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Title: A SAMPLING CRITERION ENABLING SIGNAL RECONSTRUCTION WITH SPECIFIED LIMIT ERROR

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Robert R. Michael

A sampling criterion which provides a numerical relationship between sampling rate and worst-case peak error for linear interpolation of sample points is presented. The criterion, based upon the second derivative of a waveform, is derived, its properties are observed for a sine wave, and its applicability to complex signals is discussed.

An approximate measure of the second derivative for an amplitude-time function is implemented using a linear analog circuit, and this device in conjunction with an analog computer is used to confirm the validity of the sampling criterion. Possible application to on-line variable-rate sampling control for data compression is discussed in the conclusion.

A Sampling Criterion Enabling Signal Reconstruction with Specified Limit Error

by

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A SAMPLING CRITERION ENABLING SIGNAL RECONSTRUCTION WITH SPECIFIED LIMIT ERROR

I. INTRODUCTION

Today much information is handled in digital form. Digital computers process ever increasing quantities of data for the engineering and scientific disciplines. Deep-space probes and earth-bound communications links digitize and multiplex signals in a bid for greater efficiency. Yet, much physical data with which engineers and scientists are concerned is analog in nature. When continuous data is digitized, the information processing equipment must ignore the signal for finite periods of time. If the analog signal varies during such an interval, there will not be digital data taken to define the signal at times between the sample points. Higher sampling rates provide more data points at increased cost and, in the case of stored data computers, at the expense of memory space. If some practical sampling rate is chosen, the question arises as to how well and by what means can one recover the original signal from the given samples.

The sampling theorem states that if a signal is sampled at two samples per cycle of bandwidth, all the information content of the signal is captured. Thus, the signal itself can be recovered, in theory. The problem which confronts the engineer is how to actually recover the signal. It is known that a signal cannot be recovered exactly from its samples taken at two samples per cycle of bandwidth using a physically realizable linear system (14, p. 2.19). This comes about because an ideal filter cannot be built.

A practical approach to recovery of a signal is to interpolate between its sample points according to some mathematical scheme. A first-order linear interpolation is a rather easy one to accomplish, but with such a scheme the sampling theorem does not hold. Two samples per cycle of a sine wave, if linearly interpolated, will not reproduce the sine wave.

The object of this thesis is to provide a means for predicting the accuracy of the reconstruction of a signal when linear interpolation of its samples is used. An expression for the relationship of sampling rate to worst-case peak error is derived, implemented, and tested on a variety of real signals.

II. DERIVATION AND PROPERTIES OF THE SAMPLING CRITERION

A. Derivation of the Sampling Criterion

The derivative sampling criterion presented in this thesis is based upon an investigation by Professor Robert R. Michael of Oregon State University (11). As originally conceived, the study determined a method of estimating the number and distribution of taps required on a linear slidewire potentiometer in order to generate an empirical function to a prescribed limit of error. In this paper, where the criterion is applied to an amplitude-time function, the original form of the derivation applies.

A sampling criterion, to be useful, must describe numerically the relationship between sampling rate and a given specific measure of error for any physically realizable signal. The criterion here derived shall describe numerically what peak error can occur between a signal and a linear interpolation of its samples. A fixed limit of error is specified in preference to probable or mean square error because it more precisely describes error conditions when considerable significance is attributed to a single interpolation value. Limit error is taken to mean worst-case peak error. First order linear interpolation is used as the basis for reconstruction because of its widespread use and ready implementation.

It should be emphasized that this paper is concerned only with errors resulting from the linear interpolation of sample points. Errors arising from limitations of sampling equipment and from subsequent analytic procedures must be superimposed on the limit error here derived.

Figure 1 shows a region of interest of a typical amplitude-time function. A limit as the angle 2ϕ approaches zero is assumed. A result of this assumption is that the second derivative of the function must be nearly constant over the interval δ. The consequences of this limitation will be observed near the end of this chapter. Using definitions of analytic geometry, angle θ can be written

2.1 $$\theta = \arctan(y')$$

where

$$y' = \frac{dy}{dt} \quad \text{at point} \quad P.$$

Differentiating θ with respect to time,

2.2 $$\frac{d\theta}{dt} = \frac{d(\arctan(y'))}{dt},$$

2.3 $$\frac{d\theta}{dt} = \frac{y''}{1+(y')^2},$$

where

$$y'' = \frac{d^2y}{dt^2} \quad \text{at point} \quad P.$$

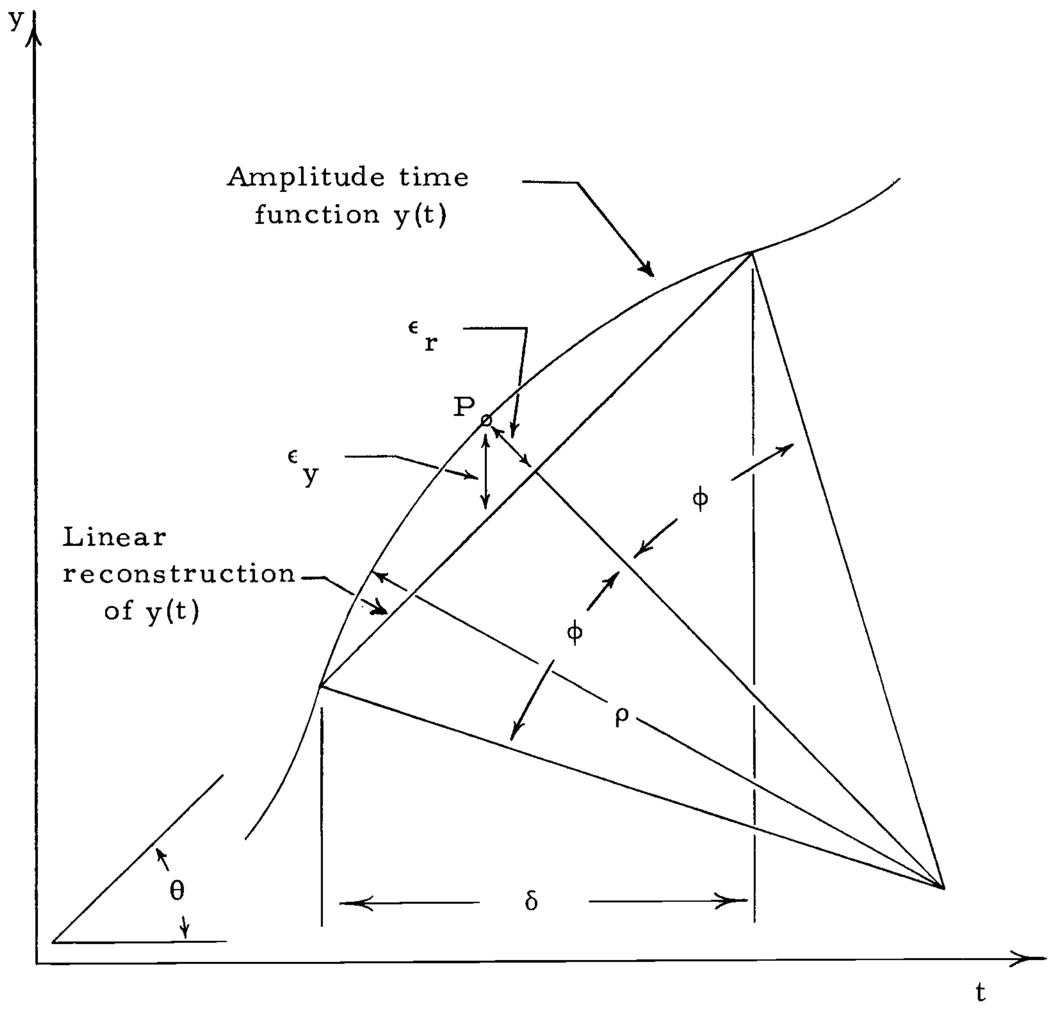


Figure 1. A region of interest of an amplitude-time function.

The angle 2ϕ can be expressed as the change of θ over the interval δ:

2.4
$$2\phi = \frac{d\theta}{dt}\delta.$$

Substituting Equation 2.3 into 2.4,

2.5
$$2\phi = \frac{y''}{1+(y')^2}\delta.$$

Referring to Figure 1, if 2ϕ is small, the radial distance ϵ_r from the chord to the curve is

2.6
$$\epsilon_r = \rho - \rho \cos \phi.$$

2.7
$$\epsilon_r = \rho(1-\cos \phi).$$

In terms of Taylor's series expansion

2.8
$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots .$$

Since this series converges rapidly for small values of ϕ,

2.9
$$\epsilon_r \cong \rho(1-1+\frac{\phi^2}{2!}) .$$

2.10
$$\epsilon_r \cong \frac{\rho\phi^2}{2} .$$

Now ρ, the radius of curvature, can be expressed in terms of derivatives as

$$2.11 \qquad \rho = \frac{[1+(y')^2]^{3/2}}{y''} \qquad \text{(Ref. 13, p. 396).}$$

Then Equation 2.5 can be expressed as

$$2.12 \qquad 2\phi = \frac{[1+(y')^2]^{1/2}}{\rho}\,\delta.$$

Dividing both sides of Equation 2.12 by two and squaring

$$2.13 \qquad \phi^2 = \frac{[1+(y')^2]}{4\rho^2}\,\delta^2\;;$$

therefore

$$2.14 \qquad \epsilon_r \cong \frac{[1+(y')^2]}{8\rho}\,\delta^2.$$

In the limit as 2ϕ approaches zero, the greatest vertical distance from the chord to the curve in Figure 1 occurs at the midpoint of the curve and designated by ϵ_y.

$$2.15 \qquad \epsilon_y \cong \frac{[1+(y')^2]}{8\rho\cos\theta}\,\delta^2$$

As a consequence of the Pythagorean Theorem $\cos\theta$ can be expressed in terms of derivatives as

$$2.16 \qquad \cos\theta = [1+(y')^2]^{-1/2}.$$

Equation 2.15 becomes

2.17 $$\epsilon_y \cong \frac{[1+(y')^2]^{3/2}}{8\rho}\delta^2$$

Let

$$\epsilon = \frac{-[1+(y')^2]^{3/2}}{8\rho}\delta^2$$

so that

$$\epsilon_y \cong -\epsilon .$$

Then

2.18 $$\frac{\epsilon}{\delta^2} = \frac{-[1+(y')^2]^{3/2}}{8\rho} .$$

Substituting the derivative form of ρ (Equation 2.11) into Equation 2.18 yields

2.19 $$\frac{8\epsilon}{\delta^2} = -y'' .$$

Setting $\frac{1}{\delta} = R$, where R is the sampling rate in samples per second, and rearranging:

2.20 $$R = \left(\frac{-y''}{8\epsilon}\right)^{1/2} .$$

Note that ϵ is positive when y'' is negative and vice versa, so that R is real. Rearranging 2.20,

2.21 $$\epsilon = \frac{-y''}{8R^2} ,$$

where δ is the sampling interval in seconds, R is the sampling rate in samples per second, ϵ is the error in units of y, y'' is the second derivative with respect to time of the waveform in units of y per $(\text{second})^2$.

B. Properties of the Sampling Criterion

Some properties of the derivative criterion may be observed by applying it to a sine function $y(t) = A \sin(\omega t)$. Using Equation 2.21

$$\epsilon_1 = \left(\frac{1}{8R^2}\right) \frac{-d^2(A \sin(\omega t))}{dt^2}$$

$$\epsilon_1 = \frac{\omega^2 A \sin(\omega t)}{8(R)^2} \ .$$

If the sampling rate is doubled,

$$\epsilon_2 = \frac{\omega^2 A \sin(\omega t)}{8(2R)^2}$$

thus,

$$\frac{\epsilon_2}{\epsilon_1} = \frac{1}{4} \ .$$

If, for a given sampling rate, the amplitude of the sine wave is doubled,

$$\epsilon_3 = \frac{\omega^2 (2A) \sin(\omega t)}{8R^2}$$

and

$$\frac{\epsilon_3}{\epsilon_1} = 2.$$

The above demonstrates two properties of the limit error predicted by the derivative criterion when applied to a sinusoidal wave: (1) The limit error is inversely proportional to the square of the sampling rate; (2) the limit error is proportional to the amplitude of the sine wave.

These properties can be tested numerically. The derivative sampling criterion indicates that the limit error for a sine function $A \sin(\omega t)$ will occur at $\omega t = \frac{\pi}{2}$, the peak of the sine wave, since the second derivative, $-\omega^2 A \sin(\omega t)$, is a maximum there. Figure 2a shows an arrangement of samples which will produce such a worst-case peak error condition. At a given sampling rate $R_1 = \frac{1}{\delta_1}$, the actual limit error can be found from

$$\epsilon_1 = A \sin\left(\frac{\pi}{2}\right) - A \sin\left(\frac{\pi}{2} - \frac{2\pi}{2R_1}\right).$$

If R_1 is equal to ten samples per second,

$$\epsilon_1 = A(0.048943).$$

The theoretical limit error can be calculated from the sine function using the derivative sampling criterion. The function is

$$y(t) = A \sin(\omega t)$$
$$= A \sin(2\pi f t).$$

Let $f = 1$ Hz, then $y'' = -4\pi^2 A \sin(2\pi t)$. The peak value of y'' occurs at $2\pi t = \frac{\pi}{2}$

$$y''_{(peak)} = -4\pi^2 A$$
$$= -39.4784A.$$

For ten samples per cycle of a one Hz sine wave, R equals ten samples per second. Using Equation 2.21

$$\epsilon = \frac{-y''}{8R^2}$$

$$\epsilon = A(0.049340).$$

This value is slightly larger than the actual limit error previously calculated. The difference between the predicted limit error and the actual maximum error is less than .05% of A (less than 1% of the actual error) even though the second derivative varies by nearly 5% during the sampling interval.

If the sampling rate is doubled to $2R_1 = \frac{2}{\delta_1} = 20$ samples per cycle of the sine wave, the actual limit error is

$$\epsilon_2 = A \sin\left(\frac{\pi}{2}\right) - A \sin\left(\frac{\pi}{2} - \frac{2\pi}{4R_1}\right)$$

$$\epsilon_2 = A(0.012312)$$

so that

$$\frac{\epsilon_2}{\epsilon_1} = 0.25153.$$

This is very nearly 1/4 as it should be.

If the sampling rate is constant and the amplitude of the sine wave is doubled the actual limit error is

$$\epsilon_3 = 2A \sin\left(\frac{\pi}{2}\right) - 2A \sin\left(\frac{\pi}{2} - \frac{2\pi}{2R_1}\right)$$

$$= 2[(A \sin\left(\frac{\pi}{2}\right) - A \sin\left(\frac{\pi}{2} - \frac{2\pi}{2R_1}\right)]$$

$$\frac{\epsilon_3}{\epsilon_1} = 2.000 \ldots \text{ as the sampling criterion predicts.}$$

If the samples fall as shown in Figure 2b, the peak error occurring between samples will not occur at the peak value of the sine wave, and thus the peak error will be somewhat less than the limit error of Figure 2a. Figure 2b represents the minimum peak error case for the given sampling rate. The limit error predicted by Equation 2. 22 using the maximum value of the derivative corresponds to the maximum peak error case of Figure 2a. If a very low sampling rate is used (less than five samples per cycle) the difference between the maximum and minimum peak errors becomes appreciable, and possibly more important, the reconstruction based on linear interpolation becomes meaningless (see Figure 9). For such a sampling rate the

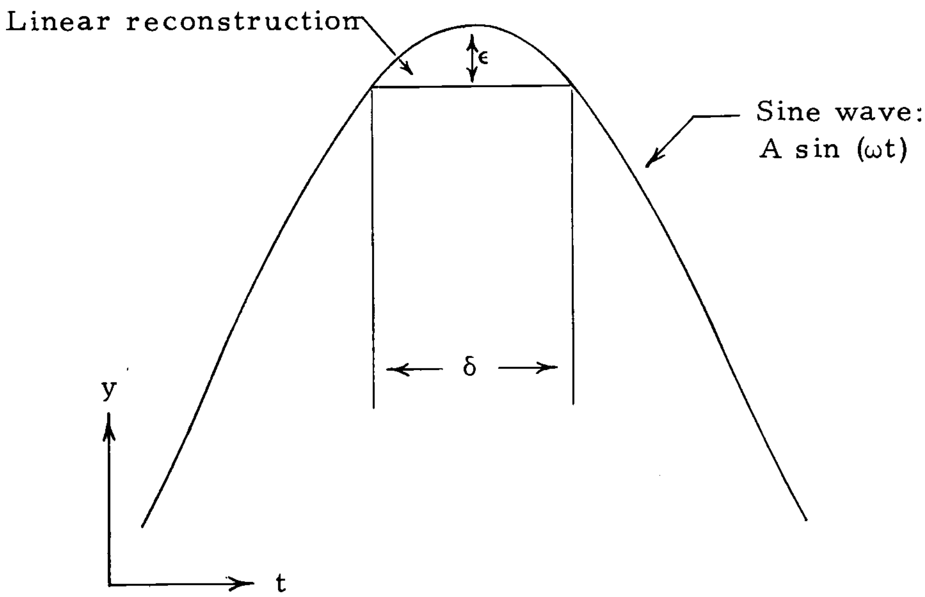


Figure 2a. Condition for maximum peak error for sine wave.

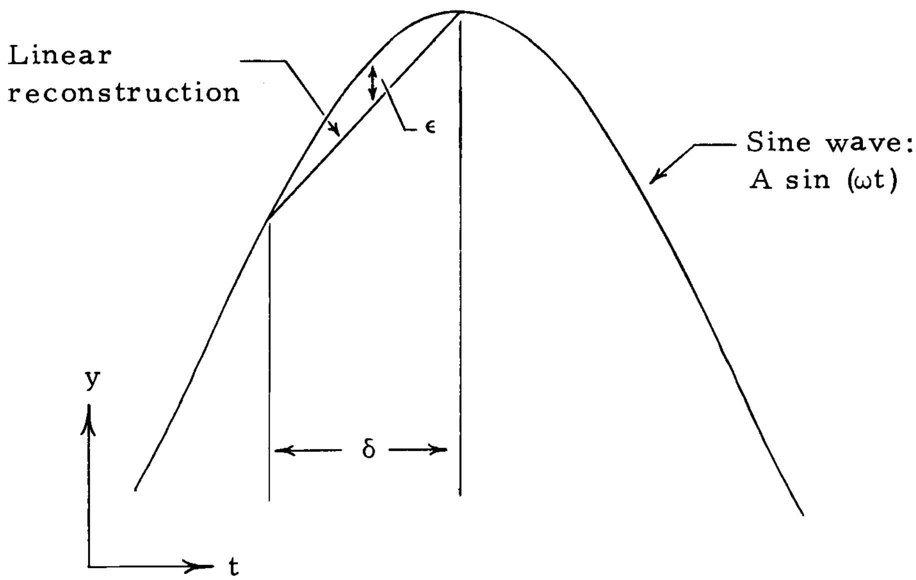


Figure 2b. Condition for minimum peak error for sine wave.

assumption that the variation of the second derivative is small over the sampling interval will not be valid. In this case the predicted limit error will exceed the actual limit error by some small amount, but in light of the breakdown of the reconstruction, this decrease in the accuracy of the predicted limit error is of little consequence (see Appendices A and B for comparison of predicted and actual maximum peak errors for a sine wave).

Using a sine wave it has been shown that the derivative sampling criterion provides an accurate prediction of the limit error for meaningful sampling rates. Both Equations 2.20 and 2.21 will be valid allowing one to determine the required sampling rate for a specified limit error or to predict the limit error for a given sampling rate. Extension of the criterion to complex waveforms will be valuable.

C. Extension of the Sampling Criterion to Complex Signals

In terms of Fourier series, any well behaved waveform can be expressed as a linear combination of sinusoidal waves (10, p. 4-96). Therefore, the properties observed in the last section should, in theory, apply to the sampling of any real signal as long as the second derivative of the waveform does not change appreciably during the interval between samples.

A real signal may have frequency components extending over many decades. The signal spectrum is one way of indicating the distribution of these components. Figure 3a shows an ideal case of flat spectrum (equal amplitude components) out to a cutoff frequency beyond which no signal power exists. For such a signal it is clear that the highest existing frequency component will determine the maximum value of the second derivative (see Figure 4b). Two possibly more realistic spectra are shown in Figures 3b and 3c. These represent realizable band limitations on the frequency components. If the sloping tail of such spectra rolls off at a rate exceeding -40 db per decade (Figure 3c) the second derivative will be finite due to its nature in the frequency domain (see Figure 4) (15, p. 458-460). A difficulty arises if the spectrum rolls off at less than -40 db per decade (Figure 3b). For the case where the tail maintains a slope of less than -40 db per decade to an infinite frequency (in theory) the second derivative will approach an infinite magnitude and the sampling criterion will specify an infinite sampling rate for any finite limit error. Only as the sampling rate approaches infinity will the assumption that the second derivative is nearly constant between samples be valid. Since this assumption is a necessary one for the validity of the derivation, the criterion will not hold for finite sampling rates of such a signal. In other words, if a signal of this nature were sampled at a finite rate, the actual limit error would be finite, not infinite as the criterion predicts. The limit error would be finite because the magnitude of the frequency components beyond the capabilities of the

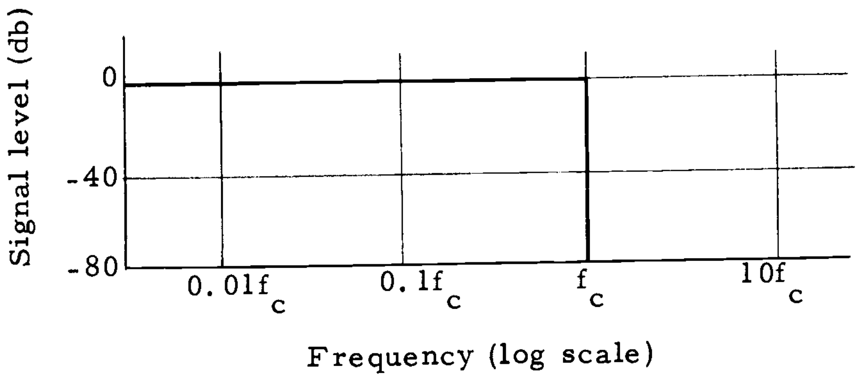


Figure 3a. An ideal bandlimited spectrum.

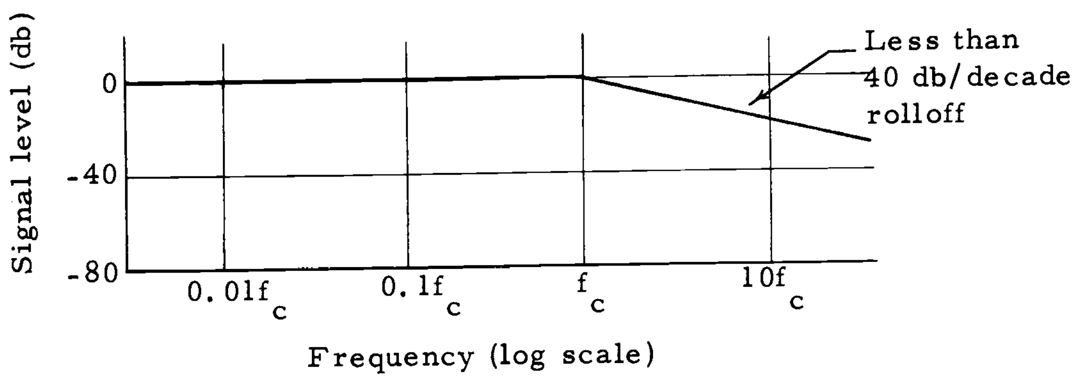


Figure 3b. A realizable bandlimited spectrum.

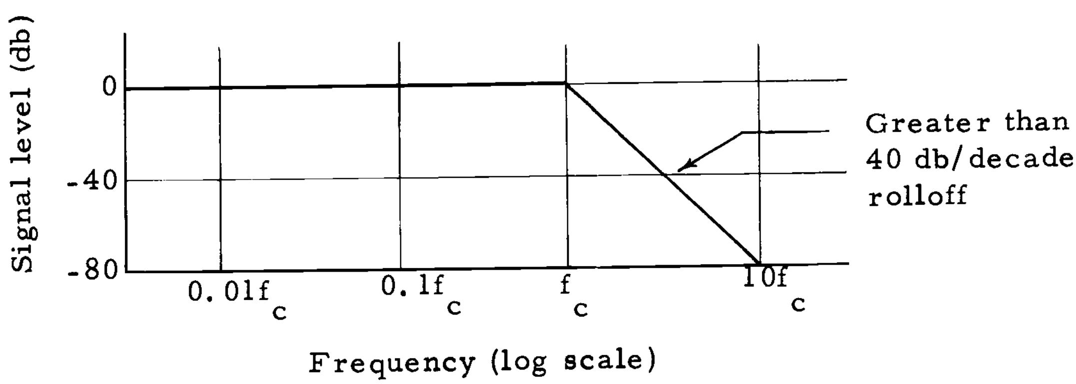


Figure 3c. A realizable bandlimited spectrum.

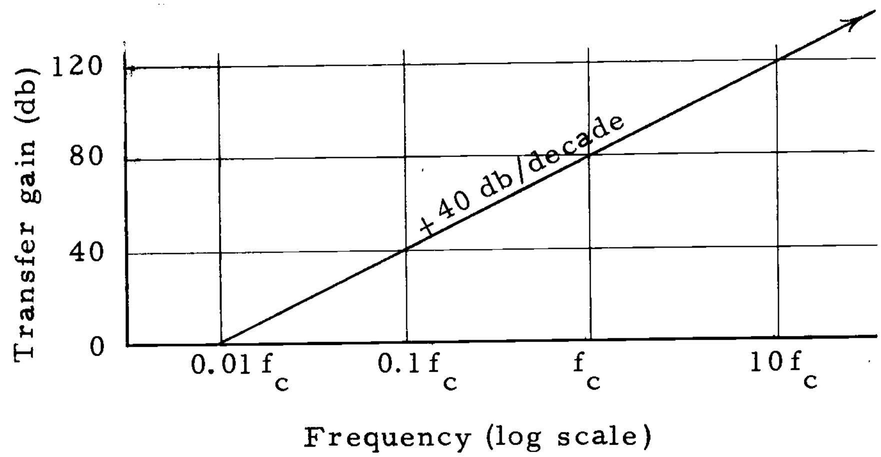


Figure 4a. A second derivative transfer function.

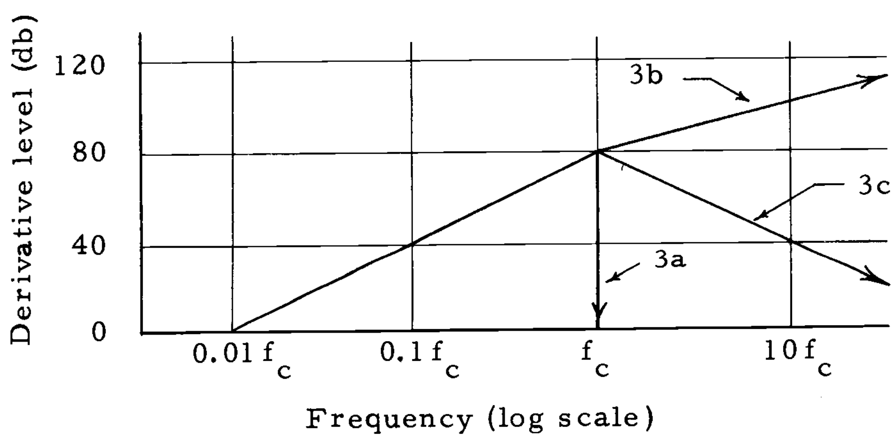


Figure 4b. Relative second derivatives for spectra of Figure 3.

sampling rate would be small.

Aliasing is a term used to denote false low frequencies arising from the sampling of a component at less than twice its frequency (14, p. 2. 6). For linear interpolation the aliasing error for a component is equal to the magnitude of that component. If a signal to be sampled contains high frequency components which are small in comparison with the desired limit error, the components can be ignored in choosing a sampling rate. Figure 5 shows how the aliasing error of small high frequency components combines with the sampling error of larger low frequency components to produce a total maximum peak error. The high frequency components create, in effect, a band of error which must be superimposed upon the error predicted by the sampling criterion. In order for the sampling criterion to predict a limit error based upon the lower frequency components, the derivative used in the criterion must be influenced by only these lower frequencies. This implies low-pass filtration of the signal before measuring the second derivative. Care must be exercised in order not to filter out significant components. If a similar low-pass filter is used as a pre-sampling filter, the effects of aliasing will be decreased. This technique is commonly used to minimize false frequency components when sampled data is to be harmonically analyzed. Such pre-filtering does modify the signal; whether or not significant distortion occurs depends upon the specific case (8).

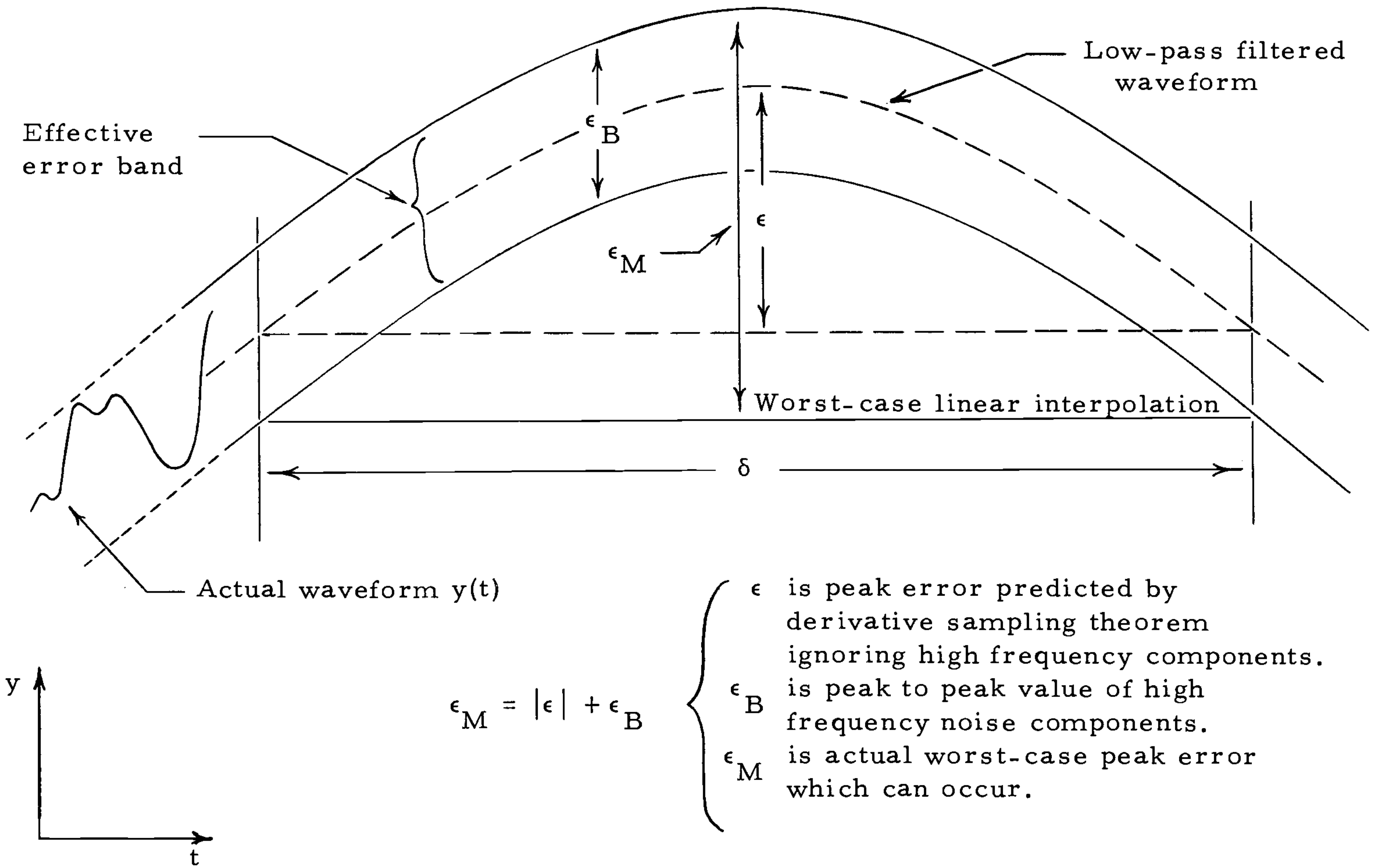


Figure 5. Effect of aliasing on maximum peak error.

Since this paper is concerned with peak errors arising from linear interpolation of sampled data, aliasing will be taken as a contribution to limit error in the reconstruction.

In this section it has been suggested that the derivative sampling criterion is useful for physically realizable signals if the limitations of the criterion are observed. The near impossibility of an analytic approach for the demonstration of the validity of the criterion for complex waveforms necessitates an experimental procedure.

III. PROCEDURE FOR VERIFICATION OF THE SAMPLING CRITERION

A. Implementation of a Differentiator

In order to carry out an experimental verification of the value of the derivative sampling criterion, a method for determining the second derivative of a real time signal must be implemented. Electronic differentiation is inherently a noisy process since high frequency components are most accentuated. Thus, it is good practice to bandlimit the signal to frequencies of interest before taking the derivative. This agrees with the suggestion of the last chapter to filter out insignificant high frequency components of a signal in order that the derivative measured not vary appreciably between samples. Figure 6 shows the kind of gain-transfer function desired for effectively ignoring frequency components above a given value. An ideal such function would be as that labeled a. A realizable transfer function might be as the line labeled b. The solid line b is, of course, the asymptotic representation of the actual response shown as a dashed line.

The transfer function of Figure 6b is implemented using a second-order high-pass filter in cascade with a second-order low-pass filter with coincident corner frequencies. Multiple-feedback active filters are used (6, p. 74-77). Figure 7 shows the filters along with

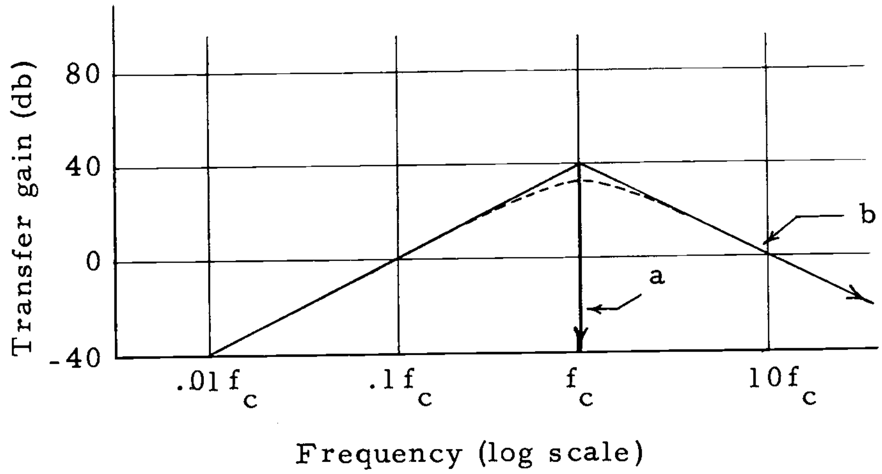
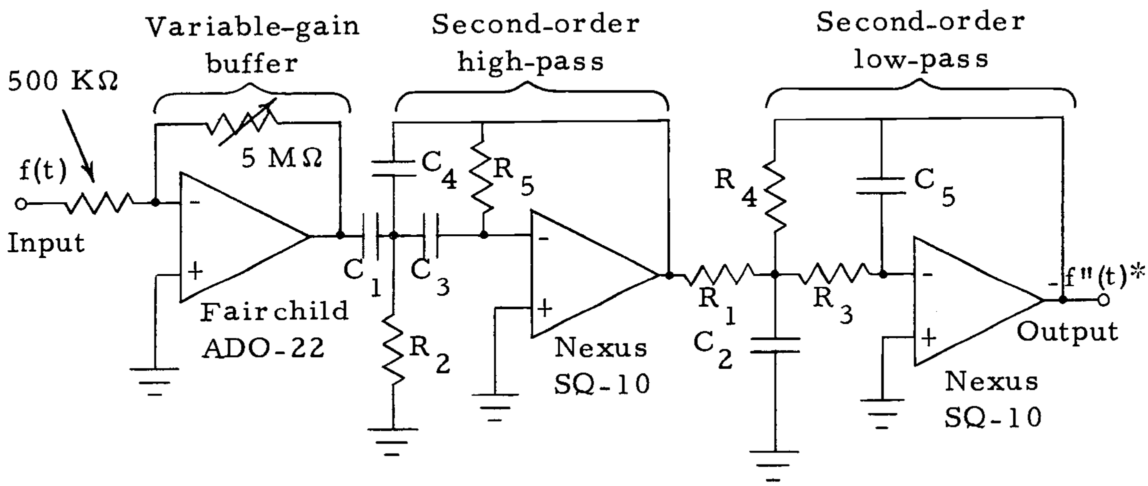


Figure 6. Transfer functions of ideal and realizable bandlimited differentiators.



*Scaled second derivative

Figure 7. Electronic second derivative estimator.

a variable-gain input buffer (4, p. 12-15). The component values for the filters are tabulated in Table 1. The basic range of the corner frequency, f_c, is 100 Hz, 1 KHz, and 10 KHz with a multiplier of 1.0 or 3.16 available. This range of f_c allows some flexibility in the range of frequencies to be measured. The gain-bandwidth product of the operational amplifiers limits the maximum value of f_c; capacitive loading of the input buffer imposes a limitation on the low value of this frequency. Internal noise of the ±10 volt operational amplifiers limits their useful dynamic output ratio to approximately 1000 : 1. This restricts the frequency range for a constant amplitude input at a given f_c to a ratio of approximately 30 : 1.

Table 1. Component values for multiple-feedback active filters.

Basic corner frequency in Hz	High pass $C_1 = C_3$ μf	High pass C_4 μf	Low pass C_2 μf	Low pass C_5 μf	
100	.1	.01	.01	.22	
1,000	.01	.001	.001	.022	
10,000	.001	.0001	.0001	.0022	
Frequency multiplier	R_2 $\Omega \times 10^3$	R_5 $\Omega \times 10^3$	R_1 $\Omega \times 10^3$	R_2 $\Omega \times 10^3$	R_3 $\Omega \times 10^3$
1	10.7	236	11.25	10.2	112
3.16	3.39	74.8	3.56	3.2	35.6

Note: This table applies to Figure 7.

With a peak gain of 100 (40 db) at the corner frequency (according to the asymptotes of the transfer function) the expected gain at

$0.1\,f_c$ is 0 db (unity gain). Since the actual response of the circuit departs from the asymptotic representation at frequencies about the corner, it is desirable to test the circuit using a sine wave. The results of this test appear in Table 2. The listed time delay is the time difference of the occurrence of the peak value of the measured derivative with respect to the expected peaking time of the ideal second derivative. The time delay (which causes an effective phase-lag of the measured derivative from the ideal) is the most apparent fault of the derivative implementation. Since both phase and magnitude must be correct in order for the transfer function to truly represent a second derivative operator, the circuit should be restricted to use at frequencies less than about one-tenth of the corner frequency for an accurate approximation. The transition region from $0.1\,f_c$ to $10\,f_c$ will cause problems if frequency components of the input signal exist in that range. Above $10\,f_c$ the gain rolls off rapidly so that higher frequency components are attenuated.

Using Appendix A, the theoretical limit error for a given normalized sampling rate of a unit amplitude sine wave can be determined. The actual value of error for a sine wave of amplitude A_1 is simply A_1 times the value listed in Appendix A. The sampling rate for a sine wave of frequency W Hz is W times the samples per cycle value corresponding to the desired limit error. Using the circuit of Figure 7 the value of the measured second derivative of a

Table 2. Test data for second derivative circuit using sine wave input.

Frequency in Hz	Input voltage in peak volts	Ideal derivative in peak volts	Derivative circuit output in peak volts	Percent magnitude error	Time delay of measured derivative in milliseconds	Phase-lag of measured derivative from ideal
50	5. 0	1. 25	1. 25	---	0. 4	7°
100	5. 0	5. 0	4. 95	1. 0%	0. 4	14°
200	1. 0	4. 0	3. 90	2. 5%	0. 4	29°
400	0. 50	8. 0	7. 78	2. 8%	0. 4	58°
800	0. 20	12. 8	9. 40	26. 5%	0. 5	144°
1000	0. 20	20. 0	10. 0	50 %	0. 5	180°

Note: Above data is for 1000 Hz corner frequency.

signal may be related to the second derivative of a sine wave and the limit error for a given sampling rate determined from data in Appendix A. An example will illustrate this procedure.

The differentiation circuit, used to estimate the derivative of a signal, outputs a peak voltage of 10 volts with f_c set to 316 Hz. Since the gain of the circuit is unity at $0.1\ f_c$, a 10 volt peak sine wave would produce an output of 10 volts if its frequency were 31.6 Hz. The signal being applied to the circuit has the same second derivative as a 10 volt peak, 31.6 Hz sine wave. Therefore, if a sampling rate of 316 samples per second is used to sample either the given signal or a 10 volt peak, 31.6 Hz sine wave, the limit error expected is the same and is 0.49340 volts (316 samples per second is 10 samples per cycle of 31.6 Hz sine wave. The error corresponding to 10 samples per cycle is 0.049340 per unit of sine wave amplitude. Thus, an error of 0.49340 volts is to be expected for a 10 volt peak sine function).

A limit on the validity of the above procedure is the accuracy of the second derivative measured. Significant components of frequency above one-tenth of the f_c used will contribute to errors in this estimation.

B. Simulation of a Sampled Data System

To actually test the sampling criterion an analog computer simulation of a sampled data system is used. A block diagram of the circuit is shown in Figure 8. Repetitive mode is used for control of four of the lower six amplifiers to sample the input and construct a delayed-time linear interpolation of the samples. The three track-store units are alternately switched between states to provide successive samples at points S_2 and S_3--the difference between these samples is applied to the input of the reconstruction integrator (the time constant of which must be properly adjusted) so that it may ramp from the value at S_2 to that of S_3 during the operate time of the computer. The interval between samples is equal to the operate time plus the reset time of the computer; the reset time is necessary (so that the track-store units can track to new values accurately) but should be minimized.

In order to compare the reconstruction (which is delayed by a time equal to one sampling interval) with the original signal, a delayed version of the input must be provided. The upper six amplifiers of Figure 8 are connected as a fourth-order Padé time delay approximation (7, p. 285-287). With this circuit any time delay T may be obtained by proper selection of coefficients and integrator time constants if frequency limitations are observed. A test of this circuit shows it to be precise within 2% delay time with better than 0.1% amplitude accuracy for frequencies up to 1/T Hz, which is adequate for the present treatment.

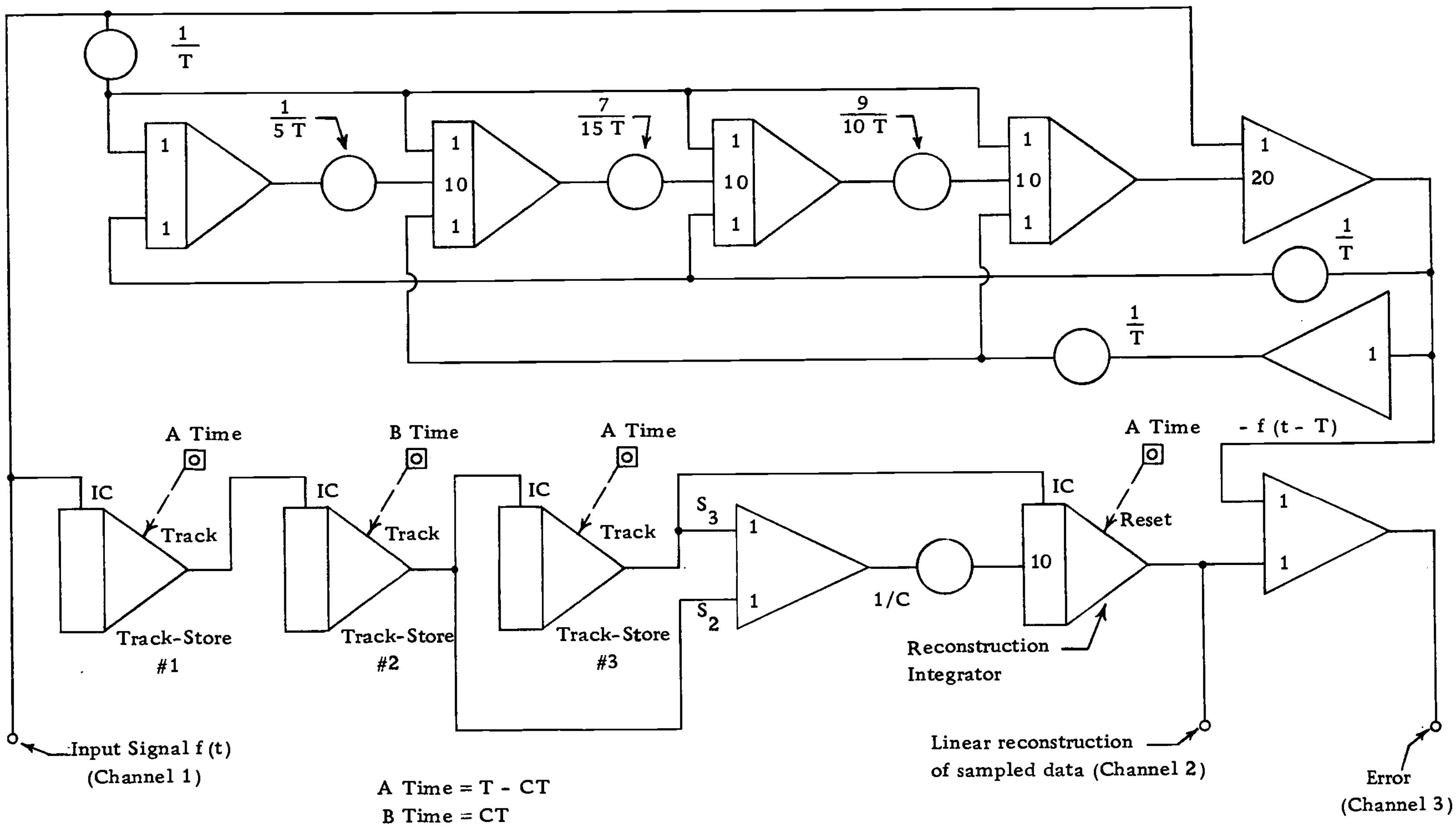


Figure 8. Analog computer simulation of sampled data system. (Channel numbers correspond to those of Figures 9, 10, 11, and 12.)

IV. EXPERIMENTAL VERIFICATION OF THE SAMPLING CRITERION

A. Signals Used for Tests

Three signals are applied to the circuitry of Chapter III to verify the properties of the sampling criterion: a sine wave, bandlimited random noise, and a signal of biological origin similar to an electrocardiograph (but of no intended medical significance). Each of these signals is first recorded on an Ampex FR 1300 analog tape recorder along with the corresponding derivative as measured by the circuit of Figure 7. Use of the tape recorder limits the signal-to-noise ratio to a maximum of approximately 40 db (100:1) for all three signals. The frequency range of this noise extends to 20 KHz at a tape speed of 60 ips and is proportionally lower for slower tape speeds reaching minimum bandwidth of 625 Hz at 1 7/8 ips. Even at this slowest speed the noise contains frequency components far beyond the 10 and 100 samples per second sampling rates used in the circuit of Figure 8. The variable-speed feature of the recorder is used to change the effective sampling rate for the signals since the simulation circuit which samples and reconstructs these signals is a carefully adjusted circuit which can be easily changed from 10 samples per second to 100 samples per second by time scaling, but cannot be practically varied by any factor less than this without extensive and time consuming readjustment of the circuit parameters.

The chart recordings of Figures 10, 11, and 12 are portions of output data for the sine, bandlimited noise and biological signal respectively. In each case the uppermost trace (channel 1) is a record of the signal itself; the second trace is of a linear reconstruction of the sample points; the third trace is a plot of the error between the linear reconstruction and the original signal; the lower trace is a record of the second derivative of the signal scaled to show the proper magnitude and phase of the limit error for each effective sampling rate used.

B. Tests with a Sine Wave

The sine wave tests afford a view of some properties of the sampling criterion and several effects of linear interpolation. The signal is recorded as a 5 Hz sine wave from a Hewlett-Packard 200 CD oscillator at 60 ips tape speed so that a minimum frequency of 5/32 Hz can be obtained at 1 7/8 ips. This allows a real-time sampling rate of 10 samples per second to be used for effective sampling rates of from 2 samples per cycle to 64 samples per cycle. Figure 9 shows some results for very low sampling rates. Figure 10a shows signal, reconstruction, and error, along with predicted limit error, for 8 samples per cycle. The straight-line segments of the linear interpolation are clearly visible on channel 2. A comparison of the two lower tracks shows that the actual limit error is approximately equal to the predicted limit error plus the peak-to-peak value of the high frequency noise (a measurement made with a Tektronix 565 oscilloscope shows this noise to be about 0. 18 volt peak-to-peak for the 7. 5 volt peak amplitude sine wave). The scaled value of the derivative was determined by the method of Chapter III, Section A.

One property of the error for linear reconstruction is that it returns to zero at each sampled data point--this is observed on channel 3 of Figure 10a (pen dynamics causes some departure from an ideal trace of this phenomena). Another property visible in Figure 10 is that the peak error varies periodically due to the positions of the samples changing relative to the signal. The samples can actually be observed drifting through the sine wave in Figure 10a, channel 2.

As the sampling rate is increased, the limit error decreases in theory, and in practice as Figures 10b and 10c demonstrate. The high frequency noise is observed to take over as the major contributor of error as the sampling rate is increased. Since the frequency components of this noise are beyond the capabilities of the simulation circuit it is not possible to try to sample at a rate adequate to reconstruct the noise itself.

C. Tests with Bandlimited Noise

The noise used for the tests of Figure 11 is a recording of the output of a General Radio 1390-B random noise generator filtered through a fourth-order low-pass filter with an upper corner frequency (-12 db point) of 48 Hz and a first-order bass-boost circuit (7, p. 112-113) with 20 db of boost below 5 Hz (to compensate for a first-order rolloff of the noise generator below 5 Hz as determined from its circuit configuration--below 0. 5 Hz the noise generator rolls off more rapidly). Thus, the bandlimited noise as recorded is approximately flat from 0. 5 Hz to 24 Hz (-4 db point). The actual spectrum is not known and is not needed in order to test the derivative sampling criterion.

Figure 11a shows the results of sampling the bandlimited noise at a rate equivalent to 400 samples per second. The striking aspect of channels 3 and 4 is their similarity. The second derivative of track 4 is scaled to reflect the limit error to the same scale as track 3 plots the actual error. It is observed that the actual error never exceeds the limit error (even though the high frequency noise components on the signal are 0. 2 volts peak-to-peak), and in some cases the actual error is quite small where the limit error is appreciable. This indicates that the second derivative of the bandlimited noise is changing somewhat between samples. Figure 11b shows the results for an equivalent of 800 samples per second. In this case the predicted limit error plus the peak-to-peak high frequency noise is closer to the observed actual limit error. Figure 11c shows the results of using an equivalent of 1600 samples per second. Here the error due to the high frequency components of the noise almost obscures the error due to the linear interpolation of the samples. Note that in Figure 11, and in Figure 12, the real-time sampling rate used is 100 samples per second so that the dynamics of the pen recorder do not allow it to return to zero at each sample point. However, it can be seen in these Figures that the pen does try to return to zero one hundred times a second: it can be observed with an oscilloscope that the error does behave as expected.

D. Tests with Biological Signal

The signal displayed in Figure 12 is of a biological potential derived from a measuring procedure similar to that of taking an electrocardiogram. However the waveform of Figure 12 is not intended to imply anything of medical significance. It is a kind of signal which is inherently different from a sine wave (deterministic in theory) or random noise (purely random in theory). It is not actually periodic, nor is it truly random. It is a real signal which is representative of a type of physically occurring phenomena.

Figure 12a presents a set of traces for an equivalent sampling rate of 200 samples per second for the biological signal. The predicted limit error is observed to be a good indication of the actual limit error, though conservative in that the true limit error never reaches the value predicted by the second derivative (this was observed to be the case for a much greater length of signal record than is included in this thesis). The second derivative must therefore change somewhat during the interval between samples. Notice at what location the major positive and negative errors occur: the positive error corresponds to the positive peak of the biological waveform, but the negative error corresponds to the steep section of signal just previous to the negative peak of the waveform. The second derivative predicts this behavior.

Figure 12b displays the results of sampling the biological signal 400 times per second. The predicted limit error plus the peak-to-peak value of the high frequency components of noise agrees very well with the observed limit error. Figure 12c shows the traces corresponding to 800 samples per second. Again the predicted limit error taking into account the higher frequency noise level agrees very well with the maximum occurring peak error.

Table 3 lists the predicted limit errors, noise levels, and actual worst-case errors observed for the Figures 10, 11, and 12.

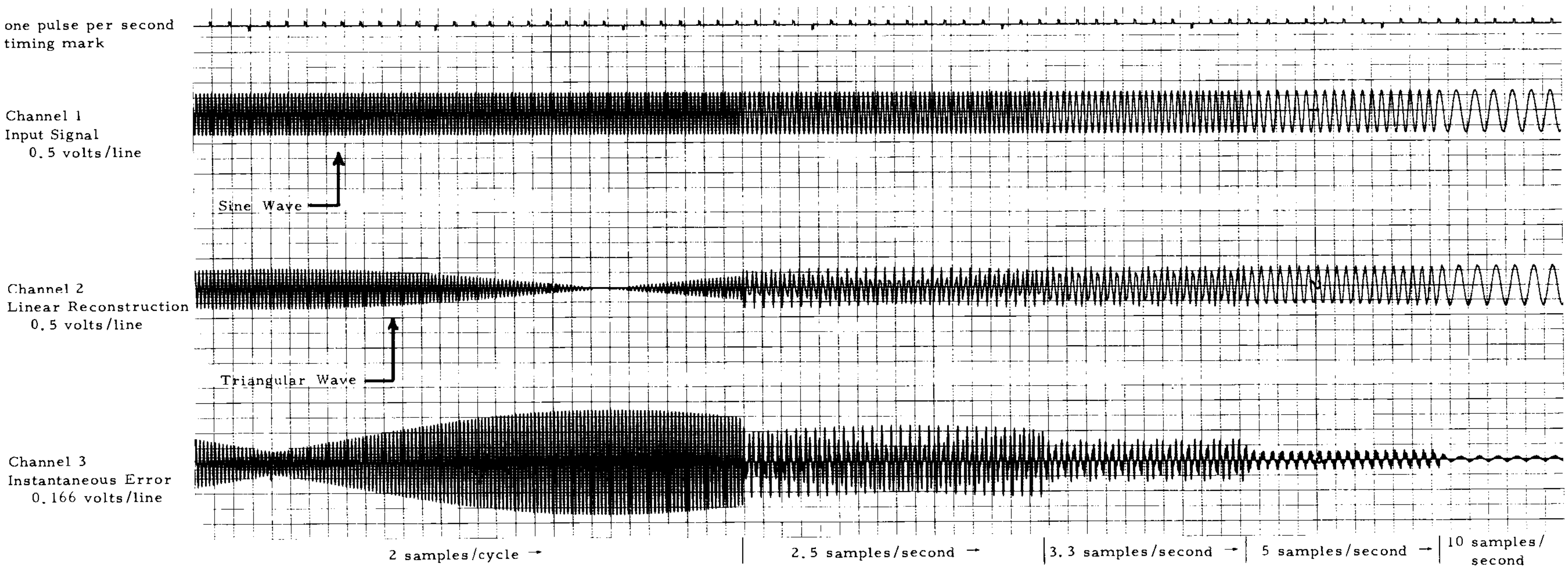


Figure 9. Signal, reconstruction, and error for very low sampling rates of a sine wave.

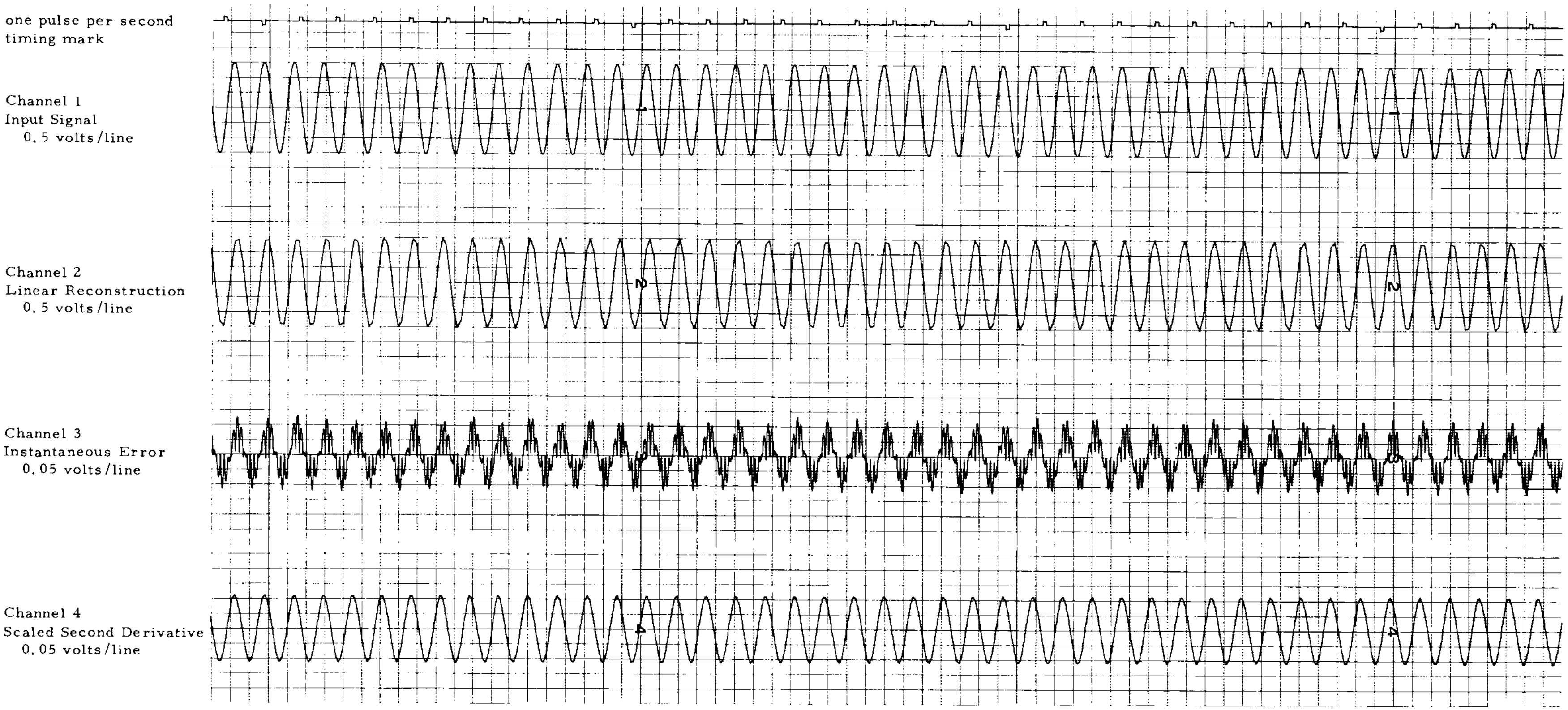


Figure 10a. Data plot for 8 samples per cycle of a sine wave.

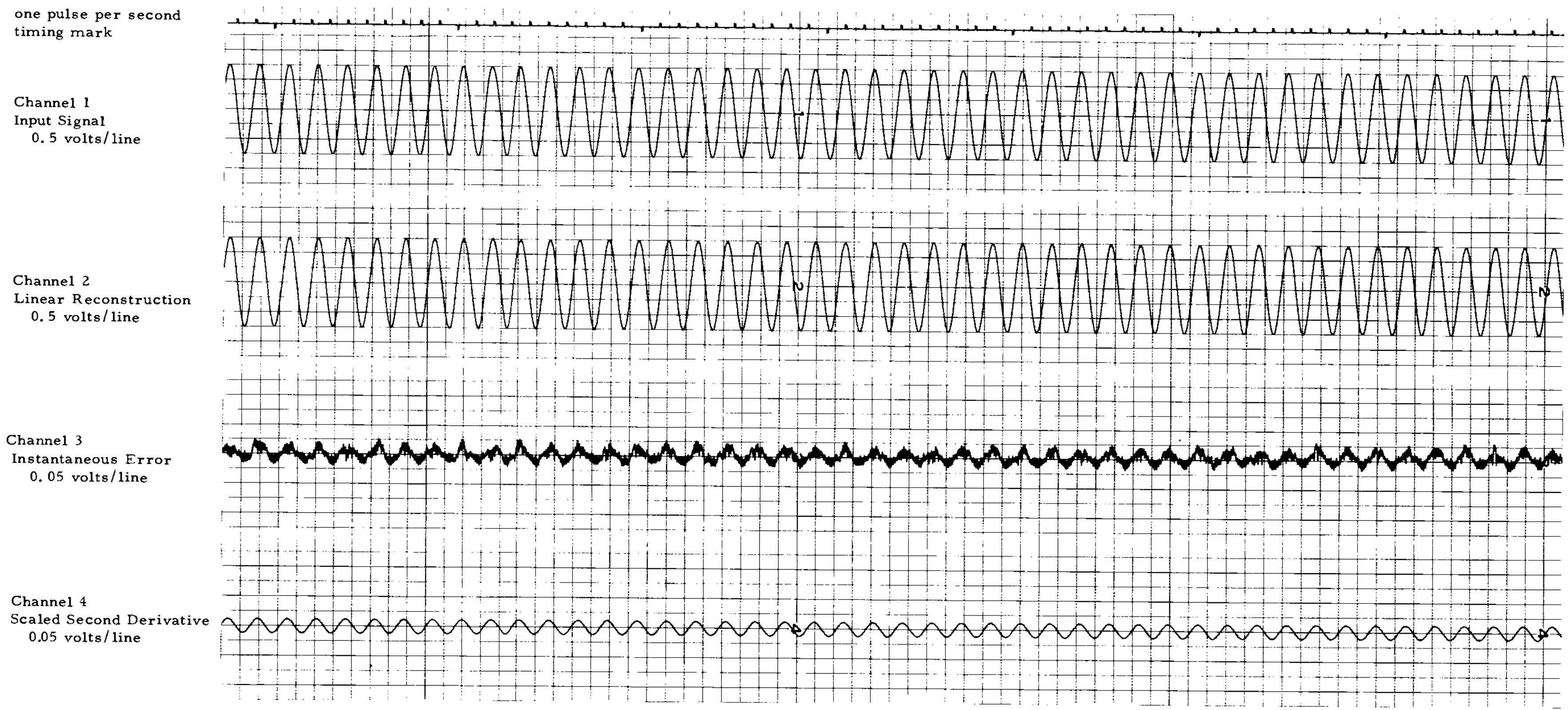


Figure 10b. Data plot for 16 samples per cycle of a sine wave.

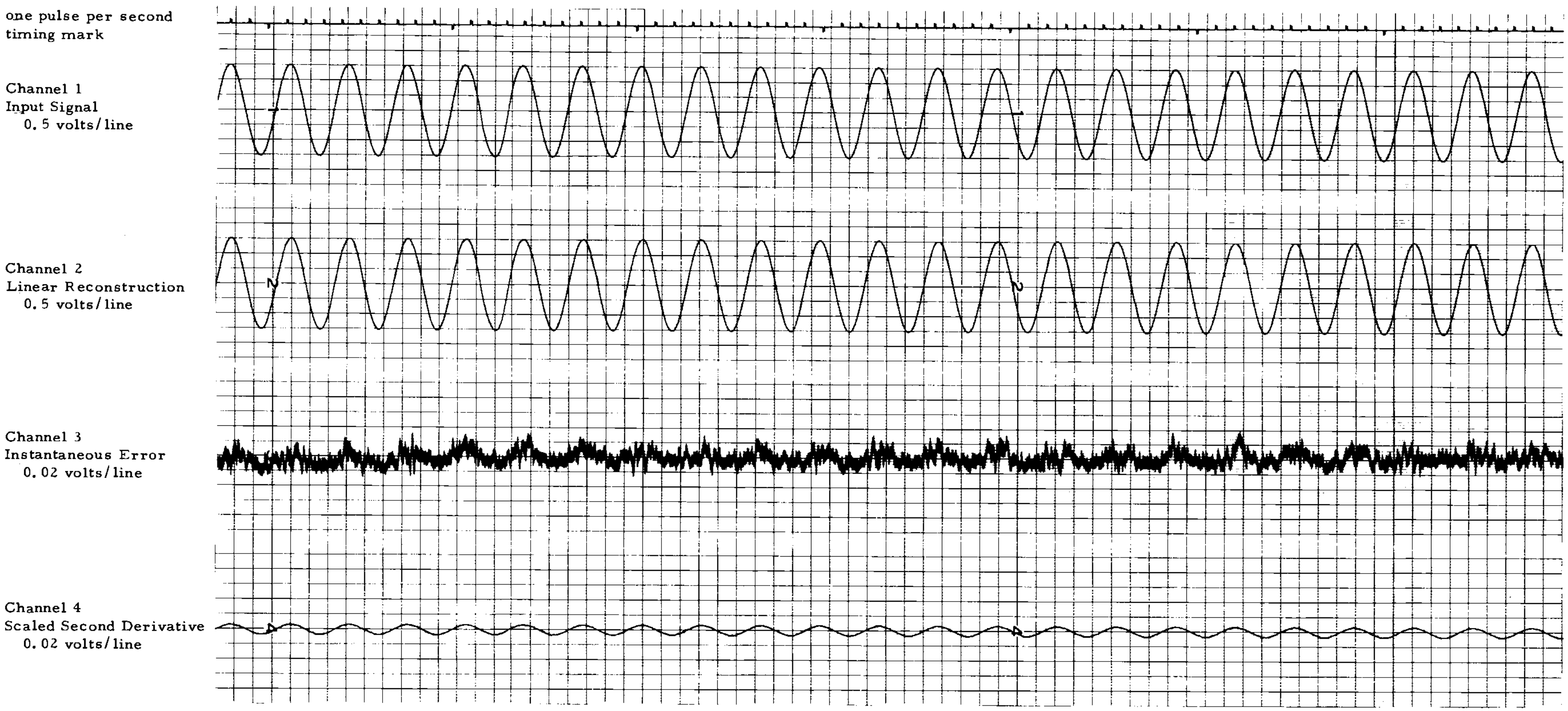


Figure 10c. Data plot for 32 samples per cycle of a sine wave.

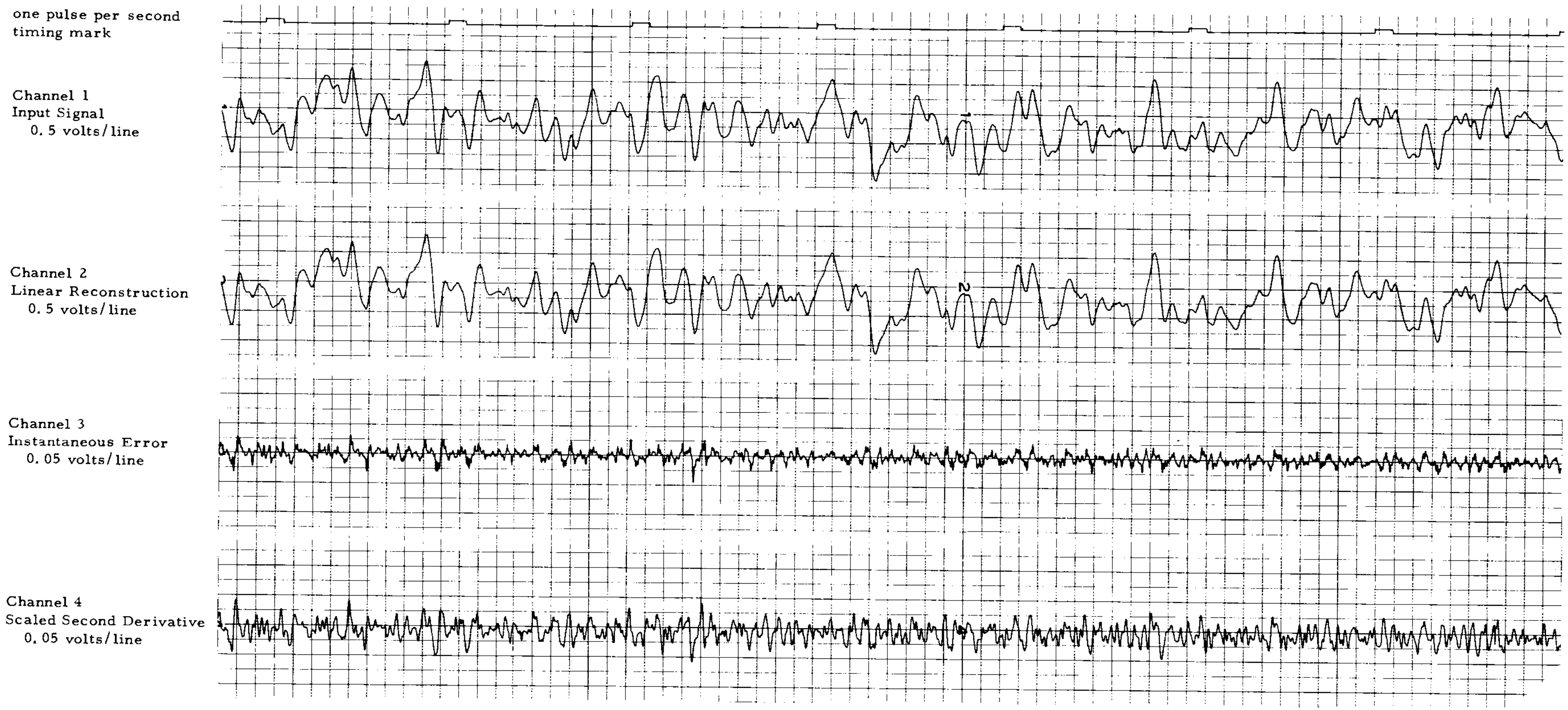


Figure 11a. Data plot for 400 samples per second of bandlimited noise.

Figure 11b. Data plot for 800 samples per second of bandlimited noise.

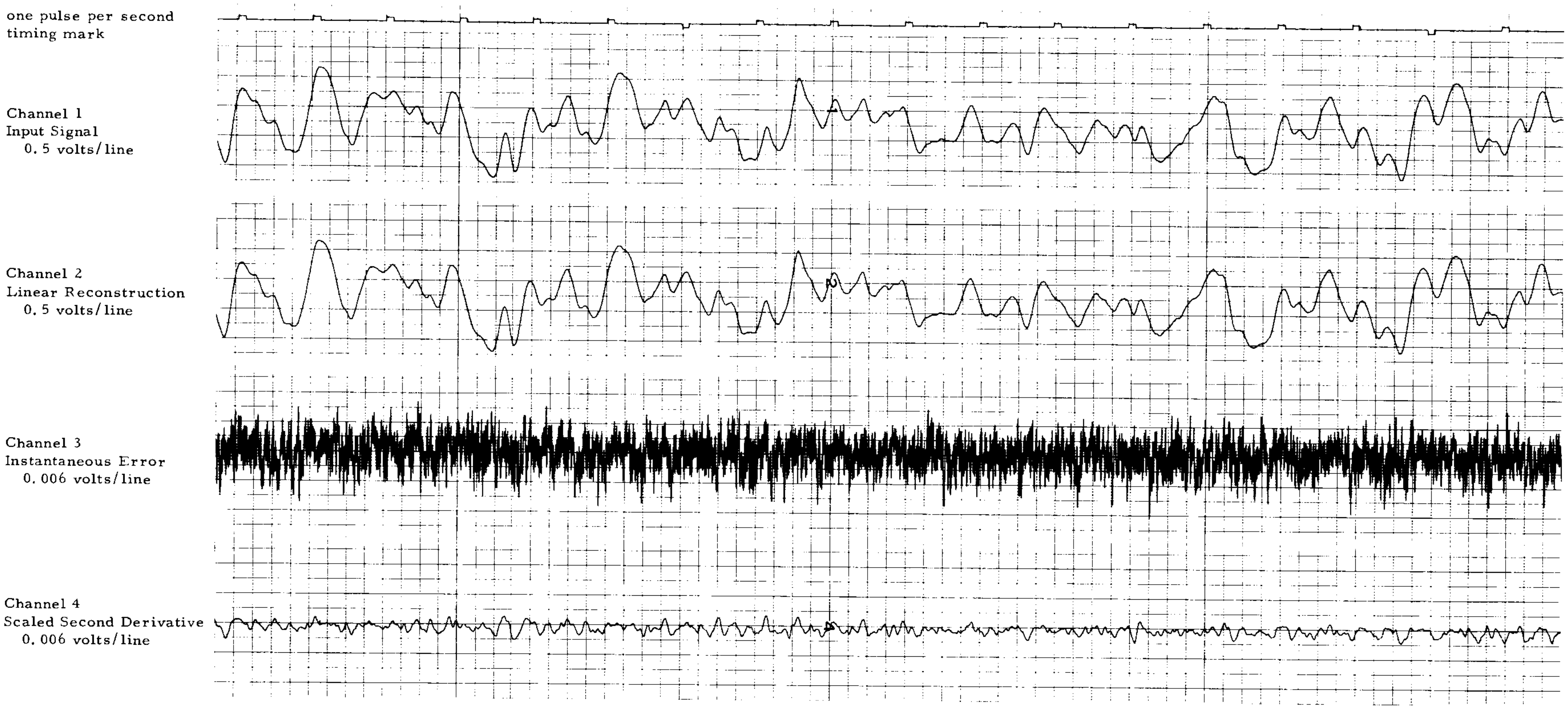


Figure 11c. Data plot for 1600 samples per second of bandlimited noise.

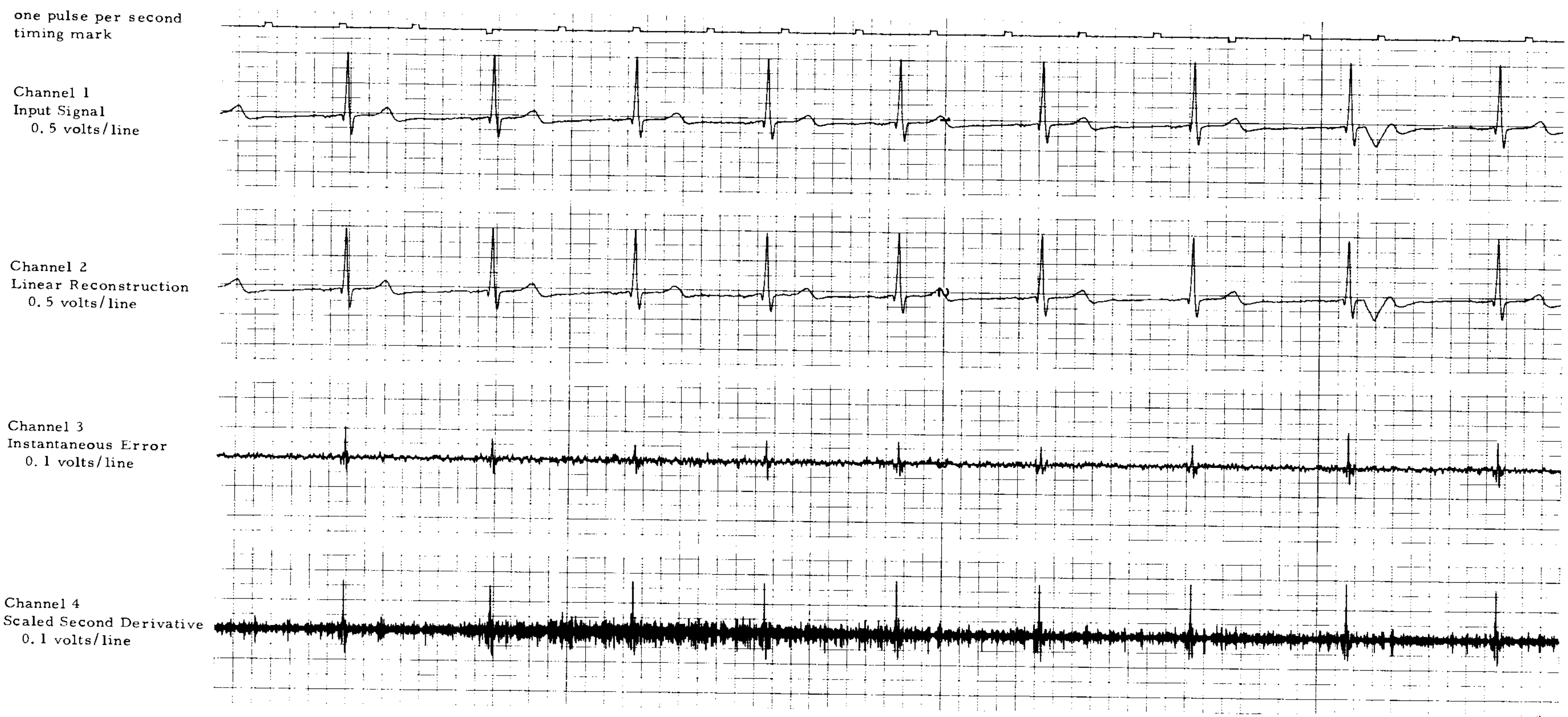


Figure 12a. Data plot for 200 samples per second of biological signal.

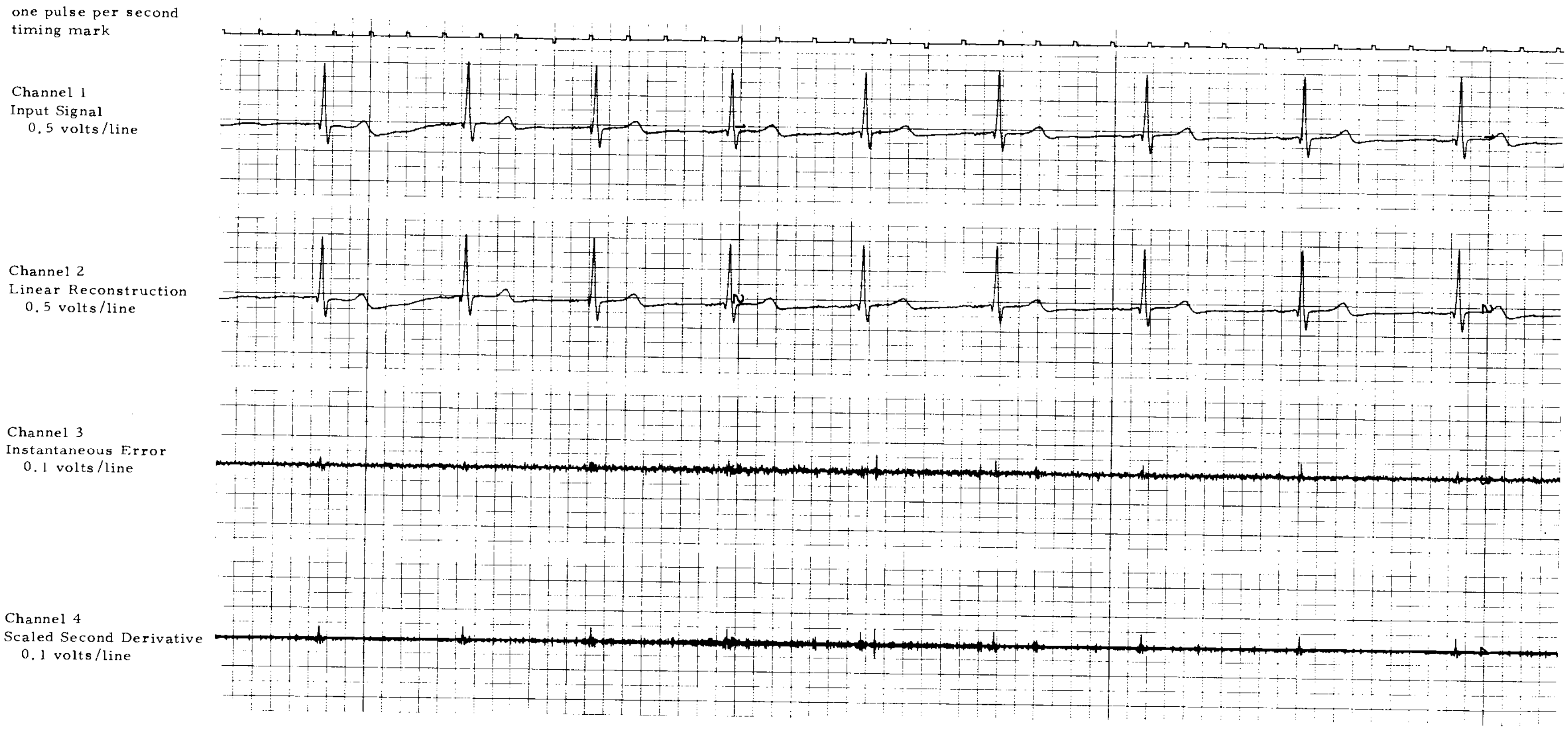


Figure 12b. Data plot for 400 samples per second of biological signal.

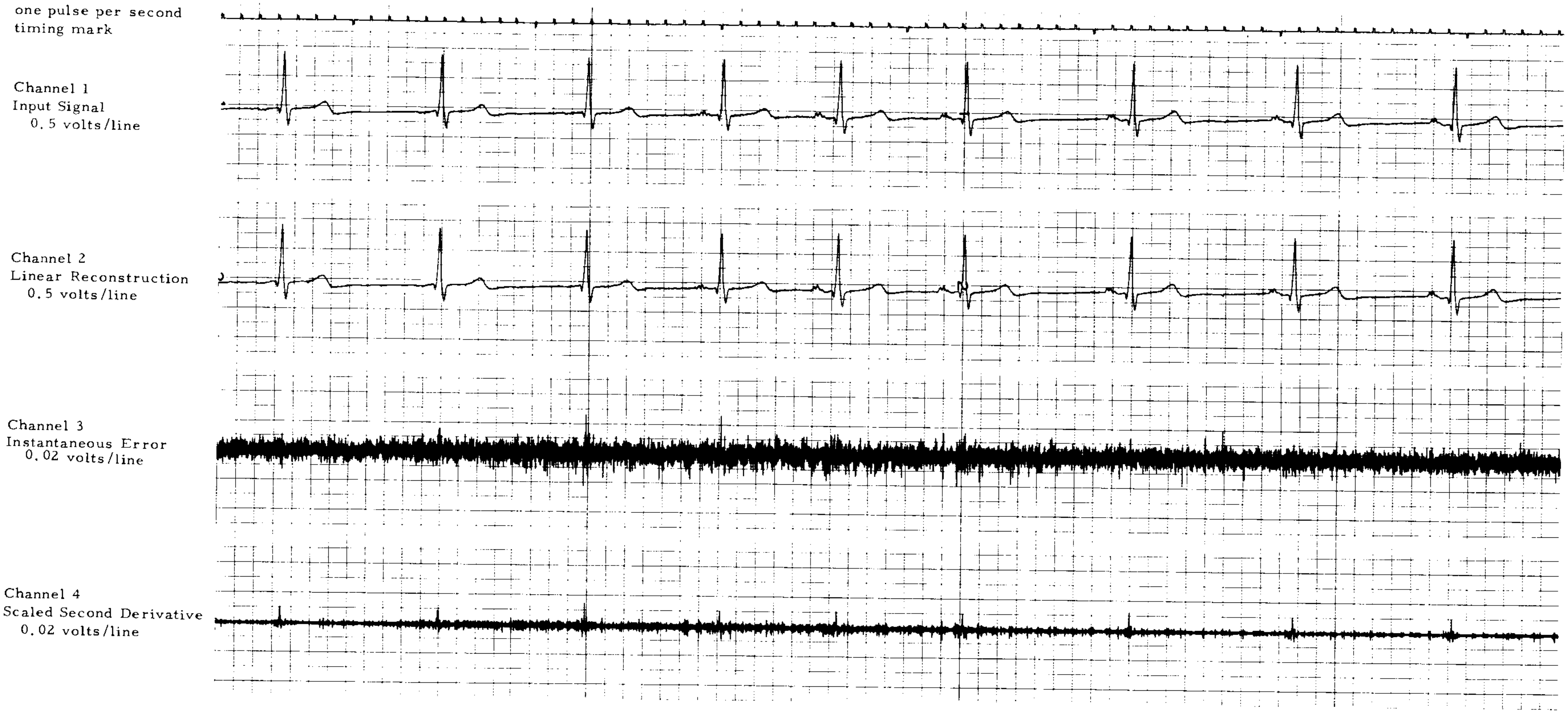


Figure 12c. Data plot for 800 samples per second of biological signal.

Table 3. Observed results for the experimental verification of the derivative sampling criterion.

Figure	Predicted limit error ϵ in volts	Error band ϵ_B* in volts	Total expected limit error ϵ_M* in volts	Observed worst-case error in volts
10				
a	0. 56	0. 18	0. 74	0. 66
b	0. 14	0. 18	0. 320	0. 225
c	0. 035	0. 18	0. 215	0. 20
11				
a	-0. 60	0. 2	0. 80	0. 45
b	0. 042	0. 2	0. 242	0. 145
c	-0. 006	0. 2	0. 206	0. 125
12				
a	1. 7	0. 2	1. 90	1. 2
b	0. 50	0. 2	0. 70	0. 60
c	0. 15	0. 2	0. 35	0. 265

* As defined in Figure 5.

V. CONCLUSION AND DISCUSSION

This thesis has presented a sampling criterion which provides a conservative estimate of possible errors arising from use of linear interpolation between sample points. The criterion can be used to determine the necessary sampling rate for a given signal in order that the worst-case peak error between a linear reconstruction of its samples and the signal itself will not exceed a specified limit. Conversely, if a sampling rate is chosen, the criterion will indicate what limit error is to be expected. The sampling criterion is based upon an instantaneous parameter, the second derivative, and is equally applicable to statistically stationary and non-stationary waveforms.

It has been shown that a measurement of the second derivative of an amplitude-time function can be approximated conveniently and economically using linear analog circuits so that the criterion may be implemented. Since this approximation to the second derivative of a signal can be made in delayed real-time, the methods developed in this paper might be extended to the control of a variable-rate sampling scheme for data compression where linear interpolation of sample points is used. Such a system would not necessitate a costly digital data processor for the control of the sampling rate.

As the bandwidth and noise specifications for operational amplifiers are improved, it will become possible to more accurately estimate the second derivative of a signal over a wider frequency range than has been accomplished in this project. Use of non-linear feedback techniques might lead to direct implementation for the square root of the absolute value of the second derivative as needed in variable-rate sampling control.

The advantage of variable-rate sampling is apparent from the tests of the biological signal. Most of the time signal activity is low and the error is small. When the signal moves, the error can become quite large for a short duration of time. A high sampling rate will minimize the error during the period of high activity, but will greatly over-sample the waveform most of the time. Variable-rate sampling could provide greater efficiency than is possible with fixed-rate techniques.

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APPENDIX

APPENDIX A

Theoretical Limit Errors for a Sine Wave

```
1.100    LF=2
1.200    ROR=4.9340/(LF**2)
1.300    "    SAMPLES PER CYC"LF,"  LIMIT ER"ROR:
1.400    LF=LF+1,   1.2.
```

THIS PROGRAM IS IN HYTRAN OPERATIONS INTERPRETER, AN INTER-ACTIVE ON-LINE LANGUAGE SYSTEM AVAILABLE ON THE EAI 690 HYBRID COMPUTER.

```
SAMPLES PER CYCLE =  2.00000,   LIMIT ERROR =  1.23350
SAMPLES PER CYCLE =  3.00000,   LIMIT ERROR =  .548222
SAMPLES PER CYCLE =  4.00000,   LIMIT ERROR =  .308375
SAMPLES PER CYCLE =  5.00000,   LIMIT ERROR =  .197360
SAMPLES PER CYCLE =  6.00000,   LIMIT ERROR =  .137056
SAMPLES PER CYCLE =  7.00000,   LIMIT ERROR =  .100694
SAMPLES PER CYCLE =  8.00000,   LIMIT ERROR =  .077093
SAMPLES PER CYCLE =  9.00000,   LIMIT ERROR =  .060913
SAMPLES PER CYCLE =  10.0000,   LIMIT ERROR =  .049340
SAMPLES PER CYCLE =  11.0000,   LIMIT ERROR =  .040776
SAMPLES PER CYCLE =  12.0000,   LIMIT ERROR =  .034263
SAMPLES PER CYCLE =  13.0000,   LIMIT ERROR =  .029195
SAMPLES PER CYCLE =  14.0000,   LIMIT ERROR =  .025173
SAMPLES PER CYCLE =  15.0000,   LIMIT ERROR =  .021928
SAMPLES PER CYCLE =  16.0000,   LIMIT ERROR =  .019273
SAMPLES PER CYCLE =  17.0000,   LIMIT ERROR =  .017072
SAMPLES PER CYCLE =  18.0000,   LIMIT ERROR =  .015228
SAMPLES PER CYCLE =  19.0000,   LIMIT ERROR =  .013667
SAMPLES PER CYCLE =  20.0000,   LIMIT ERROR =  .012335
SAMPLES PER CYCLE =  21.0000,   LIMIT ERROR =  .011188
SAMPLES PER CYCLE =  22.0000,   LIMIT ERROR =  .010194
SAMPLES PER CYCLE =  23.0000,   LIMIT ERROR =  .009327
SAMPLES PER CYCLE =  24.0000,   LIMIT ERROR =  .008565
SAMPLES PER CYCLE =  25.0000,   LIMIT ERROR =  .007894
SAMPLES PER CYCLE =  26.0000,   LIMIT ERROR =  .007298
SAMPLES PER CYCLE =  27.0000,   LIMIT ERROR =  .006768
SAMPLES PER CYCLE =  28.0000,   LIMIT ERROR =  .006293
SAMPLES PER CYCLE =  29.0000,   LIMIT ERROR =  .005866
SAMPLES PER CYCLE =  30.0000,   LIMIT ERROR =  .005482
```

```
SAMPLES PER CYCLE =  31.0000,   LIMIT ERROR =  .005134
SAMPLES PER CYCLE =  32.0000,   LIMIT ERROR =  .004818
SAMPLES PER CYCLE =  33.0000,   LIMIT ERROR =  .004530
SAMPLES PER CYCLE =  34.0000,   LIMIT ERROR =  .004268
SAMPLES PER CYCLE =  35.0000,   LIMIT ERROR =  .004027
SAMPLES PER CYCLE =  36.0000,   LIMIT ERROR =  .003807
SAMPLES PER CYCLE =  37.0000,   LIMIT ERROR =  .003604
SAMPLES PER CYCLE =  38.0000,   LIMIT ERROR =  .003416
SAMPLES PER CYCLE =  39.0000,   LIMIT ERROR =  .003243
SAMPLES PER CYCLE =  40.0000,   LIMIT ERROR =  .003083
SAMPLES PER CYCLE =  41.0000,   LIMIT ERROR =  .002935
SAMPLES PER CYCLE =  42.0000,   LIMIT ERROR =  .002797
SAMPLES PER CYCLE =  43.0000,   LIMIT ERROR =  .002668
SAMPLES PER CYCLE =  44.0000,   LIMIT ERROR =  .002548
SAMPLES PER CYCLE =  45.0000,   LIMIT ERROR =  .002436
SAMPLES PER CYCLE =  46.0000,   LIMIT ERROR =  .002331
SAMPLES PER CYCLE =  47.0000,   LIMIT ERROR =  .002233
SAMPLES PER CYCLE =  48.0000,   LIMIT ERROR =  .002141
SAMPLES PER CYCLE =  49.0000,   LIMIT ERROR =  .002054
SAMPLES PER CYCLE =  50.0000,   LIMIT ERROR =  .001973
SAMPLES PER CYCLE =  51.0000,   LIMIT ERROR =  .001896
SAMPLES PER CYCLE =  52.0000,   LIMIT ERROR =  .001824
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SAMPLES PER CYCLE =  55.0000,   LIMIT ERROR =  .001631
SAMPLES PER CYCLE =  56.0000,   LIMIT ERROR =  .001573
SAMPLES PER CYCLE =  57.0000,   LIMIT ERROR =  .001518
SAMPLES PER CYCLE =  58.0000,   LIMIT ERROR =  .001466
SAMPLES PER CYCLE =  59.0000,   LIMIT ERROR =  .001417
SAMPLES PER CYCLE =  60.0000,   LIMIT ERROR =  .001370
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SAMPLES PER CYCLE =  63.0000,   LIMIT ERROR =  .001243
SAMPLES PER CYCLE =  64.0000,   LIMIT ERROR =  .001204
SAMPLES PER CYCLE =  65.0000,   LIMIT ERROR =  .001167
SAMPLES PER CYCLE =  66.0000,   LIMIT ERROR =  .001132
SAMPLES PER CYCLE =  67.0000,   LIMIT ERROR =  .001099
SAMPLES PER CYCLE =  68.0000,   LIMIT ERROR =  .001067
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SAMPLES PER CYCLE =  71.0000,   LIMIT ERROR =  .000978
SAMPLES PER CYCLE =  72.0000,   LIMIT ERROR =  .000951
SAMPLES PER CYCLE =  73.0000,   LIMIT ERROR =  .000925
SAMPLES PER CYCLE =  74.0000,   LIMIT ERROR =  .000901
SAMPLES PER CYCLE =  75.0000,   LIMIT ERROR =  .000877
```

APPENDIX B

Actual Limit Errors for a Sine Wave

```
2.100    LE=2
2.200    X=6.28318/LE
2.300    RCR=1-COS(X/2)
2.400    "    SAMPLES PER CYC"LE," LIMIT ER"ROR:
2.500    LE=LE+1,   2.2.
```

THIS PROGRAM IS IN HYTRAN OPERATIONS INTERPRETER, AN INTER-ACTIVE ON-LINE LANGUAGE SYSTEM AVAILABLE ON THE EAI 690 HYBRID COMPUTER.

```
SAMPLES PER CYCLE =  2.00000,   LIMIT ERROR =  .999999
SAMPLES PER CYCLE =  3.00000,   LIMIT ERROR =  .500000
SAMPLES PER CYCLE =  4.00000,   LIMIT ERROR =  .292893
SAMPLES PER CYCLE =  5.00000,   LIMIT ERROR =  .190983
SAMPLES PER CYCLE =  6.00000,   LIMIT ERROR =  .133975
SAMPLES PER CYCLE =  7.00000,   LIMIT ERROR =  .099031
SAMPLES PER CYCLE =  8.00000,   LIMIT ERROR =  .076120
SAMPLES PER CYCLE =  9.00000,   LIMIT ERROR =  .060307
SAMPLES PER CYCLE =  10.0000,   LIMIT ERROR =  .048943
SAMPLES PER CYCLE =  11.0000,   LIMIT ERROR =  .040507
SAMPLES PER CYCLE =  12.0000,   LIMIT ERROR =  .034074
SAMPLES PER CYCLE =  13.0000,   LIMIT ERROR =  .029058
SAMPLES PER CYCLE =  14.0000,   LIMIT ERROR =  .025072
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SAMPLES PER CYCLE =  16.0000,   LIMIT ERROR =  .019215
SAMPLES PER CYCLE =  17.0000,   LIMIT ERROR =  .017027
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SAMPLES PER CYCLE =  19.0000,   LIMIT ERROR =  .013639
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SAMPLES PER CYCLE =  21.0000,   LIMIT ERROR =  .011169
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SAMPLES PER CYCLE =  27.0000,   LIMIT ERROR =  .006762
SAMPLES PER CYCLE =  28.0000,   LIMIT ERROR =  .006288
SAMPLES PER CYCLE =  29.0000,   LIMIT ERROR =  .005862
SAMPLES PER CYCLE =  30.0000,   LIMIT ERROR =  .005478
```

```
SAMPLES PER CYCLE =  31.0000,   LIMIT ERROR =  .005131
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SAMPLES PER CYCLE =  33.0000,   LIMIT ERROR =  .004528
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SAMPLES PER CYCLE =  73.0000,   LIMIT ERROR =  .000926
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SAMPLES PER CYCLE =  75.0000,   LIMIT ERROR =  .000877
```