ERRORS IN LINEAR NONHOMOGENEOUS ALGEBRAIC SYSTEMS

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CHAPTER I

INTRODUCTION

STATEMENT OF THE PROBLEM. The purpose of this thesis is to consider a system of linear nonhomogeneous algebraic equations in n unknowns, $x_1, x_2, ----, x_n$,

$$\sum_{j=1}^{n} a_{ij}x_j = y_i, \; i = 1, 2, ----, n,$$

where the n^2 coefficients a_{ij} are subject to error, and to find by how much the solutions $\bar{x}_1, \bar{x}_2, ----, \bar{x}_n$, of the approximating equations

$$\sum_{j=1}^{n} (a_{ij} + \epsilon_{ij}) \bar{x}_j = y_i, \; i = 1, 2, ----, n,$$

may differ from the original x_j's. Errors in the coefficients frequently occur in the application of mathematics to natural phenomena, where they may appear as errors of observation; they may also arise when decimal coefficients are "rounded off."

METHOD OF INVESTIGATION. This problem has been solved before and partial or complete solutions may be found in papers by Etherington(3), Lonseth(5) and Moulton(6). The results of Lonseth and Moulton are compared with the results of this paper in a numerical example in Chapter four.

The method of this paper is quite different from those mentioned above. It follows Tricomi's solution(8) of the analogous problem with respect to a linear integral equation of Fredholm type and second kind.

Tricomi considered the integral equation for $x(s)$

$$1)\quad x(s) - \lambda\int_0^1 K(s,t)x(t)dt = y(s), \quad 0 \le s \le 1,$$

where $K(s,t)$ and $y(s)$ are known functions. Also, $K(s,t)$ is continuous on the square $0 \le s,t \le 1$ and $y(s)$ is continuous on $0 \le s \le 1$. The parameter λ is restricted to values such that the Fredholm formula (9, p.214) for $x(s)$ holds. If the kernel $K(s,t)$ is replaced by $\overline{K}(s,t)$, where

$$\left|\overline{K}(s,t) - K(s,t)\right| < \epsilon, \quad \epsilon > 0, \text{ and "small,"}$$

on the square $0 \le s,t \le 1$, and if λ is correctly restricted, the equation

$$2)\quad \overline{x}(s) - \lambda\int_0^1 \overline{K}(s,t)\overline{x}(t)dt = y(s), \quad 0 \le s \le 1,$$

is solvable for $\overline{x}(s)$ by Fredholm's formula. Tricomi derives an expression which limits the maximum value of

$$\left|\overline{x}(s) - x(s)\right|, \quad 0 \le s \le 1.$$

Suppose $D(\lambda)$ is the Fredholm denominator for $K(s,t)$, $\overline{D}(\lambda)$ that for $\overline{K}(s,t)$ and

$$\Omega(x) = \sum_{n=0} \frac{n^{n/2}}{n!} x^n \quad .$$

If K and Y are upper bounds for $\left|K(s,t)\right|$ and $\left|y(s)\right|$, respectively, and $L = \left|\lambda\right|(K + \epsilon)$, ϵ being some "small" positive number, then Tricomi proves the following theorem:

Tricomi's Theorem. If ϵ is a positive number such that

$$\epsilon < \frac{|\overline{D}(\lambda)|}{|\lambda|\Omega'(L)} \quad \text{and} \quad \left|\overline{K}(s,t) - K(s,t)\right| < \epsilon \quad ,$$

then

$$|\bar{x}(s) - x(s)| < |\lambda| Y \frac{\Omega(L)\Omega'(L) + L[\Omega'^2(L) + \Omega(L)\Omega''(L)]}{|\bar{D}(\lambda)|[|\bar{D}(\lambda)| - |\lambda|\Omega'(L)\,\epsilon]}\,\epsilon$$

The coefficients a_{ij} in the algebraic system can always be written in the form

$$a_{ij} = \delta_{ij} - k_{ij}\text{ , where}$$

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

When the k_{ij}'s are real numbers, the two systems

$$3)\quad x_i - \sum_{j=1}^{n} k_{ij}x_j = y_i$$

$$4)\quad \bar{x}_i - \sum_{j=1}^{n} \bar{k}_{ij}\bar{x}_j = y_i\text{ , } i = 1, 2, ----, n,$$

where $\bar{k}_{ij} = k_{ij} + \epsilon_{ij}$, ϵ_{ij} "small," are algebraic analogues of the two integral equations 1) and 2), with $\lambda = 1$.

It is supposed that the determinants $D_n(k)$ and $\bar{D}_n(k)$ of the two systems are not zero so that both systems can be solved uniquely. Instead of the transcendental integral function $\Omega(x)$, two polynomials $\psi_n(x)$ and $\varphi_n(x)$ are used, where

$$5)\quad \psi_n(x) = \sum_{m=o}^{n} \left(\frac{n!\,m^{m/2}}{(m!)^2(n-m)!}\right) x^m$$

and

$$6)\quad \varphi_n(x) = x\sum_{m=o}^{n} \left(\frac{n!\,(m+1)^{(m+1)/2}}{(m!)^2(n-m)!}\right) x^m \quad .$$

When $|k_{ij}| < K$, $|\epsilon_{ij}| < \epsilon$ and $|y_i| < Y$, the principal result of this paper may be stated as

Theorem 1. If ϵ is a positive number such that

$$\epsilon < \frac{|\bar{D}_n(k)|}{\psi'_n(K+\epsilon)},$$

and if $|\bar{k}_{ij} - k_{ij}| < \epsilon$, then

$$|\bar{x}_i - x_i| < \frac{nY\epsilon\left[\phi_n(K)\,\psi'_n(K+\epsilon) + \psi_n(K)\,\phi'_n(K+\epsilon)\right]}{|\bar{D}_n(k)|\left[|\bar{D}_n(k)| - \epsilon\,\psi'_n(K+\epsilon)\right]}.$$

Corollary 1. If

(a) the coefficients in equation 4) satisfy

$$1 - \bar{k}_{ii} \geq \sum_{j\neq i}^{n} |\bar{k}_{ij}|, \text{ for } i = 1, 2, \text{——}, n;$$

(b) $\epsilon < \dfrac{(1-\bar{k}_{11})\,\Pi_n(k)}{\psi'_n(K+\epsilon)}$, where $\Pi_n(k) = \prod_{i=2}^{n}\left[(1-\bar{k}_{ii}) - \sum_{j=1}^{i-1}|\bar{k}_{ij}|\right]$;

then

$$|\bar{x}_i - x_i| < \frac{nY\epsilon\left[\phi_n(K)\,\psi'_n(K+\epsilon) + \psi_n(K)\,\phi'_n(K+\epsilon)\right]}{(1-\bar{k}_{11})\Pi_n(k)\left[(1-\bar{k}_{11})\Pi_n(k) - \epsilon\,\psi'_n(K+\epsilon)\right]}.$$

Corollary 2. If

(a) the coefficients in equation 4) satisfy

$$1 - \bar{k}_{ii} \geq \sum_{j\neq i}^{n} |\bar{k}_{ij}|, \text{ for } i = 1, 2, \text{——}, n;$$

(b) $\epsilon < \dfrac{\theta_n(K)}{\psi'_n(K+\epsilon)}$, where $\theta_n(K) = (1-K)(1-2K)\ldots(1-(n-1)K)$,
$(n-1)K < 1$;

then

$$|\bar{x}_i - x_i| < \frac{nY\epsilon\left[\phi_n(K)\,\psi'_n(K+\epsilon) + \psi_n(K)\,\phi'_n(K+\epsilon)\right]}{\theta_n(K)\left[\theta_n(K) - \epsilon\,\psi'_n(K+\epsilon)\right]}.$$

A table of values for $\psi_3(x)$, $\psi'_3(x)$, $\phi_3(x)$ and $\phi'_3(x)$ will be found at the end of Chapter IV.

AN INEQUALITY OF K.O. FRIEDRICHS.[1] This thesis includes a proof of an inequality of K.O. Friedrichs which specifies a positive lower bound for a certain type of determinant. It is the belief of the writer that a proof of this inequality has never been published and the one given in this paper may be new.

RELATED INVESTIGATIONS. Mrs. Adams(1) considers the homogeneous case, $y(s) = 0$, of equations 1) and 2) and derives bounds for the errors in the characteristic values and characteristic functions associated with the problem.

Lonseth derives the inequality

$$7)\quad |\bar{x}_i - x_i| < \frac{\epsilon}{|\Delta|}\sum_{j=1}^{n}|A_{ji}|\left\{1 + \sum_{j=1}^{n}|x_j|\right\} \Big/ \left\{1 - \frac{\epsilon}{|\Delta|}\sum_{j=1}^{n}\sum_{k=1}^{n}|A_{jk}|\right\}$$

where Δ is the nonvanishing determinant of the system

$$\sum_{j=1}^{n} a_{ij}x_j = y_i\,, \quad i = 1, 2, \text{——}, n,$$

$\bar{x}_i$ satisfies the system

$$\sum_{j=1}^{n}(a_{ij} + \epsilon_{ij})x_j = y_i + \eta_i, \quad i = 1, 2, \text{——}, n,$$

A_{ij} is the cofactor of a_{ij} in Δ, and

$$|\epsilon_{ij}|;|\eta_i| < \epsilon < \frac{|\Delta|}{\sum_{i=1}^{n}\sum_{j=1}^{n}|A_{ij}|}\,.$$

1. Suggested by Dr. A. T. Lonseth, Professor of Mathematics, Oregon State College.

CHAPTER II

UPPER AND LOWER BOUNDS FOR A DETERMINANT

In order to prove Theorem 1 essential use is made of an inequality of Hadamard which specifies an upper bound for a determinant. A proof is presented below. Also shown in this chapter is a sufficient condition for a determinant to be not zero and a sufficient condition for a determinant to be positive or zero. This enables the writer to prove Friedrichs inequality which gives a positive lower bound for a certain type of determinant.

INEQUALITY OF HADAMARD. Theorem 2. If D is the determinant whose elements are a_{ij}, $i,j = 1, 2, \text{——}, n$, then

$$D^2 \leq \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 .$$

Proof (4, p.34): Suppose that $\sum_{i,j=1}^{n} c_{ij}x_ix_j$, where $c_{ij} = c_{ji}$, is a positive definite quadratic form[1] and let Δ be the determinant whose elements are c_{ij}. Then the determinant

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22}-\lambda & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nn}-\lambda \end{vmatrix} = 0$$

is a polynomial of degree n in λ ,$P_n(\lambda)$. It can be shown (2, p.171)

1. Positive for all real values of the variables x_i except for $x_1 = x_2 = \ldots = x_n = 0$.

that $P_n(\lambda)$ has n positive roots, $\lambda_1, \lambda_2, \text{----}, \lambda_n$, and that $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} c_{ii}$ and $\prod_{i=1}^{n} \lambda_i = \Delta$. Making use of the fact that the geometric mean of n positive numbers is less than or equal to the arithmetic mean, it follows that

$$8) \quad \Delta \leq \left(\frac{\sum_{i=1}^{n} c_{ii}}{n} \right)^n$$

If $c_{ii} > 0$ for all i, then the form

$$\sum_{i,j=1}^{n} \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}} x_i x_j = \sum_{i,j=1}^{n} B_{ij} x_i x_j \quad ,$$

is also positive definite. Applying 8) to this form it follows that

$$\Delta_B \leq 1,$$

where Δ_B is the determinant consisting of elements B_{ij}, $i,j = 1, 2, \text{----}, n$. Now

$$\Delta_B = \frac{1}{c_{11}c_{22} \cdots c_{nn}} \Delta \leq 1,$$

therefore

$$\Delta \leq c_{11}c_{22} \cdots c_{nn} \,.$$

Now suppose that the form

$$\sum_{i,j=1}^{n} c_{ij} x_i x_j = \sum_{k=1}^{n} (a_{1k}x_1 + a_{2k}x_2 + \dots + a_{nk}x_n)^2,$$

where

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = D,$$

so that $c_{ii} = a_{i1}^2 + a_{i2}^2 + \ldots + a_{in}^2$. Also, if $D \neq 0$, $D^2 = \Delta$ is the determinant of a positive definite form and

$$D^2 \leq c_{11}c_{22} \ldots c_{nn} = \sum_{k=1}^{n} a_{1k}^2 \sum_{k=1}^{n} a_{2k}^2 \ldots \sum_{k=1}^{n} a_{nk}^2$$

or

$$D^2 \leq \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 .$$

SUFFICIENT CONDITION THAT A DETERMINANT BE NOT ZERO. Lemma 1. If D is the determinant whose elements are a_{ij}, $i,j = 1, ——, n$, and

$$|a_{ii}| > \sum_{k \neq i}^{n} |a_{ik}| = A_i ,$$

then $D \neq 0$.

Proof (7, p.672): assume that $D = 0$, then the system of equations

$$\begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ \vdots \qquad\qquad\quad \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = 0 \end{array}$$

has a nontrivial solution $x_1, x_2, ——, x_n$. Let $|x_r|$ be the $\max\limits_{(i)} |x_i|$, $i = 1, 2, ——, n$, and consider the r^{th} equation in the system:

$$a_{1r}x_1 + \ldots + a_{rr}x_r + \ldots + a_{nr}x_n = 0$$

or

$$a_{rr}x_r = - \sum_{k \neq r}^{n} a_{kr}x_k ;$$

whence

$$|a_{rr}||x_r| \leq \sum_{k \neq r}^{n} |a_{kr}||x_r| \leq |A_r||x_r| .$$

But since $|x_r| > 0$, this contradicts the hypothesis. Therefore $D \neq 0$.

POSITIVE DETERMINANTS. Lemma 2. If D is a determinant whose elements are a_{ij}, $i,j = 1, 2, ---, n$, and

$$a_{ii} \geq \sum_{k=i}^{n} |a_{ik}|,$$

then $D \geq 0$.

Proof (7, p.674): The lemma is obviously true if $a_{ij} = 0$ for $i \neq j$. Since D is a continuous function of n^2 variables, $D \nless 0$ by lemma 1.

INEQUALITY OF FRIEDRICHS. Theorem 3. If D_n is a determinant whose elements are a_{ij}, $i,j = 1, ---, n$, and

$$a_{ii} \geq \sum_{k \neq i}^{n} |a_{ik}| ,$$

then

$$D_n \geq a_{11}(a_{22} - |a_{21}|) \;\ldots\ldots\; (a_{nn} - \sum_{k=1}^{n-1} |a_{nk}|) .$$

Proof: Consider the determinant

$$D_n = \begin{vmatrix} a_{11}\cdots a_{1n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}\cdots a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13}\cdots a_{1n} \\ 0 & a_{22}-|a_{21}| & a_{23}\cdots a_{2n} \\ a_{31} & a_{32} & a_{33}\cdots a_{3n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3}\cdots a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13}\cdots a_{1n} \\ a_{21} & |a_{21}| & 0 \ldots 0 \\ a_{31} & a_{32} & a_{33}\cdots a_{3n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3}\cdots a_{nn} \end{vmatrix}$$

By lemma 2. the second determinant in the sum is positive or zero. Therefore

$$D_n \geq \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ 0 & a_{22}-|a_{21}| & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix}$$

By repeating the step outlined above to the remaining n-2 rows, the point is reached, after a finite number of steps, where

$$D_n \geq \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-|a_{21}| & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33}-|a_{31}|-|a_{32}| & \cdots & a_{3n} \\ 0 & 0 & 0 & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a_{nn}-\sum_{k=1}^{n-1}|a_{nk}| \end{vmatrix}$$

Q.E.D.

CHAPTER III

THE ALGEBRAIC CASE

FORM OF SOLUTION OF ALGEBRAIC SYSTEMS. By Cramer's Rule, if $D_n(k) \neq 0$ and $\overline{D}_n(k) \neq 0$, then

$$3) \quad x_i - \sum_{j=1}^{n} k_{ij} x_j = y_i$$

and

$$4) \quad \overline{x}_i - \sum_{j=1}^{n} \overline{k}_{ij} \overline{x}_j = y_i, \quad i = 1, 2, \text{-----}, n,$$

have unique solutions of the form

$$x_j = \frac{\sum_{i=1}^{n} y_i D_{ij}}{D_n(k)} \qquad \text{and} \qquad \overline{x}_j = \frac{\sum_{i=1}^{n} y_i \overline{D}_{ij}}{\overline{D}_n(k)}$$

where

$$D_n(k) = \begin{vmatrix} 1-k_{11} & -k_{12} \cdots & -k_{1n} \\ -k_{21} & 1-k_{22} \cdots & -k_{2n} \\ \vdots & \vdots & \\ -k_{n1} & -k_{n2} \cdots & 1-k_{nn} \end{vmatrix} \quad \text{and } \overline{D}_n(k) = \begin{vmatrix} 1-\overline{k}_{11} & -\overline{k}_{12} \cdots & -\overline{k}_{1n} \\ -\overline{k}_{21} & 1-\overline{k}_{22} \cdots & -\overline{k}_{2n} \\ \vdots & \vdots & \\ -\overline{k}_{n1} & -\overline{k}_{n2} \cdots & 1-\overline{k}_{nn} \end{vmatrix}$$

D_{ik} and $\overline{D}_{ik}$ are the cofactors of $\;\;_{ik} - k_{ik}$ in $D_n(k)$ and $\;\;_{ik} - \overline{k}_{ik}$ in $\overline{D}_n(k)$ respectively.

To repeat, the purpose of this paper is to find a bound for

$$9) \quad \left|\overline{x}_i - x_i\right| = \left| \frac{\overline{D}_n^i(k,y)}{\overline{D}_n(k)} - \frac{D_n^i(k,y)}{D_n(k)} \right| ,$$

where

$$D_n^j(k,y) = \sum_{i=1}^{n} y_i D_{ij} \quad \text{and} \quad \overline{D}_n^j(k,y) = \sum_{i=1}^{n} y_i \overline{D}_{ij} .$$

Inequality 9) may be written as

$$10)\quad \left|\bar{x}_i - x_i\right| \leq \frac{\left|D_n^i(k,y)\left|\bar{D}_n(k)-D_n(k)\right|\right| + \left|D_n^i(k)\left|D_n(k,y)-\bar{D}_n^i(k,y)\right|\right|}{\left|\bar{D}_n(k)\right|\left|\left|\bar{D}_n(k)\right| - \left|\bar{D}_n(k) - D_n(k)\right|\right|}$$

To find a bound for the maximum value of 14) it will be necessary to determine bounds for the expressions $|D_n^i(k,y)|$, $|\bar{D}_n(k) - D_n(k)|$, $|D_n(k)|$ and $|D_n^i(k,y) - \bar{D}_n^i(k,y)|$. The following material will develop these bounds and complete the proof of Theorem 1, Corollary 1, and Corollary 2.

ERROR IN A DETERMINANT. Theorem 4. If $D_n(a)$ is a determinant whose elements are a_{ij} and $\bar{D}_n(a)$ is a determinant whose elements are $a_{ij} + \epsilon_{ij}$; if further

$$|a_{ij}| < K, \quad |\epsilon_{ij}| < \epsilon \quad ,$$

then

$$\left|\bar{D}_n(a) - D_n(a)\right| < n^{n/2}\left[(K + \epsilon)^n - K^n\right]$$

Proof:

$$\bar{D}_n(a) = \begin{vmatrix} \epsilon_{11} & a_{12}+\epsilon_{12} & \cdots & a_{1n}+\epsilon_{1n} \\ \epsilon_{21} & a_{22}+\epsilon_{22} & \cdots & a_{2n}+\epsilon_{2n} \\ \vdots & \vdots & & \\ \epsilon_{n1} & a_{n2}+\epsilon_{n2} & \cdots & a_{nn}+\epsilon_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12}+\epsilon_{12} & \cdots & a_{1n}+\epsilon_{1n} \\ a_{21} & a_{22}+\epsilon_{22} & \cdots & a_{2n}+\epsilon_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2}+\epsilon_{n2} & \cdots & a_{nn}+\epsilon_{nn} \end{vmatrix}$$

By continuing the decomposition started above, the point is reached where

$$\overline{D}_n(a) = D_n(a) + \begin{vmatrix} \epsilon_{11} & a_{12} & \cdots & a_{1n} \\ \epsilon_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ \epsilon_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & \epsilon_{1n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & \cdots & a_{n,n-1} & \epsilon_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} \epsilon_{11} & {}_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \\ \epsilon_{n1} & {}_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \epsilon_{11} & \cdots & \epsilon_{1n} \\ \vdots & & \\ \epsilon_{n1} & \cdots & \epsilon_{nn} \end{vmatrix} .$$

That is, $\overline{D}_n(a)$ is equal to $D_n(a)$ plus n determinants in each of which just one column of a's has been replaced by the corresponding ϵ-column, plus $n(n-1)/2$ determinants in which two columns of a's have been replaced by ϵ's, etc. Consequently, with the aid of Theorem 2,

$$\left|\overline{D}_n(a)-D_n(a)\right| < n^{n/2}\left[nK^{n-1}\epsilon + \frac{n(n-1)}{2!}K^{n-2}\epsilon^2 + \cdots + \frac{n(n-1)\ldots(n-m+1)\,\epsilon^m}{m!}\right]$$

$$\left|\overline{D}_n(a)-D_n(a)\right| < n^{n/2}\left[(K+\epsilon)^n - K^n\right].$$

LIMITATION OF $\left|\overline{D}_n(k) - D_n(k)\right|$. It can be shown (9, p.214) that $D_n(k)$ may be expanded in the following manner.

$$D_n(k) = 1 - \sum_{i=1}^{n} k_{ii} + \frac{1}{2!}\sum_{i=j=1}^{n} \begin{vmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{vmatrix} + \cdots + \begin{vmatrix} -k_{11} & \cdots & -k_{1n} \\ \vdots & & \\ -k_{n1} & \cdots & -k_{nn} \end{vmatrix} .$$

Let $|k_{ij}| < K$, then by Theorem 2,

$$\left|D_n(k)\right| < 1 + nK + \frac{n(n-1)2K^2}{(2!)^2} + \cdots + \frac{n(n-1)\ \ldots\ (n-m+1)m^{m/2}K^m}{(m!)^2} .$$

Since $\mathop{\mathrm{Lim}}_{m\quad 0} m^{m/2} = 1$ we may define $0^{o/2}$ to be 1, and write

$$\left|D_n(k)\right| < \sum_{m=o}^{n} \left(\frac{n!\, m^{m/2}}{(m!)^2 (n-m)!} \right) K^m$$

or, by 5)

11) $\left|D_n(k)\right| < \psi_n(K)$.

In the same manner, expand the determinant $\overline{D}_n(k)$ to get

$$\left|\overline{D}_n(k) - D_n(k)\right| = \sum_{i=1}^{n} \epsilon_{ii} + \frac{1}{2!} \sum_{i=j=1}^{n} \left|\begin{matrix} k_{ii}+\epsilon_{ii} & k_{ij}+\epsilon_{ij} \\ k_{ji}+\epsilon_{ji} & k_{jj}+\epsilon_{jj} \end{matrix}\right| - \left|\begin{matrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{matrix}\right| + \ldots$$

$$+ \left|\begin{matrix} -k_{11}-\epsilon_{11} & \cdots & -k_{1n}-\epsilon_{1n} \\ \vdots & & \\ -k_{n1}-\epsilon_{n1} & \cdots & -k_{nn}-\epsilon_{nn} \end{matrix}\right| - \left|\begin{matrix} -k_{11} & \cdots & -k_{1n} \\ \vdots & & \\ -k_{n1} & \cdots & -k_{nn} \end{matrix}\right| .$$

Let $|k_{ij}| < K$ and $|\epsilon_{ij}| < \epsilon$, by Theorem 4,

$$\left|\overline{D}_n(k) - D_n(k)\right| < n\epsilon + \frac{n(n-1)\; 2\left[(K+\epsilon)^2 - K^2\right]}{(2!)^2} + \ldots$$

$$+ \frac{n(n-1)\ \ldots\ (n-m+1) m^{m/2}\left[(K+\epsilon)^m - K^m\right]}{(m!)^2} .$$

With the aid of the mean value theorem of differential calculus, write

$$(K+\epsilon)^m - K^m = \epsilon\, m(K + \theta_n \epsilon)^{m-1}, \quad 0 < \theta_n < 1$$

$$(K+\epsilon)^m - K^m < \epsilon\, m(K+\epsilon)^{m-1}$$

to get

$$\left|\overline{D}_n(k) - D_n(k)\right| < \epsilon \sum_{m=o}^{n} \left(\frac{n!\, m^{m/2}}{(m!)^2 (n-m)!} \right) m(K+\epsilon)^{m-1}$$

or

12) $\left|\overline{D}_n(k) - D_n(k)\right| < \epsilon\, \psi'_n(K+\epsilon)$.

LIMITATION OF $\left|\overline{D}^i_n(k,y) - D^i_n(k,y)\right|$. The determinants D_{ik} may be expanded (9, p.214) as

$$D_{ik} = k_{ik} - \sum_{j=1}^{n} \begin{vmatrix} k_{ik} & k_{ij} \\ k_{jk} & k_{jj} \end{vmatrix} + \frac{1}{2!} \sum_{j=\ell=1}^{n} \begin{vmatrix} k_{ik} & k_{ij} & k_{i\ell} \\ k_{jk} & k_{jj} & k_{j\ell} \\ k_{\ell k} & k_{\ell j} & k_{\ell\ell} \end{vmatrix} - \dots$$

to a finite number of terms. Since $|k_{ij}| < K$, by Theorem 2,

$$|D_{ik}| < K + 2nK^2 + \frac{n(n-1)\ 3^{3/2}\ K^3}{(2!)^2} + \dots$$
$$+ \frac{n(n-1)\ \dots\ (n-m+1)(m+1)^{(m+1)/2}K^{m+1}}{(m!)^2}$$

$$|D_{ik}| < K \sum_{m=o}^{n} \left(\frac{n!(m+1)^{(m+1)/2}}{(m!)^2\ (n-m)!} \right) K^m$$

or, by 6)

13) $|D_{ik}| < \phi_n(K)$.

Since $D^j_n(k,y) = \sum_{i=1}^{n} y_i D_{ij}$, if $|y_i| < Y$, then

14) $|D^i_n(k,y)| < nY\phi_n(K)$.

In a manner analogous to that of deriving inequality 12),

$$|\overline{D}_{ik} - D_{ik}| < |\epsilon_{ik}| + \left| \sum_{j=1}^{n} \begin{vmatrix} k_{ik}+\epsilon_{ik} & k_{ij}+\epsilon_{ij} \\ k_{jk}+\epsilon_{jk} & k_{jj}+\epsilon_{jj} \end{vmatrix} - \begin{vmatrix} k_{ik} & k_{ij} \\ k_{jk} & k_{jj} \end{vmatrix} \right| + \dots$$

to a finite number of terms. Now $|k_{ij}| < K$, $|\epsilon_{ij}| < \epsilon$, so by Theorem 4,

$$|\overline{D}_{ik} - D_{ik}| < \epsilon + 2n\left\{(K+\epsilon)^2 - K^2\right\} + \dots$$
$$+ \frac{n(n-1)\ \dots\ (n-m+1)(m+1)^{(m+1)/2}}{(m!)^2} \left\{(K+\epsilon)^{m+1} - K^{m+1}\right\}$$

$$\left|\overline{D}_{ik} - D_{ik}\right| < \sum_{m=o}^{n} \left(\frac{n!\,(m+1)^{(m+1)/2}}{(m!)^2(n-m)!}\right)\left((K+\epsilon)^{m+1} - K^{m+1}\right).$$

Again, with the aid of the mean value theorem of differential calculus, write

$$(K+\epsilon)^{m+1} - K^{m+1} = \epsilon\,(m+1)(K+\theta_n\epsilon)^m, \quad 0 < \theta_n < 1$$

$$(K+\epsilon)^{m+1} - K^{m+1} < \epsilon\,(m+1)(K+\epsilon)^m .$$

Therefore

$$\left|\overline{D}_{ik} - D_{ik}\right| < \sum_{m=o}^{n} \left(\frac{n!\,(m+1)^{(m+1)/2}}{(m!)^2(n-m)!}\right) \epsilon\,(m+1)(K+\epsilon)^m$$

$$\left|\overline{D}_{ik} - D_{ik}\right| < \epsilon\,\phi'_n(K+\epsilon).$$

If $|y_i| < Y$, then

15) $\left|\overline{D}_n^i(k,y) - D_n^i(k,y)\right| < nY\epsilon\,\phi'_n(K+\epsilon).$

LIMITATION OF $|\overline{x}_i - x_i|$. The inequalities which will establish the proof of Theorem 1 are,

11) $|D_n(k)| < \psi_n(K)$,

12) $|\overline{D}_n(k) - D_n(k)| < \epsilon\,\psi'_n(K+\epsilon)$,

14) $|D_n^i(k,y)| < nY\phi_n(K)$ and

15) $|\overline{D}_n^i(k,y) - D_n^i(k,y)| < nY\epsilon\phi'_n(K+\epsilon)$.

If

$$\epsilon < \frac{|\overline{D}_n(k)|}{\psi'_n(K+\epsilon)},$$

substitution of 11), 12), 14) and 15) in 10) results in

16) $$|\overline{x}_i - x_i| < \frac{nY\epsilon\left[\phi_n(K)\,\psi'_n(K+\epsilon) + \psi_n(K)\,\phi'_n(K+\epsilon)\right]}{|\overline{D}_n(k)|\left[|\overline{D}_n(k)| - \epsilon\psi'_n(K+\epsilon)\right|},$$

which is the inequality of Theorem 1.

To prove Corollary 1, assume that $\overline{D}_n(k)$ is a determinant that satisfies the inequality of Friedrichs, i.e.

17) $\overline{D}_n(k) \geq (1 - \bar{k}_{11}) \prod_n(k)$, where

$$\prod_n(k) = \prod_{i=2}^{n} \left[(1 - \bar{k}_{ii}) - \sum_{j=1}^{i-1} |\bar{k}_{ij}| \right] ,$$

and that

$$\epsilon < \frac{(1 - \bar{k}_{11}) \prod_n(k)}{\psi'_n(k + \epsilon)} ,$$

then replace $\overline{D}_n(k)$ in 16) by the right hand side of 17) to get

$$18) \quad |\bar{x}_i - x_i| < \frac{nY\epsilon \left[\phi_n(K)\, \psi'_n(K+\epsilon) + \psi_n(K)\, \phi'_n(K+\epsilon) \right]}{(1-\bar{k}_{11}) \prod_n(k) \left[(1-\bar{k}_{11}) \prod_n(k) - \epsilon\, \psi'_n(K+\epsilon) \right]} ,$$

which is the inequality of Corollary 1.

To prove Corollary 2, assume that $\overline{D}_n(k)$ is a determinant that satisfies the inequality of Friedrichs, $|\bar{k}_{ij}| < K$, $(n-1)K < 1$, and

$$\epsilon < \frac{\theta_n(K)}{\psi'_n(K + \epsilon)} \quad , \text{ where } \theta_n(K) = (1 - K)(1 - 2K) \ldots (1 - (n-1)K),$$

then

19) $\overline{D}_n(k) \geq \theta_n(K)$.

Now replace $\overline{D}_n(k)$ in 16) by the right hand side of 19) to get

$$20) \quad |\bar{x}_i - x_i| < \frac{nY\epsilon \left[\phi_n(K)\, \psi'_n(K+\epsilon) + \psi_n(K)\, \phi'_n(K+\epsilon) \right]}{\theta_n(K) \left[\theta_n(K) - \epsilon\, \psi'_n(K+\epsilon) \right]} ,$$

which is the inequality of Corollary 2.

CHAPTER IV

NUMERICAL EXAMPLE

EXAMPLE. Consider the system of equations,

$$3.99x_1 - 2.01x_2 + .99x_3 = 1$$

$$2.00x_1 + 3.01x_2 - 3.01x_3 = 1$$

$$1.01x_1 + 2.00x_2 + 3.99x_3 = 1$$

Rewrite this system as

$$4.00\bar{x}_1 - 2.00\bar{x}_2 + 1.00\bar{x}_3 = 1$$

$$2.00\bar{x}_1 + 3.00\bar{x}_2 - 3.00\bar{x}_3 = 1$$

$$1.00\bar{x}_1 + 2.00\bar{x}_2 + 4.00\bar{x}_3 = 1$$

Then, $n = 3$, $K = 3.99$, $Y = 1$, and $\epsilon = 10^{-2}$.

The solutions, by Cramer's rule, to four places are;

$$x_1 = .3272 \qquad x_2 = .1882 \qquad x_3 = .0734$$

$$\bar{x}_1 = .3263 \qquad \bar{x}_2 = .1894 \qquad \bar{x}_3 = .0736$$

and

$$|\bar{x}_1 - x_1| = .0009$$

$$|\bar{x}_2 - x_2| = .0012$$

$$|\bar{x}_3 - x_3| = .0002$$

Lonseth's inequality gives,

$$|\bar{x}_1 - x_1| < .0052$$

$$|\bar{x}_2 - x_2| < .0111$$

$$|\bar{x}_3 - x_3| < .0075$$

For $n = 3$, and first approximation, Moulton writes,

$$|\bar{x}_1 - x_1| \sim \frac{\epsilon}{|\Delta|}\left\{1 + \sum_{j=1}^{3} |x_j|\right\}\left\{\sum_{j=1}^{3} |A_{ji}|\right\},$$

where ϵ is the largest error in the system, Δ is the non-vanishing determinant of the system and A_{ji} is the cofactor of a_{ji} in Δ.

For this example,

$$|\bar{x}_1 - x_1| \sim .0052$$

$$|\bar{x}_2 - x_2| \sim .0067$$

$$|\bar{x}_3 - x_3| \sim .0045.$$

By Theorem 1, i.e. inequality 20), of this paper, and the table on the following page,

$$|\bar{x}_i - x_i| < .6466, \; i = 1, 2, 3.$$

The example indicates that inequality 20) of this paper does not give as close a bound for the maximum value of $|\bar{x}_i - x_i|$ as the one derived by Lonseth, nor the approximation of Moulton. However, if a table of values for the polynomial functions $\psi_n(x)$, $\phi_n(x)$ and their derivatives were made available, the calculation of 16) would be quite easily done. The methods of Lonseth and Moulton require the calculation of the determinant of the coefficients in the approximating system in addition to n^2 cofactors, in Lonseth's, and n in Moulton's, of this determinant. Also, the method of this paper gives a bound for the maximum value of $|\bar{x}_i - x_i|$, for all i = 1, 2, - - -, n, in a single computation.

TABLE OF VALUES

$$\psi_n(x) = \sum_{m=o}^{n} \left(\frac{n!\, m^{m/2}}{(m!)^2 (n-m)!} \right) x^m$$

$$\phi_n(x) = x \sum_{m=o}^{n} \left(\frac{n!\, (m+1)^{(m+1)/2}}{(m!)^2 (n-m)!} \right) x^m$$

x	$\psi_3(x)$	$\psi'_3(x)$	$\phi_3(x)$	$\phi'_3(x)$
.00	1.0000	3.0000	0.0000	1.0000
.05	1.1576	3.3065	.0657	2.2598
.10	1.3309	3.6260	.1655	2.4445
.15	1.5204	3.9585	.3039	3.3621
.20	1.7269	4.3039	.4858	4.4206
.25	1.9531	4.6624	.7166	5.6281
.30	2.2401	5.0338	1.0018	6.9924
.35	2.4546	5.4183	1.3478	8.5220
.40	2.7354	5.8157	1.7608	10.2243
.45	3.0364	6.2261	2.6979	12.1070
.50	3.2608	6.6495	2.8162	14.1790
.55	3.7016	7.0860	3.4735	16.4479
.60	4.0671	7.5353	4.2280	18.9218
.65	4.4553	7.9977	5.0880	21.6085
.70	4.8670	8.4730	6.0626	24.5162
.75	5.3029	8.9614	7.1609	27.6527
.80	5.7634	9.4627	8.3927	31.0262
.85	6.2493	9.9771	9.7681	34.6447
.90	6.7613	10.5044	11.2976	38.5160
.95	7.3000	11.0447	12.9921	42.6482
1.00	7.8660	11.5980	14.8628	47.0493
2.00	25.5282	25.3923	110.2359	203.8640
3.00	59.0325	44.3826	413.2961	534.4440
4.00	113.2253	68.5691	1115.2204	1105.7896
5.00	193.0025	97.9518	2471.1855	2978.9003

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