AN ABSTRACT OF THE THESIS OF

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Title: The Violin's Sound: A Mathematical Exploration Employing Principles of Continuum Mechanics and Numerical Methods

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This thesis explores the vibrational behavior of the main components of sound production in the violin using a continuum mechanics approach. The author provides a mathematical description of the regions in the vibrating continuum, and begins to develop a system of equations governing their behavior, focusing on the air in the resonant chamber. Later chapters, contain discussion of issues involved in solving the system of equations, and examples involving both formal and numerical methods. The existence of a unique formal solution would allow mathematicians to make predictive models for sound waves of instruments based on physical characteristics such as size, shape, density and elasticity.

The Violin's Sound: A Mathematical Exploration
Employing Principles of Continuum Mechanics
and Numerical Methods

by
Nina J. Gydé

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Nina J. Gydé, Author

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by
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Chapter 1

INTRODUCTION

The mysteries of the Stradivarius have defied the efforts of violin makers and scientists for more than two centuries. Though there have been many skilled luthiers over the years who have produced instruments of excellent quality and beautiful tone, none have been able to reproduce the supremely resonant sound achieved by late 17th and early 18th century masters such as Stradivari, Guarneri, and Amati. Most instruments made today are patterned after the work of one of these masters. In

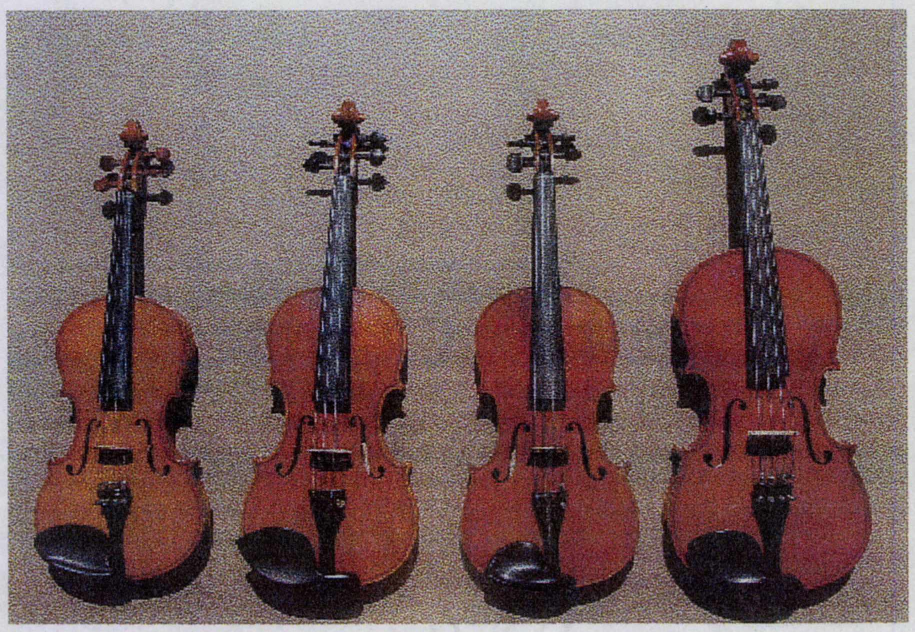


FIGURE 1.1. Modern violins. Note the similarities in shape and construction.

fact, many makers include the name of one these Italian masters on their instrument label, assumedly in an effort to increase the value of their work. But makers who have sought to reproduce the Golden Age instruments precisely by using the same measurements and materials have still failed to produce instruments with the same quality of tone. Thus makers have learned that tone production in the violin involves more than just the dimensions and type of wood used. More recent studies involving the chemical composition of the undercoat and varnish of historic instruments have produced decidedly more headway in recreating the sound of the Italian masters.

During the 1850s and early 1860s Hermann Helmholtz spent a good deal of time and effort studying musical sound and human perception. Since the publication of his findings (especially in the book *Die Lehre von den Tonempfindungen*, though there are also many related lectures and articles), the relationship between musical sound and mathematics has been well established. The work of Helmholtz helps us to understand why we prefer the tone of one instrument over another. Because of his work, we are able to describe a musical sound mathematically and determine whether or not that sound is likely to have a pleasing quality to our ear. Yet even a century after his time, the physical acoustics of the violin remain elusive from a mathematical point of view. Any progress we have made toward achieving the "perfect" violin has been by two centuries of trial and error, aided by the observations of science. Though even the most basic physics texts discuss the mathematics of the vibrating string, few authors venture into the complexities of the resonant chamber. There is no mathematical model which can be used to predict the output of a violin before it is made, thus crafters must often work for several years before they know if their ideas produce the desired effects on tone.

Since initial attempts to replicate instruments involved size, shape, and proportion, the geometry of the violin has been studied extensively. Scientists have also studied the sound waves produced by violins, and the vibrations of various separate parts of the instrument (e.g. the vibrating string and the front and back plates). The sound waves have been analyzed and described mathematically through spectral analysis. But the complexity of the instrument as a vibrational system, due to both the number of vibrating parts and the interaction of those parts, has thus far prevented a complete mathematical description of how different properties of a particular instrument influence the sound it produces. The purpose of this work is to explore how one might approach making and solving a mathematical model for the sound produced by a violin using the concepts of continuum mechanics and numerical methods, and to generate a limited mathematical model which incorporates some of the defining features of the instrument. The ultimate goal of this research, should it be continued beyond the scope of this work, would be to produce a complete mathematical model, where the particulars about an instrument's measurements and composition could be entered into the equation as constants, and a sound curve for the instrument could be produced. If a complete model could be formulated, spectral analysis on existing instruments would allow us to test its validity. This work, however, will be but a stepping stone on the path toward this ultimate goal.

Chapter 2

The Big Picture

To begin, we must look at what comprises the acoustical system of a violin. An analysis of the system's components will allow us to choose which are most important to sound production, and therefore which to concentrate on describing mathematically in this work. Anyone who has played or studied a violin will realize that this is no small task. Even the bow with which the instrument is played has an influence on the sound produced. Selecting the right bow for an instrument is like a young wizard shopping for wands at Olivander's. (Rowlings, 1997) The bow is the wand through which the musician works his magic. Though other subtleties of technique (the flick

FIGURE 2.1. Violin Bow

of a wrist, the tilt of an arm) are indicative of the skill level of the performer, it is the bow that transmits the energy from the artist to the medium. Hairs from horse tails may not have the romantic appeal of phoenix feathers, but the powerful violinist wields his bow of penumbucco and horse hair to fill the music hall with the magic of sound. The weight of the bow, and the materials from which it is crafted, combined with the skill level of the player, have a profound influence on the tone an instrument produces. But in this work, we are interested in what the instrument itself does with this initial stimulus, that is, what frequencies are added or emphasized by various parts of the instrument before the sound wave finally issues from the f-holes on its way to our ears. So, to simplify matters, let us first consider the instrument alone, with a single pizzicato note being played at a known frequency (pitch).

When the string is initially displaced, energy is stored, which causes the string to vibrate when it is released.The string is fixed at one end, where it is held against the

FIGURE 2.2. The initial stimulus.

nut by the force of tension, and wrapped around a tuning peg. Near the other end, the string is held (again by tension) against the bridge, which rests on the top plate of the body of the instrument. The bridge is not attached, but is held against the top plate by the tension of the string. A couple inches to the other side of the bridge, the string is attached to the tailpiece, which does not touch the face of the violin, but is, instead, looped around the button at the end of the instrument and held there by the force of tension. The end of the tailpiece where the string is attached is free to vibrate, but since the ebony tailpiece is much more dense, and much less elastic than the string, we will consider that end of the string to be fixed as well. The behavior of a vibrating string with fixed ends is a well-known elementary physics problem. The motion of the string at its point of contact with the bridge causes the bridge to rock, thus transferring the vibrations from the string to the top plate through the vertical motion of the bridge's feet.

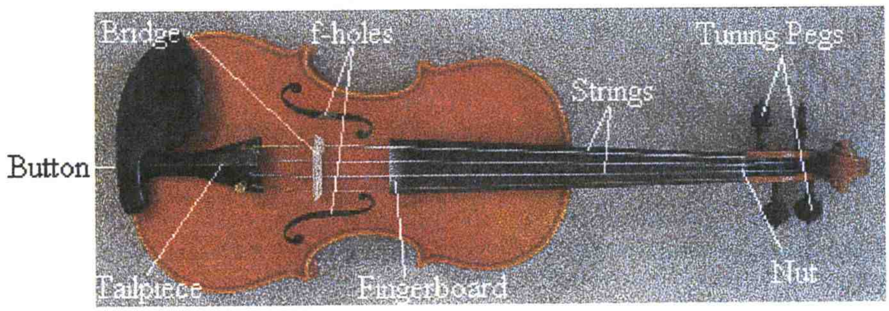


FIGURE 2.3. Violin top view.

Some significant ways in which the vibrations of the top plate of the violin differ from those of the string are:

1. The string has only one point where its motion is being influenced (the location where it was originally displaced by the finger of the player), whereas the top plate of the instrument's body has two points of displacement (the two feet of the bridge);
2. The string has one significant spatial dimension (length), while the plate has two (length and width);
3. The vibrations of the string (assuming a string of perfect quality) are uninfluenced by any changes in density or discontinuities (breaks in the string). The top plate, however, has two f-holes. These holes affect the progress of vibrations issuing outward from the feet of the bridge.

It is important to notice that the link between the string and the plate transmits motion in both directions. We can think of the bridge as a two way conduit. There is one instant of stimulus to the plucked string–that moment in time when it was displaced and released. The plate continues to receive stimuli from the bridge for as long as the string is still in motion, but in turn, the vibrations of the plate are transmitted via the bridge back to the string. The string, having less mass, then begins to vibrate in sympathy with the plate, and its motion continues as long as the plate is still vibrating. In this way, certain vibrations of both components are reinforced, and last longer than they would in either component alone.

The bass bar and soundpost also influence the vibration of the top plate. They are inside the resonant chamber, and so are not readily visible unless one knows of their existence. The bass bar is a comparatively thick bar of wood that is glued to the underside of the top plate between the left foot of the bridge and the left f-hole. It runs lengthwise, damping vibration of part of the top plate in that direction, and tapers to an end before reaching the edge of the plate. The soundpost is near the right foot of the bridge. Its exact placement varies, and is determined empirically by the maker when the violin is completed. It is held in place by the tension of the front and back plates, and serves to transmit vibrations between the two. If the strings are loosened, and tension on the top plate is thereby reduced, the soundpost can easily be jarred out of place. It requires a special tool and skill to reposition correctly. The placement of the soundpost is important because it provides the initial displacement of the back plate. Its French name, *âme* (meaning "soul"), is also indicative of its profound influence on the instrument's tone. As with the bridge, the soundpost is a two-way vibrational conduit.

Since the back plate does not have a bass bar or f-holes, one would expect its vibrations to spread outward in a different pattern than those of the top plate. The vibrations of the back plate are much less pronounced than those of the top for several reasons:

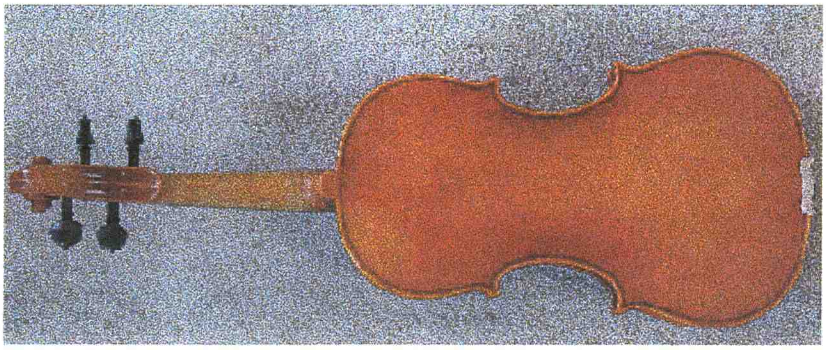


FIGURE 2.4. Violin back view.

1. The top is usually made from spruce, which has a good deal of flexibility across the grain, but not along the grain. The back plate is usually made from a less flexible variety of wood such as maple.
2. The back plate is thicker than the top in the area between the C-bouts where it supports the soundpost. The top is generally thinned to approximately 3mm, whereas the back may be nearly twice that thickness near the soundpost.
3. The back plate also does not touch the bridge, and therefore does not receive any direct stimulus from the strings. At the location of the soundpost, however, the back plate must always be in phase with the top plate.

The only other direct influences on the vibration of the back plate are the motion of the air in the resonant chamber, and whatever vibrations may be transmitted from the front plate through the sides of the body (ribs). Due to rigid properties of wood, the ribs are not free to vibrate vertically. Any horizontal components of vibration they may experience would have to be small, since the ribs are less than two inches high, and have nodes (fixed points) where they are glued to the plates at the top and bottom. Motion of the ribs, therefore, does not significantly affect the motion of the back plate.

The volume of air in the resonant chamber has three spatial dimensions (length, width and height). As the plates vibrate, small changes in the volume of the chamber ensue. Since the vibrations occur in a regular pattern over time, so, also, do the changes in volume. When the volume of the chamber becomes smaller, it causes the air inside to be compressed. The increase in pressure inside the chamber forces some air out the f-holes to the lower pressure region of air outside the instrument. When the volume of the chamber becomes larger, the pressure on the air inside decreases, and air is sucked in through the f-holes from the higher pressure region of air outside the instrument. The concept is similar to breathing, though on a much smaller scale. The studies of Helmholtz show that musical sound is what we experience whenever a regular pattern of air pressure changes stimulates the sensory membranes in our ears. This regular pattern of compressions issuing from the f-holes of the violin is the sound wave which we seek to describe mathematically. The number of compression patterns (cycles) which occur each second is called the frequency of the sound. Musicians identify this as pitch. Small patterns within the main cycle add overtones. The combination of overtones which occur with a given pitch varies from instrument to instrument. Overtones determine the richness of the instruments tone. The amplitude of the vibrations of the string (and therefore the plates) determines the amount of change in volume of the chamber, which in turn determines the force with which air is expelled from the f-holes. Therefore, the amplitude determines how loud the sound is when perceived by the listener.

Since the air inside the resonant chamber is directly in contact with the two plates and the ribs, its motion has an influence on the motion of the plates. As with the top plate and string, once a vibrational pattern is established inside the resonant chamber, it is transmitted directly to the plates, and indirectly (via the bridge) to the strings, thus reinforcing certain frequencies in the entire system. The string, where the initial stimulus occurs, shares its energy with the plates. Because of its fixed ends, and the location of the initial stimulus, it is only possible for the string to vibrate at certain frequencies. The plates then begin to vibrate at frequencies which are consonant with those introduced by the string. Since the fundamental tone introduced by the string has the most energy, it will also be the fundamental tone of the vibration in the plates if the plates are able to sustain a vibration at that frequency. Otherwise, the fundamental tone in the plates will be the closest frequency at which they can readily vibrate. The same relationship holds between the plates and the air. The air will only vibrate at frequencies which are introduced to it by the plates. But the plates may have added overtones which were not present in the vibration of the string, so the sound wave becomes richer as it is passed from component to component, but the string provides the fundamental frequency for the whole system. When energy is passed back to the strings by the motion of the top plate, all strings may vibrate in sympathy, but the string which was originally displaced naturally vibrates longer and with more amplitude than the other strings because it has the most stored energy, and is already vibrating in consonance with the air and plates.

Physicists have determined that the motion of each separate component of the system (string, plates and air) can be described by a wave equation of the form

$$u_{tt} = a^2 \Delta u$$

(in one, two, and three dimensions respectively), though the shape of the body of the instrument, both in outline and in cross section, poses difficulties for mathematicians attempting to describe the two- and three-dimensional components. Another difficulty arises when we try to link the components and describe their influence on one another mathematically. This would be much easier if vibrations were only transmitted in one direction, say from string to plates to air and not back from air to plates to string. The construction of the top plate with f-holes and bass bar also complicates the mathematics. Once a vibrational pattern is established, however, the separate components are vibrating in sympathy with one another. Therefore, the important thing to consider is which vibrations are strong enough to reach the ear of the listener. Since the air in the resonant chamber is less dense and more elastic than the other components, it responds most readily when force is applied. We expect the air issuing from the f-holes to have the most overtones, and the greatest force of vibration. It is also well known from experiment that without the resonant chamber, the sound of the instrument would be pitifully weak. Thus, the motion of the air in the resonant chamber is key to our description of the vibrational system. But the vibrations of the string and plates are the stimulus for the pressure changes of the air in the resonant chamber. Since the violin body provides the boundary for the chamber in which the vibrating air is contained, its shape and properties continuously influence the motion of the air. In other words, the volume and motions of the air are continuously changing as the plates move. We assume that once a pattern of air pressure changes is established, the other components continue to vibrate in sympathy, and therefore do not add any new frequencies to the sound wave. In this work, therefore, we focus our attention on the behavior of the air inside the resonant chamber. The string and plates provide what we call the "initial conditions" and "boundary conditions" for our equation of vibration.

Chapter 3

PROPERTIES OF THE VIBRATING MEDIUM

Now that we have decided where to begin, let us turn our attention to mathematical considerations. Our quest is to explore the possibility of making a mathematical model that will describe the motion of the air which issues from the f-holes of a violin (the sound wave) due to the regular variations in pressure inside of the resonant cavity. We want to keep this model as general as possible so that any individual could change parameters such as the exact shape and dimensions, and the physical properties of the instrument such as density, elasticity, and thickness of its parts. But we also want to make the model specific enough so that it is representative of the vibrational behavior of a violin, and not just any generic resonant box. There is a good deal of skepticism as to whether such a model is possible. The following quotation from Lothar Cremer's *The Physics of the Violin* is a good representative of this skepticism:

> "An aviation engineer, say, can generate predictive models out of elements such as beams, plates, and shells, with the aid of a computer. No such models will ever be possible for the violin; in any case, predictive models are indispensable only when human lives are at stake. In the realm of art, in which instruments are 'played,' there is more freedom in design. If an engineering firm equipped with all of today's knowledge and instrumentation had been given the task of developing a string instrument, the resulting design would not be the same as the actual, empirically developed one."(Cremer, 1984)

Cremer also later notes that "Since...the shapes of the boundaries are not amenable to calculation, the exact position of the air cavity resonances can be determined only by experiment."(Cremer, 1984) But what is the purpose of mathematics if not to describe physical phenomenon in the interest of predicting without doing? Of what use would such predictions be if there were not some freedom of design? Since the basic features of today's empirically developed violin evolved over more than three centuries, there must be some reason that this model produces a sound which we perceive as superior to previous instrument designs. A mathematical model may shed some light on why this design is superior, or if a better one could be developed. Helmholtz found that our perception of a sound as "superior" has to do with what combination of individual frequencies are present in the sound wave, and which frequencies are present with the most strength. We know mathematics can describe the sound wave because it has been done through spectrograms and Fourier analysis. But these methods do not describe the wave in terms of the physical properties of the instrument. The complexity of the mathematics should not prevent us from making a model if we are willing to put in the time and thought required, though solving the model may be another matter entirely if the proper technique has not yet been discovered. Spectrograms do, however, give us an idea as to the general nature of the solutions.

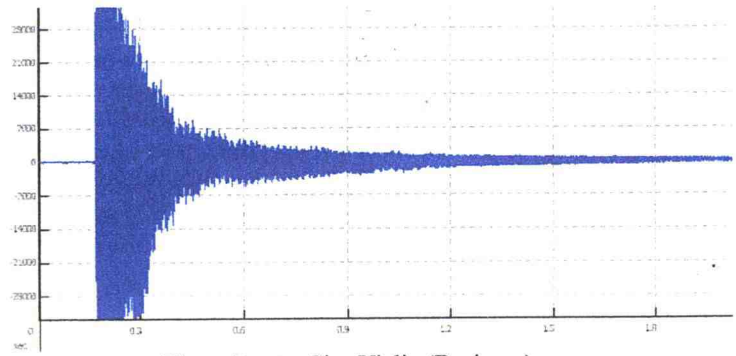
It is reasonable to expect that our problem is well posed because:

1. There is a sound wave produced each time a string is displaced and released. This sound wave can be recorded and mathematically described.
2. It is reasonable to assume that the sound wave produced by a given instrument is unique if the same exact physical conditions are present. These conditions would include environmental conditions such as temperature and humidity, as well as conditions governing the vibrating system such as force and direction of initial displacement, and tension, density and elasticity of the string.

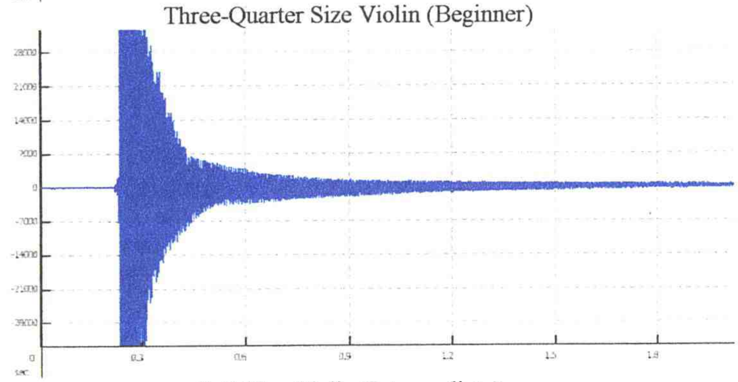
3. It is reasonable to assume that the sound wave depends continuously on the motion of the plates, which depends continuously on the environmental conditions, the physical qualities of the instrument, and the input from the player.

Having established that a model is theoretically feasible, though complicated, we turn to the question of what features one might consider to be defining characteristics of the violin. Visually, the body shape seems most important, since this is how many people distinguish the violin from other stringed instruments. Size and proportion are also important, as we see that cellos and violas have qualities of tone distinguishable from those of violins. The bass bar, soundpost, and f-holes directly and substantially affect the vibrations of the plates, and therefore can not be ignored as major contributors to tone quality. Some of these may have more affect on our ability to distinguish a violin from a clarinet or a guitar, while others may have more affect on distinguishing one violin from another. There are many other factors to consider, but we will begin by devising a mathematical description which takes these features into consideration.

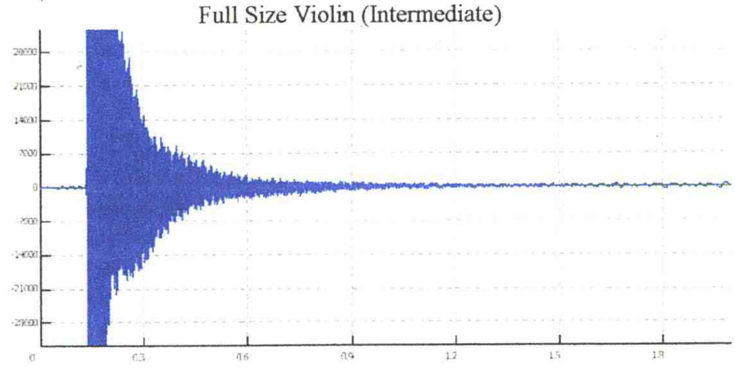
The sound waves pictured in plates 1, 2, and 3 show a single pizzicato note on the open A string of each of the four instruments pictured in Figure 1.1. The sound waves correspond to the pictured instruments in order, from left to right. Notice in Plate 1 that the four sound waves have a similar overall appearence, but the detailed views in Plate 2 show that their composition is actually quite different. Plate 3 shows which frequencies occur in each sound wave, and their relative strengths. In the interest of detail, the top portion of the peaks corresponding to the fundamental and first overtone are not shown. These two peaks are substantially higher than any of the others. There were also higher frequencies present in all four tones, but we have limited the horizontal axis to a maximum of 3000 cycles per second in order to show more detail of the stronger frequencies (those nearer to the fundamental).



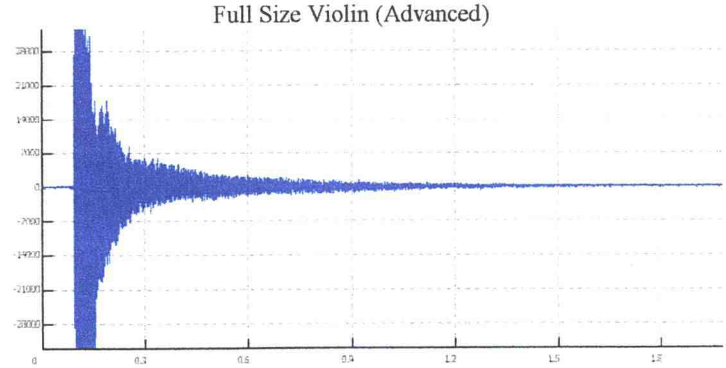
Three-Quarter Size Violin (Beginner)



Full Size Violin (Intermediate)

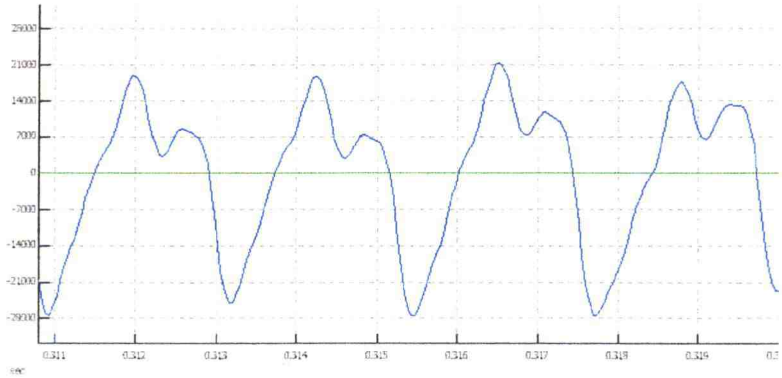


Full Size Violin (Advanced)

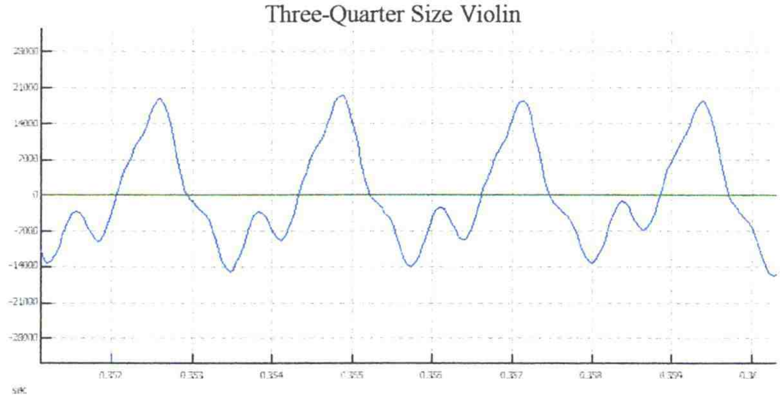


Viola (Intermediate)

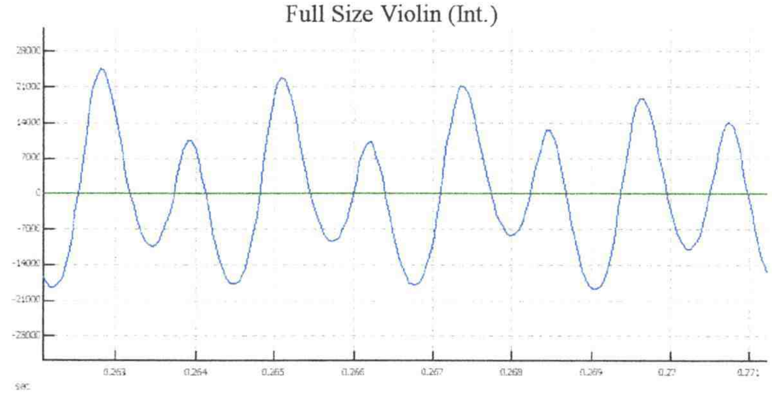
Plate 1 - Sound waves from the four instruments in Figure 1.1.
Single pizzicato note on the open A string.



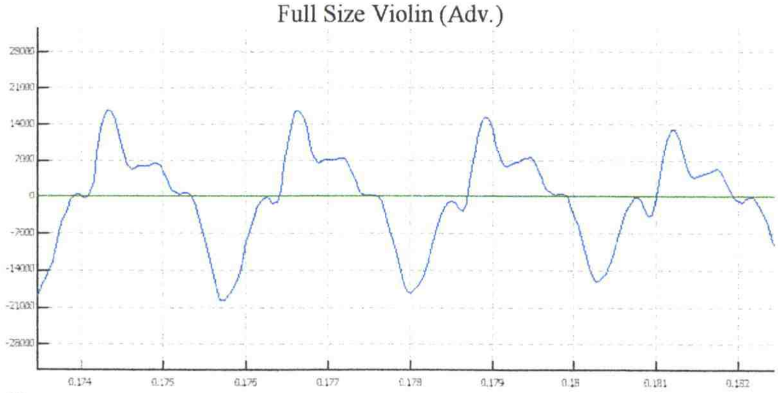
Three-Quarter Size Violin



Full Size Violin (Int.)

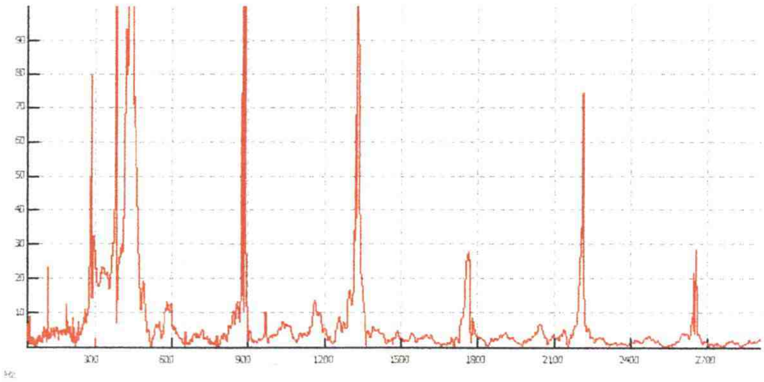


Full Size Violin (Adv.)

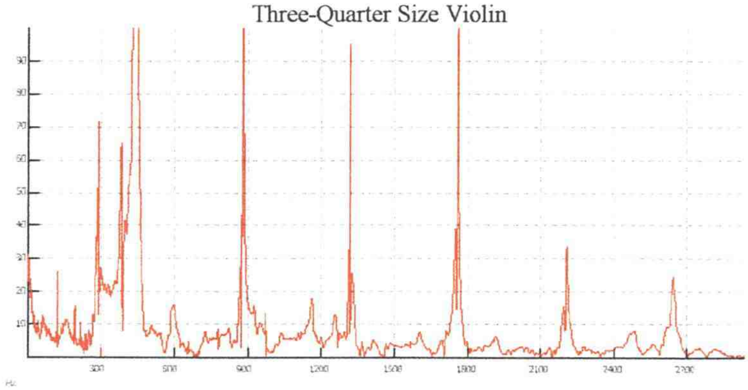


Viola

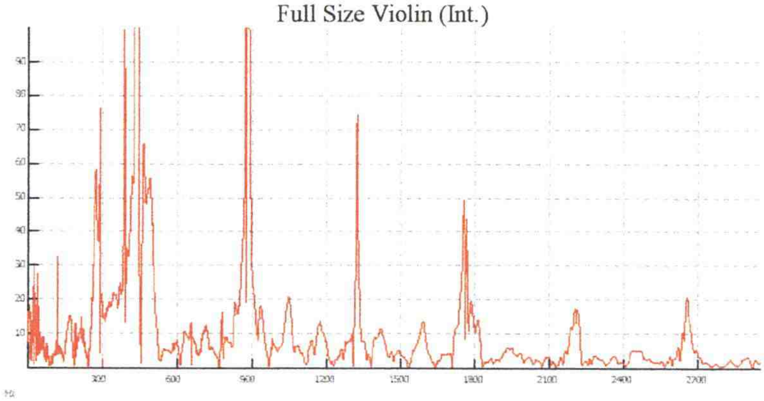
Plate 2 - Zoom View. Sound waves from the same four instruments.



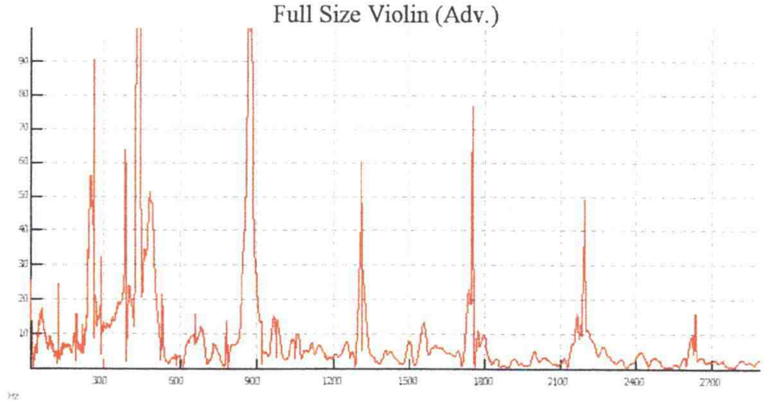
Three-Quarter Size Violin



Full Size Violin (Int.)



Full Size Violin (Adv.)



Viola

Plate 3 - FFT spectrum showing relative strengths of various overtones.

We will model the air in the violin's resonant chamber as a continuum with mass density (ρ), pressure (p), and velocity (v), all of which depend on time (t) and position (x). We assume that the air inside the violin is at the same constant temperature as the outside air. We expect, as with all forces, that the motion of the air inside the resonant chamber of the violin is governed by Newton's Laws. Thus, we describe the properties of the air as follows.

3.1 Notation

position	x = (x, y, z)
displacement of plates	$u(x, y, z, t)$
velocity of air	$d\mathrm{x}/dt = v(x, y, z, t)$
mass density	$\rho = \rho(\mathrm{x}, t) =$ mass per unit volume
mass flux	$q = q(\mathrm{x}, t) = \rho v$
pressure	$p = p(\mathrm{x},t)$
volume	$V = V(t)$ (denotes region or quantity, depending on context)
surface region	S
area of f-holes	$A(t)$
element of surface	dS
outward unit normal to surface	$\hat{n}$
element of volume	$dV = dzdydx$

3.2 Conservation of Mass

The rate of change of mass is equal to the net rate at which mass enters or leaves the resonant chamber through the f-holes plus the rate of mass produced or lost within the chamber. Let $\mathcal{V}$ represent an arbitrary volume of air inside our resonant chamber with surface region $\mathcal{S}$. Since no mass is produced or lost inside the violin, we can translate this law of physics mathematically, using elementary calculus, to

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, dV = -\int_{\mathcal{S}} q \cdot \hat{n} \, dS.$$

Using the Gauss divergence theorem, this equation can be written as

$$\int_{\mathcal{V}} \rho_t \, dV = -\int_{\mathcal{V}} div \; q \, dV, \text{ or}$$

$$\int_{\mathcal{V}} (\rho_t \; + div \; q \;) \, dV = 0$$

Since this law holds for any arbitrary volume, it follows that the integrand is identically zero. Hence we write the equation more simply as the Continuity Equation:

$$\rho_t + div \; q = 0 \qquad or \qquad \rho_t + div(\rho v) = 0.$$

3.3 Conservation of Momentum

Newton's second law tells us that the rate of change of momentum is equal to the sum of the forces acting on a region. Let V_t represent the region occupied by the resonant chamber at an arbitrary time t. In the case of the violin, the surface force acting on V_t is pressure from the moving plates. The body force is gravity. Since the violin is small, gravity has nearly the same effect at every point in the continuum. We therefore expect its effect on the relative motion of the particles of air to be negligable. Thus we translate Newton's second law mathematically as

$$\frac{d}{dt}\int_{V_t} \rho v dV = -\int_{S_t} p\hat{n}\, dS = -\int_{V_t} \nabla p\, dV.$$

Via the transport theorem, we can also write this as

$$\int_{V_t} \rho\left[v_t + (v\cdot\nabla)v\right] dV = -\int_{V_t} \nabla p\, dV,$$

and since this holds for arbitrary V_t, we can write the Momentum Equation:

$$\rho\left[v_t + (v\cdot\nabla)v\right] = -\nabla p.$$

In the case of small deformations, which we assume for the vibrations of the violin, we can write

$$\rho\left[v_t + (v\cdot\nabla)v\right] = \rho_o v_t,$$

where ρ_o is the equilibrium density of the air, and the Momentum Equation becomes

$$\rho_o v_t = -\nabla p.$$

3.4 Equation of State

The sound wave we seek to describe is a regular variation in pressure inside the violin relative to the outside air pressure. According to Boyle's Law, air pressure is inversely proportional to volume. Since $\rho =$ mass per unit volume, it follows that air pressure is directly proportional to density. So for air at a given constant temperature, $p = \rho R$ for all t, where R is the gas constant for air at that temperature. This gives the result that

$$\frac{p}{\rho} = \frac{p_o}{\rho_o} = R, \text{ or } \frac{p}{p_o} = \frac{\rho}{\rho_o}.$$

Since we assume the violin is being played in a room with contstant air temperature, one might initially expect the vibrations to be governed by this equation. Upon further study, however, one will find that rapid compressions and expansions of air (such as those caused by the vibrations of the plates) cause temperature variance, which changes the elasticity of the air. The modern air conditioner is one familiar example of how rapid compression and expansion affect air temperature. In an air conditioner the air is rapidly compressed, allowed to dissipate the resulting heat energy, and then is released (expanded) back into the room. The re-expanded air is much cooler than it was initially. Thus we see that the vibrations inside the violin's chamber are affected by temperature even if the temperature of the outer environment is held constant.

To determine the correct equation of state, therefore, we instead assume that entropy, not temperature, is constant over time. We take internal energy of the chamber to be a function of entropy and volume, and we express temperature as the change in internal energy with respect to entropy

$$T = \frac{\partial e}{\partial \eta},$$

and pressure as the opposite of the change in internal energy with respect to volume

$$p = -\frac{\partial e}{\partial V}.$$

We can then translate the differential equation

$$de = \frac{\partial e}{\partial \eta} d\eta + \frac{\partial e}{\partial V} dV$$

into

$$de = T d\eta - p dV.$$

Solving this for $d\eta$ gives

$$d\eta = \frac{de}{T} + \frac{pdV}{T}.$$

By Boyle's Law, we know that

$$\begin{aligned} pV &= nRT, \text{ or} \\ \frac{p}{T} &= \frac{nR}{V}. \end{aligned}$$

Let C_V represent the specific heat of the air at constant volume, and C_P the specific heat at constant pressure. We then use the identities

$$e = C_V T \text{ and } nR = C_P - C_V$$

to get

$$de = C_V \, dT, \text{ and } \frac{p}{T} = \frac{C_P - C_V}{V},$$

and hence by substitution,

$$d\eta = \frac{C_V}{T} dT + \frac{C_P - C_V}{V} dV.$$

By integrating this equation to eliminate the differentials, we get

$$\begin{aligned}
\eta &= C_V \ln T + (C_P - C_V) \ln V + \kappa \\
&= \ln \left(T^{C_V} V^{C_P} V^{-C_V} \right) + \kappa \\
&= \ln \left(\frac{T}{V} V^{\frac{C_P}{C_V}} \right)^{C_V} + \kappa \\
&= C_V \ln \left(\frac{p}{C_P - C_V} V^{\frac{C_P}{C_V}} \right) + \kappa
\end{aligned}$$

Examining this equation for entropy leads to the conclusion that entropy can only be constant if pV^{λ} is constant for $\lambda = C_P/C_V$. Since density is inversely proportional to volume, this also implies that p/ρ^{λ} is constant over time, thus yielding the equation

$$\begin{aligned}
\frac{p}{\rho^{\lambda}} &= \frac{p_o}{\rho_o^{\lambda}}, \text{ or equivalently} \\
\frac{p}{p_o} &= \left(\frac{\rho}{\rho_o} \right)^{\lambda}.
\end{aligned}$$

This equation describes the relationship between air pressure and density when rapid compressions such as sound waves are considered. The constant, λ, which represents the ratio of the specific heat of air at constant pressure to that at constant volume, has an empirically determined value of approximately 1.4.

3.5 Equation of Motion

Let $\sigma(\mathrm{x},t)$ represent the relative change in density of the air inside the chamber to the outside air, so

$$\frac{\rho}{\rho_o} = 1 + \sigma(\mathrm{x}, t).$$

We then have

$$\frac{p}{p_o} = (1 + \sigma(\mathrm{x}, t))^{\lambda}.$$

By the binomial theorem,

$$(1 + \sigma)^{\lambda} = \sum_{n=0}^{\infty} \binom{\lambda}{n} \sigma^n = 1 + \lambda\sigma + \frac{\lambda(\lambda - 1)}{2!}\sigma^2 + \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!}\sigma^3 \ldots$$

Since the vibrations are small, the changes in mass density are small, which means σ is small. We therefore neglect non-linear terms and approximate p/p_o by $1 + \lambda\sigma$. This gives $p = p_o + p_o\lambda\sigma$, which means

$$\nabla p = p_o \lambda \nabla \sigma. \qquad (*)$$

It is useful to note here that the relative change in pressure, $\lambda\sigma$, is what our ears sense, and we perceive as sound. Thus if we can determine the mathematical nature of σ, expressed in terms of known quantities, we will have a usable equation for the sound wave. Since we seek a solution for σ, our equation of motion should be in terms of σ.

Notice that

$$\rho = \rho_o (1 + \sigma) \Rightarrow \rho_t = \rho_o \sigma_t.$$

So by the Continuity Equation,

$$\rho_o \sigma_t + div\,(\rho v) = 0.$$

Since $div\ (\rho v) = div\ (\rho_o v)$, the Continuity Equation becomes $\rho_o(\sigma_t + div\ v) = 0$, which implies $\sigma_t = -div\ v$. Taking the time derivative of both sides then gives

$$\sigma_{tt} = -div\ v_t. \qquad (**)$$

If we take the divergence of both sides of the momentum equation, we have

$$\rho_o div(v_t) = -div(\nabla p).$$

To obtain an equation of motion in terms of σ, substitute $(*)$ and $(**)$ into this equation to get

$$\begin{aligned} -\rho_o \sigma_{tt} &= -div(p_o \lambda \nabla \sigma), \text{ or} \\ \sigma_{tt} &= \frac{p_o}{\rho_o} \lambda \Delta \sigma. \end{aligned}$$

Since p_o, ρ_o, and λ are all positive, we can write

$$\sigma_{tt} = a^2 \Delta \sigma, \quad \text{where } a^2 = \frac{p_o}{\rho_o} \lambda.$$

This is known as the Wave Equation. Since the properties by which we derived this equation were specific to the air, this equation applies to $\sigma(x, y, z, t)$ for x inside the resonant cavity, and $t \geq 0$.

3.6 Initial Conditions

Notice that when $t = 0$, $\rho = \rho_o$ so we have $\sigma(\mathrm{x}, 0) = 0$ for all x in the resonant cavity. This is the initial relative change in density for our system. At time $t = 0$, the Continuity Equation imposes the initial condition $\sigma_t(x, 0) = -div\ v(\mathrm{x}, 0)$ for all x in the resonant cavity. Since this second condition is related to the displacement of the particles of the continuum, we would expect it relate, also, to the displacement of the plates. We will discuss this further in Chapter 4.

Since $t = 0$ is the moment when the string is displaced, only points of the continuum near the bridge have initial velocity. The displacement of the top plate is concentrated at the right foot of the bridge because the string is initially displaced to the right. Therefore the initial value for σ_t should describe a sudden intense pressure change at the foot of the bridge, with intensity decreasing to zero outside of a small neighborhood centered at that foot. Since the elastic properties of air are the same in all directions, we expect the effects of this initial disturbance in pressure to eminate spherically over time until they meet with resistance from the boundaries. This will not be the case for the wood plates, however, since wood is more elastic across its grain than along it.

Chapter 4

THE BOUNDARY

The Wave Equation and initial conditions for σ found in Chapter 3 depend only on the physical properties of the medium contained inside and around the resonant chamber (in our case, air). This means that the general solution to the wave equation will describe the motion of air in any generic container under the same initial conditions. To make the solution (or sound wave) specific to the violin, we must describe the boundary which contains the air. This is where it becomes important which features we choose to emphasize as defining characteristics of the instrument. The boundary conditions for σ depend on the physical properties, and therefore the vibrational properties, of the materials from which the violin is crafted. Changes in air density are inversely proportional to changes in the volume of the resonant chamber. Using the definition of density as mass per unit volume,

$$\begin{aligned}
\frac{\rho}{\rho_o} &= \frac{(m/V)}{(m_o/V_o)} = \frac{m}{V}\cdot\frac{V_o}{m_o} = \frac{m}{m_o}\cdot\frac{V_o}{V} \Rightarrow \sigma = \frac{m}{m_o}\cdot\frac{V_o}{V} - 1, \text{ so} \\
\frac{p}{p_o} &= 1 + \lambda\sigma = 1 - \lambda + \lambda\left(\frac{m}{m_o}\right)\left(\frac{V_o}{V}\right).
\end{aligned}$$

Thus we see that variations in the volume of the resonant chamber directly affect pressure variations of the air inside. The volume of any region is determined by the shape of its boundary, hence the study of the geometry of various instruments is necessary for an understanding of their vibrational behavior, and we are justified in choosing shape as a defining characteristic of an instrument. The ratio V_o/V is affected by the amount of change in volume caused by the motion of the boundaries (in our case, wood plates and ribs), while m/m_o depends on the amount of mass gained or lost through the f-holes over a period of time. The displacement of the plates which causes the relative change in volume is determined in part by the elasticity of the wood, which is affected by wood type, curvature and thickness, and treatments such as aging and varnish. These factors will enter into calculations as constants, since they do not change for a given instrument once it is crafted. The area of the f-holes and the velocity at the f-holes determine the relative change in mass, since, for the violin, $q = 0$ everywhere on the boundary except at the f-holes. The speed, or frequency, of the sound waves is therefore dependent on the velocity of motion at the boundaries. Other factors affecting change in volume are force of initial stimulus (from the player), temperature, and pressure on the top plate caused by tension of the strings. More energy from these sources implies more amplitude in vibrations, and hence greater variance in volume. As we seek a mathematical description for the motion of the plates, we expect these factors to come into play.

It also becomes necessary at this point to recall our discussion from Chapter 2 about how vibrations are transmitted in both directions (not only from plates to air, but also from air to plates). It is true that the plates initiate the vibrations of the air, but once the air is in motion, it also has an influence on the motion of the plates. Thus we are dealing with a coupled system. This means that the equations governing the wood and those governing the air must be solved simultaneously. The same will be the case between the wood and the string. In this work, we assume that the displacement function for the wood plates is known, since exploring the nature of motion of the plates would merit its own volume of work, but we still need to consider the properties of the plates to some extent in order to discuss the behavior of the air at the boundary of the resonant chamber.

Because boundary conditions for σ are determined by the motion of the wood plates and ribs, they will be most easily described in terms of the velocity (v). As we have seen, the velocity of the air inside the chamber is related to σ in the Continuity Equation by

$$\sigma_t = -div\ v.$$

Also, the Momentum Equation can be written $\rho_o v_t = p_o \lambda \nabla \sigma$ by substituting $(*)$ into the right side, thus

$$\nabla \sigma = \frac{\rho_o}{p_o \lambda} v_t.$$

The laws of conservation of mass and momentum also apply to the plates, the difference being that the plates have a different equilibrium density than the air.

Let $\tilde{\rho}$ represent the mass density of the wood. If displacement of the wood (u) is known, we can find $velocity = (u_t)$ for $\mathrm{x} \in S$. It then remains to relate velocity of the plates (u_t) to velocity of the air in the resonant chamber (v). We do this by equating components of velocity in the direction of the outward unit normal to the surface $(\hat{n})$, which can be found using the cross product of the partial derivatives of the instrument's surface function. Notice that $\hat{n}$ is a function of time, since the shape of the instruments surface varies slightly as it vibrates. We then have the condition

$$v \cdot \hat{n} = u_t \cdot \hat{n} \text{ for } \mathrm{x} \in S \text{ and } t \geq 0.$$

We have no information about the motion of the air in tangential directions except at $t = 0$, when both air and boundary are assumed to be still.

The coupled system must also satisfy Conservation of Energy at the boundary. To this end, consider Newton's third law that every action has an equal but opposite reaction. This law tells us that the outward pressure at the surface of the chamber must at all times be balanced by an equal inward pressure from the boundary. To satisfy this law, we take the pressure from the boundary inward to be $\tilde{p}\hat{n}$, which, following the same reasoning we used in the case of the air, makes the Momentum Equation for the wood

$$\tilde{\rho}_o u_{tt} = \nabla \tilde{p}.$$

Thus balancing forces at the boundary produces the condition

$$\rho_o v_t = p_o \lambda \nabla \sigma = -\tilde{\rho}_o u_{tt} \text{ for x} \in S.$$

In order to notate violin-specific boundary conditions, it is necessary to first describe the boundary of the resonant chamber mathematically. There are many published works available which discuss the geometry of the violin. Since the time of the Cremonese masters, people have studied instruments with exemplary tone in an effort to understand, and possibly replicate the work of their makers. But violin makers are artists. Violin geometry varies from maker to maker. Individual templates are generally created empirically, an in any case, are usually not published or shared. From the studies available, one will find that a general outline of a violin can be created by drawing a series of tangent arcs, and switching from one arc to another at the point of tangency. Violin maker David Gusset, of Eugene, Oregon is kind enough to share two examples of such construction on his website. He developed these constructions by studying instruments made by the Amati family. The design of intersecting and tangent circles in Figure 4.1 appears commonly in geometric studies.

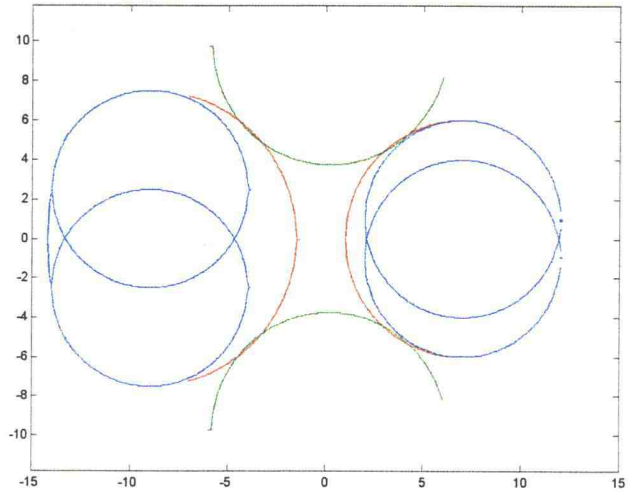


FIGURE 4.1. Circles. Note how the circles form a violin outline.

For the sake of discussion, let us assume that the top and back plates have outline shape described by $y = \pm F(x, t)$, where the origin is at the geometric center of the resonant chamber (see diagram below for position of coordinate axes). Assuming geometric construction similar to that shown above, $F(x, t)$ is smooth and continuous for all x in its domain, and can be defined piecewise using the tangent arcs which form the violin's edges. It is sensible to describe the outline as a function of x because if we let the direction parallel to the strings be the x direction, as is commonly done when finding the equation of motion for a vibrating string, the instrument is symmetric about the x-axis, but not about the y-axis. The lower region (the end with the tailpiece and chinrest) is generally wider than the upper region. Let the length of the violin be $L = L_1 + L_2$ with lower extreme at $x = L_1 < 0$ and upper extreme at $x = L_2 > 0$. For the scope of this work, we will assume that the ribs are vertical and fixed, so $\pm F(x, t) = \pm F(x, 0) = \pm F(x)$ for all $x \in [L_1, L_2]$.

With this notation, the boundary of our chamber can be described piecewise in

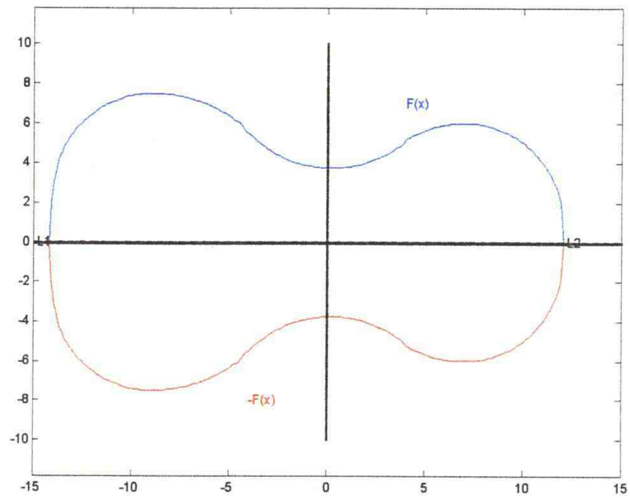


FIGURE 4.2. Violin outline in 2-D.

three regions: the top plate (S_T), the back plate(S_B), and the ribs(S_R). The entire boundary is $S = S_T \cup S_B \cup S_{R.}$

4.1 Top Plate

Let $z_T(x_o, y_o)$ denote the height of the top plate when $t = 0$. In other words, a particle (x, y, z) of the continuum is considered to be part of the top plate if its initial position can be described by $(x_o, y_o, z_T(x_o, y_o))$. Thus the displacement of the top plate can be described as

$$u_T(x, y, t) = (x(t) - x_o, y(t) - y_o, z_T(x, y, t) - z_T(x_o, y_o)).$$

Since our origin is at the geometric mean when $t = 0$, we can conclude that $z_T(x_o, y_o) > 0$ for all x_o and y_o. In fact, since the vibrations are small compared

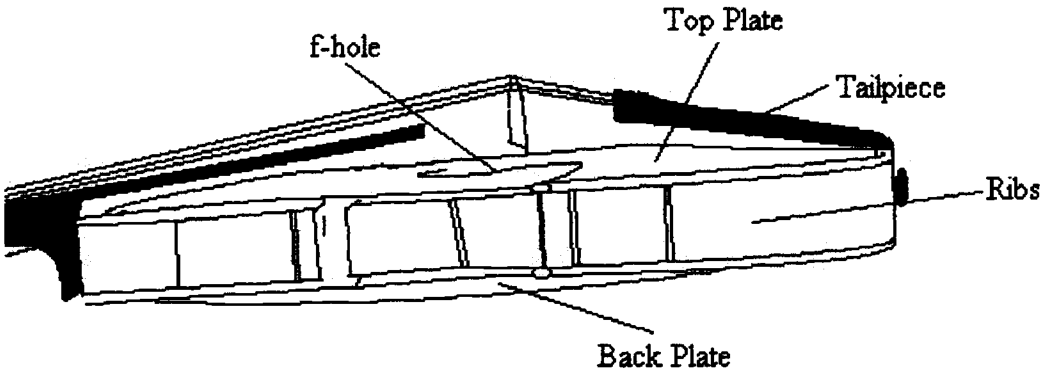


FIGURE 4.3. Violin side view.

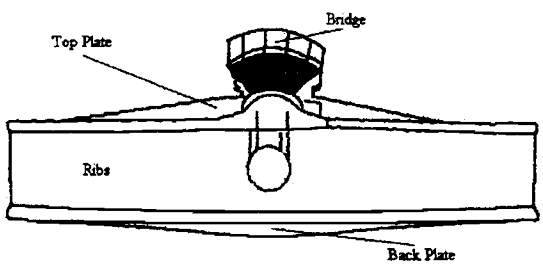


FIGURE 4.4. Violin end view.

to the height of the ribs, we expect that if a particle of the continuum is in the top plate, $z > 0$ for all t. We describe the top plate by the region

$$S_T = \{(x, y, z)/\ L_1 \leq x \leq L_2; -F \leq\ y \leq F;\ z = z_T = u_T \cdot \hat{e}_3 + z_T(x_o, y_o) > 0\}\,.$$

For our purposes, we will consider the bass bar to be a region of the top plate with a specified fixed y coordinate:

$$\text{Bass Bar} = \left\{(x, y, z) \in S_T\ /\ b_1 < x < b_2;\ y = \beta \text{ constant};\ 0 < \beta < \min_{b_1 \leq x \leq b_2} F(x)\right\}.$$

Consider the feet of the bridge to be single fixed points of contact at $(\tilde{x}, \pm\tilde{y}, z_T(\tilde{x}, \pm\tilde{y}))$.

The f-holes are also regions on the top plate. They are closed curves which mirror one another across the x-axis. Let $f_1(x, y, z_T)$ and $f_2(x, y, z_T)$ represent the functions which bound the f-hole above the x-axis. Then $-f_1(x, y, z_T)$ and $-f_2(x, y, z_T)$ bound the lower f-hole. The four points of intersection–those of f_1 with f_2, and those of $-f_1$ with $-f_2$ –usually all lie on a circle. The outward unit normal to the top plate at a point (x, y) at time t is

$$\hat{n}_T(x, y, t) = \frac{\frac{\partial}{\partial x}\langle x, y, z_T\rangle \times \frac{\partial}{\partial y}\langle x, y, z_T\rangle}{\left\|\frac{\partial}{\partial x}\langle x, y, z_T\rangle \times \frac{\partial}{\partial y}\langle x, y, z_T\rangle\right\|}.$$

4.2 Back Plate

Let $z_B(x_o, y_o)$ denote the height of the back plate when $t = 0$. In other words, a particle (x, y, z) of the continuum is considered to be part of the back plate if its initial position can be described by $(x_o, y_o, z_B(x_o, y_o))$. Thus the displacement of the back plate can be described as

$$u_B(x, y, t) = (x(t) - x_o, y(t) - y_o, z_B(x, y, t) - z_B(x_o, y_o)).$$

As with the top plate, vibrations of the back plate are small, so we expect $z < 0$ for all t if a particle is in the back plate. We describe the back plate as

$$S_B = \{(x, y, z)/\ L_1 \leq x \leq L_2;\ \ -F \leq y \leq F;\ z = z_B = u_B \cdot \hat{e}_3 + z_B(x_o, y_o)\}.$$

The outward unit normal to the back plate at a point (x, y) at time t is

$$\hat{n}_B(x, y, t) = \frac{\frac{\partial}{\partial x}\langle x, y, z_B\rangle \times \frac{\partial}{\partial y}\langle x, y, z_B\rangle}{\left\|\frac{\partial}{\partial x}\langle x, y, z_B\rangle \times \frac{\partial}{\partial y}\langle x, y, z_B\rangle\right\|}.$$

4.3 Ribs

We describe the ribs by the region

$$S_R = \{(x, y, z)/ \ L_1 \leq x \leq L_2; \ y = \pm F(x); \ z_B(x, \pm F, t) \leq z \leq z_T(x, \pm F, t)\}$$

Since we have chosen to view the ribs as fixed, we have

$$u(x, \pm F(x), z, t) = 0.$$

The outward unit normal to ribs at a point $(x, \pm F(x))$ at time t is

$$\hat{n}_R(x, \pm F(x)) = \frac{\langle 1, \pm F', 0 \rangle \times \hat{e}_3}{\|\langle 1, \pm F', 0 \rangle \times \hat{e}_3\|}.$$

4.4 Sound Post

In this work, we will treat the soundpost as one-dimensional, though it is actually a small cylinder.

$$\text{SoundPost} = \{(x, y, z)/ \ x = x_P; \ y = y_P; \ z_B(x_P, y_P, t) \leq z \leq z_T(x_P, y_P, t)\}.$$

We will assume that the soundpost is rigid so that $u(x_P, y_P, z, t)$ is constant with respect to z, and is therefore only dependent on time.

4.5 An Example

In order to consider a specific example of a possible region, let us suppose that the top and back plates of the instrument are flat and fixed. We then have $\hat{n}_T = -\hat{n}_B = \hat{e}_3$. Let $-z_B = z_T = \frac{h}{2}$ so h represents the height of the ribs, and in this case, the constant height of the resonant chamber. We let $F(x)$ be piecewise defined as

$$
\begin{aligned}
F(x) &= \sqrt{14.23^2 - x^2} && \text{for } x \in [-14.23, -14] \\
F(x) &= \sqrt{25 - (x+9)^2} \ + \ 2.5 && \text{for } x \in [-14, -9] \\
F(x) &= \sqrt{7.5^2 - (x+9)^2} && \text{for } x \in [-9, -4.5] \\
F(x) &= -\sqrt{36 - (x-0.2)^2} \ + \ 9.75 && \text{for } x \in [-4.5, 4] \\
F(x) &= \sqrt{36 - (x-7)^2} && \text{for } x \in [4, 7] \\
F(x) &= \sqrt{25 - (x-7)^2} \ + \ 1 && \text{for } x \in [7, 12] \\
F(x) &= \sqrt{145 - x^2} && \text{for } x \in \left[12, \sqrt{145}\right]
\end{aligned}
$$

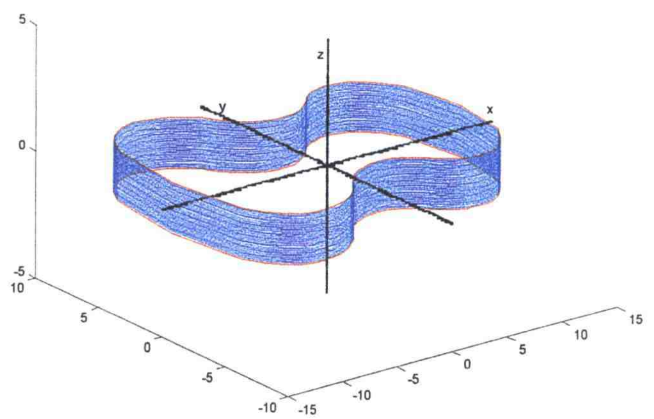


FIGURE 4.5. Violin ribs in a 3-D coordinate system.

This outline shape is based on David Gusset's model for an Amati viola. It can be used to model any size instrument by varying the unit of length. For example, if the unit is half-inches, the model fits quite well to a $\frac{3}{4}$-size violin owned by the author. Based on the same model, we choose the location of the right foot of the bridge to be $\tilde{x} = -2$, $\tilde{y} = -1$, $z_T(\tilde{x}, -\tilde{y}) = \frac{h}{2}$. We have $L_1 = -14.23$ and $L_2 = \sqrt{145} \approx 12.042$. The rib height at the edges of the same $\frac{3}{4}$-size violin is approximately one inch, so let $h = 2$. These values will be used in our numerical computations.

4.6 Boundary Conditions

Based on the assumptions and notation established in this chapter, we can now notate the boundary conditions for our region of interest. Assuming the ribs are fixed provides the condition

$$\sigma(x, \pm F, z, t) = 0 \text{ for all } x \in [L_1, L_2],\ \ z \in [z_B(x, \pm F, 0), z_T(x, \pm F, 0)], \text{ and } t \geq 0.$$

The condition on the air at the top and back plates are conditions of compatibility with the motion of the plates, and are given by

$$v \cdot \hat{n} = u_t \cdot \hat{n}$$

and

$$\rho_o v_t = p_o \lambda \nabla \sigma = -\tilde{\rho}_o u_{tt}$$

for x $\in S$ and $t \geq 0$.

In order for the boundary displacement function to be continuous where the plates meet the ribs, and compatible with the boundary conditions for the air, we use the conditions $u(x, \pm F, z_T, t) = 0$ and $u(x, \pm F, z_B, t) = 0$ as boundary conditions for the motion of the two plates.

Chapter 5

The IBVP

We have now arrived at a system of equations which models the behavior of the changes in air density relative to initial (equilibrium) density:

$$\begin{array}{ll} \sigma_{tt} = a^2 \Delta \sigma & \text{for } \mathrm{x} \in V \text{ and } t \geq 0, \\ \sigma(\mathrm{x}, 0) = 0 \text{ and } \sigma_t(\mathrm{x}, 0) = -div\ v(\mathrm{x}, 0) & \text{for } \mathrm{x} \in V, \\ \sigma(\mathrm{x}, t) = 0 & \text{for } \mathrm{x} \in S_R,\ t \geq 0, \\ v \cdot \hat{n} = u_t \cdot \hat{n} \text{ and } \rho_o v_t = p_o \lambda \nabla \sigma = -\tilde{\rho}_o u_{tt} & \text{for } \mathrm{x} \in S_B \cup S_T \end{array}$$

This system of equations describes an IBVP (initial boundary value problem). For the sake of generality, let us first see what we can determine about the nature of our solution before the boundary conditions are imposed. We will then have a general solution which can be applied to find a sound wave for any instrument simply by modifying the boundary conditions. We will, for now, suppose $v(\mathrm{x}, 0)$ to be unknown. If $v(\mathrm{x}, 0)$ is known, we can use numerical methods to find values for σ. We will discuss this option further in Chapter 6. For now, we seek a formal solution for σ using the method of separation of variables. There are many sources where the reader may find the details of this method. In this chapter, we will summarize the method to determine the general form for σ. The value of λ is also important, since $\lambda\sigma$ represents the sound wave we seek to describe. We will use $\lambda = 1.4$ in numerical calculations, but will leave λ in our general calculations as an unknown constant in case the reader wishes to consider a different empirical value for λ, or an instrument played in a gaseous medium other than air. We expect $\lambda\sigma(\mathrm{x}, t)$, and therefore $\sigma(\mathrm{x}, t)$, to be periodic due to the discoveries made by Helmholtz that all musical sound is periodic in nature, and the frequencies of the specific solutions for σ will tell us what overtones the specific resonant cavity will emphasize. Amplitude will provide information on proportional loudness of the various overtones. In particular, information about frequency and amplitude at points in the f-holes will be of interest, since these holes are where the sound affected by resonance escapes from the chamber.

It is also important to observe that without continued input from the player, there is a finite point in time when the violin ceases to vibrate, and the sound dies away–that is, the pressure at each point in the violin returns to its equilibrium state. Since we are using an idealized model which neglects the effects of air friction, our solution for σ may not converge to zero over time. We will, however, seek solutions for σ which do not diverge, as divergent solutions obviously would not closely model the actual behavior of the vibrations.

The general solution to the wave equation is well-known in one spatial dimension, and formal solutions have been found under certain boundary conditions in two- and three-dimensional space, but few are eager take on the difficulties presented by the shape of the violin. Since the focus of this work is to discuss the how motion of the air in the resonant chamber is affected by changes in the construction of the instrument, we will devote some space to the solution process in three dimensions, and discuss where some of the difficulties lie. The solutions in one and two dimensions needed for our discussion of boundary conditions will be assumed known. The same general principles can be employed to derive them. For details beyond what is offered here, there are numerous other sources to which the interested reader may refer.

To find a formal solution for σ, assume we can express σ as a product of space-dependent and time-dependent functions,

$$\sigma(\mathrm{x}, t) = \Phi(x, y, z)T(t).$$

Then $\sigma_{tt} = \Phi T''$ and $\Delta\sigma = T\Delta\Phi$ The wave equation from Chapter 3, which governs the motion of the air, then becomes

$$\Phi T'' = a^2 T \Delta\Phi.$$

Since we seek a non-trivial solution for σ (otherwise, we would hear no sound when the violin is played), we will assume that neither Φ nor T is identically zero. We can, therefore, divide our revised wave equation through by the product of Φ and (a^2T) to get

$$\frac{T''}{a^2T} = \frac{\Delta\Phi}{\Phi}.$$

Since the two sides are dependent on different variables, we can use a separation constant, K, to form the separated equations

$$\frac{T''}{a^2T} = K \qquad \text{and} \qquad \frac{\Delta\Phi}{\Phi} = K.$$

Thus, this method results in two separated equations:

$$T''(t) - Ka^2T(t) = 0 \qquad \text{and} \qquad \Delta\Phi(x, y, z) - K\Phi(x, y, z) = 0.$$

Since we have already established initial conditions, let us first consider, the equation in t, a single-variable wave equation.

If $K = 0$, the separated equations become $T'' = \Delta\Phi = 0$ (which then implies that $\sigma_{tt} = 0$ and $\Delta\sigma = 0$). We can solve the equation $T'' = 0$ by integrating twice with respect to t, to get solutions of the form $T(t) = C_1 t + C_2$.

If $K \neq 0$, the solution process becomes a bit more complicated. Let $w = T' - a\sqrt{K}T$, allowing for the possibility that $\sqrt{K}$ may be imaginary if $K < 0$. Then

$$w' + a\sqrt{K}w = \left(T' - a\sqrt{K}T\right)' + a\sqrt{K}\left(T' - a\sqrt{K}T\right) = T'' - Ka^2T = 0.$$

Multiplication by $e^{a\sqrt{K}t}$ gives

$$\begin{aligned} e^{a\sqrt{K}t}w' + a\sqrt{K}e^{a\sqrt{K}t}w &= 0 \\ (e^{a\sqrt{K}t}w)' &= 0 \\ e^{a\sqrt{K}t}w &= C_1 \\ w &= C_1e^{-a\sqrt{K}t} \\ T' - a\sqrt{K}T &= C_1e^{-a\sqrt{K}t}. \end{aligned}$$

Multiplication by $e^{-a\sqrt{K}t}$ then gives

$$\begin{aligned} e^{-a\sqrt{K}t}T' - a\sqrt{K}e^{-a\sqrt{K}t}T &= C_1e^{-2a\sqrt{K}t} \\ (e^{-a\sqrt{K}t}T)' &= C_1e^{-2a\sqrt{K}t}. \end{aligned}$$

Integrating both sides with respect to t,

$$\begin{aligned} e^{-a\sqrt{K}t}T &= \frac{C_1}{-2a\sqrt{K}}e^{-2a\sqrt{K}t} + C_2, \text{ so} \\ T(t) &= \frac{C_1}{-2a\sqrt{K}}e^{-a\sqrt{K}t} + C_2e^{a\sqrt{K}t}. \end{aligned}$$

This is the general form of the solution for T if $K \neq 0$. Note that the value of the separation constant, K, determines whether the exponents are real or complex. In the case of the violin, knowing that the sound wave is periodic, we expect to get complex values, but let us justify this mathematically.

Based on physical considerations, we are not interested in divergent solutions for σ, since they will not closely model the actual behavior of the air pressure as we observe it. We can express this conclusion mathematically as

$$\lim_{t\to\infty} \sigma = \lim_{t\to\infty} (\Phi(x,y,z)T(t)) = \Phi(x,y,z) \lim_{t\to\infty} T(t) < \infty.$$

Therefore, we must have $\lim_{t\to\infty} T(t) < \infty$ if our solution for σ is to be viable. In the two formal solutions above, we can use this fact to help narrow the set of possible solutions.

If $K = 0$ and $C_1 \neq 0$, then $\lim_{t\to\infty} T(t) = \lim_{t\to\infty}(C_1 t + C_2) = \pm\infty$ (depending on the sign of C_1). This can not be the case. If $K = 0$ and $C_1 = 0$, then $T(t) = C_2$, which would imply that σ is independent of time. Observation tells us that this is also not the case for a sound wave. Since $K > 0 \Longrightarrow T \to \pm\infty \Rightarrow \sigma \to \infty$ unless $C_2 = 0$, we require that either $K > 0$ and $C_2 = 0$, or $K < 0$. The initial condition on σ gives us some additional insight about the time-dependent portion of our separated solution.

$$\sigma\,(\mathrm{x}, 0) = \Phi(x,y,z)T(0) = 0 \Rightarrow T(0) = \frac{C_1}{-2ai\sqrt{-K}} + C_2 = 0 \Rightarrow \frac{C_1}{2ai\sqrt{-K}} = C_2,$$

so $C_2 = 0 \Longrightarrow C_1 = 0 \Longrightarrow T \equiv 0$. Since we seek a non-trivial solution, we must conclude that $K < 0$.

With $K < 0$, the formal solution for T can be written as

$$\begin{aligned} T(t) &= \frac{C_1}{-2ai\sqrt{-K}} e^{-ai\sqrt{-K}t} + C_2 e^{ai\sqrt{-K}t} \\ &= \frac{C_1}{-2ai\sqrt{-K}} \left(\cos(-a\sqrt{-K}t) + i\sin(-a\sqrt{-K}t)\right) \\ &\qquad + C_2\left(\cos(a\sqrt{-K}t) + i\sin(a\sqrt{-K}t)\right) \\ &= \left(\frac{C_1}{-2ai\sqrt{-K}} + C_2\right)\cos(a\sqrt{-K}t) + \left(\frac{C_1}{2ai\sqrt{-K}} + C_2\right) i\sin(a\sqrt{-K}t). \end{aligned}$$

$$T(t) = 2\left(\frac{C_1}{2ai\sqrt{-K}}\right) i\sin(a\sqrt{-K}t) = \frac{C_1}{a\sqrt{-K}}\sin(a\sqrt{-K}t)$$

for all $t \geq 0$, where $K < 0$ and $C_1 \neq 0$.

The initial condition on σ_t then gives some information about Φ.

$$\sigma_t(\mathrm{x}, 0) = \Phi(x,y,z)T'(0) = \Phi(x,y,z)C_1\cos(0) = C_1\Phi(x,y,z) = -div\ v(\mathrm{x},0), \text{ so}$$

$$\Phi(x,y,z) = -\frac{1}{C_1} div\ v(\mathbf{x},0).$$

The equation for our sound wave now looks like

$$\lambda\sigma = \frac{-\lambda C_1}{a\sqrt{-K}}\Phi(x,y,z)\sin(a\sqrt{-K}t) \qquad \text{for } t > 0, K < 0 \text{ and } \lambda \approx 1.4,$$

$$\text{where } \Phi(x,y,z) = -\frac{1}{C_1} div\ v(\mathrm{x},0) \text{ and } \Delta\Phi(x,y,z) - K\Phi(x,y,z) = 0.$$

The amplitude of this wave is a constant multiple of $|\Phi(x,y,z)|$ (with the constant depending on the value of K), and its frequency is $a\sqrt{-K}$, where a is a known constant for the given set of environmental conditions where the instrument is being played. To determine what frequencies are emphasized by the violin, we need to determine the nature of $v(\mathrm{x},0)$, and find possible values for K. With this purpose in mind, we now turn to the second separated equation,

$$\Delta\Phi(x, y, z) - K\Phi(x, y, z) = 0,$$

which is known as the Helmholtz equation, and for which we have found that

$$\Phi(x, y, z) = -\frac{1}{C_1} div\ v(\mathrm{x}, 0).$$

As with the time dependent portion of the solution, we expect non-divergent solutions for Φ as any of its variables tend to $\pm\infty$. Common experience supports this conclusion because if we station several listeners at varying distances from the violin in a given direction at a given moment in time, the listeners who are farther from the instrument will experience a weaker sound wave. This is why many concert goers prefer front row seats. Possible eigenvalues, K, for this system of equations depend on the boundary conditions, though we have established that $K < 0$ based on physical considerations. As we will find in the next chapter, the relation between Φ and $v(\mathrm{x}, 0)$ will play a role in determining our boundary conditions.

Applying the boundary condition at the ribs gives

$$\sigma(x, \pm F, z, t) = 0 \implies \Phi(x, \pm F, z)T(t) = 0 \implies \Phi(x, \pm F, z) = 0$$

for all $x \in [L_1, L_2]$, $z \in [z_B(x, \pm F, 0), z_T(x, \pm F, 0)]$, and $t \geq 0$.

5.1 Motion of the surface regions

As freely vibrating surfaces (with no forcing), we expect the motion of both plates to satisfy the wave equation in two spatial dimensions, that is

$$u_{tt} = \tilde{a}^2 \Delta u \qquad \text{for } \mathrm{x} \in S_T \cup S_B, t > 0, \text{ and } \tilde{a}^2 = \frac{\tilde{p}_o}{\tilde{\rho}_o}\tilde{\lambda} \text{ as before.}$$

This equation can be derived and solved in the same way as for the air in the three-dimensional case, except that the constants on which $\tilde{a}$ depends are determined by the elastic properties of the wood. The initial conditions of the wood are the same as those of the air, since the top plate is also subject to the influence of the string via the bridge's feet. If we let $u(x, y, z, t) = \tilde{\Phi}(x, y, z)\tilde{T}(t)$, with $\lim_{t\to\infty} |u(x, y, z, t)| < \infty \Rightarrow \lim_{t\to\infty} \tilde{T}(t) < \infty$, we can follow the same method as before to arrive at the separated solution for $\mathrm{x} \in S$ and $t \geq 0$,

$$\tilde{T}(t) = \frac{\tilde{C}_1}{\tilde{a}\sqrt{-\tilde{K}}} \sin(\tilde{a}\sqrt{-\tilde{K}}t), \text{ and}$$

$$\Delta\tilde{\Phi}(x, y, z) - \tilde{K}\tilde{\Phi}(x, y, z) = 0, \text{ where } \tilde{K} < 0.$$

Since our surface region is piecewise defined, u and $\tilde{\Phi}$ are also piecewise defined, but since all surface regions are stationary at their edges, u is continuous over the surface. Without considering specific initial or boundary conditions on the wood, we can use the compatibility conditions to write σ in terms of u as follows. According to the Gauss Divergence Theorem,

$$\sigma_t = -div\ v \Rightarrow \int_V \sigma_t dV = -\int_V div\ v dV = -\int_S v \cdot \hat{n} dS \text{ for } x \in V \text{ and } t \geq 0.$$

Since we have determined that $v \cdot \hat{n} = u_t \cdot \hat{n}$ for $\mathrm{x} \in S$ and $t \geq 0$, we have

$$\int_V \sigma_t dV = -\int_S u_t \cdot \hat{n} dS.$$

The second compatibility condition gives

$$p_o \lambda \nabla \sigma = -\tilde{p}_o u_{tt} = -\tilde{p}_o \tilde{\lambda} \Delta u \text{ for } \mathrm{x} \in S.$$

Using our separated solutions from before, we obtain

$$\begin{array}{ll} \sigma_t = \Phi(\mathrm{x}) T'(t) & \text{and } u_t = \tilde{\Phi}(\mathrm{x}) \tilde{T}'(t), \text{ where} \\ T'(t) = C_1 \cos(a\sqrt{-K}t),\ C_1 \neq 0 & \text{and } \tilde{T}'(t) = \tilde{C}_1 \cos(\tilde{a}\sqrt{-\tilde{K}}t),\ \tilde{C}_1 \neq 0 \\ p_o \lambda \nabla \sigma = p_o \lambda T(t) \nabla \Phi(\mathrm{x}) & \text{and } -\tilde{p}_o \tilde{\lambda} \Delta u = -\tilde{p}_o \tilde{\lambda} \tilde{T}(t) \Delta \tilde{\Phi}(\mathrm{x}). \end{array}$$

It follows that

$$\begin{aligned} T'(t) \int_V \Phi(\mathrm{x}) dV &= -\tilde{T}'(t) \int_S \tilde{\Phi}(\mathrm{x}) \cdot \hat{n}\ dS, \text{ and} \\ T(t) \nabla \Phi(\mathrm{x}) &= -\frac{\tilde{p}_o \tilde{\lambda}}{p_o \lambda} \tilde{T}(t) \tilde{\Phi}(\mathrm{x}). \end{aligned}$$

Fixing $t = 0$ gives $T'(0) = C_1$ and $\tilde{T}'(0) = \tilde{C}_1$, which makes the first condition

$$\int_V \Phi(\mathrm{x}) dV = -\frac{\tilde{C}_1}{C_1} \int_S \tilde{\Phi}(\mathrm{x}) \cdot \hat{n}\ dS.$$

These equations show that the vibrations of the air and wood are codependent at the boundaries. As with the air, the initial velocity of the plates is zero except in a small neighborhood of the right foot of the bridge. The size and shape of this small neighborhood depend on the resistance of the top plate.

The boundary conditions on the wood plates are

$$u_T(x, \pm F(x), t) = u_B(x, \pm F(x), t) = 0 \text{ for } x \in [L_1, L_2]\ , t > 0.$$

Since $u_T(x, \pm F(x), t) = \tilde{\Phi}_T(x, \pm F(x))\tilde{T}(t)$ and $\tilde{T}(t)$ is not identically zero, this gives $\tilde{\Phi}_T(x, \pm F(x)) = 0$. Likewise, $\tilde{\Phi}_B(x, \pm F(x) = 0$. We must also impose the conditions $u_T(x_P, y_P, t) = u_B(x_P, y_P, t)$ for a rigid soundpost, $u_T(x, \beta, t) = h(t)$ for the bass bar, and if we assume that the bridge transfers all of the motion from the string into normal forces at the two points where its feet contact the top plate, $u_T(\tilde{x}, -\tilde{y}, t) \cdot \hat{n}_T = -u_T(\tilde{x}, \tilde{y}, t) \cdot \hat{n}_T = \mathrm{y}_S(\tilde{x}, t)$ where $\mathrm{y}_S(\tilde{x}, t)$ represents the displacement of the string at its point of contact with the bridge. The change in sign results from the assumption of rigidity on the bridge, which mandates that the feet are always moving in opposite directions.

5.2 Motion of the string

Let the endpoints of the string be at l_1 and l_2 with $L_1 < l_1 < \tilde{x} < L_2 < l_2$. Denote the length of the string by $l = l_2 - l_1$. Let $\mathrm{y}_S(x, 0) = 0$ represent the initial displacement of the string, and $\frac{\partial}{\partial t}\mathrm{y}_S(x, 0) = g(x)$ be its initial velocity. These initial conditions also give the initial conditions for the motion of the top plate via the equation

$$u_T(\tilde{x}, -\tilde{y}, t) \cdot \hat{n}_T = -u_T(\tilde{x}, \tilde{y}, t) \cdot \hat{n}_T = \mathrm{y}_S(\tilde{x}, t).$$

The initial displacement and velocity functions for u_B are both zero, because the soundpost transfers the displacement of the top plate at (x_P, y_P) to the back plate. Since (x_P, y_P) is not equal to $(\tilde{x}, -\tilde{y})$, we would expect a short time to pass before initial displacement occurs at (x_P, y_P).

Both ends of the string are fixed, so $\mathrm{y}_S(l_1,t) = \mathrm{y}_S(l_2,t) = 0$ represents the string's boundary condition. Note that the string being fixed at both ends also means that $g(x)$ can be extended to be $2l$-periodic. Finding the displacement function for a string under these conditions is a well-known elementary physics problem, with solution

$$\mathrm{y}_S(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{cn\pi t}{l} + B_n \sin\frac{cn\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

where $c^2 = \frac{\tau}{\rho_s}$, $\tau(x,t)$ is the tension of the string, and $\rho_s(x,t)$ is its mass density. In this solution, A_n and B_n are the Fourier coefficients:

$A_n = \frac{2}{l}\int_0^l \mathrm{y}_S(x,0)\sin\frac{n\pi x}{l}dx = 0$, and

$B_n = \frac{2}{c\pi n}\int_0^l \frac{\partial}{\partial t}\mathrm{y}_S(x,0)\sin\frac{n\pi x}{l}dx = \frac{2}{c\pi n}\int_0^l g(x)\sin\frac{n\pi x}{l}dx.$

Thus the displacement of the string from its equilibrium position of $\mathrm{y} = 0$ has equation

$$\mathrm{y}_S(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{c\pi n}\int_0^l g(x)\sin\frac{n\pi x}{l}dx\right)\sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}.$$

As we stipulated in Chapter 1, the frequency of the fundamental vibration of the string is known. It is determined by the pitch of the note being plucked. The fundamental vibration is that for which $n = 1$, so its frequency is $c\pi/l$. With known fundamental frequency and known string length, we can easily determine a value for c. We can then use the solution for the displacement of the string to determine the boundary conditions for the top and back plates, which then determine the boundary conditions for the air pressure changes in our original PDE. The initial conditions on the string's displacement and velocity also affect the initial conditions of the plates and the air, for if pressure is initially applied to the string, it is transferred by the bridge into pressure normal to the top plate.

Chapter 6

Issues in Solving the IBVP

Thus far, the equations we have found which govern the motion of the components of the violin are

$$
\begin{array}{ll}
\sigma_{tt} = a^2 \Delta \sigma & \text{for x} \in V,\ t \geq 0, \\
\sigma(\text{x}, 0) = 0 \text{ and } \sigma_t(\text{x}, 0) = -div\ v(\text{x}, 0) & \text{for x} \in V, \\
\sigma(\text{x}, t) = u(\text{x}, t) = 0 & \text{for x} \in S_R,\ \ t \geq 0, \\
v \cdot \hat{n} = u_t \cdot \hat{n} \text{ and } \rho_o v_t = p_o \lambda \nabla \sigma = -\tilde{\rho}_o u_{tt} & \text{for x} \in S_B \cup S_T,\ t \geq 0, \\
u_{tt} = \tilde{a}^2 \Delta u & \text{for x} \in S_T \cup S_B,\ t > 0, \\
u(\text{x}, 0) = 0 & \text{for x} \in S, \\
u_t(\text{x}, 0) = 0 & \text{for x} \in S_R \cup S_B, \\
u_T(x, \pm F(x), t) = u_B(x, \pm F(x), t) = 0 & \text{for } x \in [L_1, L_2],\ \ t \geq 0, \\
u(x_P, y_P, z_B, t) = u(x_P, y_P, z_T, t) & \text{for } t \geq 0, \\
u(x, \beta, z_T, t) = \ constant & \text{for } x \in (b_1, b_2),\ t \text{ fixed}, \\
u_T\left(\tilde{x}, -\tilde{y}, t\right) \cdot \hat{n}_T = -u_T\left(\tilde{x}, \tilde{y}, t\right) \cdot \hat{n}_T = \text{y}_S(\tilde{x}, t) & \text{for } t \geq 0, \\
\text{y}_S(x, t) = \sum_{n=1}^{\infty} \left(\frac{2}{c\pi n} \int_0^l g(x) \sin \frac{n\pi x}{l} dx \right) \sin \frac{cn\pi t}{l} \sin \frac{n\pi x}{l} & \text{for } x \in [l_1, l_2],\ \ t \geq 0, \\
\text{y}_S(x, 0) = 0 \text{ and } \frac{\partial}{\partial t}\text{y}_S(x, 0) = g(x) & \text{for } x \in [l_1, l_2]
\end{array}
$$

A study of the wood plates will add to this system the constitutive equations for orthotropic linear elastic behavior of the top and back plates, and an initial condition on u_t for the top plate, which will be determined by the initial conditions of the string and the elastic behavior of the wood. There will also be an equation balancing pressure where the outer surface of the wood meets the outside air. The outward pressure of the wood at the feet of the bridge must equal the inward pressure exerted on the surface by the displacement of the string. Hence if we were to assume the wood is rigid in the normal direction, we might write

$$-p\left(\tilde{x}, -\tilde{y}, z_T, t\right) \cdot \hat{n} = \tau_y.$$

The pressure which the wood exerts on the air in the chamber will be affected by the curvature and elasticity of the plates because these determine how the tension from the point of contact with the bridge is distributed to nearby points. The various compatibility conditions at the boundaries and certain points of interest such as the soundpost and the feet of the bridge show how the system is coupled, and require that the above equations must be solved simultaneously.

We have reduced the problem of solving formally for σ to finding a solution to the Helmholtz equation with boundary conditions:

$$\begin{aligned}
&\Delta\Phi(\mathrm{x}) - K\Phi(\mathrm{x}) = 0 && \text{for } \mathrm{x} \in V \\
&\Phi(\mathrm{x}) = 0 && \text{for } \mathrm{x} \in S_R, \text{ and} \\
&T(t)\nabla\Phi(\mathrm{x}) = -\tfrac{\tilde{p}_o\tilde{\lambda}}{p_o\lambda}\tilde{T}(t)\tilde{\Phi}(\mathrm{x}) && \text{for } \mathrm{x} \in S_T \cup S_B
\end{aligned}$$

Here, all of the information on the right of the last equation is presumed known. Among authors who discuss the violin from a mathematical point of view, there is a general consensus that our problem of mathematically describing the sound wave can not be solved, but few of them touch on why. It is probable that solving the Helmholtz equation is the kink in the hose for most who set out on this mission. The difficulty in solving the Helmholtz equation often leads mathematicians seeking a solution for the wave equation to guess at a reasonable approximation for an initial time derivative. Looking at well-known problems involving the wave equation, one will find that the initial time derivative (initial velocity in a displacement problem) is usually given as part of the IBVP. In fact, if our problem were one of infinite spatial extent, we could try something similar to the approximation

$$\sigma_t(\mathrm{x},0) = \delta_\varepsilon(\mathrm{x}) = (4\pi\varepsilon)^{-\frac{3}{2}} e^{-\frac{r^2}{4\varepsilon}}, \quad \text{where } r = |\mathrm{x}| = \left((x-0)^2 + (y-0)^2 + (z-0)^2\right)^{\frac{1}{2}}$$

as is done in the derivation of Kirchhoff's solution to the IVP

$$\begin{aligned} u_{tt} &= c^2 u \qquad \mathrm{x} \in \mathbb{R}^3,\ t > 0 \\ u(\mathrm{x},0) &= 0, \qquad u_t(\mathrm{x},0) = \delta_\varepsilon(\mathrm{x}), \qquad \mathrm{x} \in \mathbb{R}^3. \end{aligned}$$

It may be that this approximation would serve us as well in our bounded domain. The approximation makes sense, since there is only one point of initial disturbance of the air. There are, however some fundamental differences if we were to choose the approach of making an estimate for $\sigma_t(\mathrm{x},0)$ based on intuition. In the Kirchhoff case, the coordinate system is chosen so that the origin is the point where the initial change in density takes place. In our violin, we would need our δ function $(C_1\Phi)$ to be centered at the point $(\tilde{x}, -\tilde{y}, z_T(\tilde{x}, -\tilde{y}))$, and have intensity of 1 inside V. That is, we would like $\Phi(\mathrm{x})$ to possess the qualities $\int_V \Phi(\mathrm{x})dV = \frac{1}{C_1}$ and $\lim_{\varepsilon\to 0}\Phi(\mathrm{x}) = 0$ for all $\mathrm{x} \neq (\tilde{x}, -\tilde{y}, z_T(\tilde{x}, -\tilde{y}))$. Further study of Kirchhoff's formulas and their derivations may well provide the means to finding a formal solution to σ in the case of the violin, but since we ultimately seek a mathematical means of finding Φ using our boundary condition, we will not make any guesses at present. We will, instead, look at a region for which the Helmholtz equation is easily solvable. Three features by which the violin defies conventional approaches to solving the Helmholtz equation are a curved outline with limited symmetry, a bounded domain, and lack of symmetry around the point of initial disturbance. The latter two issues can be dealt with if we look at a slightly more symmetric domain. Such an example will show how the boundary conditions are applied, and how the results are interpreted. Let us, then, take a look at two common methods of approaching the Helmholtz equation for a region with much simpler geometry: separation of variables, and a numerical method using difference approximations applied to a rectangular box. While these methods fail to produce a viable solution for the region we ultimately wish to consider, they will serve as an example of how one might utilize a formal solution to understand the sound wave. They will also serve to show the reader exactly where difficulties arise in attempting to find a solutions in a violin-shaped region.

6.1 Separation of Variables

Suppose $\Phi(x, y, z)$ can be written as a product of three functions, each depending on one spatial variable, as

$$\Phi(x, y, z) = X(x)Y(y)Z(z).$$

We can then follow the same reasoning used in separating T and Φ to get

$$\begin{aligned}
\Delta\Phi - K\Phi &= X''YZ + XY''Z + XYZ'' - KXYZ = 0 \\
\frac{X''}{X} &= \frac{-Y''Z - YZ'' + KYZ}{YZ} = K_1 \qquad \text{(constant)} \\
X'' - K_1X &= 0 \quad \text{and} \quad Y''Z = (K - K_1)YZ - YZ''
\end{aligned}$$

The equation in Y and Z can then be separated as

$$\begin{aligned}
\frac{Y''}{Y} &= \frac{(K-K_1)Z - Z''}{Z} = K_2 \\
Y'' - K_2 Y &= 0 \quad \text{and} \quad Z'' - (K - K_1 - K_2)Z = 0
\end{aligned}$$

Thus the separation leads to three independent wave equations. Since we expect σ not to diverge as any of its variables $\to \infty$, we can assume that, as with K, we also have $K_1 < 0$, $K_2 < 0$, and $(K - K_1 - K_2) < 0$. Thus the solutions to these equations take the same form as the solution for T before the initial conditions were applied:

$$\begin{aligned}
X(x) &= A_1 \cos(\sqrt{-K_1}x) + B_1 \sin(\sqrt{-K_1}x) \\
Y(y) &= A_2 \cos(\sqrt{-K_2}y) + B_2 \sin(\sqrt{-K_2}y) \\
Z(z) &= A_3 \cos(\sqrt{-(K - K_1 - K_2)}z) + B_3 \sin(\sqrt{-(K - K_1 - K_2)}z)
\end{aligned}$$

For the sake of example, we choose a rectangular box with fixed boundaries. This shape is not entirely unrelated to the violin, as nearly rectangular stringed instruments do exist, though the fixed boundaries will do no favors for the prospective tone quality. Two such examples are the 18th-century Welsh crwth, which is one of the violin's precursors, and the Mongolian morin kuhr, or "horsehead violin". One difference between this region and the violin is that our calculations can be greatly simplified by choosing to view one corner of the box as the origin of our coordinate system. There is no location in the violin which affords this opportunity, and when the violin's boundary conditions are applied to the above solution forms, the values for the constants, K, end up being dependent on variables. Thus we determine that the solution in the case of the violin can not be separated in this way. In order to look at the effects of the boundary conditions on the rectangular box, let the edges of the box be bounded by the lines $x = 0$, $x = 26$, $y = 0$, $y = 14$, $z = 0$, and $z = 2$ for all $t \geq 0$. This gives boundary conditions as follows:

$$\begin{aligned} \sigma(0, y, z, t) &= \Phi(0, y, z)T(t) = 0 \Longrightarrow \Phi(0, y, z) = 0 \\ &\Longrightarrow X(0)Y(y)Z(z) = 0, \\ &\Longrightarrow X(0) = 0 \end{aligned}$$

and likewise,

$$\begin{aligned} X(26) &= 0, \\ Y(0) &= Y(14) = 0, \text{ and} \\ Z(0) &= Z(2) = 0. \end{aligned}$$

Applying these boundary conditions,

$$X(0) = 0 \Longrightarrow A_1 \cos(0) + B_1 \sin(0) = A_1 = 0.$$

Likewise,

$$Y(0) = 0 \Longrightarrow A_2 = 0 \text{ and } Z(0) = 0 \Longrightarrow A_3 = 0, \text{ so}$$

$$\begin{aligned} X(x) &= B_1 \sin(\sqrt{-K_1}x) \\ Y(y) &= B_2 \sin(\sqrt{-K_2}y), \text{ and} \\ Z(z) &= B_3 \sin(\sqrt{-(K - K_1 - K_2)}z) \end{aligned}$$

Notice that $B_1, B_2, B_3 \neq 0$, since we seek a non-trivial solution for Φ. It follows that

$$X(26) = 0 \Longrightarrow \sin(26\sqrt{-K_1}) = 0 \Longrightarrow 26\sqrt{-K_1} = k_1\pi \text{ for } k_1 = 1, 2, 3...$$

$$\text{so } K_1 = -\left(\frac{k_1\pi}{26}\right)^2,$$

$$Y(14) = 0 \Longrightarrow \sin(14\sqrt{-K_2}) = 0 \Longrightarrow 14\sqrt{-K_2} = k_2\pi \text{ for } k_2 = 1, 2, 3...$$

$$\text{so } K_2 = -\left(\frac{k_2\pi}{14}\right)^2, \text{ and}$$

$$\begin{aligned} Z(2) &= 0 \Longrightarrow \sin(2\sqrt{-(K - K_1 - K_2)}) = 0 \\ &\Longrightarrow 2\sqrt{-(K - K_1 - K_2)} = k_3\pi \text{ for } k_3 = 1, 2, 3... \end{aligned}$$

$$\text{so } K - K_1 - K_2 = -\left(\frac{k_3\pi}{2}\right)^2.$$

The result of these boundary conditions is that we are able to determine a value for K, which tells us what frequencies of vibration are possible inside the box. In this case,

$$K = -\left(\frac{\pi}{2}\right)^2 \left[(k_3)^2 + \left(\frac{k_2}{7}\right)^2 + \left(\frac{k_1}{13}\right)^2\right].$$

Let $\omega = \sqrt{(k_3)^2 + (k_2/7)^2 + (k_1/13)^2}$. Our formal solution for σ then looks like

$$\sum_{k=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_1=1}^{\infty} B_1 B_2 B_3 \left(\frac{2C_1 \sin(\frac{k_1\pi}{26}x)\sin(\frac{k_2\pi}{14}y)\sin(\frac{k_3\pi}{2}z)}{a\pi\omega}\right) \sin(\frac{a\pi}{2}\omega t).$$

Of course, we can see that the amplitude of these superposed waves is bounded by $1/\omega$, and therefore decreases as $k_i \longrightarrow \infty$. In fact, the constants B_1, B_2, and B_3

have the same form as the coefficient of the time-dependent solution,

$$B_i = \frac{C_{2i+1}}{\sqrt{-K_i}} \text{ for } i = 1, 2, 3 \text{ if we let } K_3 = K - K_1 - K_2.$$

So, since each of the K_i contain a factor of k_i^2, the amplitude is actually bounded by $1/\left(k_1^2 k_2^2 k_3^2 \omega\right)$. We can then write σ as

$$\frac{1456 C_1 C_3 C_5 C_7}{a\pi^4} \sum_{k_3=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_1=1}^{\infty} \frac{\sin(\frac{k_1\pi}{26}x)\sin(\frac{k_2\pi}{14}y)\sin(\frac{k_3\pi}{2}z)}{(k_1k_2k_3)^2\,\omega} \sin(\frac{a\pi}{2}\omega t)$$

Because the fraction inside the sum decreases as any of the $k_i \longrightarrow \infty$, there is some finite value for each k_i for which the overtones will become so weak that they are inaudible to the human ear. Thus for all practical purposes, the sums are finite. The lowest possible frequency of vibration for the air inside this box would occur when $k_1 = k_2 = k_3 = 1$,and is approximately $1.59a$ cycles per unit of time. We can also see from the above formula that a larger box would have a lower fundamental frequency because the denominators of the frequency would be larger.

Since we are interested mainly in relative strength of the various overtones, we need not seek values for the constants C, though one might learn something about them from Fourier analysis if one so desired. Also, multiplication by λ does not affect which overtones are present, or their relative strengths. Multiplying by a constant affects the amplitude, or loudness, of all overtones equally.

In particular, we are interested in the behavior of the sound wave at x $\in f-holes$. We can look at which frequencies are strongest in these regions by inserting coordinates of locations within the holes. For example, perhaps one end of an f-hole is at x $= (10, 1, 2)$. We would then have (ingnoring the constant in front of the summation)

$$
\begin{aligned}
\text{relative amplitudes} &= \left| \frac{\sin(\frac{k_1\pi}{26}10)\sin(\frac{k_2\pi}{14}1)\sin(\frac{k_3\pi}{2}2)}{(k_1k_2k_3)^2\omega} \right| \\
&= \left| \frac{\sin(\frac{5k_1\pi}{13})\sin(\frac{k_2\pi}{14})\sin(k_3\pi)}{(k_1k_2k_3)^2\omega} \right| .
\end{aligned}
$$

It is important to note that in this problem, no initial stimulus was given to the air in the box, so the solution represents only which modes of vibration are possible. A specific initial condition for σ_t (or $C_1\Phi$) would limit which values of k actually occur, and would therefore affect which overtones are present and their relative strengths. As we have seen, the initial conditions on the plates determine the initial condition for σ_t.

6.2 Numerical Methods

In the absence of a closed-form solution for the Helmholtz equation as the one above, one might wish to turn to numerical methods. This will likely be the case for the violin because its shape is not simple enough for separation of variables to work. There are two choices for IBVPs we might wish to solve using numerical methods.

The first possible choice is

$$
\begin{aligned}
\sigma_{tt} &= a^2\Delta\sigma \qquad \mathrm{x} \in V,\ t \geq 0 \\
\sigma(\mathrm{x},0) &= 0, \qquad \sigma_t(\mathrm{x},0) = \delta_\varepsilon(r), \qquad \mathrm{x} \in V \\
\sigma(\mathrm{x},t) &= 0 \qquad \mathrm{x} \in S,\ t \geq 0.
\end{aligned}
$$

where $r^2 = (x+2)^2 + (y+1)^2 + (z-1)^2$ and $a^2 = \frac{p_o}{\rho_o}\lambda$. To find a reasonable value for a at room temperature (20^oC), we use the empirical value for velocity of sound (a) in air at 0^oC, and multiply by $\sqrt{1+\frac{1}{273}(20)}$.

$$a = \left(33,156\frac{cm}{\sec}\right)\sqrt{1+\frac{1}{273}(20)} = 34,349\frac{cm}{\sec}$$

Divide this by 1.27 cm per half-inch to get $a = 27,046$ half-inches per second.

We are particularly interested in the behavior of the solution, σ, at the f-holes, since this is where the violin disturbs the outer air with greatest force due to resonance. It is, therefore, interesting to examine the behavior of numerical solutions at progressive time levels for points in the f-holes. One might choose to look at the geometric center of the holes, or perhaps at the centers of the four small circles at the ends of the holes. Based on Gusset's model, with our flat plates, we could locate these four points at $(-4, 4.5, 2)$, $(0, 2, 2)$, $(0, -2, 2)$, and $(-4, -4.5, 2)$.

The second IBVP we might wish to solve is given at the beginning of this chapter. Two advantages of considering the Hemholtz IBVP over the IBVP involving σ is that there is one fewer variable, and we need not guess at initial conditions if we have a known solution for the motion of the boundary.

One example of a numerical method is to use difference approximations for the partial derivatives in the Helmholtz equation. To do this, we discretize in space. Let ϕ_{ijk} denote an approximation to $\Phi(x_i, y_j, z_k)$. Using difference quotients to approximate the derivatives in the Helmholtz equation, we get

$$D_+^{(x)}D_-^{(x)}\phi_{ijk} + D_+^{(y)}D_-^{(y)}\phi_{ijk} + D_+^{(z)}D_-^{(z)}\phi_{ijk} = K\phi_{ijk},$$

which is equivalent to

$$\frac{\phi_{i+1jk} - 2\phi_{ijk} + \phi_{i-1jk}}{(\Delta x)^2} + \frac{\phi_{ij+1k} - 2\phi_{ijk} + \phi_{ij-1k}}{(\Delta y)^2} + \frac{\phi_{ijk+1} - 2\phi_{ijk} + \phi_{ijk-1}}{(\Delta z)^2} = K\phi_{ijk},$$

or

$$\begin{aligned} \phi_{ijk+1} &= \left[2 + K(\Delta z)^2 + 2\left(\frac{\Delta z}{\Delta x}\right)^2 + 2\left(\frac{\Delta z}{\Delta y}\right)^2\right]\phi_{ijk} \\ &\quad + (\Delta z)^2\left[\frac{\phi_{i+1jk} + \phi_{i-1jk}}{(\Delta x)^2} + \frac{\phi_{ij+1k} + \phi_{ij-1k}}{(\Delta y)^2}\right] - \phi_{ijk-1}. \end{aligned}$$

According to this equation, in order to find a solution for ϕ at a given height, we need data from the two previous heights. If M represents the number of steps taken in the x, y, and z directions, then the boundary condition can be represented as

$$\phi_{0jk} = \phi_{Mjk} = \phi_{i0k} = \phi_{iMk} = \phi_{ij0} = \phi_{ijM} = 0.$$

If our boundary condition also included information for Φ_z, we would be able to find numerical approximations for Φ at different locations inside the box by using the approximation

$$\Phi_z = \frac{\phi_{ij1} - \phi_{ij0}}{\Delta z}$$

to express ϕ_{ij1} in terms of ϕ_{ij0}.

There are several difficulties which arise when applying numerical methods to a violin-shaped box. The space is not easily discretized in the y-direction because of the curved boundary. When graphing the region, it works nicely to allow Δy to vary with x, but this does not work well for solving. Since our equations were all obtained using the Cartesian coordinate system, each variation of Δy would require a change of variables. In order to satisfy the Courant-Fredrichs-Lewy condition for stability in three-dimensional space, we need

$$\left(\frac{a\Delta t}{\Delta x}\right)^2 + \left(\frac{a\Delta t}{\Delta y}\right)^2 + \left(\frac{a\Delta t}{\Delta z}\right)^2 < 1.$$

Since the spatial dimensions of the violin are small, we are forced to choose small values for Δx, Δy, and Δz. But a (the speed of sound in air) is quite large. Time steps must, therefore, be extremely small if we choose to seek a solution for σ via numerical methods. This makes sense, however, since the frequency of the sound wave for the fundamental vibration of the open A string is close to 440 cycles per second, and this is only a mid-range frequency for the violin. In other words, the speed of sound is rapid, so waves traverse the small space inside the violin quickly. If we want to examine the behavior of a wave in any detail, we need to choose a small time increment. Specific initial and boundary conditions are needed for numerical calculations, and can only be determined after finding equations to represent the motion of the plates. Thus finding suitable conditions necessitates a thorough study of the behavior of the plates.

Once the boundary conditions can be specified, there are some numerical methods which show promise in overcoming some of the difficulties we mentioned above. Finite element methods and boundary element methods provide a way of discretizing the space (or boundary) which should work with the violin's shape, and would be worth exploring as methods for finding the elusive eigenvalues.

Chapter 7

Conclusion

We do not have a complete model of the violin at this point, but we have made some progress in setting up basic equations and processes which may be modified to reflect more of the violin's complexities. We have provided a mathematical description of the violin's major components at rest, and have discussed how the motion of one component affects the motion of the others. We have also discussed some methods for solving the equations of motion, and some issues that arise in trying to apply those methods to the space inside the violin. Our findings in this work suggest many interesting paths for future research.

We have not yet explored the motion of the plates, which is key in determining the motion of the air. The effects of outline shape and curvature are important. It also remains to apply the restrictions imposed by the bass bar and sound post to the motion of the plates, but if a general solution is already known, these will simply damp or encourage certain modes of vibration. Though we chose to view the ribs as fixed in this work, there are horizontal components to the vibration of the air which set the ribs in motion. The change in density at the f-holes, and their size and placement need to be considered as well. Since the equation for the motion of the vibrating string is well known, one might use a particular frequency of vibration in it to get initial values for the plates. Perhaps the next step in research should be to study these issues in order to learn more about our boundary conditions.

Another possible approach would be to gather empirical data on either the sound wave or the motion of the plates, and use the data to determine possibilities for the nature of the solution. There is a wealth of empirical data out there, since people have been studying the violin for centuries in the attempt to solve the mysteries of the Golden Age instruments. Working "backward" in this way may shed some light on whether or not a closed form solution for σ exists. Empirical data will also be useful in testing mathematical results once they are obtained.

It is important to remember, as we are mired in the complexities of the mathematical problems associated with the violin, that the instrument has flourished for centuries without us. Robert Adair, in *The Physics of Baseball,* makes a statement about analyzing sports, which applies as well to our mathematical analysis of the violin if one considers it as a metaphor (one might substitute "music" for "sports", "artists" for "athletes", and "violin" for "baseball").

> In all sports analyses, it is important for a scientist to avoid hubris and pay careful attention to the athletes. Major league players are serious people, who are intelligent and knowledgeable about their livelihood. Specific, operational conclusions held by a consensus of players are seldom wrong, though–since baseball players are athletes, not engineers or physicists–their analyses and rationale may be imperfect.... Honed by a century of intelligent trial and error, baseball must surely be played correctly–though not everything said about the play, by players and others, is impeccable. Hence, if a contradiction arises concerning some aspect of my analyses and the way the game is actually played, I would presume it likely that I have either misunderstood that aspect myself or that my description of my conclusion was inadequate and subject to misunderstanding. (Adair, 1994)

As we seek our own form of mathematical beauty in understanding the physical process of creating musical sound, we must appreciate the instinct and skill of the artists and craftsmen in producing beautiful music, and we can not help but acknowledge that the art has no need of a scientific understanding. Many artists may even view our mathematical descriptions as blasphemy. As mathematicians, however, we seek not to discount the skill or beauty of the craft, but merely to describe it. Describing beauty does not diminish it, but more often enhances its experience for those who did not perceive it in the same way.

FIGURE 7.1. Artist.

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