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William A. Bogley

In 1941 J.H.C. Whitehead posed the question whether asphericity is a hereditary property for two-dimensional CW complexes. This question remains unanswered. Out of its study developed the formulation of several combinatorial properties for group presentations that are sufficient (but not necessary) for asphericity of the associated two-dimensional model. The logical relationships between these flavors of asphericity are just partially understood. The main result of this dissertation is that two of these flavors of asphericity are in fact distinct. As a consequence, all of the flavors are distinct. An argument of R. C. Lyndon and P. E. Schupp (1977) shows that if a two-dimensional CW complex K is Cohen–Lyndon aspherical (CLA), then K is also diagrammatically aspherical (DA). We resolve the status of the converse implication in the negative by showing that the two-dimensional model of the presentation

$$(a, b \, : \, a, b^{-2}aba^{-1})$$

is DA but not CLA. This settles a question that has been addressed in [7], [28] and [35].

Logical Implications Between Different Flavors of Asphericity

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Igor Marko Biskup

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Major Professor, representing Mathematics

Redacted for Privacy

Chair of Department of Mathematics

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Dean of Graduate School

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Igor M. Biskup, Author

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

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Marko and Nevenka Biškup, my brother

Tomislav Biškup and my uncle

Marijan Biškup.

Logical Implications Between Different Flavors of Asphericity

1 HISTORY

In part IV of his article on homotopy groups published in 1935/36 W. Hurewicz [19] called an arcwise connected space X *aspherical* if all its higher homotopy groups vanish. That is to say that X is *aspherical* if for all $k \geq 2$, each map of the k–sphere into X extends to a map of the $(k+1)$–ball into X, see E. Dyer and A.T. Vasquez [12]. Hurewicz furthermore discovered that for a finite aspherical simplicial complex X, its homotopy type is fully determined by its fundamental group $\pi_1 X$.

In 1941, J.H.C. Whitehead [40] posed the question whether asphericity is a hereditary property for two-dimensional CW complexes. That is, given an aspherical connected two-dimensional CW–complex, is every connected subcomplex also aspherical? This question remains unanswered. The assumption that the answer will be positive is known as the *Whitehead Conjecture.*

Previously, in 1910, Max Dehn [10] had already realized that a finitely presented group is naturally associated to a two-dimensional CW complex modeled after its presentation. Thus, an intrinsic connection between group presentations and two-dimensional CW complexes was given. The interplay between algebra (group presentations) and topology (two-dimensional CW complexes) is one characteristic of combinatorial group theory and low dimensional topology.

Thus, the Whitehead Conjecture is of central importance in the study of combinatorial group theory and low dimensional topology. Out of its study developed the formulations of several combinatorial properties for group presentations that are sufficient (but not necessary) for asphericity of the associated model. These different flavors of asphericity, which we will define precisely in the next two chapters, are known as combinatorial asphericity (CA), diagrammatic asphericity (DA), Cohen-Lyndon asphericity (CLA) and diagrammatic reducibility (DR).

The notion of a presentation to be CA is derived from a long exact sequence of the universal cover of the standard two-dimensional CW complex modeled after that presentation. The question whether CA is hereditary is equivalent to Whitehead's question, as recently shown by S.V. Ivanov [21].

Notions of asphericity which all imply CA and are known to be hereditary have been investigated. These are DA, CLA and DR, see Chiswell, Collins and Huebschmann [7] and the references cited there for DA, CLA, and A.J. Sieradski [35] for DR. The DA property resulted from correcting a mistake in a proof for asphericity in Lyndon and Schupp [24] and was introduced by D.J. Collins and J. Huebschmann [9].

Important in these developments have been the publications by R. Peiffer [27] and K. Reidemeister [32], which introduced the concepts of identities between relations, in 1949. Out of it came the notions of Peiffer elements, Peiffer identities and (identity) sequences. It is in that terminology that DA was first formulated.

In 1979, K. Igusa [20] and C.P. Rourke [33] introduced pictures, although the key ideas where already present in the article by E.R. van Kampen [39] published in 1933. Pictures give a combinatorial representation of spherical maps into a two-dimensional CW complex X, and are therefore of great interest when studying elements of the second homotopy group $\pi_2 X$.

The properties CA, DA and DR can all be formulated in terms of pictures, which reflects their combinatorial nature.

In 1963, D.E. Cohen and R.C. Lyndon [8] published a purely algebraic notion of asphericity, known as *Cohen–Lyndon Asphericity*. A presentation $\mathcal{P} = (X : R)$ is Cohen–Lyndon aspherical (CLA) if there exists a certain free basis for the relation module N^{ab} which lifts to a free basis for N, where N denotes the normal closure of R in the free group on X. A theorem in Lyndon and Schupp [24] first showed that CLA implies DA. Whether this implication can be reversed has been an open question, since it was first posed in article by Chiswell, Collins and Huebschmann [7], that gives an account of the developments until 1981. In that article, a presentation was given by I.M. Chiswell which is CA but not DA. Another such example is given by A.J. Sieradski [34]. Both examples show the same picture theoretic characteristics, as mentioned by J. Huebschmann [18].

The logical relationships between CA, DA, CLA and DR that were known prior to this writing, are exhibited in the following diagram:

$$\begin{tikzcd}
\mathbf{CLA} \arrow[dr] \arrow[dd, "/" marking] & & \\
& \mathbf{DA} \arrow[r, shift left] \arrow[dl, shift left, "/" marking] & \mathbf{CA} \arrow[l, shift left, "/" marking] \\
\mathbf{DR} \arrow[ur, shift left] & &
\end{tikzcd}$$

This leaves the two open questions:

$$\text{DA} \stackrel{?}{\Longrightarrow} \text{CLA} \qquad\qquad \text{and} \qquad\qquad \text{DR} \stackrel{?}{\Longrightarrow} \text{CLA}$$

We will show that the first of these questions, the reverse implication of the Theorem by R.C. Lyndon and P.E. Schupp, has a negative answer by exhibiting an example of a group presentation that is DA but not CLA.

2 ASPHERICITY

2.1 Definitions

A *group presentation* $\mathcal{P} = (X : R)$ consists of a set X, called *generators*, and a set R, called *relators*. The elements of R are words in the semigroup $W(X)$ on the alphabet $X \cup X^{-1}$.

We then define the free group on X:

$$F = F(X) = \frac{W(X)}{xx^{-1} \sim x^{-1}x \sim \emptyset}$$

and the normal closure of R in F, i.e. the smallest normal subgroup in F containing R, called the *consequences* of R:

$$N = \langle\langle R \rangle\rangle_F$$

We thus can formulate the following

Definition 2.1 *The group defined by the presentation $\mathcal{P} = (X : R)$ is $G = G(P) = F/N$.*

Furthermore, we have the

Theorem 2.1 *For each group G there exists a presentation $\mathcal{P}$ such that $G \cong G(\mathcal{P})$.*

Next, we will examine the relationship between certain algebraic and topological notions in combinatorial group theory and low dimensional topology. Let $\mathcal{P} = (X : R)$ be a group presentation. We construct a two-dimensional CW complex as follows. Let $K^0 = c^0$ be a single 0-cell, called *basepoint*, and for each element $x \in X$ attach an oriented 1-cell, denoted by c_x^1, so that both endpoints are identified with c^0 and for $x \neq y$ we have $c_x^1 \cap c_y^1 = c^0$. The resulting set K^1 is then one point union of circles S_x^1, $x \in X$, that is

$$K^1 = \bigvee_{x \in X} S_x^1 \qquad (S_x^1 = c^0 \cup c_x^1).$$

The set K^1 is given the weak topology with respect to the family $\{S_x^1 : x \in X\}$ of circles.

Theorem 2.2 *[37] The fundamental group of the space K^1 is isomorphic to the free group on X.*

Now, for each element in R we define a map $\dot{\varphi}_r$ from the circle S_r^1 into K^1 as follows. Each relator $r \in R$ can be uniquely written as a finite reduced word of the form $x_1^{\pm 1} \ldots x_{m(r)}^{\pm 1}$, where the positive integer $m(r)$ denotes the *length* of r. We subdivide $\partial D_r^2 = S_r^1$ accordingly and label the segments by $x_1^{\pm 1} \ldots x_{m(r)}^{\pm 1}$, where $\dot{\varphi}_r$ identifies each labeled segment $x_i^{\pm 1}$ of the circle S_r^1 with the loop $S_{x_i}^1$ of K^1, respecting the orientation of K^1, i.e. x_i^{+1} is positively oriented.

The standard complex / model for $\mathcal{P} = (X : R)$ is given by

$$K^2 = K(\mathcal{P}) = c^0 \cup c_x^1 \cup c_r^2, \quad x \in X, r \in R.$$

For this, we extend $\dot{\varphi}_r$ to a map φ_r from the pair (D_r^2, S_r^1) to (K^2, K^1), where for each $r \in R$ the boundary S_r^1 of D_r^2 spells the relator $r = x_1^{\pm 1} \ldots x_{m(r)}^{\pm 1}$.

Theorem 2.3 *[37] The fundamental group of the standard complex modeled after the presentation $\mathcal{P}$ is isomorphic to the group $G(\mathcal{P})$ obtained from $\mathcal{P}$.*

And, in fact,

Theorem 2.4 *[37][Theorem 1.9] Every connected two-dimensional CW complex has the homotopy type of the model of some group presentation.*

2.2 Motivation

Our theme is to study groups by realizing them as fundamental group of a two-complex K^2, modeled after a given presentation, that is, using algebraic topology and homotopy theory of two-complexes to study group theory.

From the above construction, we realize that different group presentations give rise to different models K^2. Difficulties in homotopy of two-complexes will then cause problems in our group theoretic investigations.

It is important to note that homotopy invariants of aspherical CW complexes are group theoretic invariants of the fundamental group π_1:

Theorem 2.5 (Hurewicz [19]) *If K is an aspherical CW complex, that is $\pi_n K = 0, \forall 2 \leq n$, then the homotopy type of K is determined by the fundamental group $\pi_1 K$.*

Our goal thus becomes to build a CW complex K such that K is aspherical and $\pi_1 K \cong G$. Such a space is called a *Eilenberg–Mac Lane space* of G, denoted by $K(G, 1)$.

For this, we need to "kill" all higher homotopy groups of K^2 by attaching cells in dimensions three and higher, i.e. build

$$K = K^2 \cup \bigcup c^n, \quad 3 \leq n$$

That is, for each non–trivial element $[\alpha] \in \pi_2 K^2$ attach a three cell via

$$\begin{array}{ccc} S^2 & \xrightarrow{\alpha} & K^2 \\ \cap\vert & & \cap\vert \\ B^3 & \longrightarrow & K^2 \cup c^3 \subseteq K^3 \end{array}$$

so $[\alpha] = 0$ in $\pi_2 K^3$. To do this, we can use generators of $\pi_2 K^2$ to attach three cells, in order to obtain:

$$\pi_2 K^2 \xrightarrow{0} \pi_2 K^3$$

From the cellular approximation theorem (see Fuks and Rokhlin [14]) we furthermore get that:

$$G \cong \pi_1 K^2 \xrightarrow{\cong} \pi_1 K^3$$

$$\pi_2 K^2 \xrightarrow{surj.} \pi_2 K^3$$

Thus the three-dimensional CW complex K^3 has fundamental group G and trivial second homotopy group. We say that K^3 is obtained from K^2 by attaching 3-cells to "kill" π_2. One continues in this way, attaching $(n+1)$-cells to kill π_n to obtain a CW complex K, possibly infinite-dimensional, to serve as a $K(G,1)$. Explicit computations of the homotopy invariants of a $K(G,1)$ require more explicit knowledge of the cell structure.

The challenge then becomes to find generators for $\pi_2 K^2$, as a $\mathbf{Z}G$-module. In particular, given a two-complex K^2, we would like to be able to determine whether $\pi_2 K^2 = 0$. But, $\pi_2 K^2$ is not well enough understood. In particular, the unresolved statue of following question indicates the difficulty of the above problem:

Whitehead's Question [40]: *Is any connected subcomplex of an aspherical two complex itself aspherical?*

2.3 Flavors of Asphericity: CA

Before we enter into an analysis of the different flavors of asphericity, we will for the remainder always assume that the two following conventions hold:

Standard Model By $K = K(X : R)$ we always denote the model of $\mathcal{P} = (X : R)$.

Relator Hypothesis No element $r \in R$ is freely trivial nor conjugate to any other relator or its inverse.

In the following, we summarize the description of the equivariant world for two-dimensional CW complexes given in [37]. Consider the universal covering space $(\tilde{K}, \rho, K)$ for $K = K^2$. The cell structure of K lifts through the covering projection $\rho : \tilde{K} \to K$ to a cell structure on $\tilde{K}$, making $\tilde{K}$ into a two-complex with one-skeleton $\tilde{K}^1$. The long exact sequence (LES) for the pair $(\tilde{K}, \tilde{K}^1)$ has the following form:

$$\ldots \to H_2\tilde{K}^1 \to H_2\tilde{K} \to H_2(\tilde{K}, \tilde{K}^1) \to H_1\tilde{K}^1 \to H_1\tilde{K} \to \ldots$$

A closer investigation of the above LES yields:

(1) $H_2\tilde{K}^1 = 0$ (since $\tilde{K}^1$ is one-dimensional)

(2) Since $\tilde{K}$ is simply connected we obtain that $H_1\tilde{K} = 0$ and hence the Hurewicz homomorphism $h : \pi_2\tilde{K} \to H_2\tilde{K}$ is an isomorphism. From covering space theory, we have that the covering projection ρ induces an isomorphism $\rho_\sharp : \pi_2\tilde{K} \xrightarrow{\cong} \pi_2 K$ of homotopy groups.

(3) If $\tilde{e}_r^2$ denotes a preferred lift of e_r^2, then: $H_2(\tilde{K}, \tilde{K}^1) = \bigoplus_{r \in R} \mathbf{Z}G \cdot \tilde{e}_r^2$ is the free $\mathbf{Z}G$-module with basis $\{\tilde{e}_r^2 \,|\, r \in R\}$

(4) The covering situation for $\tilde{K}$ is given by:

$$\begin{array}{ccccc} \rho^{-1}(K^1) \subseteq & \tilde{K}^1 & \subseteq & \tilde{K} \\ & \downarrow & & \downarrow \\ & K^1 & \subseteq & K \end{array}$$

so that

$$\begin{aligned} (\rho_{|_{\tilde{K}^1}})_\sharp(\pi_1(K^1)) &= (\rho_{|_{\tilde{K}^1}})_\sharp(\pi_1\rho^{-1}(\tilde{K}^1)) \\ &= ker[\pi_1 K^1 \to \pi_1 K] \\ &= ker[F \to G] \\ &= N. \end{aligned}$$

Whence, $\pi_1\tilde{K}^1 \cong N$ and $H_1\tilde{K}^1 \cong (\pi_1\tilde{K}^1)^{ab} \cong N^{ab}$.

(5) Finally, since $\pi_1\tilde{K} = 0$, it follows that $H_1\tilde{K} = 0$.

Altogether, we get a short exact sequence of $\mathbf{Z}G$–modules described in the following

Theorem 2.6 *There exists a short exact sequence of* $\mathbf{Z}G$*–modules*

$$0 \to \pi_2 K \xrightarrow{\beta} \bigoplus_{r\in R} \mathbf{Z}G \cdot \tilde{e}^2_r \xrightarrow{\partial} N^{ab} \xrightarrow{\alpha} 0$$

where ∂ is defined through $\tilde{e}^2_r \longmapsto r[N,N]$ $\forall r \in R$. This short exact sequence is compatible with the following actions:

(a) $\pi_2 K$ is a $\mathbf{Z}G$–module via the homotopy action of $\pi_1 K \cong G$.

(b) G acts on $\bigoplus_{r\in R} \mathbf{Z}G \cdot \tilde{e}^2_r$ by permuting the cells, i.e. by Deck Transformation.

(c) G acts on N^{ab} via conjugation in F, e.g. $(wN)\cdot(n[N,N]) = wnw^{-1}[N,N]$.

With this action, N^{ab} is called the relation module for $\mathcal{P}$.

We can now formulate the following two results:

Corollary 2.1 *The second homotopy group $\pi_2 K$ is trivial if and only if N^{ab} is a free (left) $\mathbf{Z}G$–module with basis $\{r[N,N] \,|\, r \in R\}$.*

Proof: This follows immediately from the short exact sequence in Theorem 2.6.

$\square$

Lemma 2.1 *If $\pi_2 K = 0$, then no relator $r \in R$ is a proper power.*

Proof: Under the assumption that $\pi_2 K = 0$, suppose $r = s^k$ where $s \in F$ is not a proper power and $k \in \mathbf{Z}$. Show that $k = \pm 1$.

Step 1: $s \in N$ We have that $ker\, \partial \cong \pi_2 K = 0$, and that

$$srs^{-1}[N,N] \;-\; r[N,N] \;=\; 0$$

since $srs^{-1} = r$ in F.

We therefore conclude that s is in fact an element on N, that is

$$\begin{aligned} & (sN - 1_G)\tilde{e}_r^2 \in ker\, \partial \;\; \cong \pi_2 K \\ \Rightarrow \quad & (sN - 1_G)\tilde{e}_r^2 = 0 \\ \Rightarrow \quad & sN = 1_G \qquad\qquad \text{since } H_2(\tilde{K}, \tilde{K}^1) \text{ is free} \\ \Rightarrow \quad & s \in N \end{aligned}$$

Step 2: $k = \pm 1$ If we consider the LES:

$$\begin{array}{ccccccc} \pi_2 K & \xrightarrow{j_\sharp} & \pi_2(K, K^1) & \xrightarrow{\partial_\sharp} & \pi_1 K^1 & \xrightarrow{\alpha} & \pi_1 K \\ & & [D] & \longmapsto & s & \longmapsto & 0 \end{array}$$

Since $s \in N \cong ker\, \alpha = im\, \partial_\sharp$ we find that there exist $[D] \in \pi_2(K, K^1)$ such that $\partial_\sharp([D]) = s$. Thus, $\partial_\sharp([D]^k) = \partial_\sharp([D])^k = s^k = r$.

If we now consider the characteristic map for $e_r^2 \subseteq K$:

$$\begin{cases} \varphi_r^2 : \; B^2 \longrightarrow K \\ \varphi^2_{r|_{S^1}} = \dot{\varphi}_r^2 : \; S^1 \longrightarrow K^1 \end{cases}$$

we then obtain that $[\varphi_r^2] \in \pi_2(K, K^1)$ and $\partial_\sharp([\varphi_r^2]) = [\varphi^2_{r|_{S^1}}] = r \in N \trianglelefteq F \cong \pi_1 K^1$.

Whence,

$$\begin{aligned} \partial_\sharp([\varphi_r^2][D]^{-k}) &= \partial_\sharp([\varphi_r^2])\,\partial_\sharp([D])^{-k} \\ &= [\dot{\varphi}_r^2]\,s^{-k} \\ &= r\,s^{-k} \\ &= 1 \end{aligned}$$

Therefore, $[\varphi_r^2]\,[D]^{-k} \in ker\,\partial_\sharp = im\,j_\sharp = 1$. So, we obtain that $[\varphi_r^2] = [D]^k$. Next, from the Hurewicz homomorphism $h : \pi_2(K, K^1) \longrightarrow H_2(K, K^1)$ it follows that $c_r^2 = h([\varphi_r^2]) = h([D]^k) = k \cdot h([D])$.

Now, c_r^2 is an element of a basis for the free abelian group $H_2(K, K^1)$, i.e. $k \cdot h([D])$ is a basis element. Therefore $k = \pm 1$.

□

Here the last step follows from the

Lemma 2.2 *If A is a free abelian group and $a \in A$, $k \in \mathbf{Z}$ are such that $k \cdot a$ is a member of a basis for A, then $k = \pm 1$.*

Proof: Assume A has basis $\{b_1, \ldots, b_m\}$ that contains ka.

Then, $a = k_1 b_1 + \ldots + k_m b_m$ where $k_1, \ldots, k_m \in \mathbf{Z}$. Moreover, if ka is some basis element, say b_1, we have that $ka = kk_1 b_1 + \ldots + kk_m b_m = b_1$.

Thus, $kk_1b_1 = b_1$, so $kk_1 = 1$, and hence $k = k_1 = \pm 1$.

$\square$

We note here that if $r \in R$ and $r = s^k$, then $(sN - 1_G) \cdot \tilde{e}^2_r \in ker\partial$.

Definition 2.2 *The presentation* $\mathcal{P} = (X : R)$ *is called* combinatorially aspherical, *(CA), if the kernel of the map*

$$\partial : \bigoplus_{r \in R} \mathbf{Z}G \cdot \tilde{e}^2_r \longrightarrow N^{ab}$$

is generated (as $\mathbf{Z}G$*–module) by* $\mathcal{D} = \{ (sN - 1_G) \cdot \tilde{e}^2_r \}_{r \in R}$*, where* s *is the root of* r*, that is* $r = s^k$ *with* k *maximal.*

In this formulation, every CA presentation satisfies the relator hypothesis. This can be seen as follows. First, if $r_0 \in R$ is freely trivial, then $\tilde{e}^2_{r_0} \in ker\partial$ since

$$\partial(\tilde{e}^2_{r_0}) = r_0[N, N] = 1[N, N] = 0.$$

<u>Claim:</u> If $r_0 \in R$, then $\tilde{e}^2_{r_0} \notin \mathbf{Z}G \cdot \mathcal{D}$.

<u>Proof:</u> Consider the (augmentation) G-homomorphism

$$\varepsilon : \bigoplus_{r \in R} \mathbf{Z}G \cdot \tilde{e}^2_r \longrightarrow \bigoplus_{r \in R} \mathbf{Z}$$

defined by $\varepsilon(g \cdot \tilde{e}^2_r) = 1$. Then, $\varepsilon((sN - 1) \cdot \tilde{e}^2_r) = 0$, whence $\mathbf{Z}G \cdot \mathcal{D} \subseteq \ker(\varepsilon)$. Since $\varepsilon(\tilde{e}^2_{r_0}) = 1 \neq 0$, we find that $\tilde{e}^2_{r_0} \notin ker(\varepsilon)$. Thus, $\tilde{e}^2_{r_0} \notin \mathbf{Z}G \cdot \mathcal{D}$.

$\square$

Next, if r_1 is conjugate to r_2^δ, where $r_1 \neq r_2$ and $\delta \in \{-1, +1\}$, say

$$r_1 = w r_2^\delta w^{-1},$$

then,

$$\begin{aligned}\partial(\tilde{e}^2_{r_1} - \delta w N \tilde{e}^2_{r_2}) &= r_1 w r_2^{-\delta} w^{-1}[N.N] \\ &= 1[N.N] \\ &= 0\end{aligned}$$

Claim: If r_1 and r_2 are distinct elements of R, $w \in F$ and $\delta \in \{-1,+1\}$, then $\tilde{e}^2_{r_1} - \delta w N \tilde{e}^2_{r_2} \notin \mathbf{Z}G \cdot \mathcal{D}$.

Proof: Let ε be defined as above, then

$$\varepsilon(\tilde{e}^2_{r_1} - \delta w N \tilde{e}^2_{r_2}) = (0,\ldots,0,1,0,\ldots,0,-\delta,0,\ldots,0) \neq 0$$

Thus, $\tilde{e}^2_{r_1} - \delta w N \tilde{e}^2_{r_2} \notin ker(\varepsilon)$, whence $\tilde{e}^2_{r_1} - \delta w N \tilde{e}^2_{r_2} \notin \mathbf{Z}G \cdot \mathcal{D}$.

□

Combinatorial asphericity is also referred to as the Identity Property. A large class of groups which are CA is given by the following result, which is known as Lyndon's Simple Identity Theorem,

Theorem 2.7 (R. Lyndon [23]) *All one-relator presentations are CA.*

The relationship between asphericity and the CA property is the following.

Theorem 2.8 *[7][Proposition 1.3] Let K be the model of the presentation $\mathcal{P} = (X : R)$. The second homotopy group $\pi_2 K$ is trivial if and only if the presentation $\mathcal{P}$ is CA and no relator is a proper power.*

Proof: We assume first that $\pi_2 K$ is trivial. Then, from Lemma 2.1 we conclude that no relator is a proper power.

Moreover, it follows that the map ∂ in Theorem 2.6 is an isomorphism, whence $\mathcal{P}$ is CA.

On the other hand, assume that $\mathcal{P}$ is CA and that no relator is a proper power, that is the kernel of the map

$$\partial : \bigoplus_{r \in R} \mathbf{Z}G \cdot \tilde{e}_r^2 \longrightarrow N^{ab}$$

is generated by $\{\, (rN - 1_G) \cdot \tilde{e}_r^2 \,\}_{r \in R} = \{\, (1_G - 1_G) \cdot \tilde{e}_r^2 \,\}_{r \in R}$.

Consequently, $\pi_2 K = 0$, by Theorem 2.6.

□

2.4 Sequences and Peiffer Operations

We will now introduce the notions of sequences, identity sequences and Peiffer operations.

If $\mathcal{P} = (X : R)$ is a given presentation, let $F = F(X)$ denote the free group on the set of generators and $\langle\langle R \rangle\rangle_F = N \trianglelefteq F$, so $G = F/N$. Moreover, $\mathsf{F}(X : R)$ is the free group on the set $F \times R$, and $\vartheta : \mathsf{F}(X : R) \to F$ is the homomorphism defined by $(w, r) \mapsto wrw^{-1}$.

Elements $\sigma \in \mathsf{F}(X : R)$ are called *sequences*; they are viewed as formal consequences of the relators R. The free group F acts on $\mathsf{F}(X : R)$ on the left, through:

$$x \cdot (w, r) \;=\; (xw, r)$$

Furthermore, $\vartheta(w \cdot \sigma) = w\vartheta(\sigma)w^{-1}$, so $ker\,\vartheta$ is F-invariant.

Elements of $\mathsf{E}(X : R) = ker\,\vartheta$ are called identity sequences for the presentation $\mathcal{P}$, and more precisely

Definition 2.3 *An* identity sequence *over $\mathcal{P} = (X : R)$ is a sequence of the form*

$$(w_1, r_1)^{\epsilon_1} \dots (w_n, r_n)^{\epsilon_n}$$

where $0 \leq n$, $\epsilon_i \in \{-1, +1\}$, $w_i \in F(X)$, $r_i \in R$ for all $i = 1, \dots, n$ and $\prod_{i=1}^{n} w_i r_i^{\epsilon_i} w_i^{-1} = 1$ in $F(X)$. If $n = 0$, we speak of the empty sequence.

Lemma 2.3 *There is a short exact sequence of left F-groups*

$$1 \longrightarrow \mathsf{E}(X : R) \longrightarrow \mathsf{F}(X : R) \xrightarrow{\vartheta} N \longrightarrow 1$$

The set of elements of the form

$$\{(w, r)^{\delta}(v, t)^{\varepsilon}(w, r)^{-\delta}(wr^{\delta}w^{-1}v, t)^{-\varepsilon} \mid w, v \in F, r, t \in R, \delta, \varepsilon \in \{-1, +1\}\}$$

is called the set of *Peiffer elements* and their normal closure in $\mathsf{F}(X : R)$, denoted by $\mathsf{P} = \mathsf{P}(X : R)$, is called the group of *Peiffer identities.*

Lemma 2.4 *The group of Peiffer identities $\mathsf{P}(X : R)$ is a subgroup of $\mathsf{E}(X : R)$ and $\mathsf{P}(X : R)$ is F-invariant.*

Proof: We observe first that

$$\begin{aligned}
\vartheta\left((w, r)^{\delta}(v, t)^{\varepsilon}(w, r)^{-\delta}(wr^{\delta}w^{-1}v, t)^{-\varepsilon}\right) &= \\
wr^{\delta}w^{-1}vt^{\varepsilon}v^{-1}wr^{-\delta}\underbrace{w^{-1}w}r^{\delta}w^{-1}vt^{-\varepsilon}v^{-1}wr^{-\delta}w^{-1} &= \\
wr^{\delta}w^{-1}vt^{\varepsilon}v^{-1}w\underbrace{r^{-\delta}r^{\delta}}w^{-1}vt^{-\varepsilon}v^{-1}wr^{-\delta}w^{-1} &= \\
&\vdots \\
&= 1
\end{aligned}$$

Thus, $\mathsf{P}(X:R)$ is a subgroup of $\mathsf{E}(X:R)$.

To show that $\mathsf{P}(X:R)$ is F-invariant, we check that

$$\begin{aligned}
x\cdot\Big((w,r)^{\delta}(v,t)^{\varepsilon}(w,r)^{-\delta}(wr^{\delta}w^{-1}v,t)^{-\varepsilon}\Big) &= \\
\Big((xw,r)^{\delta}(xv,t)^{\varepsilon}(xw,r)^{-\delta}(xwr^{\delta}w^{-1}v,t)^{-\varepsilon}\Big) &= \\
\Big((xw,r)^{\delta}(xv,t)^{\varepsilon}(xw,r)^{-\delta}(xwr^{\delta}w^{-1}x^{-1}xv,t)^{-\varepsilon}\Big) &\in \mathsf{P}(X:R)
\end{aligned}$$

since xw and xv are in F.

□

We can perform the following *Peiffer operations* on sequences

$$\sigma = \prod_{i=1}^{n}(w_i,r_i)^{\epsilon_i} \in \mathsf{F}(X:R),$$

Substitution: Replace any w_i by a word freely equal to it.

Exchange: Replace $(w_i,r_i)^{\epsilon_i}(w_{i+1},r_{i+1})^{\epsilon_{i+1}}$

by $(w_i r_i^{\epsilon_i} w_i^{-1} w_{i+1}\ ,\ r_{i+1})^{\epsilon_{i+1}}(w_i,r_i)^{\epsilon_i}$

or $(w_{i+1},r_{i+1})^{\epsilon_{i+1}}(w_{i+1}r_{i+1}^{-\epsilon_{i+1}}w_{i+1}^{-1}w_i\ ,\ r_i)^{\epsilon_i}$

Deletion: Delete two consecutive terms $(w_i,r_i)^{\epsilon_i}(w_{i+1},r_{i+1})^{\epsilon_{i+1}}$, whenever

$w_i r_i^{\epsilon_i} w_i^{-1} w_{i+1} r_{i+1}^{\epsilon_{i+1}} w_{i+1}^{-1} = 1$ in $F(X)$.

Insertion: Opposite of **Deletion**.

Then, two sequences σ and σ' are *Peiffer equivalent* if σ can be transformed into σ' via the above operations. We write $[\sigma]$ to denote the equivalence class of all sequences that are Peiffer equivalent to σ.

Theorem 2.9 *If the presentation $\mathcal{P} = (X : R)$ satisfies the relator hypothesis, then $\mathcal{P}$ is CA provided every identity sequence over $\mathcal{P}$ is Peiffer equivalent to the empty sequence $1 \in \mathsf{F}(X : R)$.*

Proof: Suppose σ is an identity sequence, and that σ can be reduced to the empty sequence via a finite number of Peiffer operations. From Lemma 2.3, Theorem 2.6 and the map

$$\eta : \mathsf{F}(X : R) \longrightarrow \bigoplus_{r \in R} \mathbf{Z}G \cdot \tilde{e}_r^2$$

defined by $\eta(w, r) = wN \cdot \tilde{e}_r^2$, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \to & \mathsf{E}(X : R) & \to & \mathsf{F}(X : R) & \xrightarrow{\vartheta} & N & \to & 1 \\
 & & & & \eta \downarrow & & \downarrow \text{nat} & & \\
0 & \to & \pi_2 K & \to & \bigoplus_{r \in R} \mathbf{Z}G \cdot \tilde{e}_r^2 & \xrightarrow{\partial} & N^{\text{ab}} & \to & 0
\end{array}$$

We note that η respects the F-actions, since:

$$\begin{aligned}
\partial\eta(x \cdot (w, r)) &= \eta(xw, r) \\
&= (xwN) \cdot \tilde{e}_r^2 \\
&= xN(wN \cdot \tilde{e}_r^2) \\
&= xN \cdot \eta(w, r)
\end{aligned}$$

Moreover, $\partial \circ \eta = \text{nat} \circ \vartheta$, since:

$$\begin{aligned}
\partial\eta(w, r) &= \partial(wN \cdot \tilde{e}_r^2) \\
&= wrw^{-1}\,[N, N] \\
&= \vartheta(w, r)\,[N, N]
\end{aligned}$$

Finally, we will make use of the fact that η is surjective. This follows since $\bigoplus_{r\in R} \mathbf{Z}G\cdot\tilde{e}_r^2$ has $\mathbf{Z}G$-basis $\{\tilde{e}_r^2 : r \in R\}$ and given any $r \in R$, we have $\eta(1,r) = \tilde{e}_r^2$, i.e. $\{\tilde{e}_r^2 : r \in R\}$ is contained in $im(\eta)$.

We need to show that $ker\,\partial$ is generated by the set $\{ (sN-1)\cdot\tilde{e}_r^2 \,|\, r \in R \}$, where s is the root of $r \in R$.

Assume that $\Sigma \in ker\,\partial$. Since η is surjective, there exist $\sigma' \in \mathsf{F}(X:R)$ such that $\eta(\sigma') = \Sigma$. From $\partial\eta = \text{nat}\vartheta$ it follows that $\vartheta(\sigma') \in ker(\text{nat}) = [N,N]$. Moreover, since ϑ is surjective, there exist $\tau \in [\mathsf{F}(X:R), \mathsf{F}(X:R)]$ such that $\vartheta(\tau) = \vartheta(\sigma')$. Then, $\vartheta(\sigma'\tau^{-1}) = 1$, so $\sigma'\tau^{-1} \in \mathsf{E}(X:R)$. Since $\bigoplus_{r\in R} \mathbf{Z}G\cdot\tilde{e}_r^2$ is abelian, it follows that $[\mathsf{F}(X:R), \mathsf{F}(X:R)] \subseteq ker\,\eta$ and so $\eta(\tau) = 0$. Now, we compute as follows.

$$\begin{aligned}
\eta(\sigma'\tau^{-1}) &= \eta(\sigma') - \eta(\tau) \\
&= \eta(\sigma') \\
&= \Sigma
\end{aligned}$$

Thus, for all $\Sigma \in ker\,\partial$ there exist $\sigma = \sigma'\tau^{-1} \in \mathsf{E}(X:R)$ such that $\eta(\sigma) = \Sigma$. By hypothesis, σ is Peiffer equivalent to the empty sequence $1 \in \mathsf{F}(X:R)$, that is there exist a finite sequence $\sigma = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n = 1$, where σ_{i+1} is obtained from σ_i by a Peiffer move.

Next, we show that if an (identity) sequence σ_i is Peiffer equivalent to a sequence σ_{i+1}, then $\eta(\sigma_i) \equiv \eta(\sigma_{i+1})$ modulo the submodule of $\bigoplus_{r\in R} \mathbf{Z}G\cdot\tilde{e}_r^2$ generated by $\mathcal{D} = \{(sN-1)\cdot\tilde{e}_r^2 \,|\, r \in R\}$.

If σ_{i+1} is obtained from σ_i by a Substitution, then $\eta(\sigma_{i+1}) = \eta(\sigma_i)$, since

$$\begin{aligned}
\eta(w'x^{\epsilon}x^{-\epsilon}w'', r) &= w'x^{\epsilon}x^{-\epsilon}w''N \cdot \tilde{e}_r^2 \\
&= wN \cdot \tilde{e}_r^2 \\
&= \eta(w, r)
\end{aligned}$$

Similarly, if σ_{i+1} is obtained from σ_i by an Exchange, e.g. $\sigma_i = \sigma_i'(w,r)^{\delta}(v,t)^{\epsilon}\sigma_i''$ and $\sigma_{i+1} = \sigma_i'(wr^{\delta}w^{-1}v, t)^{\epsilon}(w,r)^{\delta}\sigma_i''$, then $\eta(\sigma_{i+1}) = \eta(\sigma_i)$, since

$$\begin{aligned}
\eta(\sigma_{i+1}) &= \eta(\sigma_i') + \epsilon\eta(wr^{\delta}w^{-1}v, t) + \delta\eta(w,r) + \eta(\sigma_i'') \\
&= \eta(\sigma_i') + \epsilon\,\underbrace{wr^{\delta}w^{-1}}_{\in N}vN \cdot \tilde{e}_t^2 + \delta\eta(w,r) + \eta(\sigma_i'') \\
&= \eta(\sigma_i') + \epsilon vN \cdot \tilde{e}_t^2 + \delta\eta(w,r) + \eta(\sigma_i'') \\
&= \eta(\sigma_i') + \delta\eta(w,r) + \epsilon\eta(v,t) + \eta(\sigma_i'') \\
&= \eta(\sigma_i)
\end{aligned}$$

Next, suppose σ_{i+1} is obtained from σ_i by an Insertion, e.g. $\sigma_i = \sigma_i'\sigma_i''$ and $\sigma_{i+1} = \sigma_i'(w,r)^{\delta}(v,t)^{\epsilon}\sigma_i''$, where $wr^{\delta}w^{-1}vt^{\epsilon}v^{-1} = 1$ in $F(X)$. Then, $r^{\delta} = w^{-1}vt^{\epsilon}v^{-1}w$, that is r^{δ} is conjugate to $t^{-\epsilon}$ in F. From the relator hypothesis, it follows that $r = t$, where $\epsilon = -\delta$ (no relator is freely trivial). Moreover, we have that wv^{-1} centralizes r, and so $wv^{-1} = s^k$ where $r = s^{k_0}$ and $k, k_0 \in \mathbf{Z}$.

With this, we have

$$\begin{aligned}
\sigma_{i+1} &= \sigma_i'(w,r)^{\delta}(v,t)^{\epsilon}\sigma_i'' \\
&= \sigma_i'(w,r)^{\delta}(ws^k, r)^{-\delta}\sigma_i'' \\
&= \sigma_i'w \cdot (1,r)^{\delta}ws^k \cdot (1,r)^{-\delta}\sigma_i'' \\
&= \sigma_i'w \cdot ((1,r)^{\delta}s^k \cdot (1,r)^{-\delta})\sigma_i''
\end{aligned}$$

Now,

$$\begin{aligned}
\eta(\sigma_{i+1}) &= \eta(\sigma_i') + \delta w N \cdot (\tilde{e}_r^2 - s^k N \cdot \tilde{e}_r^2) + \eta(\sigma_i'') \\
&= \eta(\sigma_i') + \delta w N \cdot (1 - s^k N) \cdot \tilde{e}_r^2) + \eta(\sigma_i'') \\
&= \eta(\sigma_i') + \delta w N \cdot (1 - s^{k \bmod (k_0)} N) \cdot \tilde{e}_r^2) + \eta(\sigma_i'') \\
&= \eta(\sigma_i') + \underbrace{\delta w N(1 + N + \ldots + s^{k_0 - 1} N)}_{\in \mathbf{Z}G} \cdot \underbrace{(1 - sN)}_{\in \mathcal{D}} \cdot \tilde{e}_r^2) + \eta(\sigma_i'')
\end{aligned}$$

Therefore, $\eta(\sigma_{i+1}) - \eta(\sigma_i)$ lies in the submodule generated by $\mathcal{D}$ and the theorem is proved.

□

The converse to Theorem 2.9 is also true. That is, if $\mathcal{P}$ is CA, then every identity sequence over $\mathcal{P}$ is Peiffer equivalent to the empty sequence. We will not need this.

2.5 Flavors of Asphericity: DA

One problem that arises in this context is that, in practice, one has no control over the number of insertions in a sequence of Peiffer operations. It is therefore desirable to find criteria which guarantee that an identity sequence can be reduced to the empty sequence in a controlled number of Peiffer operations. An essentially stronger condition is not to allow for any insertions at all. We therefore state the following definition given by Chiswell, Collins and Huebschmann [7]:

Definition 2.4 *The presentation* $\mathcal{P} = (X : R)$ *is called* diagrammatically aspherical *(DA) if every identity sequence over* $\mathcal{P}$ *can be transformed into the empty sequence using substitution, exchange and deletion operations, only.*

Thus, from Theorem 2.9 we obtain the

Corollary 2.2 *If the presentation $\mathcal{P} = (X : R)$ is diagrammatically aspherical and satisfies the relator hypothesis, then $\mathcal{P}$ is also combinatorially aspherical.*

2.6 Flavors of Asphericity: CLA

Let $\mathcal{P} = (X : R)$ be a presentation for a group $G = G(\mathcal{P}) = F/N$, where N is the normal closure $\langle\langle R \rangle\rangle_F$ of R in the free group $F = F(X)$ with basis X. From the fact that

$$N = \{\prod_{i=0}^{m} w_i r_i^{\pm 1} w_i^{-1} \,|\, w_i \in F, r_i \in R\} \trianglelefteq F \quad \text{(free)}$$

we obtain that

$$N^{ab} = \{\sum_{i=0}^{m} g_i \cdot r_i[N, N] \,|\, g_i \in G, r_i \in R\} \quad \text{(free abelian)}$$

with the natural map $N \longrightarrow N^{ab}$. Certainly, any free basis for N will project down to a $\mathbf{Z}$–basis for N^{ab}.

Theorem 2.10 (Cohen–Lyndon [8]) *Let $\mathcal{P} = (X : r)$ be a one-relator presentation, where $F = F(X)$, $N = \langle\langle r \rangle\rangle_F$, $r \neq 1$ in F, and C_r is the centralizer of r in F. Then there exists a transversal U, that is a choice of coset representatives, for NC_r in F such that N is freely generated by the set of all elements uru^{-1} for $u \in U$.*

As shown in Theorem 2.1, when $\pi_2 K = 0$, N^{ab} has the $\mathbf{Z}$-basis:

$$\{g \cdot r[N, N] \,|\, g \in G, r \in R\}.$$

Now, let U be a transversal for N in F. Then, G can be written as the disjoint union of cosets uN, that is $G = \mathring{\bigcup_{u \in U}} uN$. Furthermore, N^{ab} has the $\mathbf{Z}$–basis: $\{uru^{-1}[N, N] \,|\, u \in U, r \in R\}$.

We can now pose the

Question: *When is it possible to lift the $\mathbf{Z}$–basis, with G-action, for N^{ab} to a free basis, with F–action (conjugation), for N?*

Definition 2.5 *The presentation $\mathcal{P} = (X : R)$ is called* Cohen–Lyndon aspherical *(CLA) if N, the normal closure of R in the free group F, has a Cohen–Lyndon basis of conjugates of elements of R. That is, N has a basis of the form*

$$B = \bigcup_{r \in R} \{uru^{-1} \,|\, u \in U(r)\}$$

where $U(r)$ is a full left transversal for NC_r in F, $r \in R$, where C_r is the centralizer of r in F.

As pointed out by Chiswell, Collins and Huebschmann, Lyndon and Schupp gave an argument to show that all CLA presentations are DA. We give a picture-theoretic proof of this fact in 4.1.

2.7 Whitehead's Question

Two important facts concerning the CLA property are that, first of all, CLA implies asphericity in the usual sense ($\pi_2 = 0$) if no proper powers occur in R, and secondly, the CLA property is hereditary. Thus, the Whitehead Conjecture holds for all two-complexes modeled after CLA presentations:

Theorem 2.11 *[7][Proposition 2.4] Let $\mathcal{Q} = (Y : S)$ be a subpresentation of $\mathcal{P} = (X : R)$, that is $Y \subseteq X$ and $S \subseteq R \cap F(Y)$. Suppose $\mathcal{P}$ is CLA. Then $\mathcal{Q}$ is CLA.*

The fact that DA is hereditary follows immediately from the definition, and for completeness we quote the

Lemma 2.5 *[7][Lemma 2.2] Let $\mathcal{Q} = (Y : S)$ be a subpresentation of $\mathcal{P} = (X : R)$. If $\mathcal{P}$ is DA, then $\mathcal{Q}$ is DA.*

For the CA property the question regarding heredity turns out to be much more difficult. In fact, as recently shown by S. V. Ivanov, we have the following

Theorem 2.12 *[21][Theorem 4] Whitehead's Conjecture is equivalent to the conjecture that combinatorial asphericity is hereditary.*

3 PICTURES

3.1 Basic Definitions

We have introduced three algebraic flavors of asphericity. The CA condition is homological, the DA and CLA conditions are combinatorial group theoretic. Our next goal is to reinterpret diagrammatic asphericity as a combinatorial geometric flavor of asphericity. First, we present the basic definitions and facts about the theory of pictures, mostly what will be used later on. For a more rigorous and extensive account we refer to the articles by W.A. Bogley and S.J. Pride [3] and S.J. Pride [28] and the references given there.

A *picture* **P** consists of

1. An ambient disc D with boundary ∂D.
2. A finite collection of pairwise disjoint closed discs $\triangle_1, \ldots, \triangle_n$ in $int(D)$. If $n = 0$, we speak of a one-dimensional picture.
3. A finite collection of pairwise disjoint compact 1–manifolds $e_1, \ldots, e_m$, called *arcs*, properly embedded in $D - \bigcup\limits_{i=1}^{n} int(\triangle_i)$, i.e. $\partial e_j = e_j \cap \partial\left(D - \bigcup int(\triangle_i)\right)$.

The picture **P** is *spherical* if no arc of **P** touches ∂D. Given a group presentation $\mathcal{P} = (X : R)$, we say **P** is a *picture over* $\mathcal{P}$, if

4. Each arc is labeled by a transverse arrow and an element of X.
5. Each interior disc $\triangle_i$ is labeled by a relator $r \in R$ and a sign $\epsilon = \pm 1$.

6. Each boundary $\partial\Delta_i$ has a marked *basepoint*, that does not lie on any arc of $\mathbf{P}$.

7. The boundary ∂D has a *basepoint*, that does not lie on any arc of $\mathbf{P}$.

The labellings are required to satisfy the following continuity condition:

8. For any interior disc Δ_i with label $r \in R$ and sign $\epsilon = \pm 1$, if we start at the basepoint of Δ_i and read clockwise around the boundary $\partial\Delta_i$ recording in order the occurrences of generators x^δ, $\delta = \pm 1$, as we cross the ends of labeled and oriented arcs that touch Δ_i, the word is identically equal to r^ϵ

Given any path α in $D - \bigcup int(\Delta_i)$ that meets all arcs transversely, we similarly associate a *labelling* word $W(\alpha)$.

In particular, the labelling word for the path that circumnavigates ∂D in the clockwise direction, starting at the basepoint, is the *boundary word* $W(\mathbf{P})$ of $\mathbf{P}$. Pictures over a presentation $\mathcal{P} = (X : R)$ provide geometric representations of the consequences of the relators of $\mathcal{P}$.

Lemma 3.1 (van Kampen) *[26][Lemma 11.1] Let w be an arbitrary nonempty word in $F(X)$. Then $w = 1$ in the group with presentation $\mathcal{P} = (X : R)$ if and only if there exists a picture $\mathbf{P}$ over $\mathcal{P}$ such that $W(\mathbf{P})$ spells exactly w.*

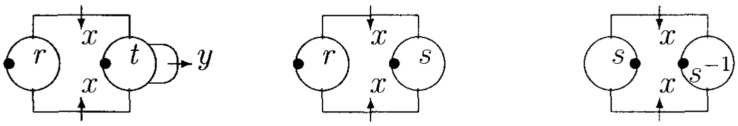
Corollary 3.1 *If $\mathcal{P}$ is a one-dimensional picture, then $W(\mathbf{P})$ is freely trivial.*

An important special class of spherical pictures is described in the following

Definition 3.1 *A (based) spherical picture over the presentation $\mathcal{P} = (X : R)$ consisting of exactly two discs is called a* dipole. *Furthermore, if there exits a region such* *that reading around the two discs (clockwise for one and counter clockwise for the other) beginning in that region spells exactly the same word, we refer to a* cancelling pair. *A cancelling pair with basepoints in the same region is called a* folding pair.

Example: Consider the following presentation for the free group on two generators

$$\mathcal{P} = (x, y : r = xx^{-1}, s = x^{-1}x, t = x^{-1}y^{-1}yx)$$



dipole | cancelling pair | folding pair

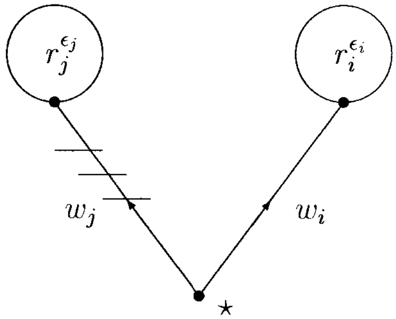
Lemma 3.2 *(1) If no relator is freely trivial nor conjugate to any other relator or its inverse, then every dipole is a cancelling pair. (2) If, in addition, no relator is a proper power, then every dipole is a folding pair.*

We note that, under the Relator Hypothesis, cancelling pairs that are not folding pairs come from proper powers.

3.2 Pictures and Identity Sequences

The relationship between pictures and identity sequences is via sprays. A *spray* Σ in a given picture $\mathbf{P}$ is a collection of arcs $\gamma_1, \ldots, \gamma_n$ connecting the basepoint of each disc Δ_i to the basepoint $\star$ of the ambient disc D. These arcs may intersect the edges $e_1, \ldots, e_m$ (transversely) but not each other, that is $\gamma_i \cap \gamma_j = \star$ for $i \neq j$.

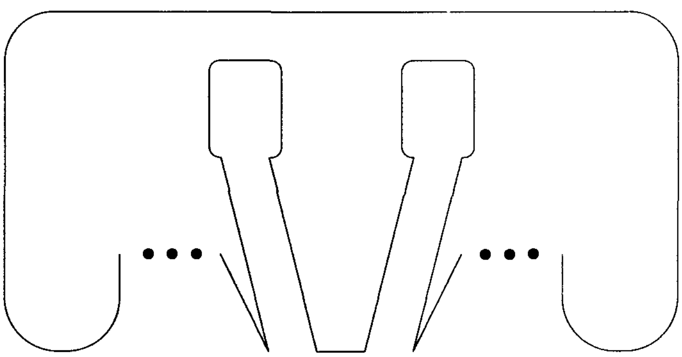
By $\sigma\langle \mathbf{P}, \Sigma \rangle$ we denote the (identity) sequence derived from the spray Σ in $\mathbf{P}$, in the following way:



For each arc γ_j of the spray we obtain a word $w_j = W(\gamma_j)$ in $F(X)$ as we list the label of each (picture) arc that γ_j traverses. We then read the disc label $r_j^{\epsilon_j}$ and as we read back along γ_j, we write down w_j^{-1}. Altogether each *balloon on a string* contributes $w_j r_j^{\epsilon_j} w_j^{-1} = (w_j, r_j)^{\epsilon_j}$. This is done in clockwise order around the basepoint $\star$ to give a sequence

$$\sigma\langle \mathbf{P}, \Sigma \rangle = \prod_{i=1}^{n} (w_i, r_i)^{\epsilon_i}.$$

If we cut the picture along the arcs of the spray and delete $int(\triangle_i)$ we obtain a one-dimensional picture with exterior label freely equal to 1 in F:



Moreover, we have the following results (see S.J. Pride [28][Section 2.1]:

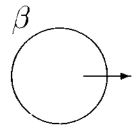
Theorem 3.1 *Given any sequence σ there is a based picture* $\mathbf{P}$ *over* $\mathcal{P} = (X : R)$ *and a spray* Σ *in* $\mathbf{P}$ *such that* $\sigma\langle \mathbf{P}, \Sigma \rangle = \sigma$. *If σ is an identity sequence, then the picture* $\mathbf{P}$ *can be chosen to be spherical.*

Corollary 3.2 *If* $\mathbf{P}$ *is a based spherical picture over* $\mathbf{P}$ *and* Σ *is a spray in* $\mathbf{P}$, *then* $\sigma\langle \mathbf{P}, \Sigma \rangle$ *is an identity sequence.*

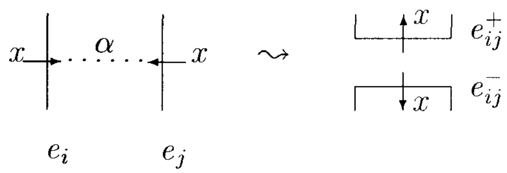
3.3 Picture Moves

The following operations can be performed on a picture $\mathbf{P}$ over $\mathcal{P} = (X : R)$:

Float: Insertion or deletion of a floating arc which is disjoint from all interior discs and arcs.



Bridge: Operation on two arcs of the following type:



Dipole: Insertion or deletion of dipoles.

We remark that an arc is a floating arc if all other discs and arcs of $\mathbf{P}$ are outside of β, that is in the connected component of $D - \beta$ that contains the basepoint of D. More specifically, a bridge move is performed in the following fashion.

1. Connect the head (tail) of a labeled arrow of the arc e_i to another, equally labeled, arrow head (tail) of some arc e_j by a path α, such that α does not intersect any arcs or discs.

2. Thicken α to an ε-tube, denoted by U_ε, where $0 < \varepsilon$ is so that U_ε is disjoint from all arcs and discs. Then $U_\varepsilon \approx \alpha \times [-\varepsilon, \varepsilon]$.

3. Connect e_i to e_j via the paths $\alpha^+ \approx \alpha \times \{\varepsilon\}$ and $\alpha^- \approx \alpha \times \{-\varepsilon\}$.

4. Delete $U_\varepsilon^* \approx \alpha \times (-\varepsilon, \varepsilon)$.

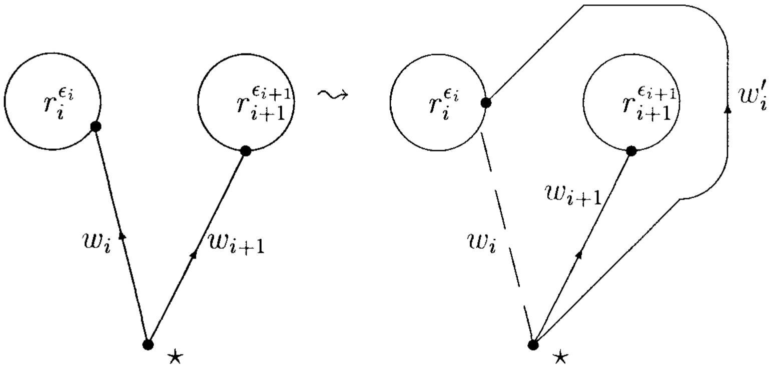
5. The new arcs, replacing e_i and e_j, are then e_{ij}^+ and e_{ij}^-.

We say that two pictures over $\mathcal{P} = (X : R)$, $\mathbf{P}$ and $\mathbf{Q}$, are *equivalent* if, up to isotopy, $\mathbf{P}$ can be transformed into $\mathbf{Q}$ using a finite number of the above operations.

Before we proceed to investigate asphericity in terms of pictures, we will show how picture moves relate to the Peiffer operations mentioned previously. This is described in the following three Lemmata.

Lemma 3.3 *Given a picture* $\mathbf{P}$ *over the presentation* $\mathcal{P} = (X : R)$ *and a spray* Σ *in* $\mathbf{P}$. *Then a Peiffer exchange can be realized by a change of spray.*

Proof: We will give the argument for the case where the change of spray arcs corresponds to the Peiffer exchange where $(w_i, r_i)^{\epsilon_i}, (w_{i+1}, r_{i+1})^{\epsilon_{i+1}}$ is replaced by $(w_{i+1}, r_{i+1})^{\epsilon_{i+1}}$, $(w_{i+1} r_{i+1})^{-\epsilon_{i+1}} (w_i, r_i)^{\epsilon_i} (w_{i+1}, r_{i+1})^{\epsilon_{i+1}}$. This can be illustrated as follows:



The label of the new spray arc w_i' is obtained by reading along w_{i+1}, counterclockwise around the disc $\triangle_{i+1}$, back along w_{i+1} and finally along w_i. Since $W = w_{i+1}r_{i+1}^{-\epsilon_{i+1}}w_{i+1}^{-1}w_i(w_i')^{-1}$ represents the label of a spherical picture, by the van Kampen Lemma, $W = 1$. Thus, $w_{i+1}r_{i+1}^{-\epsilon_{i+1}}w_{i+1}^{-1}w_i = w_i'$

The corresponding Peiffer exchange is

$$(w_i, r_i)^{\epsilon_i}(w_{i+1}, r_{i+1})^{\epsilon_{i+1}}$$

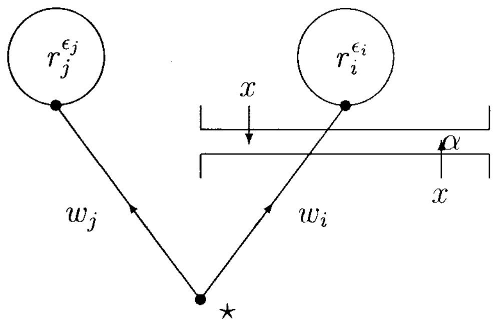
$$= (w_{i+1}, r_{i+1})^{\epsilon_{i+1}}, (w_i', r_i)^{\epsilon_i}$$

$$= (w_{i+1}, r_{i+1})^{\epsilon_{i+1}}, (w_{i+1}r_{i+1}^{\epsilon_{i+1}}w_{i+1}^{-1}w_i, r_i)^{\epsilon_i}$$

□

Lemma 3.4 *If $\mathbf{P}'$ is obtained from $\mathbf{P}$ by a bridge move and Σ is a spray in $\mathbf{P}$, then $\sigma\langle\mathbf{P}', \Sigma\rangle$ is obtained from $\sigma\langle\mathbf{P}, \Sigma\rangle$ by a finite number of Peiffer substitutions.*

Proof: The proof can be illustrated as follows:



Consider the two interior discs $\triangle_j$ and $\triangle_i$, labeled by $r_j^{\epsilon_j}$ and $r_i^{\epsilon_i}$ respectively. We label the spray arcs connecting the basepoint $\star$ of ∂D to the basepoints of $\triangle_j$ and $\triangle_i$ by w_j and w_i. As the bridge α, labeled by an element $x \in X \cup X^{-1}$, traverses w_i the word spelled when we read along w_i is changed to the word $w_i' x x^{-1} w_i''$, freely equal to w_i. In terms of operations on identity sequences this is precisely a substitution. A spray arc w_j that is not traversed by the bridge remains uneffected.

□

We note that a Peiffer substitution on a sequence can be realized by an insertion of a floating circle intersecting the corresponding spray arc.

Lemma 3.5 *There is a one-to-one correspondence between a deletion of a dipole and a Peiffer deletion. In particular, we have (a) If* $\mathbf{P}'$ *is obtained from* $\mathbf{P}$ *via a deletion of a dipole, then there exist sprays* Σ' *in* $\mathbf{P}'$ *and* Σ *in* $\mathbf{P}$ *such that* $\sigma\langle \mathbf{P}', \Sigma' \rangle$ *is obtained from* $\sigma\langle \mathbf{P}, \Sigma \rangle$ *via a Peiffer deletion. (b) If* σ' *is obtained from* σ *via a Peiffer deletion, then there exist sprays* Σ' *in* $\mathbf{P}'$ *and* Σ *in* $\mathbf{P}$ *such that* $\sigma' = \sigma\langle \mathbf{P}', \Sigma' \rangle$, $\sigma = \sigma\langle \mathbf{P}, \Sigma \rangle$ *and* $\mathbf{P}'$ *is obtained from in* $\mathbf{P}$ *via a deletion of a dipole.*

Proof: (a) Suppose $\mathbf{P}'$ is obtained from $\mathbf{P}$ via a deletion of a dipole consisting of discs Δ_i and Δ_{i+1} and let γ_i and γ_{i+1} be the corresponding spray arcs with label w_i and w_{i+1}, respectively. Then, the dipole yields two consecutive pairs

$$(w_i, r_i)^{\epsilon_i}(w_{i+1}, r_{i+1})^{\epsilon_{i+1}}$$

and since every dipole is a spherical subpicture, $w_i r_i^{\epsilon_i} w_i^{-1} w_{i+1} r_{i+1}^{\epsilon_{i+1}} w_{i+1}^{-1}$ is freely trivial. Whence, a Peiffer deletion applies.

(b) Suppose σ' is obtained from σ via a Peiffer deletion, that is

$$\sigma = \sigma_1(w_i, r_i)^{\epsilon_i}(w_{i+1}, r_{i+1})^{\epsilon_{i+1}}\sigma_2$$

where $\sigma' = \sigma_1\sigma_2$ and $w_i r_i^{\epsilon_i} w_i^{-1} w_{i+1} r_{i+1}^{\epsilon_{i+1}} w_{i+1}^{-1}$ is freely trivial. Then,

$$(w_i, r_i)^{\epsilon_i}(w_{i+1}, r_{i+1})^{\epsilon_{i+1}}$$

corresponds to a spherical subpicture with exactly two discs, a dipole. We choose the spray arcs so that γ_i connects to the disc with label $r_i^{\epsilon_i}$ and γ_{i+1} to the disc with label $r_{i+1}^{\epsilon_{i+1}}$. Now, a deletion of the dipole applies.

□

Finally, we will show the relation between (identity) sequences, pictures and the chain map

$$\partial : C_2(\tilde{K}, \tilde{K}^1) \longrightarrow C_1(\tilde{K}^1, \tilde{K}^0).$$

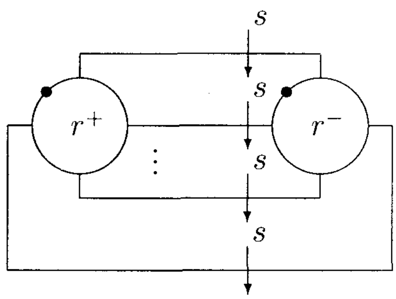
If $\mathbf{P}$ is a picture over $\mathcal{P}$ and Σ is a spray in $\mathbf{P}$ giving rise to the sequence $\sigma\langle\mathbf{P}, \Sigma\rangle$, then

$$\prod_{i=1}^{n} (w_i, r_i)^{\epsilon_i} \in \mathsf{F}(X : R).$$

We then obtain the chain

$$\sum_{i=1}^{n} \epsilon_i w_i N \cdot \tilde{e_{r_i}}^2 \in C_2(\tilde{K}, \tilde{K}^1).$$

If $\mathbf{P}$ is a spherical picture over $\mathcal{P}$, by Corollary 3.2, $\sigma\langle \mathbf{P}, \Sigma\rangle$ is an identity sequence. Thus, $\sum\limits_{i=1}^{n} \epsilon_i w_i N \cdot \tilde{e_{r_i}}^2 \in ker[C_2\tilde{K} \to C_1\tilde{K}]$. Then, in terms of pictures, the element $(sN - 1_G) \cdot \tilde{e}_r^2 \in ker(\partial)$ is represented by a dipole with basepoints offset by one, where $r = s^k$:



This follows from the fact that, in the case where $r = s^k \in R$ as in the picture above, we can choose a spray Σ such that $\sigma\langle \mathbf{P}, \Sigma\rangle = (1, r), (s, r)^{-1}$. Then, $1r1^{-1}sr^{-1}s^{-1} = s^k s s^{-k} s^{-1} = 1$.

3.4 Flavors of Asphericity: DA

For our purpose, we need to interpret the DA property in terms of pictures. This is done through the next

Theorem 3.2 *The presentation $\mathcal{P} = (X : R)$ is diagrammatically aspherical if every spherical picture* $\mathbf{P}$ *over* $\mathcal{P}$ *can be converted to the empty picture without insertions of dipoles. That is,* $\mathbf{P}$ *can be reduced using bridge moves, insertion or deletion of floating arcs and deletions of dipoles, only.*

Proof: We assume that any given spherical picture over $\mathcal{P}$ can be reduced without insertion of dipoles, and moreover that we are given an identity sequence σ over $\mathcal{P}$. Then, by Theorem 3.1, there exist a spherical picture $\mathbf{P}$ over $\mathcal{P}$ and a spray Σ in $\mathbf{P}$ such that $\sigma = \sigma\langle \mathbf{P}, \Sigma\rangle$. Now, after a finite sequence of bridge moves, we obtain a dipole with discs $\triangle_i$ and $\triangle_{i+k}$, where $1 \leq k$.

The effect of these bridge moves on σ, according to Lemma 3.4, is a finite number of Peiffer substitutions. Furthermore, we can rearrange the arcs of the spray Σ, so that the spray arcs connecting to the dipole are in consecutive order. This requires $k-1$ changes of spray arcs as described in Lemma 3.3 and we have thus performed $k-1$ exchanges in σ.

Finally, the deletion of the dipole $\mathbf{P}_1$, corresponding to a Peiffer deletion of consecutive terms as in Lemma 3.5, is possible. Inductively, we obtain the empty sequence.

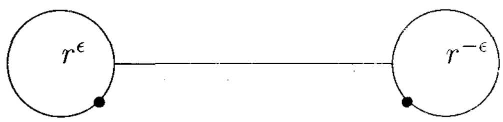
□

The converse also holds, but we will not need it. Since creating a dipole, which then can be removed, in a finite number of steps has to be possible for any given picture, this can easily lead into complicated structure analysis for the spherical pictures under consideration.

3.5 Flavors of Asphericity: DR

An even stronger condition is if we require that each spherical picture comes already equipped with a *folding arc*, that is any connecting arc between two discs that would form a folding pair if all arcs were connected to each other. In that case, we can always complete such a connection to a folding pair via bridge moves and apply a deletion.

This connection is between to discs labeled by r^{ϵ} and $r^{-\epsilon}$ such that it is part of a folding pair with respect to the basepoints:



This condition is formulated in the next

Definition 3.2 *The presentation $\mathcal{P} = (X : R)$ is called* diagrammatically reducible *(DR) if every spherical picture over $\mathcal{P}$ contains a folding arc.*

Another great advantage of the DR condition is that it is detectable by so called *Weight Test*, see S.M. Gersten [15]. The DR condition was first studied by A.J. Sieradski [35].

It now follows from the above definitions that we have the logical implications:

$$DR \implies DA \implies CA$$

We will show that none of these implications can be reversed.

4 LOGICAL RELATIONSHIPS

4.1 CLA $\Rightarrow$ DA

We have already established that $DR \Rightarrow DA \Rightarrow CA$, and will now proceed to place the CLA condition in that logical chain and investigate the reverse implications. In Lyndon and Schupp [24][Chapter III, Proposition 10.6] the fact that CLA implies DA was first observed. The proof given uses the algebraic notion of identity sequences. We will translate this into the terminology of pictures and prove the following

Theorem 4.1 *Whenever $\mathcal{P} = (X : R)$ is CLA, then every spherical picture over $\mathcal{P}$ can be transformed into the empty picture without using insertions of dipoles.*

Proof: Let $\mathcal{P} = (X : R)$ be a CLA presentation, and assume furthermore that $\mathbf{P}$ is a spherical picture over $\mathcal{P}$. We need to show how to reduce $\mathbf{P}$ to the empty picture without using insertions of dipoles.

We will make use of the following

Lemma 4.1 *Suppose $\mathbf{P}$ and $\mathbf{Q}$ are one-dimensional spherical pictures over the presentation $\mathcal{P} = (X : R)$ with identically equal boundary words. Then $\mathbf{P}$ and $\mathbf{Q}$ have equivalent cancellation patterns. That is, we can transform one cancellation pattern into the other using bridge moves and insertion/deletion of floating circles.*

Proof: Since we assumed that $W(\mathbf{P}) \equiv W(\mathbf{Q})$, it is possible to identify the two discs $D_{\mathbf{P}}$ and $D_{\mathbf{Q}}$ along their respective boundaries. Thus, we obtain a sphere with equator equal to $W(\mathbf{P}) \equiv W(\mathbf{Q})$. Moreover, if we represent the cancellation patterns of $W(\mathbf{P})$ and $W(\mathbf{Q})$ by half circles, we obtain a new one-dimensional spherical picture.

Since both $W(\mathbf{P})$ and $W(\mathbf{Q})$ freely reduce in F, we obtain a collection of closed half circles, each traversing the equator, possibly containing but not intersecting each other. Moreover, there must be at least one minimal half circle on each side of the equator, representing a free cancellation.

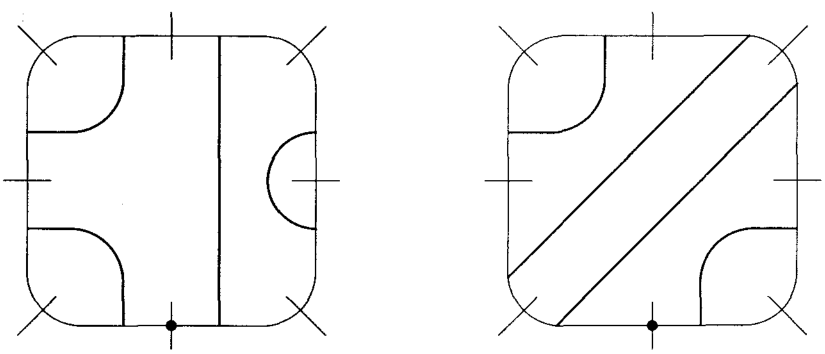
We can now apply a bridge moves to close the open side of one of the minimal half circles, if necessary. This closed minimal circle represents identical free cancellation in both, $W(\mathbf{P})$ and $W(\mathbf{Q})$, hence we can delete this minimal circle together with the corresponding part of the equator.

The proof now follows by induction.

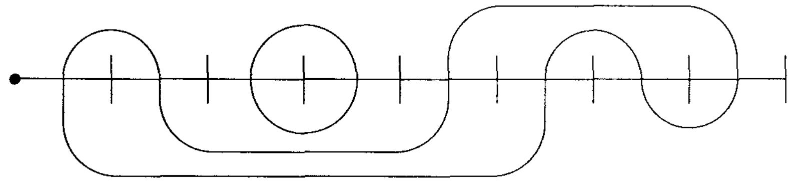
$\square$

Example:

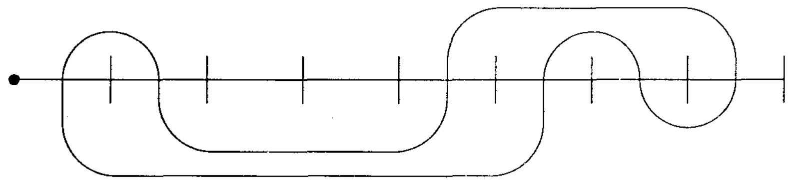
P **Q**



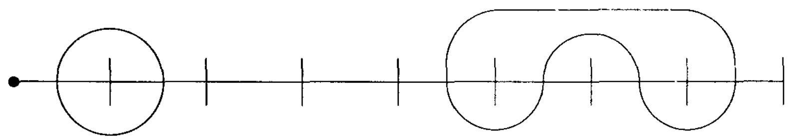
Example of two one-dimensional pictures with identical boundary words but different cancellation patterns. We can represent the common boundary edge, the equator, as interval, including both cancellation patterns represented by half circles:



Delete a minimal circle:



Apply a bridge move to close a minimal half circle:



In this fashion, we can completely reduce all half circles.

□

Recall that if $\mathcal{P} = (X : R)$ is a CLA presentation then N has a free basis B of the form $\{\, uru^{-1} \,|\, u \in U(r), r \in R \,\}$ where $U(r)$ is a full left transversal for NC_r in $F = F(X)$, that is

$$F = \mathop{\mathring{\bigcup}}_{u \in U(r)} N\, u\, C_r$$

and C_r is the normalizer for $r \in R$. Moreover, let Σ be a spray in $\mathbf{P}$ and $\sigma = \sigma\langle \mathbf{P}, \Sigma\rangle$ the identity sequence derived from Σ. Then, $\sigma = (w_1, r_1)^{\epsilon_1} \dots (w_n, r_n)^{\epsilon_n}$, where

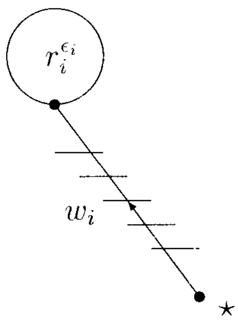
$$\prod_{i=1}^{n} w_i r_i^{\epsilon_i} w_i^{-1} = 1 \in F(X).$$

Since $w_i \in F = \bigcup_{u \in U(r)} N\,u\,C_r$ we can write

$$w_i = n_i\,u_i\,c_i \qquad \forall\, i = 1, \dots, n$$

where $u_i \in U(r_i), n_i \in N$ and $c_i \in C_r$.

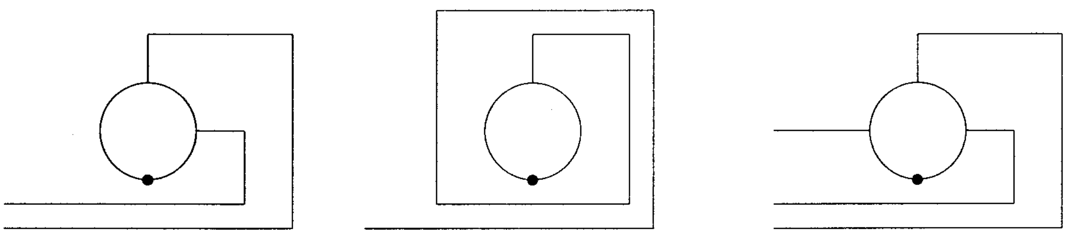
Then each $w_i r_i^{\epsilon_i} w_i^{-1} = n_i\,u_i\,c_i r_i^{\epsilon_i} c_i^{-1}\,u_i\,n_i^{-1}$ is represented by



Since $C_r = \langle s^k \rangle$, where s is the root of r

$$c_i r_i^{\epsilon_i} c_i^{-1} = s_i^{k_i} s_i^{m_i} s_i^{-k_i}$$

where $k_i, m_i \in \mathbf{Z}$. Then, we can connect the c_i-arcs via bridge moves to the $r_i^{\epsilon_i}$-arcs



$m_i = k_i$ $\qquad$ $m_i < k_i$ $\qquad$ $k_i < m_i$

Then, each r_i-disc is contained in a larger disc, denoted as R_i-disc, where reading around R_i yields the relator $r_i^{\epsilon_i}$.

We can now apply Lemma 4.1 to that effect that we replace all **P** arcs to obtain a new picture $\mathbf{P}'$ with $W(\mathbf{P}) = W(\mathbf{P}')$, where $\mathbf{P}'$ reflects a cancellation pattern in

$$B = \{ uru^{-1} \mid u \in U(r), r \in R \}$$

Here, for each $i = 1, \ldots, n$, the parallel arcs for $u_i r_i^{\epsilon_i} u^{-1}$ on R_i proceed through the picture $\mathbf{P}'$ and end all on some R_j, thus forming a dipole. The picture now reduces to the empty picture via deletions of dipoles.

□

4.2 CA $\not\Rightarrow$ DA

The counterexample that answers the question, whether the implication DA $\Rightarrow$ CA can be reversed, in the negative was given by I.M. Chiswell [7] and A.J. Sieradski [34]. I. M. Chiswell shows that the presentation

$$\mathcal{C} = (a, b \,:\, a, b^{-2}aba)$$

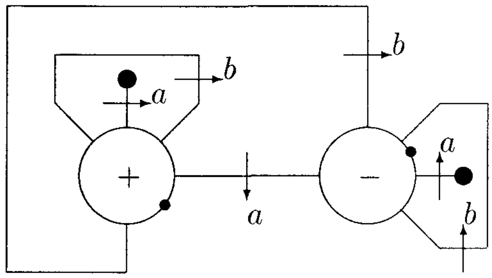
is not DA. As noted in the article by Chiswell, Collins and Huebschmann [7], the presentation $\mathcal{C}$ is CA. This follows from [7][Lemma 1.6].

To show that $\mathcal{C}$ cannot be DA, it is observed that the identity sequence:

$$\sigma = (b^{-2}aba, (ba)^{-1}\, a^{-1}\, (ba), b\, a\, b^{-1}, b\, (b^{-2}aba)^{-1}\, b^{-1})$$

cannot be transformed into the empty sequence by exchange and deletions only.

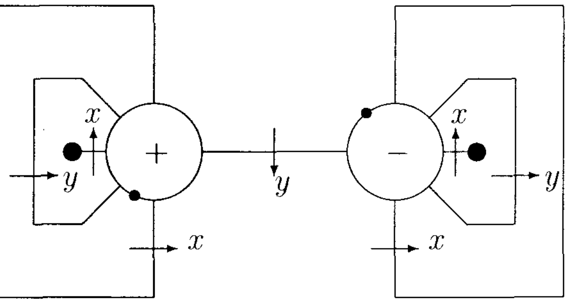
We observe that this identity sequence can be derived from the spherical picture over $\mathcal{C}$:



The above identity sequence σ can be read off a spray from this picture, and the fact that σ cannot be reduced using exchanges and deletions corresponds to the observation that the above picture does not contain a dipole nor does it allow for any nontrivial bridge moves, i.e. further reductions will only be possible after inserting a dipole (CA but not DA). Since CLA implies DA, the presentation $\mathcal{C} = (a, b : a, b^{-2}aba)$ cannot be CLA, either.

A purely algebraic argument for the fact that σ cannot be reduced with exchanges and deletions only, is given by I.M. Chiswell [7]. J. Huebschmann redoes this in [18], using the converse to Theorem 3.2.

A.J. Sieradski [34] in his article showed that the presentation $(x, y : yxy^{-1}xyx^{-1})$ does not admit a geometrically split null homotopy. Again, we find this fact represented by a spherical picture that does not allow any nontrivial bridge moves:

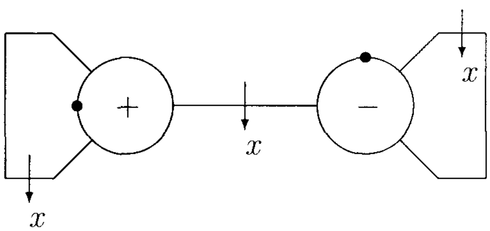


4.3 CLA & DA $\not\Rightarrow$ DR

An example for a presentation that is CLA but not DR is that of the Dunce Hat, namely:

$$\mathcal{D} = (x \,:\, xxx^{-1})$$

It follows immediately from the Cohen–Lyndon Theorem that $\mathcal{D}$, being a one–relator presentation, is CLA. Yet, the following spherical picture over $\mathcal{D}$ contains no folding arc:



Whence, D is not DR.

As shown above, the Dunce Hat presentation $\mathcal{D}$ is CLA, hence DA, but not DR. This establishes that the implication DR $\Rightarrow$ DA cannot be reversed, either.

5 A COUNTEREXAMPLE

5.1 The Presentation $\mathcal{C}$ Revisited

As mentioned in the previous section, I.M. Chiswell [7] presented an example of a group presentation which is CA but not DA, namely $\mathcal{C} = (a, b \,:\, a, b^{-2}aba)$. We alter this presentation to form a new one:

$$\mathcal{B} = (a, b \,:\, a, b^{-2}aba^{-1})$$

and put basepoints according to the relator $b^{-1}aba^{-1}b^{-1}$.

5.2 $\mathcal{B}$ is neither CLA nor DR

Essential in Chiswell's picture is the fact that we can have arcs, labeled by a, connecting the two S discs ($S = b^{-2}aba$) of different polarity without folding arcs. This property can be destroyed if we modify the presentation as indicated above:

$$\mathcal{C} = (a, b \,:\, a, b^{-2}aba) \quad \rightsquigarrow \quad \mathcal{B} = (a, b \,:\, a, b^{-2}aba^{-1}) \;=\; (a, b \,:\, a, b^{-1}aba^{-1}b^{-1})$$

We note here that the presentation $\mathcal{B}$ belongs to a certain class of balanced two generator presentations of the trivial group investigated by C.F. Miller III and P.E. Schupp [25]. These presentations are of the form:

$$(a, b \,:\, a = w, b^{k+1}a^{-1}b^{k}a)$$

where $w \in F(a, b)$.

In the new presentation, we label the relators $R = a$ and $S = b^{-1}aba^{-1}b^{-1}$. The new presentation $\mathcal{B}$ remains CA, being another balanced presentation of the trivial group.

Theorem 5.1 *The presentation $\mathcal{B} = (a, b \,:\, a, b^{-1}aba^{-1}b^{-1})$ is not CLA.*

Proof: Suppose that $\mathcal{B}$ is CLA. Then, since $C_r, C_s \leq N = F(a, b)$, we find that $F(a, b)$ has a free basis consisting of elements

$$\{u\,a\,u^{-1}, v\,b^{-1}aba^{-1}b^{-1}\,v^{-1}\}$$

with $u, v \in F(a, b)$. Then, we conclude that the quotient $F(a, b)/\langle\langle b^{-1}aba^{-1}b^{-1}\rangle\rangle$ is cyclic. Now, if the presentation

$$(a, b \,:\, b^{-1}aba^{-1}b^{-1}) \;=\; (a, b \,:\, ab = b^2a)$$

were cyclic, then it would also be abelian. Since

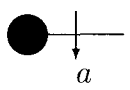
$$(a, b \,:\, ab = b^2a) \;=\; \langle b\rangle_\infty \star_{a, b \to b^2}$$

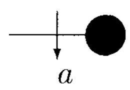
is an HNN-Extension of the infinite cyclic group and thus $\langle b\rangle_\infty$ embeds in $(a, b \,:\, ab = b^2a)$, we find that $b \neq 1$. But, $b = [a, b] \neq 1$ shows that $(a, b \,:\, ab = b^2a)$ is not abelian, hence not cyclic. Thus, $\mathcal{B}$ is not CLA.

□

5.3 Pictures over $\mathcal{B}$

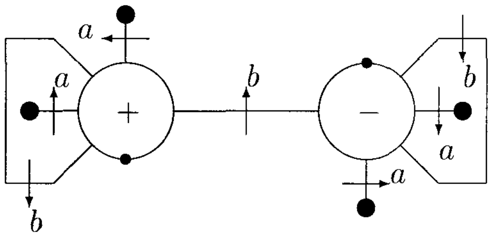
We now turn to the structure analysis of spherical pictures over $\mathcal{B}$. The two R discs corresponding to the relation $R : a = 1$ will be indicated by filled in discs without indication of basepoints, i.e.

R^+: 

R^-: 

The two S discs in any spherical picture over $\mathcal{B}$ are:

The basepoints are indicated here. For sake of clarity later on, the basepoints placed between the two b-arcs, will be omitted. Each S disc has two *double* b-arcs adjacent to the basepoints and a *single* b-arc that is not adjacent to the basepoint. We will speak of *arcs* whether we mean a connecting arc between two discs or just the end piece of it. We immediately realize that $\mathcal{B}$ is not DR, since



represents a spherical picture over $\mathcal{B}$ that does not contain a folding arc. This follows from the fact that the basepoints are on different sides of the connecting b-arc.

We will now show that $\mathcal{B}$ is in fact DA. For this it suffices to show that every spherical picture over $\mathcal{B}$ can be reduced without insertions of dipoles. This is guaranteed by giving a prescription of how to apply a certain finite sequence of bridge moves that will lead to a folding arc. This guarantees the reducibility of every identity sequence over $\mathcal{B}$ via Peiffer exchanges and deletions, only.

5.4 $\mathcal{B}$ is DA

Theorem 5.2 *The presentation $\mathcal{B} = (a, b \, : \, a, b^{-1}aba^{-1}b^{-1})$ is diagrammatically aspherical.*

Proof: We assume that we are given a nonempty spherical picture **P** over $\mathcal{B}$ that is reduced in the sense that **P** does not contain any folding arcs.

We note first that if **P** does not contain any S discs, then it must contain a R dipole (folding pair), contrary to our assumption. Thus, we conclude that **P** contains at least one S disc.

(0) Every connected spherical picture over $\mathcal{B}$ must contain an equal number of discs of opposite polarity for each relator. This follows from the fact that $H_2(\mathcal{B}) = 0$. That is, we have

$$\begin{array}{rcll} \pi_2 K(\mathcal{B}) & \longrightarrow & H_2 K(\mathcal{B}) & \leq \; C_2(K(\mathcal{B})) \\ [\mathbf{P}] & \longmapsto & (R^+ - R^-) \;\; e^2_R & \\ & & (S^+ - S^-) \;\; e^2_S & \end{array}$$

and furthermore $C^2K(\mathcal{B}) \longrightarrow C^1K(\mathcal{B})$ given by the matrix

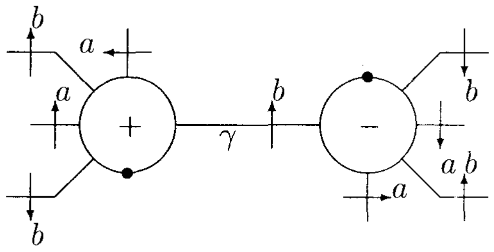
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is an isomorphism, whence $R^+ - R^- = S^+ - S^- = 0$.

Lemma 5.1 *(1) Every spherical picture over $\mathcal{B}$ contains an arc joining a S^+ disc to a S^- disc. (2) Any a-arc joining a S^+ to a S^- disc is a folding arc. (3) For any b-arc joining a S^+ disc to a S^- disc that is not a folding arc the basepoints are adjacent and on opposite sides of that arc.*

Proof: (1) There exist an arc γ in $\mathbf{P}$ joining a S^+ disc to a S^- disc, because every S disc is contained in some connected component which itself is a spherical picture $\mathbf{P}'$ over $\mathcal{B}$. Then, by (0), $\mathbf{P}'$ contains an equal number of S^+ and S^- discs, and being a connected component, some arc γ joins a S^+ disc to a S^- disc.

(2) This connection cannot happen via an a-arc or a single b-arc, without being a folding arc, contradicting our assumption of $\mathbf{P}$ being reduced. Thus the connection must be between two offset double b-arcs, for example:



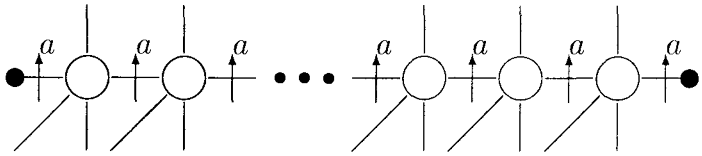
Again, we observe that the two basepoints are on different sides of the connecting arc γ. We will refer to this configuration as an *offset.*

For the remainder of the proof, we will omit the basepoints and the b-arcs in order to keep the pictures simple. The basepoints should be imagined between all the double b-arcs.

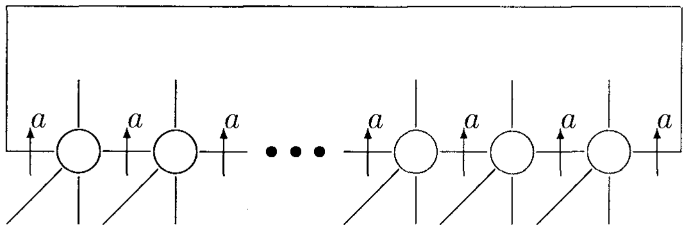
(3) It follows from (2) that an a-arc joining S discs of opposite polarity is a folding arc. Hence, any a-arc must join S discs of same polarity.

□

The a-arcs therefore form either a-strings:



or a-loops:



We will call the region involving the single b-arcs the *interior* region of the a-loop and the region involving the double b-arcs the *exterior* region of the a-loop (as the picture suggests).

Lemma 5.2 *Every S disc is contained in a unique a-string or a-loop.*

Proof: This follows immediately from the fact that exactly two a-arcs emanate from each S disc.

□

Next, we will show that one can always transform an a-loop into an a-string.

Lemma 5.3 *Any a-loop can be converted into an a-string via a finite sequence of bridge moves. Furthermore, this process does not effect any of the exterior double b-arcs on the a-loop.*

Proof: Assume we have an a-loop. We will make use of the following three facts:

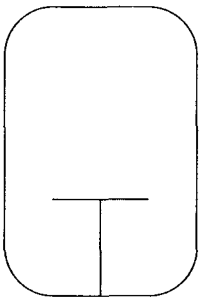
(4) None of the interior b-arcs can connect directly to each other. This follows from the orientation of the b-arcs.

(5) An interior b-arc cannot connect to a S disc of opposite polarity, since we assumed no folding arc exists.

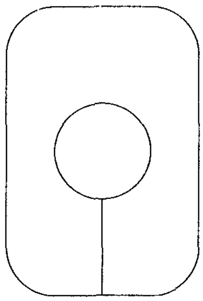
(6) Any spherical picture over $\mathcal{B}$ involves only a finite number of discs.

By (4) we conclude that the interior region must contain other S discs. Pick any single b-arc emanating from the a-loop. It must connect to another S disc of same polarity, by (5). This interior S disc is itself part of an a-string or an a-loop.

The two possible scenarios are:



Case 1



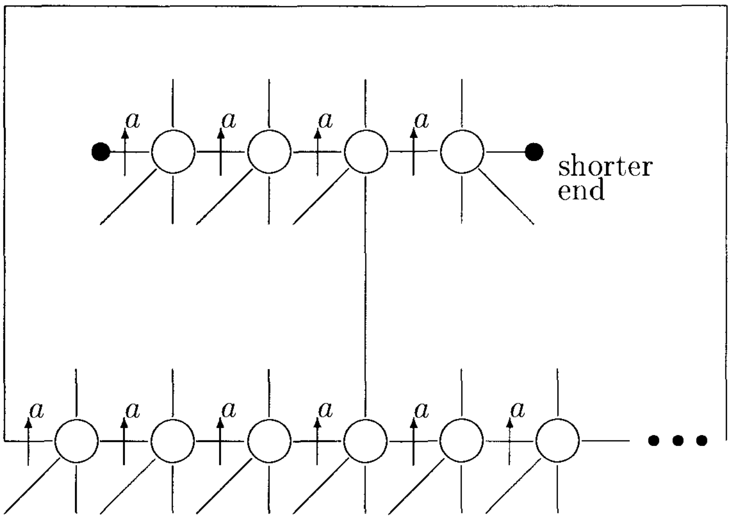
Case 2

Here, case 1 represents the case that a single b-arc connects the a-loop to an a-string in its interior region, where in the other case, case 2, a single b-arc connects the a-loop to another a-loop in its interior region. We note that this two cases are not mutually exclusive, but depend merely on our choice of which single b-arc we follow into the interior region of the a-string.

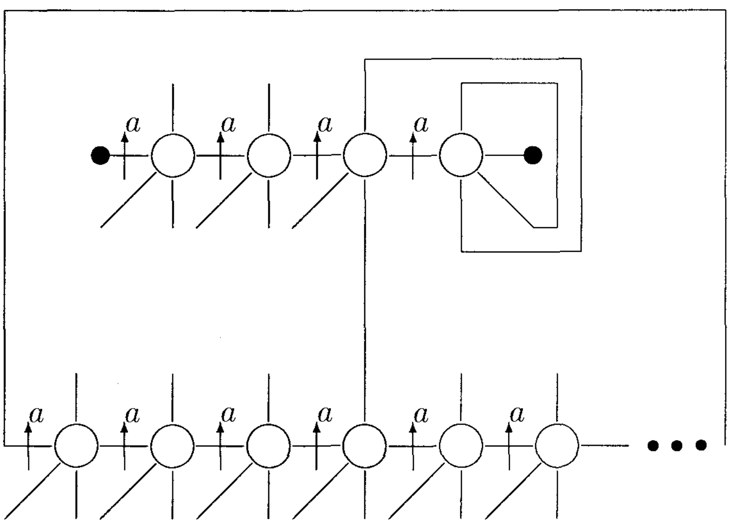
In case 1, suppose the interior a-string involves n different S discs of same polarity and therefore $2n$ double b-arcs. One end of the a-string will count at most $\frac{2n-1}{2} < n$

double b-arcs from the b-arc connecting the interior a-string to the a-loop.

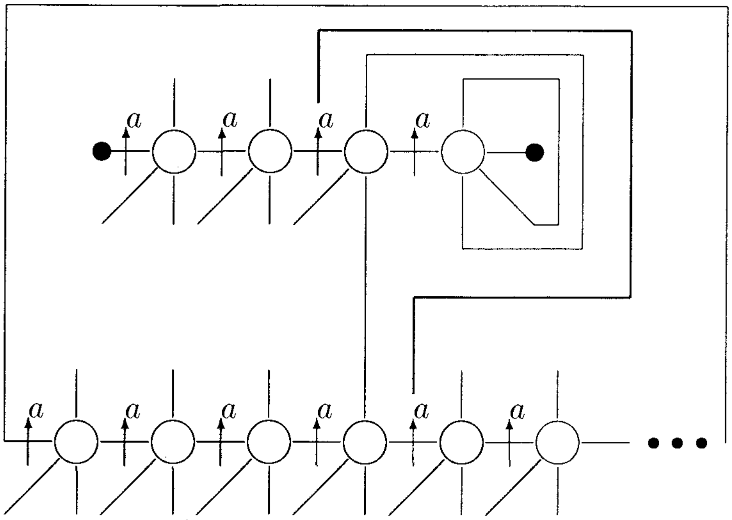
Thus, we can perform the following sequence of bridge moves which we will refer to as *capping off an end*:



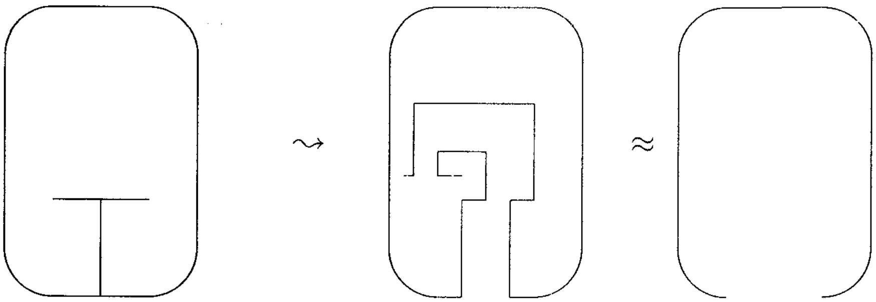
Cap-Off the shorter end:



This finally opens up the possibility to break the a-loop via the following bridge move:



The effect of this last bridge move can be illustrated as follows (the two pictures on the right being isotopic):



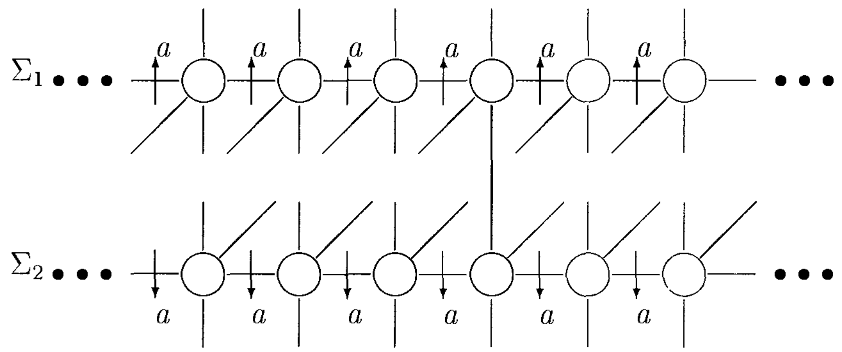
It is important to note that this process of breaking up a loop does not effect the double b-arcs in the exterior region (none of the double b-arcs where involved in any of the bridge moves).

In case 2, we can shift our focus onto the interior region a-circle. Note that this interior circle may include more boundary discs than the original. For this new a-circle we have again one of two cases, case 1 or case 2. Because of (6), we cannot continue with case 2 indefinitely. That is, case 1 must occur after a finite number of concentric a-loops. Then a cascade of the bridge moves will break up the first a-circle, eventually.

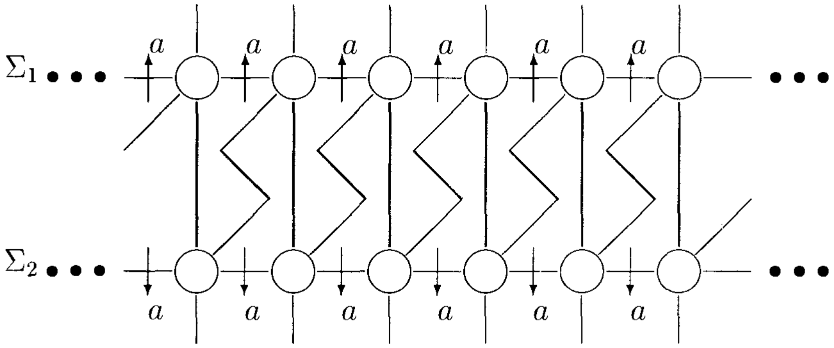
This completes the proof of the Lemma.

□

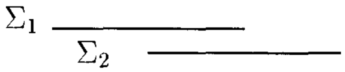
We now turn back to the situation where two S discs of opposite polarity connect via an offset. That is, we have a nontrivial reduced spherical picture **P**, and by Lemma 5.3 a finite number of bridge moves can be applied to break up all a-loops into a-strings. Two of these a-strings will be of opposite polarity and joined by an offset. Hence, we may assume that we face the situation where a S string Σ_1 is connected to a S string Σ_2 via at least one b-arc, and Σ_1 and Σ_2 are of opposite polarity:



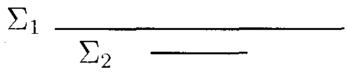
The first observation we make is that we can always apply a sequence of bridge moves that will align the two a-strings as far as possible, i.e. think of "zipping" a zipper:



This can result in two situations. Either with both or only one a-string overlapping:

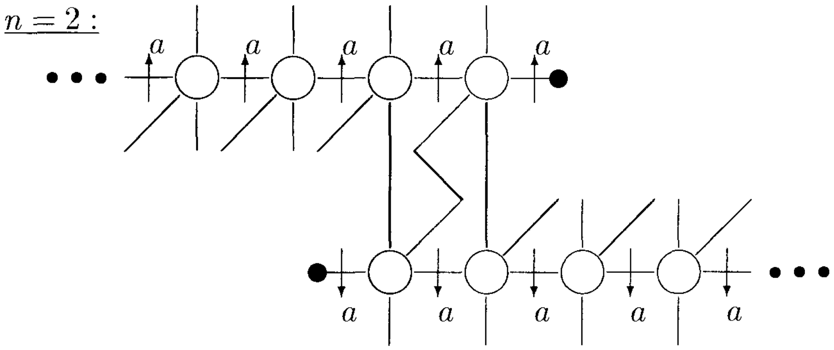


Both Σ_1 and Σ_2 have unzipped double b-arcs

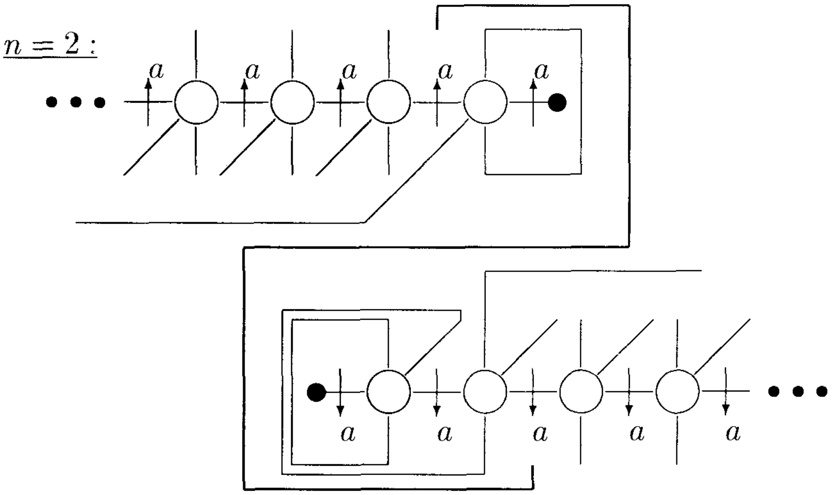


All double b-arcs on Σ_2 are zipped

Note that, in the case when both a-strings overlap, we have at least one unzipped b-arc on both Σ_1 and Σ_2, whereas when only Σ_1 overlaps, all b-arcs of Σ_2 are zipped. In the case that both a-strings overlap, we can apply a process we call *Snake-move*. Let n be the number of aligned discs after the above "zipping" process. We then face the situation below (example for $n = 2$):



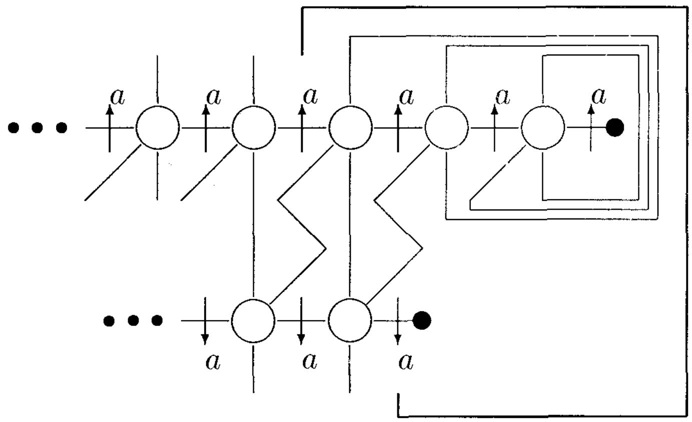
The connection will happen via $2n-1 = n+(n-1)$ double b-arcs. We now cap off n b-arcs on the zipped end of the the a-strings and $n-1$ on the zipped end of the other a-string. Thus, we break off the $2n-1$ connections, making way for a bridge move connecting a head of an a-arc to another head of an a-arc of different polarity. Hence, we created a folding arc:



We remark that at least n of the S discs in both a-strings were involved and we have not used more than that many single b-arcs for the capping off process. Therefore, we are guaranteed to have made available two a-arcs to be connected via a Snake-move.

This connection between S discs of different polarity represents a folding arc.

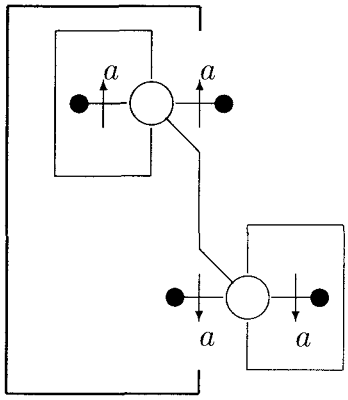
On the other hand, suppose we face the situation that only one of the two a-strings is overlapping. Then, the longer a-string involves n different S discs of same polarity and the shorter a-string m different S discs of opposite polarity ($m + 1 \leq n$). After zipping the two strings together, one end of the longer a-string will have at most $n - 1$ unzipped double b-arcs. Thus, we can completely cap off that end and find a bridge move connecting the head of an a-arc to another head of an a-arc representing a folding arc between two S discs of opposite polarity:



This exhausts all possible situations, completing the proof that $\mathcal{B}$ is DA.

$\square$

Finally, looking back at the spherical picture over $\mathcal{B}$:

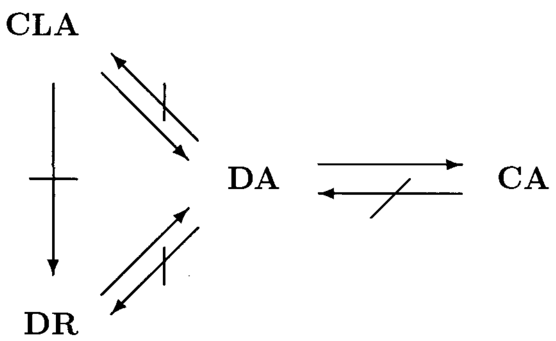


we now recognize the case of two a-strings (of length 1) of opposite polarity connecting via an offset. Here both strings are overlapping and completely aligned. Note that we can apply a Snake move right away, since both ends are already capped-off.

6 CONCLUSION

It seems noteworthy to compare a presentation that has the CLA property with the presentation $\mathcal{B}$ which is DA but not CLA. In both cases, spherical pictures reduce through a finite sequence of bridge moves and deletions of dipoles. Whereas in the case of being CLA any given picture over such a presentation reduces in a fairly tame manner, such as in the proof of Theorem 4.1, we observed in the proof of Lemma 5.3 that a reduction may include a swirl (like some cyclone). CLA presentations do not exhibit such behavior.

Now the known logical relations are:



The question whether DR $\Longrightarrow$ CLA or not remains open. The DR property can be detected via *weight tests*, and it is therefore fairly easy to generate a large number of DR presentations. One would then test these examples on having the CLA property, hoping to find a suitable counterexample. As it turns out, to see whether a given presentation is CLA or not, except in the case of a one–relator presentation, is rather difficult. In fact, it seems more likely that the following is true:

Conjecture: *If the presentation $\mathcal{P} = (X : R)$ satisfies the weight test, then $\mathcal{P}$ is CLA.*

Further investigations concerning the CLA property have been published, e.g. Cohen–Lyndon Theorems for locally indicable groups and one–relator products and the characterization of $\langle\langle S \rangle\rangle_{(X:R)}$ being the free product of maximally many conjugates (*fpmmc*) introduced by A. Karrass and D. Solitar [22]. An interesting topological observation has been made by A.J. Sieradski [36], that for a CLA presentation the universal cover of the standard model is contractible in a very special manner.

Even though it seems rather unlikely that DR implies CLA, a counterexample is difficult to produce. This is mostly due to the fact, that for a DR presentation $\mathcal{P}$, it is very difficult to show that $\mathcal{P}$ is not CLA. It may be impossible.

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