ALGEBRAIC METHODS of ANALYSIS and SYNTHESIS of CONTACT NETWORKS

by

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T A B L E of C O N T E N T S

A C K N O W L E D G M E N T

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INTRODUCTION

The purpose of this paper is to discuss some of the theory of Boolean matrices and apply this theory to construct contact networks by using algebraic methods.

The first section of this paper (Chapter I to III) describes Boolean algebra and its properties, the theory of Boolean matrices, and the generalized associative and commutative laws.

The second part of this paper (Chapter IV to VII) discusses the application of Boolean matrices to switching theory.

CHAPTER I

Boolean Algebra

Definition 1.1: (Whitesett) A class of elements B together with two binary operations (+) and (·) is a Boolean Algebra if and only if the following postulates hold for elements $a,b,c \in B$:

P_1 The operations (+) and (·) are commutative,

ie. $a+b=b+a$ and $a \cdot b=b \cdot a$.

P_2 There exist in B distinct identity elements 0 and 1 relative to the operations (+) and (·), respectively,

ie. there exist elements $0,1 \in B$ such that $a+0=a$ and $a \cdot 1=a$.

P_3 Each operation is distributive over the other,

ie. $(a+b)c=a \cdot c+b \cdot c$ and $a+bc=(a+b)(a+c)$.

P_4 For every a in B there exixts an element a' in B such that

$$a+a'=1 \quad \text{and} \quad aa'=0.$$

The following properties of Boolean Algebra follow from the above.

Definition 1.2: If a,b,c B then we have:

(i) $a+a=a$ and $a \cdot a=a$

(ii) $a+1=1$ and $a \cdot 0=0$

(iii) $a+a \cdot b=a$ and $a(a+b)=a$

(iv) $a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c$

(v) $(a')'=a$

(vi) $(a \cdot b)'=a'+b'$ and $(a+b)'=a'b'$

(vii) $a+a'b=a+b$

<u>Definition 1.3</u>: If $a, b \in B$ and there exists c such that $a = bc$ then a is less than or equal to b (<u>ie.</u> $a \leq b$).

Example of Boolean Algebra: The set $\{0, 1\}$ with operations (+) and ($\cdot$) defined below is a Boolean Algebra.

+	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

x	x'
0	1
1	0

CHAPTER II

Definition 2.1: A Boolean algebra expression is a meaningful string of Boolean variables, values, operation signs, and parentheses which has a value obtained by applying the operators with the precedence rules as modified by the parentheses.

In a Boolean algebra we have the following precedence table for operations:

1^{st}	'
2^{nd}	$\cdot$
3^{rd}	+

Definition 2.2: Let $x_1, x_2, \ldots, x_n$ be n variables where x_i takes the value 1 or 0. Then the product of n variables is

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n = \begin{cases} 1 & \text{if all } x_i = 1, \\ 0 & \text{if at least one } x_i = 0. \end{cases}$$

Definition 2.3: Let $x_1, x_2, \ldots, x_n$ be n variables where x_i takes the value 1 or 0. Then the sum of n variables is

$$x_1 + x_2 + \ldots + x_n = \begin{cases} 0 & \text{if all } x_i = 0, \\ 1 & \text{if at least one } x_i = 1. \end{cases}$$

We shall let B_o be the Boolean algebra consisting of two elements, 0 and 1. The operations of (+) and ($\cdot$) for 0 and 1 from B_o are defined by the tables on the preceding page.

If to the Boolean algebra B_o are adjoined the variables $x_1, x_2, \ldots, x_n$, and the Postulates I-IV and Definition 1.2 are extended to the set then we shall have a new algebra $B_o(x_1, \ldots, x_n)$. The elements of $B_o(x_1, \ldots, x_n)$ are called the functions on the algebra B_o. A function is denoted by $f(x_1, \ldots, x_n)$.

Definition 2.4: By a representation of a Boolean function $f(x_1, \ldots, x_n)$ we will mean any expression which is the combining of a finite set of symbols each representing a constant or a variable by the operation of (+), (•), or ('), so that the result is a Boolean expression.

Example: (a) $f(x,y,z)=xy'+xy+xz$.

(b) $f(x,y,z)=x+x'y$.

Definition 2.5: Two functions $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are said to be equal if f has the same value as g when $x_1, x_2, \ldots, x_n$ take each set of values from B_o.

Generalized Associative Law

Let $a_1, a_2, \ldots, a_n$ be n distinct elements and let m_n be the number of distinct possible meanings for a product of $a_1, a_2, \ldots, a_n$ (taken in this order). M_n is a number of ways one can insert parentheses in the product formula for n factors.

If n=3

The possible products of a_1, a_2, a_3 are:

$$(a_1a_2)a_3 \,,\; a_1(a_2a_3).$$

$m_3=2$.

If n=4

The possible products of a_1, a_2, a_3, a_4 are:

$$((a_1a_2)a_3)a_4 \,,\; (a_1(a_2a_3))a_4 \,,\; (a_1a_2)(a_3a_4) \,,$$

$$a_1(a_2(a_3a_4)) \quad , \text{ and } \quad a_1((a_2a_3)a_4).$$

$m_4 = 5.$

Now we will show that if our operation is associative, then all the possible products of $a_1, a_2, \ldots, a_n$ taken in this order are equal.

Let us define a particular product $\prod_{i=1}^{n} a_i$ by the formula

$$\prod_{i=1}^{1} a_i = a_1, \quad \prod_{i=1}^{r+1} a_i = (\prod_{i=1}^{r} a_i) a_{r+1}.$$

<u>Lemma 2.6</u>: (Jacobson) $\prod_{i=1}^{n} a_i \prod_{j=1}^{m} a_{n+j} = \prod_{k=1}^{n+m} a_k.$

<u>Proof</u>: It is true for m=1 (by definition). Suppose it is true for m=r, then we are going to show that it is also true for m=r+1.

$$\prod_{i=1}^{n} a_i \prod_{j=1}^{r+1} a_{n+j} = \prod_{i=1}^{n} a_i ((\prod_{j=1}^{r} a_{n+j}) a_{n+r+1})$$

$$= (\prod_{i=1}^{n} a_i \prod_{j=1}^{r} a_{n+j}) a_{n+r+1}$$

$$= (\prod_{k=1}^{n+r} a_k) a_{n+r+1}$$

$$= \prod_{k=1}^{n+r+1} a_k$$

By suing the above lemma, any product associated with $(a_1, a_2, \ldots, a_n)$ can be obtained.

Set u is the product associated with $(a_n, \ldots, a_m)$, $1 < m < n$, and v is the product associated with $(a_{m+1}, \ldots, a_n)$.

$$u = \prod_{i=1}^{m} a_i$$

$$v = \prod_{j=1}^{n-m} a_{m+j}$$

$$u \circ v = \prod_{k=1}^{n} a_k \qquad \text{(by above lemma)}$$

Therefore all products determined by $(a_1, \ldots, a_n)$ are equal.

Since all the products determined by $(a_1, \ldots, a_n)$ are equal, we will omit all parentheses and denote this unique product as:

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n.$$

If all $a_i = a$ then we denote $a_1 \cdot a_2 \cdot \ldots \cdot a_n$ by a^n.

$$a^n a^m = a^{n+m}, \quad (a^n)^m = a^{nm} \tag{1}$$

If (+) is used, then $a_1 + a_2 + \ldots + a_n = na$ if all $a_i = a$.

The rules for multiples na:

$$ma + na = (m+n)a = a(m+n), \quad m(na) = (mn)a \tag{1'}$$

Generalized Commutative Law

Let $a_1, a_2, \ldots, a_n$ be n elements and $a_i a_j = a_j a_i$ for all i, j. Then we say that a_i and a_j commute.

Let $a_{1'} \cdot a_{2'} \cdot \ldots \cdot a_{n'}$ be a product where $1', 2', 3', \ldots, n'$ is some permutation of the numbers $1, 2, 3, \ldots, n$.

Suppose a_n occurs in the k^{th} position in this product,

<u>ie</u>. $a_{k'} = a_n$.

We can write

$$a_{1'} \cdot a_{2'} \cdot \ldots \cdot a_{n'} = a_{1'} \cdot a_{2'} \cdot \ldots \cdot a_{(k-1)'} a_{(k+1)'} \cdot \ldots \cdot a_{(n-1)'} a_{n'}.$$

Since a_i and a_j commute, we can assume by induction

$$a_{1'} \cdot a_{2'} \cdot \ldots \cdot a_{(k-1)'} \cdot a_{(k+1)'} \cdot \ldots \cdot a_{(n-1)'} = a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}.$$

Therefore,

$$a_{1'} \cdot a_{2'} \cdot \ldots \cdot a_{n'} = a_1 \cdot a_2 \cdot \ldots \cdot a_n.$$

If $a_1a_2=a_2a_1$, then we can write

$$(a_1a_2)^n=a_1^na_2^n=a_2^na_1^n. \tag{2}$$

CHAPTER III

Definition 3.1: Let B be any Boolean algebra. A Boolean matrix is an array whose components are functions over B.

For Boolean matrices the operation of addition and matrix multiplication are defined as they are defined for matrices over a Field.

We can write for Boolean matrices A and C the addition and multiplication as:

$$A+C\ ,\ AXC.$$

For these matrices the associative law and commutative law (for addition) and distributive law will also hold. The operation indicated by "X" is not a commutative operation. Thus Boolean matrix algebra can not be a Boolean algebra with "X" as the multiplication operation (see P_1 of Definition 1.1). If two Boolean matrices are written in order we shall understand that the matrix indicated is the above matrix product of the two.

Let S be the set of all nxn matrices where each element of the matrix is a function over B, an arbitrary algebra. The set of such matrices with the above operations will be denoted by S.

The following properties and definitions are obvious extensions of the situation for ordinary matrices and the above definition.

(i) If $A=(a_{ij})$ and $B=(b_{ij})$ for Boolean matrices with n rows and n columns then $A=B$ iff $a_{ij}=b_{ij}$ for all i and j.

(ii) The product AB is equal to C in which

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj},\ \text{for } i,j=1,2,\ldots,n,$$

the symbol $\sum_{k=1}^{n} t_k$ standing for the Boolean sum

$$t_1+t_2+ \ . \ . \ .+t_n.$$

(iii) The inequality $A \leqslant B$ means $a_{ij} \leqslant b_{ij}$ for all i and j, with inequality in the Boolean sense.

(iv) Determinant $|A| = \sum a_{1r_1} \cdot a_{2r_2} \cdot . \ . \ . \cdot a_{nr_n}$ where the summation is taken over all permutations $(r_1, \ . \ . \ ., r_n)$ of $(1, \ . \ . \ ., n)$.

(v) Transpose $A^T=(a_{ij})$ if $A=(a_{ij})$, $i,j=1, \ . \ . \ , n$.

(vi) $E=(a_{ij})$ where $a_{ij}=1$ if $i=j$ and $a_{ij}=0$ for $i \neq j$.

(vii) We shall call the matrix A "normal" if its diagonal elements are equal to 1, <u>ie</u>. $A \geqslant E$.

(viii) The n^{th} power of a Boolean matrix A is $A \cdot A \cdot . \ . \ . \cdot A$, with n factors in the indicated product. We write for it, A^n is in the ordinary matrix theory.

One useful theorem due to A.G. Lunts is given below.

<u>Theorem 3.2</u>: (Lunts) If A is a "normal" matrix of order n then there exists a positive integer $m \leqslant n-1$ such that

$$E \leqslant A \leqslant A^2 \leqslant A^3 \ . \ . \ . \leqslant A^m = A^{m+1}.$$

<u>Properties 3.3</u>: For $A \in S$ we have the following properties:

(I) The determinant is not changed by interchanging rows and columns.

(II) The determinant is not changed by interchanging two rows.

(III) The determinant may be expanded with respect to the elements of an arbitrary row or column.

$$|A| = \sum_{k=1}^{n} a_{ik} |A_{ik}| = \sum_{k=1}^{n} a_{kj} |A_{kj}| \ ,$$

where $|A_{ik}|$ is a minor of the determinant A of a matrix of (n-1) rows and (n-1) columns obtained by deleting the i^{th} row and the j^{th} column from the matrix A.

(IV) If all elements of a row have a common factor, then it may be taken outside the determinant sign.

(V) If each element of an arbitrary row is the sum of two terms, then the determinant is the sum of two determinants.

(VI) The matrix whose element in the i^{th} row and j^{th} column is the minor of the element in the j^{th} row and i^{th} column of A is called the adjoint of A.

CHAPTER IV

Let B_o be the Boolean algebra consisting of only two elements, 0 and 1. We shall say that a variable conductance of a contact with meaning from B_o, is equal to 1, when the contact is closed, and 0 when the contact is open.

If $x_1, x_2, \ldots, x_n$ are conductances of closed contacts of relays $X_1, X_2, \ldots, X_n$ respectively, they are independent variable quantities with values from the algebra B_o. Likewise for the conductances $x_1', x_2', \ldots, x_n'$ of open contacts. For brevity we shall speak of contacts x_i and x_i' rather than conductances of closed or open relay contacts.

Consider a set of n points and a network of paths joining these n points. These n points are called the nodes of a network. If there is a path that does not go through another node, starting at node P_i and ending at node P_j, we call its directly joining path P_{ij}.

If we consider a network having two or more nodes and if we fix two nodes of a network as terminals, then this contact network is called a two terminal network.

In the figure (a) given below, the network consists of four nodes and node 1 and 2 represent the therminals T_1 and T_2:

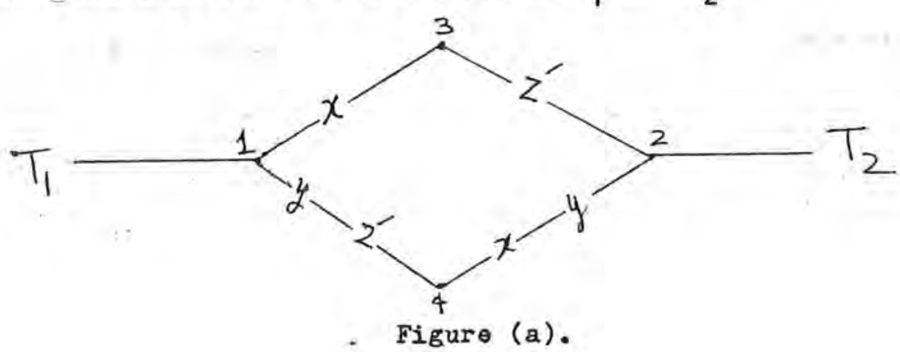

Figure (a).

We shall call a path directly joining the nodes P_i to P_j a <u>link</u> of a network. The link has conductance 1 (<u>ie</u>. closed contact) if the Boolean variable x_i associated with path P_i to P_j has the value <u>1</u> and it has conductance 0 (<u>ie</u>. open contact) if the Boolean variable x_i has the value <u>0</u>. If the above node P_i is connected to P_j through two paths one defined by p_{ij} and the other by q_{ij} in parallel, then the Boolean variable r_{ij} defining the compound connection from P_i to P_j satifies $r_{ij}=p_{ij}+q_{ij}$.

We shall call each uninterrupted path of the network connecting the terminals T_1 and T_2 an <u>elementary chain</u>. An elementary chain will be closed when and only when all its links are closed. The conductance of an elementary chain is equal to the product of the conductances of the links composing it. This product will be equal to 1 when all factors are equal to 1.

The conductance of a two terminal network is equal to the sum of the conductances of all its elementary chains. This sum is equal to 1 only when at least one elementary chain has conductance whose value is 1.

The given figure (a) has four elementary chains and the conductances corresponding to these chains are

$$xz'+xyxy+yz'xy+yz'yz'=xz'+xy+xyz'+yz'.$$

Obtaining the conductances of a given network is called <u>analysis</u> of the network and construction of a network from the given conductances is called the <u>synthesis</u>.

A <u>diode</u> element in the network passes the current in one definite direction. The conductance of a diode is equal to 1 in the direction of current and the conductance is equal to 0 in the opposite direction.

Contact Multipoles

Let us consider a given network N with an ordered sub set of its nodes $P_1, P_2, \ldots, P_n$. These nodes are also called its poles (ie. the words are interchangeable) and the network is called an n-pole network. Let a_{ij} denote the conductances of the links joining the nodes P_i and P_j. We set $a_{ij}=0$ if there is no link connecting P_i and P_j and set $a_{ij}=1$ if $i=j$. We shall call a_{ij} the immediate conductance from the pole P_i to P_j. We can easily write the "immediate conductances" in the form of a normal matrix of n rows A. This matrix will be symmetric if there is no diode element in the network. For the general case, of an arbitrary, perhaps non-symmetric matrix, for some poles P_i and P_j, a_{ij} may be different from a_{ji} and the only way to realize such a multipole is to use diodes.

Example: The immediate conductances matrix of Figure (b) is given below:

$$P= \begin{bmatrix} 1 & x & xy \\ x & 1 & yz \\ xy & yz & 1 \end{bmatrix}$$

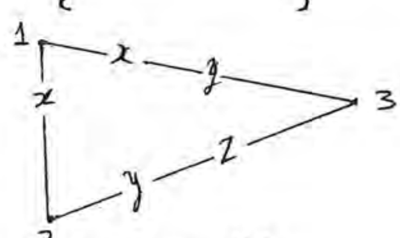

Figure (b).

In order to represent a contact network in matrix form, it is necessary to consider it as a multipole. Therefore, we will consider all nodes as poles.

Let $C_{ij}(A)$ denote the sum of the conductances of all elementary chains of the contact network N, leading from the pole P_i to P_j. $C_{ij}(A)=0$ if there is no such chain and $C_{ij}(A)=1$ if i=j. We shall call $C_{ij}(A)$ the complete conductance from the pole P_i to P_j of multipole A. The square matrix C(A) composed of the n^2 quantities $C_{ij}(A)$ whose (i,j) element is $C_{ij}(A)$ is termed the characteristic of the multipole A.

Remark 4.1: The complete conductance can be written as

$$C_{ij}(A)=\sum_{k_1,k_2,\ldots,k_m} a_{ik_1}a_{k_1k_2}\cdots a_{k_mj},$$

in which $(k_1,k_2,\ldots,k_m)$ run through all possible collections of elements $1,2,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,n$ for $m=0,1,2,\ldots,(n-2)$ elements.

Example: Let us consider the network consisting of four poles:

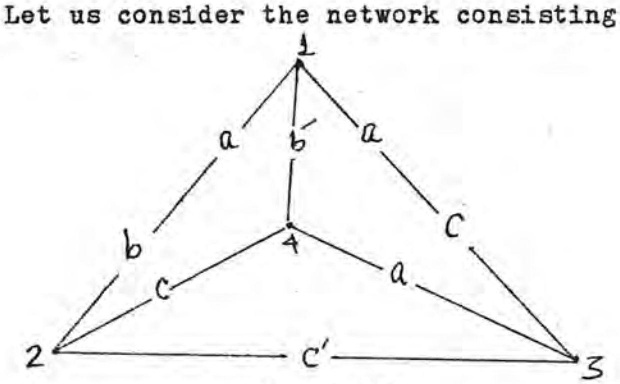

Figure (c)

$C_{11}(A)=C_{22}(A)=C_{33}(A)=C_{44}(A)=1.$

$C_{12}(A)=ab+b'c+ac+ab'c'=ab+b'c+ac+ab'=a+b'c=C_{21}(A).$

$C_{13}(A)=ac+b'a+abc'+abc=ac+a(b'+bc')=ac+ab'+ac'=a=C_{31}(A).$

$C_{14}(A)=b'+abc+ac+abc'=b'+ab+ac=b'+a+ac=a+b'=C_{41}(A).$

$C_{23}(A)=c'+abc+ac+ab'c=c'+ac+ac=c'+ac=c'+a=C_{32}(A).$

$C_{24}(A)=c+c'a=a+c=C_{42}(A).$

$C_{34}(A)=a+ab'c+abc=a+ac=a=C_{43}(A).$

The characteristic matrix of the multipole is equal to:

$$C(A)=\begin{bmatrix} 1 & a+b'c & a & a+b' \\ a+b'c & 1 & a+c' & a+c \\ a & a+c' & 1 & a \\ a+b' & a+c & a & 1 \end{bmatrix}.$$

Two n-poles A and B are said to be <u>equivalent</u>, written as $A\sim B$, if $C(A)=C(B)$, <u>i.e.</u> A and B have identical characteristics.

CHAPTER V

Analysis of Contact Networks

The fundamental problem of the analysis of contact networks is to obtain characteristic C(A) of the network from the given immediate conductance matrix. In this chapter we present an algebraic method of analysis of contact networks.

The following theorem gives the method of analysis of a multipole.

Theorem 5.1: (Lunts) If A is an nxn immediate conductance matrix corresponding to some contact network, then the characteristic matrix $C(A)=|A_{ji}|$, where $|A_{ji}|$ is the adjoint of A.

Let us apply the above theorem to the contact network of figure (c) (pg.15) in Chapter IV. The immediate conductance of figure (c) is given by:

$$A=\begin{bmatrix} 1 & ab & ac & b' \\ ab & 1 & c' & c \\ ac & c' & 1 & a \\ b' & c & a & 1 \end{bmatrix}$$

$|A_{11}|=|A_{22}|=|A_{33}|=|A_{44}|=1.$

$$|A_{12}|=\begin{vmatrix} ab & c' & c \\ ac & 1 & a \\ b' & a & 1 \end{vmatrix} \begin{aligned} &=ab(1+a)+c'(ac+ab')+c(ac+b') \\ &=ab+ab'c'+ac+b'c=a(b'c'+c)+b'c+ab \\ &=ab+ab'+ac+b'c=a+ac+b'c=a+b'c=|A_{21}|. \end{aligned}$$

$$\left|A_{13}\right| = \begin{vmatrix} ab & 1 & c \\ ac & c' & a \\ b' & c & 1 \end{vmatrix} \begin{aligned} &= ab(c'+ac)+(ac+ab')+c(ac+b'c') \\ &= abc'+abc+ac+ab' \\ &= abc'+ac+ab' = a = \left|A_{31}\right| . \end{aligned}$$

$$\left|A_{14}\right| = \begin{vmatrix} ab & 1 & c' \\ ac & c' & 1 \\ b' & c & a \end{vmatrix} \begin{aligned} &= ab(ac'+c)+(ac+b')+c'(ac+b'c') \\ &= ab+abc+ac+b'+b'c' \\ &= ab+ac+b'+b'c' \\ &= a+b'+ac+b'c' = a+b' = \left|A_{41}\right| . \end{aligned}$$

$$\left|A_{23}\right| = \begin{vmatrix} 1 & ab & b' \\ ac & c' & a \\ b' & c & 1 \end{vmatrix} \begin{aligned} &= (c'+ac')+ab(ac+ab')+b'(ac+b'c') \\ &= c'+ac'+abc+ab'c+b'c' \\ &= c'+ac'+ac+b'c' = c'+a+b'c' \\ &= c'+a = \left|A_{32}\right| . \end{aligned}$$

$$\left|A_{24}\right| = \begin{vmatrix} 1 & ab & ac \\ ac & c' & 1 \\ b' & c & a \end{vmatrix} \begin{aligned} &= (ac'+c)+ab(ac+b')+ac(ac+b'c') \\ &= a+c+abc+ac \\ &= a+c+ac = a+c = \left|A_{42}\right| . \end{aligned}$$

$$\left|A_{34}\right| = \begin{vmatrix} 1 & ab & ac \\ ab & 1 & c' \\ b' & c & a \end{vmatrix} \begin{aligned} &= a+ab(ab+b'c')+ac(abc+b') \\ &= a+ab+abc+ab'c \\ &= a+ab+ab'c = a = \left|A_{43}\right| . \end{aligned}$$

Since matrix A is a symmetric matrix, therefore the transpose of $A^T = A$, since $a_{ij} = a_{ji}$.

Therefore, the adjoint of A is given by:

$$\begin{bmatrix} |A_{11}| & |A_{12}| & |A_{13}| & |A_{14}| \\ |A_{21}| & |A_{22}| & |A_{23}| & |A_{24}| \\ |A_{31}| & |A_{32}| & |A_{33}| & |A_{34}| \\ |A_{41}| & |A_{42}| & |A_{43}| & |A_{44}| \end{bmatrix}$$

Thus the adjoint of A is:

$$\begin{bmatrix} 1 & a+b'c & a & a+b' \\ a+b'c & 1 & a+c' & a+c \\ a & a+c' & 1 & a \\ a+b' & a+c & a & 1 \end{bmatrix}$$

which is equivalent to C(A) given in Chapter IV.

<u>Definition 5.2</u>: (Lunts) The characteristic function of the immediate conductance matrix A is the Boolean function in n variables $x_1, x_2, \ldots, x_n$,

$$f_A(x_1,\ldots,x_n) = \sum_{i,j=1}^{n} a_{ij}x_i x_j'. \tag{1}$$

The characteristic function of the immediate conductance matrix of Figure (c) (pg. 15) in Chapter IV is given by:

$$f_A(x_1,x_2,x_3,x_4) = ab(x_1x_2'+x_2x_1') + ac(x_1x_3'+x_3x_1') + b'(x_1x_4'+x_4x_1') +$$
$$c'(x_2x_3'+x_3x_2') + c(x_2x_4'+x_4x_2') + a(x_3x_4'+x_4x_3').$$

<u>Theorem 5.3</u>: (Lunts) If A and B are the immediate conductance matrices of two networks, then the characteristic matrices

$$C(A) \leqslant C(B) \tag{2}$$

$$\text{iff } f_A(x_1,\ldots,x_n) \leqslant f_B(x_1,\ldots,x_n). \tag{3}$$

<u>Lemma 5.4</u>: (Lunts) The equation:

$$f_A(x_1,\ldots,x_n) = f_{C(A)}(x_1,\ldots,x_n) \tag{4}$$

always holds.

Lemma 5.5: (Lunts) Every row of the characteristic matrix $C(A)$ appears as a solution of the characteristic equation $f_A(x_1,\ldots,x_n)$, ie. $f_A(C_{k1}(A),C_{k2}(A),\ldots,C_{kn}(A))=0$ (5)

for $k=1,2,\ldots,n$.

In order to show A and B are equivalent networks, we need only to show

$$f_A(x_1,\ldots,x_n)=f_B(x_1,\ldots,x_n).$$

Definition 5.6: The greatest lower bound of Boolean function $f(x)$ is $f(0)\cdot f(1)$.

Definition 5.7: The process of replacing $f(x)$ by $f(0)\cdot f(1)$ is called the exclusion of the variable x from $f(x)$. We will write $(Ex)f(x)=f(0)\cdot f(1)$.

The reverse process of finding $f(x)$ is called the introduction of variable x. The exclusion process yields a unique result whereas the introduction process does not.

Example: (i) If $f(x)=ax+bx'+c$, (ii) If $f(x)=ax+bx'+c+ab$;

Both above functions give

$$(Ex)f(x)=ab+c.$$

Introduction of x to ab+c yields $f(x)=ax+bx'+c$. Thus the function $f(x)$ is not unique.

In his paper A. G. Lunts shows that if from the characteristic function $f_A(x_1,\ldots,x_n)$ we exclude the variable x_n, then we get the characteristic function of (n-1) pole:

$$(Ex_n)f_A(x_1,\ldots,x_n)=\sum_{i,j=1}^{n-1} a_{ij}x_ix'_j+\sum_{i=1}^{n-1}a_{in}x_i\cdot\sum_{j=1}^{n-1}a_{nj}x'_j$$

$$=\sum_{i,j=1}^{n-1}(a_{ij}+a_{in}a_{nj})x_i x_j' = f_B(x_1,\ldots,x_n) \qquad (6)$$

where $b_{ij}=a_{ij}+a_{in}a_{nj}$, $(i,j=1,\ldots,n-1)$. (7)

We shall say that the matrix B is obtained by the exclusion of pole P_n from the characteristic function of A.

<u>Theorem 5.8</u>: (Lunts) In the exclusion of the pole P_n from the multipole A, the elements of the characteristic corresponding to the remaining poles remain unchanged.

<u>ie</u>. $C_{ij}(B)=C_{ij}(A)$, $(i,j=1,\ldots,n-1)$.

The immediate conductance matrix A for figure (c) (pg. 15) in Chapter IV is given below:

$$A=\begin{bmatrix} 1 & ab & ac & b' \\ ab & 1 & c' & c \\ ac & c' & 1 & a \\ b' & c & a & 1 \end{bmatrix}$$

If we exclude variable x_4, then we get the characteristic function of 3-pole B:

$$\begin{aligned}(Ex_4)f_A(x_1,\ldots,x_4) &= ab(x_1x_2'+x_1'x_2)+ac(x_1x_3'+x_1'x_3)+c'(x_2x_3'+x_2'x_3)\\ &\quad +(b'x_1+cx_2+ax_3)(b'x_1'+cx_2'+ax_3')\\ &=(ab+b'c)(x_1x_2'+x_1'x_2)+(ac+b'a)(x_1x_3'+x_1'x_3)\\ &\quad +(ac+c')(x_2x_3'+x_2'x_3).\end{aligned}$$

The immediate conductance matrix B of the 3-pole can be written as:

$$B=\begin{bmatrix} 1 & ab+b'c & ac+ab' \\ ab+b'c & 1 & a+c' \\ ac+ab' & a+c' & 1 \end{bmatrix}$$

The characteristic matrix B can be obtained using the adjoint method:

$$|B_{11}|=|B_{22}|=|B_{33}|=1.$$

$$\begin{aligned}
|B_{12}| &= ab+b'c+(a+c')(ac+ab') \\
&= ab+b'c+ac+ab'+ab'c \\
&= a+b'c+ab'c=a+b'c=|B_{21}|. \\
|B_{13}| &= (ab+b'c)(a+c')+ac+ab' \\
&= ab+abc'+ab'c+ac+ab' \\
&= a+ac=a=|B_{31}|. \\
|B_{23}| &= (a+c')+(ab+b'c)(ac+ab') \\
&= a+c'+abc+ab'c=a+c'=|B_{32}|.
\end{aligned}$$

$$C(B)=\begin{bmatrix} 1 & a+b'c & a \\ a+b'c & 1 & a+c' \\ a & a+c' & 1 \end{bmatrix}.$$

The elements of the characteristic matrix corresponding to the remaining poles are the same as the elements of the characteristic of A,

$$C_{ij}(B)=C_{ij}(A),(i,j=1,2,3)$$

for the above example when we exclude variable x_4. Thus the Theorem 5.8 is illustrated for the above example.

When we exclude the variable x_3 from B. we get the characteristic function of the 2-pole D:

$$\begin{aligned}
(Ex_3)f_A(x_1,\ldots,x_4) &= (ab+b'c)(x_1x_2'+x_1'x_2)+((ac+ab')x_1+(a+c')x_2) \\
&\quad \cdot((ac+ab')x_1'+(a+c')x_2') \\
&= (ab+b'c)(x_1x_2'+x_1'x_2)+(ac+ab')(x_1x_2'+x_1'x_2) \\
&= (ab+b'c+ac+ab')(x_1x_2'+x_1'x_2) \\
&= (a+b'c)(x_1x_2'+x_1'x_2)
\end{aligned}$$

Matrix D is given by:

$$D= \begin{bmatrix} 1 & a+b'c \\ a+b'c & 1 \end{bmatrix} =C(D) .$$

<u>Corollary 5.9</u>: If A is a "normal" matrix of order n then there exists an integer $m \leqslant n-1$ such that

$$A^m=A^{m+1}= . . . =C(A) .$$

Algorithm for analysis of contact networks:

Given any finite network with an ordered sub set of its nodes $P_1, P_2, . . ., P_n$:

(1) Write an immediate conductances matrix A for the given network.

(2) Set $i=1$.

(3) Compute $AA^i=B$.

(4) If $B=A^i$, go to step 6.

(5) If $B\neq A^i$, replace A^i by B and go to step 3.

(6) Stop. B is the characteristic matrix C(A).

Let us apply the above algorithm to the network of Figure (c) (pg. 15) whose immediate conductances matrix A is:

$$A= \begin{bmatrix} 1 & ab & ac & b' \\ ab & 1 & c' & c \\ ac & c' & 1 & a \\ b' & c & a & 1 \end{bmatrix} \qquad \text{(Step 1)}$$

$i=1.$ (Step 2)

$$B=AA=A^2=\begin{bmatrix} 1 & ab+b'c & a & ac+b' \\ ab+b'c & 1 & a+c' & a+c \\ a & a+c' & 1 & a \\ ac+b' & a+c & a & 1 \end{bmatrix} \quad \text{(Step 3)}$$

$B \not\leq A^1$, replace A^1 by B and go to step 3. (Step 5)

$$B=AA^2=\begin{bmatrix} 1 & a+b'c & a & a+b' \\ a+b'c & 1 & a+c' & a+c \\ a & a+c' & 1 & a \\ a+b' & a+c & a & 1 \end{bmatrix} =C(A). \quad \text{(Step 3)}$$

CHAPTER VI

Synthesis of Contact Networks

Introduction of a new variable into the characteristic function can be done by transforming the characteristic function to the form:

$$f_A(x_1,\ldots,x_n)=\sum_{i=1}^{n} b_{i,n=1}x_i \circ \sum_{j=1}^{n} b_{n+1,j}x'_j + \sum_{i,j=1}^{n} b_{ij}x_i x'_j \qquad (8)$$

where b_{ij} is the coefficient of the (n+1)-pole B.

By using the above formula we can obtain an (n+1)-pole B from the n-pole A by introduction of a new pole P_{n+1}. If the original multipole A is symmetric and we want the (n+1)-pole B to be symmetric, then it is necessary that the coefficients of the right side of (8) satisfy the symmetry condition:

$$b_{ij}=b_{ji}\ ,\ (i,j=1,2,\ .\ .\ .,n+1).$$

One may introduce a new pole into the multipole in many ways since the chatacteristic function f_A can be represented in many ways in the form of (8).

The transformation of multipoles in a more general sense than that given above can be obtained by the method given below using characteristic functions.

Suppose $P_1,P_2,\ .\ .\ .,P_m,P_{m+1},\ .\ .\ .,P_n$ are the n-poles of A in which $P_1,\ .\ .\ .,P_m$ are the <u>essential poles</u> and the remaining poles are the <u>auxiliary poles</u>.

Essential poles are those poles in the network which coincide with the poles of the electric network considered; the remaining poles which we have called auxiliary poles play a less important role. The n-poles of A solve the given problem and so does every q-pole B ($q \geqslant m$) for which:

$$C_{ij}(B)=C_{ij}(A)\ ,\ (i,j=1,2,\ .\ .\ .,m). \tag{9}$$

Let $f_B(x_1,\ldots,x_m,x_{m+1}^{(1)},\ldots,x_q^{(1)})$ be the characteristic function corresponding to the poles $P_1,P_2,\ .\ .\ .,P_m,P'_{m+1},\ .\ .\ .,P'_q$ of B and let $f_A(x_1,\ldots,x_n)$ be the characteristic function corresponding to the multipole A.

It is not difficult to show that, by the exclusion and introduction of new poles the multipole A may be transformed into the multipole B by using equation (10).

$$(Ex_{m+1})\circ\ldots\circ(Ex_n)f_A(x_1,\ldots,x_n)$$
$$=(Ex_{m+1}^{(1)})\circ\ldots\circ(Ex_q^{(1)})f_B(x_1,\ldots,x_m,x_{m+1}^{(1)},\ldots,x_q^{(1)}). \tag{10}$$

By exclusion of auxiliary poles $x_{m+1},\ .\ .\ .,x_n$ the function $f_A(x_1,\ldots,x_n)$ is transformed into the left side of equation (10). Furthermore, the left side may be transformed into f_B by introduction of the parameters $x_{m+1}^{(1)},\ .\ .\ .,x_q^{(1)}$.

Example: We shall construct two terminal networks with given conductances as stated. (Lunts) constructed a two terminal network with the conductances:

$C_{12}(A)=bd+ac+ad;$

$C_{21}(A)=bd+ac+bc.$

In this example we start with a non-symmetric core. Hence, there must be diodes in the original multipole and the transformed multipole will necessarily contain a diode since its characteristic matrix will be non-symmetric.

The two pole network corresponding to the above conductance is shown in figure (A).

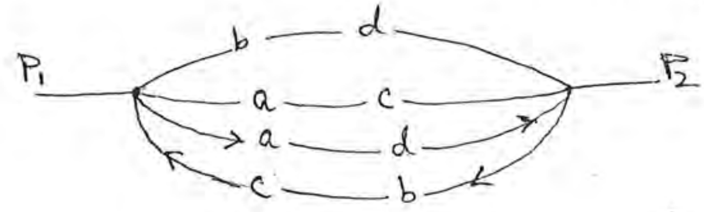

Figure (A)

We transform the characteristic function of the two pole:

$$f(x_1,x_2)=(bd+ac+ad)x_1x_2'+(bd+ac+bc)x_2x_1'$$

$$=(ax_1+cx_2)(ax_1'+bx_1'+cx_2'+dx_2')+bd(x_1x_2'+x_2x_1').$$

Introducing the variable x_3:

$$f(x_1,x_2,x_3)=(ax_1+cx_2)x_3'+(ax_1'+bx_1'+cx_2'+dx_2')x_3+bd(x_1x_2'+x_2x_1').$$

By using mod 2 addition:

$$x_ix_j'+x_j'x_i=x_i\oplus x_j.$$

$$f(x_1,x_2,x_3)=a(x_1\oplus x_3)+c(x_2\oplus x_3)+dx_2'x_3+bx_1'x_3+bd(x_1\oplus x_2).$$

But $bd(x_1\oplus x_2)=bd(x_1x_2'+x_2x_1')=(bx_1+dx_2)(bx_1'+dx_2')$.

Therefore:

$$f(x_1,x_2,x_3)=a(x_1\oplus x_3)+c(x_2\oplus x_3)+(bx_1+dx_2+x_3)(bx_1'+dx_2').$$

Introducing x_4:

$$f(x_1,x_2,x_3,x_4)=a(x_1\oplus x_3)+c(x_2\oplus x_3)+(bx_1+dx_2+x_3)x_4'+(bx_1'+dx_2')x_4.$$

$$(bx_1+dx_2)x_4'+(bx_1'+dx_2')x_4=b(x_1\oplus x_4)+d(x_2\oplus x_4).$$

$$f(x_1,x_2,x_3,x_4)=a(x_1\oplus x_3)+c(x_2\oplus x_3)+b(x_1\oplus x_4)+d(x_2\oplus x_4)+x_3x_4'.$$

The four pole network is:

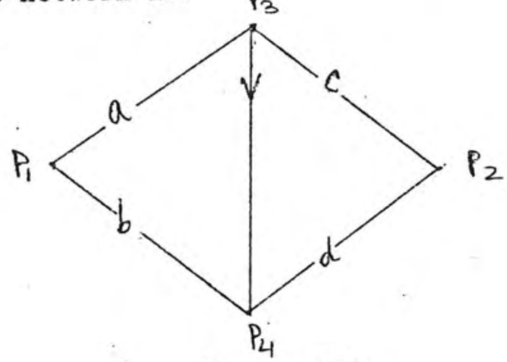

Figure (B).

CHAPTER VII

A Matrix Method for the Design of Multipole Contact Networks

Let us consider the circuit having n inputs $x_1, x_2, \ldots, x_n$ and t outputs in which the state of the connection between any two output terminals depends only upon the values of all or some of the input variables.

This can be written as:

$$f_{ij}=f_{ij}(x_1,\ldots,x_n)$$

since $f_{ii}=1$ for $i=j$,

the t^2 such switching functions $f_{ij}(x_1,\ldots,x_n)$ which can be used as components of a txt Boolean matrix is called the <u>output matrix</u> $F=[f_{ij}]$. It is the same as the characteristic matrix of the immediate conductance matrix of the multipole.

<u>Example</u>: The output matrix of figure (1) is:

(1)

$$F=\begin{bmatrix} 1 & x & z+xy' \\ x & 1 & y'z'+xz \\ z+xy' & y'z'+xz & 1 \end{bmatrix}$$

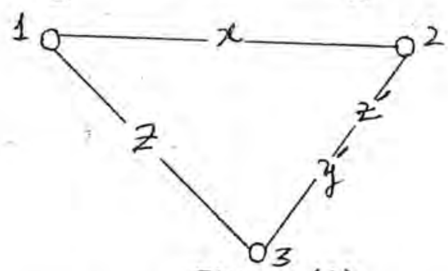

Figure (1)

In a given circuit, we can associate a sufficient number of non-terminal nodes t+1, t+2, . . ., t+k such that between any two of the t+k nodes of the circuit there appears at most a single contact or a group of single contacts in parallel. Let p_{ij} represent the direct connection (immediate conductance) between nodes i and j if $i \neq j$, and $p_{ij}=0$ if there is no direct connection between nodes i and j, and $p_{ij}=1$ if $i=j$.

If there are k non-terminal nodes, then the matrix $P=[p_{ij}]$ is called the <u>primitive connection</u> matrix of order t+k.

We will use dots for non-terminal nodes and circles for terminal nodes in examples.

<u>Example</u>: The primitive connection matrix for Figure (2) is
(2)
given below:

$$P=\begin{bmatrix} 1 & x & 0 & x' & 0 \\ x & 1 & 0 & 0 & z \\ 0 & 0 & 1 & y & x \\ x' & 0 & y & 1 & 0 \\ 0 & z & x & 0 & 1 \end{bmatrix}$$

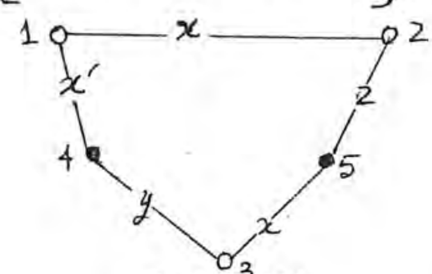

Figure (2).

There is a third type of Boolean matrix described by Semmon as <u>connective matrix</u> in which the components are switching functions describing the logical links connecting pairs of nodes; terminal and non-terminal, but the number of non-terminal nodes selected need not be large to lead a primitive connection matrix in the sense of Hohn and Schissler, connection matrix is the same as the immediate conductance matrix of the multipole.

<u>Example</u>: Connection matrix of Figure (3) is

(3)

$$C=\begin{bmatrix} 1 & x & x'y & x \\ x & 1 & z & y \\ x'y & z & 1 & y' \\ x & y & y' & 1 \end{bmatrix}.$$

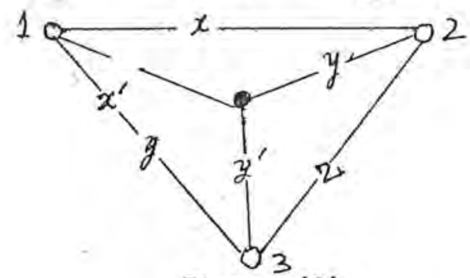

Figure (3).

Evidently, the connection matrix is usually not unique. It is also worth noting that the matrices we use are all connection matrices, the output matrix and primitive connection matrices simply being extreme cases.

A procedure for analysis of Boolean matrices is as follows:

We start out with the matrix of a given circuit which has non-terminal and terminal nodes. Let us consider a matrix C whose entry in the i^{th} row and j^{th} column is denoted by C_{ij}. Suppose this matrix C has k non-terminal nodes and t terminal nodes. Let r be any non-terminal node of the (t+k)x(t+k) connection matrix C. Then we say that there exists a path from node i to node j containing only the node r if and only if $C_{ir}C_{rj}=1$. If in C we replace each entry C_{ij} by the function $C_{ij}+C_{ik}C_{kj}$ and cross out all the entries C_{ik} and C_{kj}, we will get a connection matrix of order one less, <u>ie</u>. t+k-1. We will denote this matrix by C_{-k}. We will repeat this process until only terminal nodes remain. The resulting txt matrix is called the reduced connection matrix C_o.

<u>Example</u>: Let us consider the non-terminal removal process for
(4)
the circuit in figure (2).

The connection matrix is:

$$C = \begin{bmatrix} 1 & x & 0 & x' & 0 \\ x & 1 & 0 & 0 & z \\ 0 & 0 & 1 & y & x \\ x' & 0 & y & 1 & 0 \\ 0 & z & x & 0 & 1 \end{bmatrix}$$

The steps of the procedure for eliminating node 4 and 5 is illustrated by the equations:

$$C_{-5} = \begin{bmatrix} 1 & x & 0 & x' \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ x' & 0 & y & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ z \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & z & x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x & 0 & x' \\ x & 1 & xz & 0 \\ 0 & xz & 1 & y \\ x' & 0 & y & 1 \end{bmatrix} ;$$

$$C_o = C_{-5,-4} = \begin{bmatrix} 1 & x & 0 \\ x & 1 & xz \\ 0 & xz & 1 \end{bmatrix} + \begin{bmatrix} x' \\ 0 \\ y \end{bmatrix} \begin{bmatrix} x' & 0 & y \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & x & x'y \\ x & 1 & xz \\ x'y & xz & 1 \end{bmatrix} .$$

The above C_o matrix is the reduced connection matrix.

The relationship between connection and output matrices was first stated by Lunts and later generalized by Hohn and Schissler; it may be stated as the following theorem:

Theorem 7.1: If C is any connection matrix of a t-terminal circuit, C_o the corresponding reduced connection matrix, and if F is the output matrix of matrix C, then there exists an integer $k, 1 \leqslant k < t$, such that $C_o^{t-k} = F$.

Proof: The generalized form of the relationship as stated above can be justified by reference to the theorem 3.2. From this theorem (ie. Theorem 3.2) it follows that there exists an integer $k, 1 \leqslant k < t$ such that $C_o^{t-k} = C_o^{t-k+1} = C_o^{t-k+2} = \ldots$; it is therefore only necessary to show that $C_o^{t-1} = F$. If the components of C_o are denoted by C_{ij}, then the component of C_o^2 in row i and column j is

$$C_{i1}C_{1j} + C_{i2}C_{2j} + \ldots + C_{it}C_{tj}.$$

This function is equal to one for $i \neq j$ only if the input variables are in condition such that there exists a path from node i to node j or else a path via one intermediate node. In a similar manner, the component of C_o^3 in row i and column j is equal to 1 only when there is either a direct path from node i to node j, or a path via one intermediate node or two intermediate nodes. Since no path requires more than t-2 intermediate nodes, we see that ij-entry of C_o^{t-1} is a function which is 1 when and only when the circuit variables are such as to interconnect i and j. That is, $C_o^{t-1} = F$.

The output matrix of example (2) is given by the theorem 7.1.

$$F=C_o^{3-1}=C_o^2=\begin{bmatrix} 1 & x & x'y+[xz] \\ x & 1 & xz \\ x'y+[xz] & xz & 1 \end{bmatrix}.$$

The above output matrix F is the same as the characteristic matrix of the intermediate conductance matrix.

A corollary to Theorem 7.1 is:

Corollary 7.2: The reduced connection matrix of a two terminal circuit is the output matrix, and also the component of an output matrix F in row i and column j may be determined by considering a circuit as a two terminal circuit connecting node i and j with all other nodes removed.

In the synthesis of a circuit it is often helpful to detect and remove or to insert redundant elements. These are terms or factors whose replacement by open circuits (in parallel case) or by short circuits (in the series case) will not alter the output of the circuits.

To illustrate these notation, we first consider the matrix given below in which redundant elements are bracketed:

$$F=\begin{bmatrix} 1 & x & x'y+[xz] \\ x & 1 & xz \\ x'y+[xz] & xz & 1 \end{bmatrix}.$$

The terms $[xz]$ in the matrix F are redundant, since there is a path from node 1 to 2 if x=1 (ie. $f_{12}=x=1$) and also there is a path from 2 to 3 if xz=1 (ie. $f_{23}=xz=1$). Therefore, we have a combined path containing node 2 from 1 to 3 if xz=1. If we replace the terms $[xz]$ by zero, we have the connection matrix given on the next page:

$$\begin{bmatrix} 1 & x & x'y \\ x & 1 & xz \\ x'y & xz & 1 \end{bmatrix}.$$

We add a redundant term xx' in 1,3 entry, then we have the matrix given below:

$$\begin{bmatrix} 1 & x & x'(x+y) \\ x & 1 & xz \\ x'(x+y) & xz & 1 \end{bmatrix}.$$

By insertion of a node the product x'y can be separated and it will provide the matrix given below:

$$\begin{bmatrix} 1 & x & 0 & [x]+y \\ x & 1 & xz & x \\ 0 & xz & 1 & x' \\ [x]+y & x & x' & 1 \end{bmatrix}.$$

Since there is a path from node 1 to 2 if x=1 and also there is a path from node 2 to 4 if x=1. Therefore, we have a combined path containing node 2 from 1 to 4 if x=1. Thus x in (1,4) entries of the above matrix is redundant and can be replaced by zero. Thus we have the matrix given below:

$$\begin{bmatrix} 1 & x & 0 & y \\ x & 1 & xz & x \\ 0 & xz & 1 & x' \\ y & x & x' & 1 \end{bmatrix}.$$

The above matrix is not a primitive connection matrix, since the entry (2,3) (which is also the entry (3,2)) is a product of two contacts (<u>ie</u>. xz), not satisfying the condition of <u>primitivity</u> (<u>ie</u>.

single contact in series case and group of contacts in parallel case). By insertion of another node the product xz can be separated and it will give the primitive connection matrix. However, this operation is unnecessary since no contacts would be saved. This illustrates why we often are satisfied with the above kind of matrix in the synthesis process. We shall call this matrix a <u>near primitive</u> connection matrix, because the insertion of one node in a near primitive connection matrix provides a primitive connection matrix.

We had said earlier that in the synthesis of a circuit we often remove or insert redundant elements in the matrix, it is illustrated in the previous example, and also the use of redundant elements in the node insertion process. It is important that this process should be carried out in individual successive steps since each operation may alter the conditions for redundancy of the other terms.

In order to show that the node insertion process is not unique, we will insert a node in the near primitive connection matrix which will provide a primitive connection matrix:

$$P = \begin{bmatrix} 1 & x & 0 & y & 0 \\ x & 1 & 0 & 0 & x \\ 0 & 0 & 1 & x' & z \\ y & 0 & x' & 1 & 0 \\ 0 & x & z & 0 & 1 \end{bmatrix}.$$

The matrix P and C of example (3) are not the same in components, but when we carry out the node removal process, both matrices lead the same output matrix. This shows that the node removal process is unique (<u>ie</u>. gives the same output matrix) but the node insertion process is not unique.

Insertion of a node can also be achieved in the presence of the proper terms in the matrix. For example:

$$\begin{bmatrix} 1 & a+\alpha\beta & b+\alpha\gamma \\ a+\alpha\beta & 1 & c+\beta\gamma \\ b+\alpha\gamma & c+\beta\gamma & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & b & \alpha \\ a & 1 & c & \beta \\ b & c & 1 & \gamma \\ \alpha & \beta & \gamma & 1 \end{bmatrix},$$

we say that the above two matrices are equivalent (ie. symbolized by $\sim$) in the sense that both give the same output matrix. Thus one can insert a node directly in the given matrix either with the use of redundant elements or if in the given matrix has proper terms as indicated by the above example.

The Truth Table Method of Synthesis

If the output of an n terminal circuit is given in the form of a truth table, then the output matrix F may be obtained. One can obtain a primitive connection matrix from this output matrix by the node insertion process. First it is necessary and sufficient to test the required condition for consistency. This can be done by checking that whenever $f_{ij}=f_{jk}=1$, then $f_{ik}=1$.

Let us consider an example given in the Table 4. (This example is a problem from the book Boolean Algebra and Its Application by Flegg, H. Grahm; ref. 1.)

Table 4.

y	z	f_{12}	f_{13}	f_{14}	f_{23}	f_{24}	f_{34}
0	0	0	1	0	0	1	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	0	0
1	1	0	0	0	1	1	1

First let us check the consistency for this table.

In the second row, $f_{12}=1=f_{23}$; it is therefore necessary to check $f_{13}=1$ similarly one can check for other rows. It is found that the rows of the table given are consistent.

We have output functions as:

$$f_{12}=y'z$$

$$f_{13}=y'z'+y'z=y'$$

$$f_{14}=yz'$$

$$f_{23}=y'z+yz=z$$

$$f_{24}=y'z'+yz$$

$$f_{34}=yz$$

Hence:

$$F=\begin{bmatrix} 1 & [y'z] & y' & yz' \\ [y'z] & 1 & z & y'z'+[yz] \\ y' & z & 1 & yz \\ yz' & y'z'+[yz] & yz & 1 \end{bmatrix}$$

Since there is a path from node 1 to 3 if $y'=1$ and 3 to 2 if $z=1$, therefore, we have a path from 1 to 2 if $y'z=1$. Therefore, the term $y'z$ is redundant and may be replaced by zero. Similary, term $[yz]$

is redundant since there is a path from node 2 to 3 if z=1 and 3 to 4 if yz=1, thus we have a path from 2 to 4 if yz =1. Since the yz term appears in entries f_{24} and f_{34}, but only one of the yz terms may be replaced by zero. Thus we have a reduced connection matrix,

$$F=\begin{bmatrix} 1 & 0 & y' & yz' \\ 0 & 1 & z & y'z' \\ y' & z & 1 & yz \\ yz' & y'z' & yz & 1 \end{bmatrix}$$

The resultant network is shown in figure (4).

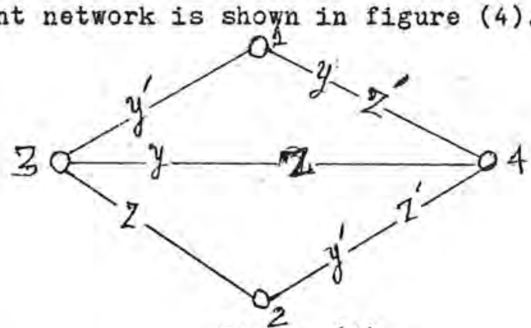

Figure (4).

Let us consider the output matrix F of the above network figure (4) as a connection matrix and try to convert it into a primitive connective matrix. In order to separate the product yz' and y'z' we insert node 5 to yeild matrix given below:

$$\begin{bmatrix} 1 & 0 & y' & 0 & y \\ 0 & 1 & z & 0 & y' \\ y' & z & 1 & yz & 0 \\ 0 & 0 & yz & 1 & z' \\ y & y' & 0 & z' & 1 \end{bmatrix}$$

In order to separate the product yz, we insert node 6 and we get the primitive connection matrix:

$$\begin{bmatrix} 1 & 0 & y' & 0 & y & 0 \\ 0 & 1 & z & 0 & y' & 0 \\ y' & z & 1 & 0 & 0 & y \\ 0 & 0 & 0 & 1 & z' & z \\ y & y' & 0 & z' & 1 & 0 \\ 0 & 0 & y & z & 0 & 1 \end{bmatrix}$$

The resultant network is shown in figure (5).

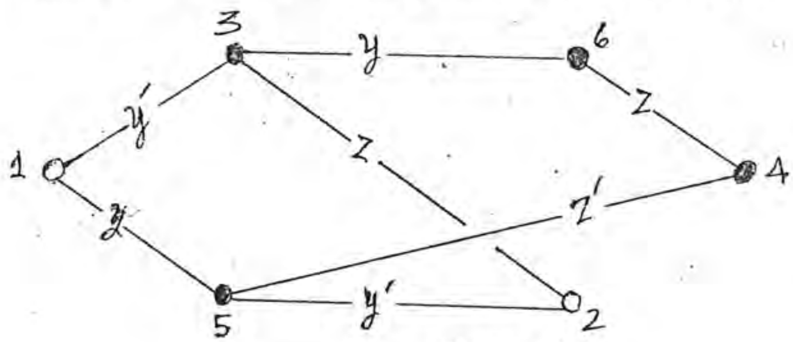

Figure (5).

The insertion of node 6 replaces two z' contacts by one contact. From this one may be satisfied with the primitive connection matrix in the synthesis process.

It is possible to synthesize two terminal circuits for a given switching function f, starting with an output matrix,

$$F = \begin{bmatrix} 1 & f \\ f & 1 \end{bmatrix},$$

and inserting nodes to convert it into a near primitive or a primitive matrix.

For example, let us consider:

Example: (5) $f = y'(w+x') + xy(w'+z')$.

$$F = \begin{bmatrix} 1 & [y'(w+x')] + [xy(w'+z')] \\ [y'(w+x')] + [xy(w'+z')] & 1 \end{bmatrix}.$$

(The previous example is also a problem from ref. 1).

The term $y'(w+x')$ of F can be removed by inserting a third node giving the matrix:

$$\begin{bmatrix} 1 & [xy(w'+z')] & y' \\ [xy(w'+z')] & 1 & (w+x') \\ y' & (w+x') & 1 \end{bmatrix}.$$

The term $xy(w'+z')$ can be removed by inserting the fourth node giving the matrix:

$$\begin{bmatrix} 1 & 0 & y' & [xy] \\ 0 & 1 & (w+x') & (w'+z') \\ y' & (w+x') & 1 & 0 \\ [xy] & (w'+z') & 0 & 1 \end{bmatrix}.$$

Finally a fifth node separates the xy term and gives the primitive connection matrix:

$$\begin{bmatrix} 1 & 0 & y' & 0 & x \\ 0 & 1 & (w+x') & (w'+z') & 0 \\ y' & (w+x') & 1 & 0 & 0 \\ 0 & (w'+z') & 0 & 1 & y \\ x & 0 & 0 & y & 1 \end{bmatrix}.$$

The resultant network is shown in figure (6).

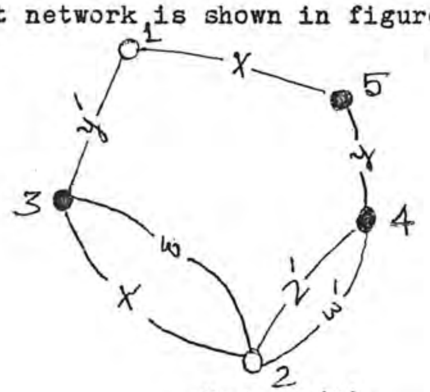

Figure (6).

Instead of copying the matrix at every step of node insertion, this work can be performed without recopying. Thus the matrix of example 5 can be written as:

$$\begin{bmatrix} 1 & [y'(w+x')]+[xy(w'+z')] & y' & [xy] & x \\ [y'(w+x')]+[xy(w'+z')] & 1 & (w+x') & (w'+z') & 0 \\ y' & (w+x') & 1 & 0 & 0 \\ [xy] & (w'+z') & 0 & 1 & y \\ x & 0 & 0 & y & 1 \end{bmatrix}.$$

The brackets are drawn around the redundant elements, which can be replaced by zero and gives the primitive connection matrix.

CONCLUSION

In this paper we have described the basic properties of Boolean matrices and applied them to both analysis and synthesis of relay circuits.

We have given two different methods of analysis and synthesis of networks. Chapter V to VI describe the analysis and synthesis with the aid of the characteristic function. Chapter VII also describes the analysis and synthesis using redundant elements and non-terminals a node removal and a node insertion process respectively. It is interesting to note that both removal of nodes and insertion of nodes can be reversed in synthesis.

The methods of synthesis do not guarantee that the circuit obtained by the node insertion process represents the optimal circuit for a given function or group of functions. "Optimal Circuit" may be interpreted in a sense that one is interested in the circuit which has a minimal number of contacts. It also may be interpreted in the sense of minimal number of nodes or in the sense of a minimum for some function of the number of nodes and contacts or in the minimum total length of wire in the connections, or in some other sense.

A set of circuits representing a given function or set of functions can be obtained by using different methods of synthesis (ie. with the aid of a characteristic or a node insertion process with the aid of redundant elements). From this set of circuits we can select the circuit which best realizes the given function or set of functions among the given set of solutions. But still this way of obtaining the circuits does not guarantee that we can always get the absolutely best circuit although it might in certain cases.

The first attempt to synthesize multiterminal contact networks in general was made by A. G. Lunts who suggested a method based on the use of the characteristic function. Hohn and Schissler suggested the method based on the use of redundant elements and non-terminal nodes. Lunts never discussed the redundant elements arising out of the method of synthesis. Although we can work with the method given by Lunts instead of that of Hohn and Schissler, but we will have a large set of matrices due to the presence of redundant elements. On the other hand, matrices obtained by node insertion process as discussed by Hohn and Schissler may have some redundant elements, but we remove these elements before inserting other nodes. Instead of copying the matrix every step of node insertion, using the Hohn and Schissler method this work can be performed without recopying the matrices. We can say that the synthesis method given by Hohn and Schissler is the easier of the two possible methods.

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ALGEBRAIC METHODS OF ANALYSIS

AND SYNTHESIS OF CONTACT NETWORKS*

by A. G. Lunts
(Translated by H. Gohean)

In this paper is treated a special mathematical apparatus for the solution of the problems of the theory of relay-contact schemes. In particular, an algebraic method is indicated for the construction of a contact scheme with given conditions of operation.

The distinguishing feature of the present day state of automation is the complicated automation of production processes. Not only the separate production processes are automated, but so also is the control of these automatic processes. The solution of the problems of automation is to a significant degree founded on the application of relay-contact networks. With the degree of automation the structure of the relay contact networks used becomes complicated, and the question of the creation of a theory of the construction of networks is placed above all. V. I. Shestakov [(3) (4)], demonstrated that as a mathematical apparatus for the theory of networks, having some special structure (π- networks) Boolean algebra may serve. [see (1) (2)]

Carrying out the theoretical results of Shestakov, M. A. Gavrilov treated methods of application of the theory to the practical problem of engineering practice. To Gavrilov belongs a series of theoretical and practical results, related to the theory of general and special networks (6). The present article contains a summary exposition of part of the results of the author's dissertation devoted both to investigations by an algebraic method and also to construction of contact networks. As apparatus for these methods appear matrices and functions over a Boolean algebra.

1. Boolean Algebra

Not repeating the theory of Boolean algebras which the reader may note in the works (1) (2), we shall indicate only that by a Boolean algebra is denoted a set of elements A, on which two binary operations are defined: "addition" and "multiplication" subject to the same law as the set-theoretical sum and intersection of subsets of an arbitrary set.

These are, namely:

I. $(a+b)+c = a+(b+c)$, $(ab)c = a(bc)$.

II. $a+b = b+a$, $ab = ba$.

III. $(a+b)c = ac+bc$.

IV. Existence of elements "0" and "1" with the properties

$$a+0=a\ ,\ a\cdot 1=a.$$

* Izvestia Akad. Nauk SSSR, Ser. Mat. 16 (1952), 405-426.

V. $a+a=a$.

VI. For every element a there exists a dual element a' with the properties,

$$a+a'=1;\ aa'=0$$

Here a, b, c, represent arbitrary elements of the algebra A. One writes $a \leq b$, if there exists some element $c \in A$ such that $b=a+c$.

The simplest Boolean algebra which we shall denote in the following by B consists of the two elements o and 1. The addition and multiplication tables in this algebra have the form,

$$0+0=0,\ 1+0=0+1=1,\ 1+1=1,$$
$$0\cdot 0=0,\ 1\cdot 0=0\cdot 1=0,\ 1\cdot 1=1,$$
$$0'=1,\ 1'=0.$$

If to some Boolean algebra A are adjoined the symbols $x_1, x_2, x_3, \dots, x_n$ (and likewise the symbols $x_1', x_2', \dots, x_n'$) extending to them the rules I - VI, then we shall have a new Boolean algebra $A(x_1, x_2, \dots, x_n)$, in which the original algebra A comes in as a subalgebra. The elements of the algebra $A(x_1, x_2, \dots x_n)$ are called functions on the algebra A and designated by $f(x_1, x_2, \dots, x_n)$, $g(x_1, x_2, \dots x_n)$. Every function $f(x_1, x_2, \dots x_n)$, in accordance with the definition, appears as some expression composed of elements of the algebra A and the symbols $x_1, x_2, \dots x_n$ joined together by the rules of addition, multiplication, and taking duals. From this, one and the same function may be represented by expressions differing in form. For example, the function $ax_1x_2' + x_1x_2$ (in which $a \in A$) may be expressed also as $x_1(a+x_2)$, since from axioms I - VI may be developed the equation,

$$ax_1x_2' + x_1x_2 = x_1(a+x_2).$$

It is interesting to note, that if the values of two functions $f(x_1, x_2, \dots x_n)$ and $g(x_1, x_2, \dots x_n)$ coincide on the algebra B (i.e. when $x_1, x_2, \dots x_n$ take on the set of all values of B) then

$$f(x_1, x_2, \dots x_n) = g(x_1, x_2, \dots, x_n).$$

Hence it follows that in order to prove the validity of an arbitrary formula, written by means of the laws of addition, multiplication, duality of elements and equality, for example, $(ab)' = a'+b'$, in an arbitrary Boolean algebra, it is sufficient to verify it only for the algebra B, which, clearly, does not present difficulty.

2. Contact Networks

Contacts, which we shall consider, may exist in only one of two states: closed or open. In the closed state the contact closes a chain, in which it consequently is contained; in the open it interrupts this chain.

We shall denote by B the Boolean algebra consisting of two elements: o and 1.

[illegible] that a variable conductance of a contact with meaning from $\mathcal{B}$, is equal to 1, when the contact is closed, and 0, when the contact is open.

We shall call the junction of two points the link of a contact, and the conductance of a link, the conductance of a contact found in it. A contact network represents its entire set of points in space, (the nodes of the network), some pairs of which are connected by one or more links. Among the contacts of a network are found relays which operate on the contacts of the system.

By this means each contact of the network appears as either a "closed" or "open" contact of a definite relay. Each relay, independent of the other relays, may either execute or not execute; in the first case, it closes all of its closed contacts and opens all of its open; in the second case, on the other hand, it opens its closed contacts and closes its open ones.

We shall agree to denote the contacts by the same symbols as their conductances. In fact, the letter a shall designate each closed contact of the relay A, and the symbol a' each open contact of the same relay (as some conductance of closed or open contacts of one and the same relay appear dual quantities). Thus, if $A_1, A_2, \cdots, A_n$ are relays of the network under consideration, then the conductances $a_1, a_2, \cdots, a_n$ of closed contacts of these relays are independent variable quantities with values from the algebra B. The accepted meaning for us of the independence of the particular working of a relay allows us to speak only of the contact network and the contacts entering into it (i.e. of the variables $a_1, a_2, \ldots a_n$) without each reference to a relay. Therefore for the networks considered in the present paper we adopt the adjective "contact" in place of relay-contact.

If two fixed nodes of the network are accepted as terminals, then the contact network is called a two-terminal one. As conductance of a two-terminal network is denoted the variable quantity with value from the algebra B, equal to 1, if for the given state of the contacts of the network is had a closed (in the electrical sense) chain connecting the terminals M_1 and M_2, and equal to 0 if this chain is not. The conductance of a two-terminal network appears as completely described by a definite function of the variables i.e. as a Boolean variable of the algebra $B(a_1, a_2, a_3 \cdots a_n)$

We shall call an elementary chain each uninterrupted path of the network connecting the terminals M_1 and M_2. An elementary chain appears as the successive union of its links, since it will be closed when and only when all its links are closed.

Consequently, the conductance of an elementary chain is equal to the product of the conductances of the links composing it (the product in the algebra B is equal to 1 when and only when all its factors are equal to 1).

The conductance of a two-terminal network is equal to the sum of the conductances of all its elementary chains, since the network closes its terminals then and only then when in it is closed though but one elementary chain (the sum in the algebra B is equal to 1 then and only then when it has an addend equal to 1). For example, the network of Fig. 1a has three elementary chains and its conductance is equal to

$$ab + ac' + bc',$$

Fig. 1a

Fig. 1b

(by a, b, c, are denoted closed contacts of three relays). The network in Fig. 1b has four elementary chains and then also the conductance

$$ab + ac'a + ba + bc'b = ab + ac' + bc'.$$

Obtaining the conductance of a given (as, for example, by a graph) contact network is called the analysis of the network, and the construction of a network with given conductance, synthesis.

In the practical construction of contact networks a great economy of contacts is obtained by the use of diode elements. By a diode element is meant the adaptation which as consequence of the closing of the chain, passes current only in one definite direction. In the nature of diode elements may be the uses, for example, of rectifying metals, electron tubes, etcetera. The conductance of a diode element depends on the direction under consideration in the chain, in which, consequently, one closes it. In one direction this conductance is equal to 1 (the conducting direction), in the other direction - 0 (the blocked direction). Thus just as contacts a diode element of a network will have its conductance (i.e. 0 or 1) and an arrow to denote the direction in which this conductance has place. Besides this we shall allow that in a contact network, in addition to relay contacts and diode elements may be encountered "always closed" and "always open" contacts as the values of which will be 1 and 0 respectively.

The representation of a link given in Fig. 2 following is a suitable means to use:

Fig. 2

The use of diode elements in two terminal contact networks enforces use of two conductances of the network; from one terminal to the other and in the opposite direction.

3. Matrices Over a Boolean Algebra

Let A be an arbitrary Boolean algebra. We shall consider matrices over A, (i.e, with elements from A). As for an ordinary Matrix (over a field) for a Matrix over A the operations of addition and (Matrix) multiplication may be introduced, which we shall write as

$$A + B, \quad A \times B$$

for this, the associative, commutative, (for addition) and distributive laws will also hold. Besides this, we shall introduce the operation of "Boolean" product: AB=C of matrices A and B, obtaining the elements of the matrix C from the elements of the matrices A and B by the following means:

$$c_{\alpha\beta} = a_{\alpha\beta} b_{\alpha\beta}\,, \quad \alpha, \beta = 1, 2, \dots n.$$

The totality $\mathfrak{a}_n$ of quadratic matrices of n - rows over a Boolean algebra relative to the operations of addition and Boolean multiplication appears as a Boolean algebra. As zero of this algebra appears the matrix 0, all elements of which are zero; and as unit the matrix I all elements of which are equal to unity. The inequality $A \leq B$ in the algebra $\mathfrak{a}_n$ means that $a_{\alpha\beta} \leq b_{\alpha\beta}$ for $\alpha, \beta = 1, 2, \dots, n$. Likewise, as the dual of A appears the matrix A', elements of which are dual with respect to the corresponding element of A, i.e.

$$\{A'\}_{\alpha\beta} = a'_{\alpha\beta}\,.$$

If $A \leq B$, and C is an arbitrary matrix of $\mathfrak{a}_n$, then $A \times C \leq B \times C$ and $C \times A \leq C \times B$. In fact, from $A \leq B$, follows, that A + B = B; furthermore

$$A \times C + B \times C = (A + B) \times C = B \times C,$$

whence results, that

$$A \times C \leq B \times C\,.$$

Taking advantage only of the proof given, it is easy to obtain that, if $A_1 \leq A_2$ and $B_1 \leq B_2$, then $A_1 \times B_1 \leq A_2 \times B_2$.

We shall call the matrix A "normal" if its diagonal elements are equal to 1, that is $A \geq E$. From the above inequalities the inequalities for normal matrices follow,

$$E \leq A \leq A^2 \leq A^3 \leq \cdots \tag{1}$$

where $A^2 = A \times A$, $A^3 = A^2 \times A$, and so forth.

We shall prove that for normal matrices the equation always holds

$$A^{n-1} = A^n, \tag{2}$$

where there are n rows of the matrix A. Denoting the elements of the matrix A^k by $a^k_{\alpha\beta}$, we shall have,

$$a^n_{k_0 k_n} = \sum_{k_1, k_2 \dots k_{n-1} = 1}^{n} a_{k_0 k_1} a_{k_1 k_2} \cdots a_{k_{n-1} k_n}\,. \tag{3}$$

We shall consider the term,

$$a_{k_0 k_1} a_{k_1 k_2} \cdots a_{k_{n-1} k_n}$$

Among the n + 1 numbers $k_0, k_1, \ldots k_n$ there must be a coincidence. Set $k_i = k_j$; then omitting from (3) a part of the factors, we shall obtain the inequality,

$$a_{k_0 k_1} \cdots a_{k_{n-1} k_n} \leq a_{k_0 k_1} \cdots a_{k_{i-1} k_i} a_{k_j k_{j+1}} \cdots a_{k_{n-1} k_n}$$

$$\leq a_{k_0 k_n}^{n-j+i} \leq a_{k_0 k_n}^{n-1} .$$

In this manner, $a_{k_0 k_n}^{n} \leq a_{k_0 k_n}^{n-1}$, i.e. $A^n \leq A^{n-1}$.

From this inequality and the inequalities (1) follows the equation to be proved (2).

The least natural number r for which $A^r = A^{r+1}$ we shall call the "index" of the normal matrix A. From (2) follows that always, (taking $n > 1$), $r < n$.

From $A^r = A^{r+1}$ follows $A^k = A^r$ for every $k \geq r$.

4. Contact Multipoles

Consider a given contact network F. If an ordered set $M_1, M_2, \ldots, M_n$ of the nodes of the network F is given, then the contact network F is called an n - pole, and the given nodes, its poles. We denote by $a_{\alpha\beta}$ the sum of the conductances of the totality of elementary chains of the network F, leading from the pole M_α to the pole M_β, avoiding the remaining poles (having in view the conductances of chains in the direction from M_α to M_β.) We shall set $a_{\alpha\beta} = 0$ if there is no such chain in the network and $a_{\alpha\beta} = 1$ if M_α and M_β coincide. We shall call the variable $a_{\alpha\beta}$ the "immediate" conductance from the pole M_α to the pole M_β. Immediate conductances between poles we shall write in the form of a normal quadratic matrix with n rows A. If in the contact network F diode elements are lacking, then the matrix A will be symmetric. The matrix A does not uniquely define a many-terminal network. For example, for the three-pole networks of Fig. 3 will answer one and the same matrix.

$$A = \begin{bmatrix} 1 & d+ab & abc \\ d+ab & 1 & bc \\ abc & bc & 1 \end{bmatrix} .$$

Fig. 3

Every n-poled contact network, the immediate conductances of which form the matrix A, we shall call a contact n-pole A. The idea of a multipole may be used for algebraic representation of contact networks. For this, since to represent a contact network in matrix form it is necessary to consider it as a multipole; we shall take as poles all nodes of the given network. If for poles we take not all the nodes, then we shall obtain an approximate description of the structure of the network.

We shall represent by $\chi_{\alpha\beta}(A)$ the sum of the conductances of all elementary chains of the network F, leading from the pole M_α to the pole M_β. We shall set $\chi_{\alpha\beta}(A) = 0$ if there is no such chain of the network; and $\chi_{\alpha\beta}(A) = 1$ if M_α and M_β coincide. We shall call the variable $\chi_{\alpha\beta}(A)$ the "complete" conductance from the pole M_α to the pole M_β of the multipole A (this quantity does not depend on the possibility of the realization of the multipole in the form of the multipole F.). A quadratic matrix $\chi(A)$ composed of n^2 quantities $\chi_{\alpha\beta}(A)$ we shall call a "characteristic" of the multipole A. For example, the characteristic of the three-pole A realization of which are the representations of Fig. 3, is equal to

$$\chi(A) = \begin{bmatrix} 1 & d+ab & bc(a+d) \\ d+ab & 1 & bc \\ bc(a+d) & bc & 1 \end{bmatrix}.$$

Two n-poles A and B having identical characteristics $\chi(A) = \chi(B)$ will be called "equivalent," written $A \sim B$. The matrix $\chi(A)$ in a certain sense characterizes the operation of the multipole A. Let for example part of some electric network relatively to the points of contact of that part with the remaining parts of the network exhibit the contact multipole A. Then, not changing the work of the network at all, we may substitute the multipole A for an arbitrary other equivalent multipole.

For the complete conductance we may write the following expression arising out of its definition

$$\chi_{\alpha\beta}(A) = \sum_{k_1, k_2, \ldots, k_m} a_{\alpha k_1} a_{k_1 k_2} \ldots a_{k_m \beta}, \tag{4}$$

in which $(k_1, k_2, \ldots, k_m)$ runs through the totality of all possible collections of elements $1, 2, \ldots, \alpha-1, \alpha+1, \ldots, \beta-1, \beta+1, \ldots, n$ for $m = 0, 1, 2, \ldots,$ n-2 elements. In this the "collection of 0 elements" in the sum corresponds to the term $a_{\alpha\beta}$.

Using formula 4, we may translate the meaning of "characteristic" as normal matrix over an arbitrary Boolean algebra A. In the following we shall understand by A an arbitrary Boolean algebra; nevertheless, having in view only applications to a contact network, we shall, in the extent of the whole paper, make use of the terminology corresponding to this network. In particular normal matrices over A we shall often call many-poles. In applications to a contact network, ... ground algebra A, ... understand the algebra B ($a_1, a_2, \ldots, a_m$) of all functions over B with respect to the conductances $a_1, a_2, \ldots, a_m$ of the closed contacts, ...

Two fundamental problems arise:

a) To obtain the characteristic $\chi(A)$ given the multipole A (analysis of a multipole).

b) To find a multipole A, having given characteristic $\chi(A)$ (synthesis of a multipole).

Formula (4), expressing the elements of $\chi(A)$ in terms of the elements of the matrix A, although it gives the possibility of analysis of a multipole, for practical applications is entirely useless. In the work is given some solutions of the problems posed, by which the problem of synthesis is considered in more general formulation than that given above, which is called forth by the necessities of electrical engineering practice.

5. First Method of Analysis of a Multipole

Let A be an arbitrary n-rowed square matrix over the algebra A. As "determinant $|A|$ of the matrix" we shall mean the function of its elements, composed in the following way:

$$|A| = \sum_{S \in G_n} a_{1S_1} a_{2S_2} \cdots a_{nS_n} \tag{5}$$

in which S takes on all the permutations of n numbers, 1, 2,... , n, of the symmetric group G_n, and "Sk" stands for that number into which the permutation S transforms the number k. The formula (5) is distinguished from the definition of the ordinary determinant only with respect to the Kronecker symbol; therefore it is natural to expect that the theory of these determinants in some respects is given by a theory analogous to the theory of ordinary determinants. We shall enumerate some of the relationships of these determinants.

I. The determinant is not changed by interchanging rows and columns.

II. The determinant is not changed by interchange of two rows (columns).

III. The determinant may be expanded with respect to the elements of an arbitrary row or column

$$|A| = \sum_{k=1}^{n} a_{\alpha k} A_{\alpha k} = \sum_{k=1}^{n} a_{k\beta} A_{k\beta}, \tag{6}$$

in which $A_{\alpha\beta}$ is a minor of the determinant A, i.e., a determinant of (n-1) rows obtained from the determinant A by suppressing the α^{th} row and β^{th} column. There also holds the more general expansion (of the theorem of Laplace.)

IV. If all elements of a row have a common factor then it may be taken outside the determinant sign.

V. If the elements of an arbitrary row be the sum of two terms, then the determinant appears as sum of two determinants.

VI. The atrix composed of the minors of the matrix A we shall call the adjoint of the matrix A. The following theorem gives the first method of analysis of a multipole.

Theorem. The characteristic of a normal matrix A is equal to the matrix adjoint to A, i.e.

$$\chi_{\alpha\beta}(A) = A_{\beta\alpha} \tag{7}$$

Proof. First of all let us consider the expression for the minor $A_{\beta\alpha}$, correct for an arbitrary quadratic matrix. For it we shall write formula (5) in a somewhat different form. Let α and β be fixed indices. We shall expand each permutation S into the product of non-overlapping cycles. Let $(\beta\beta_2\beta_3 \cdots \beta_m)$ be that cycle of the permutation S which contains the symbol β $(m \leq n)$. Then the term of the sum (5), corresponding to the permutation S, has the factors $a_{\beta\beta_2}a_{\beta_2\beta_3}\cdots a_{\beta_m\beta}$; denoting the product of the remaining factors by a_S, we shall rewrite formula (5) in the form

$$|A| = \sum_{S \in G_n} a_{\beta\beta_2}a_{\beta_2\beta_3}\cdots a_{\beta_m\beta}a_S , \tag{8}$$

where m, clearly, depends on the permutation S. If now in the totality (8) be substituted $a_{\beta k}$ for $k \neq \alpha$ and $a_{\beta\alpha} = 1$, then the left side appears as the minor $A_{\beta\alpha}$ (the relation of property III) and on the right side appear only the terms in which $\beta_2 = \alpha$, that is, only the terms corresponding to permutations S which replace the symbol β by the symbol α. In this fashion we obtain

$$A_{\beta\alpha} = \sum_{S\beta=\alpha} a_{\alpha\beta_3}a_{\beta_3\beta_4}\cdots a_{\beta_m\beta}a_S , \tag{9}$$

where the sum runs over only those permutations of G_n that replace β by α.

Let us now assume that A is a normal matrix. Among the permutations S having a fixed cycle $(\beta\alpha\beta_3\cdots\beta_m)$, there occurs the one-cycle permutation

$$S' = (\beta\alpha\beta_3\cdots\beta_m).$$

The term of the sum corresponding to this permutation has the form

$$a_{\alpha\beta_3}a_{\beta_3\beta_4}\cdots a_{\beta_m\beta} ,$$

since in this case, $a_{S'} = 1$ (on account of the normality of the matrix A). All terms, corresponding to the permutation S, having the fixed cycle S^1 and differing from S^1, may be omitted from the sum (9) since

$$a_{\alpha\beta_3}\cdots a_{\beta_m\beta}a_S \leq a_{\alpha\beta_3}\cdots a_{\beta_m\beta} .$$

In this fashion

$$A_{\beta\alpha} = \sum_{S'\beta=\alpha} a_{\alpha\beta_3}a_{\beta_3\beta_4}\cdots a_{\beta_m\beta} , \tag{10}$$

where $S' = (\beta\alpha\beta_3\cdots\beta_m)$ runs through the cyclic permutations, taking the symbol β into α. The right side of equation (10) coincides with the right side of equation (4); whence follows the proof of equation (7).

From the formulas (7) and (10) follows the inequality

$$\chi(A) \leq A^{n-1}. \tag{11}$$

In fact, for each term of the right side of (10) we have

$$a_{\alpha\beta_3} a_{\beta_3\beta_4} \cdots a_{\beta_m\beta} \leq a_{\alpha\beta}^{m-1} \leq a_{\alpha\beta}^{n-1},$$

if $m \leq n$. Consequently for all sums in (10) we obtain

$$\chi_{\alpha\beta}(A) = A_{\beta\alpha} \leq a_{\alpha\beta}^{n-1}.$$

6. Another Method of Analysis of Multipoles

The second method of analysis arises out of the formula

$$\chi(A) = A^{r} \tag{12}$$

in which r is the index of the normal matrix A. Since $A^{r} = A^{n-1}$ and equality (11) holds, then for the proof of formula (12) it is sufficient to show that

$$A^{n-1} \leq \chi(A). \tag{13}$$

Let k_1 and k_n be fixed indices and $k_1 \neq k_n$. We have

$$a_{k_1 k_n}^{n-1} = \sum_{k_2,\dots,k_{n-1}=1}^{n} a_{k_1 k_2} \cdots a_{k_{n-1} k_n}. \tag{14}$$

We shall consider the term $a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{n-1} k_n}$. If in the series of numbers $k_1, k_2 \dots k_n$ are equal, for example, k_i and k_j in which $i < j$, then omitting the factors $a_{k_i k_{i+1}}, a_{k_{i+1} k_{i+2}}, \dots, a_{k_{j-1} k_j}$, we shall have the inequality

$$a_{k_1 k_2} \cdots a_{k_{n-1} k_n} \leq a_{k_1 k_2} \cdots a_{k_{i-1} k_i} a_{k_j k_{j+1}} \cdots a_{k_{n-1} k_n}.$$

If among the remaining indices $k_1, k_2, \dots, k_i, k_{j+1}, \dots k_n$, are equal ones, in fashion similar to the preceding we again omit some factors and so forth until there are no equalities, and we arrive at the inequality

$$a_{k_1 k_2} \cdots a_{k_{n-1} k_n} \leq a_{k_1 l_1} a_{l_1 l_2} \cdots a_{l_m k_n},$$

where among the indices $k_1, l_1, \dots l_m, k_n$ there are no two equal. But then

$$a_{k_1 l_1} a_{l_1 l_2} \cdots a_{l_m k_n} \leq \chi_{k_1 k_n}(A),$$

and consequently for the sum (14) we have the inequality,

$$a_{k_1 k_n}^{n-1} \leq \chi_{k_1 k_n}(A),$$

true for all values of indices k_1 and k_n (among this number also for $k_1 = k_n$) i.e. inequality (13).

In particular, for two-terminal networks,

$$\chi(A) = A \quad (n = 2) \tag{15}$$

From the proved equation (12) (see also the connection with section 3) arises

$$A \times \chi(A) = \chi(A) \times A = \chi(A)^2 = \chi(\chi(A)) = \chi(A) \,. \qquad (16)$$

If $A \leq B$, in which A & B are normal matrices, then $A^n \leq B^n$; consequently also

$$\chi(A) \leq \chi(B) \,.$$

Normal matrices whose index is equal to unity, appear as their characteristics; and only such ones do.

In order that the normal matrix A be characteristic, it is necessary and sufficient that its elements satisfy the inequality

$$a_{\alpha\kappa} a_{\kappa\beta} \leq a_{\alpha\beta} \,, \qquad (17)$$

for all indices α, β, κ. In fact from (17) follows

$$a_{\alpha\beta} \geq \sum_{\kappa=1}^{n} a_{\alpha\kappa} a_{\kappa\beta} = a^2_{\alpha\beta} \,, \qquad (18)$$

$$A \geq A^2 \,,$$

whence $A = A^2$ and $r = 1$. Conversely, if A is a characteristic, then $A = A^2$, whence

$$a_{\alpha\beta} = \sum_{\kappa=1}^{n} a_{\alpha\kappa} a_{\kappa\beta} \geq a_{\alpha\kappa} a_{\kappa\beta}$$

for an arbitrary k.

7. Characteristic Functions

To every normal matrix A of $\mathcal{A}$ we associate the function of n variables $x_1, x_2, \dots, x_n$:

$$f_A(x_1, x_2, \dots x_n) = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} x_\alpha x'_\beta$$

as the coefficients of which serve the elements of A. Such a form of function we shall call a "characteristic function," in fact, the characteristic function of the matrix (or multipole) A. We shall prove two lemmas:

Lemma 1. The equation

$$f_A(x_1, x_2, \dots, x_n) = f_{\chi(A)}(x_1, x_2, \dots, x_n) \,, \qquad (19)$$

holds.

Proof

$$f_{A^2}(x_1, x_2, \dots, x_n) = \sum_{\alpha,\beta=1}^{n} a^2_{\alpha\beta} x_\alpha x'_\beta = \sum_{\alpha,\beta,\kappa=1}^{n} a_{\alpha\kappa} a_{\kappa\beta} x_\alpha x'_\beta$$

$$= \sum_{\alpha,\beta,\kappa=1}^{n} a_{\alpha\kappa} a_{\kappa\beta} x_\alpha x'_\beta (x_\kappa + x'_\kappa) = \sum_{\alpha,\beta,\kappa=1}^{n} a_{\alpha\kappa} a_{\kappa\beta} x_\alpha x'_\beta x_\kappa + \sum_{\alpha,\beta,\kappa=1}^{n} a_{\alpha\kappa} a_{\kappa\beta} x_\alpha x'_\beta x'_\kappa$$

$$\leq \sum_{\alpha,\beta,\kappa=1}^{n} a_{\kappa\beta} x_\kappa x'_\beta + \sum_{\alpha,\beta,\kappa=1}^{n} a_{\alpha\kappa} x_\alpha x'_\kappa = f_A(x_1, x_2, \dots x_n) + f_A(x_1, x_2, \dots, x_n)$$

$$= f_A(x_1, x_2, \dots, x_n) \,,$$

i.e. $f_{A^2} \leq f_A$; on the other hand, from $A \leq A^2$, follows the contrary inequality, $f_A \leq f_{A^2}$, by which means,

$$f_A = f_{A^2}$$

pplying the successively proved equalities, we obtain

$$f_A = f_{A^2} = f_{A^4} = \dots = f_{\chi(A)}$$

and the lemma is proved.

Lemma 2. Every row of the characteristic matrix $\chi(A)$ appears as a solution of the "characteristic" equation $f_A(x_1, x_2, \dots x_n) = 0$, i.e.

$$f_A(\chi_{\gamma 1}(A), \dots \chi_{\gamma n}(A)) = 0 \tag{20}$$

for arbitrary $\gamma = 1, 2, \dots, n$.

Proof. Observing that $\chi(A) \times A = \chi(A)$, we shall obtain

$$f_A(\chi_{\gamma 1}(A), \dots \chi_{\gamma n}(A)) = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} \chi_{\gamma\alpha}(A) \chi'_{\gamma\beta}(A) =$$
$$= \sum_{\beta=1}^{n} \{\chi(A) \times A\}_{\gamma\beta} \chi'_{\gamma\beta}(A) = \sum_{\beta=1}^{n} \chi_{\gamma\beta}(A) \chi'_{\gamma\beta}(A) = 0.$$

The same writing (18) of the characteristic function of the multipole A appears at the same time as a way of writing the multipole A. We shall show that the characteristic function characterizes the operation of the multipole A. In fact the statement holds: For two matrices A and B to be equivalent it is necessary and sufficient that their characteristic functions be equal. This statement arises immediately from the following theorem:

Theorem. The inequality of the characteristics

$$\chi(A) \leq \chi(B) \tag{21}$$

of two multipoles A and B is equivalent to the same inequality of the characteristic functions:

$$f_A(x_1, x_2, \dots x_n) \leq f_B(x_1, x_2, \dots, x_n) \tag{22}$$

Proof. Let (21) hold; then $f_{\chi(A)} \leq f_{\chi(B)}$ and inequality (22) arises on the strength of lemma 1. Let (22) hold. We substitute into this inequality

$$x_K = \chi_{\gamma K}(B) \qquad (K = 1, 2, \dots, n) ;$$

then we obtain, in virtue of lemma 2:

$$f_A(\chi_{\gamma 1}(B), \chi_{\gamma 2}(B), \dots \chi_{\gamma n}(B)) \leq 0,$$

i.e.

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} \chi_{\gamma\alpha}(B) \chi'(B) = 0$$

or

$$\sum_{\beta=1}^{n} \{\chi(B) \times A\}_{\gamma\beta} \chi'_{\gamma\beta}(B) = 0.$$

In this manner, for arbitrary indices γ and β we have

$$\{\chi(B) \times A\}_{\gamma\beta} \leq \chi_{\gamma\beta}(B),$$

or

$$\chi(B) \times A \leq \chi(B)$$

and since

$$\chi(B) \times A \geqslant E \times A = A$$

then $A \leq \chi(B)$ and finally

$$\chi(A) \leq \chi(B)$$

and the theorem is proved.

The theorem just proved gives an algebraic method for transforming a multipole into an equivalent multipole. If A and B are equivalent multipoles, then as proof, the two characteristic functions are equal; and that betokens that the function $f_A(x_1, \ldots x_n)$ (having the representation $\sum a_{\alpha\beta} x_\alpha x'_\beta$) by means of the totality of transformations (i.e. by use of the axioms of Boolean algebra) may be transformed into the function $f_B(x_1, x_2, \ldots x_n)$ (having the representation $\sum b_{\alpha\beta} x_\alpha x'_\beta$)..

8. Exclusion of Variables from a Characteristic Function

In order to write the structure of a two-terminal (or many-terminal) network with sufficient fullness, one is led to consider that network as a multipole taking as poles of the multipole all or almost all of the nodes of the network. In such a multipole not all the poles play the same role. Essential are only those poles which coincide with the poles of the electrical network considered; the remaining poles play a less important role. For this reason, for transformation of such a multipole, the requirement of its entire lack of difference from its characteristic appears strongly superfluous. It is sufficient to restrict the requirement of invariance only of those elements of the characteristic which correspond to pairs of essential poles. It concerns the necessary poles, although not the auxiliary ones, that the number of them remain unchanged. We shall now introduce two operations on characteristic functions which permit the performance of a transformation of such a kind for multipoles by means of transformation of their characteristic functions.

The transition from a function $f(x)$ to its greatest lower bound $f(0)f(1)$ we shall call the "exclusion" of the variable x from the function $f(x)$ and write

$$(Ex)\ f(x) = f(0)\,f(1)\ .$$

The converse operation -- restoration of a function of x from its greatest lower bound we shall call the introduction of the variable x . The converse operation, in contrast to the direct one is not unique. In the following the use of the following simple formula often occurs,

$$(Ex)\,(ax + bx' + c) = ab + c\ . \tag{23}$$

If from the characteristic function $f_A(x_1, x_2, \ldots x_n)$ we exclude the variable x_n, then we shall have once more the characteristic function of some (n-1) pole B:

$$(Ex_n) f_A(x_1, \ldots x_n) = \sum_{\alpha\beta=1}^{n-1} a_{\alpha\beta} x_\alpha x'_\beta + \sum_{\alpha=1}^{n-1} a_{\alpha n} x_\alpha \cdot \sum_{\beta=1}^{n-1} a_{n\beta} x'_\beta = \\ = \sum_{\alpha,\beta=1}^{n-1} (a_{\alpha\beta} + a_{\alpha n} a_{n\beta}) x_\alpha x'_\beta = f_B(x_1, x_2, \ldots x_n), \qquad (24)$$

where

$$b_{\alpha\beta} = a_{\alpha\beta} + a_{\alpha n} a_{n\beta}, \quad (\alpha, \beta = 1, 2, \ldots, n-1).$$

We shall say that the multipole B is obtained by exclusion of the pole M_n from the multipole A.

Theorem: In the exclusion of the pole M_n from the multipole A the elements of the characteristic corresponding to the remaining poles, remain unchanged, i.e.

$$\chi_{\alpha\beta}(B) = \chi_{\alpha\beta}(A), \quad (\alpha, \beta = 1, 2, \ldots n-1) \qquad (25)$$

where B is the result of the exclusion of the pole M_n from A.

Proof. Excluding the variable x_n from the function $f_{\chi(A)}(x_1, x_2, \ldots, x_n)$, we have

$$(Ex_n) f_{\chi(A)}(x_1, x_2, \ldots, x_n) = f_C(x_1, \ldots, x_n),$$

where

$$c_{\alpha\beta} = \chi_{\alpha\beta}(A) + \chi_{\alpha n}(A) \chi_{n\beta}(A), \quad (\alpha, \beta = 1, 2, \ldots, n-1)$$

or taking into account inequality (17), there holds for the elements of the characteristic

$$c_{\alpha\beta} = \chi_{\alpha\beta}(A), \quad (\alpha, \beta = 1, 2, \ldots n-1). \qquad (26)$$

The matrix C appears as a characteristic, since its elements satisfy the inequality (17). From $f_A = f_{\chi(A)}$ and the uniqueness of the operation of exclusion we have,

$$(Ex_n) f_A = (Ex_n) f_{\chi(A)},$$

i.e.

$$f_B(x_1, \ldots x_{n-1}) = f_C(x_1, \ldots x_{n-1})$$

whence $B \sim C$, and since C is characteristic, that

$$\chi(B) = \chi(C) = C; \qquad (27)$$

$$\chi_{\alpha\beta}(A) = c_{\alpha\beta} \quad (\alpha, \beta = 1, 2, \ldots n-1).$$

Substituting (27) into (26) we shall have the equation (25).

From the theorem proved arises a third method of analysis of multipoles. If from the characteristic function $f_A(x_1, \ldots, x_n)$ we exclude successively the variables $x_n, x_{n-1}, \ldots x_3$, then we shall have the characteristic function of a two-terminal network the conductances of which we denote $\chi_{12}(A)$ and $\chi_{21}(A)$:

$$(E x_3)(E x_4) \ldots (E x_n) f_A(x_1, x_2, \ldots, x_n) = \chi_{12}^{(A)} x_1 x_2' + \chi_{21}^{(A)} x_1' x_2 .$$

By setting in the preceding equation $x_1 = 1$ and $x_2 = 0$, we shall have

$$\chi_{12}(A) = (E x_3) \cdots (E x_n) f_A(1, 0, x_3 \cdots x_n) \qquad (28)$$

or

$$\chi_{12}(A) = \prod_{x_3, x_4, \cdots x_n = 0}^{1} f_A(1, 0, x_3, \cdots x_n).$$

In the calculation of the right side of (28) we may begin by setting $x_1 = 1$ and $x_2 = 0$ and then excluding the variables; the result of such an interchange of acts does not differ. For the other elements of the characteristic, formulas analogous to formula (28) are used. These formulas give the third method of analysis.

9. Introduction of Auxiliary Poles.

The operation of exclusion of a variable from the characteristic function is unique and may be accomplished, for example by formula (24). For the introduction of a new variable into the characteristic function, that formula may also be used. For this we transform the characteristic function to the form:

$$f_A(x_1, x_2, \cdots x_n) = \sum_{\alpha=1}^{n} b_{\alpha, n+1} x_\alpha \cdot \sum_{\beta=1}^{n} b_{n+1, \beta} x_\beta' + \sum_{\alpha, \beta = 1} b_{\alpha\beta} x_\alpha x_\beta' , \qquad (29)$$

where b_{ij} $(i, j = 1, 2, \ldots, n-1)$ in a corresponding manner are the matching coefficients. Then one may transform the right side of the preceding equation as the result of exclusion of some variable x_{n+1} from the characteristic function $f_B(x_1, \cdots, x_{n+1})$:

$$f_A(x_1, \cdots, x_n) = (E x_{n+1}) f_B(x_1, \cdots x_{n+1}).$$

According to the theorem of the preceding section the elements of the characteristics of the multipoles A and B containing the original n poles will be the same:

$$\chi_{\alpha\beta}(A) = \chi_{\alpha\beta}(B), \quad (\alpha, \beta = 1, 2, \cdots, n).$$

We shall say that the (n+1)-pole B is obtained from the n-pole A by introduction of a new pole Mn+1: If the original multipole A is symmetric and we want as result of the introduction of a new pole, to obtain again a symmetric multipole B, then it is necessary to require that the coefficients of the right side of (29) satisfy the condition of symmetry:

$$b_{ij} = b_{ji} \quad (i, j = 1, 2, \cdots, n+1) \qquad (30)$$

Clearly, one may represent the characteristic function $f(A)$ in many ways in the form (29); and for this reason one may introduce a new pole into the multipole in many ways.

10. General Transformation of Multipoles

We use the introduction operation on characteristic functions to obtain an answer to the question of the transformation of multipoles in a more general sense, than that given earlier. Let A be the n-pole under consideration, with poles $M_1, \ldots, M_m, \ldots, M_n$, and let as the stipulations of the problem, the poles $M_1, \ldots, M_m$ be the "essential" ones and the remaining, the "auxiliary."

It follows in this sense that the n-pole A solves the given problem and also every p-pole B ($p \geq n$) for which

$$\chi_{\alpha\beta}(B) = \chi_{\alpha\beta}(A), \; (\alpha, \beta = 1, 2, \ldots, m). \quad (31)$$

We denote the poles of B by $M_1, M_2, \ldots, M_m, M'_{m+1}, \ldots, M'_p$, and the characteristic function of the multipoles A and B by $f_A(x_1, \ldots, x_n)$ and $f_B(x_1, \ldots, x_m, x^{(1)}_{m+1}, \ldots, x^{(1)}_p)$. The variables $x_{m+1}, \ldots, x_n, x^{(1)}_{m+1}, \ldots, x^{(1)}_p$ corresponding to the auxiliary poles of the multipoles A and B we shall call parameters. According to the previous exposition, the assumption of (31) is equivalent to the equation resulting from the exclusion of the parameters from the characteristic functions f_A and f_B :

$$(E x_{m+1}) \cdots (E x_n) f_A(x_1, \ldots x_n) = (E x^{(1)}_{m+1}) \cdots (E x^{(1)}_p) f_B(x_1, \ldots x_m, x^{(1)}_{m+1} \ldots x^{(1)}_p) \quad (32)$$

With the aid of this equation it is not difficult to show that the totality of transformations, exclusion of parameters and introduction of new the characteristic function of the multipole A may be transformed into the characteristic function of each multipole B, satisfying equation (31). In fact, by exclusion of the parameters $x_{m+1}, \ldots, x_n$ the function $f_A(x_1, \cdots x_n)$ is transformed into the left side of equation (32). By the totality of transformations the left side may be transformed into the right. And the right side of equation (32) by introduction of the parameters $x^{(1)}_{m+1}, \ldots, x^{(1)}_p$ is transformed into the characteristic function of the multipole B. In particular if in A all poles are essential (m = n), then as an urgent theoretical measure, the necessity of the operation of exclusion of parameters is lacking. The following case of the construction of the multipole B with given elements of the characteristic $\chi_{\alpha\beta}(B)$ ($\alpha, \beta = 1, 2, \ldots, m$) as a natural generalization of the problem, called by us in § 4, the synthesis of a multipole. Synthesis of two-poles in this generalized sense corresponds to the problem of construction of two-terminal networks.

Example: We shall construct a two-terminal network with the conductances

$$\chi_{12}(A) = bd + ac + ad \; ; \; \chi_{21}(A) = bd + ac + bc.$$

We transform the characteristic function of the two-pole:

$$\begin{aligned} f(x_1, x_2) &= (bd + ac + ad)\, x_1 x'_2 + (bd + ac + bc)\, x_2 x'_1 \\ &= (a x_1 + c x_2)(a x'_1 + b x'_1 + c x'_2 + d x'_2) + \\ &\quad bd\,(x_1 x'_2 + x_2 x'_1). \end{aligned}$$

We introduce the parameter x_3:

$$f(x_1, x_2, x_3) = (ax_1 + cx_2)x_3' + (ax_1' + bx_1' + cx_2' + dx_2')x_3 + bd(x_1x_2' + x_2x_1') = a(x_1 \circ x_3) + c(x_2 \circ x_3) + dx_2'x_3 + bx_1'x_3 + bd(x_1x_2' + x_2x_1') = a(x_1 \circ x_3) + c(x_2 \circ x_3) + (bx_1 + dx_2 + x_3)(bx_1' + dx_2'),$$

where is denoted*: $\quad x_\alpha x_\beta' + x_\beta x_\alpha' = x_\alpha \circ x_\beta$.

We introduce the parameter x_4:

$$f_A(x_1, x_2, x_3, x_4) = a(x_1 \circ x_3) + c(x_2 \circ x_3) + (bx_1 + dx_2 + x_3)x_4' + (bx_1' + dx_2')x_4 = a(x_1 \circ x_3) + c(x_2 \circ x_3) + b(x_1 \circ x_4) + d(x_2 \circ x_4) + x_3 x_4'.$$

The four-pole A is drawn in figure 4

Figure 4

We note one simple method of introducing a new parameter into the characteristic function of a symmetric multipole. We shall assume, that from the characteristic function $f_A(x_1, \ldots, x_n)$ a term of the form $ab(x_\alpha \circ x_\beta)$ possibly stands out, i.e.

$$ab(x_\alpha \circ x_\beta) \leq f_A(x_1, x_2, \ldots, x_n),$$

(where a and b are quantities from A). Then the term a b $(x_\alpha \circ x_\beta)$ may serve as the sum

$$a(x_\alpha \circ x_{n+1}) + b(x_{n+1} \circ x_\beta).$$

Indeed,

$$ab(x_\alpha \circ x_\beta) = ab(x_\alpha x_\beta' + x_\beta x_\alpha') = (ax_\alpha + bx_\beta)(ax_\alpha' + bx_\beta').$$

Introducing a new parameter x_{n+1} we obtain:

$$(ax_\alpha + bx_\beta)x_{n+1}' + (ax_\alpha' + bx_\beta')x_{n+1} = a(x_\alpha \circ x_{n+1}) + b(x_{n+1} \circ x_\beta).$$

* The operation $x \circ y$ in the algebra B is no more than the usual addition, modulo 2.

The remaining terms of the characteristic function $f_A(x_1, x_2, \ldots, x_n)$ in this introduction of the parameter x_{n+1} remain unchanged. If one is confined to only this simple method of introduction of a parameter, then in the result is obtained the so called π-network; and an arbitrary π-network with given conductance may be obtained by this method. We shall consider this in an example, in which we shall alter the method in such form that it exactly coincides with the method of Shestakov (4) of construction of a π-network.

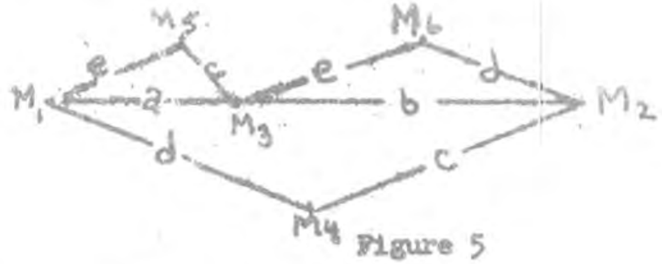

Figure 5

Example 2. We shall construct a two-terminal contact network without diode elements with the conductance

$$\chi = ab + acd + bec + dc.$$

We shall transform χ into the form of an arbitrary π-expression, for example

$$\chi = (a+ec)(b+ed) + dc$$

Into the characteristic function of the two-pole,

$$\chi(x_1 \circ x_2) = (a+ec)(b+ed)(x_1 \circ x_2) + dc(x_1 \circ x_2),$$

we shall introduce the parameters x_3 and x_4:

$$(a+ec)(x_1 \circ x_3) + (b+ed)(x_3 \circ x_2) + d(x_1 \circ x_4) + c(x_2 \circ x_4) =$$
$$a(x_1 \circ x_3) + b(x_3 \circ x_2) + d(x_1 \circ x_4) + c(x_4 \circ x_2) + ec(x_1 \circ x_3) + ed(x_3 \circ x_2)$$

We introduce parameters x_5 and x_6:

$$a(x_1 \circ x_3) + b(x_3 \circ x_2) + d(x_1 \circ x_4) + c(x_4 \circ x_2) + e(x_1 \circ x_5) + c(x_5 \circ x_3) +$$
$$e(x_3 \circ x_6) + d(x_6 \circ x_2).$$

The π-scheme obtained is drawn in figure 5. If one is not confined to this simple method of introduction of parameters then one may obtain a network containing only 5 contacts (8).

In the construction of a two-terminal network A with given conductance $\chi_{12}(A)$ one may introduce some simplification in the calculation, if at the very beginning one sets $x_1 = 1$ and $x_2 = 0$ i.e., transforms to the function $f_A(1, 0, x_3, \ldots, x_n)$ of the parameters $x_3, \ldots, x_n$. Formula (28) shows the totality of transformations of the function $f_A(1, 0, x_3, \ldots, x_n)$, as exclusion of parameters and introduction of new parameters does not alter the conductance $\chi_{12}(A)$. Conversely, by the aid of the enumerated operations the multipole A may be transformed into an arbitrary multipole B for which

$$\chi_{12}(B) = \chi_{12}(A)$$

the preceding equation is equivalent to the equation

$$\left(E x_3\right)\cdots\left(E x_n\right) f_A(1,0,x_3,\ldots,x_n)=\left(E x_3^{(1)}\right)\cdots\left(E x_p^{(1)}\right) f_B\left(1,0,x_3^{(1)}\ldots x_p^{(1)}\right),$$

whence also follows that by aid of the enumerated operations the function $f_A(1,0,x_3,\ldots,x_n)$ may be transformed into the function $f_B(1,0,x_3^{(1)},\ldots,x_p^{(1)})$. We note that in the expression of the function $f_A(1,0,x_3,\ldots,x_n)$ the conductances $a_{\alpha 1}, a_{2\beta}$ $(\alpha,\beta=1,2,\ldots,n)$ do not occur, since these coefficients in the expression for the characteristic function $f_A(x_1,x_2,\ldots,x_n)$ may be taken arbitrarily. If it is originally taken that A is symmetric (i.e. the network is constructed without diode elements) then these coefficients may be tkaen by means of symmetry

$$a_{\alpha 1}=a_{1\alpha}\,;\; a_{2\beta}=a_{\beta 2}\,.$$

In conclusion we shall look at some examples of contruction of networks with and without diode elements.

Example 3. We shall construct a two-terminal network A with conductance

$$X_{12}(A)=ac+bc+bd+bg+ge\,.$$

We have for the two-pole

$$f(1,0)=b(c+d+g)+ac+ge\,.$$

We introduce the parameter x_3:

$$\begin{aligned} f(1,0,x_3)&=bx_3'+(c+d+g)x_3+ac+ge\\ &=(x_3+e)g+bx_3'+(c+d)x_3+ac\,. \end{aligned}$$

We introduce the parameter x_4:

$$\begin{aligned} f(1,0,x_3,x_4)&=(x_3+e)x_4'+gx_4+bx_3'+(c+d)x_3+ac=\\ &=(a+x_3)c+gx_4+x_3x_4'+ex_4'+bx_3'+dx_3\,. \end{aligned}$$

We introduce the parameter x_5:

$$\begin{aligned} f_A(1,0,x_3,x_4,x_5)=(a+x_3)x_5'+cx_5+gx_4+x_3x_4'+ex_4'\\ +bx_3'+dx_3 \end{aligned}$$

We have, taking the lacking elements of the five-pole A:

$$a_{\alpha 1}=a_{1\alpha}\,;\; a_{2\beta}=a_{\beta 2}\,,$$

i.e.

$$\begin{aligned} f_A(x_1,x_2,x_3,x_4,x_5)=a(x_1\circ x_5)+x_3x_5'+c(x_2\circ x_5)\\ +g(x_2\circ x_4)+x_3x_4'+e(x_1\circ x_4)\\ +b(x_1\circ x_3)+d(x_3\circ x_2)\,. \end{aligned}$$

Figure 6

Example 4. We shall construct a two-terminal network A without diode elements, whose conductance is

$$X_{12}(A) = gh + gec + abc + adh + gdbc + abeh + adec.$$

We transform this expression:

$$(g + ad + abe)(h + ec + dbc) + abc.$$

We introduce the parameter x_3:

$$(g+ad+abe)x_3' + (h+ec+dbc)x_3 + abc = [(d+be)x_3+a]\cdot[(d+be)x_3'+bc] + gx_3' + (h+ec)x_3.$$

We introduce the parameter x_4:

$$[(d+be)x_3+a]x_4' + [(d+be)x_3'+bc] + gx_3' + (h+ec)x_3 =$$
$$=(d+be)(x_3 \circ x_4) + ax_4' + bcx_4 + gx_3' + (h+ec)x_3 =$$
$$= (bx_4+ex_3)(bx_4'+ex_3'+c) + d(x_3 \circ x_4) + ax_4' + gx_3' + hx_3$$

We introduce the parameter x_5:

$$(bx_4+ex_3)x_5' + (bx_4'+ex_3'+c)x_5 + d(x_3 \circ x_4) + ax_4' + gx_3' + hx_3$$
$$= b(x_4 \circ x_5) + e(x_3 \circ x_5) + cx_5 + d(x_3 \circ x_4) + ax_4' + gx_3' + hx_3$$

4. From this preceding representation we write the characteristic function of the symmetric five-pole A.

$$f_A = b(x_4 \circ x_5) + e(x_3 \circ x_5) + c(x_5 \circ x_2) + d(x_3 \circ x_4) + a(x_1 \circ x_4) + g(x_1 \circ x_3) + h(x_3 \circ x_2).$$

The five-pole A is drawn in Fig. 7.

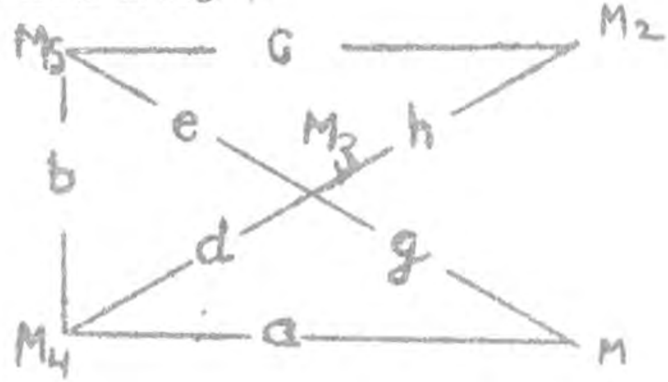

Figure 7

Example 5. We shall construct a two-terminal network A without diode elements with the conductance

$$X_{12}(A) = ab + cd + agf + chf + aghd + chgb + ef + egb + ehd =$$
$$=(a+eg+chg)(b+gf+ghd)+cd+chf+ef+ehd.$$

Introducing into the following parameters x_3, x_4, x_5, we shall have

$$(a+eg+chg)x_3' + (b+gf+ghd)x_3 + cd + chf + ef + ehd =$$
$$= (gx_3+e+ch)(gx_3'+f+hd) + ax_3' + bx_3 + cd =$$
$$= (gx_3+e+ch)x_4' + (gx_3'+f+hd)x_4 + ax_3' + bx_3 + cd =$$
$$= g(x_3 \circ x_4) + (e+ch)x_4' + (f+hd)x_4 + ax_3' + bx_3 + cd =$$
$$= (hx_4+c)(hx_4'+d) + g(x_3 \circ x_4) + ex_4' + fx_4 + ax_3' + bx_3 =$$
$$= (hx_4+c)x_5' + (hx_4'+d)x_5 + g(x_3 \circ x_4) + ex_4' + fx_4 + ax_3' + bx_3 =$$
$$= h(x_4 \circ x_5) + cx_5' + dx_5 + g(x_3 \circ x_4) + ex_4' + fx_4 + ax_3' + bx_3$$

From the previous representation one may immediately construct the five-pole A (Fig. 8)

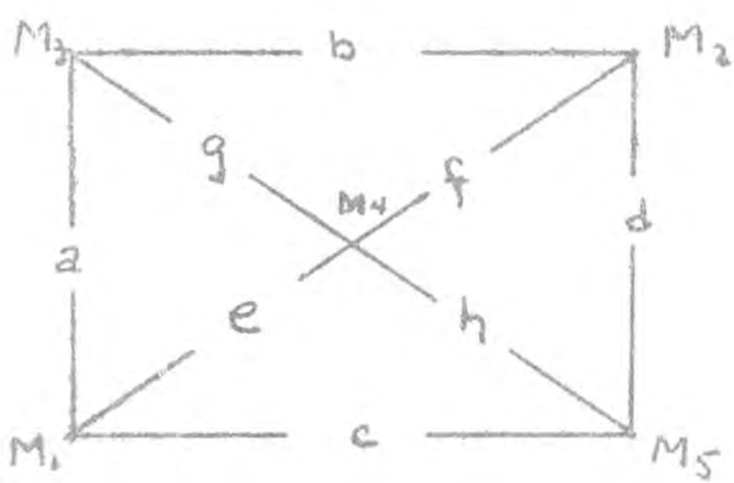

Figure 8.

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