An Abstract of the Thesis of

Charles Kevin Lyons for the degree of Doctor of Philosophy in Forest Engineering presented on September 6, 2001. Title: Mechanical Stresses in Trees Resulting from Strain Compatibility in an Anisotropic Material.

Abstract approved: Marvin R. Pyles

Consider the bole of a tree to consist of a linear elastic material that is orthotropic with respect to the cylindrical coordinates. When the bole of a tree is subjected to resultant loads in the directions of the Cartesian base vectors, the S_{11}, S_{22}, S_{33}, and S_{12} stresses in Cartesian coordinates are coupled. It is desirable to use beam elements to analyze the structural behavior of trees because of the ease with which these can be incorporated into Finite Element Models. However, elementary beam theory is not able to consider the problem where the S_{11}, S_{22}, S_{33}, and S_{12} stresses are coupled. The objective of this study was to determine the magnitudes of the normal stresses in the radial and tangential directions (S_{rr}, $S_{\theta\theta}$) and the shear stress ($S_{r\theta}$), relative to the normal stress in the x_3 direction for an element of a tree bole.

In cylindrical coordinates the strains are not unique at $r = 0$. Therefore, a constitutive equation was adopted in cylindrical coordinates where the elastic coefficients are dependent on r. An element of a tree bole was considered as a cantilever beam and posed as a Relaxed Saint-Venant's Problem in Cartesian

coordinates. It was found if the strains resulting from the generalized plane strain part of the problem were considered linear functions of the x_1 and x_2 coordinates, then the strain compatibility conditions and equilibrium equations could be satisfied.

Given the assumption that the generalized plane strains are linear in x_1 and x_2, it was proven that the S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ stresses are analytic functions of the complex variable z. It is also proven that the S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ stresses are equal to zero on the lateral surface of the element of the tree bole. Therefore, using the analyticity of the stress functions and the fact that they are zero on the lateral surface it is possible to show that the S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ stresses are zero throughout the element of a tree bole.

Mechanical Stresses in Trees Resulting from Strain Compatibility in an Anisotropic Material

by
Charles Kevin Lyons

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented September 6, 2001
Commencement June 2002

Doctor of Philosophy thesis of Charles Kevin Lyons presented on September 6, 2001

APPROVED:

Major Professor, representing Forest Engineering

Head of Department of Forest Engineering

Dean of Graduate School

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Mechanical Stresses in Trees Resulting from Strain Compatibility in an Anisotropic Material

1. Introduction

Common skyline logging systems employ a bicable ropeway (Schneigert 1966 p27, 501), which as the name suggests consists of two cables, a skyline, and a mainline. The logs are suspended from a carriage that is pulled along the skyline by the mainline, where the skyline is the carrying cable and the mainline is the hauling cable. The function of hauling logs from the cutblock to the roadside is termed yarding; the machine containing the winch set for the skyline and the mainline and usually a tower to support the skyline is called the yarder.

The skyline is a wire rope suspended between two or more points (Conway 1976, p200). The suspension points may be the yarding crane, trees, or stumps. The advantage to using a skyline system is that there is greater control over the logs when yarding, and it is possible to either partially or fully suspend the logs to minimize ground disturbance. To realize this advantage it is necessary to maintain a minimum clearance between the ground and the carriage. In some situations, it is necessary to support both ends of the skyline off the ground in order to maintain clearance between the carriage and the ground. If live trees are used to support the end of the skyline opposite from the yarder, these trees are called tailspars (Figure 1.1). It is also possible that the skyline will require midspan support; trees used for this function are called intermediate supports.

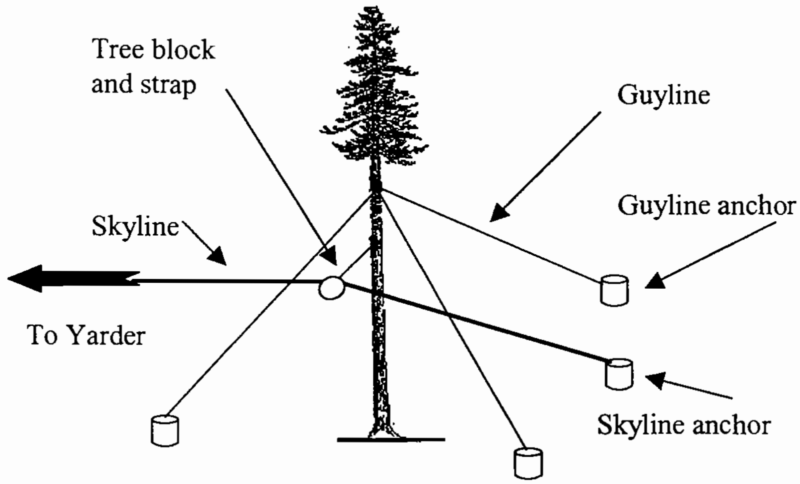

Figure 1.1, Tailspar support tree

The structural behavior of tailspar and intermediate support trees is of interest because they may limit the capacity of the logging system. The point where the cables are attached to the tree may displace laterally up to a meter or more and still be compatible with the other elements of the cable system. The displacement of the support tree becomes important when geometric nonlinear effects are considered (P-delta effect).

Stalnaker and Harris (1989, p.142) discuss the P-delta effect in beam column design, where there is a moment magnification from the product of the vertical load (P) and the deflection (delta). The support tree also has a flexible base, which can be represented as a rotational spring (Pyles 1987). The interaction of the nonlinear restraining force supplied by the guylines, the P-delta effect of the column, and the rotational stiffness of the base may combine to produce a complicated displacement field in the support tree.

Lyons (1997) used elementary beam theory to estimate the stresses in a tailspar. The results of this study indicated that compression parallel to the grain would be the limiting stress; however, the calculated maximum stress was located farther down the tree than where other trees were observed to have failed. Ammeson et al. (1987) demonstrated that the geometric nonlinear effects in the structural analysis of a support tree could be modeled in a Finite Element Model (FEM). Ammeson used beam elements to model the support tree, where the bending component of the stiffness matrix used in the model was derived from the Bernoulli-Euler law of elementary bending theory (Ugural and Fenster, 1995 p187).

Connor (1989) studied the stress distribution in trees rigged for experimental purposes. Connor used Ammeson's model to predict the displacement of the rigging point of a tailspar. Connor compared the predicted displacement of the rigging point to the actual displacement of test trees and found agreement to with in a few centimeters. However, Connor's results also showed that the maximum stress in compression was near the base of the tree. Neither Connor (1989) nor Lyons (1997) measured trees that were loaded to failure. It is possible that in these studies the applied loads were not sufficient to produce a noticeable P-delta effect.

To use a FEM to perform a detailed analysis of a support tree structure it is necessary to identify the appropriate material model and the type of element that should be used for a given analysis. For example Pellicane and Franco (1993) use a detailed solid element FEM to model the effects of grain pattern around knots in wooden poles, while Ammeson et al (1987) used beam elements to model the displacement of a support tree. Bodig and Jayne (1993, pg 110) describe a cylindrical section of a tree as being an orthotropic material with cylindrical anisotropy, where the axes of symmetry are the long axis $\underline{x}_3$, the radial axis r, and the tangential axis θ (Figure 1.2). If a cylindrical section of a tree is consider orthotropic in cylindrical coordinates then it will not be possible to estimate the normal stresses or the shear stress in the r, θ plane using elementary beam theory. Thus, in order to justify the use of beam elements derived from the Bernoulli-Euler law of elementary bending theory, it is necessary to determine if cylindrical anisotropy has a significant impact on the stress field.

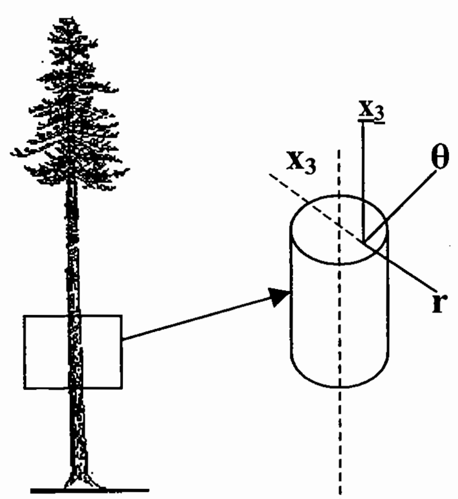

Figure 1.2, Cylindrical coordinates axes in a section of a tree

1.1 Objective

Consider a cylindrical beam with the x_3 axis as the generator of the cylinder. The beam is composed of wood where the x_3 axis passes through the center of the growth rings, and the wood is considered a linear elastic material that is orthotropic in cylindrical coordinates. If the wood is Douglas fir, the ultimate strength of the wood in compression is an order of magnitude less in the radial and tangential directions than it is in the $\underline{x}_3$ direction (USDA 1974, pg 4-46). If elementary beam theory is used to analyze the stresses in the beam, it is not possible to consider the radial or tangential stresses. If the beam is fixed at one end and loads that are independent of $\underline{x}_3$ are applied to the opposite end, is it possible that the S_{rr}, $S_{\theta\theta}$, or $S_{r\theta}$ stresses (refer to section 2.31 for definitions of these stresses) become limiting before the normal stress in the $\underline{x}_3$ direction does?

The objective of this thesis is to determine the magnitudes of the S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ stresses in a cylindrical cantilever beam, that is orthotropic in cylindrical coordinates for loads independent of $\underline{x}_3$. If the S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ stresses are always small, irrespective of the magnitude of the applied loads then they should never become limiting before the normal stress in the $\underline{x}_3$ direction does.

1.2 Organization of the dissertation

This dissertation is organized as follows. In Chapter 2, the material properties of wood are considered. The constitutive equations in cylindrical coordinates are transformed into Cartesian coordinates to simplify the derivation of the stress functions for the beam. In Chapter 3, a beam element will be considered as a Saint-Venant's Problem. Iesan (1987) proposed a solution to Saint-Venant's Problem for constitutive equations with material coefficients that are a function of the x_1 and x_2 coordinates. Iesan's proposed solution is specialized to the problem being considered in this paper. In Chapter 4, the functions for the T_{11}, T_{22}, or T_{12} generalized plain strain stresses are derived for a cylindrical beam of Douglas fir. In Chapter 5, the stress functions in Cartesian coordinates are used to estimate the magnitudes of the S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ stresses.

2. Material Properties of Wood

2.1 Defining wood as a continuum

Fung (1994, pg 3) suggests for practical application of continuum mechanics that it is necessary to relax the classical definition of a continuum. In the classical description of a continuum it is said that the densities of mass, momentum, and energy, must exist in the mathematical sense. That is, considering mass density as an example, the limit must exist

$$\rho(P) = \lim_{\substack{n\to\infty \\ V_n\to 0}} \frac{M_n}{V_n}. \tag{2.1}$$

Here: P is a point such that $P \in V_{n+1}$,

ρ is the mass density at point P,

n a positive integer,

V_n is a volume where V_{n+1} is contained within V_n,

M_n is the mass of the matter contained in volume V_n.

If the classical definition were to be rigorously imposed, it would not be possible to model wood as a continuum. Wood is composed of cells that have solid walls and voids and so the limit in equation (2.1) does not exist as a continuous function. However, if we define the mass density of the material *at P with an acceptable variability* ε *in a defining limit volume* ω then it may be possible to consider wood as a continuum, where

$$\left|\rho - \frac{M_n}{V_n}\right| < \varepsilon \quad \text{(Fung 1994, pg 4)} \tag{2.2}$$

Thus, to define a material as a continuum in a practical analysis it is necessary to consider both the acceptable error in approximating ρ, and the size of the limiting volume ω. When considering wood, defining the limiting volume also dictates the constitutive equations to be used. On a gross scale where the displacement of a cross section of a tree may be of interest, the discrete cells are not recognizable. If the strain of an individual cell is of interest then the cell walls could be considered a continuum, though they are also composed of many substructures.

2.2 The role of wood in trees

When considering the structural properties of a live tree the xylem is the most important element. Kramer and Kozlowski (1979, pg 30) discuss wood structure of Gymnosperms, which include important species such as *Pseudotsuga menziesii* (Douglas-fir). Xylem is the woody material produced on the inside of the cambium. The xylem performs two primary functions; 1) it provides the structural support for the tree, and 2) it transports sap, which is mostly water. The xylem has some cells that store nutrients, though this function is also performed by other tissues in the tree. Up to 90% of the xylem is composed of vertically stacked overlapping cells called tracheids. Tracheid cells may be up to 100 times long as they are wide and average 3 to 7 mm in length. Looking at the cross section of a tree, variation in the cells can be seen both within an individual growth ring and across the section. Bodig (1993, pg 3) reports that the specific gravity of the wood can vary three fold within a growth ring. The cells formed at the beginning of the growing season are larger and have thinner cell walls than those formed towards the end of the growing season.

The wood in the cross section of a tree can be divided into the sapwood and the heartwood. The sapwood is young xylem and it conducts as the name suggests the sap. As the sapwood ages, the living cells die off and it becomes heartwood. Heartwood can be differentiated from sapwood in that all the cells are dead; compounds such as oils, gums, resins, and tannins may be deposited in heartwood and add to its rot resistance (Kramer and Kozlowski 1979, pg 22). The sole function of heartwood is the structural support of the tree. It requires energy to maintain the living cells in the sapwood, and the tree only requires a portion of the cross sectional area to meet its sap transportation needs. Therefore, the sapwood is limited to about 10 to 20 of the outer rings.

The wood in a tree can also be divided into juvenile wood and mature wood. These terms are misleading though; as they imply that the type of wood is dependent on the age of the tree when it was formed or how old the wood is. A more precise terminology would be respectively, crown wood and bole wood. The growth hormones produced in the crown create the variation between the crown wood and the bole wood. Bruchert et al. (1997) found in *Picea abies* (Norway spruce) that Young's modulus in the x_3 direction could be 30% lower in the crown wood than in the bole wood of a given tree. When the tree is young, the crown extends to the base of the tree and so the whole tree is formed of crown wood. As the tree ages the crown may retreat up the stem and then bole wood is formed in regions distant from the crown. Thus, for regions below the crown the tree will have a core of crown wood surrounded by a cylinder of bole wood. While near the base of the crown, and higher, there will be only crown wood.

2.3 Constitutive equations

The choice of the appropriate constitutive equations depends on both the size of the defining limit volume and the intended analysis. It will be assumed in this study that the defining limit volume is large enough so that the individual cells are small in comparison. Then the error term ε will be a function of the general structure of the wood, such as early wood or late wood, and not a function of whether the point P falls with in a cell cavity or a cell wall. Constitutive equations that could be used in an analysis of wood include, isotropic, rectilinear orthotropic, curvilinear orthotropic, or completely general anisotropy. Orthotropic materials will be discussed in section 2.3.1. The choice of the constitutive equation depends not only on which one best represents the real material, but also given the analysis being performed, what is the simplest constitutive equation that produces acceptable results.

Hosford (1993, pg 21) states that it is possible to use the basic forms for the matrices of the elastic constants for materials other than crystals, which have similar symmetries of structure. Green and Zerna (1968, pg 155) note that wood may be considered an orthotropic material, though it is not an exact material property. The wood in a tree is organized into concentric tapered cones; therefore, it is logical to consider it as a material with curvilinear anisotropy. Curvilinear anisotropy is *characterized by the fact that the equivalent directions for its different points are not parallel, but obey some other laws* (Lekhnitskii 1981, pg 67). However, for rectilinear anisotropy, the equivalent directions are the rectilinear coordinate axes and these are the same for every point in the material.

2.3.1 Constitutive equations in cylindrical coordinates

Lai et al (1993, pg 221) give the constitutive equations for a linear elastic solid written in tensor form,

$$\left.\begin{aligned} S_{ij} &= C_{ijkl}E_{kl} \\ E_{ij} &= S_{ijkl}S_{kl} \end{aligned}\right\} \quad (i,j,k,l = 1,2,3) \qquad (2.5)$$

$$E_{kl} = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\right).$$

Here S_{ij} is Cauchy's stress tensor, C_{ijkl} is the elasticity tensor, S_{ijkl} is the compliance tensor, E_{kl} is the infinitesimal strain tensor, and u_k are the displacements.

Cauchy's stress tensor and the infinitesimal strain tensor are second order tensors and so by the quotient rule C_{ijkl} must be a fourth order tensor having 81 coefficients in the most general form. The number of independent coefficients is reduced by 27 due to the symmetry of the infinitesimal strain tensor ($C_{ijkl} = C_{ijlk}$), by 18 due to the symmetry of the Cauchy stress tensor ($C_{ijkl} = C_{jilk}$), and by a further 15 when assuming an elastic potential exists ($C_{ijkl} = C_{klij}$). Thus, the elasticity tensor in (2.5) has at most 21 independent coefficients. Similarly, the compliance tensor in (2.5) has at most 21 independent coefficients.

In wood, the constitutive equations may be simplified further by using the cylindrical coordinate axes as the basis for the tensors in (2.5). From now on tensors with base vectors corresponding to the cylindrical coordinate axes will be denoted with a prime (i.e. C_{ijkl}', S_{ijkl}', S_{ij}', E_{kl}'). The strain tensor and the stress tensor will be arranged as follows,

$$\begin{bmatrix} E_{11}' & E_{12}' & E_{13}' \\ E_{21}' & E_{22}' & E_{23}' \\ E_{31}' & E_{32}' & E_{33}' \end{bmatrix} = \begin{bmatrix} E_{rr} & E_{r\theta} & E_{rx_3} \\ E_{\theta r} & E_{\theta\theta} & E_{\theta x_3} \\ E_{x_3 r} & E_{x_3\theta} & E_{x_3x_3} \end{bmatrix}$$
$$\begin{bmatrix} S_{11}' & S_{12}' & S_{13}' \\ S_{21}' & S_{22}' & S_{23}' \\ S_{31}' & S_{32}' & S_{33}' \end{bmatrix} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rx_3} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta x_3} \\ S_{x_3 r} & S_{x_3\theta} & S_{x_3x_3} \end{bmatrix} \tag{2.6}$$

Here r corresponds to the radial direction, θ corresponds to the tangential direction, and x_3 corresponds to the direction parallel to the generator of a cylinder.

Given the most general type of anisotropy it is obvious in equation (2.5) that the normal stresses can produce shear strains and shear stresses can produce normal strains. This fact greatly increases the complexity of analyzing these materials. Bodig (1993, pg 112) suggests wood has three orthogonal planes of symmetry, which in a cylindrical coordinate system are 1) a plane perpendicular to the $\underline{x_3}$ axis, 2) a plane perpendicular to the radial axis, and 3) a third plane orthogonal to the first two (Figure 2.1).

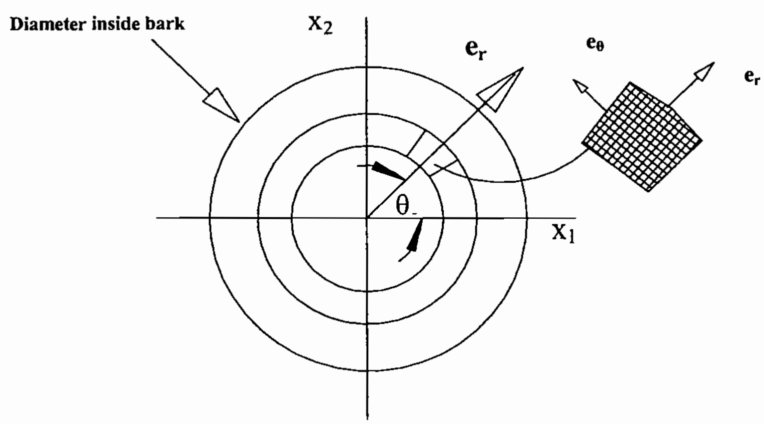

Figure 2.1, Cross section of a tree with the positive x_3 axis directed out of the page.

A plane of symmetry requires that the base vector normal to the plane can be reflected about the plane to form a new set of base vectors, without altering the compliance or elasticity tensors.

That is,

$$C_{ijkl}' = C_{ijkl}'^{\mathrm{II}} \quad (\text{or } S_{ijkl}' = S_{ijkl}'^{\mathrm{II}}) \tag{2.7}$$

Here C_{ijkl}' is the elasticity tensor in the original basis and $C_{ijkl}'^{\mathrm{II}}$ is the elasticity tensor in the new basis.

Certain coefficients must be equal to zero for the C_{ijkl}' to be invariant when a base vector is reflected about a plane of symmetry. If wood is considered to have three orthogonal planes of symmetry defined by the base vectors of the cylindrical coordinate system, then the constitutive equations take on the following form in Viogt notation.

$$\begin{bmatrix} S_{11}' \\ S_{22}' \\ S_{33}' \\ S_{23}' \\ S_{13}' \\ S_{12}' \end{bmatrix} = \begin{bmatrix} C_{1111}' & C_{1122}' & C_{1133}' & 0 & 0 & 0 \\ C_{2211}' & C_{2222}' & C_{2233}' & 0 & 0 & 0 \\ C_{3311}' & C_{3322}' & C_{3333}' & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323}' & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313}' & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212}' \end{bmatrix} \begin{bmatrix} E_{11}' \\ E_{22}' \\ E_{33}' \\ 2E_{23}' \\ 2E_{13}' \\ 2E_{12}' \end{bmatrix} \tag{2.8}$$

Note, equation (2.8) is symmetric so $C_{ijkl} = C_{klij}$ and there are only nine independent coefficients. A material with three orthogonal planes of symmetry will be called orthotropic. Equation (2.8) indicates there are no interactions between the normal strains and shear stresses or between the shear strains and the normal stresses in an orthotropic material.

2.3.2 Constitutive equations in Cartesian coordinates

Equations (2.5) are tensor equations and so are valid under any proper transformation; however, it will be necessary to take the derivatives of these equations. If equation (2.5) has a curvilinear basis, then on taking the derivative with respect to a base vector the resulting differential will have a different set of base vectors from the point where the derivative was taken (Charlier et al 1992, pg 21). The resulting matrix is no longer a tensor, and will have to be corrected in order to regain the original properties of the tensor equation. This complication can be avoided if the constitutive equations are transformed to a rectilinear basis. Then the base vectors are the same for all points in the domain and so taking the derivative of a tensor will result in a tensor.

Lai et al (1993, pg 221) give the transformation taking the fourth order tensor C_{ijkl}' from the $\mathbf{e}_i'$ basis to the $\mathbf{e}_i$ basis as

$$\begin{aligned} C_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}C_{mnrs}' \\ S_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}S_{mnrs}' \end{aligned} \tag{2.9}$$

Here Q_{ij} is the second order tensor containing the direction cosines for the rotation of interest. To convert the elasticity tensor or the compliance tensor from a cylindrical basis to a Cartesian basis Q_{ij} would be

$$Q_{ij} = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

where C_θ and S_θ are the cosine and sine of the cylindrical coordinate θ.

The rotation (2.10) transforms the positive r direction in cylindrical coordinates to the positive x_1 direction in Cartesian coordinates. Recall there are only nine independent coefficients in the C_{ijkl}' and S_{ijkl}' tensors. Refer to Appendix C for the complete list of transformation equations for C_{ijkl}' and S_{ijkl}'.

Equations (C2.11) and (C2.12) show that there are now thirteen coefficients, after transforming the C_{ijkl}' and S_{ijkl}' tensors to Cartesian coordinates. In addition, the coefficients in the new C_{ijkl} and S_{ijkl} tensors are no longer constant; instead, they are now dependent on the cylindrical coordinate θ and the nine coefficients from the C_{ijkl}' and S_{ijkl}' tensors. To view the change in dependence between the stresses and strains after the transformation (2.9) the constitutive equations can be written in Voigt notation. For example,

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{13} \\ S_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & C_{3312} \\ 0 & 0 & 0 & C_{2323} & C_{2313} & 0 \\ 0 & 0 & 0 & C_{1323} & C_{1313} & 0 \\ C_{1211} & C_{1222} & C_{1233} & 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix} \tag{2.13}$$

Equation (2.13) indicates when the constitutive equations are transformed from cylindrical coordinates to Cartesian coordinates only one plane of symmetry remains, and this plane is formed by the x_1, x_2 axes. If the material is subjected to rectilinear strains, or stresses, there is now an interaction between the E_{12} shear strain and the normal stresses, and the S_{12} shear stress and the normal strains. There is also an interaction between E_{23} and S_{13}, and E_{13} and S_{23}.

2.3.3 Constitutive equations for the bole of a tree

The Wood handbook (USDA 1974, pg 4-6) gives the engineering constants for Douglas-fir when the wood is assumed to be orthotropic in cylindrical coordinates (Table 2.1). The engineering constants reported by the Wood handbook do not produce a symmetric compliance tensor where $S_{ijkl} = S_{klij}$. To produce a symmetric compliance tensor from the published values the off diagonal terms in (2.8) were averaged (Table 2.2).

Table 2.1 Engineering constants for Douglas-fir (USDA, 1974)

E_Z^* (Pa)	E_θ (Pa)	E_R (Pa)	$G_{\theta Z}^{**}$ (Pa)	G_{RZ} (Pa)	$G_{\theta R}$ (Pa)
1.08E+10	5.40E+08	7.34E+08	6.91E+08	8.42E+08	7.56E+07
ν_{ZR}^{***}	$\nu_{Z\theta}$	$\nu_{R\theta}$	ν_{RZ}	$\nu_{\theta R}$	$\nu_{\theta Z}$
0.292	0.449	0.390	0.287	0.020	0.022

* Young's modulus, ** Shear modulus, *** Poisons ratio

Table 2.2 Compliance coefficients for Douglas-fir

S_{1111}' (Pa^{-1})	S_{1122}' (Pa^{-1})	S_{1133}' (Pa^{-1})	S_{2222}' (Pa^{-1})	S_{2233}' (Pa^{-1})	S_{3333}' (Pa^{-1})
1.362E-09	-2.842E-10	-2.090E-10	1.852E-09	-4.116E-11	9.259E-11
S_{2323}' (Pa^{-1})	S_{1313}' (Pa^{-1})	S_{1212}' (Pa^{-1})			
1.188E-09	1.447E-09	1.323E-08			

If a cylindrical section of the bole of a tree is assumed orthotropic with $\underline{x_3}$, r, and θ being the axes of anisotropy. Then, if the x_3 axis falls within the bole of the tree certain relations are required between the elastic coefficients (Lekhnitskii 1981, pg 68). In cylindrical coordinates when $r = 0$, the unit vectors $\mathbf{e_r}$ and $\mathbf{e_\theta}$ become indistinguishable. Therefore, it must be possible to interchange the r and θ directions in (2.8); this requires certain of the coefficients to be equivalent. The coefficients that must be equivalent at $r = 0$ are

$$\left.\begin{array}{l} S_{1111}' = S_{2222}', S_{1133}' = S_{2233}', S_{2332}' = S_{1313}' \\ C_{1111}' = C_{2222}', C_{1133}' = C_{2233}', C_{2332}' = C_{1313}' \end{array}\right\} at\ r = 0 \qquad (2.14)$$

However, the compliance coefficients (Table 2.2) indicate that (2.14) is not true for all points in the cross section of a tree. It is necessary to introduce constitutive equations that are a function of the cylindrical coordinate r in order to satisfy (2.14) at $r = 0$, while allowing for other combinations of coefficients where $r \neq 0$. The following constitutive equation in cylindrical coordinates will be assumed to apply to a cylindrical section of the bole of a tree,

$$\left.\begin{array}{l} E_{ij}' = S_{ijkl}' S_{kl}' = \left[\underline{S_{ijkl}} + r * \underline{M_{ijkl}}\right] S_{kl}' \\ S_{ij}' = C_{ijkl}' E_{kl}' = \left[\underline{C_{ijkl}} + r * \underline{K_{ijkl}}\right] E_{kl}' \end{array}\right\} in\ cylindrical\ coordinates \qquad (2.15)$$

where $\underline{S_{ijkl}}$, $\underline{M_{ijkl}}$, $\underline{C_{ijkl}}$, and $\underline{K_{ijkl}}$ are constants.

Before transforming the compliance and elasticity coefficients in (2.15) to Cartesian coordinates, some simplifications can be made. Equation (2.14) does not place any restrictions on S_{1122}', S_{3333}', S_{1212}', or C_{1122}', C_{3333}', C_{1212}'. Therefore, these coefficients may be independent of r and so the following simplifications can be made.

Let

$$\begin{aligned} &\underline{K_{1122}} = \underline{K_{1122}} = \underline{K_{1122}} = 0 \\ &\underline{M_{1122}} = \underline{M_{1122}} = \underline{M_{1122}} = 0 \end{aligned} \tag{2.16}$$

Equation (2.14) does place restrictions on (2.15) when $r = 0$, therefore, let

$$\left.\begin{aligned} &\underline{C_{2222}} = \underline{C_{1111}},\ \underline{C_{2233}} = \underline{C_{1133}},\ \underline{C_{2323}} = \underline{C_{1313}} \\ &\underline{S_{2222}} = \underline{S_{1111}},\ \underline{S_{2233}} = \underline{S_{1133}},\ \underline{S_{2323}} = \underline{S_{1313}} \end{aligned}\right\} at\ r = 0 \tag{2.17}$$

On substituting (2.17) into (2.15) when $r = 0$, then substituting this into the fourth and fifth equations of (C2.11) and (C2.12), the following can be noted.

$$\left.\begin{aligned} C_{2323} &= S_\theta^2 \underline{C_{1313}} + C_\theta^2 \underline{C_{2323}} = \left[S_\theta^2 + C_\theta^2\right]\underline{C_{1313}} = \underline{C_{1313}} \\ C_{1313} &= C_\theta^2 \underline{C_{1313}} + S_\theta^2 \underline{C_{2323}} = \left[C_\theta^2 + S_\theta^2\right]\underline{C_{1313}} = \underline{C_{1313}} \\ S_{2323} &= S_\theta^2 \underline{S_{1313}} + C_\theta^2 \underline{S_{2323}} = \left[S_\theta^2 + C_\theta^2\right]\underline{S_{1313}} = \underline{S_{1313}} \\ S_{1313} &= C_\theta^2 \underline{S_{1313}} + S_\theta^2 \underline{S_{2323}} = \left[C_\theta^2 + S_\theta^2\right]\underline{S_{1313}} = \underline{S_{1313}} \end{aligned}\right\} at\ r = 0 \tag{2.18}$$

To form the compliance coefficients in Cartesian coordinates substitute (2.15) into (C2.12), then take into account (2.16), (2.17), and (2.18). The resulting compliance coefficients in Cartesian coordinates are as follows.

$$
\begin{aligned}
S_{1111} &= \left[C_\theta^4 + S_\theta^4\right]\underline{S_{1111}} + r\left[C_\theta^4\,\underline{M_{1111}} + S_\theta^4\,\underline{M_{2222}}\right] + 2C_\theta^2 S_\theta^2\,\underline{S_{1122}} + 4C_\theta^2 S_\theta^2\,\underline{S_{1212}} \\
S_{1122} &= \underline{S_{1122}} \\
S_{1133} &= \underline{S_{1133}} + r\left[C_\theta^2\,\underline{M_{1133}} + S_\theta^2\,\underline{M_{2233}}\right] \\
S_{1112} &= -S_\theta C_\theta\left[C_\theta^2\left[\underline{S_{1111}} + r\underline{M_{1111}}\right] - C_\theta^2\,\underline{S_{1122}} - 2C_\theta^2\,\underline{S_{1212}} + 2S_\theta^2\,\underline{S_{1212}} + S_\theta^2\,\underline{S_{1122}} - S_\theta^2\left[\underline{S_{1111}} + r\underline{M_{2222}}\right]\right] \\
S_{2222} &= \left[C_\theta^4 + S_\theta^4\right]\underline{S_{1111}} + r\left[S_\theta^4\,\underline{M_{1111}} + C_\theta^4\,\underline{M_{2222}}\right] + 2C_\theta^2 S_\theta^2\,\underline{S_{1122}} + 4C_\theta^2 S_\theta^2\,\underline{S_{1212}} \\
S_{2233} &= \underline{S_{2233}} + r\left[S_\theta^2\,\underline{M_{1133}} + C_\theta^2\,\underline{M_{2233}}\right] \\
S_{2212} &= -S_\theta C_\theta\left[S_\theta^2\left[\underline{S_{1111}} + r\underline{M_{1111}}\right] - S_\theta^2\,\underline{S_{1122}} - 2C_\theta^2\,\underline{S_{1212}} + 2S_\theta^2\,\underline{S_{1212}} + C_\theta^2\,\underline{S_{1122}} - C_\theta^2\left[\underline{S_{1111}} + r\underline{M_{2222}}\right]\right] \\
S_{3333} &= \underline{S_{3333}} \\
S_{3312} &= -S_\theta C_\theta\left[\left[\underline{S_{1133}} + r\underline{M_{1133}}\right] - \left[\underline{S_{1133}} + r\underline{M_{2233}}\right]\right] \\
S_{2323} &= \underline{S_{1313}} + r\left[S_\theta^2\,\underline{M_{1313}} + C_\theta^2\,\underline{M_{2323}}\right] \\
S_{2313} &= -S_\theta C_\theta\, r\left[\underline{M_{1313}} - \underline{M_{2323}}\right] \\
S_{1313} &= \underline{S_{1313}} + r\left[C_\theta^2\,\underline{M_{1313}} + S_\theta^2\,\underline{M_{2323}}\right] \\
S_{1212} &= C_\theta^2 S_\theta^2\left[2\underline{S_{1111}} + r\left[\underline{M_{1111}} + \underline{M_{2222}}\right] - 2\left[\underline{S_{1122}} + \underline{S_{1212}}\right]\right] + \left[C_\theta^4 + S_\theta^4\right]\underline{S_{1212}}
\end{aligned}
\tag{2.19a}
$$

Similar equations can be formed for the elastic coefficients by replacing S_{ijkl} by C_{ijkl}, $\underline{S_{ijkl}}$ by $\underline{C_{ijkl}}$, and $\underline{M_{ijkl}}$ by $\underline{K_{ijkl}}$, and these equations would be called (2.19b).

3. A Solution for Saint-Venant's Problem

From now on in this paper, Greek indices will range from 1 to 2, while Latin indices range from 1 to 3 unless otherwise specified. Summation notation is used for repeated indices, and a comma followed by a subscript will indicate a partial derivative with respect to the coordinate. Definitions of the Kronecker delta function (δ_{ij}), the two-dimensional alternator symbol ($e_{\alpha\beta}$), and the permutation symbol (e_{ijk}), can be found in Appendix B.

To formulate a problem in elastostatics some basic equations will be required. The constitutive equations are required to link the observable strains to the corresponding stresses. The constitutive equations (2.15) are independent of θ when viewed in cylindrical coordinates. However, when the elasticity tensor ($\underline{\mathbf{C}}$) and compliance tensor ($\underline{\mathbf{S}}$) are transformed into the Cartesian frame by (2.19) they become functions of both x_1, and x_2, though they are still independent of x_3.

$$\underline{\mathbf{C}} = C_{ijkl} = C_{ijkl}(x_1, x_2)$$

$$\underline{\mathbf{S}} = S_{ijkl} = S_{ijkl}(x_1, x_2) \tag{3.1}$$

The constitutive equations written in Cartesian coordinates are,

$$S_{ij}(\mathbf{u}) = C_{ijkl}E_{kl}(\mathbf{u})$$

$$E_{ij}(\mathbf{u}) = S_{ijkl}S_{kl}(\mathbf{u}) \tag{3.2}$$

The first equation of (3.2) shows the stress tensor S_{ij} is a function of the displacement vector $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$. The displacement vector enters into the formulation of the problem through the strain tensor E_{ij}.

Only the infinitesimal strain tensor will be used in this analysis, and it may be written in vector or indicial form depending on convenience.

$$E(\mathbf{u}) = \frac{1}{2}\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right]$$
$$E_{ij}(\mathbf{u}) = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \tag{3.3}$$

The equations of motion are the differential equations that define the displacements. In a static analysis, the displacements are independent of time and so the equations of motion become the equations of equilibrium. As will be noted in section 3.1 the body forces may be ignored without restricting the generality of the analysis because if required they can be reintroduced later. The equilibrium equations when ignoring the body forces are

$$\frac{\partial S_{ij}(\mathbf{u})}{\partial x_j} = 0 \tag{3.4}$$

3.1 Problem statement and classification

Consider a cantilever beam with constant circular cross sections (Figure 3.1). Let the region B refer to the interior of the cylinder, where $\overline{B}$ denotes the closure of B and ∂B is the boundary,

$$\overline{B} = \partial B \cup B \tag{3.5}$$

Specifically let Σ_1 be the open cross section at $x_3 = 0$, Σ_2 be the open cross section at $x_3 = h$, and let Σ be an arbitrary cross section with normal x_3. The lateral surface of the cylinder will be Π, while the boundary of an arbitrary cross section is Γ. The cantilever beam in Figure 3.1 is a mixed fundamental boundary value problem of elastostatics (Muskhelishvili 1963, pg 68). Σ_2 is fixed, so the displacements are zero on this portion of the boundary, where as the surface tractions on $\partial B - \Sigma_2$ will be specified. Muskhelishvili (1963, pg 71) notes the mixed fundamental problem will have a unique displacement solution, where as the first fundamental problem where equilibrium is maintained by surface tractions may not.

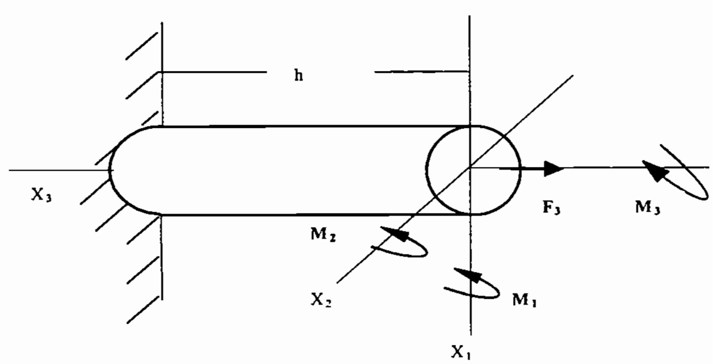

Figure 3.1, A cylindrical cantilever beam with loads independent of x_3

Sokolnikoff (1956, pg 258) notes the body forces resulting from self-weight occur in the equations of motion as a linear term. This linearity makes it possible to reduce the system to a homogeneous form where the body forces are not present, solve this system, and then reconstruct a solution to the original problem. Therefore, in this analysis the body forces will be ignored.

Gurtin (1981, pg 221) states the mixed problem of elastostatics as follows. Let $\gamma_1 \cup \gamma_2 = \partial B$, where $\gamma_1 \cap \gamma_2 = \emptyset$. Given: an elasticity tensor C_{ijkl} on $\overline{B}$, a prescribed displacement field $\hat{\mathbf{u}}$ on γ_1, and a prescribed stress field $\hat{\mathbf{s}}$ on γ_2. Find a solution $[\mathbf{u}, E_{ij}(\mathbf{u}), S_{kl}(\mathbf{u})]$ in B, such that $\mathbf{u} = \hat{\mathbf{u}}$ on γ_1 and $\mathbf{s} = S_{ij}(\mathbf{u})\mathbf{n} = \hat{\mathbf{s}}$ on γ_2, where $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$ and $\mathbf{n}$ is the unit normal to ∂B.

Iesan (1987, pg 3) notes that Saint-Venant's Problem was originally posed as a first fundamental boundary value problem (a traction problem). If $\gamma_1 = \emptyset$ and $\gamma_2 = \partial B$ then find $[\mathbf{u}, E_{ij}, S_{kl}]$ in B, such that $\mathbf{s} = S(\mathbf{u})\mathbf{n} = \hat{\mathbf{s}}$ on γ_2. However, the prescribed stress fields $\hat{\mathbf{s}}^{(1)}$ on Σ_1 and $\hat{\mathbf{s}}^{(2)}$ on Σ_2 are not known a priori for the problem in Figure 3.1. Therefore, it is necessary to formulate the problem in Figure 3.1 as a Relaxed Saint-Venant's Problem, where the prescribed stress fields $\hat{\mathbf{s}}^{(1)}$ and $\hat{\mathbf{s}}^{(2)}$ are replaced by integral functions that equal their resultants (Knowles 1966).

Iesan (1987, pg 4) classifies the relaxed Saint-Venant's problem based on assumptions concerning the resultant forces **F** and the resultant moments **M**. The relaxed problem can be decomposed into two classes.

1) P_1 (loads independent of x_3): $F_\alpha = 0$. The class of solutions to P_1 is denoted by $K_1(F_3, M_1, M_2, M_3)$.

2) P_2 (flexure): $F_3 = M_i = 0$. The class of solutions to P_2 is denoted by $K_2(F_1, F_2)$.

Only problems of class P_1 will be considered in this paper.

From Figure 3.1 the normal vectors $\mathbf{n}^{(1)}$ on Σ_1 and $\mathbf{n}^{(2)}$ on Σ_2 are seen to be

$$\mathbf{n}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{on } \Sigma_1\text{, and } \mathbf{n}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{on } \Sigma_2 \tag{3.6}$$

The stress vector on Σ_1 is given by

$$s_j^{(1)}(\mathbf{u}) = S_{ij}(\mathbf{u}) n_j^{(1)} \tag{3.7}$$

Iesan (1987, pg 3) notes, for the Relaxed Saint-Venant's Problem the prescribed stress vector $\hat{\mathbf{s}}$ will be replaced by the following functions on Σ_1 and Π for problems of class P_1.

$$\begin{gathered} f_\alpha(\mathbf{u}) = F_\alpha = 0,\ f_3(\mathbf{u}) = F_3,\ m_1(\mathbf{u}) = M_1,\ m_2(\mathbf{u}) = M_2,\ m_3(\mathbf{u}) = M_3 \text{ on } \Sigma_1 \\ s(\mathbf{u}) = \mathbf{0} \quad \text{on } \Pi \end{gathered} \tag{3.8}$$

(Note, for P_1 the resultant loads are independent of x_3, therefore, $\mathbf{f(u)} = \mathbf{F}$ and $\mathbf{m(u)} = \mathbf{M}$ on all Σ)

The integral functions **f(u)** and **m(u)** are functions of the prescribed stress vector $\hat{\mathbf{s}}^{(1)}$ on Σ_1, where

$$\mathbf{f}(\mathbf{u}) = \int_{\Sigma_1} \hat{s}^{(1)}(\mathbf{u})da, \quad \mathbf{m}(\mathbf{u}) = \int_{\Sigma_1} \mathbf{x} \times \hat{s}^{(1)}(\mathbf{u})da \tag{3.9}$$

Here **x** is the position vector directed from the origin to a point on Σ_1,

$$\mathbf{x}^T = [x_1 \quad x_2 \quad 0] \qquad \text{on } \Sigma_1$$

In expanded form the four nonzero equations from (3.9) are

$$\begin{aligned}
f_3(\mathbf{u}) &= \int_{\Sigma_1} \hat{s}_3^{(1)}(\mathbf{u})da = \int_{\Sigma_1} S_{3i}n_i^{(1)}(\mathbf{u})da = -\int_{\Sigma_1} S_{33}(\mathbf{u})da \\
m_3(\mathbf{u}) &= \int_{\Sigma_1} -x_2\hat{s}_1^{(1)}(\mathbf{u})da + \int_{\Sigma_1} x_1\hat{s}_2^{(1)}(\mathbf{u})da = \int_{\Sigma_1} -x_2S_{1i}n_i^{(1)}(\mathbf{u})da + \int_{\Sigma_1} x_1S_{2i}n_i^{(1)}(\mathbf{u})da \\
&= \int_{\Sigma_1} x_2S_{13}(\mathbf{u})da - \int_{\Sigma_1} x_1S_{23}(\mathbf{u})da \\
m_1(\mathbf{u}) &= \int_{\Sigma_1} x_2\hat{s}_3^{(1)}(\mathbf{u})da = \int_{\Sigma_1} x_2S_{3i}n_i^{(1)}(\mathbf{u})da = -\int_{\Sigma_1} x_2S_{33}(\mathbf{u})da \\
m_2(\mathbf{u}) &= \int_{\Sigma_1} -x_1\hat{s}_3^{(1)}(\mathbf{u})da = \int_{\Sigma_1} -x_1S_{3i}n_i^{(1)}(\mathbf{u})da = \int_{\Sigma_1} x_1S_{33}(\mathbf{u})da
\end{aligned} \tag{3.10}$$

Since the body forces are being ignored and the inertial forces equal zero, the necessary conditions for a solution require that the sum of forces and sum of moments on ∂B equal zero. Recall that the surface tractions on Π equal zero; therefore, the necessary conditions for a solution require that the sum of the stress fields on Σ_2 and the resultants on Σ_1 equal zero.

The sums of the stress fields on Σ_2 are

$$\int_{\Sigma_2} \hat{s}_3^{(2)}(\mathbf{u})da = \int_{\Sigma_2} S_{3i}n_i^{(2)}(\mathbf{u})da = \int_{\Sigma_2} S_{33}(\mathbf{u})da$$

$$\begin{aligned}\int_{\Sigma_2} -x_2\hat{s}_1^{(2)}(\mathbf{u})da + \int_{\Sigma_2} x_1\hat{s}_2^{(2)}(\mathbf{u})da &= \int_{\Sigma_2} -x_2S_{1i}n_i^{(2)}(\mathbf{u})da + \int_{\Sigma_2} x_1S_{2i}n_i^{(2)}(\mathbf{u})da \\ &= -\int_{\Sigma_2} x_2S_{13}(\mathbf{u})da + \int_{\Sigma_2} x_1S_{23}(\mathbf{u})da \qquad (3.11)\end{aligned}$$

$$\int_{\Sigma_2} x_2\hat{s}_3^{(2)}(\mathbf{u})da = \int_{\Sigma_2} x_2S_{3i}n_i^{(2)}(\mathbf{u})da = \int_{\Sigma_2} x_2S_{33}(\mathbf{u})da$$

$$\int_{\Sigma_2} -x_1\hat{s}_3^{(2)}(\mathbf{u})da = \int_{\Sigma_2} -x_1S_{3i}n_i^{(2)}(\mathbf{u})da = -\int_{\Sigma_2} x_1S_{33}(\mathbf{u})da$$

The necessary conditions for a solution require that (3.11) is equal and opposite to (3.10). The necessary conditions for a solution are

$$\begin{aligned}&\int_{\Sigma_2} S_{\alpha 3}(\mathbf{u})da = -f_\alpha(\mathbf{u}) = 0 \quad \text{(from (3.9))} \\ &\int_{\Sigma_2} S_{33}(\mathbf{u})da = -f_3(\mathbf{u}) = -F_3 \\ &\int_{\Sigma_2} e_{\alpha\beta}x_\alpha S_{3\beta}(\mathbf{u})da = -m_3(\mathbf{u}) = -M_3 \\ &\int_{\Sigma_2} x_\alpha S_{33}(\mathbf{u})da = e_{\alpha\beta}m_\beta(\mathbf{u}) = e_{\alpha\beta}M_\beta\end{aligned} \qquad (3.12)$$

Saint-Venant's Principle has been used to pose the mixed fundamental boundary value problem in Figure 3.1 as a Relaxed Saint-Venant's Problem. Iesan (1987, pg 4) notes that a solution to the problem in Figure 3.1, posed as a Relaxed Saint-Venant's Problem, is any displacement field $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$ that satisfies (3.8). In posing the problem as a Relaxed Saint-Venant's Problem the boundary condition $\mathbf{u} = \mathbf{0}$ on Σ_2 has been replaced by (3.12). The stresses in (3.12) are related to the first derivatives of the displacements by (3.2). Recalling that rigid body motions may have zero first partial derivatives with respect to the coordinates

(A14), the displacement field defined by (3.8) is unique only up to an arbitrary rigid body motion.

3.2 Statement and proof of Theorem 3.1

Theorem 3.1 and Corollary 3.1 were proposed by Iesan (1987, pg 45). The following proof of Theorem 3.1 and Corollary 3.1 parallels that given by Iesan. Assume, $C_{ijkl} = C_{ijkl}(x_1, x_2)$ *where* $(x_1, x_2) \in \Sigma$. Let D denote the set of all equilibrium displacement fields **u** that satisfy the second equation (3.8).

Theorem 3.1

If $\mathbf{u} \in D$ and $\mathbf{u},_3 \in C^1(\overline{B}) \cap C^2(B)$

Then $\mathbf{u}_{,3} \in D$ and

$$f(\mathbf{u},_3) = 0,\ m_\alpha(\mathbf{u},_3) = e_{\alpha\beta} f_\beta(\mathbf{u}),\ m_3(\mathbf{u},_3) = 0$$

Proof of Theorem 3.1

Recall from (3.2) and (3.3) that

$$S(\mathbf{u}) = \underline{\mathbf{C}}\,\mathbf{E}(\mathbf{u}) = \frac{1}{2}\underline{\mathbf{C}}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \frac{1}{2}(\underline{\mathbf{C}}\,\nabla\mathbf{u} + \underline{\mathbf{C}}\,\nabla\mathbf{u}^T)$$

Since $\underline{\mathbf{C}} = \underline{\mathbf{C}}(x_1, x_2)$ is independent of x_3, and $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$, then

$$\frac{\partial}{\partial x_3}[\nabla\mathbf{u}] = [\nabla\mathbf{u},_3] \text{ and } \frac{\partial}{\partial x_3}\{\underline{\mathbf{C}}[\nabla\mathbf{u}]\} = \underline{\mathbf{C}}[\nabla\mathbf{u},_3]$$

Therefore, $S(\mathbf{u},_3) = \underline{\mathbf{C}}\,\mathbf{E}(\mathbf{u},_3) = \dfrac{\partial}{\partial x_3} S(\mathbf{u})$

On the lateral surface Π, where $\mathbf{n}$ is the unit normal, introducing $\mathbf{u},_3$ results in

$$s(\mathbf{u},_3) = S(\mathbf{u},_3)\mathbf{n} = \left\{\frac{\partial}{\partial x_3}[S(\mathbf{u})]\right\}\mathbf{n} \qquad \text{on } \Pi$$

The beam being considered is a right cylinder; therefore, on Π the following is true.

$$\mathbf{n} = \mathbf{n}(x_1, x_2),\ \mathbf{n},_3 = 0$$

$$\frac{\partial}{\partial x_3}[S(\mathbf{u})\mathbf{n}] = \mathbf{n}\frac{\partial}{\partial x_3}[S(\mathbf{u})] + S(\mathbf{u})\frac{\partial}{\partial x_3}\mathbf{n} = \mathbf{n}\frac{\partial}{\partial x_3}[S(\mathbf{u})]$$

Therefore,

If $\mathbf{u} \in D$ then,

$$s(\mathbf{u}) = S(\mathbf{u})\mathbf{n} = \mathbf{0} \qquad \text{on } \Pi$$

$$s(\mathbf{u},_3) = \left\{\frac{\partial}{\partial x_3}[S(\mathbf{u})]\right\}\mathbf{n} = \frac{\partial}{\partial x_3}[S(\mathbf{u})\mathbf{n}] = \mathbf{0} \qquad \text{on } \Pi \tag{3.13}$$

This proves the first part of Theorem 3.1, if $\mathbf{u} \in D$ then also $\mathbf{u},_3 \in D$.

To prove the last part of Theorem 3.1 the equilibrium equations (3.4) must be considered. Substitute $S_{ij} = S_{ij}(\mathbf{u},_3)$ into the equilibrium equations,

$$S_{11},_1 + S_{12},_2 + S_{13},_3 = 0$$

$$S_{21},_1 + S_{22},_2 + S_{23},_3 = 0$$

$$S_{31},_1 + S_{32},_2 + S_{33},_3 = 0$$

Note S_{ij} is symmetric and that $S_{3i}(\mathbf{u},_3) = \frac{\partial}{\partial x_3} S_{3i}(\mathbf{u})$, therefore, the equilibrium equations can be re-written as

$$S_{31}(\mathbf{u},_3) = -S_{\rho 1}(\mathbf{u}),_\rho$$

$$S_{32}(\mathbf{u},_3) = -S_{\rho 2}(\mathbf{u}),_\rho$$

$$S_{33}(\mathbf{u},_3) = -S_{\rho 3}(\mathbf{u}),_\rho$$

These equilibrium equations can be written more compactly as

$$S_{3i}(\mathbf{u},_3) = -S_{\rho i}(\mathbf{u}),_\rho \tag{3.14}$$

To prove the last part of Theorem 3.1 it is necessary to prove that $\mathbf{u},_3$ satisfies the first three equations of the necessary conditions for a solution. Substitute $\mathbf{u},_3$ into the first of (3.8), substitute this into the first three of (3.12), and then take into account (3.14).

$$f_i(\mathbf{u},_3) = -\int_{\Sigma_1} S_{3i}(\mathbf{u},_3)da = \int_{\Sigma_1} S_{\rho i}(\mathbf{u}),_\rho \, da$$

Using the divergence theorem,

$$f_i(\mathbf{u},_3) = \int_{\Sigma_1} S_{1i}(\mathbf{u}),_1 + S_{2i}(\mathbf{u}),_2 \, da$$

$$= \int_{\Gamma} S_{1i}(\mathbf{u})n_1 + S_{2i}(\mathbf{u})n_2 ds$$

On Γ $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix}$. Then $S_{ij}n_j = S_{i1}n_1 + S_{i2}n_2$, therefore,

$$f_i(\mathbf{u},_3) = \int_{\Gamma} S_{ij}n_j ds = \int_{\Gamma} s_i(\mathbf{u})ds \tag{3.15}$$

To prove the last part of Theorem 3.1 it is necessary to prove that $\mathbf{u}_{,3}$ satisfies the fifth and sixth of the necessary conditions for a solution. Substitute $\mathbf{u}_{,3}$ into the first of (3.8) and this into the fifth and sixth of (3.12) .

$$m_\beta(\mathbf{u}_{,3}) = \int_{\Sigma_2} e_{\alpha\beta} x_\alpha S_{33}(\mathbf{u}_{,3}) da$$

Using (3.14),

$$e_{\alpha\beta} x_\alpha S_{33}(\mathbf{u}_{,3}) = -e_{\alpha\beta} x_\alpha S_{\rho 3}(\mathbf{u})_{,\rho}$$

$$e_{\alpha\beta}[x_\alpha S_{\rho 3}(\mathbf{u})]_{,\rho} = e_{\alpha\beta}[x_\alpha S_{13,1} + x_\alpha S_{13,2} + S_{\alpha 3}]$$

Therefore,

$$e_{\alpha\beta} x_\alpha S_{33}(\mathbf{u}_{,3}) = -e_{\alpha\beta}[[x_\alpha S_{\rho 3}(\mathbf{u})]_{,\rho} - S_{\alpha 3}(\mathbf{u})] \tag{3.16}$$

Substitute (3.16) into $m_\beta(\mathbf{u}_{,3})$,

$$m_\beta(\mathbf{u}_{,3}) = -\int_{\Sigma_2} e_{\alpha\beta}[[x_\alpha S_{\rho 3}(\mathbf{u})]_{,\rho} - S_{\alpha 3}(\mathbf{u})] da$$

Using the divergence theorem,

$$m_\beta(\mathbf{u}_{,3}) = -\int e_{\alpha\beta} x_\alpha S_{\rho 3}(\mathbf{u}) n_\rho ds + \int_{\Sigma_2} e_{\alpha\beta} S_{\alpha 3}(\mathbf{u}) da$$

Recall from (3.12) that

$$f_\alpha(\mathbf{u}) = -\int_{\Sigma_2} S_{\alpha 3}(\mathbf{u}) da$$

Therefore,

$$m_\beta(\mathbf{u}_{,3}) = -\int e_{\alpha\beta} x_\alpha S_{\rho 3}(\mathbf{u}) n_\rho ds - e_{\alpha\beta} f_\alpha(u) \tag{3.17}$$

To prove the last part of Theorem 3.1 it is necessary to prove that $\mathbf{u}_{,3}$ satisfies the fourth of the necessary conditions for a solution. Substitute $\mathbf{u}_{,3}$ into the first of (3.8) and this into the fourth of (3.12).

$$m_3(\mathbf{u}_{,3}) = -\int_{\Sigma_2} e_{\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u}_{,3})da$$

Again using (3.14),

$$e_{\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u}_{,3}) = -e_{\alpha\beta} x_\alpha S_{\rho\beta}(\mathbf{u})_{,\rho}$$

$$e_{\alpha\beta}[x_\alpha S_{\rho\beta}(\mathbf{u})]_{,\rho} = e_{\alpha\beta}[x_\alpha S_{1\beta,1} + x_\alpha S_{2\beta,2} + S_{\alpha\beta}]$$

Therefore,

$$e_{\alpha\beta} x_\beta S_{3\beta}(\mathbf{u}_{,3}) = -e_{\alpha\beta}[[x_\alpha S_{\rho\beta}(\mathbf{u})]_{,\rho} - S_{\alpha\beta}(\mathbf{u})] \tag{3.18}$$

Substitute (3.18) into $m_3(\mathbf{u}_{,3})$,

$$m_3(\mathbf{u}_{,3}) = \int_{\Sigma_2} e_{\alpha\beta}[[x_\alpha S_{\rho\beta}(\mathbf{u})]_{,\rho} - S_{\alpha\beta}(\mathbf{u})]da$$

Using the divergence theorem,

$$m_3(\mathbf{u}_{,3}) = \int_\Gamma e_{\alpha\beta} x_\alpha S_{\rho\beta}(\mathbf{u})n_\rho ds - \int_{\Sigma_2} e_{\alpha\beta} S_{\alpha\beta}(\mathbf{u})da$$

However,

$$e_{\alpha\beta} S_{\alpha\beta}(\mathbf{u}) = S_{12}(\mathbf{u}) - S_{21}(\mathbf{u}) = 0$$

Therefore,

$$m_3(\mathbf{u}_{,3}) = \int_\Gamma e_{\alpha\beta} x_\alpha S_{\rho\beta}(\mathbf{u})n_\rho ds \tag{3.19}$$

The proof of Theorem 3.1 is summarized by the following observations. Consider (3.15),

$$\text{on } \Gamma \ s(\mathbf{u}) = \mathbf{0}, \text{ then } f_i(\mathbf{u}_{,3}) = \int_\Gamma s(\mathbf{u})ds = 0 \tag{3.20}$$

Consider (3.17),

on Γ $n_3 = 0$ *and* $s(\mathbf{u}) = \mathbf{0}$, then $S_{\rho 3}(\mathbf{u})n_\rho = 0$, therefore,

$$m_\alpha(\mathbf{u},_3) = e_{\alpha\beta} f_\beta(\mathbf{u}) \tag{3.21}$$

Consider (3.19),

on Γ $n_3 = 0$ *and* $s(\mathbf{u}) = \mathbf{0}$, then $S_{\rho\beta}(\mathbf{u})n_\rho = 0$, therefore,

$$m_3(\mathbf{u},_3) = 0 \tag{3.22}$$

Thus, by equations (3.13), (3.20), (3.21), and (3.22) Theorem 3.1 is proven.

Corollary 1.1

If $\mathbf{u} \in K_1(F_3, M_1, M_2, M_3)$ and $\mathbf{u},_3 \in C^1(\overline{B}) \cap C^2(B)$

Then $\mathbf{u},_3 \in D$ and

$f(\mathbf{u},_3) = \mathbf{0}$, $m(\mathbf{u},_3) = \mathbf{0}$

Proof of Corollary 3.1

Recall that D is the set of all equilibrium displacement fields $\mathbf{u}$ that satisfy $s(\mathbf{u}) = \mathbf{0}$ on Π. K_1 is the set of solutions to P_1 and since P_1 requires $s(\mathbf{u}) = \mathbf{0}$ on Π, then

$$K_1(F_3, M_1, M_2, M_3) \subset D$$

Therefore, noting in P_1 that $F_1 = F_2 = 0$, then from (3.20), (3.21), and (3.22)

$$f_i(\mathbf{u},_3) = \int s_i(\mathbf{u})ds = 0$$

$$m_\alpha(\mathbf{u},_3) = 0,\ m_3(\mathbf{u},_3) = 0$$

Thus, proving Corollary 3.1 is true.

3.3 Statement and proof of Theorem 3.2

Theorem 3.2 was proposed by Iesan (1987, pg 48). The following proof of Theorem 3.2 parallels that given by Iesan. Let R be the set of all rigid body displacement fields. Recall from Corollary 1.1,

If $\mathbf{u} \in K_1(F_3, M_1, M_2, M_3)$ and $\mathbf{u},_3 \in C^1(\overline{B}) \cap C^2(B)$

Then $\mathbf{u},_3 \in D$ and

$f(\mathbf{u},_3) = 0$, $m(\mathbf{u},_3) = 0$

This suggests that $\mathbf{u},_3$ is a rigid body displacement and that $\mathbf{u}$ is a solution to P_1.

Theorem 3.2

Let J be the set of all vector fields $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$ such that $\mathbf{u},_3 \in R$. Then there exists a vector field $\mathbf{u} \in J$ that is a solution to the problem P_1.

Proof of Theorem 3.2

To prove Theorem 3.2 a system of equations will be derived in sections 3.3.1 to 3.3.3 that uniquely define $\mathbf{u} \in J$ as a solution to P_1.

3.3.1 Displacement equations

Let $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$ such that $\mathbf{u},_3$ is a rigid body displacement given by,

$$\mathbf{u},_3 = \mathbf{A} + \mathbf{B} \times \mathbf{x} \qquad \text{(See Appendix A)} \qquad (3.23)$$

Here **A, B** are constant vectors, and **x** is the position vector originating from the origin.

Integrate (3.23) with respect x_3,

$$u_1 = A_1x_3 - B_3x_2x_3 + \frac{B_2x_3^2}{2} + U_1(x_1,x_2)$$

$$u_2 = A_2x_3 - B_3x_1x_3 - \frac{B_1x_3^2}{2} + U_2(x_1,x_2) \tag{3.24}$$

$$u_3 = A_3x_3 - B_2x_1x_3 + B_1x_2x_3 + U_3(x_1,x_2)$$

Here the $\mathbf{U}(x_1, x_2)$ are the functions of integration with respect to x_3.

Let **u'** be an arbitrary rigid body displacement

$$u_1' = -w_3x_2 + w_2x_3 + u_{10}$$

$$u_2' = w_3x_1 - w_1x_3 + u_{20} \tag{3.25}$$

$$u_3' = w_1x_2 - w_2x_1 + u_{30}$$

Here w_i are rotations about the x_i axes, u_{i0} are translations in the x_i directions, and

$$A_1x_3 = w_2x_3$$

$$A_2x_3 = -w_1x_3$$

Recognize in (3.24) that the U_i may include parts of a rigid displacement that are not a function of x_3. Rewrite (3.24) by separating out an arbitrary rigid body displacement such that

$$u_1 = \frac{B_2x_3^2}{2} - B_3x_2x_3 + W_1(x_1,x_2) + u_1'$$

$$u_2 = -\frac{B_1x_3^2}{2} + B_3x_1x_3 + W_2(x_1,x_2) + u_2' \tag{3.26}$$

$$u_3 = [A_3 - B_2x_1 + B_1x_2]x_3 + W_3(x_1,x_2) + u_3'$$

In (3.26) the W_i are

$$W_1(x_1,x_2) = U_1 - (-w_3 x_2 + u_{10})$$
$$W_2(x_1,x_2) = U_2 - (w_3 x_1 + u_{20})$$
$$W_3(x_1,x_2) = U_3 - (w_1 x_2 - w_2 x_1 - u_{30})$$

Note, $\mathbf{W}$ is a displacement field that is independent of x_3 and that produces strains, where

$$\mathbf{W}(x_1,x_2) \in C^1(\overline{\Sigma}) \cap C^2(\Sigma) \tag{3.27}$$

3.3.2 Forming the generalized plane strain problem

Recall $C_{ijkl} = C_{ijkl}(x_1, x_2)$, and the stresses are given by

$$\begin{aligned} S_{ij}(\mathbf{u}) &= C_{ijkl} E_{kl} \\ E_{kl} &= \frac{1}{2}\left[\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right] \end{aligned} \tag{3.28}$$

Substitute (3.26) into the second equation of (3.28) and note the rigid body displacement does not contribute to the strain.

$$\begin{aligned} E_{11} &= \frac{\partial u_1}{\partial x_1} = W_{1,1} & E_{23} &= \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right] = \frac{1}{2}\left[\mathrm{B}_3 x_1 + W_{3,2}\right] \\ E_{22} &= \frac{\partial u_2}{\partial x_2} = W_{2,2} & E_{13} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] = \frac{1}{2}\left[-\mathrm{B}_3 x_2 + W_{3,1}\right] \\ E_{33} &= \frac{\partial u_3}{\partial x_3} = (\mathrm{A}_3 - \mathrm{B}_2 x_1 + \mathrm{B}_1 x_2) & E_{12} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] = \frac{1}{2}\left[W_{1,2} + W_{2,1}\right] \end{aligned} \tag{3.29}$$

Recall from equation (C2.11) that

$$C_{1123} = C_{1113} = C_{2223} = C_{2213} = C_{3323} = C_{3313} = C_{1223} = C_{1213} = 0$$

$$C_{2311} = C_{1311} = C_{2322} = C_{1322} = C_{2333} = C_{l333} = C_{2312} = C_{1312} = 0$$

Substituting (3.29) in to the first of (3.28),

$$\begin{aligned}
S_{11} &= C_{1111}W_{1},_{1} + C_{1122}W_{2},_{2} + C_{1133}(\mathrm{A}_3 - \mathrm{B}_2 x_1 + \mathrm{B}_1 x_2) + C_{1112}(W_{1},_{2} + W_{2},_{1}) \\
S_{22} &= C_{2211}W_{1},_{1} + C_{2222}W_{2},_{2} + C_{2233}(\mathrm{A}_3 - \mathrm{B}_2 x_1 + \mathrm{B}_1 x_2) + C_{2212}(W_{1},_{2} + W_{2},_{1}) \\
S_{33} &= C_{3311}W_{1},_{1} + C_{3322}W_{2},_{2} + C_{3333}(\mathrm{A}_3 - \mathrm{B}_2 x_1 + \mathrm{B}_1 x_2) + C_{3312}(W_{1},_{2} + W_{2},_{1}) \\
S_{23} &= C_{2332}(\mathrm{B}_3 x_1 + W_{3},_{2}) + C_{2313}(-\mathrm{B}_3 x_2 + W_{3},_{1}) \\
S_{13} &= C_{1332}(\mathrm{B}_3 x_1 + W_{3},_{2}) + C_{1313}(-\mathrm{B}_3 x_2 + W_{3},_{1}) \\
S_{12} &= C_{1211}W_{1},_{1} + C_{1222}W_{2},_{2} + C_{1233}(\mathrm{A}_3 - \mathrm{B}_2 x_1 + \mathrm{B}_1 x_2) + C_{1212}(W_{1},_{2} + W_{2},_{1})
\end{aligned} \tag{3.30}$$

Four coefficients from the constant vectors **A** and **B** occur in (3.30). These coefficients can be combined in the single vector **a**.

Let

$$a_1 = -B_2,\ a_2 = B_1,\ a_3 = A_3,\ a_4 = B_3 \tag{3.31}$$

Equation (3.30) can be written compactly using (3.31)

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(\mathbf{W}) \tag{3.32}$$

where

$$T_{ij}(\mathbf{W}) = C_{ijk\alpha} W_{k},_{\alpha} \tag{3.33}$$

Iesan (1987 pg 45) defines the state of generalized plane strain for a cylinder to be the state where the displacement vector is only a function of x_1 and x_2. Equation (3.33) can be considered a state of generalized plane strain, where the stresses (T_{ij}) are a function of the displacement vector $\mathbf{W}=\mathbf{W}(x_1, x_2)$. Substituting (3.32) into the equilibrium equations $S_{ij}(\mathbf{u}),_j = 0$, and the boundary conditions on the lateral surface $s(\mathbf{u})\mathbf{n} = \mathbf{0}$, results in six equations that are independent of x_3. The three equations resulting from the equilibrium equations are independent of x_3 because (3.32) is independent of x_3. The three equations resulting from the boundary conditions are independent of x_3 because (3.32) is independent of x_3, and because the normal vector on the lateral surface has only x_1 and x_2 components. The following six equations resulting from equation (3.32), the equilibrium equations, and the boundary conditions define a problem of generalized plane strain.

$$
\begin{aligned}
&\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = -a_1 C_{1133} - a_2 C_{1233} - (a_3 + a_1 x_1 + a_2 x_2)\left[\frac{\partial C_{1133}}{\partial x_1} + \frac{\partial C_{1233}}{\partial x_2}\right] \\
&\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} = -a_1 C_{1233} - a_2 C_{2233} - (a_3 + a_1 x_1 + a_2 x_2)\left[\frac{\partial C_{1233}}{\partial x_1} + \frac{\partial C_{2233}}{\partial x_2}\right] \\
&\frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} = -a_4 x_1\left[\frac{\partial C_{1323}}{\partial x_1} + \frac{\partial C_{2323}}{\partial x_2}\right] + a_4 x_2\left[\frac{\partial C_{1313}}{\partial x_1} + \frac{\partial C_{2313}}{\partial x_2}\right] \\
&T_{11}n_1 + T_{12}n_2 = -(a_3 + a_1 x_1 + a_2 x_2)[C_{1133}n_1 + C_{1233}n_2] \\
&T_{12}n_1 + T_{22}n_2 = -(a_3 + a_1 x_1 + a_2 x_2)[C_{1233}n_1 + C_{2233}n_2] \\
&T_{13}n_1 + T_{23}n_2 = a_4[x_2 C_{1313} - x_1 C_{1323}]n_1 + a_4[x_2 C_{2313} - x_1 C_{2323}]n_2
\end{aligned}
\tag{3.34}
$$

The right hand side of the first three equations in (3.34) can be interpreted as body loads, while the terms on the right hand side of the last three equations are surface tractions. The necessary conditions for a solution to the traction problem are given by the laws of momentum balance (Gurtin 1981, pg 100).

The equations for conservation of momentum are

$$\int_{\partial B} \mathbf{t}_{\mathbf{n}} da + \int_{B} \mathbf{b} dV = 0$$
$$\int_{\partial B} \mathbf{r} \times \mathbf{t}_{\mathbf{n}} da + \int_{B} \mathbf{r} \times \mathbf{b} dV = 0 \tag{3.35}$$

Here $\mathbf{t}_{\mathbf{n}}$ are the surface tractions acting on the body, $\mathbf{b}$ are the body loads, and $\mathbf{r}$ are the position vectors from the origin to points in $\overline{B}$.

The first equation of (3.35) is the conservation of linear momentum, while the second equation is the conservation of angular momentum. For the generalized plane strain problem defined by (3.34) the necessary conditions for a solution are,

$$\int_{\Sigma} g_i da + \int_{\Gamma} H_i ds = 0$$
$$\int_{\Sigma} e_{\alpha\beta} x_\alpha g_\beta da + \int_{\Gamma} e_{\alpha\beta} x_\alpha H_\beta ds = 0 \tag{3.36}$$
$$\int_{\Sigma} x_\alpha g_3 da + \int_{\Gamma} x_\alpha H_3 ds = \int_{\Sigma} T_{\alpha 3} da$$

where

g_i are the right hand sides of the first three equations of (3.34), and

H_i are the right hand sides of the last three equations of (3.34).

The body loads (g_i) and surface tractions (H_i) can be written in the following form,

$$g_\alpha = a_1 C_{\alpha 133} + a_2 C_{\alpha 233} + (a_\rho x_\rho + a_3)\left[\frac{\partial C_{\alpha 133}}{\partial x_1} + \frac{\partial C_{\alpha 233}}{\partial x_2}\right]$$
$$g_3 = a_4 x_\beta e_{\beta\alpha}\left[\frac{\partial C_{13\alpha 3}}{\partial x_1} + \frac{\partial C_{23\alpha 3}}{\partial x_2}\right] \tag{3.37}$$
$$H_\alpha = -(a_\rho x_\rho + a_3)[C_{\alpha\beta 33} n_\beta]$$
$$H_3 = [e_{\alpha\beta} x_\beta a_4 C_{\rho 3\alpha 3}] n_\rho$$

The area integrals in (3.36) can be converted into boundary integrals by noting from (3.37) that

$$g_{\alpha} = \frac{\partial}{\partial x_{\beta}}\left[(a_{\rho}x_{\rho} + a_3)C_{\alpha\beta 33}\right]$$

$$g_3 = \frac{\partial}{\partial x_{\rho}}\left[a_4 x_{\beta} e_{\beta\alpha} C_{\rho 3\alpha 3}\right] \tag{3.38}$$

Substitute equations (3.38) into (3.37) and these into (3.36), then use the divergence theorem to convert the area integrals into boundary integrals. The first three equations of (3.36) become

$$\int_{\Gamma} (a_3 + a_1 x_1 + a_2 x_2)[C_{\alpha\beta 33} n_{\beta}]ds + \int_{\Gamma} -(a_{\rho}x_{\rho} + a_3)[C_{\alpha\beta 33} n_{\beta}]ds = 0$$

$$\int_{\Gamma} [e_{\beta\alpha} x_{\beta} a_4 C_{\rho 3\alpha 3}]n_{\rho} ds + \int_{\Gamma} [e_{\alpha\beta} x_{\beta} a_4 C_{\rho 3\alpha 3}]n_{\rho} ds = 0 \tag{3.39}$$

(Note $e_{\alpha\beta} = -e_{\beta\alpha}$)

Recall the fourth equation of (3.36),

$$\int_{\Sigma} [e_{\alpha\beta} x_{\alpha} g_{\beta}]da + \int_{\Gamma} [e_{\alpha\beta} x_{\alpha} H_{\beta}]ds = 0 \tag{3.40}$$

Substitute the fourth and fifth of (3.37) into (3.40). Collect the terms in the second integral of (3.40) by n_1 and n_2, and then use the divergence theorem to convert the second integral into an area integral.

$$\int_{\Gamma} -e_{\gamma\beta} x_{\gamma} (a_{\rho}x_{\rho} + a_3)[C_{\beta\alpha 33} n_{\alpha}]ds = \int_{\Gamma} (a_{\rho}x_{\rho} + a_3)[e_{\alpha\beta} x_{\beta} C_{\alpha\gamma 33} n_{\gamma}]ds$$

$$\int_{\Gamma} (a_{\rho}x_{\rho} + a_3)[e_{\alpha\beta} x_{\beta} C_{\alpha\gamma 33} n_{\gamma}]ds = \int_{\Sigma} \frac{\partial}{\partial x_{\gamma}} (a_{\rho}x_{\rho} + a_3)[e_{\alpha\beta} x_{\beta} C_{\alpha\gamma 33}]da$$

Substituting this and (3.38) back into (3.40) results in,

$$\int_{\Sigma} e_{\alpha\beta} x_{\alpha} \frac{\partial}{\partial x_{\gamma}}[(a_{\rho}x_{\rho} + a_3)C_{\beta\gamma 33}]da + \int_{\Sigma} \frac{\partial}{\partial x_{\gamma}}(a_{\rho}x_{\rho} + a_3)[e_{\alpha\beta} x_{\beta} C_{\alpha\gamma 33}]da = 0 \tag{3.41}$$

However, in the second integral of (3.41) the following terms cancel.

$$(a_\rho x_\rho + a_3)C_{1233}\frac{\partial}{\partial x_1}(-x_1) + (a_\rho x_\rho + a_3)C_{1233}\frac{\partial}{\partial x_2}(x_2) = 0$$

Therefore, $e_{\alpha\beta}x_\beta$ can be moved outside of the derivative in the second integral and (3.41) is seen to equal zero.

$$\begin{aligned}&\int_\Sigma e_{\alpha\beta}x_\alpha\frac{\partial}{\partial x_\gamma}[(a_\rho x_\rho + a_3)C_{\beta\gamma 33}]da + \int_\Sigma e_{\alpha\beta}x_\beta\frac{\partial}{\partial x_\gamma}(a_\rho x_\rho + a_3)C_{\alpha\gamma 33}]da \\ &= \int_\Sigma e_{\alpha\beta}x_\alpha\frac{\partial}{\partial x_\gamma}[(a_\rho x_\rho + a_3)C_{\beta\gamma 33}]da - \int_\Sigma e_{\alpha\beta}x_\alpha\frac{\partial}{\partial x_\gamma}(a_\rho x_\rho + a_3)C_{\beta\gamma 33}]da = 0\end{aligned} \tag{3.42}$$

Consider the fifth and sixth equations of (3.36). The equilibrium equations for generalized plane strain are,

$$T_{i\alpha},_\alpha + g_i = 0 \tag{3.43}$$

Insert (3.43) into the right hand side of the fifth and sixth equations of (3.36),

$$\int_\Sigma T_{\alpha 3}da = \int_\Sigma T_{\alpha 3} + x_\alpha(T_{3\beta} + g_3)da \tag{i}$$

Note,

$$\frac{\partial}{\partial x_\beta}(x_\alpha T_{3\beta}) = T_{3\alpha} + x_\alpha T_{31},_1 + x_\alpha T_{32},_2 \tag{ii}$$

Substituting (ii) into (i), using the divergence theorem to convert the area integral to a boundary integral, and then recognizing that $x_\alpha T_{3\beta}n_\beta = x_\alpha H_3$ results in,

$$\int_\Sigma T_{\alpha 3}da = \int_\Gamma x_\alpha H_3 ds + \int_\Sigma x_\alpha g_3 da \tag{3.44}$$

Substituting (3.44) into the fifth and sixth equations of (3.36) results in the following equation that is identically satisfied,

$$\int_{\Sigma} x_{\alpha} g_3 da + \int_{\Gamma} x_{\alpha} H_3 ds = \int_{\Gamma} x_{\alpha} H_3 ds + \int_{\Sigma} x_{\alpha} g_3 da \tag{3.45}$$

Thus, from equations (3.39), (3.42), and (3.45) the necessary conditions for a solution to the generalized plane strain problem, are satisfied for any values of **a**. The generalized plane strain problem is defined by (3.34), and the necessary conditions for a solution are given in (3.36).

3.3.3 Forming the system of equations for determining a_p

Substituting **a** into the displacement equations and ignoring the rigid body displacements u_i', since these do not contribute to the strains, (3.26) can be written in the following form.

$$\begin{aligned} u_i &= \delta_{i\alpha}[-a_{\alpha}\frac{x_3^2}{2} + e_{\beta\alpha} a_4 x_{\beta} x_3] + \delta_{i3}[a_{\rho} x_{\rho} + a_3]x_3 + W_i \\ u_i - W_i &= \delta_{i\alpha}[-a_{\alpha}\frac{x_3^2}{2} + e_{\beta\alpha} a_4 x_{\beta} x_3] + \delta_{i3}[a_{\rho} x_{\rho} + a_3]x_3 \end{aligned} \tag{3.46}$$

The right hand side of the difference u_i - W_i can be written as the dot product of two vectors,

$$u_i - W_i = \begin{bmatrix} -\delta_{i1}\frac{x_3^2}{2} + \delta_{i3} x_1 x_3 \\ -\delta_{i2}\frac{x_3^2}{2} + \delta_{i3} x_2 x_3 \\ \delta_{i3} x_3 \\ (-\delta_{i1} x_2 x_3 + \delta_{i2} x_1 x_3) \end{bmatrix} \bullet \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{3.47}$$

The coefficients a_p ($p = 1\ldots4$) form the vector **a**,

$$\mathbf{a} = a_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + a_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = a_p\mathbf{e}_p \tag{3.48}$$

The vector **W** is a solution to the boundary value problem (3.34). Equations (3.39), (3.42), and (3.45) show that the necessary conditions for a solution to the generalized plane strain problem are satisfied for any values of a_p. Let $\mathbf{W}^{(p)}$ be the solution to the generalized plane strain problem for a particular set of a_p. Let $n = 1$, 2, 3, 4, then in (3.48) set $a_p = 1$ for $p = n$, and $a_p = 0$ for $p \neq n$ to form the base vectors for (3.47). The displacements $u_i^{(p)}$ in the directions of the base vectors then form four auxiliary problems.

$$u_i^{(p)} - W_i^{(p)} = \begin{bmatrix} -\delta_{i1}\dfrac{x_3^2}{2} + \delta_{i3}x_1x_3 \\ -\delta_{i2}\dfrac{x_3^2}{2} + \delta_{i3}x_2x_3 \\ \delta_{i3}x_3 \\ (-\delta_{i1}x_2x_3 + \delta_{i2}x_1x_3) \end{bmatrix} \bullet \mathbf{e}_p \tag{3.49}$$

$$u_i^{(p)} = \begin{bmatrix} -\delta_{i1}\dfrac{x_3^2}{2} + \delta_{i3}x_1x_3 \\ -\delta_{i2}\dfrac{x_3^2}{2} + \delta_{i3}x_2x_3 \\ \delta_{i3}x_3 \\ (-\delta_{i1}x_2x_3 + \delta_{i2}x_1x_3) \end{bmatrix} \bullet \mathbf{e}_p + W_i^{(p)}$$

$$u_i = \sum_p^4 a_p u_i^{(p)} \tag{3.50}$$

To show that the stresses, $S_{ij} = S_{ij}(\mathbf{u})$, can also be written as linear combinations of their components in the directions of the base vectors, substitute (3.50) into (3.28).

$$S_{ij}(\mathbf{u}) = C_{ijkl}E_{kl}$$

$$S_{ij}(\mathbf{u}) = \frac{1}{2}C_{ijkl}\left[\frac{\partial}{\partial x_l}\sum_{p=1}^{4} a_p u_k^{(p)} + \frac{\partial}{\partial x_k}\sum_{p=1}^{4} a_p u_l^{(p)}\right] \tag{3.51}$$

Since the a_p are independent of the derivatives, the derivatives can be moved inside the summations in (3.51).

$$\begin{aligned} S_{ij}(\mathbf{u}) &= \sum_{p=1}^{4} a_p \frac{1}{2}C_{ijkl}\left[\frac{\partial u_k^{(p)}}{\partial x_l} + \frac{\partial u_l^{(p)}}{\partial x_k}\right] \\ S_{ij}(\mathbf{u}) &= \sum_{p=1}^{4} a_p S_{ij}(\mathbf{u}^{(p)}) \end{aligned} \tag{3.52}$$

Therefore, the stresses $S_{ij} = S_{ij}(\mathbf{u})$ can also be written as linear combinations of their components in the directions of the base vectors. Recall equation (3.32)

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(\mathbf{W})$$

$$S_{ij}(\mathbf{u}) - T_{ij}(\mathbf{W}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta \tag{3.53}$$

The right hand side of the difference $S_{ij} - T_{ij}$ in (3.53) can be written as the dot product of two vectors,

$$S_{ij}(\mathbf{u}) - T_{ij}(\mathbf{W}) = \begin{bmatrix} C_{ij33} x_1 \\ C_{ij33} x_2 \\ C_{ij33} \\ -C_{ij\alpha 3} e_{\alpha\beta} x_\beta \end{bmatrix} \bullet \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{3.54}$$

Substituting $\mathbf{e_p}$ for $\mathbf{a}$ in (3.54) results in,

$$S_{ij}(\mathbf{u}^{(p)}) - T_{ij}(\mathbf{W}^{(p)}) = \begin{bmatrix} C_{ij33}x_1 \\ C_{ij33}x_2 \\ C_{ij33} \\ -C_{ij\alpha 3}e_{\alpha\beta}x_\beta \end{bmatrix} \bullet \mathbf{e}_p$$

$$S_{ij}(\mathbf{u}^{(p)}) = \begin{bmatrix} C_{ij33}x_1 \\ C_{ij33}x_2 \\ C_{ij33} \\ -C_{ij\alpha 3}e_{\alpha\beta}x_\beta \end{bmatrix} \bullet \mathbf{e}_p + T_{ij}(\mathbf{W}^{(p)}) \qquad (3.55)$$

In equation (3.55) the stresses $T_{ij}^{(p)} = T_{ij}(\mathbf{W}^{(p)})$ indicate the generalized plane strain problem can be separated into four auxiliary problems corresponding to the base vectors. The six equations of (3.34) define a generalized plane strain boundary value problem for the unknown stresses $T_{ij} = T_{ij}(\mathbf{W})$. Replacing $\mathbf{a}$ in (3.34) with the base vectors $\mathbf{e_p}$ will result in four systems of six equations that define the stresses $T_{ij}^{(p)}$. Iesan (1987 pg 49) writes these four systems of equations compactly as the three equilibrium equations on Σ,

$$\begin{aligned} &T_{i\alpha}^{(\beta)},_{\alpha} + (C_{i\alpha 33}x_\beta),_{\alpha} = 0 \\ &T_{i\alpha}^{(3)},_{\alpha} + (C_{i\alpha 33}),_{\alpha} = 0 \\ &T_{i\alpha}^{(4)},_{\alpha} - e_{\rho\beta}(C_{i\alpha\rho 3}x_\beta),_{\alpha} = 0 \end{aligned} \qquad (3.56)$$

and the three boundary conditions on Γ,

$$\begin{aligned} &T_{i\alpha}^{(\beta)}n_\alpha = -C_{i\alpha 33}x_\beta n_\alpha \\ &T_{i\alpha}^{(3)}n_\alpha = -C_{i\alpha 33}n_\alpha \\ &T_{i\alpha}^{(4)}n_\alpha = e_{\rho\beta}C_{i\alpha\rho 3}x_\beta n_\alpha \end{aligned} \qquad (3.57)$$

Recall that D represents the set of all equilibrium displacement fields **u** that satisfy the conditions $s(\mathbf{u}) = \mathbf{0}$ on Π, where $s_i(\mathbf{u}) = S_{ij}(\mathbf{u})n_j$. Substitute (3.55) into the equilibrium equations $S_{ij}(\mathbf{u}^{(p)})_{,j} = 0$, noting (3.56) it can be seen that the equilibrium equations are identically satisfied. Substitute (3.55) into the boundary conditions on Π, $s_i(\mathbf{u}^{(p)}) = S_{ij}(\mathbf{u}^{(p)})n_j = 0$, noting (3.57) it can be seen that the boundary equations are identically satisfied. Therefore, $\mathbf{u}^{(p)} \in D$.

Recall that Saint-Venant's principle was used to replace the resultant loads on Σ_I with the stress functions $\mathbf{f} = \mathbf{f}(\mathbf{u})$ and $\mathbf{m} = \mathbf{m}(\mathbf{u})$, where $\mathbf{u} \in K_1(F_3, M_1, M_2, M_3)$. Equation (3.8) gives six equations relating **u** to the applied loads,

$$\begin{aligned} &f_1(\mathbf{u}) = 0, f_2(\mathbf{u}) = 0, f_3(\mathbf{u}) = F_3 \\ &m_1(\mathbf{u}) = M_1,\ m_2(\mathbf{u}) = M_2,\ m_3(\mathbf{u}) = M_3 \end{aligned} \tag{3.58}$$

Recall from Theorem 3.1 that $m_\alpha(\mathbf{u},_3) = e_{\alpha\beta} f_\beta(\mathbf{u})$. However, from Corollary 1.1 $m(\mathbf{u},_3) = 0$ because $\mathbf{u},_3 \in R$. Therefore, the first two equations of (3.58) are satisfied. The last four equations of (3.58) will be used to form a system of equations that determine the constants a_p (p = 1,2,3,4). Substitute the last four equations of (3.58) into the last four equations of the necessary conditions for a solution (3.12).

$$\begin{aligned} &\int_{\Sigma_2} S_{33}(\mathbf{u})da = -F_3 \\ &\int_{\Sigma_2} x_2 S_{33}(\mathbf{u})da = M_2 \\ &\int_{\Sigma_2} x_1 S_{33}(\mathbf{u})da = M_1 \\ &\int_{\Sigma_2} \left[x_1 S_{32}(\mathbf{u}) - x_2 S_{31}(\mathbf{u})\right]da = -M_3 \end{aligned} \tag{3.59}$$

Substitute (3.52) into the first of (3.59) and then replace $S_{33}(\mathbf{u}^{(p)})$ with (3.55).

$$\int_{\Sigma_2} S_{33}(\mathbf{u})da = \int_{\Sigma_2} \sum_{p=1}^{4} a_p S_{33}(\mathbf{u}^{(p)})da = -F_3$$

$$\int_{\Sigma_2} [C_{3333}(a_1x_1 + a_2x_2 + a_3) - a_4C_{3313}x_2 + a_4C_{3323}x_1 + T_{33}]da = -F_3$$

From (C2.11) $C_{3313} = C_{3323} = 0$, therefore,

$$\int_{\Sigma_2} [C_{3333}(a_1x_1 + a_2x_2 + a_3) + T_{33}]da = -F_3 \tag{i}$$

Following a similar method, three more equations can be formed by substituting (3.52) and (3.55) into the last three equations of (3.59). These three equations and (i) form the following system of equations that determine the constants a_i.

$$\begin{aligned}
&\int_{\Sigma_2} [C_{3333}(a_1x_1 + a_2x_2 + a_3) + T_{33}]da = -F_3 \\
&\int_{\Sigma_2} x_2[C_{3333}(a_1x_1 + a_2x_2 + a_3) + T_{33}]da = M_2 \\
&\int_{\Sigma_2} x_1[C_{3333}(a_1x_1 + a_2x_2 + a_3) + T_{33}]da = M_1 \\
&\int_{\Sigma_2} a_4x_1[C_{2323}x_1 - C_{2313}x_2] + x_1T_{32}da - \int_{\Sigma_2} a_4x_2[C_{2313}x_1 - C_{1313}x_2] + x_2T_{31}da = -M_3
\end{aligned} \tag{3.60}$$

Rewrite the system of equations in (3.60) in matrix form,

$$\begin{bmatrix} \int_{\Sigma_2} x_1^2C_{3333}da & \int_{\Sigma_2} x_1x_2C_{3333}da & \int_{\Sigma_2} x_1C_{3333}da & 0 \\ \int_{\Sigma_2} x_1x_2C_{3333}da & \int_{\Sigma_2} x_2^2C_{3333}da & \int_{\Sigma_2} x_2C_{3333}da & 0 \\ \int_{\Sigma_2} x_1C_{3333}da & \int_{\Sigma_2} x_2C_{3333}da & \int_{\Sigma_2} C_{3333}da & 0 \\ 0 & 0 & 0 & \int_{\Sigma_2} [-2x_1x_2C_{2313} + x_1^2C_{2323} + x_2^2C_{1313}]da \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \tag{3.61}$$

In (3.61) the G_i are

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} M_1 - \int_{\Sigma_2} x_1 T_{33} da \\ M_2 - \int_{\Sigma_2} x_2 T_{33} da \\ -F_3 - \int_{\Sigma_2} T_{33} da \\ -M_3 - \int_{\Sigma_2} x_1 T_{32} + x_2 T_{31} da \end{bmatrix}$$

There are two independent problems in (3.61). As will be shown in section 5.1 the three coefficients a_1, a_2, and a_3 correspond to the $S_{\alpha\beta}$ stresses, while a_4 corresponds to the $S_{\alpha 3}$ stresses.

The zero terms in the coefficient matrix allow the separation of (3.61) into the following two problems.

$$\begin{bmatrix} \int_{\Sigma_2} x_1^2 C_{3333} da & \int_{\Sigma_2} x_1 x_2 C_{3333} da & \int_{\Sigma_2} x_1 C_{3333} da \\ \int_{\Sigma_2} x_1 x_2 C_{3333} da & \int_{\Sigma_2} x_2^2 C_{3333} da & \int_{\Sigma_2} x_2 C_{3333} da \\ \int_{\Sigma_2} x_1 C_{3333} da & \int_{\Sigma_2} x_2 C_{3333} da & \int_{\Sigma_2} C_{3333} da \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \tag{3.62a}$$

$$\int_{\Sigma_2} [-2\mathrm{x}_1\mathrm{x}_2\mathrm{C}_{2313} + \mathrm{x}_1^2\mathrm{C}_{2323} + \mathrm{x}_2^2\mathrm{C}_{1313}]\mathrm{da} * \mathrm{a}_4 = G_4 \tag{3.62b}$$

Consider (3.62a), note from (2.19b) that $C_{3333} = \underline{C_{3333}}$ is a constant and can be moved outside of the integrals. With C_{3333} taken outside the integrals and since Σ_2 is circular, the off diagonal terms in the coefficient matrix are seen to equal zero, while the first two diagonal terms are the moment of inertia (I) and the third is the area of Σ_2 (A).

Equation (3.63) shows that a_1, a_2, and a_3 are uniquely defined if a function can be found for T_{33}, since I, A, and C_{3333} are real and never zero.

$$\begin{bmatrix} C_{3333}I & 0 & 0 \\ 0 & C_{3333}I & 0 \\ 0 & 0 & C_{3333}A \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \tag{3.63}$$

Iesan (1987, pg 51) uses a strain energy argument to prove that (3.61) uniquely defines a_p for a more general elasticity tensor. Equation (3.62b) can be shown to uniquely define a_4 when considering equation (C2.11) and the engineering constants in Table 2.1. Since the objective of this paper is to consider the $S_{\alpha\beta}$ stresses, equation (3.62b) is not required and it will be assumed with out further proof to uniquely define a_4.

Equation (3.61) uniquely defines the coefficients a_p ($p = 1, 2, 3, 4$) so the displacement field (3.26) exists. Therefore, a displacement field $\mathbf{u}_{,3} \in R$ exists such that $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$ is a solution to the problem P_1. Thus, Theorem 3.2 is proven.

4. Solving the Generalized Plane Strain Problem

Recall for generalized plane strain that $E_{ij} = S_{ijmn}T_{mn}$. Chirita (1979) notes this process must be reversible, therefore, $T_{kl} = C_{klrs}E_{rs}$. This results in $E_{ij} = S_{ijmn}C_{mnrs}E_{rs}$, and

$$S_{ijmn}C_{mnrs} = \frac{1}{2}[\delta_{ir}\delta_{js} + \delta_{ij}\delta_{jr}] = \begin{cases} (i = s, j = r) \Rightarrow 1/2 \\ (i = r, j = s) \Rightarrow 1/2 \\ (i = r, j = s; i = s, j = r) \Rightarrow 1 \\ all\ other\ i, j, r, s \Rightarrow 0 \end{cases} \tag{4.1}$$

where the S_{ijkl}, and C_{ijkl} tensors are defined by (2.19).

For the generalized plane strain problem, the stresses and strains are a function of the displacement vector $\mathbf{W}$.

$$T_{ij} = T_{ij}(\mathbf{W}(x_1, x_2)) = C_{ijkl}E_{kl}$$

$$E_{ij} = E_{ij}(\mathbf{W}(x_1, x_2)) = S_{ijkl}T_{kl} \tag{4.2}$$

Equation (4.2) results in the following three sets of constraints on the strains, the first is a result of $\mathbf{W}$ being independent of x_3, the second and third are obtained from equation (D11) in Appendix D.

$$E_{33}^{(p)}(\mathbf{W}) = \frac{1}{2}\left[\frac{\partial W_3^{(p)}}{\partial x_3} + \frac{\partial W_3^{(p)}}{\partial x_3}\right] = 0 \tag{4.3}$$

$$E_{\rho\sigma}^{(p)}(\mathbf{W}) = \alpha_{\rho\sigma}^{(p)}x_1 + \beta_{\rho\sigma}^{(p)}x_2 \tag{4.4}$$

$$E_{\rho 3}^{(p)}(\mathbf{W}) = \alpha_{\rho 3}^{(p)}x_1 + \beta_{\rho 3}^{(p)}x_2 \tag{4.5}$$

(Here, $\alpha_{\rho j}^{(p)}$ and $\beta_{\rho j}^{(p)}$ are constants.)

4.1 Determining the stress functions for $T_{ij}^{(1)}$

Let $p = 1$, then (3.56) and (3.57) form a generalized plane strain boundary value problem that defines $T_{ij}^{(1)} = T_{ij}(\mathbf{W}^{(1)})$.

$$\frac{\partial T_{11}^{(1)}}{\partial x_1} + \frac{\partial T_{12}^{(1)}}{\partial x_2} + C_{1133} + x_1 * \left[\frac{\partial C_{1133}}{\partial x_1} + \frac{\partial C_{1233}}{\partial x_2}\right] = 0$$

$$\frac{\partial T_{12}^{(1)}}{\partial x_1} + \frac{\partial T_{22}^{(1)}}{\partial x_2} + C_{1233} + x_1 * \left[\frac{\partial C_{1233}}{\partial x_1} + \frac{\partial C_{2233}}{\partial x_2}\right] = 0$$

$$\frac{\partial T_{13}^{(1)}}{\partial x_1} + \frac{\partial T_{23}^{(1)}}{\partial x_2} = 0 \tag{4.6}$$

$$T_{11}^{(1)} n_1 + T_{12}^{(1)} n_2 = -x_1 * [C_{1133} n_1 + C_{1233} n_2]$$

$$T_{12}^{(1)} n_1 + T_{22}^{(1)} n_2 = -x_1 * [C_{1233} n_1 + C_{2233} n_2]$$

$$T_{13}^{(1)} n_1 + T_{23}^{(1)} n_2 = 0$$

Introduce the stress function $\varphi^{(1)} = \varphi^{(1)}(x_1, x_2)$ such that

$$T_{11}^{(1)} = -x_1 C_{1133} + \phi^{(1)}{}_{,22}$$

$$T_{22}^{(1)} = -x_1 C_{2233} + \phi^{(1)}{}_{,11} \tag{4.7}$$

$$T_{12}^{(1)} = -x_1 C_{1233} - \phi^{(1)}{}_{,12}$$

Let

$$T_{13}^{(1)} = T_{23}^{(1)} = 0 \tag{4.8}$$

Substitution of (4.7) into (4.6) shows that the first two equations of (4.6) are satisfied. Substitution of (4.8) into (4.6) shows that the third and sixth equations of (4.6) are satisfied. The fourth and fifth equations of (4.6) will be used to form a boundary condition for $\varphi^{(1)}{}_{,\alpha\beta}$.

Equations (4.2) and (4.3) will be used to form an equation that defines $T_{33}^{(1)}$. First, substitute the second of (4.2) into (4.3), while taking into account (4.8).

$$E_{33}^{(1)} = S_{3311}T_{11}^{(1)} + S_{3322}T_{22}^{(1)} + S_{3333}T_{33}^{(1)} + 2S_{3312}T_{12}^{(1)} = 0 \qquad \text{(i)}$$

Substitute (4.7) into (i)

$$\begin{aligned} -x_1[S_{3311}C_{1133} + S_{3322}C_{2233} + 2S_{3312}C_{1233}] + S_{3311}\frac{\partial^2\phi^{(1)}}{\partial x_2^2} + S_{3322}\frac{\partial^2\phi^{(1)}}{\partial x_1^2} \\ -2S_{3312}\frac{\partial^2\phi^{(1)}}{\partial x_1 \partial x_2} + S_{3333}T_{33}^{(1)} = 0 \end{aligned} \qquad \text{(ii)}$$

Note from (4.1), when $i = j = 3$ and $r = s = 3$,

$$S_{3311}C_{1133} + S_{3322}C_{2233} + 2S_{3312}C_{1233} = 1 - S_{3333}C_{3333} \qquad \text{(iii)}$$

Substitute (iii) into (ii) and solve for $T_{33}^{(1)}$.

$$T_{33}^{(1)} = x_1 S_{3333}^{-1} - x_1 C_{3333} - \frac{1}{S_{3333}}\left[S_{3311}\frac{\partial^2\phi^{(1)}}{\partial x_2^2} + S_{3322}\frac{\partial^2\phi^{(1)}}{\partial x_1^2} - 2S_{3312}\frac{\partial^2\phi^{(1)}}{\partial x_1 \partial x_2}\right] \qquad (4.9)$$

Thus, (4.7), (4.8), and (4.9) provide six equations that define $T_{ij}^{(1)}$.

4.2 Satisfying the strain constraint equations given $T_{ij}^{(1)}$

The constraint equation (4.5) is automatically satisfied by (4.8), while the constraint equation (4.4) will be used to derive an equation that defines the stress function $\varphi^{(1)}$.

Let L_1 and L_2 be the differential operators

$$L_1 = S_{3311}\frac{\partial^2}{\partial x_2^2} + S_{3322}\frac{\partial^2}{\partial x_1^2} - 2S_{3312}\frac{\partial^2}{\partial x_1 \partial x_2}$$

$$L_2 = S_{1111}\frac{\partial^2}{\partial x_2^2} + S_{1122}\frac{\partial^2}{\partial x_1^2} - 2S_{1112}\frac{\partial^2}{\partial x_1 \partial x_2}$$

Substitute $T_{ij}^{(1)}$ into the second of (4.2) for $i = j = 1$, then

$$E_{11}^{(1)} = S_{1111}T_{11}^{(1)} + S_{1122}T_{22}^{(1)} + S_{1133}T_{33}^{(1)} + 2S_{1112}T_{12}^{(1)} \tag{i}$$

Substitute (4.7) and (4.9) into (i), then

$$\begin{aligned} E_{11}^{(1)} = {} & -x_1\left[S_{1111}C_{1133} + S_{1122}C_{2233} + S_{1133}C_{3333} + 2S_{1112}C_{1233}\right] \\ & + x_1\frac{S_{1133}}{S_{3333}} - \frac{S_{1133}}{S_{3333}}L_1\phi^{(1)} + L_2\phi^{(1)} \end{aligned} \tag{ii}$$

Note from (4.1) that

$$S_{11mn}C_{mn33} = 0 \tag{iii}$$

Substitute (iii) into (ii), expand the operators L_1 and L_2, and group the terms by the derivatives of $\varphi^{(1)}$.

$$\begin{aligned} E_{11}^{(1)} = {} & \frac{x_1 S_{1133}}{S_{3333}} + \frac{\partial^2\phi^{(1)}}{\partial x_2^2}\left[S_{1111} - \frac{S_{1133}^2}{S_{3333}}\right] + \frac{\partial^2\phi^{(1)}}{\partial x_1^2}\left[S_{1122} - \frac{S_{1133}S_{3322}}{S_{3333}}\right] \\ & - 2\frac{\partial^2\phi^{(1)}}{\partial x_1 \partial x_2}\left[S_{1112} - \frac{S_{1133}S_{3312}}{S_{3333}}\right] \end{aligned} \tag{iv}$$

The equations for $E_{22}^{(1)}$ and $E_{12}^{(1)}$ can be derived in a similar fashion as for $E_{11}^{(1)}$.

Lekhnitskii (1981, pg 104) identifies the reduced strain coefficients for generalized Hooke's law in matrix form; the equivalent tensor forms of the reduced strain coefficients are,

$$B_{ijkl} = S_{ijkl} - \frac{S_{ij33} S_{kl33}}{S_{3333}} \tag{v}$$

Substituting (v) into the equations for $E_{11}{}^{(1)}$, $E_{22}{}^{(1)}$, and $E_{12}{}^{(1)}$ results in the following three equations.

$$\begin{aligned}
E_{11}^{(1)} &= \frac{x_1 S_{1133}}{S_{3333}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1111} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1122} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{1112} \\
E_{22}^{(1)} &= \frac{x_1 S_{2233}}{S_{3333}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{2211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{2222} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{2212} \\
E_{12}^{(1)} &= \frac{x_1 S_{1233}}{S_{3333}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1222} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{1212}
\end{aligned} \tag{4.10}$$

Substitute the transformation equations (2.19 a) into the first term on the R.H.S. of (4.10),

$$\begin{aligned}
E_{11}^{(1)} &= \frac{x_1 \{\underline{\underline{S_{1133}}} + R[C_\theta^2 \underline{\underline{M_{1133}}} + S_\theta^2 \underline{\underline{M_{2233}}}]\}}{\underline{S_{3333}}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1111} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1122} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{1112} \\
E_{22}^{(1)} &= \frac{x_1 \{\underline{\underline{S_{1133}}} + R[S_\theta^2 \underline{\underline{M_{1133}}} + C_\theta^2 \underline{\underline{M_{2233}}}]\}}{\underline{S_{3333}}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{2211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{2222} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{2212} \\
E_{12}^{(1)} &= \frac{x_1 \{-S_\theta C_\theta R[\underline{\underline{M_{1133}}} - \underline{\underline{M_{2233}}}]\}}{\underline{S_{3333}}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1222} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{1212}
\end{aligned} \tag{4.11}$$

Assume the strains are not identically equal to zero. To ensure that the strains in (4.11) satisfy the constraint equation (4.4), $\varphi^{(1)}$ may be selected so that the strains are linear functions of x_1 and x_2.

To select $\varphi^{(1)}$ so that the strains in (4.11) are linear functions of x_1 and x_2 the following may be assumed,

$$\begin{aligned}
&\frac{x_1 R[C_\theta^2 \underline{M}_{1133} + S_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1111} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1122} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{1112} = 0 \\
&\frac{x_1 R[S_\theta^2 \underline{M}_{1133} + C_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{2211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{2222} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{2212} = 0 \qquad (4.12) \\
&\frac{-x_1 S_\theta C_\theta R[\underline{M}_{1133} - \underline{M}_{2233}]}{\underline{S}_{3333}} + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1222} - 2\frac{\partial^2 \phi^{(1)}}{\partial x_1 \partial x_2} B_{1212} = 0
\end{aligned}$$

Let

$$\begin{aligned}
A^{(1)} &= \frac{x_1 R[C_\theta^2 \underline{M}_{1133} + S_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \\
B^{(1)} &= \frac{x_1 R[S_\theta^2 \underline{M}_{1133} + C_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \qquad (4.13) \\
C^{(1)} &= \frac{-x_1 S_\theta C_\theta R[\underline{M}_{1133} - \underline{M}_{2233}]}{\underline{S}_{3333}}
\end{aligned}$$

Multiply the third equation of (4.12) by B_{1112}/B_{1212} and then subtract it from the first equation of (4.12). Multiply the third equation of (4.12) by B_{2212}/B_{1212} and then subtract it from the second equation of (4.12). This results in

$$\begin{aligned}
&\left[A^{(1)} - \frac{B_{1112}C}{B_{1212}}\right] + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{1111} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{1122} - \frac{\partial^2 \phi^{(1)}}{\partial x_2^2}\frac{B_{1211}^2}{B_{1212}} - \frac{\partial^2 \phi^{(1)}}{\partial x_1^2}\frac{B_{1222}B_{1112}}{B_{1212}} = 0 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(i)} \\
&\left[B^{(1)} - \frac{B_{2212}C}{B_{1212}}\right] + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2} B_{2211} + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2} B_{2222} - \frac{\partial^2 \phi^{(1)}}{\partial x_2^2}\frac{B_{1211}B_{2212}}{B_{1212}} - \frac{\partial^2 \phi^{(1)}}{\partial x_1^2}\frac{B_{1222}^2}{B_{1212}} = 0
\end{aligned}$$

Multiply the second of (i) by B_{1112}/B_{2212} and subtract from the first of (i), then

$$\left[A^{(1)} - B^{(1)}\frac{B_{1112}}{B_{2212}}\right] + \frac{\partial^2 \phi^{(1)}}{\partial x_2^2}\left[B_{1111} - \frac{B_{2211}B_{1112}}{B_{2212}}\right] + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2}\left[B_{1122} - \frac{B_{2222}B_{1112}}{B_{2212}}\right] = 0 \quad \text{(ii)}$$

Substitute (4.7) into the fourth and fifth of (4.6), then

$$\frac{\partial^2\phi^{(1)}}{\partial x_2^2}n_1 - \frac{\partial^2\phi^{(1)}}{\partial x_1\partial x_2}n_2 = 0$$

$$-\frac{\partial^2\phi^{(1)}}{\partial x_1\partial x_2}n_1 + \frac{\partial^2\phi^{(1)}}{\partial x_1^2}n_2 = 0 \qquad \text{(iii)}$$

Therefore, on the boundary Γ, $\varphi^{(1)}$ is defined by

$$\frac{\partial^2\phi^{(1)}}{\partial x_2^2} = \frac{\partial^2\phi^{(1)}}{\partial x_1^2}\frac{n_2^2}{n_1^2} \qquad \text{(iv)}$$

Equations (ii) and (iv) form the following boundary value problem,

$$\frac{\partial^2\phi^{(1)}}{\partial x_2^2}\left[B_{1111} - \frac{B_{2211}B_{1112}}{B_{2212}}\right] + \frac{\partial^2\phi^{(1)}}{\partial x_1^2}\left[B_{1122} - \frac{B_{2222}B_{1112}}{B_{2212}}\right] = \left[B^{(1)}\frac{B_{1112}}{B_{2212}} - A^{(1)}\right] \text{on } \overline{\Sigma}$$

$$\frac{\partial^2\phi^{(1)}}{\partial x_2^2} = \frac{\partial^2\phi^{(1)}}{\partial x_1^2}\frac{n_2^2}{n_1^2} \text{ on } \Gamma \qquad (4.14)$$

4.3 Determining the stress functions for $T_{ij}^{(4)}$

Let $p = 4$, then (3.56) and (3.57) form a plane strain boundary value problem that defines $T_{ij}^{(4)} = T_{ij}(\mathbf{W}^{(4)})$.

$$\frac{\partial T_{11}^{(4)}}{\partial x_1} + \frac{\partial T_{12}^{(4)}}{\partial x_2} = 0$$

$$\frac{\partial T_{12}^{(4)}}{\partial x_1} + \frac{\partial T_{22}^{(4)}}{\partial x_2} = 0$$

$$\frac{\partial T_{13}^{(4)}}{\partial x_1} + \frac{\partial T_{23}^{(4)}}{\partial x_2} = -x_1\left[\frac{\partial C_{1323}}{\partial x_1} + \frac{\partial C_{2323}}{\partial x_2}\right] + x_2\left[\frac{\partial C_{1313}}{\partial x_1} + \frac{\partial C_{2313}}{\partial x_2}\right] \qquad (4.15)$$

$$T_{11}^{(4)}n_1 + T_{12}^{(4)}n_2 = 0$$

$$T_{12}^{(4)}n_1 + T_{22}^{(4)}n_2 = 0$$

$$T_{13}^{(4)}n_1 + T_{23}^{(4)}n_2 = \left[C_{1313}x_2 - C_{1323}x_1\right]n_1 + \left[C_{2313}x_2 - C_{2323}x_1\right]n_2$$

Introduce the stress function $\psi = \psi(x_1, x_2)$ such that

$$T_{13}^{(4)} = C_{1313}x_2 - C_{1323}x_1 + \frac{\partial \psi}{\partial x_2}$$
$$T_{23}^{(4)} = C_{2313}x_2 - C_{2323}x_1 - \frac{\partial \psi}{\partial x_2} \tag{4.16}$$

Let

$$T_{11}^{(4)} = T_{22}^{(4)} = T_{12}^{(4)} = 0 \tag{4.17}$$

Substitution of (4.16) into (4.15) shows the third equation of (4.15) is satisfied. Substitution of (4.17) into (4.15) shows the first, second, fourth, and fifth equations of (4.15) are satisfied. The sixth equation of (4.15) will be used to form a boundary condition for ψ. Equations (4.2) and (4.3) will be used to form an equation that defines $T_{33}^{(4)}$. Substitute the second of (4.2) into (4.3), while taking into account (4.17).

Therefore,

$$E_{33}^{(4)} = S_{3311}T_{11}^{(4)} + S_{3322}T_{22}^{(4)} + S_{3333}T_{33}^{(4)} + S_{3312}T_{12}^{(4)} = 0$$
$$= S_{3333}T_{33}^{(4)} = 0$$

$$T_{33}^{(4)} = 0 \tag{4.18}$$

Thus, (4.16), (4.17), and (4.18) provide six equations that define $T_{ij}^{(4)}$.

4.4 Satisfying the strain constraint equations given $T_{ij}^{(4)}$

The stress function ψ will be defined by satisfying the strain constraint equation (4.5). Substitute $T_{ij}^{(4)}$ into the second of (4.2) for $i = 2$, $j = 3$, and for $i = 1$, $j = 3$.

$$\begin{aligned} E_{23}^{(4)} &= 2S_{2313}T_{13}^{(4)} + 2S_{2323}T_{23}^{(4)} \\ E_{13}^{(4)} &= 2S_{1313}T_{13}^{(4)} + 2S_{1323}T_{23}^{(4)} \end{aligned} \tag{i}$$

Substitute (4.16) into (i),

$$\begin{aligned} E_{23}^{(4)} &= 2x_2S_{2313}C_{1313} - 2x_1S_{2313}C_{1323} + 2S_{2313}\frac{\partial\psi}{\partial x_2} + 2x_2S_{2323}C_{2313} \\ &\qquad - 2x_1S_{2323}C_{2323} - 2S_{2323}\frac{\partial\psi}{\partial x_1} \\ E_{13}^{(4)} &= 2x_2S_{1313}C_{1313} - 2x_1S_{1313}C_{1323} + 2S_{1313}\frac{\partial\psi}{\partial x_2} + 2x_2S_{1323}C_{2313} \\ &\qquad - 2x_1S_{1323}C_{2323} - 2S_{1323}\frac{\partial\psi}{\partial x_1} \end{aligned} \tag{ii}$$

Recall from (4.1), when taking into account (4.17) and (4.18) that

$$\begin{aligned} S_{23mn}C_{mn23} &= 2S_{2313}C_{1323} + 2S_{2323}C_{2323} = \frac{1}{2} \\ S_{13mn}C_{mn13} &= 2S_{1313}C_{1313} + 2S_{1323}C_{2313} = \frac{1}{2} \\ S_{23mn}C_{mn13} &= 2S_{2313}C_{1313} + 2S_{2323}C_{2313} = 0 \\ S_{13mn}C_{mn23} &= 2S_{1313}C_{1323} + 2S_{1323}C_{2323} = 0 \end{aligned} \tag{iii}$$

Substitute (iii) into (ii), then

$$E_{23}^{(4)} = -4x_1\frac{1}{2} + 2S_{2313}\frac{\partial\psi}{\partial x_2} - 2S_{2323}\frac{\partial\psi}{\partial x_1}$$
$$E_{13}^{(4)} = 4x_2\frac{1}{2} + 2S_{1313}\frac{\partial\psi}{\partial x_2} - 2S_{1323}\frac{\partial\psi}{\partial x_1} \tag{4.19}$$

If ψ is selected so $E_{23}{}^{(4)}$ is linear in x_1 and so $E_{13}{}^{(4)}$ is linear in x_2 then the constraint equation (4.5) will be satisfied. In order to linearize the strains, ψ may be selected such that,

$$2S_{2313}\frac{\partial\psi}{\partial x_2} - 2S_{2323}\frac{\partial\psi}{\partial x_1} = 0$$
$$2S_{1313}\frac{\partial\psi}{\partial x_2} - 2S_{1323}\frac{\partial\psi}{\partial x_1} = 0 \tag{i}$$

Multiply the second equation of (i) by S_{2323}/S_{1323} and subtract it from the first equation of (i), then

$$2S_{2313}\frac{\partial\psi}{\partial x_2} - 2\frac{S_{1313}S_{2323}}{S_{1323}}\frac{\partial\psi}{\partial x_2} = 0$$
$$\frac{\partial\psi}{\partial x_2} = 0 \quad\Rightarrow\quad \psi \text{ is independent of } \mathrm{x}_2 \tag{ii}$$

Similarly, multiply the second equation of (i) by S_{2313}/S_{1313} and subtract it from the first equation of (i), then

$$-2S_{2323}\frac{\partial\psi}{\partial x_1} + 2\frac{S_{1323}^2}{S_{1313}}\frac{\partial\psi}{\partial x_1} = 0$$
$$\frac{\partial\psi}{\partial x_1} = 0 \quad\Rightarrow\quad \psi \text{ is independent of } \mathrm{x}_1 \tag{iii}$$

Substitute (4.16) into the sixth equation of (4.15), then

$$\frac{\partial\psi}{\partial x_2}n_1 - \frac{\partial\psi}{\partial x_1}n_2 = 0$$
$$\frac{\partial\psi}{\partial x_2} = \frac{\partial\psi}{\partial x_1}\frac{n_2}{n_1} \tag{iv}$$

Equations (ii) and (iii) indicate ψ is constant. Equation (iv) gives the condition for ψ on the boundary, and this is automatically satisfied by the results from equations (ii) and (iii). Therefore, equation (4.16) becomes,

$$T_{13}^{(4)} = C_{1313}x_2 - C_{1323}x_1$$
$$T_{23}^{(4)} = C_{2313}x_2 - C_{2323}x_1 \tag{4.20}$$

4.5 Summarizing the generalized plane strain stresses $T_{ij}^{(p)}$

The stresses that are a function of $\mathbf{W}^{(1)}$ are defined in sections 4.1 and 4.2. The stresses that are a function of $\mathbf{W}^{(2)}$ or $\mathbf{W}^{(3)}$ can be defined following methods similar to those shown in sections 4.1 and 4.2. The stresses that are a function of $\mathbf{W}^{(4)}$ were defined in sections 4.3 and 4.4. The four systems of stresses corresponding to the four displacement vectors $\mathbf{W}^{(p)}$ are as follows.

Consider $\mathbf{W}^{(1)}$; the corresponding stresses defined by (4.7), (4.8), and (4.9) are

$$T_{11}^{(1)} = -x_1C_{1133} + \phi^{(1)}{}_{,22}\,, \quad T_{22}^{(1)} = -x_1C_{2233} + \phi^{(1)}{}_{,11}$$
$$T_{12}^{(1)} = -x_1C_{1233} - \phi^{(1)}{}_{,12}\,, \quad T_{23}^{(1)} = T_{13}^{(1)} = 0 \tag{4.21a}$$
$$T_{33}^{(1)} = x_1S_{3333}{}^{-1} - x_1C_{3333} - \frac{1}{S_{3333}}\left[S_{3311}\frac{\partial^2\phi^{(1)}}{\partial x_2^2} + S_{3322}\frac{\partial^2\phi^{(1)}}{\partial x_1^2} - 2S_{3312}\frac{\partial^2\phi^{(1)}}{\partial x_1\partial x_2}\right]$$

The unknown function $\varphi^{(1)}$ is defined by the boundary value problem (4.14),

$$\frac{\partial^2 \phi^{(1)}}{\partial x_2^2}\left[B_{1111} - \frac{B_{2211}B_{1112}}{B_{2212}}\right] + \frac{\partial^2 \phi^{(1)}}{\partial x_1^2}\left[B_{1122} - \frac{B_{2222}B_{1112}}{B_{2212}}\right] = \left[B^{(1)}\frac{B_{1112}}{B_{2212}} - A^{(1)}\right] on\ \overline{\Sigma}$$

$$\frac{\partial^2 \phi^{(1)}}{\partial x_2^2} = \frac{\partial^2 \phi^{(1)}}{\partial x_1^2}\frac{n_2^2}{n_1^2}\ on\ \Gamma \tag{4.21b}$$

The coefficients $A^{(1)}$ and $B^{(1)}$ are defined by equation (4.13),

$$A^{(1)} = \frac{x_1 R[C_\theta^2 \underline{M}_{1133} + S_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}}$$

$$B^{(1)} = \frac{x_1 R[S_\theta^2 \underline{M}_{1133} + C_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \tag{4.21c}$$

Consider $\mathbf{W}^{(2)}$; the corresponding stresses derived similarly as for $\mathbf{W}^{(1)}$ are

$$T_{11}^{(2)} = -x_2 C_{1133} + \phi^{(2)}{}_{,22}\,, \quad T_{22}^{(2)} = -x_2 C_{2233} + \phi^{(2)}{}_{,11}$$

$$T_{12}^{(2)} = -x_2 C_{1233} - \phi^{(2)}{}_{,12}\,, \quad T_{23}^{(2)} = T_{13}^{(2)} = 0 \tag{4.22a}$$

$$T_{33}^{(2)} = x_2 S_{3333}^{-1} - x_2 C_{3333} - \frac{1}{S_{3333}}\left[S_{3311}\frac{\partial^2 \phi^{(2)}}{\partial x_2^2} + S_{3322}\frac{\partial^2 \phi^{(2)}}{\partial x_1^2} - 2S_{3312}\frac{\partial^2 \phi^{(2)}}{\partial x_1 \partial x_2}\right]$$

The unknown function $\varphi^{(2)}$ is defined by a boundary value problem similar to

(4.14),

$$\frac{\partial^2 \phi^{(2)}}{\partial x_2^2}\left[B_{1111} - \frac{B_{2211}B_{1112}}{B_{2212}}\right] + \frac{\partial^2 \phi^{(2)}}{\partial x_1^2}\left[B_{1122} - \frac{B_{2222}B_{1112}}{B_{2212}}\right] = \left[B^{(2)}\frac{B_{1112}}{B_{2212}} - A^{(2)}\right] on\ \overline{\Sigma}$$

$$\frac{\partial^2 \phi^{(2)}}{\partial x_2^2} = \frac{\partial^2 \phi^{(2)}}{\partial x_1^2}\frac{n_2^2}{n_1^2}\ on\ \Gamma \tag{4.22b}$$

The coefficients $A^{(2)}$ and $B^{(2)}$ are defined by equations similar to (4.13),

$$A^{(2)} = \frac{x_2 R[C_\theta^2 \underline{M}_{1133} + S_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}}$$

$$B^{(2)} = \frac{x_2 R[S_\theta^2 \underline{M}_{1133} + C_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \tag{4.22c}$$

Consider $\mathbf{W}^{(3)}$; the corresponding stresses derived similarly as for $\mathbf{W}^{(1)}$ are

$$T_{11}^{(3)} = -C_{1133} + \phi^{(3)}{}_{,22}\,, \quad T_{22}^{(3)} = -C_{2233} + \phi^{(3)}{}_{,11}$$

$$T_{12}^{(3)} = -C_{1233} - \phi^{(3)}{}_{,12}\,, \quad T_{23}^{(3)} = T_{13}^{(3)} = 0 \tag{4.23a}$$

$$T_{33}^{(3)} = S_{3333}{}^{-1} - C_{3333} - \frac{1}{S_{3333}}\left[S_{3311}\frac{\partial^2\phi^{(3)}}{\partial x_2^2} + S_{3322}\frac{\partial^2\phi^{(3)}}{\partial x_1^2} - 2S_{3312}\frac{\partial^2\phi^{(3)}}{\partial x_1 \partial x_2}\right]$$

The unknown function $\varphi^{(3)}$ is defined by a boundary value problem similar to (4.14),

$$\frac{\partial^2\phi^{(3)}}{\partial x_2^2}\left[B_{1111} - \frac{B_{2211}B_{1112}}{B_{2212}}\right] + \frac{\partial^2\phi^{(3)}}{\partial x_1^2}\left[B_{1122} - \frac{B_{2222}B_{1112}}{B_{2212}}\right] = \left[B^{(3)}\frac{B_{1112}}{B_{2212}} - A^{(3)}\right] on\ \overline{\Sigma}$$

$$\frac{\partial^2\phi^{(3)}}{\partial x_2^2} = \frac{\partial^2\phi}{\partial x_1^2}^{(3)}\frac{n_2^2}{n_1^2}\ on\ \Gamma \tag{4.23b}$$

The coefficients $A^{(3)}$ and $B^{(3)}$ are defined by equations similar to (4.13),

$$A^{(3)} = \frac{R[C_\theta^2 \underline{M}_{1133} + S_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}}$$

$$B^{(3)} = \frac{R[S_\theta^2 \underline{M}_{1133} + C_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \tag{4.23c}$$

Consider $\mathbf{W}^{(4)}$; the corresponding stresses from (4.17), (4.18), and (4.20) are

$$T_{11}^{(4)} = T_{22}^{(4)} = T_{12}^{(4)} = T_{33}^{(4)} = 0$$

$$T_{13}^{(4)} = C_{1313}x_2 - C_{1323}x_1 \tag{4.24a}$$

$$T_{23}^{(4)} = C_{2313}x_2 - C_{2323}x_1$$

5. Determining the Magnitude of $S_{\alpha\beta}$

5.1 Defining $S_{\alpha\beta}$ in terms of S_{33}

Recall the S_{ij} can be written as linear combinations of their components in the directions of the base vectors $\mathbf{e_p}$ (p = 1, 2, 3, 4).

$$S_{ij}(\mathbf{u}) = \sum_{p=1}^{4} a_p S_{ij}(\mathbf{u}^{(p)}) \tag{3.52}$$

$$S_{ij}(\mathbf{u}^{(p)}) = \begin{bmatrix} C_{ij33}x_1 \\ C_{ij33}x_2 \\ C_{ij33} \\ -C_{ij\alpha 3}e_{\alpha\beta}x_\beta \end{bmatrix} \bullet \mathbf{e}_p + T_{ij}(\mathbf{W}^{(p)}) \tag{3.55}$$

Here the $T_{ij}^{(p)} = T_{ij}(\mathbf{W}^{(p)})$ are determined by equations (4.21), (4.22), (4.23), (4.24), and the a_p have yet to be determined.

Consider the case where $i = j = 1$, substitute equations (4.21) to (4.24) into (3.55) and this into (3.52), then

$$\begin{aligned} S_{11}(\mathbf{u}) = \{a_1 C_{1133}x_1 + a_2 C_{1133}x_2 + a_3 C_{1133} - a_4 C_{1113}x_2 + a_4 C_{1123}\} \\ + \{a_1[-C_{1133}x_1 + \phi^{(1)},_{22}] + a_2[-C_{1133}x_2 + \phi^{(2)},_{22}] \\ + a_3[-C_{1133} + \phi^{(3)},_{22}] - a_4[0]\} \end{aligned}$$

$$S_{11}(\mathbf{u}) = \{-a_4 C_{1113}x_2 + a_4 C_{1123}x_1\} + \{a_1\phi^{(1)},_{22} + a_2\phi^{(2)},_{22} + a_3\phi^{(3)},_{22}\}$$

Recall from (C2.11) that $C_{1113} = C_{1123} = 0$, then

$$S_{11}(\mathbf{u}) = \{a_1\phi^{(1)},_{22} + a_2\phi^{(2)},_{22} + a_3\phi^{(3)},_{22}\} \tag{i}$$

The stresses S_{22} and S_{12} can be determined in a similar fashion as for S_{11}, giving

$$S_{22}(\mathbf{u}) = \{a_1\phi^{(1)}{}_{,11} + a_2\phi^{(2)}{}_{,11} + a_3\phi^{(3)}{}_{,11}\}$$
$$S_{12}(\mathbf{u}) = -\{a_1\phi^{(1)}{}_{,12} + a_2\phi^{(2)}{}_{,12} + a_3\phi^{(3)}{}_{,12}\} \qquad \text{(ii)}$$

Consider the case where $i = 1,2$ and $j = 3$. Recall from (C2.11) that $C_{2333} = C_{1333} = 0$. Substitute equations (4.21) to (4.24) into (3.55) and these into (3.52), then

$$S_{23}(\mathbf{u}) = -a_4 C_{2313} x_2 + a_4 C_{2323} x_1$$
$$S_{13}(\mathbf{u}) = -a_4 C_{1313} x_2 + a_4 C_{1323} x_1 \qquad \text{(iii)}$$

Consider the case where $i = j = 3$. Substitute equations (4.21) to (4.24) into (3.55) and these into (3.52), then

$$S_{33}(\mathbf{u}) = \{C_{3333}[a_1 x_1 + a_2 x_2 + a_3] - C_{3313} x_2 + C_{3323} x_1\} + \{C_{3333}[-a_1 x_1 - a_2 x_2 - a_3]$$
$$+ S_{3333}{}^{-1}[a_1 x_1 + a_2 x_2 + a_3] + S_{3333}{}^{-1}[a_1 L_1\phi^{(1)} + a_2 L_1\phi^{(2)} + a_3 L_1\phi^{(3)}]\}$$

Recall L_1 is the differential operator defined in section 4.2,

$$L_1 = S_{3311}\frac{\partial^2}{\partial x_2^2} + S_{3322}\frac{\partial^2}{\partial x_1^2} - 2S_{3312}\frac{\partial^2}{\partial x_1 \partial x_2}$$

Recall from (C2.11) that $C_{3313} = C_{3323} = 0$, therefore,

$$S_{33}(\mathbf{u}) = S_{3333}{}^{-1}\{a_1[x_1 + L_1\phi^{(1)}] + a_2[x_2 + L_1\phi^{(2)}] + a_3[1 + L_1\phi^{(3)}]\} \qquad \text{(iv)}$$

The stresses defined by equations (i) to (iv) are,

$$
\begin{aligned}
&S_{11}(\mathbf{u}) = \{a_1\phi^{(1)}{}_{,22} + a_2\phi^{(2)}{}_{,22} + a_3\phi^{(3)}{}_{,22}\} \\
&S_{22}(\mathbf{u}) = \{a_1\phi^{(1)}{}_{,11} + a_2\phi^{(2)}{}_{,11} + a_3\phi^{(3)}{}_{,11}\} \\
&S_{12}(\mathbf{u}) = -\{a_1\phi^{(1)}{}_{,12} + a_2\phi^{(2)}{}_{,12} + a_3\phi^{(3)}{}_{,12}\} \\
&S_{23}(\mathbf{u}) = -a_4 C_{2313} x_2 + a_4 C_{2323} x_1 \\
&S_{13}(\mathbf{u}) = -a_4 C_{1313} x_2 + a_4 C_{1323} x_1 \\
&S_{33}(\mathbf{u}) = S_{3333}{}^{-1}\{a_1[x_1 + L_1\phi^{(1)}] + a_2[x_2 + L_1\phi^{(2)}] + a_3[1 + L_1\phi^{(3)}]\}
\end{aligned}
\tag{5.1}
$$

Recall from (4.21), (4.22), and (4.23) that $\varphi^{(p)}$ $(p = 1, 2, 3)$ are defined by

$$
\begin{aligned}
&\frac{\partial^2\phi^{(p)}}{\partial x_2^2}\left[B_{1111} - \frac{B_{2211}B_{1112}}{B_{2212}}\right] + \frac{\partial^2\phi^{(p)}}{\partial x_1^2}\left[B_{1122} - \frac{B_{2222}B_{1112}}{B_{2212}}\right] = \left[B^{(p)}\frac{B_{1112}}{B_{2212}} - A^{(p)}\right] \text{ on } \overline{\Sigma} \\
&\frac{\partial^2\phi^{(p)}}{\partial x_2^2} = \frac{\partial^2\phi^{(p)}}{\partial x_1^2}\frac{n_2^2}{n_1^2} \quad \text{on } \Gamma
\end{aligned}
\tag{5.2}
$$

Substitute the second equation of (5.2) into the first, then on the boundary Γ the second derivatives of $\varphi^{(p)}$ are

$$
\begin{aligned}
&\frac{\partial^2\phi^{(p)}}{\partial x_1^2} = \frac{n_1^2[B_{1112}B^{(p)} - B_{2212}A^{(p)}]}{n_2^2[B_{1111}B_{2212} - B_{2211}B_{1112}] + n_1^2[B_{1122}B_{2212} - B_{2222}B_{1112}]} \\
&\frac{\partial^2\phi^{(p)}}{\partial x_2^2} = \frac{n_2^2[B_{1112}B^{(p)} - B_{2212}A^{(p)}]}{n_2^2[B_{1111}B_{2212} - B_{2211}B_{1112}] + n_1^2[B_{1122}B_{2212} - B_{2222}B_{1112}]}
\end{aligned}
\tag{5.3}
$$

Notice from 4.21c, 4.22c, and 4.23c that

$$A^{(\alpha)} = x_\alpha A^{(3)}, \qquad B^{(\alpha)} = x_\alpha B^{(3)} \tag{i}$$

Substitute (i) into (5.3), then

$$\phi^{(\alpha)}{}_{,\beta\beta} = x_\alpha \phi^{(3)}{}_{,\beta\beta} \quad \text{on } \Gamma \tag{5.4}$$

Substitute (5.4) into the first two equations of (5.1) then, S_{11} and S_{22} become

$$\left.\begin{aligned} S_{11}(\mathbf{u}) &= \phi^{(3)},_{22}(a_1x_1 + a_2x_2 + a_3) \\ S_{22}(\mathbf{u}) &= \phi^{(3)},_{11}(a_1x_1 + a_2x_2 + a_3) \end{aligned}\right\} \text{on } \Gamma \tag{5.5}$$

Recall the conditions on the lateral surface Π from equation (3.3),

$$s(\mathbf{u}) = S_{ij}(\mathbf{u})n_j = \mathbf{0}$$

Substitute (5.5) into the boundary conditions on Π for $i = 1$ and solve for S_{12}.

$$S_{11}(\mathbf{u})n_1 + S_{12}(\mathbf{u})n_2 = 0$$

$$S_{12}(\mathbf{u}) = -\frac{n_1}{n_2}\phi^{(3)},_{22}(a_1x_1 + a_2x_2 + a_3) \quad \text{on } \Pi \tag{5.6}$$

Note, if (5.6) and the second of (5.5) are substituted into the boundary conditions on Π for $i = 2$, after taking into account the second equation of (5.2), it can be seen that this boundary condition is also satisfied.

Consider the sixth equation of (5.1). After expanding the differential operator L_1 and then grouping the terms by the coefficients S_{ijkl}, the equation for S_{33} becomes,

$$\begin{aligned} S_{33}(\mathbf{u}) = \frac{1}{S_{3333}}\{&[a_1x_1 + a_2x_2 + a_3] + S_{3311}[a_1\phi^{(1)},_{22} + a_2\phi^{(2)},_{22} + a_3\phi^{(3)},_{22}] \\ &S_{3322}[a_1\phi^{(1)},_{11} + a_2\phi^{(2)},_{11} + a_3\phi^{(3)},_{11}] - 2S_{3312}[a_1\phi^{(1)},_{12} + a_2\phi^{(2)},_{12} + a_3\phi^{(3)},_{12}] \end{aligned} \tag{i}$$

Substitute the equations for S_{11}, S_{22}, and S_{12} from (5.1) into (i), then substitute (5.5) and (5.6) for S_{11}, S_{22}, and S_{12}.

$$S_{33}(\mathbf{u}) = \frac{1}{S_{3333}}[a_1x_1 + a_2x_2 + a_3][1 + S_{3311}\phi^{(3)},_{22} + S_{3322}\phi^{(3)},_{11} - 2S_{3312}\frac{n_1}{n_2}\phi^{(3)},_{22}] \tag{ii}$$

Recall the second equation of (5.2), then write (ii) in terms of $\varphi^{(3)},_{11}$ on Π.

$$S_{33}(\mathbf{u}) = \frac{[a_1x_1 + a_2x_2 + a_3]}{n_1^2 S_{3333}}\{n_1^2 + \phi^{(3)},_{11}[n_2^2 S_{3311} + n_1^2 S_{3322} - 2n_1n_2S_{3312}]\} \quad (5.7)$$

Rearrange (5.7) so the parameter group ($a_1x_1 + a_2x_2 + a_3$) is a function of $S_{33}(\mathbf{u})$ on Π, then

$$[a_1x_1 + a_2x_2 + a_3] = \frac{n_1^2 S_{3333}}{\{n_1^2 + \phi^{(3)},_{11}[n_2^2 S_{3311} + n_1^2 S_{3322} - 2n_1n_2S_{3312}]\}} S_{33}(\mathbf{u}) \quad (5.8)$$

Substitute (5.8) into (5.5) and (5.6) so that $S_{11}(\mathbf{u})$, $S_{22}(\mathbf{u})$, and $S_{12}(\mathbf{u})$ can be written as functions of $S_{33}(\mathbf{u})$ on the boundary Π. After substituting the definitions for $\varphi^{(3)},_{\alpha\alpha}$ from (5.3) and canceling terms, the functions for $S_{11}(\mathbf{u})$, $S_{22}(\mathbf{u})$, and $S_{12}(\mathbf{u})$, as functions of $S_{33}(\mathbf{u})$ are as follows.

$$\left.\begin{aligned}
&S_{12}(\mathbf{u}) = -\frac{n_1}{n_2} S_{11}(\mathbf{u}) \\
&S_{\alpha\alpha}(\mathbf{u}) = \{n_\alpha^2[B_{1112}B^{(3)} - B_{2212}A^{(3)}]S_{33}S_{3333}\}/\{[B_{1111}B_{2212} - B_{2211}B_{1112}]n_2^2 \\
&\qquad + [B_{1122}B_{2212} - B_{2222}B_{1112}]n_1^2 + [B_{1112}B^{(3)} - B_{2212}A^{(3)}] \\
&\qquad * [n_2^2 S_{3311} + n_1^2 S_{3322} - 2n_1n_2S_{3312}]\}
\end{aligned}\right\} \text{ on } \Pi \quad (5.9)$$

5.2 Examining the linear approximation of the strain field

In section 4.2 the strain field $E_{ij}(\mathbf{W}^{(p)})$ was linearized to ensure that the strain constraint equation (4.4) was satisfied. The question remains, where are equations (4.12), which were used to linearize the strains, valid. Since only the second derivatives of $\varphi^{(p)}$ are required to calculate the stresses in (5.1), the equations (4.12) can be considered as a system of three algebraic equations

$$\begin{bmatrix} B_{1111} & B_{1122} & -2B_{1112} \\ B_{1122} & B_{2222} & -2B_{2212} \\ B_{1112} & B_{2212} & -2B_{1212} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \dfrac{-r[C_\theta^2 \underline{M}_{1133} + S_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \\ \dfrac{-r[S_\theta^2 \underline{M}_{1133} + C_\theta^2 \underline{M}_{2233}]}{\underline{S}_{3333}} \\ \dfrac{S_\theta C_\theta r[\underline{M}_{1133} - \underline{M}_{2233}]}{\underline{S}_{3333}} \end{bmatrix} \tag{5.10}$$

where

$$k_1 = \frac{\partial^2 \phi^{(p)}}{\partial x_2^2}, \quad k_2 = \frac{\partial^2 \phi^{(p)}}{\partial x_1^2}, \quad k_3 = \frac{\partial^2 \phi^{(p)}}{\partial x_1 \partial x_2} \qquad (p = 1, 2, 3)$$

The system (5.10) has a solution when the coefficient matrix is nonsingular. The system (5.10) is dependent on the cylindrical coordinates r and θ, and will be nonsingular when the determinant of the coefficient matrix is not equal to zero. In order to calculate the determinant of the coefficient matrix in (5.10), it will be necessary to calculate the coefficients required for the constitutive equations (2.15).

Values for the coefficients in the constitutive equations are not currently available in a form suitable for (2.15). Ultimately, the values of $\varphi^{(p)}$ of the most interest in this analysis will be on the boundary Π. The compliance coefficients from Table 2.2 will be used to approximate the constants necessary for the constitutive equations (2.15) on the boundary of a circular cross section. That is, it will be assume that the reported values for the compliance coefficients in cylindrical coordinates came from an element on the boundary Γ. The constants $\underline{S}_{ijkl}$, and $\underline{M}_{ijkl}$ will be calculated so the compliance coefficients in cylindrical coordinates agree with the reported values in Table 2.2 when the radius of the cross section is entered into the first equation of (2.15).

Let the boundary Γ, of an arbitrary cross section Σ, be the circle $r = 0.3$m. This radius is in the size range of many trees used for tailspars and intermediate supports in cable logging. Recall the following simplifications, which were determined to be allowable in section 2.33. The compatibility conditions at $r = 0$ do not place any restrictions on S_{1122}', S_{3333}', and S_{1212}'. Therefore, these coefficients may be set equal to the published values over the whole cross section, and then

$$\underline{M_{1122}} = \underline{M_{3333}} = \underline{M_{1212}} = 0 \tag{2.16}$$

The fact that the cylindrical base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not unique at $r = 0$ does place the following restrictions on S_{1111}', S_{2222}', S_{1133}', S_{2233}', S_{2323}', and S_{1313}' at $r = 0$

$$\underline{S_{1111}} = \underline{S_{2222}},\ \underline{S_{1133}} = \underline{S_{2233}},\ \underline{S_{2323}} = \underline{S_{1313}} \tag{2.17}$$

To form the coefficients necessary for (2.17) from the values presented in Table 2.2 let

$$\begin{aligned} \underline{S_{1111}} &= \underline{S_{2222}} = \frac{S_{1111}' + S_{1111}'}{2} \\ \underline{S_{1133}} &= \underline{S_{2233}} = \frac{S_{1133}' + S_{2233}'}{2} \\ \underline{S_{2323}} &= \underline{S_{1313}} = \frac{S_{2323}' + S_{1313}'}{2} \end{aligned} \tag{5.11}$$

To form the coefficients $\underline{M_{ijkl}}$, which are necessary for (2.15), let

$$\underline{M_{ijkl}} = \frac{S_{ijkl}' - \underline{S_{ijkl}}}{r}. \quad (r = 0.3\text{m on } \Gamma) \tag{5.12}$$

The coefficients, in cylindrical coordinates, resulting from substituting the values from Table 2.2 into equations (5.11) and (5.12), when taking (2.16) and (2.17) into account, are presented in Table 5.1.

Table 5.1, Compliance coefficients in cylindrical coordinates at $r = 0.3$ m

	Constant at $r = 0$	Coefficient of r	Compliance coefficients calculated by (2.15) at $r = 0.3$ m
	S_{ijkl} (Pa^{-1})	M_{ijkl} (Pa^{-1}m^{-1})	S_{ijkl}' (Pa^{-1})
i=j=k=l= 1	1.607e-09	-8.158e-10	1.362e-09
i=j= 1, k=l= 2	-2.842e-10	0.000	-2.842e-10
i=j= 1, k=l= 3	-1.251e-10	-2.798e-10	-2.090e-10
i=j=k=l= 2	1.607e-09	8.158e-10	1.852e-09
i=j= 2, k=l= 3	-1.251e-10	2.798e-10	-4.116e-11
i=j=k=l= 3	9.259e-11	0.000	9.259e-11
i=k= 2, j=l= 3	1.317e-09	-4.325e-10	1.188e-09
i=k= 1, j=l= 3	1.317e-09	4.325e-10	1.447e-09
i=k= 1, j=l= 2	1.323e-08	0.000	1.323e-08

Recall the reduced strain coefficients are

$$B_{ijkl} = S_{ijkl} - \frac{S_{ij33} S_{kl33}}{S_{3333}} \tag{5.13}$$

The S_{ijkl} are the compliance coefficients transformed into the Cartesian frame by substituting the compliance coefficients from Table 5.1 into the transformation equations (2.19a). To take the determinant of the coefficient matrix of (5.10), substitute the compliance coefficients in Cartesian coordinates into (5.13) and this into (5.10). The reduced strain coefficients will be a function of the cylindrical coordinates r and θ. Therefore, the determinant of the coefficient matrix in (5.10) can be calculated along circles of radius r, where $0 \le r \le 0.3$ (Figure 5.1).

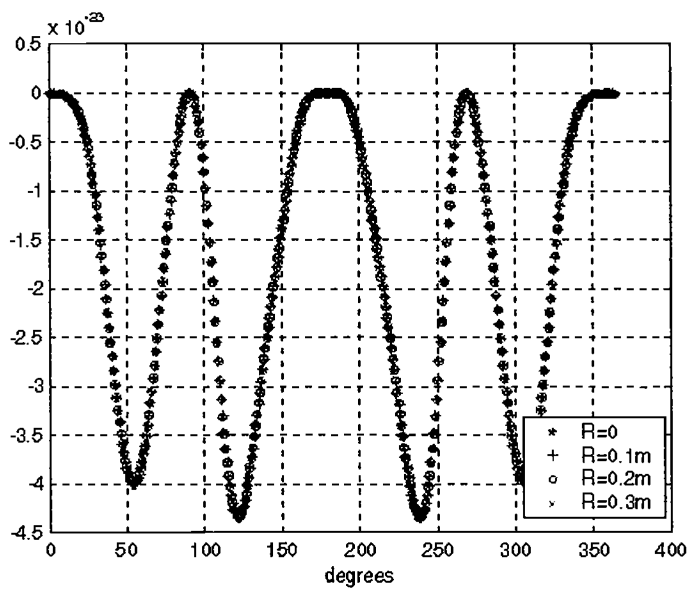

Figure 5.1, The determinant of the coefficient matrix from equation (5.10)

Figure 5.1 indicates that the dependence of the determinant on the cylindrical coordinate r is very small. Let Σ_s be the region where the coefficient matrix is singular, and Σ_d be the region where the coefficient matrix is nonsingular, then

$$\begin{aligned} &\Sigma_s : \left[\frac{n\pi}{2} - c_n\right] \le \theta \le \left[\frac{n\pi}{2} + c_n\right] \quad (n = 1, 2, 3, 4), \\ &\Sigma_d \cap \Sigma_s = \emptyset, \quad \Sigma_d \cup \Sigma_s = \Sigma, \quad \textit{and} \quad \Sigma \cup \Gamma = \overline{\Sigma} \end{aligned} \tag{5.14}$$

In equation (5.14) the c_n are selected so the determinant of the coefficient matrix in (5.10) is zero for θ. The solution for the function $\varphi^{(p)}$ may not be valid in the region where the determinant approaches zero since equation (5.10) is singular there.

Theorem 5.1

If $\varphi_{,\alpha\beta}{}^{(p)} = 0$ at $\mathbf{x} = \mathbf{x}(m_1, m_2, m_3) \in \Sigma_d$, then $\varphi_{,\alpha\beta}{}^{(p)} = 0$ on the closed circular path $x_1^2 + x_2^2 = b^2$, where $m_1^2 + m_2^2 = b^2$, $0 \le m_3 \le h$, and $b \ne 0$.

Proof of Theorem 5.1

Consider the original Cartesian coordinate system ($\mathbf{x}^I$) (Figure 3.1). A second Cartesian coordinate system can be formed by a counterclockwise rotation about the x_3 axis.

$$x_i^{II} = Q_{ij} x_j^I \tag{5.15}$$

Here γ is a positive angle such that $0 < \gamma \le 2\pi$, and

$$Q_{ij} = \begin{bmatrix} Cos(\gamma) & Sin(\gamma) & 0 \\ -Sin(\gamma) & Cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When the problem was originally defined the cross section Σ was assumed circular, and the constitutive equations in the original cylindrical coordinates were independent of θ. Therefore, only the applied loads are dependent on the original placement of the $x_1{}^I$ axis. Equation (5.2) was derived as part of the solution to the auxiliary generalized plane strain problems $T_{ij}{}^{(p)}$ (p = 1, 2, 3), in which the applied loads do not occur.

Consider the $T_{ij}^{(p)}$ (p = 1, 2, 3) problems in cylindrical coordinates. In cylindrical coordinates the transformation (5.15) results in

$$Q_{ij}x_j^I(r,\theta^I,x_3) = \begin{bmatrix} r[Cos(\gamma)Cos(\theta^I) + Sin(\gamma)Sin(\theta^I)] \\ r[Sin(\gamma)Cos(\theta^I) + Cos(\gamma)Sin(\theta^I)] \\ 1 \end{bmatrix} = \begin{bmatrix} rCos(\theta^I + \gamma) \\ rSin(\theta^I + \gamma) \\ 1 \end{bmatrix} \quad (5.16)$$

Consider the case where $\theta^{II} = \theta^I$, then because the applied loads do not occur in the $T_{ij}^{(p)}$ (p = 1, 2, 3) problems,

$$T_{ij}^{(p)II} = T_{ij}^{(P)I} \quad (5.17)$$

However, if $\theta^{II} = \theta^I$ then

$$x_i^{II}(r,\theta^{II},x_3) \neq Q_{ij}x_j^I(r,\theta^I,x_3) \quad (5.18)$$

Thus, the $T_{ij}^{(p)}$ (p = 1, 2, 3) problems when considered in cylindrical coordinates must be independent of θ.

$$\frac{\partial T_{ij}^{(p)}}{\partial\theta} = 0 \quad \text{(in cylindrical coordinates)} \quad (5.19)$$

Recall from equations (4.21a), (4.22a), and (4.23a) that $\varphi,_{\alpha\beta}^{(p)}$ are portions of the $T_{ij}^{(p)}$ stresses. Therefore, by equation (5.19) and still considering the problem in cylindrical coordinates, the functions $\varphi,_{\alpha\beta}^{(p)}$ must also be independent of θ.

$$\frac{\partial \phi,_{\alpha\beta}^{(p)}}{\partial\theta} = 0 \quad \text{(in cylindrical coordinates)} \quad (5.20)$$

Let $\mathbf{R}_d^I$ be a ray extending from the origin to a point on the boundary Γ, where the superscript I indicates it is in the $\mathbf{x}^I$ frame and the subscript d indicates it is in the region Σ_d. Since $\mathbf{R}_d^I$ is in the region where (5.10) is nonsingular, $\varphi,_{\alpha\beta}^{(p)}$ will be defined at all points on $\mathbf{R}_d^I$ except at $r = 0$. The singular point at $r = 0$ is an isolated singular point, which is removable when the problem is considered in the complex domain (section 5.3). Thus, by equation (5.20) if the value of $\varphi,_{\alpha\beta}^{(p)}$ (in cylindrical coordinates) is known at the point on $\mathbf{R}_d^I$ where $r = b$, then the value of $\varphi,_{\alpha\beta}^{(p)}$ (in cylindrical coordinates) will be known for all points on the closed circular path $r = b$.

Consider the particular case where $\varphi,_{\alpha\beta}^{(p)}$ (in cylindrical coordinates) is equal to zero on $\mathbf{R}_d^I$. Equation (5.10) was derived using only the definitions of Cauchy's stress tensor (S_{ij}), the infinitesimal strain tensor (E_{ij}), the elasticity tensor (C_{ijkl}), the compliance tensor (S_{ijkl}), and the constitutive equations (2.6), where all these were in Cartesian coordinates. Fung (1965, pg 48) notes that a tensor equation established in one coordinate system *must hold for all coordinate systems obtained by admissible transformations*. The transformation from cylindrical coordinates to Cartesian coordinates is a rotation, which is an admissible transformation because it is one to one (except at $r = 0$), has continuous first partial derivatives, and the Jacobian determinant is nonzero. Therefore, if

$$\begin{aligned} &\phi,_{\alpha\beta}^{(p)} = 0, \quad \text{on } r = b, \quad b \neq 0 \quad \text{(in cylindrical coordinates)} \quad \text{then,} \\ &\phi,_{\alpha\beta}^{(p)} = 0, \quad \text{on } x_1^2 + x_2^2 = b^2 \quad \text{(in Cartesian coordinates)} \end{aligned} \tag{5.21}$$

Thus, Theorem 5.1 is proven for all $\mathbf{x} \in \overline{\Sigma}$ except at the origin.

5.3 Determining the maximum values of $S_{\alpha\beta}'$

Equation (5.2) is a second order PDE in two independent variables (x_1, x_2). In the general form given by Guenther and Lee (1988, pg 40) equation (5.2) with p = 3 is

$$a(x_1,x_2)\phi^{(3)},_{11}+b(x_1,x_2)\phi^{(3)},_{12}+c(x_1,x_2)\phi^{(3)},_{22}=f(x_1,x_2,\phi^{(3)},\phi^{(3)},_1,\phi^{(3)},_2)\ (5.22)$$

Recall from (5.2) that

$$b=0,\quad a=B_{1111}B_{2212}-B_{2211}B_{1112}$$

$$c=B_{1122}B_{2212}-B_{2222}B_{1112},\quad f=B^{(3)}B_{1112}-A^{(3)}B_{2212}$$

The classification of (5.2) will depend on the sign of the discriminant b^2-ac. Therefore, it will be necessary to calculate the values of the coefficients in (5.2) using values in the range of interest for the compliance coefficients and the radius of the cross section.

Substitute the S_{ijkl}' from Table 5.1 into equations (2.19) and these into (5.13), then calculate a, c, and f on the boundary Γ (Figure 5.2). Figure 5.2 shows that the coefficients a and c always have the same sign except for small regions about $n\pi/2$ (n is an integer), which coincide with Σ_s from (5.14). Since a and c always have the same sign in Σ_d, the discriminant $b^2 - ac$ (where $b = 0$), will always be negative and so equation (5.2) is elliptic in Σ_d.

Note that $a(x_\alpha)$ and $c(x_\alpha)$ are combinations of the trigonometric functions, the coordinates, and the material constants. Since the trigonometric functions are continuously differentiable and may be represented by convergent power series over the whole domain $\overline{B}$, the coefficients $a(x_\alpha)$ and $c(x_\alpha)$ must be analytic in $\overline{B}$.

Therefore, since equation (5.2) is elliptic and $a(x_\alpha)$ and $c(x_\alpha)$ are analytic, then the general equation (5.22) can be reduced to normal (canonical) form by calculating with the complex variables ζ, and η, in which case (5.22) is reduced to,

$$\phi^{(3)}{}_{,12} = G(\varsigma,\eta,\phi,\phi_{,\varsigma},\phi_{,\eta}) \quad \text{on } \Sigma_d \quad \text{(Guenther and Lee 1988 pg 45) (5.23)}$$

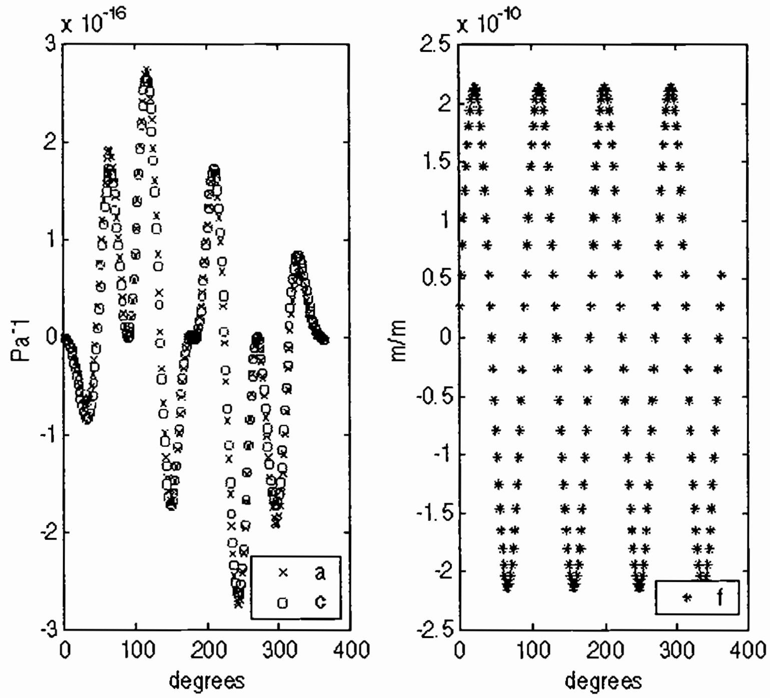

Figure 5.2 Coefficients of equation (5.22) on Γ, $r = 0.3$

Equation (5.23) shows that the $\varphi,_{\alpha\beta}^{(3)}$ can be defined as functions of the complex variables ζ, and η, therefore, they can also be represented by analytic functions of the complex variable $z = x_1 + ix_2$.

$$\phi_{,\alpha\beta}^{(3)} = \phi_{,\alpha\beta}^{(3)}(z) \quad \text{on } \Sigma_d \tag{5.24}$$

Equations (5.1) show that $S_{\alpha\beta}$ are multiples of $\varphi,_{\alpha\beta}^{(3)}$, therefore, by (5.24)

$$S_{\alpha\beta} = S_{\alpha\beta}(z) \qquad \text{on } \Sigma_d \tag{5.25}$$

Recall the objective of this paper is to determine the magnitudes of S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ relative to the normal stress on a transverse cross section. A counter clockwise rotation about the x_3 axis will transform the stress tensor in Cartesian coordinates ($S_{\alpha\beta}$) back to cylindrical coordinates ($S_{\alpha\beta}'$).

$$S_{\alpha\beta}' = Q_{\alpha m} Q_{\beta n} S_{mn} \tag{5.26}$$

Here $Q_{ij} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and θ is the cylindrical coordinate measured from the positive x_1 axis.

Consider $i = 1, 2$ and $j = 3$, then $Q_{ij} = 0$. Therefore (5.26) may be written as

$$S_{\alpha\beta}' = Q_{\alpha\rho} Q_{\beta\gamma} S_{\rho\gamma} \tag{5.27}$$

Equation (5.27) shows that $S_{\alpha\beta}'$ are functions of only S_{11}, S_{22}, and S_{12}. Consider the boundary condition on Π in cylindrical coordinates

$$s(\mathbf{u})' = S_{ij}' n_j = 0 \qquad \text{on } \Pi \tag{5.28}$$

Recall on Π that $n_3 = 0$, therefore, substituting (5.27) into (5.28) results in

$$S_{\alpha\beta}' n_\beta = 0 \quad \text{on } \Pi \tag{5.29}$$

Recall from (2.6) that $S_{11}' = S_{rr}$, and $S_{12}' = S_{r\theta}$, then from (5.29)

$$S_{r\theta} = -\frac{n_1}{n_2} S_{rr} \quad \text{on } \Pi \tag{5.30}$$

Expanding (5.27) for S_{rr} results in

$$S_{rr} = C_\theta^2 S_{11} + 2C_\theta S_\theta S_{12} + S_\theta^2 S_{22} \tag{5.31}$$

Substituting equations (5.5) and (5.6) into (5.31) results in

$$S_{rr} = (a_\rho x_\rho + a_3)(C_\theta^2 \phi^{(3)},_{22} - 2C_\theta S_\theta \frac{n_1}{n_2} \phi^{(3)},_{22} + S_\theta^2 \phi^{(3)},_{11}) \quad \text{on } \Pi$$

Note that $n_1 = C_\theta$ and $n_2 = S_\theta$, therefore, S_{rr} becomes

$$\begin{aligned} S_{rr} &= (a_\rho x_\rho + a_3)(C_\theta^2 \phi^{(3)},_{22} - 2C_\theta^2 \phi^{(3)},_{22} + S_\theta^2 \phi^{(3)},_{11}) \\ &= (a_\rho x_\rho + a_3)(-C_\theta^2 \phi^{(3)},_{22} + S_\theta^2 \phi^{(3)},_{11}) \quad \text{on } \Pi \end{aligned} \tag{5.32}$$

Substitute the second equation of (5.2) into (5.32), rewriting S_{rr} in terms of $\varphi^{(3)},_{11}$ alone

$$S_{rr} = (a_\rho x_\rho + a_3)\frac{(-C_\theta^2 S_\theta^2 \phi^{(3)},_{11} + C_\theta^2 S_\theta^2 \phi^{(3)},_{11})}{C_\theta^2} = 0 \quad \text{on } \Pi \tag{5.33}$$

Substitute (5.33) into (5.30) then

$$S_{r\theta} = -\frac{n_1}{n_2} S_{rr} = 0 \quad \text{on } \Pi \tag{5.34}$$

Recall from (2.6) that $S_{22}' = S_{\theta\theta}$, and $S_{12}' = S_{r\theta}$, then from (5.29) and (5.34)

$$S_{\theta\theta} = -\frac{n_1}{n_2} S_{r\theta} = 0 \quad \text{on } \Pi \tag{5.35}$$

Thus, from (5.33), (5.34), (5.35), and noting from (5.25) that $S_{\alpha\beta}$ maybe written as a function of the complex variable z, then

$$S_{rr}(z) = S_{\theta\theta}(z) = S_{r\theta}(z) = 0 \quad \text{on } \Pi \tag{5.36}$$

Recall that $\varphi^{(3)}{}_{\alpha\beta}$ is not defined on some segments of Γ, however, by Theorem 5.1 the results of (5.36) can be extended to all of Γ. Thus, the maximum value of

$S_{\alpha\beta}'$ is zero on Π. Sokolnikoff and Redheffer (1958, pg 547) note that the modulus of the sum of complex numbers is never greater than the sum of the moduli. Therefore, if $S_{\alpha\beta}'$ has a maximum value M on the boundary Γ, then

$$\left|\oint_{\Gamma} S_{\alpha\beta}'(z)dz\right| \leq \oint_{\Gamma}\left|S_{\alpha\beta}'(z)\right|\left|dz\right| \leq M\oint_{\Gamma}\left|dz\right| = M\oint_{\Gamma} ds = ML \tag{5.37}$$

where L is the length of the boundary Γ.

Let the path Γ be the circle $\left|z - z_0\right| = R$, where R is the radius of Σ, then by Cauchy's Integral Formula

$$\left|S_{\alpha\beta}'(z_0)\right| \leq \frac{1}{2\pi}\frac{M}{R}2\pi R = M \text{ (Sokolnikoff and Redheffer 1958, pg 558)} \tag{5.38}$$

Sokolnikoff and Redheffer further suggest that (5.38) is true for all points z within the circle Γ if in (5.22) $f = B^{(3)}B_{1112} - A^{(3)}B_{2212}$ is always positive. However, Figure 5.2 shows that f is not always positive. Sperb (1981, pg 19) notes when the maximum principle is violated by $f < 0$, it is still possible to show that $\varphi^{(3)}(z)$ may not attain a maximum on the interior of Γ provided the coefficients are bounded. Figure 5.2 does show that the coefficients a, c, and f are bounded, therefore, by (5.36) and (5.38)

$$\left|S_{\alpha\beta}'(z)\right| \leq 0 \quad \text{on } \overline{\Sigma} \tag{5.39}$$

Equation (5.39) indicates that S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$ equal zero in $\overline{B}$ given the assumptions made in this analysis.

6. Summary

Consider a cylindrical section of a tree (Figure 2.1) that is solid and orthotropic in cylindrical coordinates. The x_3 axis is an axis of symmetry in cylindrical coordinates and it falls within $\overline{B}$. The cylindrical base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not unique at $r = 0$, therefore, the constitutive equations must allow for non-unique strains in these directions at $r = 0$. However, published values of the engineering constants for Douglas-fir (Table 2.1) do not produce a compliance tensor that allows for non-unique strains at $r = 0$. If the compliance coefficients are assumed to depend on r then it is possible to propose constitutive equations that allow non-unique strains at $r = 0$, while still allowing orthotropic behavior at points where $r \neq 0$ (equation (2.15)).

The constitutive equation (2.15) may be transformed into Cartesian coordinates. Only the plane perpendicular to the x_3 axis remains as a plane of symmetry after transforming (2.15) into Cartesian coordinates. With the only plane of symmetry being perpendicular to the x_3 axis, there will now be interactions between the E_{12} shear strain and the S_{11}, S_{22}, and S_{33} normal stresses. In addition, the compliance and elasticity tensors become functions of x_1, and x_2.

The generalized plane strain problem $T_{ij} = T_{ij}(\mathbf{W})$ where $\mathbf{W} = \mathbf{W}(x_1, x_2)$, can be separated from the three-dimensional stress problem $S_{ij} = S_{ij}(\mathbf{u})$ where $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$. The unknown coefficients a_p ($p = 1, 2, 3, 4$) in the equations for the S_{ij}, may be found from the system (3.61) if the functions for the T_{ij} can be found. It is possible to find functions for the T_{ij} that satisfy the boundary conditions, stress equilibrium equations, and strain constraints if $E_{ij} = E_{ij}(\mathbf{W})$ are assumed to be linear functions of x_1 and x_2. Equation (5.2), which defines the unknown component of $T_{\alpha\beta}$, is shown to be elliptic when using the engineering constants for Douglas-fir. The result of (5.2) being elliptic is that the $T_{\alpha\beta}$, and therefore the $S_{\alpha\beta}$, maybe represented as analytic functions of the complex variable z in regions where (5.2) is defined.

By Theorem 5.1 and equation (5.25) the functions for $S_{\alpha\beta}$ are shown to be analytic over $\overline{\Sigma}$, except possibly at $r = 0$. Equation (5.36) shows that $S_{\alpha\beta}' = 0$ on Γ. Since $S_{\alpha\beta}' = 0$ on Γ the Maximum Modulus Theorem, Cauchy's Integral Formula, and the Maximum Value Theorem may be employed to prove that $S_{\alpha\beta}' = 0$ on $\overline{\Sigma}$.

Thus, for a cylindrical cantilever beam given: a constitutive equation in cylindrical coordinates that is a linear function of r, generalized plane strains that are linear in x_1 and x_2, the engineering constants for Douglas fir (Table 2.1), and ignoring body loads, then $S_{\alpha\beta}' = 0$ on $\overline{\Sigma}$. This result indicates it is unlikely that the S_{rr}, $S_{\theta\theta}$, or $S_{r\theta}$ stresses will become limiting before the S_{33} stress does in a tree under loads independent of x_3. Therefore, S_{33} calculated by elementary beam theory will be a satisfactory estimate of the load bearing capacity of a tree given small strains.

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Appendices

Appendix A: A rigid body motion given two constant vectors

Let $\mathbf{u}_{,3}$ be a smooth vector valued function on $\overline{B}$. If $\mathbf{u}_{,3}$ is also a rigid body displacement field then $\nabla\mathbf{u}_{,3}$ is constant (Gurtin 1981, pg 36). A deformation $\mathbf{u}_{,3}$ with $\nabla\mathbf{u}_{,3}$ constant is a homogeneous deformation, and admits the form

$$\mathbf{u}_{,3}(\mathbf{p}) = \mathbf{u}_{,3}(\mathbf{q}) + \nabla\mathbf{u}_{,3}(\mathbf{p}-\mathbf{q}) \tag{A1}$$

for all points $\mathbf{p}$ and $\mathbf{q}$ in $\overline{B}$ (Gurtin 1981, pg 42).

Proof

Choose $\mathbf{p},\mathbf{q} \in \overline{B}$. Since $\overline{B}$ is connected there is a curve $\mathbf{c}$ in $\overline{B}$ from $\mathbf{q}$ to $\mathbf{p}$, where $\mathbf{c} = \mathbf{c}(\sigma)$ and σ is some parameter. At the point $\mathbf{q}$ on $\mathbf{c}$, $\sigma = 0$, and at the point $\mathbf{p}$ on $\mathbf{c}$, $\sigma = 1$ Therefore,

$$\mathbf{u}_{,3}(\mathbf{p}) - \mathbf{u}_{,3}(\mathbf{q}) = \int_0^1 \frac{d}{d\sigma}\mathbf{u}_{,3}(\mathbf{c}(\sigma))\,d\sigma \tag{A2}$$

Note, since $\mathbf{u}_{,3}$ is continuous

$$\frac{d}{d\sigma}\mathbf{u}_{,3}(\mathbf{c}(\sigma)) = \begin{bmatrix} \frac{\partial u_{1,3}}{\partial c_1}\frac{\partial c_1}{\partial \sigma} & \frac{\partial u_{1,3}}{\partial c_2}\frac{\partial c_2}{\partial \sigma} & \frac{\partial u_{1,3}}{\partial c_3}\frac{\partial c_3}{\partial \sigma} \\ \frac{\partial u_{2,3}}{\partial c_1}\frac{\partial c_1}{\partial \sigma} & \frac{\partial u_{2,3}}{\partial c_2}\frac{\partial c_2}{\partial \sigma} & \frac{\partial u_{2,3}}{\partial c_3}\frac{\partial c_3}{\partial \sigma} \\ \frac{\partial u_{3,3}}{\partial c_1}\frac{\partial c_1}{\partial \sigma} & \frac{\partial u_{3,3}}{\partial c_2}\frac{\partial c_2}{\partial \sigma} & \frac{\partial u_{3,3}}{\partial c_3}\frac{\partial c_3}{\partial \sigma} \end{bmatrix}$$

$$\frac{d}{d\sigma}\mathbf{u}_{,3}(\mathbf{c}(\sigma)) = [\nabla\mathbf{u}_{,3}(\mathbf{c}(\sigma))]\dot{\mathbf{c}}(\sigma) \quad \left(where\ \dot{\mathbf{c}} = \frac{d}{d\sigma}\mathbf{c}(\sigma)\right) \tag{A3}$$

Since $\nabla \mathbf{u},_3$ is constant, then

$$\mathbf{u},_3(\mathbf{p}) - \mathbf{u},_3(\mathbf{q}) = \nabla \mathbf{u},_3 \int_0^1 \dot{\mathbf{c}}(\sigma)\, d\sigma$$

$$\mathbf{u},_3(\mathbf{p}) - \mathbf{u},_3(\mathbf{q}) = \nabla \mathbf{u},_3 \left[\mathbf{c}(1) - \mathbf{c}(0)\right]$$

$$\mathbf{u},_3(\mathbf{p}) = \mathbf{u},_3(\mathbf{q}) + \nabla \mathbf{u},_3 \left[\mathbf{p} - \mathbf{q}\right] \qquad \text{(A4)}$$

To write the second term on the R.H.S of (A4) as a cross product using the axial vector $\boldsymbol{\omega}$, $\nabla \mathbf{u},_3$ must be skew. To prove $\nabla \mathbf{u},_3$ is skew assume there are only small deformations (Gurtin 1981, pg 54).

Let **C** and **B** be the right and left Cauchy-Green strain tensors, then

$$\begin{aligned} \mathbf{C} &= \mathbf{I} + \nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u}^T \nabla \mathbf{u} \\ \mathbf{B} &= \mathbf{I} + \nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u} \nabla \mathbf{u}^T \end{aligned} \qquad \text{(A5)}$$

where $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$.

For the rigid body displacement field $\mathbf{u},_3$, $\mathbf{C} = \mathbf{B} = \mathbf{I}$, therefore,

$$\nabla \mathbf{u},_3 + \nabla \mathbf{u},_3{}^T + \nabla \mathbf{u},_3 \nabla \mathbf{u},_3{}^T = \mathbf{0} \qquad \text{(A6)}$$

Assuming infinitesimal strains, where the infinitesimal strain tensor is

$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u},_3 + \nabla \mathbf{u},_3{}^T)$, then

$$\begin{aligned} \mathbf{C} &= \mathbf{I} + 2\mathbf{E} + \nabla \mathbf{u}_3{}^T \nabla \mathbf{u}_3 \\ \mathbf{B} &= \mathbf{I} + 2\mathbf{E} + \nabla \mathbf{u}_3 \nabla \mathbf{u}_3{}^T \end{aligned} \qquad \text{(Gurtin 1981, pg 54)} \qquad \text{(A7)}$$

Let $f_\varepsilon (0 < \varepsilon < \varepsilon_0)$ be a one-parameter family of deformations with $|\nabla \mathbf{u},_3| = \varepsilon$ (Gurtin 1981, pg 55), then

$$2\mathbf{E}_\varepsilon = \mathbf{C}_\varepsilon - \mathbf{I} + O(\varepsilon) = \mathbf{B}_\varepsilon - \mathbf{I} + O(\varepsilon) \; as \; \varepsilon \to 0 \tag{A8}$$

If f_ε is rigid then $\mathbf{C}_\varepsilon = \mathbf{B}_\varepsilon = \mathbf{I}$, therefore, from (A7)

$$\nabla \mathbf{u},_3{}^T \nabla \mathbf{u},_3 = \mathbf{C} - \mathbf{I} - 2\mathbf{E} = -2\mathbf{E} = O(\varepsilon) \tag{A9}$$

Substitute (A9) into (A6),

$$\nabla \mathbf{u},_3 = -\nabla \mathbf{u},_3{}^T + O(\varepsilon) \tag{A10}$$

Thus, to within a given error O(ε) the terms $2\mathrm{E}_\varepsilon$, $\mathbf{C}_\varepsilon$**-I**, and $\mathbf{B}_\varepsilon$**-I** coincide; therefore, $\nabla \mathbf{u},_3$ must be skew symmetric.

$$\nabla \mathbf{u},_3 = \begin{bmatrix} 0 & (u,_3(x_1)),_2 & -(u,_3(x_1)),_3 \\ -(u,_3(x_2)),_1 & 0 & (u,_3(x_2)),_3 \\ (u,_3(x_3)),_1 & -(u,_3(x_3)),_2 & 0 \end{bmatrix} \tag{A11}$$

The axial vector of (A11), when noting (A10), is

$$\omega_i = -\frac{1}{2} e_{ijk} \nabla \mathbf{u},_{3jk} \, \mathbf{e}_i$$

$$\omega = -\frac{1}{2}\begin{bmatrix} (\mathbf{u},_3(x_2)),_3 + (\mathbf{u},_3(x_3)),_2 \\ (\mathbf{u},_3(x_3)),_1 + (\mathbf{u},_3(x_1)),_3 \\ (\mathbf{u},_3(x_1)),_2 + (\mathbf{u},_3(x_2)),_1 \end{bmatrix} = -\begin{bmatrix} (\mathbf{u},_3(x_2)),_3 \\ (\mathbf{u},_3(x_3)),_1 \\ (\mathbf{u},_3(x_1)),_2 \end{bmatrix} \tag{A12}$$

(Note, $(\mathbf{u},_3(x_2)),_3 = (\mathbf{u},_3(x_3)),_2$ by (A10))

Using (A12), (A4) can be re-written as

$$\begin{aligned} \mathbf{u},_3(\mathbf{p}) &= \mathbf{u},_3(\mathbf{q}) + \nabla u,_3 [\mathbf{p} - \mathbf{q}] \\ \mathbf{u},_3(\mathbf{p}) &= \mathbf{u},_3(\mathbf{q}) + \omega \times [\mathbf{p} - \mathbf{q}] \end{aligned} \tag{A13}$$

If $\mathbf{q}$ is the position vector $\mathbf{x}_0 = \mathbf{0}$ then $\mathbf{p} - \mathbf{q} = \mathbf{x}$, and since $\mathbf{u}_{,3}$ is a rigid body displacement $\mathbf{u}_{,3}(0)$ is a constant vector. Note, $\boldsymbol{\omega}$ is constant because $\nabla\mathbf{u}_{,3}$ is constant. Thus, (A13) can be re-written as

$$\mathbf{u}_{,3}(\mathbf{p}) = \boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{x} \tag{A14}$$

where $\boldsymbol{\alpha} = \mathbf{u}_{,3}(0)$ and $\boldsymbol{\omega}$ are constant vectors.

Appendix B: Notation Conventions

Mathematical Conventions	Symbol
Continuous first partial derivatives	C^1
Continuous second partial derivatives	C^2
Cross product	$\times$
Cylindrical basis	'
Dot product	$\bullet$
Gradient	∇
Indices	
(1, 2)	(Greek symbols)
(1, 2, 3, …)	(Latin symbols)
Kronecker delta function	$\delta_{ij} = \begin{cases} i = j \Rightarrow 1 \\ i \neq j \Rightarrow 0 \end{cases}$
Partial derivatives	$\dfrac{\partial u}{\partial x_1} = u,_1$
Summation notation	A_{ii} (i = 1 to 3) = $A_{11} + A_{22} + A_{33}$
Tensor (vector notation)	$\underline{\mathbf{A}}$ (bold type)
Tensor (indicial notation)	A_{ij}
Permutation symbol	$e_{ijk} = \begin{cases} e_{123} = e_{312} = e_{231} = 1 \\ e_{321} = e_{132} = e_{213} = -1 \\ e_{111} = e_{112} = \ldots = 0 \end{cases}$

Mathematical Conventions	Symbol
Set notation	
Element of	$\in$
Empty set	$\emptyset$
Intersection	$\cap$
Subset of	$\subset$
Union	$\cup$
Total derivative	$\dfrac{du(s)}{ds} = \dot{u}$
Transpose of a matrix	$A_{ij}{}^{T}$
Trigonometric functions	
Cos(θ)	C_θ
Sin(θ)	S_θ
Two-dimensional alternator	$e_{\alpha\beta} = \begin{cases} \alpha = 1, \beta = 2 \Rightarrow 1 \\ \alpha = 2, \beta = 1 \Rightarrow -1 \\ \alpha = \beta \Rightarrow 0 \end{cases}$
Vector (vector notation)	$\mathbf{u}$ (bold type)
Vector (indicial notation)	u_i

Constants and variables	Symbol	Page introduced
Applied loads		
Force (resultant)	F_i, $\mathbf{F}$	25
Force (vector function of $\mathbf{u}$)	$f_i(\mathbf{u})$, $\mathbf{f}(\mathbf{u})$	25
Moment (resultant)	M_i, $\mathbf{M}$	25
Moment (vector function of $\mathbf{u}$)	$m_i(\mathbf{u})$, $\mathbf{m}(\mathbf{u})$	25
Cartesian coordinates	x_i, $\mathbf{x}$	
Cauchy's Stress tensor	S_{ij}, $S_{ij}(\mathbf{u})$, $\mathrm{S}(\mathbf{u})$	11
Complex variable	z	77
Compliance tensor	S_{ijkl}, $\underline{\mathbf{S}}$	11
Constitutive equation constants		
First of (2.15)	$\underline{S_{ijkl}}, \underline{M_{ijkl}}$	18
Second of (2.15)	$\underline{C_{ijkl}}, \underline{K_{ijkl}}$	18
Constant vectors	A_i, $\mathbf{A}$, B_i, $\mathbf{B}$	34
Cross section (transverse)		
Area	A	48
Moment of inertia	I	48
Interior	Σ	22
Boundary	Γ	23
Closure	$\overline{\Sigma}$	71
Interior where (5.2) defined	Σ_d	71
Interior where (5.2) not defined	Σ_s	71

Constants and variables	Symbol	Page introduced
Cylinder		
Interior	B	22
Boundary	∂B	22
Closure	$\overline{B}$	22
Lateral surface	Π	23
Length	h	22
Segment of boundary	γ_i	24
Cylindrical coordinates	$r, \theta, \underline{x_3}$	4
Displacement vector		
Function of x_i	u_i, $\mathbf{u}$	11
Function of x_α	W_α, $\mathbf{W}$	35
Differential operators	L_1, L_2	53
Elasticity tensor	C_{ijkl}, $\underline{\mathbf{C}}$	11
Generalized plain strain stresses	T_{ij}, $T_{ij}(\mathbf{W})$	33
Imaginary unit	i	77
Infinitesimal strain tensor	E_{ij}, $E_{ij}(\mathbf{u})$, $\mathrm{E}(\mathbf{u})$	11
Normal vector	n_i, $\mathbf{n}$	25
Parameter groups from (4.12)		
(p = 1, 2, 3)	$A^{(p)}, B^{(p)}, C^{(p)}$	55
Prescribed displacement field	$\hat{\mathbf{u}}$	24
Prescribed stress field	$\hat{\mathbf{s}}$	24
Problem (loads independent of x_3)	P_1	24

Constants and variables	Symbol	Page introduced
Problem (flexure)	P_2	24
Reduced Strain Coefficients	B_{ijkl}	54
Right hand side of (3.34)		
First three equations	g_i	39
Last three equations	H_i	39
Right hand side of (3.61)	G_i	47
Rigid body displacements		
Translation	u_{i0}	35
Rotation angle about an axis	w_i	35
Sets		
Equilibrium displacement fields	D	27
Rigid body displacement fields	R	33
Continuous displacement field	J	34
Solution to P_1	K_1	24
Solution to P_2	K_2	24
Stress function for $T_{ij}^{(p)}$ ($p = 1, 2, 3$)	$\varphi^{(p)}$, $\phi^{(p)}$	51
Stress function for $T_{ij}^{(p)}$ ($p = 4$)	$\psi^{(p)}$, $\psi^{(p)}$	57
Stress vector	s_i, $s_i(\mathbf{u})$, $\mathbf{s}$	24
Transformation tensor (rotation)	Q_{ij}	15
Unit base vector	$\mathbf{e_p}$	42
Vector of constants ($p = 1, 2, 3, 4$)	a_p, $\mathbf{a}$	37
Zero vector	$\mathbf{0}$	25

Appendix C: Transformation equations

The equations transformation the elasticity coefficients in cylindrical coordinates (C_{ijkl}') to Cartesian coordinates (C_{ijkl}) are

$$C_{1111} = C_\theta^4 C_{1111}' + 2C_\theta^2 S_\theta^2 C_{1122}' + 4C_\theta^2 S_\theta^2 C_{1212}' + S_\theta^4 C_{2222}'$$

$$C_{2222} = S_\theta^4 C_{1111}' + 2C_\theta^2 S_\theta^2 C_{1122}' + 4C_\theta^2 S_\theta^2 C_{1212}' + C_\theta^4 C_{2222}'$$

$$C_{3333} = C_{3333}'$$

$$C_{2323} = S_\theta^2 C_{1313}' + C_\theta^2 C_{2323}'$$

$$C_{1313} = C_\theta^2 C_{1313}' + S_\theta^2 C_{2323}'$$

$$C_{1212} = C_\theta^2 S_\theta^2 [C_{1111}' - 2C_{1122}' + C_{2222}' - 2C_{1212}'] + [C_\theta^4 + S_\theta^4] C_{1212}'$$

$$C_{1122} = C_\theta^2 S_\theta^2 C_{1111}' + C_\theta^4 C_{1122}' - 4C_\theta^2 S_\theta^2 C_{1212}' + S_\theta^4 C_{2211}' + C_\theta^2 S_\theta^2 C_{2222}'$$

$$C_{1133} = C_\theta^2 C_{1133}' + S_\theta^2 C_{2233}'$$

$$C_{1123} = 0$$

$$C_{1113} = 0$$

$$C_{1112} = -C_\theta S_\theta [C_\theta^2 C_{1111}' - C_\theta^2 C_{1122}' - 2C_\theta^2 C_{1212}' + 2S_\theta^2 C_{1212}' + S_\theta^2 C_{1122}' - S_\theta^2 C_{2222}']$$

$$C_{2233} = S_\theta^2 C_{1133}' + C_\theta^2 C_{2233}'$$

$$C_{2223} = 0$$

$$C_{2213} = 0$$

$$C_{2212} = -C_\theta S_\theta [S_\theta^2 C_{1111}' - S_\theta^2 C_{1122}' - 2S_\theta^2 C_{1212}' + 2C_\theta^2 C_{1212}' + C_\theta^2 C_{1122}' - C_\theta^2 C_{2222}']$$

$$C_{3323} = 0$$

$$C_{3313} = 0 \qquad \text{(C2.11)}$$

$$C_{3312} = -C_\theta S_\theta [C_{3311}' - C_{3322}']$$

$$C_{2313} = -C_\theta S_\theta [C_{1313}' - C_{2323}']$$

$$C_{2312} = 0$$

$$C_{1312} = 0$$

The equations transformation the compliance coefficients in cylindrical coordinates (S_{ijkl}') to Cartesian coordinates (S_{ijkl}) are

$$S_{1111} = C_\theta^4 S_{1111}' + 2C_\theta^2 S_\theta^2 S_{1122}' + 4C_\theta^2 S_\theta^2 S_{1212}' + S_\theta^4 S_{2222}'$$

$$S_{2222} = S_\theta^4 S_{1111}' + 2C_\theta^2 S_\theta^2 S_{1122}' + 4C_\theta^2 S_\theta^2 S_{1212}' + C_\theta^4 S_{2222}'$$

$$S_{3333} = S_{3333}'$$

$$S_{2323} = S_\theta^2 S_{1313}' + C_\theta^2 S_{2323}'$$

$$S_{1313} = C_\theta^2 S_{1313}' + S_\theta^2 S_{2323}'$$

$$S_{1212} = C_\theta^2 S_\theta^2 [S_{1111}' - 2S_{1122}' + S_{2222}' - 2S_{1212}'] + [C_\theta^4 + S_\theta^4] S_{1212}'$$

$$S_{1122} = C_\theta^2 S_\theta^2 S_{1111}' + C_\theta^4 S_{1122}' - 4C_\theta^2 S_\theta^2 S_{1212}' + S_\theta^4 S_{2211}' + C_\theta^2 S_\theta^2 S_{2222}'$$

$$S_{1133} = C_\theta^2 S_{1133}' + S_\theta^2 S_{2233}'$$

$$S_{1123} = 0$$

$$S_{1113} = 0$$

$$S_{1112} = -C_\theta S_\theta [C_\theta^2 S_{1111}' - C_\theta^2 S_{1122}' - 2C_\theta^2 S_{1212}' + 2S_\theta^2 S_{1212}' + S_\theta^2 S_{1122}' - S_\theta^2 S_{2222}']$$

$$S_{2233} = S_\theta^2 S_{1133}' + C_\theta^2 S_{2233}'$$

$$S_{2223} = 0$$

$$S_{2213} = 0$$

$$S_{2212} = -C_\theta S_\theta [S_\theta^2 S_{1111}' - S_\theta^2 S_{1122}' - 2S_\theta^2 S_{1212}' + 2C_\theta^2 S_{1212}' + C_\theta^2 S_{1122}' - C_\theta^2 S_{2222}']$$

$$S_{3323} = 0$$

$$S_{3313} = 0 \qquad \text{(C2.12)}$$

$$S_{3312} = -C_\theta S_\theta [S_{3311}' - S_{3322}']$$

$$S_{2313} = -C_\theta S_\theta [S_{1313}' - S_{2323}']$$

$$S_{2312} = 0$$

$$S_{1312} = 0$$

Appendix D: Constraints on the strains resulting from the generalized plane strain problem

Recall the three-dimensional displacement field $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$, where $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$. Since the displacements have continuous second partial derivatives with respect to the coordinates, the order that the derivatives are taken may be interchanged for derivatives up to the third partial derivative. Therefore, given the three dimensional displacement field $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$, the strain compatibility equations are

$$\begin{aligned}
&E_{11}(\mathbf{u})_{,23} = -E_{23}(\mathbf{u})_{,11} + E_{13}(\mathbf{u})_{,21} + E_{12}(\mathbf{u})_{,13} \\
&E_{22}(\mathbf{u})_{,31} = -E_{31}(\mathbf{u})_{,22} + E_{12}(\mathbf{u})_{,23} + E_{23}(\mathbf{u})_{,12} \\
&E_{33}(\mathbf{u})_{,12} = -E_{12}(\mathbf{u})_{,33} + E_{23}(\mathbf{u})_{,13} + E_{13}(\mathbf{u})_{,23} \\
&2E_{12}(\mathbf{u})_{,12} = E_{11}(\mathbf{u})_{,22} + E_{22}(\mathbf{u})_{,11} \\
&2E_{23}(\mathbf{u})_{,23} = E_{22}(\mathbf{u})_{,33} + E_{33}(\mathbf{u})_{,22} \\
&2E_{13}(\mathbf{u})_{,13} = E_{33}(\mathbf{u})_{,11} + E_{11}(\mathbf{u})_{,33} \quad \text{(Fung 1994, pg 149)}
\end{aligned} \tag{D1}$$

Substituting the strain equations (3.29) into (D1) results in

$$\begin{aligned}
&0 = -\frac{1}{2}W_{3,211} + \frac{1}{2}W_{3,121} + 0 \\
&0 = -\frac{1}{2}W_{3,122} + 0 + \frac{1}{2}W_{3,212} \\
&0 = 0 + 0 + 0 \\
&[W_{1,212} + W_{2,112}] = [W_{1,122} + W_{2,211}] \\
&0 = 0 + 0 \\
&0 = 0 + 0
\end{aligned} \tag{D2}$$

Equation (D2) indicates that the displacement vector $\mathbf{W} = \mathbf{W}(x_1, x_2)$ will satisfy the strain compatibility equations (D1) when represented by any function such that $\mathbf{W} \in C^1(\overline{B}) \cap C^2(B)$. Therefore, it is possible to consider $\mathbf{W}$ as the following second-degree polynomial in two variables (x_1 and x_2),

$$\begin{aligned} W_1 &= \alpha_1 x_1^2 + \beta_1 x_1 x_2 + \lambda_1 x_2^2 \\ W_2 &= \alpha_2 x_1^2 + \beta_2 x_1 x_2 + \lambda_2 x_2^2 \\ W_3 &= \alpha_3 x_1^2 + \beta_3 x_1 x_2 + \lambda_3 x_2^2 \end{aligned} \tag{D3}$$

Substituting (D3) into the strains from (3.29) and recalling (3.31) results in

$$\begin{aligned} E_{11}(\mathbf{u}) &= E_{11}(\mathbf{W}) \quad \Rightarrow & E_{11}(\mathbf{W}) &= 2\alpha_1 x_1 + \beta_1 x_2 \\ E_{22}(\mathbf{u}) &= E_{22}(\mathbf{W}) \quad \Rightarrow & E_{22}(\mathbf{W}) &= \beta_2 x_1 + 2\lambda_2 x_2 \\ E_{33}(\mathbf{u}) &= a_\rho x_\rho + a_3 \quad \Rightarrow & E_{33}(\mathbf{W}) &= 0 \\ E_{23}(\mathbf{u}) &= \frac{1}{2} a_4 x_1 + E_{23}(\mathbf{W}) \quad \Rightarrow & E_{23}(\mathbf{W}) &= \frac{1}{2}[\beta_3 x_1 + 2\lambda_3 x_2] \\ E_{13}(\mathbf{u}) &= -\frac{1}{2} a_4 x_2 + E_{13}(\mathbf{W}) \quad \Rightarrow & E_{13}(\mathbf{W}) &= \frac{1}{2}[2\alpha_3 x_1 + \beta_3 x_2] \\ E_{12} &= E_{12}(\mathbf{W}) \quad \Rightarrow & E_{12}(\mathbf{W}) &= \frac{1}{2}[\beta_1 x_2 + 2\lambda_1 x_2 + 2\alpha_2 x_1 + \beta_2 x_2] \end{aligned} \tag{D4}$$

Equation (D4) can be simplified by combining coefficients, then

$$\begin{aligned} E_{11}(\mathbf{W}) &= \alpha_{11} x_1 + \beta_{11} x_2 \\ E_{22}(\mathbf{W}) &= \alpha_{22} x_1 + \beta_{22} x_2 \\ E_{33}(\mathbf{W}) &= \alpha_{33} x_1 + \beta_{33} x_2 \\ E_{23}(\mathbf{W}) &= \alpha_{23} x_1 + \beta_{23} x_2 \\ E_{13}(\mathbf{W}) &= \alpha_{13} x_1 + \beta_{13} x_2 \\ E_{12}(\mathbf{W}) &= \alpha_{12} x_1 + \beta_{12} x_2 \end{aligned} \tag{D5}$$

Writing (D5) more compactly,

$$E_{ij}(\mathbf{W}) = \alpha_{ij}x_1 + \beta_{ij}x_2 \quad (\alpha_{33} = \beta_{33} = 0) \tag{D6}$$

Thus, the strains that are functions of **W** can be written as linear functions of x_1 and x_2.

Recall the displacement equations (3.50), and the definition of the infinitesimal strain tensor. Then the strains can be written as

$$E_{ij}(\mathbf{u}) = \frac{1}{2}[u_i,_j + u_j,_i]$$

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\left[\sum_{p=1}^{4} a_p u_i^{(p)}\right],_j + \frac{1}{2}\left[\sum_{p=1}^{4} a_p u_j^{(p)}\right],_i$$

However, since $\mathbf{u} \in C^1(\overline{B}) \cap C^2(B)$ and the a_p are constant, the derivatives may be moved inside the summations.

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\sum_{p=1}^{4} a_p[u_i^{(p)}],_j + \frac{1}{2}\sum_{p=1}^{4} a_p[u_j^{(p)}],_i$$

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\sum_{p=1}^{4} a_p\{[u_i^{(p)}],_j + [u_j^{(p)}],_i\} \tag{D7}$$

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\sum_{p=1}^{4} a_p 2E_{ij}(\mathbf{u}^{(p)}) = \sum_{p=1}^{4} a_p E_{ij}^{(p)}$$

Recognize from (3.46) that W_i occurs as a linear term in u_i, therefore, W_i may be separated from the summation in (D7). Equating the components of (D7) that are functions of **W** to (D6) results in

$$E_{ij}(\mathbf{W}) = \alpha_{ij}x_1 + \beta_{ij}x_2 = \sum_{p=1}^{4} a_p E_{ij}^{(p)}(\mathbf{W}) \tag{D8}$$

Now since the α_{ij} and β_{ij} are still arbitrary constants they can be separated into four auxiliary problems that correspond to the R.H.S. of (D8), that is

$$\alpha_{ij} = \sum_{p=1}^{4} a_p \alpha_{ij}^{(p)} \text{ and } \beta_{ij} = \sum_{p=1}^{4} a_p \beta_{ij}^{(p)} \tag{D9}$$

Substitute (D9) into (D8) and combine under a single summation,

$$\sum_{p=1}^{4} a_p [\alpha_{ij}^{(p)} x_1 + \beta_{ij}^{(p)} x_2 - E_{ij}^{(p)}(\mathbf{W})] = 0 \tag{D10}$$

The coefficients $\alpha_{ij}^{(p)}$ and $\beta_{ij}^{(p)}$ are still arbitrary except for the requirement (D9). Therefore, it is possible to select $\alpha_{ij}^{(p)}$ and $\beta_{ij}^{(p)}$ so that

$$\alpha_{ij}^{(p)} x_1 + \beta_{ij}^{(p)} x_2 = E_{ij}^{(p)}(\mathbf{W}) \tag{D11}$$

Equation (D11) shows that $E_{ij}^{(p)}(\mathbf{W})$ may be represented as linear functions of x_1 and x_2. Equation (D11) provides six constraint equations for the strains resulting from the generalized plane strain problems.