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The general theory of characteristics is reviewed for hyperbolic partial differential equations of n independent variables. The application of the theory of characteristics is made to unsteady, two-dimensional, rotational, inviscid flows; unsteady, two-dimensional, irrotational, inviscid flows; and unsteady, axial symmetric, inviscid flows. The characteristic surfaces and the compatibility relations are found. Moreover, calculating schemes are set up for numerical solutions for the conditions at a general point, a boundary point, and a constant pressure point. The calculations for the particle paths are also developed.

Solution of Unsteady, Two-dimensional, Inviscid Flows

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TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	GENERAL THEORY OF CHARACTERISTICS	3
	Characteristic Surfaces	3
	Compatibility Relations	6
III.	ROTATIONAL FLOWS	7
	Characteristic Surfaces and Bi-characteristics	8
	Compatibility Relations	15
	Calculation Scheme	17
	Particle Paths	23
IV.	IRROTATIONAL FLOWS	27
	Compatibility Relations	27
	Calculation Scheme	32
V.	AXIAL SYMMETRIC FLOWS	36
VI.	SUMMARY AND CONCLUSION	40
	BIBLIOGRAPHY	42

LIST OF FIGURES

Figure		Page
1.	The characteristic surfaces.	12
2.	Mach conoid for two-dimensional, unsteady flows.	14
3.	The Mach conoid for a general point.	18
4.	Points used to determine the coordinates of the point P_7.	25
5.	Two directions in compatibility relation.	28

SOLUTION OF UNSTEADY, TWO-DIMENSIONAL, INVISCID FLOWS

I. INTRODUCTION

The method of characteristics has been used to solve hyperbolic partial differential equations. It is well developed for the system of equations in two independent variables, for instance, steady, two-dimensional, supersonic, inviscid flows and unsteady, one-dimensional, inviscid flows. For the system of two independent variables there is a family of curves along which the derivatives of dependent variables exist, and are discontinuous or vanishing in the direction normal to them. The curves are called characteristic curves or characteristics. The equation which relates the dependent variables along the characteristic direction is called the compatibility relation. In this equation only one direction is involved.

The concept of characteristics can also be applied to the partial differential equations of n $(n > 2)$ independent variables whenever it is a hyperbolic system. Instead of a family of curves, there is a family of n-dimensional surfaces on which the derivatives of the dependent variables exist,but are discontinuous or vanishing in the normal direction. The n-dimensional surfaces are called characteristic surfaces and the compatibility equation relates the derivatives on these surfaces. In general, there are $n-1$ directions in this compatibility equation.

Several methods are found in the application of the method of characteristics to the systems of equations governing three independent variables. Coburn & Dolph (1949) suggest a method, further developed by Holt (1956), which involves two bi-characteristic curves and one ordinary curve to find the conditions at the point of intersection. The ordinary curve is the intersection of the two characteristic surfaces through these two bi-characteristics. Thornhill (1948) suggests another method, in which three characteristic surfaces are used to determine the conditions at a particular point of the intersection of those three surfaces. Butler (1960) suggests a method in which the conditions at one point are found by using the four bi-characteristics and the streamline through that point.

Applications of characteristics in the solutions of unsteady two-dimensional, inviscid flows, either rotational or irrotational, and axial symmetric flows are developed here. Following Butler's method the calculation scheme is set up for a general point, a boundary point and a constant pressure point in fluids, when $a = a_s >> |V|$. That is, the speed of sound, a, is a constant and much larger than the velocity, V.

II. GENERAL THEORY OF CHARACTERISTICS

Characteristic Surfaces

Consider a system of k first-order differential equations for k unknowns $u_1, u_2, \ldots, u_k$, of n independent variables $x_1, x_2, \ldots, x_n$,

$$L_i(u) = a_{ijm} \frac{\partial u_j}{\partial x_m} + b_i = 0 \qquad (2.1)$$

$$i, j = 1, 2, \ldots, k$$

$$m = 1, 2, \ldots, n$$

Here $L_i(u)$ is a linear operator on u_j.

Define the n-dimensional vector quantities as

$$A_{ij} = (a_{ij1}, a_{ij2}, \ldots, a_{ijn}),$$

and

$$\text{grad } u_j = \left(\frac{\partial u_j}{\partial x_1}, \frac{\partial u_j}{\partial x_2}, \ldots, \frac{\partial u_j}{\partial x_n}\right),$$

Then, any one of the k differential equations in (2.1) can be regarded as the n-dimensional scalar product of a vector A_{ij} and the vector grad u_j. In vector form, the system of equations in (2.1) becomes

$$L_i(u) = A_{ij} \cdot \text{grad } u_j + b_i = 0 \qquad (2.2)$$

where

$$i = 1, 2, \ldots\ldots, k,$$

here the summation convention is used. Now the system is independent of the choice of coordinate axes when writing it in vector form.

Constructing the linear combination of the system of equations in (2.2) by multiplying the multipliers c_i, $i = 1, 2, \ldots., k$ and then summing to get

$$c_i L_i(u) = A_j \cdot \text{grad } u_j + c_i b_i = 0 , \qquad (2.3)$$

where

$$A_j = c_i A_{ij} .$$

In Equation (2.3) the expression $A_j \cdot \text{grad } u_j$ for each value of j can be considered geometrically as the derivative of u_j along the direction A_j. So, there are k directions concerned in Equation (2.3) for each variable u_j.

In order to finding the multipliers c_i such that in the expression of Equation (2.3) the derivatives of variables $u_1, u_2, \ldots., u_k$, have the same directions lying on the surfaces $f(x_1, x_2, \ldots., x_n) = 0$ in n-dimensional space, the conditions which should be satisfied are

$$A_j \cdot N = c_i A_{ij} \cdot N = 0, \qquad (2.4)$$

where

$$j = 1, 2, \ldots, k,$$

and where N is the normal vector of the surfaces $f(x_1, x_2, \ldots, x_n) = 0$. The surfaces of which the normal vectors satisfy the Equation (2.4) are called the characteristic surfaces. This normal vector can be denoted by

$$N = (f_{x_1}, f_{x_2}, \ldots\ldots, f_{x_n}),$$

because the vector

$$T = (dx_1, dx_2, \ldots\ldots, dx_n)$$

represents the tangent vector of the surface $f(x_1, x_2, \ldots, x_n) = 0$, which when taking the total derivative would become

$$df = f_{x_1} dx_1 + f_{x_2} + \ldots\ldots + f_{x_n} dx_n = 0 ,$$

i.e.,

$$T \cdot N = 0.$$

The non-trivial solution of the multipliers c_i in the system of k Equations in expression (2.4) exists if the determinant of coefficients vanishes, i.e., if

$$|N \cdot A_j| = 0. \qquad (2.5)$$

This is the characteristic equation, which denotes the relations among the components of the normal vector to the characteristic surface $f(x_1, x_2, \ldots\ldots, x_n) = 0$. Any surfaces in the x_n n-dimensional space satisfying the Equation (2.5) are characteristic surfaces.

Compatibility Relations

The multipliers c_i can be found to satisfy the Equation (2.4), while the values of N satisfy the Equation (2.5). In other words, find the values of N from the Equation (2.5) and then substitute into the Equation (2.4) to find the values of the multipliers c_i. Therefore, the differential equation $c_i L_i(u) = 0$ in the Equation (2.3) implies differentiations of all variables $u_1, u_2, \ldots\ldots\ldots, u_k$, in the directions lying on the characteristic surface. The Equation (2.3) so obtained is called the compatibility relation.

III. ROTATIONAL FLOWS

When the system of partial differential equations describe the two-dimensional, unsteady flows neglecting heat conduction and viscosity, it is a hyperbolic system. The method of characteristics can be used in this fluid field for numerical solution. There exist characteristic surfaces in t, x, y-space. They are the surfaces that exist when the transition from one region to another involves discontinuities of some derivatives.

Ferri (1954) mentioned that if heat conductivity and viscosity can not be neglected in some fluid region, then the system is no longer hyperbolic. Therefore, the characteristic surfaces can not exist and the method of characteristics can not be applied in this system.

Consider now the system of differential equations describing the motion of compressible inviscid flows in the x-y plane. The velocity component of the fluid in x-direction is $u = u(t, x, y)$ and that in y-direction is $v = v(t, x, y)$. The governing equations are: the equation of continuity,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

the momentum equations of an inviscid flow,

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} , \tag{3.2}$$

and

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} . \tag{3.3}$$

Assume that the equation of state can be expressed as

$$dp = a^2 d\rho \tag{3.4}$$

where a is the speed of sound in the fluid.

Using this equation of state to substitute the pressure terms in the momentum Equations (3.2) and (3.3), the results are

$$a^2 \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = 0 , \tag{3.5}$$

and

$$a^2 \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = 0 . \tag{3.6}$$

Characteristic Surfaces and Bi-characteristics

The system of Equations (3.1), (3.5), (3.6) with variables $\rho, u, v,$ and independent variables $t, x, y,$ can be written in the vector forms of the Equation (2.2) as:

$$A_{1\rho} \cdot \text{grad } \rho + A_{1u} \cdot \text{grad } u + A_{1v} \cdot \text{grad } v = 0 , \tag{3.7}$$

$$A_{2\rho} \cdot \text{grad } \rho + A_{2u} \cdot \text{grad } u + A_{2v} \cdot \text{grad } v = 0 , \tag{3.8}$$

$$A_{3\rho} \cdot \text{grad } \rho + A_{3u} \cdot \text{grad } u + A_{3v} \cdot \text{grad } v = 0 , \tag{3.9}$$

where

$$A_{1\rho} = (1, u, v) \ , \quad A_{1u} = \rho(0, 1, 0) \ , \quad A_{1v} = \rho(0, 0, 1) \ ,$$

$$A_{2\rho} = (0, a^2, 0), \quad A_{2u} = \rho(1, u, v) \ , \quad A_{2v} = (0, 0, 0) \ ,$$

$$A_{3\rho} = (0, 0, a^2), \quad A_{3u} = \ (0, 0, 0) \ , \quad A_{3v} = \rho(1, u, v) \ ,$$

and

$$\text{grad} = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \ .$$

Therefore, the Equation (2.3) becomes

$$A_\rho \cdot \text{grad}\ \rho + A_u \cdot \text{grad}\ u + A_v \cdot \text{grad}\ v = 0 \ , \tag{3.10}$$

where

$$A_\rho = c_1 A_{1\rho} + c_2 A_{2\rho} + c_3 A_{3\rho} = (c_1, c_1 u + c_2 a^2, c_1 v + c_3 a^2) \ ,$$

$$A_u = c_1 A_{1u} + c_2 A_{2u} + c_2 A_{3u} = \rho(c_2, c_1 + c_2 u, c_2 v) \ ,$$

and

$$A_v = c_1 A_{1v} + c_2 A_{2v} + c_3 A_{3v} = \rho(c_3, c_3 u, c_1 + c_3 v) \ .$$

Let the characteristic surface be $f(t, x, y) = 0$, and the normal vector to this surface is denoted by

$$N = (f_t, f_x, f_y) \ .$$

Hence, the conditions in the Equation (2.4) become

$$A_\rho \cdot N = A_u \cdot N = A_v \cdot N = 0 \tag{3.11}$$

These conditions are equivalent to

$$c_1 f_t + (c_1 u + c_2 a^2) f_x + (c_1 v + c_3 a^2) f_y = 0 ,$$

$$c_2 f_t + (c_1 + c_2 u) f_x + c_2 v f_y = 0 ,$$

and

$$c_3 f_t + c_3 u f_x + (c_1 + c_3 v) f_y = 0 .$$

Rearrange these equations to get

$$c_1(f_t + uf_x + vf_y) + c_2(a^2 f_x) + c_3(a^2 f_y) = 0 , \qquad (3.12)$$

$$c_1 f_x + c_2(f_t + uf_x + vf_y) = 0 ,$$

and

$$c_1 f_y + c_3(f_t + uf_x + vf_y) = 0 .$$

The condition in the Equation (2.5) for non-trivial solution of c_1, c_2, c_3, in the Equation (3.12) becomes

$$\begin{vmatrix} f_t + uf_x + vf_y & a^2 f_x & a^2 f_y \\ f_x & f_t + uf_x + vf_y & 0 \\ f_y & 0 & f_t + uf_x + vf_y \end{vmatrix} = 0 ,$$

or

$$0 = (f_t + uf_x + vf_y)^3 - a^2 f_y^2 (f_t + uf_x + vf_y) - a^2 f_x^2 (f_t + uf_x + vf_y) ,$$

or

$$= (f_t + uf_x + vf_y) \{(f_t + uf_x + vf_y)^2 - a^2(f_x^2 + f_y^2)\} . \qquad (3.13)$$

These characteristic surfaces $f(t,x,y) = 0$ in t, x, y-space which satisfy the Euqation (3.13) have the property of the vanishing gradient components normal to them. Corresponding to the first factor of the Equation (3.13), i.e.

$$f_t + uf_x + vf_y = 0 ,$$

the characteristic surface is a plane in t, x, y-space. This plane contains the streamline and therefore its projection on the x, y-plane is nothing but the streamline of the flow.

Corresponding to the second factor of the Equation (3.13), i.e.

$$(f_t + uf_x + vf_y)^2 - a^2 (f_x^2 + f_y^2) = 0 , \qquad (3.14)$$

the characteristic surfaces passing through any one point in space envelop a conoid through that point and are called a "cone" of second order or a quadratic cone in t, x, y-space. The lines of contact of the surfaces and the conoid are bi-characteristics. The characteristic surfaces, Mach conoids and bi-characteristics are shown in Figure 1.

Consider two points P and Q. There are two Mach conoids PAC and QBD vertex at P and Q respectively. The tangent planes PABQ and PCDQ are called characteristic surfaces. Curves PA, PC, QB and QD are called the bi-characteristics.

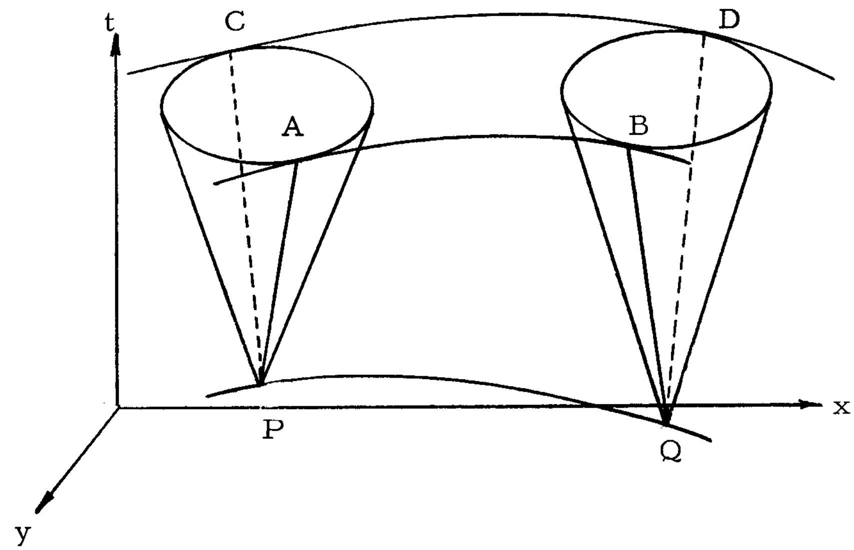

Figure 1. The characteristic surfaces.

With t as parameter and using the relations

$$F = (f_t + uf_x + vf_y)^2 - a^2(f_x^2 + f_y^2) = 0 ,$$

$$\frac{dx}{dt} = \frac{F_{f_x}}{F_{f_t}} = u - \frac{a^2 f_x}{f_t + uf_x + vf_y} \tag{3.15}$$

and

$$\frac{dy}{dt} = \frac{F_{f_y}}{F_{f_t}} = v - \frac{a^2 f_x}{f_t + uf_x + vf_y} \tag{3.16}$$

to eliminate function f and its derivatives f_t, f_x, f_y, one obtains

$$\left(\frac{dx}{dt} - u\right)^2 + \left(\frac{dy}{dt} - v\right)^2 = \frac{a^4(f_x^2 + f_y^2)}{(f_t + uf_x + vf_y)^2} = a^2 . \tag{3.17}$$

The Equation (3.17) is called the Monge differential equation. It states a condition for the direction of the bi-characteristic of the Mach conoid at point (x, y, t). But the Equation (3.14) states a condition for the tangent planes of the Mach conoid.

Multiplying through by dt in the Equation (3.15) yields

$$(dx-udt)^2 + (dy-vdt)^2 = a^2 dt^2 . \qquad (3.18)$$

The Equation (3.18) can be interpreted geometrically in the following way: suppose at time t_o a fluid particle is at point P_o in t, x, y-space with coordinates t_o, x_o, y_o. After a time interval dt the particle moves to P_1 with coordinates

$$x = x_o + udt, \; y = y_o + vdt, \; t = t_o + dt.$$

Hence an infinitesimal disturbance at point P_o at $t = t_o$ propagates inside a circle of center $x = x_o + udt$, $y = y_o + vdt$, and radius of adt after a time interval dt. The relating equation is obtained from the Equation (3.18),

$$(x-x_o - udt)^2 + (y-y_o - vdt)^2 = a^2 dt^2 . \qquad (3.19)$$

This equation defines the Mach conoid shown on the following page.

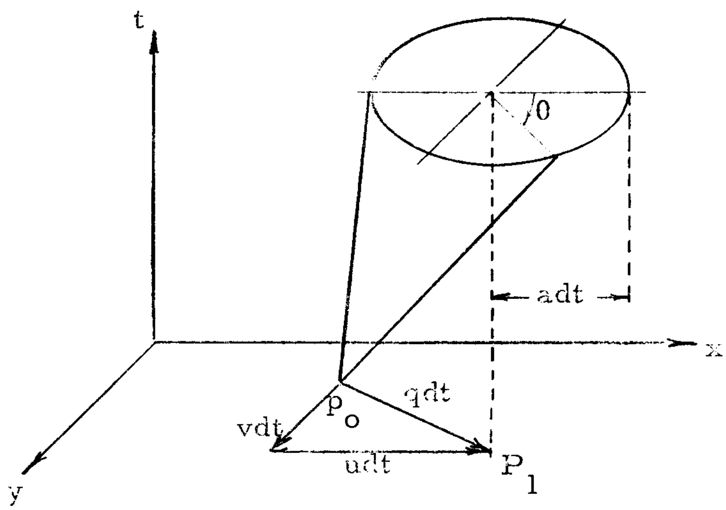

Figure 2. Mach conoid for two-dimensional, unsteady flows.

Write the Equation (3. 18) in one parametric form to give

$$dx = (u + a \cos \theta)\, dt , \tag{3.20}$$

and

$$dy = (v + a \sin \theta)\, dt ,$$

where

θ is the angle shown in Figure 2.

The Equation (3. 20) determines the bi-characteristic directions in parametric form. The vector $(1,\ u + a \cos \theta,\ v + a \sin \theta)$ denotes the direction of a family of curves through the vertex of the conoid and lying on the surface of the conoid and that of the characteristic surface.

Compatibility Relations

Substitute the relations in the Equation (3.20) into the Equations (3.15) and (3.16) to get

$$a \cos\theta = -\frac{a^2 f_x}{f_t + uf_x + vf_y},$$

and

$$a \sin\theta = -\frac{a^2 f_y}{f_t + uf_x + vf_y},$$

from which the parametric form of the Equation (3.14) is obtained:

$$\frac{f_x}{\cos\theta} = \frac{f_y}{\sin\theta} = \frac{-f_t}{a + u\cos\theta + v\sin\theta}. \tag{3.21}$$

This is equivalent to

$$f_x = k\cos\theta,$$

$$f_y = k\sin\theta,$$

$$f_t = -k(a + u\cos\theta + v\sin\theta),$$

and

$$f_t + uf_x + vf_y = -ak.$$

Using these relations the Equation (3.12) becomes

$$(-ak)c_1 + (a^2 k\cos\theta)c_2 + (a^2 k\sin\theta)c_3 = 0, \tag{3.22}$$

$$(k\cos\theta)c_1 + (-ak)c_2 = 0,$$

and

$$(k\sin\theta)c_1 + (-ak)c_3 = 0.$$

Solving the simultaneous Equations (3.22), the ratio between the values of c_1, c_2, c_3, can be obtained

$$c_2 = \frac{\cos\theta}{a} c_1 , \tag{3.23}$$

and

$$c_3 = \frac{\sin\theta}{a} c_1 .$$

From those ratios calculate the following vectors:

$$A_\rho = c_1 (1, u + a\cos\theta, v + a\sin\theta) ,$$

$$A_u = c_1 \rho \left(\frac{\cos\theta}{a}, 1 + \frac{\cos\theta}{a} u, \frac{\cos\theta}{a} v\right)$$

$$= c_1 \rho \left\{\frac{\cos\theta}{a} (1, u + a\cos\theta, v + a\sin\theta) + \sin\theta\, (0, \sin\theta, -\cos\theta)\right\}$$

$$A_v = c_1 \rho \left(\frac{\sin\theta}{a}, \frac{\sin\theta}{a} u, 1 + \frac{\sin\theta}{a} v\right)$$

$$= c_1 \rho \left\{\frac{\sin\theta}{a} (1, u + a\cos\theta, v + a\sin\theta) - \cos\theta\, (0, \sin\theta, -\cos\theta)\right\}$$

Finally the Equation (3.10) becomes:

$$(1, u + a\cos\theta, v + a\sin\theta) \cdot \left(\operatorname{grad}\rho + \frac{\rho\cos\theta}{a} \operatorname{grad} u + \frac{\rho\sin\theta}{a} \operatorname{grad} v\right) + \rho(0, \sin\theta, -\cos\theta) \cdot (\sin\theta \operatorname{grad} u - \cos\theta \operatorname{grad} v) = 0. \tag{3.24}$$

Let $\frac{d}{dt} = (1, u + a\cos\theta, v + a\sin\theta) \cdot \operatorname{grad}$ represent the derivative along the bi-characteristic defined by the vector

$(1, u+a\cos\theta, v+a\sin\theta)$ and the Equation (3.24) becomes

$$a d\rho + \rho \sin\theta\, du + \rho \cos\theta\, dv = -\rho a G dt ,$$

or

$$\frac{1}{a} dp + \rho \sin\theta\, du + \rho \cos\theta\, dv = -\rho a G dt , \qquad (3.25)$$

where

$$G = (0, \sin\theta, -\cos\theta)\cdot(\sin\theta \operatorname{grad} u - \cos\theta \operatorname{grad} v)$$

$$= \sin^2\theta \frac{\partial u}{\partial x} - \sin\theta\cos\theta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \cos^2\theta \frac{\partial v}{\partial y} .$$

The Equation (3.25) is called the compatibility relation along the bi-characteristic direction given by the Equation (3.20). This equation can be used in difference form to find out the states at the point $P(t_o, x_o, y_o)$ from the lower time surface $f(t_o-h, x, y) = 0$, which is known.

Calculation Scheme

The following calculation scheme is developed for the case of rotational subsonic flow with the assumption

$$a = a_s >> |V| = (u^2+v^2)^{\frac{1}{2}}$$

The equation of state gives the fact that the order of magnitude of $d\rho$ is $0(1/a_s^2)$, if the order of magnitude of dp is unity. Therefore, by knowing that $a_s > 1000$fps for most liquids, the density ρ can be considered as a constant, say, $\rho = \rho_s$, throughout the whole fluid region. Then, the Equation (3.25) becomes

$$\frac{1}{a_s} dp + \rho_s \sin\theta \, du + \rho_s \cos\theta \, dv = - \rho_s a_s G dt \, , \qquad (3.26)$$

along the bi-characteristic direction

$$dx = a_s \cos\theta \, dt, \quad \text{and} \quad dy = a_s \sin\theta \, dt \qquad (3.27)$$

Using Butler's method by choosing the four bi-characteristics through the point $P(t_o, x_o, y_o)$. Practically, $\theta = (n-1)\pi/2$, $n = 1, 2, 3, 4$, are chosen for convenience. Thus the only terms involved in the right hand side of the equation (3.26)along those four bi-characteristics are $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$. The subscripts 1, 2, 3, 4 denote the conditions at the points where the chosen bi-characteristics meet the plane $t = t_o - h$ in t, x, y-space. This can be shown in Figure 3.

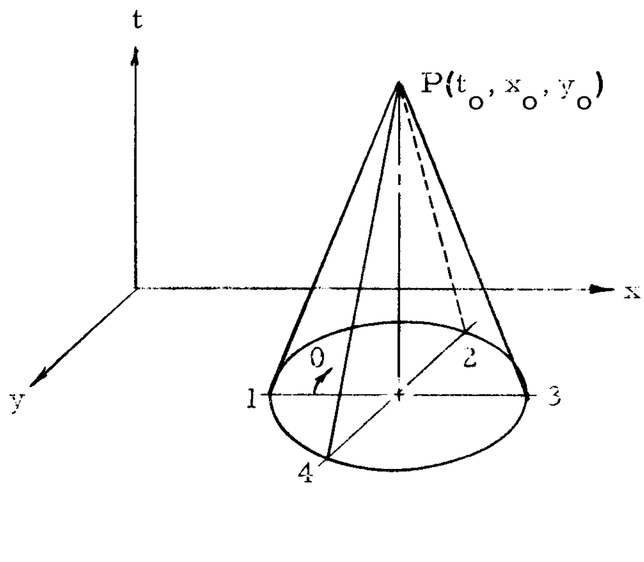

Figure 3. The Mach conoid for a general point.

The circle 1-2-3-4 is the domain of dependence for general point $P(t_o, x_o, y_o)$. The chosen bi-characteristics are P1, P2, P3, and P4. The cone P12341 is called the rear cone or backward cone vertex at point P.

The finite difference forms of the Equation (3.26) and (3.27) for $a = a_s >> |V|$ and $\rho = \rho_s$ are

when

$$\theta = 0 , \quad x_o = x_1 + a_s h \quad y_o = y_1 \tag{3.28}$$

$$\frac{1}{a_s}(p_o - p_1) + \rho_s(u_o - u_1) = -\frac{1}{2} a_s \rho_s h\{(\frac{\partial v}{\partial y})_o + (\frac{\partial v}{\partial y})_1\} \tag{3.29}$$

when

$$\theta = \pi/2 , \quad x_o = x_2 \quad y_o = y_2 + a_s h \tag{3.30}$$

$$\frac{1}{a_s}(p_o - p_2) + \rho_s(v_o - v_2) = -\frac{1}{2} a_s \rho_s h\{(\frac{\partial u}{\partial x})_o + (\frac{\partial u}{\partial x})_1\} \tag{3.31}$$

when

$$\theta = \pi , \quad x_o = x_3 - a_s h \quad y_o = y_3 \tag{3.32}$$

$$\frac{1}{a_s}(p_o - p_3) - \rho_s(u_o - u_3) = -\frac{1}{2} a_s \rho_s h\{(\frac{\partial v}{\partial y})_o + (\frac{\partial v}{\partial y})_3\} \tag{3.33}$$

when

$$\theta = 2\pi/3 ,$$

$$x_o = x_4$$

$$y_o = y_4 - a_s h \tag{3.34}$$

$$\frac{1}{a_s}(p_o - p_4) - \rho_s(v_o - v_4) = -\frac{1}{2}a_s\rho_s h\{(\frac{\partial u}{\partial x})_o + (\frac{\partial u}{\partial x})_4\} \tag{3.35}$$

The system of Equations (3.29), (3.31) (3.33) and (3.35) consists of unknown terms: p_o, u_o, v_o, $(\frac{\partial u}{\partial x})_o$, $(\frac{\partial v}{\partial y})_o$. One more relation, which is required to solve these five unknown values, can be obtained from the continuity Equation (3.1). Along the streamline

$$dx = udt ,$$

and

$$dy = vdt .$$

Therefore, the Equation (3.1) becomes

$$d\rho = -\rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})dt$$

or in term of pressure gradient,

$$(\frac{1}{a_s})^2 dp = -\rho_s(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})dt$$

The difference form of this equation is

$$(\frac{1}{a_s})^2(p_o - p_5) = -\frac{1}{2}\rho_s\{(\frac{\partial u}{\partial x})_o + (\frac{\partial v}{\partial y})_o + (\frac{\partial u}{\partial x})_5 + (\frac{\partial v}{\partial y})_5\}h \tag{3.36}$$

The coordinates $(t_o - h,\ x_5,\ y_5)$ define the point where the stream-line passing through $(t_o,\ x_o,\ y_o)$ meets the plane $t = t_o - h$, and can be determined by the relations

$$x_o = x_5 + \frac{1}{2}(u_o + u_4)h\ , \qquad (3.37)$$

and

$$y_o = y_5 + \frac{1}{2}(v_o + v_5)h\ .$$

Now the problem is how to find the values of p_o, u_o, and v_o at the given point $(t_o,\ x_o,\ y_o)$ supposing that conditions at all points on the plane $t = t_o - h$ are known. The calculation process can be summarized as following:

1. Assume the values of p_o, u_o, and v_o. For example, one may choose

$$p_o = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)\ ,$$

$$u_o = \frac{1}{4}(u_1 + u_2 + u_3 + u_4)\ ,$$

and

$$v_o = \frac{1}{4}(v_1 + v_2 + v_3 + v_4)\ .$$

2. Determine coordinates x_5, y_5 from the Equation (3.37).
3. Calculate the coordinates $x_1, y_1, x_2, y_2, x_3, y_3, x_4,$ and y_4 from the Equations (3.28), (3.30), (3, 32), and (3.34).
4. Get the values of $p_o, u_o,$ and v_o from the Equations

(3.29), (3.31), (3.33), (3.35), and (3.36).

5. Use the calculated values of p_o, u_o, and v_o in the last step as the assumed values in the first step. Repeat the process until the convergence is sufficient.

Next, consider a point $P(t_o, x_o, y_o)$, which is a boundary point. The given condition on the boundary at this point is

$$\left(\frac{dy}{dx}\right)_{\text{boundary at P}} = \tan \alpha \tag{3.38}$$

where α is the angle of the tangent line to the boundary curve at P measured from x-axis. The inviscid flow theory requires that the velocity at a solid boundary point has the property of vanishing velocity component normal to the surface. Therefore the condition (3.38) gives the relation between the velocity components u_o and v_o at the point P, that is,

$$\frac{v_o}{u_o} = \tan \alpha . \tag{3.39}$$

From this relation, the five unknowns for a general point in fluid now reduce to four independent unknowns, for instance, p_o, u_o, $\left(\frac{\partial u}{\partial x}\right)_o$, $\left(\frac{\partial u}{\partial y}\right)_o$. This leads to the fact that only three bi-characteristics with the streamline through the point P are required to find its conditions.

If the point P is a constant pressure point at which the pressure is known, then p_o is given without calculation. Therefore, again the three bi-characteristics and the streamline through the point P are used to solve the unknowns u_o, v_o, $(\frac{\partial u}{\partial x})_o$, $(\frac{\partial v}{\partial y})_o$.

Particle Paths

A fluid particle is given at the point $P_6(t_o - h, x_6, y_6)$ in t, x, y-space. After a time interval h, the particle moves to the point $P_7(t_o, x_7, y_7)$. The coordinates of the point P_7, namely, x_7 and y_7, can be found by integrating along the streamline expressed by the differential equations: $dx = udt$, and $dy = vdt$. So, an equation similar to the Equation (3.37) is used to determine the coordinates x_7 and y_7. The equation is of the form:

$$x_7 = x_6 + \frac{1}{2}(u_7 + u_6)h \,, \tag{3.40}$$

and

$$y_7 = y_6 + \frac{1}{2}(v_7 + v_6)h \,.$$

Let $P_o(t_o, x_o, y_o)$ be the point determined by the four points: $P_1(t_o - h, x_1, y_1)$, $P_2(t_o - h, x_2, y_2)$, $P_3(t_o - h, x_3, y_3)$, and $P_4(t_o - h, x_4, y_4)$. The point P_6 should lie inside the square grid $P_1P_2P_3P_4$. Then, from the former calculation one can obtain

$$x_o = \frac{1}{2}(x_1 + x_3) \,, \tag{3.41}$$

$$y_o = \frac{1}{2}(y_2 + y_4) \,,$$

and the values of u_o, v_o, $(\frac{\partial u}{\partial x})_o$ and $(\frac{\partial v}{\partial y})_o$. Now, the conditions u_7 and v_7 at the point P_7 can be expanded by Taylor's series at point p_o as

$$u_7 = u_o + (\frac{\partial u}{\partial x})_o(x_7 - x_o) + (\frac{\partial u}{\partial y})_o(y_7 - y_o) , \tag{3.42}$$

and

$$v_7 = v_o + (\frac{\partial v}{\partial x})_o(x_7 - x_o) + (\frac{\partial v}{\partial y})_o(y_7 - y_o) .$$

where the higher order derivative terms are neglected. In this Equation (3.42), the derivative terms $(\frac{\partial u}{\partial y})_o$ and $(\frac{\partial v}{\partial x})_o$ need to be determined. On the $t = t_o$ plane the values of u and v at the points: $P_{10}(t_o, x_1, y_1)$, $P_{20}(t_o, x_2, y_2)$, $P_{30}(t_o, x_3, y_3)$, and $P_{40}(t_o, x_4, y_4)$ are calculated. These points are shown in Figure 4. So the derivative terms $(\frac{\partial u}{\partial y})_o$, and $(\frac{\partial v}{\partial y})_o$ can be approximated by

$$(\frac{\partial u}{\partial y})_o = \frac{u_{40} - u_{20}}{2a_s h} , \tag{3.43}$$

and

$$(\frac{\partial v}{\partial x})_o = \frac{v_{30} - v_{10}}{2a_s h} .$$

Substituting the Equation (3.42) into the Equation (3.40) and rearranging to obtain

$$D_1 x_7 + E_1 y_7 = F_1 , \tag{3.44}$$

and

$$D_2 x_7 + E_2 y_7 = F_2 ,$$

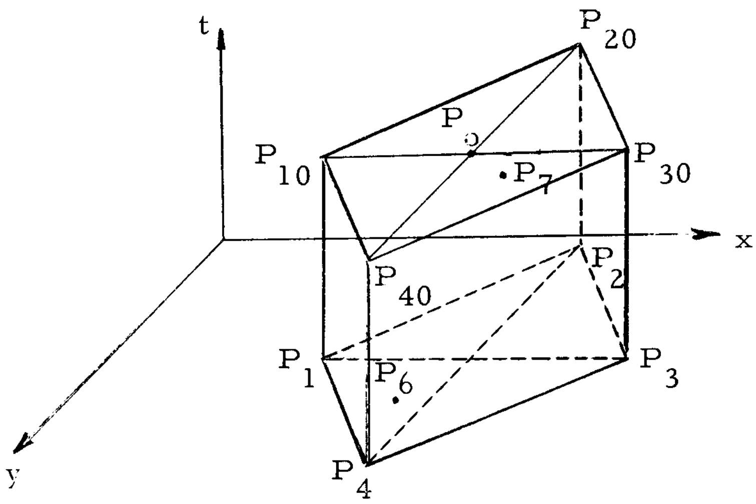

Figure 4. Points used to determine the coordinates of the point P_7.

where,

$$D_1 = 2 - (\frac{\partial u}{\partial x})_o h ,$$

$$D_2 = - (\frac{\partial v}{\partial x})_o h ,$$

$$E_1 = - (\frac{\partial u}{\partial y})_o h ,$$

$$E_2 = 2 - (\frac{\partial v}{\partial y})_o h ,$$

$$F_1 = 2x_6 + [u_o - (\frac{\partial u}{\partial x})_o x_o - (\frac{\partial u}{\partial y})_o y_o + u_6]h ,$$

and

$$F_2 = 2y_6 + [v_6 - (\frac{\partial v}{\partial x})_o x_o - (\frac{\partial v}{\partial y})_o y_o + v_6]h .$$

Then the solutions of the Equation (3.44) are

$$x_7 = \frac{F_1E_2 - F_2E_1}{D_1E_2 - D_2E_1}, \tag{3.45}$$

and

$$y_7 = \frac{F_2D_1 - F_1D_2}{D_1E_2 - D_2E_1}$$

The calculation steps for the coordinates x_7 and y_7 are as follows:

1. Choose the square grid $P_1P_2P_3P_4$ on $t = t_o - h$ plane, such that the given point P_6 lies inside the grid.
2. Calculate the data on $t = t_o$ plane, i.e., x_o, y_o, u_o, v_o, $(\frac{\partial u}{\partial x})_o$, $(\frac{\partial v}{\partial y})_o$, v_{10}, u_{20}, v_{30}, and u_{40}.
3. Obtain $(\frac{\partial u}{\partial y})_o$ and $(\frac{\partial v}{\partial x})_o$ from the Equation (3.43).
4. Evaluate D_1, D_2, E_1, E_2, F_1 and F_2 in the Equation (3.44).
5. Find the coordinates x_7 and y_7 from the Equation (3.45).

Therefore the coordinates of the point P_7 are determined. In this manner, one can trace the particle paths approximated by the straight line connecting the given point P_6 and the determined point P_7, if h is small enough.

IV. IRROTATIONAL FLOWS

For irrotational flow the characteristics and bi-characteristics developed in rotational flows are still valid. From the condition for irrotationality

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \ . \qquad (4.1)$$

The auxiliary equation to the compatibility relation derived in the Equation (3.25) can be obtained with the aid of this equation.

Compatibility Relations

Looking in the Equation (3.20), there are two directions concerned in the derivatives of the variables $\rho, u,$ and v. One is $(1, u + a\cos\theta, v + a\sin\theta)$, and the other is $(0, \sin\theta, -\cos\theta)$. Therefore, the compatibility relation involves the changes of the quantities $\rho, u,$ and v along these two directions.

Let $\frac{\partial}{\partial m}$ denote the derivatives along the bi-characteristic direction given by $(\frac{1}{a}, \frac{u}{a} + \cos\theta, \frac{v}{a} + \sin\theta)$, and $\frac{\partial}{\partial n}$ denote the derivatives along the direction given by $(0, \sin\theta, -\cos\theta)$. Express this definition mathematically

$$\frac{\partial}{\partial m} = (\frac{1}{a}, \frac{u}{a} + \cos\theta, \frac{v}{a} + \sin\theta) \ . \ \text{grad} \ , \qquad (4.2)$$

and

$$\frac{\partial}{\partial n} = (0, \sin\theta, -\cos\theta) \ . \ \text{grad} \ .$$

Those two directions which the two derivatives are defined here are shown in Figure 5. PQ refers to the direction of derivatives $\frac{\partial}{\partial m}$ and QR to $\frac{\partial}{\partial n}$.

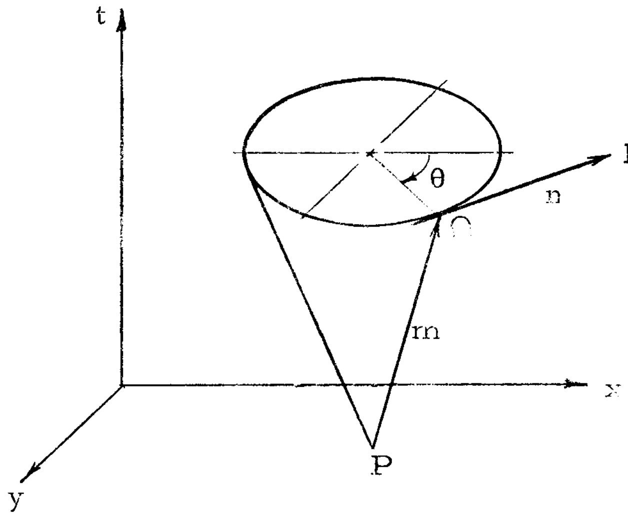

Figure 5. Two directions in compatibility relation.

Using the two derivatives defined in (4. 2), the compatibility relation in Equation (3. 24) becomes

$$a\frac{\partial \rho}{\partial m} + \rho(\cos\theta\frac{\partial u}{\partial m} + \sin\theta\frac{\partial u}{\partial n}) + \rho(\sin\theta\frac{\partial v}{\partial m} - \cos\theta\frac{\partial v}{\partial n}) = 0 \ . \qquad (4.3)$$

As for the two derivatives (i.e. $\frac{\partial}{\partial m}$ and $\frac{\partial}{\partial n}$) appearing in this compatibility relation, another differential equation should be developed. The equation must be the equation relating to the derivatives $\frac{\partial}{\partial m}$ and $\frac{\partial}{\partial n}$, and is used as an auxiliary equation to the compatibility Equation (4. 3). If the Equation (3. 7) is multiplied by $a\cos\theta$, Equation (3. 8) by $\frac{v}{a}\sin\theta + 1$ and Equation (3. 9) by $-\frac{v}{a}\cos\theta$, then added together to obtain

$$a\cos\theta\{A_{1\rho}\cdot \operatorname{grad}\rho + A_{1u}\cdot \operatorname{grad}u + A_{1v}\cdot \operatorname{grad}v\}$$
$$+ (\frac{v}{a}\sin\theta + 1)\{A_{2\rho}\cdot \operatorname{grad}\rho + A_{2u}\cdot \operatorname{grad}u + A_{2v}\cdot \operatorname{grad}v\}$$
$$- \frac{v}{a}\cos\theta\{A_{3\rho}\cdot \operatorname{grad}\rho + A_{3u}\cdot \operatorname{grad}u + A_{3v}\cdot \operatorname{grad}v\} = 0 ,$$

which may be rearranged.

$$a^2(\frac{1}{a}\cos\theta, \frac{u}{a}\cos\theta + \frac{v}{a}\sin\theta + 1, 0) \cdot \operatorname{grad}\rho \tag{4.4}$$
$$+ \rho\,(1 + \frac{v}{a}\sin\theta, u + a\cos\theta + \frac{uv}{a}\sin\theta, v + \frac{v^2}{a}\sin\theta) \cdot \operatorname{grad}u$$
$$+ \rho\,(-\frac{v}{a}\cos\theta, -\frac{uv}{a}\cos\theta, a\cos\theta - \frac{v^2}{a}\cos\theta) \cdot \operatorname{grad}v = 0.$$

Multiply through by $v + a\sin\theta$ in Equation (4.1) to get

$$(v + a\sin\theta)(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) = 0$$

or in vector form

$$(0, 0, v + a\sin\theta) \cdot \operatorname{grad}u - (0, v + a\sin\theta, 0) \cdot \operatorname{grad}v = 0 . \tag{4.5}$$

Add Equation (4.4) to Equation (4.5) to obtain

$$a^2\left(\frac{1}{a}\cos\theta, \frac{u}{a}\cos\theta + \frac{v}{a}\sin\theta + 1, 0\right). \text{ grad}\,\rho \tag{4.6}$$

$$+ \rho\left(1 + \frac{v}{a}\sin\theta, u + a\cos\theta + \frac{uv}{a}\sin\theta, 2v + a\sin\theta + \frac{v^2}{a}\sin\theta\right)$$

$$. \text{ grad}\,u + \rho\left(-\frac{v}{a}\cos\theta, -a\sin\theta - v - \frac{uv}{a}\cos\theta, a\cos\theta - \frac{v^2}{a}\cos\theta\right)$$

$$. \text{ grad}\,v = 0 ,$$

or

$$a^2\{\cos\theta\left(\frac{1}{a}, \frac{u}{a} + \cos\theta, \frac{v}{a} + \sin\theta\right) + \left(\frac{v}{a} + \sin\theta\right)(0, \sin\theta, -\cos\theta\}. \text{ grad}\,\rho$$

$$+ \rho\{(a + v\sin\theta)\left(\frac{1}{a}, \frac{u}{a} + \cos\theta, \frac{v}{a} + \sin\theta\right) - v\cos\theta(0, \sin\theta, -\cos\theta)\}$$

$$. \text{ grad}\,u + \rho\{-(a + v\sin\theta)(0, \sin\theta, -\cos\theta)$$

$$- v\cos\theta\left(\frac{1}{a}, \frac{u}{a} + \cos\theta, \frac{v}{a} + \sin\theta\right)\} .. \text{ grad}\,v = 0 . \tag{4.7}$$

In terms of the derivatives $\frac{\partial}{\partial m}$ and $\frac{\partial}{\partial n}$, the Equation (4.7) becomes

$$a\,\{a\cos\frac{\partial\rho}{\partial m} + (v + a\sin\theta)\frac{\partial\rho}{\partial n}\} + \rho\{(a + v\sin\theta)\frac{\partial u}{\partial m} - v\cos\theta\frac{\partial u}{\partial n}\}$$

$$- \rho\{(a + v\sin\theta)\frac{\partial v}{\partial n} + v\cos\theta\frac{\partial v}{\partial m}\} = 0 \tag{4.8}$$

Dividing through by a and rearranging

$$a\cos\theta\frac{\partial\rho}{\partial m} + (v + a\sin\theta)\frac{\partial\rho}{\partial n} + \rho\left(1 + \frac{v}{a}\sin\theta\right)\left(\frac{\partial u}{\partial m} - \frac{\partial v}{\partial n}\right)$$

$$- \rho\frac{v}{a}\cos\theta\left(\frac{\partial u}{\partial n} + \frac{\partial v}{\partial m}\right) = 0 . \tag{4.9}$$

This equation is then used with the compatibility equation obtained in the Equation (4.3). From those two equations a system of differential equations relating to the derivatives $\frac{\partial}{\partial m}$ and $\frac{\partial}{\partial n}$ can be obtained,

$$a\left(\cos\theta\frac{\partial \rho}{\partial m} + \sin\theta\frac{\partial \rho}{\partial n}\right) + \rho\left(\frac{\partial u}{\partial m} - \frac{\partial v}{\partial n}\right) = 0 ,$$

and

$$a\left(\sin\theta\frac{\partial \rho}{\partial m} - \cos\theta\frac{\partial \rho}{\partial n}\right) + \rho\left(\frac{\partial u}{\partial n} + \frac{\partial v}{\partial m}\right) = 0 .$$

In terms of pressure gradients, they become

$$\frac{1}{a}\left(\cos\theta\frac{\partial p}{\partial m} + \sin\theta\frac{\partial p}{\partial n}\right) + \rho\left(\frac{\partial u}{\partial m} - \frac{\partial v}{\partial n}\right) = 0 , \qquad (4.10)$$

$$\frac{1}{a}\left(\sin\theta\frac{\partial p}{\partial m} - \cos\theta\frac{\partial p}{\partial n}\right) + \rho\left(\frac{\partial u}{\partial n} + \frac{\partial v}{\partial m}\right) = 0 . \qquad (4.11)$$

Finally Equations (4.10) and (4.11) can be used for numerical computation.

If $v = 0$, and $\theta = 0, \pi$, the Equations (4.10) and (4.11) become

$$dp = \mp \rho a du \qquad (4.12)$$

along the directions

$$dx = (u \pm a)dt$$

These are the equations for the compatibility relations and their corresponding characteristics in one-dimensional unsteady inviscid flows.

Calculation Scheme

Writing the Equations (4.10) and (4.11) in difference forms along the bi-characteristic directions

$$\frac{1}{a}\cos\theta\, dp + \rho du = \left(-\frac{1}{a}\sin\theta\frac{\partial p}{\partial n} + \rho\frac{\partial v}{\partial n}\right)dm, \tag{4.13}$$

and

$$\frac{1}{a}\sin\theta\, dp + \rho dv = \left(\frac{1}{a}\cos\theta\frac{\partial p}{\partial n} + \rho\frac{\partial u}{\partial n}\right)dm . \tag{4.14}$$

These two equations may contain six unknowns, i.e., p_o, u_o, v_o, $(\frac{\partial p}{\partial n})_o$, $(\frac{\partial u}{\partial n})_o$, $(\frac{\partial v}{\partial n})_o$. Six unknowns need six relations to solve out. Therefore, three bi-characteristics passing through the vertex $P\,(t_o, x_o, y_o)$ of the rear Mach conoid are chosen arbitrarily, say, $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$, $\theta_3 = \pi$, for convenience. If the three bi-characteristics meet the plane $t = t_o - h$ at the points of which the conditions are known and denoted by subscripts 1, 2, 3, corresponding to the values θ_1, θ_2, θ_3, then the difference forms of the Equations (4.10) and (4.11) under the assumption $a = a_s >> |V|$, and $\rho = \rho_s$, can be written as:

when

$$\begin{aligned} \theta &= 0 \\ x_o &= x_1 + a_s h \\ y_o &= y_1 \\ m_1 &= \{(x_o - x_1)^2 + (y_o - y_1)^2\}^{\frac{1}{2}} \\ &= a_s h \end{aligned} \tag{4.15}$$

$$\frac{1}{a_s}(p_o - p_1) + \rho_s(u_o - u_1) = \frac{1}{2}\rho_s\{(\frac{\partial v}{\partial n})_o + (\frac{\partial v}{\partial n})_1\}m_1 \; .$$

(4.16)

$$\rho_s(v_o - v_1) = \frac{1}{2}\frac{1}{a_s}\{(\frac{\partial p}{\partial n})_o + (\frac{\partial p}{\partial n})_1\}m_1 + \frac{1}{2}\rho_s\{(\frac{\partial u}{\partial n})_o + (\frac{\partial u}{\partial n})_1\}m_1$$

when

$$\begin{aligned} \theta &= \frac{1}{2}\pi \\ x_o &= x_2 \\ y_o &= y_2 + a_s h \\ m_2 &= \{(x_o - x_2)^2 + (y_o - y_2)^2\}^{\frac{1}{2}} \\ &= a_s h \end{aligned} \tag{4.17}$$

$$\rho_s(u_o - u_2) = -\frac{1}{2}\frac{1}{a_s}\{(\frac{\partial p}{\partial n})_o + (\frac{\partial p}{\partial n})_2\}m_2 + \frac{1}{2}\rho_s\{(\frac{\partial v}{\partial n})_o + (\frac{\partial v}{\partial n})_2\}m_2$$

(4.18)

$$\frac{1}{a_s}(p_o-p_2) + \rho_s(v_o-v_2) = -\frac{1}{2}\rho_s\{(\frac{\partial u}{\partial n})_o + (\frac{\partial u}{\partial n})_2\}m_2$$

when

$$\begin{aligned} \theta &= \pi \\ x_o &= x_3 - a_s h \\ y_o &= y_3 \\ m_3 &= \{(x_o-x_3)^2 + (y_o-y_3)^2\}^{\frac{1}{2}} \\ &= a_s h \end{aligned} \tag{4.19}$$

$$\begin{aligned} -\frac{1}{a_s}(p_o-p_3) + \rho_s(u_o-u_1) &= \frac{1}{2}\rho_s\{(\frac{\partial v}{\partial n})_o + (\frac{\partial v}{\partial n})_3\}m_3 \\ \rho_s(v_o-v_3) &= -\frac{1}{2}\frac{1}{a_s}\{(\frac{\partial p}{\partial n})_o+(\frac{\partial p}{\partial n})_3\}m_3 - \frac{1}{2}\rho_s\{(\frac{\partial u}{\partial n})_o+(\frac{\partial u}{\partial n})_3\}m_3 . \end{aligned} \tag{4.20}$$

Equations (4.15), (4.17) and (4.19) are used to determine the coordinates at points 1, 2, 3. Equations (4.16), (4.18) and (4.20) are used to determine the unknowns: u_o, v_o, p_o, $(\frac{\partial u}{\partial n})_o$, $(\frac{\partial v}{\partial n})_o$, $(\frac{\partial p}{\partial n})_o$ at a general point $P(t_o, x_o, y_o)$. The conditions at points 1, 2, 3, are supposed to be known, or, the conditions on the surface $g(t_o - h, x, y) = 0$ are everywhere known. The calculation process is the same as laid down for rotational flows.

If the point $P(t_o, x_o, y_o)$ is the boundary point, then the unknowns decrease from six to four. The condition that the velocity vector should be tangent to the solid wall gives the relation between the two velocity components, u and v. Finally the independent unknowns are: p_o, u_o, $(\frac{\partial p}{\partial n})_o$, $(\frac{\partial u}{\partial n})_o$. Therefore only two bicharacteristics are properly chosen to find the conditions at point P. The same result for the point P, when P is a constant pressure point.

The particle paths can be found by integrating along the streamline between the given point at $t = t_o - h$ and the point at $t = t_o$.

V. AXIAL SYMMETRIC FLOWS

Let the velocity component in r-direction be $u = u(t, r, z)$, and the velocity component in z-direction be $v = v(t, r, z)$. Now consider the equations governing the unsteady axial symmetric, inviscid flows:

the equation of continuity

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial z} + \rho \frac{u}{r} = 0 , \tag{5.1}$$

the momentum equations

$$\rho(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z}) = - \frac{\partial p}{\partial r} , \tag{5.2}$$

and

$$\rho(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z}) = - \frac{\partial p}{\partial z} . \tag{5.3}$$

Assume that the fluid obeys the equation of state

$$dp = a^2 d\rho \tag{5.4}$$

Write in the vector forms of the Equation (2.2) for the Equations (5.1) through (5.4)

$$A_{1\rho} \cdot \text{grad}\,\rho + A_{1u} \cdot \text{grad}\,u + A_{1v} \cdot \text{grad}\,v + \frac{u}{r} = 0 \,, \tag{5.5}$$

$$A_{2\rho} \cdot \text{grad}\,\rho + A_{2u} \cdot \text{grad}\,u + A_{2v} \cdot \text{grad}\,v = 0 \,, \tag{5.6}$$

and

$$A_{3\rho} \cdot \text{grad}\,\rho + A_{3u} \cdot \text{grad}\,u + A_{3v} \cdot \text{grad}\,v = 0 \,. \tag{5.7}$$

where,

$$A_{1\rho} = (1, u, v), \;\; A_{1u} = \rho(0, 1, 0), \;\; A_{1v} = \rho(0, 0, 1)$$

$$A_{2\rho} = a^2(0, 1, 0), \;\; A_{2u} = \rho(1, u, v), \;\; A_{2v} = (0, 0, 0)$$

$$A_{3\rho} = a^2(0, 0, 1), \;\; A_{3u} = (0, 0, 0), \;\; A_{3v} = \rho(1, u, v)$$

and $\text{grad} = (\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial z})$.

The Equation (2.3) then becomes

$$A_\rho \cdot \text{grad}\,\rho + A_u \cdot \text{grad}\,u + A_v \cdot \text{grad}\,v = 0 \tag{5.8}$$

where,

$$A_\rho = c_1 A_\rho + c_2 A_{2\rho} + c_3 A_{3\rho} = (c_1, c_1 u + c_2 a^2, c_1 v + c_3 a^2) \,,$$

$$A_u = c_1 A_{1u} + c_2 A_{2u} + c_3 A_{3u} = \rho(c_2, c_1 + c_2 u, c_2 v) \,,$$

$$A_v = c_1 A_{1v} + c_2 A_{2v} + c_3 A_{3v} = \rho(c_3, c_3 u, c_1 + c_2 v)$$

Let the characteristic surface be $f(t, r, z) = 0$ in t, r, z-space. The conditions in Equation (2.4) become,

$$A_\rho \cdot (f_t, f_r, f_z) = A_u \cdot (f_t, f_r, f_z) = A_v \cdot (f_t, f_r, f_z) = 0$$

Substituting the calculated values of A_ρ, A_u, and A_v into it and get

$$c_1f_t + (c_1u+c_2a^2)f_r + (c_1v+c_2a^2)f_z = 0 \, ,$$

$$c_2f_t + (c_1+c_2u)f_r + c_2vf_z = 0 \, ,$$

and

$$c_3f_t + c_3uf_r + (c_1+c_3v)f_z = 0 \, .$$

or,

$$c_1(f_t + uf_r + vf_z) + c_2a^2f_r + c_3a^2f_r = 0 \, , \qquad (5.9)$$

$$c_1f_r + c_2(f_r + uf_r + vf_z) = 0 \, ,$$

and

$$c_1f_z + c_3(f_t + uf_r + vf_z) = 0 \, .$$

and the Equation (2.5) becomes

$$\begin{vmatrix} f_t + uf_r + vf_z & a^2f_r & a^2f_z \\ f_r & f_t + uf_r + vf_z & 0 \\ f_z & 0 & f_t + uf_r + vf_z \end{vmatrix} = 0$$

$$= (f_t + uf_r + vf_z)^3 - a^2f_z^2(f_t+uf_r+vf_z) - a^2f_r^2(f_t+uf_r+vf_z)$$

$$= (f_t+uf_r+vf_z)\{(f_r+uf_r+vf_z)^2 - a^2(f_z^2+f_r^2)\} \qquad (5.10)$$

This equation is the characteristic equation for unsteady axial symmetric flows. Compare this Equation (5.10) with (3.13). They are in the same form. Therefore there exists a quadratic cone for characteristic surfaces, and the bi-characteristic direction is

$$dr = (u + a\cos\theta)dt \qquad (5.11)$$

$$dz = (v + a\sin\theta)dt$$

The compatibility relation is obtained from the Equation (3.25)

$$a d\rho + \sin\theta\, du + \rho \sin\theta\, dv = -\rho a G dt \qquad (5.12)$$

where,

$$G = \sin^2\theta \frac{\partial u}{\partial r} - \sin\theta\cos\theta\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) + \cos^2\theta \frac{\partial v}{\partial z} + \frac{u}{r}$$

The numerical calculating scheme can be developed in the same way as for the rotational flows.

VI. SUMMARY AND CONCLUSION

In applying the general theory of characteristics for the system of hyperbolic partial differential equations to the unsteady, two-dimensional inviscid flows, the numerical methods of solutions are developed. The bi-characteristic curves play the role in numerical integrations. This method is called bi-characteristic method.

Suppose that there is a point $P(t_o, x_o, y_o)$ to be determined its conditions, i.e., p_o, u_o, v_o, and the given data are the known conditions of all points at the stage $t = t_o - h$. For unsteady, two-dimensional, inviscid flow, there are three cases: rotational flows, irrotational flows in cartesian coordinate, and axial symmetric flows. In the case of rotational flows four bi-characteristics lying on the rear branch of the Mach conoid with vertex at P and the streamline through P are used in computations. If P is a boundary point or a constant pressure point, then only three bi-characteristics and the streamline through the point P are enough to solve the unknown conditions at that point.

In the case of irrotational flows three bi-characteristics through the point P give six relations, which can be used to determine the conditions at that point. If the point P is a boundary point or a constant pressure point, then only two bi-characteristics are used.

In the case of unsteady axial symmetric flows, there are quite similar forms in the equation of characteristics and the compatibility relations. Therefore, the methods developed in rotational flows case can also be applied to the axial symmetric case.

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