$\underline{W}_x(0,1) = \begin{bmatrix} \frac{1}{4} - \frac{5}{4}e^{-2} & \frac{1}{2}e^{-2} \\ \frac{1}{2}e^{-2} & \frac{1}{4} - \frac{e^{-2}}{4} \end{bmatrix} \tag{3.45}$$

From (3.45), it is easy to see that $\underline{W}_x(0,1)$ is a symmetric and positive definite matrix. Furthermore

$$d_1 = \frac{1}{4} - \frac{e^{-2}}{4}(3-2\sqrt{2}) \tag{3.46a}$$

and

$$d_2 = \frac{1}{4} - \frac{e^{-2}}{4}(3+2\sqrt{2}) \tag{3.46b}$$

In this case $d_1 > d_2$. Using standard matrix theory techniques, the (normalized) modal matrix $\underline{T}$ is obtained

$$\underline{T} = \begin{bmatrix} -\frac{1}{\sqrt{2}}\left(1-\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} & \frac{(\sqrt{2}+1)^{\frac{1}{2}}(\sqrt{2})^{\frac{1}{2}}}{2} \\ -\frac{1}{2}\frac{1}{\left(1-\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}} & -\frac{1}{2}\frac{(\sqrt{2})^{\frac{1}{2}}}{(1+\sqrt{2})^{\frac{1}{2}}} \end{bmatrix} \tag{3.47}$$

The two orthonormal columns of $\underline{T}$ are called $\underline{t}_1$ and $\underline{t}_2$.

Also,

$$\underline{T}'\underline{W}_x(0,1)\underline{T} = \begin{bmatrix} \frac{1}{4} - \frac{e^{-2}}{4}(3-2\sqrt{2}) & 0 \\ 0 & \frac{1}{4} - \frac{e^{-2}}{4}(3+2\sqrt{2}) \end{bmatrix} \tag{3.48}$$

In the spirit of (3.30) and (3.31), one can form the following maximization problem

$$\max_{\underline{c},\mu} \left[f(\underline{c},\mu) = \sum_{j=1}^{2} d_j < \underline{c},\underline{t}_j >^2 - \mu(c_1^2+c_2^2-1) \right] \tag{3.49}$$

In (3.49), μ, of course, represents the Lagrange Multiplier. While usual methods can be used to solve (3.49), the problem at hand is readily solvable geometrically. Thus, from Bodewig (27).

$$\max_{\|\underline{c}\| = 1} < \underline{c},\underline{W}_x(0,1)\underline{c} > = d_1 \tag{3.50}$$

and, this maximum in (3.50) is attained by taking $\underline{c}$ along $\underline{t}_1$. Therefore,

$$\underline{c}'^* = \underline{t}_1' = \left[-\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^{\frac{1}{2}} \quad -\frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^{\frac{1}{2}} \right] \tag{3.51}$$

3.4. Control Interval and Optimization

In the previous sections, the problem of maximizing det.$\underline{W}_y(t_o,t_1)$ by an appropriate choice of $\underline{C}(t_1)$ has been considered. Since the optimal $\underline{C}(t_1)$ is a function of the structure of $\underline{W}_x(t_o,t_1)$, one notices the disadvantage of this optimization. In the general time-varying case, $\underline{C}^*$ would depend on both t_o and t_1. In case the state equations are stationary, the optimal $\underline{C}^*$ would depend on $(t_1-t_o) > 0$. This means even in the time-invariant case, the optimal choice of the observation matrix $\underline{C}$ depends on the control interval.

For the time-invariant case, setting $t_o = 0$, the following analysis shows how the eigenvalues and eigenvectors of $\underline{W}_x(0,t_1)$ change with the upper limit t_1. In case the control interval is slightly different from t_1, one can compute the performance degradation in terms of control energy. As will be seen shortly, the eigenvalues and eigenvectors satisfy coupled, first-order nonlinear differential equations. The eigenvalue equation is a scalar one and the eigenvector equation is a vector one. Except in the simplest of cases, these equations can only be solved by numerical techniques. Thus, for fixed coefficient matrices $\underline{A}$ and $\underline{B}$, the optimal observation matrix $\underline{C}^*$ can be found for all possible control intervals.

Recall

$$\underline{W}_x(t_o,t_1) = \int_{t_o}^{t_1} e^{\underline{A}(t_1-\sigma)}\underline{B}\ \underline{B}' e^{\underline{A}'(t_1-\sigma)} d\sigma \tag{3.52}$$

In (3.52), let $t_o = 0$ and $t_1 = t$. Also, let λ_1 be an eigenvalue of $\underline{W}_x(0,t)$ and $\underline{v}_1$ the corresponding (normalized) eigenvector.[3] Then

$$\underline{W}_x\underline{v}_1 = \lambda_1\underline{v}_1 \tag{3.53}$$

[3] This notation is different from that of section 3.2.

Differentiate (3.53) with respect to t and obtain

$$\dot{\underline{W}}_x \underline{v}_1 + \underline{W}_x \dot{\underline{v}}_1 = \dot{\lambda}_1 \underline{v}_1 + \lambda_1 \dot{\underline{v}}_1$$

or

$$(\underline{W}_x - \underline{I}\lambda_1)\dot{\underline{v}}_1 = -\ (\dot{\underline{W}}_x - \underline{I}\dot{\lambda}_1)\underline{v}_1 \tag{3.54}$$

Premultiply both sides of (3.54) by $\underline{v}_1'$ and use the transpose of (3.53) to get

$$\underline{v}_1'(\dot{\underline{W}}_x - \underline{I}\dot{\lambda}_1)\underline{v}_1 = 0 \tag{3.55}$$

Now, by straight differentiation of $\underline{W}_x(0,t)$ one has

$$\dot{\underline{W}}_x = \underline{A}\ \underline{W}_x + \underline{W}_x A' + \underline{B}\ \underline{B}' \tag{3.56}$$

Therefore Equation (3.55) yields

$$\underline{v}_1'(\underline{A}\ \underline{W}_x + \underline{W}_x \underline{A}' + \underline{B}\ \underline{B}' - \dot{\lambda}_1 \underline{I})\underline{v}_1 = 0$$

or

$$\underline{v}_1'\dot{\lambda}_1 \underline{v}_1 = \underline{v}_1'\underline{A}\ \underline{W}_x \underline{v}_1 + \underline{v}_1'\underline{W}_x \underline{A}'\underline{v}_1 + \underline{v}_1'\underline{B}\ \underline{B}'\underline{v}_1$$

or

$$\dot{\lambda}_1 = \lambda_1 \underline{v}_1'(\underline{A}+\underline{A}')\underline{v}_1 + \underline{v}_1'\underline{B}\ \underline{B}'\underline{v}_1 \tag{3.57}$$

So, (3.57) is the differential equation satisfied by λ_1 subject to the initial condition $\lambda_1(0) = 0$. Since at $t = 0$, $\underline{W}_x = \underline{0}$ and $\lambda_1 = 0$, from (3.54) and (3.57) one gets

$$[\dot{\underline{W}}_x - \underline{I}\ \underline{v}_1'\underline{B}\ \underline{B}'\underline{v}_1]\underline{v}_1 = \underline{0} \tag{3.58}$$

Using (3.56) in (3.58) one has

$$[\underline{B}\ \underline{B}' - \underline{I}\ \underline{v}_1'\underline{B}\ \underline{B}'\underline{v}_1]\ \underline{v}_1 = 0 \tag{3.59}$$

From (3.59) it is seen that $\underline{v}_1(0)$ is an eigenvector of $\underline{B}\ \underline{B}'$ corresponding to the eigenvalue $\underline{v}_1'\underline{B}\ \underline{B}'\underline{v}_1$. This is a nonlinear eigenvalue-eigenvector problem. Finally, (3.54) is the differential equation satisfied by $\underline{v}_1$ subject to initial condition given by (3.59).

Illustration: Consider example 2 again. One has

$$\underline{b}\ \underline{b}' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Obviously, the two eigenvalues of $\underline{b}\ \underline{b}'$ are 0 and 1. The linearly independent and orthogonal eigenvectors can be picked by inspection in this case.

$$\underline{v}_1(0) = \text{eigenvector corresponding to } \lambda_1 = 0$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{v}_2(0) = \text{eigenvector corresponding to } \lambda_2 = 1$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note $\underline{v}_1(0)$ and $\underline{v}_2(0)$ satisfy (3.59). Computing $\underline{W}_x(0,t)$ directly, one has

$$\underline{W}_x(0,t) = \begin{bmatrix} \frac{1}{4} - \frac{\delta t^2}{4} - \frac{\delta}{4}(1+t)^2 & \frac{\delta t^2}{2} \\ \frac{\delta t^2}{2} & \frac{1}{4} + t\delta - \frac{t^2\delta}{4} - \frac{\delta}{4}(1+t)^2 \end{bmatrix}$$

In the above matrix $\delta = \exp(-2t)$. Calling

$$\beta = \frac{1}{4} - \frac{\delta t^2}{4} - \frac{\delta}{4}(1+t)^2$$

the eigenvalues of $\underline{W}_x(0,t)$ are given by

$$\lambda_1, \lambda_2 = \beta + \frac{t\delta}{2} \mp \frac{t\delta}{2}\sqrt{1+t^2}$$

Using λ_1, the first column of the (normalized) modal matrix $\underline{T}$ is

$$\underline{v}_1(t) = \begin{bmatrix} \dfrac{1+\sqrt{(1+t^2)}}{(2+2t^2+2\sqrt{1+t^2})^{\frac{1}{2}}} \\ \\ \dfrac{-t}{(2+2t^2+2\sqrt{1+t^2})^{\frac{1}{2}}} \end{bmatrix}$$

as $t \longrightarrow 0$

$$\lim_{t\to 0} \underline{v}_1(t) = \lim_{t\to 0} \begin{bmatrix} \frac{2}{2} \\ -\frac{t}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly, using λ_2, the second column of the modal matrix $\underline{T}$ is

$$\underline{v}_2(t) = \begin{bmatrix} \dfrac{1-\sqrt{(1+t^2)}}{(2-2\sqrt{1+t^2}+2t^2)^{\frac{1}{2}}} \\ \\ \dfrac{-t}{(2-2\sqrt{1+t^2}+2t^2)^{\frac{1}{2}}} \end{bmatrix}$$

as $t \longrightarrow 0$

$$\lim_{t\to 0} \underline{v}_2(t) = \lim_{t\to 0} \begin{bmatrix} -\frac{t^2}{2t} \\ -\frac{t}{t} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

the minus sign with the second component of $\underline{v}_2(0)$ can be ignored without any damage. Therefore, as $t \longrightarrow 0$, the normalized modal matrix becomes

$$\underline{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{I}$$

Conclusion: If one calculates the eigenvectors of $\underline{W}_x(0,t)$ and lets $t \longrightarrow 0$ to find the initial vectors

$\underline{v}_1(0)$ and $\underline{v}_2(0)$, one gets the same result as obtained by solving (3.59).

3.5. Example 3

The following scalar output control problem shows how Equations (3.54) and (3.57) work in a simple case.

Consider

$$\dot{x} = ax + bu \tag{3.60a}$$

$$y = cx, \quad (|c| \leq 1) \tag{3.60b}$$

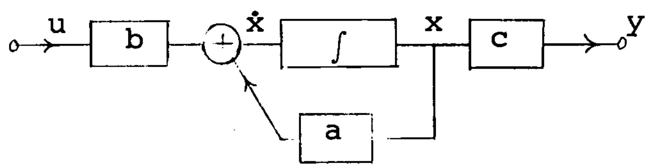


Figure. 2.

$$W_x(0,t) = \int_0^t e^{a(t-\sigma)} bb\, e^{a(t-\sigma)} d\sigma$$

$$= b^2 e^{2at} \int_0^t e^{-2a\sigma} d\sigma$$

$$= \frac{b^2}{2a}(e^{2at}-1) \tag{3.61}$$

For this (1x1) matrix, the eigenvalue of W_x is equal to W_x and $v(t) = 1$ for all t.

Applying Equation (3.57), one has

$$\dot{\lambda}_1 = 2a\lambda_1 + b^2 \ , \ \lambda_1(0) = 0$$

or

$$\lambda_1(t) = \frac{b^2}{2a}(e^{2at}-1) \tag{3.62}$$

Note (3.62) checks with (3.61).

Applying Equation (3.54), one gets

$$\dot{v}(t) = 0, \text{ trivially} \tag{3.63}$$

Therefore

$$v(t) = 1, \text{ for all } t$$

In this case, the optimal $\underline{c}^*$ is independent of control interval.

3.6. Conclusion

As a final point it should be mentioned that the idea for the work reported in this chapter came from Kalman, Ho and Narendra (8) where they introduced various measures of the quality of controllability. In connection with output-controllability, Kreindler and Sarachik (11) have mentioned the possibility of using det.$\underline{W}_y(t_o,t_1)$ as a figure of merit. More recently, the question of quality has received some attention from Brown (28), Monzingo (29), Simon and Mitter (30) and others. However, the associated problem of optimization has been largely ignored. The only significant contribution in this area appears to be that of Johnson (31) who has solved this problem for a constant, single input-single output system using the determinant of the controllability (observability) matrix as a figure of merit.

It is also appropriate to point out that the problems treated in this chapter and also by Johnson (31) are quite different from the standard minimal energy problems considered in Kalman (6), Bertram and Sarachik (18), Friedland (32), Ho (33) and Lee (34).

IV. SENSITIVITY ANALYSIS AND MEAN-SQUARED ERROR DUE TO PARAMETER UNCERTAINTY

In the first section of this chapter, a sensitivity analysis of the optimized system is made. The analysis follows the classical pattern (35) in that change in the value of det.$\underline{W}_y = \Omega$ is computed for small changes in the elements of the $\underline{C}$ matrix. This kind of sensitivity has been called absolute sensitivity by Rohrer and Sobral (36) in contrast to the notion of relative sensitivity which they introduced. Classical sensitivity was utilized by Dorato (37) in discussing the sensitivity problem of optimal control systems due to small variations in plant parameters. Among others, Pagurek (38) has made significant contribution in this area.

The remainder of the chapter is devoted to error analysis due to the uncertain nature of plant parameters. The uncertain parameters are assumed to be random variables with known distributions. In practice, such randomness of coefficients is caused either by slow aging of components, manufacturing tolerances or simply from lack of knowledge of more precise values of the parameters in question. This viewpoint appears to be a realistic one and has been adopted by several authors in recent years (39, 40, 41).

4.1. Sensitivity Analysis

Suppose for known coefficient matrices $\underline{A}(t)$, $\underline{B}(t)$ and given t_o and t_1, the system has been optimized. In other words, the optimal observation matrix $\underline{C}^*$ (subject to the constraints mentioned before) has been found such that $\det.\underline{W}_y(t_o,t_1)$ is a maximum.

Call

$$\max_{\underline{C}\in\hat{\mathcal{E}}} \det.\underline{W}_y(t_o,t_1) = \Omega^* \tag{4.1}$$

The question is, what happens if $\underline{C} \neq \underline{C}^*$?

Case 1. Scalar output q=1. For notational consistence, $\underline{C} = \underline{c}'$, a (1xn) row vector. Let the eigenvalues of $\underline{W}_x(t_o,t_1)$ be ordered and assume rank $\underline{W}_x(t_o,t_1) \geq 1$. Then

$$d_1 \geq d_2 \geq d_3 \geq \cdots\cdots d_n \geq 0 \tag{4.2a}$$

$$d_1 > 0 \tag{4.2b}$$

As shown in Chapter III,

$$\underline{c}^* = \underline{t}_1 \tag{4.3}$$

where $\underline{t}_1$ is the normalized eigenvector of $\underline{W}_x(t_o,t_1)$ corresponding to $d_1 > 0$. And

$$\Omega^* = d_1 \tag{4.4}$$

Now

$$\underline{W}_y(t_o,t_1) = \underline{c}'\underline{W}_x(t_o,t_1)\underline{c}$$

$$= \text{a scalar}$$

Because of (3.22)

$$\underline{W}_y = \underline{c}'\underline{T}\ \underline{D}\ \underline{T}'\underline{c} \tag{4.5}$$

and, one gets

$$\begin{aligned} \Omega &= \det.\underline{W}_y \\ &= d_1 <\underline{c},\underline{t}_1>^2 + d_2 <\underline{c},\underline{t}_2>^2 + \ .....\ + d_n <\underline{c},\underline{t}_n>^2 \end{aligned} \tag{4.6}$$

On replacing $\underline{c}$ by $\underline{c}+\underline{\delta}c$ in (4.6), one has for the change

$$\delta\Omega = 2d_1 <\underline{c},\underline{t}_1> <\underline{\delta}c,\underline{t}_1> + ...+ 2d_n <\underline{c},\underline{t}_n> <\underline{\delta}c,\underline{t}_n> \tag{4.7}$$

$$+ \text{ second-order terms}$$

Imposing the conditions

$$<\underline{c},\underline{t}_j> = \delta_{1j} \quad \text{(Kronecker delta)} \tag{4.8a}$$

$$\| \underline{\delta}c \| \longrightarrow 0 \tag{4.8b}$$

the following first-order approximation is obtained

$$\delta\Omega = 2d_1 <\underline{\delta}c,\underline{t}_1> \tag{4.9}$$

For the single-output case, (4.9) gives the change in the value of Ω^* for a small change $\underline{\delta}c$ in $\underline{c}^*$. It is interesting to observe that $\delta\Omega$ in (4.9) could be positive or negative since one does not require

$$\| \underline{c}^* + \underline{\delta}c \| \leq 1 \tag{4.10}$$

Case 2. Vector output $q > 1$. Using (3.23) and (3.24), one obtains by direct computation

$$\Omega = \det.\underline{W}_y(t_o, t_1) =$$

$$\begin{vmatrix} \sum_{j=1}^{n} <\underline{c}_1,\underline{t}_j>^2 d_j & \sum_{j=1}^{n} <\underline{c}_1,\underline{t}_j><\underline{c}_2,\underline{t}_j> d_j & \cdots & \sum_{j=1}^{n} <\underline{c}_1,\underline{t}_j><\underline{c}_q,\underline{t}_j> d_j \\ \sum_{j=1}^{n} <\underline{c}_1,\underline{t}_j><\underline{c}_2,\underline{t}_j> d_j & \sum_{j=1}^{n} <\underline{c}_2,\underline{t}_j>^2 d_j & \cdots & \sum_{j=1}^{n} <\underline{c}_2,\underline{t}_j><\underline{c}_q,\underline{t}_j> d_j \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} <\underline{c}_1,\underline{t}_j><\underline{c}_q,\underline{t}_j> d_j & \sum_{j=1}^{n} <\underline{c}_2,\underline{t}_j><\underline{c}_q,\underline{t}_j> d_j & \cdots & \sum_{j=1}^{n} <\underline{c}_q,\underline{t}_j>^2 d_j \end{vmatrix} \tag{4.11}$$

Letting $\underline{c}_1 \to \underline{c}_1 + \delta\underline{c}_1$ ($\|\delta\underline{c}_1\| \to 0$), one gets the new expression

$$\Omega + \delta\Omega_1 =$$

$$\begin{vmatrix} \sum_{j=1}^{n} [<\underline{c}_1,\underline{t}_j>^2 + 2<\underline{c}_1,\underline{t}_j><\delta\underline{c}_1,\underline{t}_j>] d_j & \cdots & \sum_{j=1}^{n} [<\underline{c}_1,\underline{t}_j><\underline{c}_q,\underline{t}_j> + <\delta\underline{c}_1,\underline{t}_j><\underline{c}_q,\underline{t}_j>] d_j \\ \vdots & & \vdots \\ \sum_{j=1}^{n} [<\underline{c}_1,\underline{t}_j><\underline{c}_q,\underline{t}_j> + <\delta\underline{c}_1,\underline{t}_j><\underline{c}_q,\underline{t}_j>] d_j & \cdots & \sum_{j=1}^{n} <\underline{c}_q,\underline{t}_j>^2 d_j \end{vmatrix} \tag{4.12}$$

Recalling the condition (3.41)

$$<\underline{c}_i^*, \underline{t}_j> = \delta_{ij} \text{ (Kronecker delta)}, \quad i=1,2,\ldots\ldots,q \quad j=1,2,\ldots\ldots,n$$

(4.12) simplifies to

$$\Omega^*+\delta\Omega_1 = \begin{vmatrix} [<\underline{c}_1^*,\underline{t}_1>^2+2<\underline{\delta c}_1,\underline{t}_1>]d_1 \ldots <\underline{\delta c}_1,\underline{t}_q>< \underline{c}_q^*,\underline{t}_q>d_q \\ \vdots \qquad\qquad \vdots \\ <\underline{\delta c}_1,\underline{t}_q>< \underline{c}_q^*,\underline{t}_q>d_q \ldots\ldots <\underline{c}_q^*,\underline{t}_q>^2 d_q \end{vmatrix}$$

$$= \begin{vmatrix} d_1+2d_1<\underline{\delta c}_1,\underline{t}_1> & d_2<\underline{\delta c}_1,\underline{t}_2> \ldots & d_q< \underline{\delta c}_1,\underline{t}_q> \\ d_2<\underline{\delta c}_1,\underline{t}_2> & d_2 \qquad \ldots & 0 \\ d_3<\underline{\delta c}_1,\underline{t}_3> & 0 \qquad \ldots & 0 \\ \vdots & & \vdots \\ d_q<\underline{\delta c}_1,\underline{t}_q> & 0 \qquad \ldots & d_q \end{vmatrix} \tag{4.13}$$

To evaluate (4.13),

multiply row 2 by $<\underline{\delta c}_1,\underline{t}_2>$ and subtract from row 1

multiply row 3 by $<\underline{\delta c}_1,\underline{t}_3>$ and subtract from row 1

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

multiply row q by $<\underline{\delta c}_1,\underline{t}_q>$ and subtract from row 1

Then

$$\Omega^*+\delta\Omega_1 = \left[d_1+2d_1<\underline{\delta c}_1,\underline{t}_1>-d_2<\underline{\delta c}_1,\underline{t}_2>^2 \ldots -d_q<\underline{\delta c}_1,\underline{t}_q>^2\right]d_2 d_3 \ldots d_q \tag{4.14}$$

Once again assuming that the second-order terms can be ignored, (4.14) reduces to

$$\Omega^{*}+\delta\Omega_1 = [d_1+2d_1 < \underline{\delta} c_1 , \underline{t}_1 >] \; d_2 d_3 \ldots . d_q$$

or

$$\delta\Omega_1 = 2\overset{q}{\underset{k=1}{\pi}} d_k < \underline{\delta} c_1 , \underline{t}_1 > \qquad (4.15)$$

From (4.15) one gets the first-order change $\delta\Omega_1$ in the value of Ω^* when the first row of the $\underline{C}^*$ matrix changes by $\underline{\delta} c_1$.

To derive a general expression for the change $\delta\Omega$ when all the rows of the $\underline{C}^*$ matrix change independently (and, of course, by small amounts) one proceeds as below

$$\Omega = \det.\underline{W}_y = f(\underline{c}_1, \underline{c}_2, \ldots\ldots, \underline{c}_q)$$

$$= \text{a scalar function of q vectors}$$

If the vectors $\underline{c}_1, \underline{c}_2, \ldots\ldots, \underline{c}_q$ change independently, then in terms of the gradients of Ω with respect to $\underline{c}_1$, $\underline{c}_2 \ldots\ldots, \underline{c}_q$, the change in Ω

$$\Delta\Omega = < \underline{\delta} c_1 , \underline{\Delta\Omega}_{\underline{c}_1} > + \ldots\ldots + < \underline{\delta} c_q , \underline{\Delta\Omega}_{\underline{c}_q} > \qquad (4.16)$$

To get the change in Ω^*, (4.16) has to be evaluated at $\underline{C} = \underline{C}^*$. Following the steps used in obtaining (4.15), the general expression is

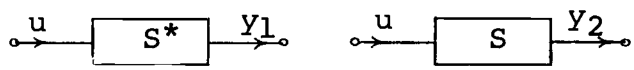
$$\delta\Omega = 2\overset{q}{\underset{k=1}{\pi}} d_k [< \underline{\delta} c_1 , \underline{t}_1 > + < \underline{\delta} c_2 , \underline{t}_2 > \ldots\ldots + < \underline{\delta} c_q , \underline{t}_q >] \qquad (4.17)$$

Finally, (4.17) shows only first-order effects and its goodness is determined by how effectively the second-order terms can be thrown away.

4.2. Mean-Squared Error (Uncertain $\underline{A}$)

In this section, the situation is considered where the $\underline{A}$ matrix is fixed but its knowledge uncertain. It is assumed that the matrix $\underline{A}$ is made up of random variables with known distribution.

Consider the two single input-single output constant systems S* and S



where

$$S^* : \quad \begin{aligned} \dot{\underline{x}} &= \underline{A}^* \underline{x} + \underline{b}u \\ y_1 &= \underline{c}'\underline{x} \end{aligned} \tag{4.18}$$

and

$$S : \quad \begin{aligned} \dot{\underline{x}} &= \underline{A}\ \underline{x} + \underline{b}u \\ y_2 &= \underline{c}'\underline{x} \end{aligned} \tag{4.19}$$

The problem is to reach a specified point $\bar{y}$ in the output space E^1 at $t = t_1$ when starting from zero initial conditions ($\underline{x}_o=\underline{0}$) at $t = t_o < t_1$. Furthermore, it is assumed that

$$\underline{A} = \underline{A}^* + \underline{\delta}a \tag{4.20}$$

where $\underline{\delta}a$ is a (nxn) matrix of random variables with mean 0 and finite variances and $\underline{A}^*$ is the nominal or mean value of $\underline{A}$.

For S^*, in order to get $y_1(t_1) = \bar{y}$, one simply chooses

$$u^*(t) = \underline{b}' e^{\underline{A}'^*(t_1-t)} \underline{c}\ \underline{W}_y^{-1}(t_o,t_1)\bar{y} \tag{4.21}$$

where

$$\underline{W}_y(t_o,t_1) = \underline{c}'\int_{t_o}^{t_1} e^{\underline{A}^*(t_1-t)}\underline{b}\ \underline{b}'e^{\underline{A}'^*(t_1-t)}dt\ \underline{c} \qquad (4.22)$$

In the single-output case, $\underline{W}_y(t_o,t_1)$ given by (4.22) turns out to be a positive real number. For the control function $u^*(t)$ in (4.21), the energy required for the transfer $y_1(t_o)=0 \longrightarrow y_1(t_1) = \bar{y}$ is minimal and given by

$$E_{min} \begin{aligned} &= \bar{y}'W_y^{-1}\bar{y} \\ &= \bar{y}^2/W_y \end{aligned} \qquad (4.23)$$

The energy expression (4.23) can be optimized in the spirit of Chapter III. Thus

$$\min_{\underline{c} \in E^n(\|\underline{c}\| \leq 1)} (E_{min}) = \bar{y}^2/d_1 \qquad (4.24)$$

where d_1 is the largest (positive) eigenvalue of $\underline{W}_x(t_o,t_1)$ and

$$\underline{W}_x(t_o,t_1) = \int_{t_o}^{t_1} e^{\underline{A}^*(t_1-t)}\underline{b}\ \underline{b}'e^{\underline{A}'^*(t_1-t)}dt \qquad (4.25)$$

In order to achieve the minimal E_{min} given by (4.24), one has to choose $\underline{c}$ along the eigenvector of $\underline{W}_x(t_o,t_1)$ corresponding to the largest eigenvalue d_1. Call it $\underline{c}^*$. It is understood that $\underline{c}^*$ has unit length. In what follows, it is assumed that in (4.18), (4.19), (4.21) and (4.22)

$$\underline{c} = c^* \qquad (4.26)$$

Suppose now one uses the control function $\overset{*}{u}(t)$ in

(4.21) to steer the output y_2 of S. Then

$$y_2(t_1) = \underline{c}'^* \int_{t_o}^{t_1} e^{\underline{A}(t_1-t)} \underline{b}\ \underline{b}' e^{\underline{A}^*(t_1-t)} \underline{c}^* \underline{W}_y^{-1}(t_o,t_1) dt\ \bar{y} \tag{4.27}$$

In order to simplify (4.27), the following assumptions are made which are somewhat restrictive

1. $\underline{A}^*$ and $\underline{\delta}a$ commute
2. $\|\underline{\delta}a\| << \|\underline{A}^*\|$ almost surely. $\|\underline{A}\|^2 = \sum_{ij}\sum a_{ij}^2$
3. Elements of $\underline{\delta}a$ are independent random variables with mean zero and finite variances.

Because of assumptions 1 and 2 and (4.20)

$$e^{\underline{A}(t_1-t)} = \underline{I}+(t_1-t)\underline{A}+(t_1-t)^2\underline{A}^2/2!+\ldots\ldots+(t_1-t)^n\underline{A}^n/n!+\ldots$$

$$\doteq \left[\underline{I}+(t_1-t)\underline{A}^*+(t_1-t)^2\underline{A}^{*2}/2!+\ldots+(t_1-t)^n\underline{A}^{*n}/n!+\ldots\right]$$

$$+\left[(t_1-t)\underline{\delta}a+(t_1-t)^2\underline{A}^*\underline{\delta}a+(t_1-t)^3\underline{A}^{*2}/2!\ \underline{\delta}a+\ldots\right]$$

In the second bracket on the right hand side, assumption 1 has been used and all higher powers of $\underline{\delta}a$ have been ignored in view of assumption 2. Therefore

$$e^{\underline{A}(t_1-t)} \doteq e^{\underline{A}^*(t_1-t)}\left[\underline{I} + \underline{\delta}a(t_1-t)\right] \tag{4.28}$$

An approximate expression similar to (4.28) has been used by Farison (42) in a different context.

Using (4.28) in (4.27) one has

$$y_2(t_1) = \underline{c}'^{*}\int_{t_o}^{t_1} e^{\underline{A}^{*}(t_1-t)}\left[\underline{I}+\underline{\delta}a(t_1-t)\right]\underline{b}\ \underline{b}'e^{\underline{A}'^{*}(t_1-t)}$$

$$\underline{c}^{*}\underline{W}_y^{-1}(t_o,t_1)\bar{y}\ dt$$

and

$$y_2(t_1)-\bar{y} = \underline{c}'^{*}\underline{\delta}a\left[\int_{t_o}^{t_1} e^{\underline{A}^{*}(t_1-t)}(t_1-t)\underline{b}\ \underline{b}'e^{\underline{A}'^{*}(t_1-t)}dt\right]\underline{c}^{*}\bar{y}/d_1 \tag{4.29}$$

Note from (4.22) and (4.25)

$$W_y(t_o,t_1) = \underline{c}'^{*}\underline{W}_x(t_o,t_1)\underline{c}^{*}$$

Calling $\underline{\gamma}$ the (nxn) matrix inside brackets in (4.29), one has

$$\underline{\gamma} = \int_{t_o}^{t_1} e^{\underline{A}^{*}(t_1-t)}(t_1-t)\underline{b}\ \underline{b}'e^{\underline{A}'^{*}(t_1-t)}dt \tag{4.30}$$

Then

$$y_2(t_1)-\bar{y} = \underline{c}'^{*}\underline{\delta}a\ \underline{\gamma}\underline{c}^{*}\bar{y}/d_1 \tag{4.31}$$

Square (4.31) to obtain

$$\begin{aligned}(y_2(t_1)-\bar{y})^2 &= \bar{y}\underline{c}'^{*}\underline{\gamma}'\underline{\delta}a'(\underline{c}^{*}\underline{c}'^{*})\underline{\delta}a\ \underline{\gamma}\underline{c}^{*}\bar{y}/d_1^2 \\ &= \bar{y}\underline{c}'^{*}\underline{\gamma}'(\underline{\delta}a'\mathscr{C}\ \underline{\delta}a)\underline{\gamma}\underline{c}^{*}\bar{y}/d_1^2;\ (\underline{c}^{*}\underline{c}'^{*}) = \mathscr{C}\end{aligned} \tag{4.32}$$

Take the expectation of (4.32) to get

$$E(y_2(t_1)-\bar{y})^2 = \bar{y}\underline{c}'^{*}\underline{\gamma}'E(\underline{\delta}a'\mathscr{C}\ \underline{\delta}a)\underline{\gamma}\underline{c}^{*}\bar{y}/d_1^2 \tag{4.33}$$

In (4.33), the matrix $\underline{\gamma}$ can be found numerically or otherwise. $\underline{\delta}a$ and $\mathscr{C}$ are (nxn) matrices. Writing

$$\underline{\delta} a = \begin{bmatrix} \underline{\alpha}_1 & \cdot & \cdot & \cdot & \cdot & \underline{\alpha}_n \\ & & & & & \\ & & & & & \end{bmatrix}, \text{ a matrix of n cols.}$$

one gets

$$\underline{\delta} a' \mathcal{C}\ \underline{\delta} a = \begin{bmatrix} \underline{\alpha}_1' \mathcal{C}\, \underline{\alpha}_1 & \underline{\alpha}_1' \mathcal{C}\, \underline{\alpha}_2 \cdot & \cdot & \cdot & \cdot\ \underline{\alpha}_1' \mathcal{C}\, \underline{\alpha}_n \\ \underline{\alpha}_2' \mathcal{C}\, \underline{\alpha}_1 & \underline{\alpha}_1' \mathcal{C}\, \underline{\alpha}_2 \cdot & \cdot & \cdot & \cdot\ \underline{\alpha}_2' \mathcal{C}\, \underline{\alpha}_n \\ & & & & \\ \underline{\alpha}_n' \mathcal{C}\, \underline{\alpha}_1 & \underline{\alpha}_n' \mathcal{C}\, \underline{\alpha}_2 \cdot & \cdot & \cdot & \cdot\ \underline{\alpha}_n' \mathcal{C}\, \underline{\alpha}_n \end{bmatrix} \quad (4.34)$$

Taking expectation of (4.34) one has

$$E(\underline{\delta} a' \mathcal{C}\ \underline{\delta} a) = \begin{bmatrix} E(\underline{\alpha}_1' \mathcal{C}\, \underline{\alpha}_1) & & & \\ & E(\underline{\alpha}_2' \mathcal{C}\, \underline{\alpha}_2) & & \\ & & \cdot & \\ & & & \cdot \\ & & & & E(\underline{\alpha}_n' \mathcal{C}\, \underline{\alpha}_n) \end{bmatrix} \quad (4.35)$$

In (4.35), all off-diagonal terms are necessarily zero because of assumption 3.

Using the fact that for any square matrix $\underline{A}$

$$\underline{x}' \underline{A}\ \underline{x} = \text{Trace}\ \underline{A} (\underline{x}\ \underline{x}')$$

one obtains from (4.35)

$$E(\underline{\delta} a' \mathcal{C} \underline{\delta} a) = \begin{bmatrix} \text{Trace}\, \mathcal{C}\, E(\underline{\alpha}_1 \underline{\alpha}_1') & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \text{Trace}\, \mathcal{C}\, E(\underline{\alpha}_n \underline{\alpha}_n') \end{bmatrix} \quad (4.36)$$

Let

$$E(\underline{\alpha}_k\underline{\alpha}_k') = \underline{\beta}_k,\ k = 1,2,\ldots\ldots,n$$
$$= (n\mathrm{x}n) \text{ covariance matrix (43)}$$

Then the diagonal matrix on the right hand side of (4.36) can be written in terms of $\underline{\beta}_1, \underline{\beta}_2, \ldots\ldots, \underline{\beta}_n$.

Finally, letting

$$y_2(t_1) - \bar{y} = \epsilon_{\bar{y}}$$
$$= \text{error in the output of S at } t = t_1$$

one notes $\epsilon_{\bar{y}}$ is a random variable of mean zero. Also, for fixed $\bar{y}$, the expression (4.33) gives the variance of $\epsilon_{\bar{y}}$ or, equivalently, the mean-squared error due to $\underline{A}$ matrix tolerance. Applying the Chebychev Inequality (44)

$$\Pr[\,|\epsilon_{\bar{y}}| \geq \sigma > 0] \leq \underline{c}'^{*}\underline{\gamma}'E(\underline{\delta}a'\mathscr{C}\,\underline{\delta}a)\underline{\gamma}\underline{c}^{*}\bar{y}^{2} / \sigma^{2} d_1^{2} \qquad (4.37)$$

Under very special conditions, for example when $\underline{W}_x(t_o,t_1)$ has repeated eigenvalues, several $\underline{c}$ vectors may be available which are all solutions to the optimization problem. Then the one that minimizes (4.37) could be picked. In general, sharp results cannot be expected through the use of the Chebychev Inequality.

4.3. Mean-Squared Error (Uncertain $\underline{b}$)

Finally, the case is considered where $\underline{b}$ is a random vector with known distribution.

Again the two single input-single output constant systems S* and S are considered. These are defined by

$$S^*: \quad \begin{aligned} \dot{\underline{x}} &= \underline{A}\ \underline{x} + \underline{b}^* u \\ y &= \underline{c}'\underline{x} \end{aligned} \tag{4.38}$$

$$S\ : \quad \begin{aligned} \dot{\underline{x}} &= \underline{A}\ \underline{x} + \underline{b}u \\ y &= \underline{c}'\underline{x} \end{aligned} \tag{4.39}$$

In this section it is assumed that

$$\underline{b} = \underline{\delta b} + \underline{b}^* \tag{4.40}$$

where $\underline{\delta b}$ is a column of independent random variables with mean zero and finite variances and $\underline{b}^*$ is the nominal or mean value of $\underline{b}$.

As before, the problem is to reach a specified point $\bar{y}$ in the output space E^1 at $t = t_1$ when starting from zero initial conditions ($\underline{x}_o=\underline{0}$) at $t = t_o \leqslant t_1$.

For S^*, in order to get $y_1(t_1) = \bar{y}$, one chooses

$$u^*(t) = \underline{b}'^* e^{\underline{A}'(t_1-t)}\underline{c}\ \underline{W}_y^{-1}(t_o,t_1)\bar{y} \tag{4.41}$$

with

$$\underline{W}_y(t_o,t_1) = \underline{c}'\int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}\underline{b}^*\underline{b}'^* e^{\underline{A}'(t_1-t)}\underline{c}\ dt \tag{4.42}$$

As discussed in the previous section, in order to optimize S^* in the sense of control energy, one chooses $\underline{c}$ along the eigenvector of $\underline{W}_x(t_o,t_1)$ corresponding to the largest positive eigenvalue d_1 and

$$\underline{W}_x(t_o,t_1) = \int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}\underline{b}^*\underline{b}'^* e^{\underline{A}'(t_1-t)}dt \tag{4.43}$$

Therefore, for the transfer $y_1(t_o) = 0 \longrightarrow y_1(t_1) = \bar{y}$

$$\min_{\underline{c}\in E^n(\|\underline{c}\|\leq 1)} (E_{min}) = \bar{y}^2/d_1 \tag{4.44}$$

and, as usual (see previous section)

$$\underline{c} = \underline{c}^* \tag{4.45a}$$

$$\|\underline{c}^*\| = 1 \tag{4.45b}$$

In the following discussion it will be assumed that in (4.38), (4.39), (4.41), and (4.42) the condition (4.45) applies.

If one uses $u^*(t)$ in (4.41) to steer the output y_2 of S, then

$$y_2(t_1) = \underline{c}'\int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}\underline{b}u^*(t)dt$$

or

$$y_2(t_1) = \underline{c}'\int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}(\underline{b}^*+\underline{\delta}b)\underline{b}'^* e^{\underline{A}'(t_1-t)}\underline{c}\ \underline{W}_y^{-1}\bar{y}\ dt$$

$$= \bar{y}+\underline{c}'\int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}\underline{\delta}b\left[\underline{b}'^* e^{\underline{A}'(t_1-t)}\underline{c}\right]dt\ \bar{y}/d_1$$

or

$$y_2(t_1)-\bar{y} = \underline{c}'\int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}\left[\underline{b}'^* e^{\underline{A}'(t_1-t)}\underline{c}\right]dt\underline{\delta}b\ \bar{y}/d_1$$

Note

$$\underline{b}'^* e^{\underline{A}'(t_1-t)}\underline{c} = \alpha(t) \tag{4.46}$$

$$= \text{a scalar function of time}$$

Let

$$y_2(t_1)-\bar{y} = \epsilon_{\bar{y}} \tag{4.47}$$

$$= \text{a random variable with mean zero}$$

Then

$$\epsilon_{\bar{y}} = \underline{c}'\int_{t_o}^{t_1} e^{\underline{A}\,(t_1-t)}\alpha(t)dt\,\underline{\delta}b\,\bar{y}/d_1$$

$$= \underline{c}'\underline{S}'\underline{\delta}b(\bar{y}/d_1) \tag{4.48}$$

where

$$\underline{S}' = \int_{t_o}^{t_1} e^{\underline{A}(t_1-t)}\alpha(t)dt \tag{4.49}$$

$$= \text{a (nxn) matrix which is linear in } \underline{c}$$

Square (4.48) to obtain

$$\epsilon_{\bar{y}}^2 = \underline{c}'\underline{S}'(\underline{\delta}b\,\underline{\delta}b')\underline{S}\;\underline{c}(\bar{y}/d_1)^2 \tag{4.50}$$

Take the expectation of (4.50) to get

$$E\epsilon_{\bar{y}}^2 = \underline{c}'\underline{S}'E(\underline{\delta}b\underline{\delta}b')\underline{S}\;\underline{c}(\bar{y}/d_1)^2 \tag{4.51}$$

Calling

$E(\underline{\delta}b\underline{\delta}b') = \underline{B}$, the (nxn) covariance matrix

one has

$$E\epsilon_{\bar{y}}^2 = (\underline{S}\;\underline{c})'\underline{B}(\underline{S}\;\underline{c})(\bar{y}/d_1)^2 \tag{4.52}$$

Finally, (4.52) gives the mean-squared error due to uncertainty in $\underline{b}$. No critical assumptions were made in deriving it and as such it is an exact expression.

V. OUTPUT-CONTROLLABILITY AND FEEDBACK

For linear time-invariant systems, the relationship between output-controllability and feedback deserves some attention. A topic of great current interest is to control the dynamics of a plant through state or output feedback (21, 45, 46, 47).

It is well known that if one starts with a controllable and observable system and applies state feedback, the feedback system need not be completely observable (21). The obvious question is: "What happens to output-controllability under state or output feedback?" Surprisingly, very little can be found in the literature in this context (48). In view of the preceding statement about loss of observability and developments in Chapter II, it appears that output-controllability (especially in the case of multi-output systems) need not be feedback invariant. In what follows, an answer to this question is provided for a particular class of systems. In this case, it is shown that output-controllability is <u>not</u> affected by (state) feedback.

5.1. Effect of State Feedback

Consider a single-input, multi-output (state) controllable system

$$S : \begin{cases} \dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{b}u & (5.1a) \\ \underline{y} = \underline{C}\ \underline{x} & (5.1b) \end{cases}$$

where $\underline{x}$ is an n-vector, u is a scalar and $\underline{y}$ is a q-vector.

Theorem: S is output-controllable under state feedback if and only if S is output-controllable under no feedback.

Before proving the theorem, the following preliminary result due to Tuel (49) is recalled.

Preliminary result: Since the pair ($\underline{A}$, $\underline{b}$) in (5.1a) is controllable, there exists a nonsingular transformation $\underline{T}$ that takes (5.1a) to the phase-variable form

$$\bar{S} : \begin{aligned} \dot{\bar{\underline{x}}} &= \bar{\underline{A}}\ \bar{\underline{x}} + \bar{\underline{b}}u \qquad (5.2a) \\ \underline{y} &= \bar{\underline{C}}\ \bar{\underline{x}} \qquad (5.2b) \end{aligned}$$

with

$$\bar{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & & -\alpha_1 \end{bmatrix}, \quad \bar{\underline{b}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \qquad (5.3)$$

The elements of the last row of $\bar{\underline{A}}$ are obtained from the characteristic polynomial of $\underline{A}$ in (5.1a).

Without loss of generality, it is assumed that $\underline{A}$ and $\underline{b}$ in (5.1a) have the form in (5.3). In case of state feedback, i.e., with u replaced by u - $\underline{k}'\underline{x}$, where $\underline{k}'$ is an arbitrary (lxn) row vector with real elements, (state) controllability is preserved.

To show this, one considers the feedback state equation

$$\dot{\underline{x}} = (\underline{A} - \underline{b}\,\underline{k}')\underline{x} + \underline{b}u \tag{5.4}$$

It is easy to check the controllability of the pair $(\underline{A} - \underline{b}\,\underline{k}', \underline{b})$ by forming the matrix

$$\underline{P} = \left[\underline{b}, (\underline{A}-\underline{b}\,\underline{k}')\underline{b}, \ldots\ldots, (\underline{A}-\underline{b}\,\underline{k}')^{n-1}\underline{b}\right] \tag{5.5}$$

and finding its rank. Let

$$\underline{k}' = \left[k_n, k_{n-1}, \ldots\ldots, k_2, k_1\right] \tag{5.6}$$

Because of the special forms of $\underline{A}$ and $\underline{b}$ given by (5.3), the columns of the matrix $\underline{P}$ are easily obtained by direct computation and are found to be linearly independent. In other words, the matrix $\underline{P}$ is nonsingular. Thus, for arbitrary $\underline{k}'$, the pair $(\underline{A} - \underline{b}\,\underline{k}', \underline{b})$ is completely controllable. Therefore,

$$(\underline{A}, \underline{b}) \text{ c. c.} \iff (\underline{A} - \underline{b}\,\underline{k}', \underline{b}) \text{ c. c.}$$

where c. c. $\triangleq$ completely controllable.

The main theorem can now be proved. Define

$$\bar{\underline{R}} = \left[\underline{C}\,\underline{b}, \underline{C}\,\underline{A}\,\underline{b}, \ldots\ldots, \underline{C}\,\underline{A}^{n-1}\underline{b}\right] = \underline{C}\,\bar{\underline{P}} \tag{5.7}$$

and

$$\underline{R} = \left[\underline{C}\,\underline{b}, \underline{C}(\underline{A}-\underline{b}\,\underline{k}')\underline{b}, \ldots\ldots, \underline{C}(\underline{A}-\underline{b}\,\underline{k}')^{n-1}\underline{b}\right] = \underline{C}\,\underline{P} \tag{5.8}$$

On account of the preliminary result, both $\bar{\underline{P}}$ and $\underline{P}$ are nonsingular.

Proof: $\bar{\underline{P}}$ has full rank n $\iff$ $\underline{P}$ has full rank n

$\underline{C}\,\bar{\underline{P}} = \bar{\underline{R}}$ has rank q $\iff$ $\underline{C}\,\underline{P} = \underline{R}$ has rank q

where q ($\leq$ n) is the number of rows of $\underline{C}$. Because of the assumptions on (5.1)

S is output-controllable under feedback

$\Longleftrightarrow$ S is output-controllable under no feedback

The next example shows how output-controllability remains invariant under feedback and what happens to the controllable observable subspace which plays a vital role in the output-controllability problem as discussed in Chapter II.

5.2. Example 4

$$S : \begin{array}{l} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}u \\ \underline{y} = \underline{C}\,\underline{x} \end{array}$$

Let

$$\underline{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

S is completely controllable and observable. It is also output-controllable. Consider state feedback with $\underline{k}' = [0, 0, 3]$. Now

$$\underline{A} - \underline{b}\,\underline{k}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$\underline{Q} = \begin{bmatrix} \underline{C} \\ \underline{C}(\underline{A}-\underline{b}\,\underline{k}') \\ \underline{C}(\underline{A}-\underline{b}\,\underline{k}')^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Note the observability matrix $\underline{Q}$ has rank 2. The feedback system is unobservable. The unobservable vector is $\underline{z}' = [0.\ 0.\ \alpha]$, $\alpha \neq 0$. The dimension of the controllable and observable subspace has been reduced from <u>three</u> <u>to</u> <u>two</u> and cannot be reduced any more no matter what $\underline{k}'$ is used. This is because of the structure of $\underline{C}$. Compute

$$\underline{R} = [\underline{C}\ \underline{b}, \underline{C}(\underline{A}-\underline{b}\ \underline{k}')\underline{b}, \underline{C}(\underline{A}-\underline{b}\ \underline{k}')^2\underline{b}]$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

Notice $\underline{R}$ has rank 2 and the feedback system is output-controllable. This is consistent with the results in Chapter II. Since the dimension of the controllable and observable subspace is never less than two (feedback or not), the system remains output-controllable under state feedback.

VI. CONCLUSIONS AND EXTENSIONS

6.1. Summary

In this research some aspects of the output-controllability problem have been studied. The main contribution of the second chapter was a characterization of output-controllability for linear time-invariant systems. In the third chapter, a way of assigning a numerical value to the quality of output-controllability was discussed and an optimization carried out in detail. A sensitivity analysis of the optimized system was made in the fourth chapter. The problem of parameter uncertainty was considered and expressions for mean-squared error for single input-single output systems were derived. The effect of state feedback on output-controllability was briefly considered in the fifth chapter.

6.2. Future Research

(1) The most important problem related to this research is an optimization based on Trace $\underline{W}_y^{-1}(t_o,t_1)$. The significance of this optimization is discussed below. Let $\underline{x}_o = 0$.

Consider the question of reaching points $\underline{y}$ on the boundary B of a closed unit hypersphere U in E^q. Since $\underline{y}$ is on the boundary

$$\langle \underline{y},\underline{y} \rangle = 1 \ , \quad \forall \ \underline{y} \in B \tag{6.1}$$

Also, $\underline{y}$ is the unit normal to the surface of B. Now, the average minimal control energy for reaching points on B is

$$\omega' = \int_B \underline{y}'\underline{W}_y^{-1}\underline{y}ds/\int_B ds \tag{6.2}$$

where ds is an element of surface area around $\underline{y}$. Using standard inner product notation, one has

$$\omega' = \int_B < \underline{W}_y^{-1}\underline{y},\underline{y}> \; ds/\int_B <\underline{y},\underline{y}> ds \tag{6.3}$$

The divergence theorem can be used to replace the surface integrals in (6.3) by volume integrals over U. Using the fact that for any square matrix $\underline{A}$

$$\text{div.}(\underline{A}\ \underline{x}) = \text{Trace } \underline{A} \tag{6.4}$$

the following expression is obtained for ω' in (6.3). Thus

$$\begin{aligned} \omega' &= (\text{Trace } \underline{W}_y^{-1})\int_U du/(\text{Trace } \underline{I}_q)\int_U du \\ &= \text{Trace } \underline{W}_y^{-1}/q \end{aligned} \tag{6.5}$$

since $\underline{I}_q$ is a (qxq) identity matrix. The significance of Trace $\underline{W}_y^{-1}$ is clear from (6.5). Other meanings can be found in Kalman, Ho and Narendra (8). Some results of Aoki and Staley (50) may be useful in this study.

(2) While the main optimization problem treated in Chapter III does not require the system to be time-invariant, there are other results which apply only to the stationary situation. An attempt to extend these to the time-varying case may be worthwhile.

(3) In section 3.4, the relationship between control

interval and optimization has been briefly investigated. This topic deserves further study. Indeed, one should be able to build an adaptive loop to take care of changing situations.

REFERENCES

1. L. A. Zadeh and C. A. Desoer. Linear System Theory - The State Space Approach, McGraw-Hill, N.Y. 1963.

2. C. A. Desoer. Notes for a Second Course on Linear Systems, Van Nostrand Reinhold, N.Y. 1970.

3. L. S. Pontriagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko. The Mathematical Theory of Optimal Processes, Interscience Publishers, N.Y. 1962.

4. J. P. LaSalle. "The time-optimal control problem," in Contributions to Nonlinear Oscillations, vol. 5, Princeton University Press, Princeton, N.J. 1960.

5. R. E. Kalman. "On the general theory of control systems," Proc. 1st International Congress on Automatic Control, Butterworth Scientific Publications, London, England, vol. 1, pp. 481-492, 1961.

6. R. E. Kalman. "Contributions to the theory of optimal control," Bol. Soc. Mat. Mex., pp. 102-119, 1960.

7. L. Weiss and R. E. Kalman. "Contributions to linear system theory," Int. J. Engrg. Sci., vol. 3, pp. 141-171, 1965.

8. R. E. Kalman, Y. C. Ho and K. S. Narendra. "Controllability of linear dynamical systems," in Contributions to Differential Equations, vol. 1, Interscience Publishers, N.Y. 1961.

9. E. Roxin. "The existence of optimal controls," Michigan Math. Journal. vol. 9, no. 2, pp. 101-119, 1962.

10. R. E. Kalman, P. L. Falb and M. A. Arbib. Topics in Mathematical System Theory, McGraw-Hill, N.Y. 1969.

11. E. Kreindler and P. E. Sarachik. "On the concepts of controllability and observability of linear systems," IEEE Trans. on Automatic Control, vol. Ac-9, pp. 129-136, April 1964.

12. E. A. Coddington and N. Levinson. Theory of Ordinary Differential Equations, McGraw-Hill, N.Y. 1955.

13. E. Kreindler. "Contributions to the theory of time-optimal control," J. Franklin Institute, vol. 275, pp. 314-344, April 1963.

14. L. M. Silverman and B. D. O. Anderson. "Controllability, observability and stability of linear systems," SIAM J. on Control, vol. 6, pp. 121-130, 1968.

15. L. Marcus and E. B. Lee. "On the existence of optimal controls," Trans. ASME, J. Basic Engrg., Ser. D, pp. 13-20, March 1962.

16. E. G. Gilbert. "Controllability and observability in multivariable control systems," SIAM J. on Control, vol. 1, pp. 128-151, 1963.

17. R. E. Kalman. "Mathematical description of linear dynamical systems," SIAM J. on Control, vol. 1, pp. 152-192, 1963.

18. J. E. Bertram and P. E. Sarachik. "On optimal computer control," Proc. 1st International Congress on Automatic Control, Butterworth Scientific Publications, London, England, vol. 1, pp. 419-422, 1961.

19. L. Weiss. "On a question related to the control of linear systems," IEEE Trans. on Automatic Control, vol. AC-9, pp. 176-177, April 1964.

20. R. Sivan. "On zeroing the output and maintaining it zero," IEEE Trans. on Automatic Control, vol. AC-10, pp. 193-194, April 1965.

21. R. W. Brockett. "Poles, zeros and feedback; State space interpretation," IEEE Trans. on Automatic Control, vol. AC-10, pp. 129-135, April 1965.

22. R. W. Brockett and M. D. Mesarovic. "The reproducibility of multivariable systems," J. Math. Analysis and Applications, vol. 11, pp. 548-563, July 1965.

23. B. Friedman. Principles and Techniques of Applied Mathematics, John Wiley and Sons, Inc., N.Y. 1956.

24. R. Bellman. Introduction to Matrix Analysis, McGraw-Hill, N.Y. 1960.

25. F. R. Gantmacher. The Theory of Matrices, vol. 1 and 2, Chelsea Publishing Company, N.Y. 1959.

26. R. Courant and D. Hilbert. Mathematical Methods of Physics, vol. 1, Interscience Publishers, Inc., N.Y. 1956.

27. E. Bodewig. Matrix Calculus, Interscience Publishers, Inc., N.Y. 1956.

28. R. G. Brown. "Not just observable, but how observable," Proc. National Electronics Conference, vol. 22, pp. 709-714, 1966.

29. R. A. Monzingo. "A note on sensitivity of system observability," IEEE Trans. on Automatic Control, vol. AC-12, pp. 314-315, June 1967.

30. J. K. Simon and S. K. Mitter. "A theory of modal control," Journal of Information and Control, vol. 13, pp. 316-353, October 1968.

31. C. D. Johnson. "Optimization of a certain quality of complete controllability and observability for linear dynamical systems," Trans. ASME, J. Basic Engrg., Ser. D, pp. 228-237, June 1969.

32. B. Friedland. "The design of optimum controllers for linear processes with energy limitations," Trans. ASME, J. Basic Engrg., Ser. D, pp. 181-192, June 1963.

33. Y. C. Ho. "Solution space approach to optimal control problems," Trans. ASME, J. Basic Engrg., Ser. D, pp. 53-58, March 1961.

34. E. B. Lee. "Design of optimum multivariable control systems," Trans. ASME, J. Basic Engrg., Ser. D, pp. 85-90, March 1961.

35. H. W. Bode. Network Analysis and Feedback Amplifier Design, Van Nostrand, N.Y. 1963.

36. R. A. Rohrer and M. Sobral. "Sensitivity considerations in optimal control design," IEEE Trans. on Automatic Control, vol. AC-10, pp. 43-48, January 1965.

37. P. Dorato. "On sensitivity in optimal control systems," IEEE Trans. on Automatic Control, vol. AC-8, pp. 256-257, July 1963.

38. B. Pagurek. "Sensitivity of performance of optimal control systems to plant parameter variations," IEEE Trans. on Automatic Control, vol. AC-10, pp. 178-180, April 1965.

39. D. A. Spaulding. "Optimizing sensitivity in the design of a class of linear filters," IEEE Trans. on Circuit Theory, vol. CT-17, pp. 459-461, August 1970.

40. L. H. Sibul. "Sensitivity analysis of linear control systems with random plant parameters," IEEE Trans. on Automatic Control, vol. AC-15, pp. 459-462, August 1970.

41. B. Liu and M. E. Van Valkenburg. "A probabilistic approach to the parameter sensitivity problem," Proc. 5th Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, pp. 701-707, October 1967.

42. J. B. Farison. "Approximation of exponential functions of random variables by statistical linearization," IEEE Trans. on Automatic Control, vol. AC-13, pp. 174-178, April 1968.

43. H. Cramer. Mathematical Methods of Statistics, Princeton University Press, Princeton, N.J. 1961.

44. M. Fisz. Probability Theory and Mathematical Statistics, John Wiley and Sons, Inc., N.Y. 1963.

45. W. M. Wonham. "On pole assignment in multi-input controllable linear systems," IEEE Trans. on Automatic Control, vol. AC-12, pp. 665-670, December 1967.

46. F. M. Brasch and J. B. Pearson. "Pole placement using dynamic compensators," IEEE Trans. on Automatic Control, vol. AC-15, pp. 34-43, February 1970.

47. A. Jameson. "Design of a single-input system for specified roots using output feedback," IEEE Trans. on Automatic Control, vol. AC-15, pp. 345-348, June 1970.

48. C. K. Liu and N. J. Bergman. "On necessary conditions for decoupling multivariable systems," IEEE Trans. on Automatic Control, vol. AC-15, pp. 131-133, February 1970.

49. W. G. Tuel. "On the transformation to phase-variable canonical form," IEEE Trans. on Automatic Control, vol. AC-11, p. 607, July 1966.

50. M. Aoki and R. M. Staley. "On input signal synthesis in parameter identification," Automatica, vol. 6, pp. 431-440, May 1970.