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ALAN ROBERT STOEBIG for the MASTER OF SCIENCE
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The following generalized generating functions, having the form

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

x and t real, are utilized in obtaining recurrence formulas, which are then applied to selected generated sets;

(1) $$F(x,t) = \phi(x,t)\psi(xt),$$

(2) $$F(x,t) = e^{xg(t)},$$

(3) $$F(x,t) = \phi(x,t)G(2xt-t^2),$$

and

(4) $$F(x,t) = \phi(x,t)(1+at)^m.$$

Methods for obtaining $F(x,t)$ from a knowledge of the generated set

$\{f_n(x)\}$ are considered. These include recurrence relations, both pure and mixed, theorems based on the F-equation [9], contour integration, and integral representations. If a polynomial generated set satisfies the pure recurrence relation

$$(5)\qquad f_n(x) + [An^3+Bn^2+Cn+D+Ex]f_{n-1}(x) + [Fn^3+Gn^2+Hn+K+Lx]f_{n-2}(x) + [Mn^3+Nn^2+Pn+R]f_{n-3}(x) = 0, \quad n = 3, 4, 5, \ldots,$$

the generating function satisfies the differential equation

$$(6)\qquad (M_1t^6+F_1t^5+A_1t^4)\frac{d^3F}{dt^3} + (N_1t^5+G_1t^4+B_1t^3)\frac{d^2F}{dt^2} + (P_1t^4+H_1t^3+C_1t^2)\frac{dF}{dt} + (t^3R+Lxt^2+K_1t^2+D_1t+Ext+1)F - [K_1t^2+Lxt^2+D_1\phi(x)t^2+C_1\phi(x)t^2+Ex\phi(x)t^2+\psi(x)t^2+Ext+\phi(x)t+D_1t+1] = 0,$$

where $f_0(x) = 1$, $f_1(x) = \phi(x)$, and $f_2(x) = \psi(x)$. Various procedures are discussed for obtaining the set $\{f_n(x)\}$ from knowledge of $F(x, t)$. In particular, if $F(x, z)$ is analytic in a circular neighborhood about $z = 0$ for all valid x,

$$(7)\qquad f_n(x) = \frac{1}{2\pi i}\oint F(x, z)z^{-n-1}dz, \quad n = 0, 1, 2, \ldots$$

for a sufficiently small contour encircling the origin. If the generated set is orthogonal over $[a, b]$, having a weight function $w(x)$,

$$(8) \qquad K_m = \int_a^b f_n(x) f_m(x) w(x)\, dx$$

$$= t^{-m} \int_a^b F(x, t) f_m(x)\, dx, \quad t \neq 0,$$

whenever the integral exists. A table of selected generating functions, some properties of uniformly convergent series in two variables, and a uniqueness theorem relating generating functions with their generated sets are found in the Appendices.

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in charge of major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

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AN INVESTIGATION OF SELECTED GENERATING FUNCTIONS

I. INTRODUCTION

The generating function concept leads to a very concise and convenient means of representing many of the special functions of mathematics, especially those encountered in mathematical physics. Many useful and valuable relationships among these functions can be determined by utilizing the properties inherent in the generating function representation.

Consider first a sequence of numbers $f_1, f_2, \ldots$ defining the power series

$$G(t) = \sum_{n=0}^{\infty} f_n t^n ,$$

$G(t)$ real and continuous. This is known as the Maclaurin series expansion of $G(t)$ with coefficients f_n. Now consider a function $F(x, t)$, x and t complex, which has the formal series expansion

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x, t) t^n. \tag{1}$$

In most cases of interest $f_n(x, t)$ is a continuous function of x only. When this condition is met for all n, $F(x, t)$ is said to be a generating function for the generated set $\{f_n(x)\}$. If, for some simple closed domain in the complex x-plane, the function $F(x,t)$ is analytic at $t = 0$, the generating series converges in some neighborhood about $t = 0$ [7, p. 129]. However, it is not necessary that the series converge in a finite circular region for the $\{f_n(x)\}$ to be defined and investigated. In some cases parameters $\alpha_1, \alpha_2, \ldots, \alpha_r$ may be present, both in the generating function $F(x,t)$ and the sequence $\{f_n(x)\}$. This can be acknowledged by rewriting the equation as

$$(2) \qquad F(x, t; \alpha_1, \ldots, \alpha_r) = \sum_{n=0}^{\infty} f_n(x; \alpha_1, \ldots, \alpha_r)t^n .$$

There are only rare examples in the literature when the number of parameters exceeds two.

Sometimes the F and $\{f_n\}$ are functions of several variables $x_1, x_2, \ldots, x_p$, say, and a relation of the form

$$(3) \qquad F(x_1, \ldots, x_p; t) = \sum_{n=0}^{\infty} f_n(x_1, \ldots, x_p)t^n$$

exists. Then F is called the generating function of the set $\{f_n(x_1, \ldots, x_p)\}$, in analogy with the one variable case.

It should be pointed out that generating functions and their attendant series are subsumed under the subject of polynomial expansions of analytic functions. Boas and Buck [3] give an extended treatment of this subject. Suffice it to say that a polynomial expansion of an analytic function $G(z, w)$ in two variables, where z and w are assumed complex, is given by

$$G(z, w) = \sum_{n=0}^{\infty} Q_n(w)p_n(z). \tag{4}$$

Comparison with a generating function of two independent variables shows that $p_n(z) = z^n$.

Unless explicitly stated this report will treat only generating functions having continuous real variables x and t. It will also be assumed the set $\{f_n(x)\}$ is continuous over the closed interval $D: [a \leq x \leq b]$ for all n. A selected list of generating functions is given in Appendix A. An investigation of that section will indicate the wide variety and forms possessed by $F(x, t)$ and $\{f_n(x)\}$. In many cases $f_n(x) = c_n g_n(x)$ where c_n is a constant for each n, dependent on n but independent of x and t, while $g_n(x)$ is called a special function.

There is some question as to what constitutes a "known" special function. As Rainville [7] points out;

> ...It is, of course, a matter of opinion or convention. We consider as known any function which has received individual attention in at least one research publication. In this terminology we follow the late Harry Bateman

(1882-1946). Bateman, who probably knew more about special functions than anyone else, is said to have known of about a thousand of them [7, p. 130].

A given special function can be expressed by more than one generating function. For example the set of Legendre polynomials of the first kind, $\{P_n(x)\}$, can be given by

$$(5)\qquad (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

or by

$$(6)\qquad e^{xt}J_0(t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x)t^n \qquad \text{[Ref. 9].}$$

Another example concerns the set of Tchebicheff polynomials of the first kind, $\{T_n(x)\}$. The most common expression is

$$(7)\qquad (1-xt)(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} T_n(x)t^n.$$

However, an equally valid relation is given by

$$(8)\qquad (1-t)(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} \varepsilon_n T_n(x)t^n;\ \varepsilon_0 = 1;\ \varepsilon_n = 2,\ n = 1,2,3,\ldots\ .$$

(ε_n is known as the Neumann constant.)

An examination of the expressions within each example lends support to the following statement: There exists a one-to-one correspondence between a generating function $F(x,t)$ and a generated set $\{f_n(x)\} = \{c_n g_n(x)\}$ if

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n.$$

This uniqueness property is proved in Appendix C under certain restrictions on the set $\{f_n(x)\}$, the generating function $F(x,t)$, and the variables x and t. Appendix A contains the $F(x,t)$ most commonly found in the literature generating a desired set of special functions $\{g_n(x)\}$.

Several generating functions listed in Appendix A have a Laurent series representation in powers of t. For example, the Bessel functions of the first kind, $\{J_n(x)\}$, are given by

$$(9) \qquad e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

For convenience such expressions are included in the Appendix.

Appendix B is a synopsis of the properties inherent in a series

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

which is uniformly convergent in x and t separately. Throughout the text of the report this property will be tacitly invoked to justify operations such as term-by-term differentiation and integration of the series. Most cases found in the literature satisfy the requirement of uniform convergence in all orders of n.

II. GENERAL RECURRENCE FORMULAS

Many generating functions can be treated under more general classifications by considering the functional dependence between x, t, and $F(x,t)$. For example the form

$$F(x,t) = \phi(x,t)G(2xt-t^2) = \sum_{n=0}^{\infty} f_n(x)t^n = \sum_{n=0}^{\infty} c_n g_n(x)t^n \tag{10}$$

is shared by Hermite polynomial sets $\{H_n(x)\}$, Legendre polynomial sets $\{P_n^m(x)\}$, and Gegenbauer polynomial sets $\{C_n^{\nu}(x)\}$. By treating these general forms it is frequently possible to derive general recurrence formulas applicable to many generating sets simultaneously. A recurrence relation is said to exist for a set $\{f_n(x)\}$ if there exists an equation or a set of equations relating two or more members of the set or their derivatives. In deriving these relations it will be assumed that the equation

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

obeys the requirements of Property 4 for $k = 2$ as listed in Appendix B. Only then can assurance be given to twice termwise differentiation of both the series and the generating functions leading to rigorously valid results. Throughout the remainder of this chapter such requirements on the generating equation will be assumed.

Functional forms considered below will include

$$F(x,t) = \phi(x,t)\psi(xt), \tag{11}$$

$$F(x,t) = e^{xg(t)}, \tag{12}$$

$$F(x,t) = \phi(x,t)G(2xt-t^2), \tag{13}$$

and

$$F(x,t) = \phi(x,t)(1+at)^m. \tag{14}$$

1. $F(x,t) = \phi(x,t)\psi(xt)$

$$= \sum_{n=0}^{\infty} f_n(x)t^n$$

If

$$F(x,t) = \phi(x,t)\psi(xt) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

$$\frac{\partial F}{\partial x} = \phi_x\psi + \phi\psi' t, \tag{15}$$

$$\frac{\partial F}{\partial t} = \phi_t\psi + \phi\psi' x, \tag{16}$$

(17) $$\frac{\partial^2 F}{\partial x^2} = \phi_{xx}\psi + 2t\phi_x\psi' + t^2\phi\psi'',$$

and

(18) $$\frac{\partial^2 F}{\partial t^2} = \phi_{tt}\psi + 2x\phi_t\psi' + x^2\phi\psi''.$$

It is understood that $\phi = \phi(x, t)$, etc.; $\phi_x = \frac{\partial\phi(x, t)}{\partial x}$, etc.; and $\psi'(xt) = \frac{d\psi(xt)}{d(xt)}$. From Eqs. (15)-(18) it is found that

(19) $$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = \phi_x\psi x - \phi_t\psi t$$

and

(20) $$x^2\frac{\partial^2 F}{\partial x^2} - t^2\frac{\partial^2 F}{\partial t^2} = x^2\phi_{xx}\psi + 2tx^2\phi_x\psi' - t^2\phi_{tt}\psi - 2xt^2\phi_t\psi'.$$

In the special case $\phi(x, t) = C$, a constant, Eqs. (19) and (20) reduce to

(21) $$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = 0$$

and

(22) $$x^2\frac{\partial^2 F}{\partial x^2} - t^2\frac{\partial^2 F}{\partial t^2} = 0.$$

Since

$$F = \sum_{n=0}^{\infty} f_n(x)t^n$$

it follows that

(23)
$$\sum_{n=0}^{\infty} xf_n'(x)t^n - \sum_{n=0}^{\infty} f_n(x)nt^n = 0$$

and

(24)
$$\sum_{n=0}^{\infty} x^2 f_n''(x)t^n - \sum_{n=0}^{\infty} f_n(x)n(n-1)t^n = 0.$$

Eq. (23) can be rewritten

(25)
$$\sum_{n=0}^{\infty} [xf_n'(x) - nf_n(x)]t^n = 0.$$

In order for (25) to hold for all valid t and n, the coefficients must vanish and therefore

(26)
$$xf_n'(x) = nf_n(x), \quad n \geq 0.$$

Similar reasoning from Eq. (24) leads to the corresponding result

(27)
$$x^2 f_n''(x) = n(n-1)f_n(x), \quad n \geq 0 .$$

These results ((26) and (27)) are seen to hold in the trivial case for which

$$F(x, t) = \frac{1}{1-xt} = \sum_{n=0}^{\infty} f_n(x)t^n$$

with $f_n(x) = x^n$.

Suppose $\phi(x,t) = A(t)$. Then

$$A(t)\psi(xt) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

the Brenke polynomial class of generating functions. Equations (19) and (20) respectively become

$$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = -A'\psi t \tag{28}$$

and

$$x^2\frac{\partial^2 F}{\partial x^2} - t^2\frac{\partial^2 F}{\partial t^2} = -t^2A''\psi - 2xt^2A'\psi' . \tag{29}$$

By making more specific choices for $A(t)$ and $\psi(xt)$, genuine recurrence relations can be derived. For example let $A(t) = e^{at}$. Then $A'(t) = ae^{at} = aA(t)$ so that (28) becomes

$$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = -aFt. \tag{30}$$

As before substitute

$$F = \sum_{n=0}^{\infty} f_n(x)t^n$$

into Eq. (30). Then,

$$(31) \qquad \sum_{n=0}^{\infty} xf_n'(x)t^n - \sum_{n=0}^{\infty} f_n(x)nt^n = -a \sum_{n=0}^{\infty} f_n(x)t^{n+1}$$

$$= -a \sum_{n=1}^{\infty} f_{n-1}(x)t^n.$$

The resulting recurrence formula is

$$(32a) \qquad xf_n'(x) - nf_n(x) = -af_{n-1}(x), \quad n \geq 1,$$

with

$$(32b) \qquad f_0'(x) = 0.$$

Other recurrence relations can be found for other forms of $\phi(x, t)$.

2. $F(x, t) = A(t)e^{xg(t)}$

This generating function form is said to be of the Sheffer class; the polynomial set it generates is called the Sheffer polynomial set of zero type [7]. The procedure is the same as before. Differentiation leads to the equations;

$$(33) \qquad \frac{\partial F}{\partial x} = Ae^{xg}g,$$

$$= Fg$$

$$(34) \qquad \frac{\partial F}{\partial t} = A'e^{xg} + Ae^{xg}xg'$$

$$= A'e^{xg} + Fxg' ,$$

$$\frac{\partial^2 F}{\partial x^2} = Age^{xg}g = Fg^2, \tag{35}$$

and

$$\frac{\partial^2 F}{\partial t^2} = A''e^{xg} + 2A'xg'e^{xg} + Fx^2g'^2 + Fxg''. \tag{36}$$

Consider the case when $g = t$. Sets generated from this class are said to be Appell polynomial sets. Then Eqs. (33) and (35) become

$$\frac{\partial F}{\partial x} = Ft \tag{37}$$

and

$$\frac{\partial^2 F}{\partial x^2} = Ft^2. \tag{38}$$

Since

$$F = \sum_{n=0}^{\infty} f_n(x)t^n,$$

Eqs. (37) and (38) can be further simplified to the following:

$$\sum_{n=0}^{\infty} f_n'(x)t^n = \sum_{n=0}^{\infty} f_n(x)t^{n+1} = \sum_{n=1}^{\infty} f_{n-1}(x)t^n \tag{39}$$

and

(40) $$\sum_{n=0}^{\infty} f_n''(x)t^n = \sum_{n=0}^{\infty} f_n(x)t^{n+2}$$

$$= \sum_{n=2}^{\infty} f_{n-2}(x)t^n.$$

After equating coefficients the equations reduce to

(41a) $$f_n'(x) = f_{n-1}(x), \quad n \geq 1,$$

with

(41b) $$f_0'(x) = 0;$$

and

(42a) $$f_n''(x) = f_{n-2}(x), \quad n \geq 2,$$

with

(42b) $$f_0''(x) = f_1''(x) = 0.$$

These are well-known formulas for Appell sets. For example, the Bernoulli polynomials form an Appell set; they are given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x)t^n.$$

Here $f_n(x) = \frac{B_n(x)}{n!}$. Substitution into Eqs. (41) and (42) leads to

(43a) $$\frac{B_n'(x)}{n!} = \frac{B_{n-1}(x)}{(n-1)!}$$

or

(43b) $$B_n'(x) = nB_{n-1}(x), \quad n \geq 1,$$

with

(43c) $$B_0'(x) = 0;$$

and

(44a) $$\frac{B_n''(x)}{n!} = \frac{B_{n-2}(x)}{(n-2)!}$$

or

(44b) $$B_n''(x) = n(n-1)B_{n-2}(x), \quad n \geq 2,$$
$$= (n^2-n)B_{n-2}(x)$$

with

(44c) $$B_0''(x) = B_1''(x) = 0.$$

These results are verified in any discussion of the Bernoulli polynomial set [1].

Another example concerns the generating function for the Euler polynomials;

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} \frac{1}{n!} E_n(x)t^n.$$

Since it is of the same functional form as the $F(x, t)$ producing the Bernoulli polynomials, the Euler generating function leads to the same recurrence formulas. Replace $f_n(x)$ in Eqs. (41) and (42) by

$\frac{E_n(x)}{n!}$. The equations reduce to

(45a) $$E_n'(x) = nE_{n-1}(x), \quad n \geq 1,$$

with

(45b) $$E_0'(x) = 0;$$

and

(46a) $$E_n''(x) = n(n-1)E_{n-2}(x)$$

with

(46b) $$E_0''(x) = E_1''(x) = 0.$$

Another important form is given by $g(t) = \frac{t}{t-1}$, which is possessed by the generating functions found in the Laguerre class. Then Eqs. (33) and (35) become

(47) $$\frac{\partial F}{\partial x} = \frac{t}{t-1} F$$

or

(48) $$(t-1) \frac{\partial F}{\partial x} = tF,$$

and

(49) $$\frac{\partial^2 F}{\partial x^2} = \left(\frac{t}{t-1}\right)^2 F$$

or

(50) $$(t^2-2t+1) \frac{\partial^2 F}{\partial x^2} = t^2 F.$$

Substitute

$$F = \sum_{n=0}^{\infty} f_n(x)t^n.$$

Equation (48) is reduced to

$$\text{(51)} \qquad \sum_{n=1}^{\infty} f'_{n-1}(x)t^n - \sum_{n=0}^{\infty} f'_n(x)t^n = \sum_{n=1}^{\infty} f_{n-1}(x)t^n,$$

while Eq. (50) leads to

$$\text{(52)} \qquad \sum_{n=2}^{\infty} f''_{n-2}(x)t^n - 2\sum_{n=1}^{\infty} f''_{n-1}(x)t^n + \sum_{n=0}^{\infty} f''_n(x)t^n = \sum_{n=2}^{\infty} f_{n-2}(x)t^n.$$

It follows that

$$\text{(53a)} \qquad f'_{n-1}(x) - f'_n(x) = f_{n-1}(x), \quad n \geq 1,$$

with

$$\text{(53b)} \qquad f'_0(x) = 0;$$

and

$$\text{(54a)} \qquad f''_{n-2}(x) - 2f''_{n-1}(x) + f''_n(x) = f_{n-2}(x), \quad n \geq 2,$$

with

$$\text{(54b)} \qquad f''_0(x) = f''_1(x) = 0.$$

Equation (53) is given in Rainville [7].

For the associated Laguerre polynomials of the first kind, the generating equation is given by

$$(1-t)^{-1-\alpha}e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n.$$

From the recurrence relations derived above, the equations

(55a) $$L_{n-1}'^{(\alpha)}(x) - L_n'^{(\alpha)}(x) = L_{n-1}^{(\alpha)}(x), \quad n \geq 1,$$

with

(55b) $$L_0'^{(\alpha)}(x) = 0;$$

and

(56a) $$L_{n-2}''^{(\alpha)}(x) - 2L_{n-1}''^{(\alpha)}(x) + L_n''^{(\alpha)}(x) = L_{n-2}^{(\alpha)}(x), \quad n \geq 2,$$

with

(56b) $$L_0''^{(\alpha)}(x) = L_1''^{(\alpha)}(x) = 0,$$

follow.

For the special case $\alpha = 0$ these become equations relating members of the polynomial set known as the Laguerre polynomials of the first kind, $\{L_n(x)\}$.

An interesting form for $g(t)$ involves the expression $g(t) = 1 - e^t$. Equations (33) and (35) reduce to

(57) $$\frac{\partial F}{\partial x} = (1-e^t)F$$

and

(58) $$\frac{\partial^2 F}{\partial x^2} = (1-2e^t+e^{2t})F.$$

Since

$$F = \sum_{n=0}^{\infty} f_n(x)t^n \quad \text{and} \quad e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} ,$$

Eq. (57) becomes

$$(59) \qquad \sum_{n=0}^{\infty} f_n'(x)t^n = \sum_{n=0}^{\infty} f_n(x)t^n - \sum_{n=0}^{\infty} f_n(x)t^n \sum_{n=0}^{\infty} \frac{t^n}{n!} .$$

For two absolutely convergent series, Mertens' theorem [2, p. 376-377] guarantees that the Cauchy product will converge to the product of the series. Under these conditions Eq. (59) reduces to

$$(60) \qquad \sum_{n=0}^{\infty} f_n'(x)t^n = \sum_{n=0}^{\infty} f_n(x)t^n - \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_k(x)\frac{1}{(n-k)!} t^n.$$

So, the recurrence relation is

$$(61) \qquad f_n'(x) = f_n(x) - \sum_{k=0}^{n} f_k(x) \frac{1}{(n-k)!} .$$

Similarly, Eq. (58) leads to

(62)

$$\sum_{n=0}^{\infty} f_n''(x)t^n = \sum_{n=0}^{\infty} f_n(x)t^n - 2\sum_{n=0}^{\infty} f_n(x)t^n \sum_{n=0}^{\infty} \frac{t^n}{n!} + \sum_{n=0}^{\infty} f_n(x)t^n \sum_{n=0}^{\infty} \frac{(2t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} f_n(x)t^n - 2\sum_{n=0}^{\infty} \sum_{k=0}^{n} f_k(x)\frac{1}{(n-k)!}t^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_k(x) \frac{2^{n-k}}{(n-k)!}t^n,$$

or

$$(63)\qquad f_n''(x) = f_n(x) - 2\sum_{k=0}^{n} f_k(x)\frac{1}{(n-k)!} + \sum_{k=0}^{n} f_k(x)\frac{2^{n-k}}{(n-k)!}.$$

The Toscano polynomial set is generated by the equation

$$e^{at+x(1-e^t)} = \sum_{n=0}^{\infty} \frac{1}{n!} g_n^{(a)}(x)t^n$$

with $f_n(x) = \dfrac{g_n^{(a)}(x)}{n!}$. Substitute into Eqs. (61) and (63) to obtain

$$(64)\qquad \frac{g_n'^{(a)}(x)}{n!} = \frac{g_n^{(a)}(x)}{n!} - \sum_{k=0}^{n} \frac{g_k^{(a)}(x)}{k!(n-k)!}$$

$$= \frac{g_n^{(a)}(x)}{n!} - \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} g_k^{(a)}(x)$$

or

$$(65)\qquad g_n'^{(a)}(x) = g_n^{(a)}(x) - \sum_{k=0}^{n} \binom{n}{k} g_k^{(a)}(x),$$

and

$$(66)\qquad \frac{g_n''^{(a)}(x)}{n!} = \frac{g_n^{(a)}(x)}{n!} - 2\frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} g_k^{(a)}(x) + \frac{2^n}{n!}\sum_{k=0}^{n} \binom{n}{k} g_k^{(a)}(x)2^{-k}$$

or

$$(67)\qquad g_n''^{(a)}(x) = g_n^{(a)}(x) - 2\sum_{k=0}^{n} \binom{n}{k} g_k^{(a)}(x) + 2^n\sum_{k=0}^{n} \binom{n}{k} g_k^{(a)}(x)2^{-k}.$$

3. $F(x,t) = \phi(x,t)G(2xt-t^2)$

The appropriate partial derivatives are

(68) $$\frac{\partial F}{\partial x} = \phi_x G + 2t\phi G' ,$$

(69) $$\frac{\partial F}{\partial t} = \phi_t G + (2x-2t)\phi G' ,$$

(70) $$\frac{\partial^2 F}{\partial x^2} = \phi_{xx} G + 4t\phi_x G' + 4t^2\phi G'' ,$$

and

(71) $$\frac{\partial^2 F}{\partial t^2} = \phi_{tt} G + 4(x-t)\phi_t G' - 2\phi G' + (2x-2t)^2\phi G'' .$$

Consider the case $\phi = 1$, applicable to the generating functions of the ultraspherical polynomials, the Hermite polynomials, the Legendre polynomials, and the Tchebicheff polynomials of the second kind. Equations (68) to (71) reduce to

(72) $$\frac{\partial F}{\partial x} = 2tG' ,$$

(73) $$\frac{\partial F}{\partial t} = (2x-2t)G' ,$$

(74) $$\frac{\partial^2 F}{\partial x^2} = 4t^2 G'' ,$$

and

(75) $$\frac{\partial^2 F}{\partial t^2} = -2G' + (2x-2t)^2 G'' .$$

Equating the coefficients of G' in Eqs. (72) and (73) leads to

(76) $$(x-t)\frac{\partial F}{\partial x} = t\frac{\partial F}{\partial t}, \qquad (t \neq 0, t \neq x)$$

and the recurrence relation which follows is

(77a) $$xf_n'(x) - f_{n-1}'(x) = nf_n(x), \quad n \geq 1,$$

with

(77b) $$f_0'(x) = 0,$$

a result given in Rainville [7, p. 131-132].

Repeating the procedure with the coefficients of G'' in Eqs. (74) and (75), and noting that Eqs. (72) and (73) give expressions for G', leads to the following results:

(78) $$t^2\frac{\partial^2 F}{\partial t^2} = -t\frac{\partial F}{\partial x} + (x^2-2xt+t^2)\frac{\partial^2 F}{\partial x^2}$$

with

$$G' = \frac{1}{2t}\frac{\partial F}{\partial x}, \quad t \neq 0,$$

and the recurrence relation

(79a) $$n(n-1)f_n(x) = f_{n-1}'(x) + x^2 f_n''(x) - 2xf_{n-1}''(x) + f_{n-2}''(x), \quad n \geq 2,$$

with

(79b) $$f_0''(x) = f_1''(x) = 0;$$

$$(80) \qquad (xt^2-t^3)\frac{\partial^2 F}{\partial t^2} = -t^2\frac{\partial F}{\partial t} + 1(x^3-3x^2t+3xt^2-t^3)\frac{\partial^2 F}{\partial x^2}$$

with

$$G' = \frac{1}{2x-2t}\frac{\partial F}{\partial t}, \qquad x \neq t,$$

and the recurrence relation

$$(80a) \quad n(n-1)xf_n(x) - (n-1)(n-2)f_{n-1}(x) = -(n-1)f_{n-1}(x) + 1x^3 f_n''(x)$$
$$- 3x^2 f_{n-1}''(x) + 3xf_{n-2}''(x)$$
$$- 1f_{n-3}''(x), \qquad n \geq 3,$$

with

$$(80b) \qquad 2xf_2(x) = -f_1(x) + 1x^3 f_2''(x).$$

For the Tchebicheff polynomials of the second kind, $\{f_n(x)\} = \{U_n(x)\}$; the Hermite polynomials,

$$\{f_n(x)\} = \{\frac{H_n(x)}{n!}\};$$

the Legendre polynomials of the first kind, $\{f_n(x)\} = \{P_n(x)\}$; and the ultraspherical polynomials,

$$\{f_n(x)\} = \{\frac{(1+2a)_n}{(1+a)_n} P_n^{(a,a)}(x)\}.$$

4. $F(x, t) = \phi(x, t)(1+at)^m$

Another interesting form of generating function occurs when

$F(x,t) = \phi(x,t)(1+at)^m$, a and m arbitrary constants. Taking derivatives leads to the following;

(81) $$\frac{\partial F}{\partial x} = \phi_x(1+at)^m,$$

(82) $$\frac{\partial F}{\partial t} = \phi_t(1+at)^m + \phi ma(1+at)^{m-1},$$

(83) $$\frac{\partial^2 F}{\partial x^2} = \phi_{xx}(1+at)^m,$$

and

(84) $$\frac{\partial^2 F}{\partial t^2} = m(m-1)a^2\phi(1+at)^{m-2} + 2am\phi_t(1+at)^{m-1} + \phi_{tt}(1+at)^m.$$

Suppose $\phi = \phi(x)$ only. Then Eqs. (82) and (84) become

(85) $$(1+at)\frac{\partial F}{\partial t} = maF$$

and

(86) $$(1+2at+a^2t^2)\frac{\partial^2 F}{\partial t^2} = m(m-1)a^2 F,$$

respectively, knowing that $F = \phi(x)(1+at)^m$. The recurrence relations are

(87a) $$(n+1)f_{n+1}(x) = a(m-n)f_n(x), \quad n \geq 1,$$

with

(87b) $$f_1(x) = maf_0(x);$$

and

(88a) $$(n+2)(n+1)f_{n+2}(x) + 2an(n+1)f_{n+1}(x) = a^2[m(m-1)-n(n-1)]f_n(x), \quad n \geq 2,$$

with

(88b)
$$6f_3(x) + 4af_2(x) = a^2 m(m-1)f_1(x)$$

and

(88c)
$$2f_2(x) = a^2 m(m-1)f_0(x).$$

Let $\phi = \phi(bxt)$. Then Eqs. (81) and (82) lead to

(89)
$$\frac{\partial F}{\partial x} = bt(1+at)^m \phi'$$

and

(90)
$$\frac{\partial F}{\partial t} = xb(1+at)^m \phi' + \frac{maF}{1+ta}, \quad t \neq -\frac{1}{a}.$$

Since $\phi' = \frac{1}{tb}(1+at)^{-m} \frac{\partial F}{\partial x}$ when $tb \neq 0$ and $at \neq -1$, Eq. (90) becomes

(91)
$$t(1+at) \frac{\partial F}{\partial t} = x(1+at) \frac{\partial F}{\partial x} + matF.$$

The derived recurrence relation is

(92a)
$$nf_n(x) = xf_n'(x) + axf'_{n-1}(x) + a(m-n+1)f_{n-1}(x), \quad n \geq 2,$$

with

(92b)
$$f_0'(x) = 0$$

and

(92c)
$$f_1(x) = xf_1'(x) + maf_0(x).$$

As an example consider the Cesáro polynomials generated by

$$(1-t)^{-k-1}(1-xt)^{-1} = \sum_{n=0}^{\infty} g_n^{(k)}(x)t^n.$$

Here $m = -k-1$, $b = -1$, and $a = -1$. Substitution into the formula leads to the result

(93a) $$ng_n^{(k)}(x) = xg_n^{'(k)}(x) - xg_{n-1}^{'(k)}(x) + (k+n)g_n^{(k)}(x), \quad n \geq 2,$$

with

(93b) $$g_0^{'(k)}(x) = 0$$

and

(93c) $$g_1^{(k)}(x) = xg_1^{'(k)}(x) + (k+1)g_0^{(k)}(x).$$

As the final consideration let $\phi(x,t) = e^{\frac{bxt}{1+ct}}$, c and b arbitrary. This form encompasses the class of Laguerre polynomials and the set of Sonine polynomials. Equations (81) and (82) reduce to

(94) $$(1+ct)\frac{\partial F}{\partial x} = btF$$

and

(95) $$(1+ct)^2(1+at)\frac{\partial F}{\partial t} = bx(1+at)F + ma(1+ct)^2F.$$

The recurrence relations are found to be

(96a) $$f_n'(x) + cf_{n-1}'(x) = bf_{n-1}(x), \quad n \geq 1,$$

with

(96b) $$f_0'(x) = 0;$$

and

(97a) $$(n+1)f_{n+1}(x) + [(2c+a)n-bx-ma]f_n(x)$$
$$+ [c(c+2a)(n-1) - abx - 2cma]f_{n-1}(x) + [ac^2(n-2)-mac^2]f_{n-2}(x) = 0,$$
$$n \geq 3,$$

with

(97b) $$3f_3(x) + 2(2c+a)f_2(x) + c(c+2a)f_1(x)$$
$$= bxf_2(x) + abxf_1(x) + maf_2(x) + 2mcaf_1(x) + mac^2f_0(x),$$

(97c) $$2f_2(x) + (2c+a)f_1(x) = bxf_1(x) + abxf_0(x) + maf_1(x) + 2mcaf_0(x),$$

and

(97d) $$f_1(x) = bxf_0(x) + maf_0(x).$$

The Sonine polynomial set is determined by the generating function

$$(1+t)^{-m-1} e^{\frac{xt}{1+t}} = \sum_{n=0}^{\infty} (m+n)!\, S_m^n(x) t^n$$

with $\{f_n(x)\} = \{(m+n)!\, S_m^n(x)\}$, $b = 1$, $c = 1$, and $m = -m-1$. Eq. (96) is reduced to

(98a) $$(m+n)S_m^{'n}(x) + S_m^{'n-1}(x) = S_m^{n-1}(x), \quad n \geq 1,$$

with

(98b) $$S_m^{'0}(x) = 0.$$

A corresponding expression holds for the second recurrence relation.

III. METHODS FOR DETERMINING $F(x,t)$ FROM $\{f_n(x)\}$

Let

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n.$$

Consider the situation in which the set $\{f_n(x)\}$ is known for all non-negative n and the generating function $F(x,t)$ is not. If such a function exists, there are several ways in which it can be formally determined.

The first method to be discussed is very elegant, yet simple in nature. It involves the concept of a pure recurrence relation; that is, equations relating two or more members of the generated set, $\{f_n(x)\}$, containing no derivatives. Such relations are guaranteed for most polynomial sets by the researches of Fasenmyer [7], who has proved that if the polynomial set $\{f_n(x)\}$ can be described by a set of hypergeometric functions of the form ${}_pF_q$, there exist constants A, B, C, D, and E, which may be functions of n, such that

$$f_n(x) + (A+Bx)f_{n-1}(x) + (C+Dx)f_{n-2}(x) + Ef_{n-3}(x) = 0. \tag{99}$$

The majority of well-known and commonly used polynomial sets can be given by hypergeometric functions; examples include

(100) $$L_n(x) = {}_1F_1(-n;\ 1;\ x),$$

the Laguerre polynomial,

(101) $$Z_n(x) = {}_2F_2(-n, n+1;\ 1, 1;\ x),$$

Bateman's $Z_n(x)$ polynomial, and

(102) $$H_n(x) = (2x)^n {}_2F_0(-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2};\ -\tfrac{1}{x^2}),$$

the Hermite polynomial.

Other representations are found in [1].

Given that a pure recurrence relation can be established among members of the generated set, the technique consists of multiplying over the recurrence formula by the power variable t^n, summing over n, and manipulating the resulting expression to obtain the desired generating function. This method requires knowledge of the first two or three members of the polynomial set in order that a unique generating function can be established; the phrase "initial values" points to the analogy such conditions have with initial value problems in ordinary differential equation theory. It should again be emphasized that this method assumes a priori that a generating function

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

exists with a uniformly convergent series in x and in t.

Furthermore, this property is assumed to be in effect through the third ~~partial~~ derivatives in x and in t, as required by the hypotheses of Property 4 (Appendix B) for $k=2$ and as mentioned generally in the Introduction.

An example should clarify the concepts involved. Suppose the following recurrence relation holds for the $\{f_n(x)\}$ under consideration;

$$nf_n(x) - 2xf_{n-1}(x) + 2f_{n-2}(x) = 0, \quad n = 2, 3, 4, \ldots \tag{103a}$$

with

$$f_0(x) = 1 \tag{103b}$$

and

$$f_1(x) = 2x, \tag{103c}$$

the "initial values."

Replace n by $n+2$ in Eq. (103a). The result is

$$(n+2)f_{n+2}(x) - 2xf_{n+1}(x) + 2f_n(x) = 0, \quad n = 0, 1, 2, \ldots . \tag{104}$$

Multiply by t^n and sum. It follows that

$$\sum_{n=0}^{\infty} (n+2)f_{n+2}(x)t^n - 2x\sum_{n=0}^{\infty} f_{n+1}(x)t^n + 2\sum_{n=0}^{\infty} f_n(x)t^n = 0. \tag{105}$$

Let

$$F[f_n(x)] = \sum_{n=0}^{\infty} f_n(x)t^n,$$

where it is understood x is held constant throughout. Then, assuming Property 4 can be applied,

(106) $$\frac{dF}{dt} = \sum_{n=1}^{\infty} nf_n(x)t^{n-1},$$

or

(107) $$t\,\frac{dF}{dt} = \sum_{n=0}^{\infty} nf_n(x)t^{n}$$

$$= F[nf_n(x)].$$

Now, the first term of Eq. (105) can be rewritten as

(108) $$\sum_{n=0}^{\infty} (n+2)f_{n+2}(x)t^n = \sum_{n=2}^{\infty} nf_n(x)t^{n-2}$$

$$= t^{-2}\sum_{n=2}^{\infty} nf_n(x)t^n$$

$$= t^{-2}\,[\sum_{n=0}^{\infty} nf_n(x)t^n - 0\cdot f_0(x)t^0 - 1\cdot f_1(x)t^1]$$

$$= t^{-2}\{F[nf_n(x)] - 0 - 1\cdot 2xt\}$$

$$= t^{-2}(t\,\frac{dF}{dt} - 2xt)$$

$$= t^{-1}\,\frac{dF}{dt} - 2xt^{-1},$$

if $t \neq 0$. Similarly, the second term of Eq. (105) can be reduced to

$$\sum_{n=0}^{\infty} f_{n+1}(x)t^n = \sum_{n=1}^{\infty} f_n(x)t^{n-1} \tag{109}$$

$$= t^{-1} \sum_{n=1}^{\infty} f_n(x)t^n$$

$$= t^{-1} \left[\sum_{n=0}^{\infty} f_n(x)t^n - f_0(x)t^0 \right]$$

$$= t^{-1}F - t^{-1}$$

if $t \neq 0$. Thus, the recurrence relation becomes

$$t^{-1} \frac{dF}{dt} - 2xt^{-1} - 2x(t^{-1}F - t^{-1}) + 2F = 0, \tag{110}$$

$$t^{-1} \frac{dF}{dt} - 2xt^{-1}F + 2F = 0,$$

$$\frac{dF}{dt} - 2xF + 2tF = 0,$$

or

$$\frac{dF}{dt} - (2x-2t)F = 0.$$

The solution is seen to be

$$\ln F = (2xt - t^2) + C_1, \tag{111}$$

or

$$F = C_1 e^{2xt-t^2}. \tag{112}$$

Since $f_0(x) = 1 = F(x, 0)$,

$$1 = C_1 e^{2x\cdot 0 - 0^2} \tag{113}$$
$$= C_1 \cdot e^0$$
$$= C_1 .$$

Therefore, the formal generating function is $F(x, t) = e^{2xt-t^2}$. A check with Appendix A will show that this result is the generating function for the set $\{\frac{H_n(x)}{n!}\}$. Further investigation will reveal that it satisfies the given pure recurrence relation. This technique is briefly discussed in [5].

Let a more general case be considered. As was stated earlier any polynomial set $\{f_n(x)\}$ described in terms of a hypergeometric function has a pure recurrence formula of the form of Eq. (99) where A, B, C, D, and E are constants independent of x but not necessarily of n. It is empirically found that the majority of polynomial sets have recurrence relations of the form

$$f_n(x) + [An^3+Bn^2+Cn+D+Ex]f_{n-1}(x) + [Fn^3+Gn^2+Hn+K +Lx]f_{n-2}(x) + [Mn^3+Nn^2+Pn+R]f_{n-3}(x) = 0, \quad n = 3,4,5,\ldots , \tag{114}$$

where $A, B, \ldots, R$ are independent of x and n, although they may be individually composed of terms with other parameters. Most of these constants are zero for specific polynomial sets. It is desired to obtain the equation from which the generating function can be found for a given set $\{f_n(x)\}$ satisfying Eq. (114). In addition it is assumed $f_0(x) = 1$, $f_1(x) = \phi(x)$, and $f_2(x) = \psi(x)$, which are the initial conditions normally used.

Replace n by $n+3$ in Eq. (114) to get

(115)

$$f_{n+3}(x) + [A(n+3)^3+B(n+3)^2+C(n+3)+D+Ex]f_{n+2}(x)$$
$$+ [F(n+3)^3+G(n+3)^2+H(n+3)+K+Lx]f_{n+1}(x)$$
$$+ [M(n+3)^3+N(n+3)^2+P(n+3)+R]f_n(x) = 0, \quad n = 0, 1, 2, \ldots .$$

Multiply by t^n and sum over n. The expression becomes

(116)

$$\sum_{n=0}^{\infty} f_{n+3}(x)t^n + \sum_{n=0}^{\infty} [A(n+3)^3+B(n+3)^2+C(n+3)+D+Ex]f_{n+2}(x)t^n$$
$$+ \sum_{n=0}^{\infty} [F(n+3)^3+G(n+3)^2+H(n+3)+K+Lx]f_{n+1}(x)t^n$$
$$+ \sum_{n=0}^{\infty} [M(n+3)^3+N(n+3)^2+P(n+3)+R]f_n(x)t^n = 0.$$

Each expression bounded in brackets remains a cubic polynomial in n after the transformations. A theorem [8, p. 40] from the calculus of finite differences exists which asserts that a polynomial expression in n of degree r can be represented by a unique factorial series of the form $A'n^{(r)} + B'n^{(r-1)} + \ldots + N'n^{(1)} + P$ where $n^{(k)} = n(n-1)\ldots(n-k+1)$. These new coefficients involve Stirling Numbers of the Second Kind. Consequently, Eq. (116) can be rewritten as

(117)

$$t^{-3}\sum_{n=3}^{\infty} f_n(x)t^n + t^{-2}\sum_{n=2}^{\infty}[A_1n^{(3)}+B_1n^{(2)}+C_1n^{(1)}+D_1+Ex]f_n(x)t^n$$

$$+\, t^{-1}\sum_{n=1}^{\infty}[F_1n^{(3)}+G_1n^{(2)}+H_1n^{(1)}+K_1+Lx]f_n(x)t^n$$

$$+\sum_{n=0}^{\infty}[M_1n^{(3)}+N_1n^{(2)}+P_1n^{(1)}+R]f_n(x)t^n = 0,$$

if $t \neq 0$. The first term can be reduced to

$$t^{-3}\sum_{n=3}^{\infty} f_n(x)t^n = t^{-3}\Big[\sum_{n=0}^{\infty} f_n(x)t^n - f_0(x)t^0 - f_1(x)t^1 - f_2(x)t^2\Big] \tag{118}$$

$$= t^{-3}\{F[f_n(x)] - 1 - \phi(x)t - \psi(x)t^2\}.$$

By repeating the procedure used in deriving Eq. (107), it can be shown that

$$t^2\frac{d^2F}{dt^2} = \sum_{n=0}^{\infty} n^{(2)}f_n(x)t^n \tag{119}$$

and

$$t^3 \frac{d^3F}{dt^3} = \sum_{n=0}^{\infty} n^{(3)} f_n(x)t^n. \tag{120}$$

With these results the second term of Eq. (117) becomes

$$\begin{aligned}(121)\quad & t^{-2}\Big\{\sum_{n=0}^{\infty} A_1 n^{(3)} f_n(x)t^n - A_1 0^{(3)} f_0(x)t^0 - A_1 (1)^{(3)} f_1(x)t^1 \\ & + \sum_{n=0}^{\infty} B_1 n^{(2)} f_n(x)t^n - B_1 (0)^{(2)} f_0(x)t^0 - B_1 (1)^{(2)} f_1(x)t^1 \\ & + \sum_{n=0}^{\infty} C_1 n^{(1)} f_n(x)t^n - C_1 (0)^{(1)} f_0(x)t^0 - C_1 (1)^{(1)} f_1(x)t^1 \\ & + \sum_{n=0}^{\infty} D_1 f_n(x)t^n - D_1 f_0(x)t^0 - D_1 f_1(x)t^1 + Ex \sum_{n=0}^{\infty} f_n(x)t^n \\ & - Exf_0(x)t^0 - Exf_1(x)t^1 \Big\} \\ = {} & t^{-2}\Big[A_1 t^3 \frac{d^3F}{dt^3} + B_1 t^2 \frac{d^2F}{dt^2} + C_1 t \frac{dF}{dt} - C_1\phi(x)t + D_1 F - D_1 - D_1\phi(x)t \\ & + ExF - Ex - Ex\phi(x)t\Big].\end{aligned}$$

The third term reduces to

$$(122)\quad t^{-1}\Big\{\sum_{n=0}^{\infty} F_1 n^{(3)} f_n(x)t^n - F_1(0)^{(3)} f_0(x)t^0 + \sum_{n=0}^{\infty} G_1 n^{(2)} f_n(x)t^n$$

$$- G_1(0)^{(2)} f_0(x)t^0 + \sum_{n=0}^{\infty} H_1 n^{(1)} f_n(x)t^n - H_1(0)^{(1)} f_0(x)t^0$$

$$+ \sum_{n=0}^{\infty} K_1 f_n(x)t^n - K_1 f_o(x)t^0 + Lx \sum_{n=0}^{\infty} f_n(x)t^n - Lxf_0(x)t^0\Big\}$$

$$= t^{-1}[F_1 t^3 \frac{d^3F}{dt^3} + G_1 t^2 \frac{d^2F}{dt^2} + H_1 t \frac{dF}{dt} + K_1 F - K_1 + LxF - Lx].$$

The last term can be rewritten as

$$(123)\quad \sum_{n=0}^{\infty} M_1 n^{(3)} f_n(x)t^n + \sum_{n=0}^{\infty} N_1 n^{(2)} f_n(x)t^n$$

$$+ \sum_{n=0}^{\infty} P_1 n^{(1)} f_n(x)t^n + \sum_{n=0}^{\infty} R f_n(x)t^n$$

$$= M_1 t^3 \frac{d^3F}{dt^3} + N_1 t^2 \frac{d^2F}{dt^2} + P_1 t \frac{dF}{dt} + RF.$$

Equation (117) is found to be

$$(124)\quad t^{-3}[F-1-\phi(x)t-\psi(x)t^2]+t^{-2}[A_1t^3\frac{d^3F}{dt^3}+B_1t^2\frac{d^2F}{dt^2}+C_1t\frac{dF}{dt}$$

$$-C_1\phi(x)t+D_1F-D_1-D\phi(x)t+ExF-Ex-Ex\phi(x)t]$$

$$+t^{-1}[F_1t^3\frac{d^3F}{dt^3}+G_1t^2\frac{d^2F}{dt^2}+H_1t\frac{dF}{dt}+K_1F-K_1+LxF-Lx]$$

$$+M_1t^3\frac{d^3F}{dt^3}+N_1t^2\frac{d^2F}{dt^2}+P_1t\frac{dF}{dt}+RF=0,$$

or

$$(125)\quad (M_1t^6+F_1t^5+A_1t^4)\frac{d^3F}{dt^3}+(N_1t^5+G_1t^4+B_1t^3)\frac{d^2F}{dt^2}$$

$$+(P_1t^4+H_1t^3+C_1t^2)\frac{dF}{dt}+(t^3R+Lxt^2+K_1t^2+D_1t+Ext+1)F$$

$$-[K_1t^2+Lxt^2+D_1\phi(x)t^2+C_1\phi(x)t^2+Ex\phi(x)t^2+\psi(x)t^2+Ext$$

$$+\phi(x)t+D_1t+1]=0.$$

Thus, the generating function, if it exists and has the requisite properties, is a solution of the third-order homogeneous linear differential equation given by Eq. (125).

Sometimes the technique can be extended to mixed recurrence relations. Here, a partial differential equation will be obtained because of the presence of derivatives in x, for if

$$F(x,t)=\sum_{n=0}^{\infty}f_n(x)t^n=F[f_n(x)];$$

$$\frac{\partial F}{\partial x} = \sum_{n=0}^{\infty} f_n'(x)t^n \tag{126}$$

$$= F[f_n'(x)].$$

Corresponding results hold for higher order derivatives. In general the resulting partial differential equation can not be solved by utilizing the Bernoulli, or separation of variables, method since the projected solution, the generating function, is not usually separable in x and t. For this reason the equation is reduced to an ordinary differential one by using transforms, the most common being Laplace.

The disadvantage of this method, as a whole, is that more information about the initial conditions is required. Besides knowledge of $f_0(x)$ and its derivatives, $F(0, t)$ is also needed in order to obtain a unique solution. Of course it is understood that the generating functions transforms, as well as their inverses, exist, which may not always be true. An example should make these objections apparent.

Consider the mixed recurrence relation given as follows;

$$f_n'(x) - f_{n-1}'(x) = -f_{n-1}(x), \quad n = 1, 2, 3, \ldots \tag{127a}$$

with

$$f_0(x) = 1 \tag{127b}$$

and

(127c) $$f_0'(x) = 0.$$

Furthermore, assume that

(127d) $$F(0,t) = (1-t)^{-1}.$$

Replace n by $n+1$ in Eq. (127a) to obtain

(128) $$f'_{n+1}(x) - f_n'(x) = -f_n(x), \quad n = 0, 1, 2, \ldots .$$

Multiply by t^n and sum over n. The result is

(129) $$\sum_{n=1}^{\infty} f_n'(x)t^{n-1} - \sum_{n=0}^{\infty} f_n'(x)t^n + \sum_{n=0}^{\infty} f_n(x)t^n = 0$$

or

(130) $$t^{-1}\left[\sum_{n=0}^{\infty} f_n'(x)t^n - f_0'(x)t^0\right] - \frac{\partial F}{\partial x} + F = 0,$$

$$\text{if} \quad t \neq 0,$$

$$t^{-1}\left[\frac{\partial F}{\partial x} - 0\right] - \frac{\partial F}{\partial x} + F = 0,$$

$$\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial x} + tF = 0,$$

$$(1-t)\frac{\partial F}{\partial x} + tF = 0.$$

Let $f(s) = \mathcal{L}\{F[f_n(x)]\}$, the Laplace transform over x. Take Laplace transforms of Eq. (130);

$$(1-t)\mathcal{L}\{\frac{\partial F}{\partial x}\} + t\,\mathcal{L}\{F\} = 0, \tag{131}$$

$$(1-t)[sf(s) - s^0 F(0,t)] + tf(s) = 0,$$

$$(1-t)[sf(s) - 1 \cdot \frac{1}{1-t}] + tf(s) = 0,$$

$$(1-t)sf(s) - 1 + tf(s) = 0,$$

$$[(1-t)s+t]f(s) = 1,$$

$$f(s) = \frac{1}{(1-t)}\left[\frac{1}{s+\frac{t}{1-t}}\right],$$

or

$$F(x,t) = \frac{1}{(1-t)}\mathcal{L}^{-1}\{\frac{1}{s+\frac{t}{1-t}}\} \,. \tag{132}$$

From a table of Laplace transforms it is found that

$$F(x,t) = \frac{1}{1-t} e^{-\frac{xt}{1-t}}, \tag{133}$$

the generating function for the Laguerre polynomial set $\{L_n(x)\}$. Investigation into any book on special functions [1, 5, 7] will show that $\{L_n(x)\}$ obeys the recurrence equation (127).

The requirement that $F(0,t)$ be known is a consequence of the relation existing between $\frac{\partial F}{\partial x}$ and its Laplace transform. Normally, this "initial value" information about F is not available. However,

$$F(0,t) = \sum_{n=0}^{\infty} f_n(0)t^n$$

is a real Maclaurin series in t. It may be possible to identify the series in a closed form by inspection or by looking into a series table.

Of course the mixed recurrence problem gets correspondingly worse as higher order derivatives are introduced into the relation. In particular, if a polynomial set is under consideration, an equation uniting second and first order derivatives with members of the set is as difficult to solve as the differential equation members of the set satisfy. It is much easier to first solve the original ordinary differential equation by techniques such as the method of Frobenius and then obtain a pure recurrence formula from the solution. The generating function can be found much more easily by this approach.

Several methods for finding the generating function are discussed by Truesdell in his book [9]. He discusses functions $F(z, \alpha)$ satisfying the F-equation;

$$\frac{\partial}{\partial z} F(z,\alpha) = F(z, \alpha+1). \tag{133}$$

The variable z and parameter α are allowed to be complex. Truesdell has amassed a table of special functions which satisfy the F-equation by attaching to them suitable coefficients in z and α.

There are exceptions, such as the generalized Bernoulli polynomials and Jacobi functions, which are not amenable to this treatment, but their number is small.

The first method he advances for determining generating functions is based upon the following theorem and corollary [9, p. 82-83];

Theorem 1. If the function $F(z, \alpha)$ satisfies the F-equation and if $F(z+y, \alpha)$ possesses a Taylor series, then this series may be put into the form

$$F(z+y, \alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} F(z, \alpha+n) y^n. \tag{134}$$

Corollary 1. 1. If for some value z_0 of z, $|F(z_0, \alpha)| < M$ when $\operatorname{Re} \alpha \geq \alpha_0$ then for all values of z and y, (134) is valid when $\operatorname{Re} \alpha \geq \alpha_0$. (See his monograph [9] for proofs of these assertions.)

For example $F(z, \alpha) = e^{i\alpha\pi} e^{-z} L_b^{(\alpha)}(z)$, where $L_b^{(\alpha)}(z)$ is the associated Laguerre polynomial, satisfies the F-equation. It can be shown [9, p. 83] that $F(z+y, \alpha) = e^{i\alpha\pi} e^{-z-y} L_b^{(\alpha)}(z+y)$ possesses a Taylor series and satisfies Corollary 1. 1. By Eq. (134) the equation above can be written as

$$(135) \qquad e^{ia\pi}e^{-z-y}L_b^{(a)}(z+y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} e^{i\pi(a+n)}e^{-z}L_b^{(a+n)}(z)$$

$$= e^{ia\pi}e^{-z} \sum_{n=0}^{\infty} \frac{y^n}{n!} e^{in\pi}L_b^{(a+n)}(z).$$

Equation (135) can be rewritten as

$$(136) \qquad e^{-y}L_b^{(a)}(z+y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} (-1)^n L_b^{(a+n)}(z).$$

Here, $L_b^{(a)}(x)$ is found in the generating function and in the generated set while the summation over n occurs in the superscript. Usually the expressions which arise from the application of the theorem are of this form, which may be undesirable. Sometimes, however, cancellations and judicious choices of parameter values will lead to more tractable expressions. If

$$F(x, a) = (a-b)!(x^2+1)^{-\frac{a+1}{2}} P_a^b\left(-\frac{x}{\sqrt{x^2+1}}\right),$$

where $P_a^b(r)$ is the associated Legendre function, is used in Eq. (134), the result turns out to be

$$(137) \qquad (t^2-2tx+1)^{-\frac{a+1}{2}} P_a^b\left(\frac{x-t}{\sqrt{t^2-2xt+1}}\right) = \sum_{n=0}^{\infty} \binom{a-b+n}{n} P_{a+n}^b(x)t^n.$$

If $b = \alpha$ is chosen Eq. (137) reduces to

$$(138) \qquad (t^2 - 2tx+1)^{-\alpha} = \sum_{n=0}^{\infty} C_n^{\alpha}(x)t^n,$$

the generating function for the Gegenbauer polynomial set $\{C_n^{\alpha}(x)\}$.

Truesdell's second method is a consequence of the following two theorems [9, p. 117-118 and p. 57-58].

Theorem 2. Suppose $F(z, \alpha)$ is a solution of the F-equation and suppose the functions $F_y(z, \alpha)$ form a set of solutions of the F-equation. Let O_y be an operator which

(i) associates itself with the variable y;

and

(ii) commutes in Eq. (140) below, with $D_z = \frac{\partial}{\partial z}$, $E_{\alpha} f(z, \alpha) = f(z, \alpha+1)$, and the operation of replacing z by z_0. Suppose, for some value z_0 of z

$$(139) \qquad F(z_0, \alpha) = O_y[F_y(z_0, \alpha)], \quad \mathrm{Re}\, \alpha \geq \alpha_0.$$

Then, for all values of z such that the expression below has a meaning,

$$(140) \qquad F(z, \alpha) = O_y[F_y(z, \alpha)], \quad \mathrm{Re}\, \alpha \geq \alpha_0.$$

Theorem 3. Suppose the complex function $\phi(\alpha) = F(z_0, \alpha)$ is absolutely bounded; $|\phi(\alpha)| < M$, $\mathrm{Re}\,\alpha \geq \alpha_0$. Then a solution $F(z,\alpha)$ of the F-equation such that $F(z_0, \alpha) = \phi(\alpha)$, $\mathrm{Re}\,\alpha \geq \alpha_0$ exists, is unique, is an integral function of z for each fixed value of α such that $\mathrm{Re}\,\alpha \geq \alpha_0$, and is represented by

$$(141) \qquad F(z, \alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi(\alpha+n)(z-z_0)^n.$$

As an example let

$$A(z, w) = \sum_{y=0}^{\infty} L_y^{(\alpha)}(z) w^y; \qquad \{L_y^{(\alpha)}(z)\}$$

is the associated Laguerre polynomial set, and $A(z, w)$ is to be found. Multiply by $e^{i\alpha\pi}e^{-z}$ to obtain

$$(142) \qquad e^{i\alpha\pi}e^{-z}A(z, w) = \sum_{y=0}^{\infty} e^{i\alpha\pi}e^{-z}L_y^{(\alpha)}(z)w^y.$$

By comparing the coefficient of w^y with Truesdell's table it is found to equal $F_y(z, \alpha)$ in the context of Theorem 2 with

$$\underset{y}{O}[\ldots] = \sum_{y=0}^{\infty} [\ldots] w^y.$$

Thus, $e^{i\alpha\pi}e^{-z}A(z)$ is a solution $F(z,\alpha)$ of the F-equation such that

(143)
$$F(0,\alpha) = \underset{y}{O}[F_y(0,\alpha)]$$

$$= \sum_{y=0}^{\infty} e^{i\alpha\pi} \frac{\Gamma(\alpha+y+1)}{\Gamma(\alpha+1)y!} w^y$$

$$= e^{i\alpha\pi}(1-w)^{-\alpha-1}$$

since

$$\sum_{y=0}^{\infty} \frac{\Gamma(\alpha+y+1)}{\Gamma(\alpha+1)y!} w^y$$

is a geometric series.

As a consequence of Theorem 3,

(144)
$$F(z,\alpha) = \sum_{n=0}^{\infty} F(0,\alpha+n) \frac{(z-z_0)^n}{n!}$$

$$= \sum_{n=0}^{\infty} e^{i(\alpha+n)\pi}(1-w)^{-\alpha-1-n} \frac{z^n}{n!}, \quad (z_0 = 0),$$

$$= e^{i\alpha\pi}(1-w)^{-\alpha-1} \sum_{n=0}^{\infty} \left(-\frac{z}{1-w}\right)^n$$

$$= e^{i\alpha\pi}(1-w)^{-\alpha-1} e^{-\frac{z}{1-w}}.$$

Comparison of Eqs. (142) and (144) leads to

$$(145) \qquad F(z, \alpha) = A(z, w)e^{i\alpha\pi}e^{-z}$$
$$= e^{i\alpha\pi}(1-w)^{-\alpha-1}e^{-\frac{z}{1-w}} \quad ;$$

that is,

$$(146) \qquad A(z, w) = (1-w)^{-\alpha-1}e^{-\frac{zw}{1-w}}$$

is the desired generating function for the set $\{L_y^{(\alpha)}(z)\}$.

The next method to be considered for determining $F(x, t)$ is applicable to a set $\{f_n(x)\}$ which is assumed orthogonal and polynomial, of degree n for each element, and the solution of the second-order differential equation

$$(147a) \qquad r(x)y'' + g(x)y' + \lambda y = 0$$

with

$$(147b) \qquad r(x) = ax^2 + bx + c,$$

$$(147c) \qquad g(x) = hx + k,$$

and

$$(147d) \qquad \lambda = \frac{n}{2}(1-n)r''(x) - ng'(x)$$
$$= n[(1-n)a-h] \qquad \text{for} \quad n = 0, 1, 2, \ldots .$$

It is further assumed at least one of a, b, or h is not zero and r, g, and $\lambda \neq 0$. Such sets are the ones normally encountered in physical problems.

It can be shown [5] that multiplying Eq. (147a) by

$$\mu(x) = \frac{1}{r(x)} \exp \left[\int^{x} \frac{g(t)}{r(t)} dt \right]$$

will make it self-adjoint, having a polynomial solution of degree n given by

(148) $$y = f_n(x)$$
$$= \frac{C_n}{\mu(x)} D^n\{[r(x)]^n \mu(x)\},$$

where C_n is a constant dependent on n and D is the differential operator $\frac{d}{dx}$. Such a solution is known as the Rodrigues formula.

Under these conditions a generating function of the form

$$F(x, t) = \sum_{n=0}^{\infty} b_n f_n(x) t^n$$

may be found if recourse to analytic function theory is made. By Cauchy's integral formula [5, 10];

(149) $$D^n F(x) = \frac{n!}{2\pi i} \int_C \frac{F(z)}{(z-x)^{n+1}} dz, \quad n = 1, 2, 3, \ldots,$$

where C is a closed contour in the Argand (x, y) plane that encloses x but does not enclose singularities of $F(z)$. C is taken in the usual counterclockwise sense while $F(z)$ is assumed to be analytic throughout the domain containing C. Substitute Rodrigues' formula into Cauchy's integral formula. Equation (149) reduces to

$$(150) \qquad \frac{C_n}{\mu(x)} D^n[r^n \mu] = \frac{n!C_n}{\mu(x)2\pi i} \int_C \frac{r^n(z)\mu(z)}{(z-x)^{n+1}} dz$$

$$= f_n(x) .$$

Since,

$$F(x,t) = \sum_{n=0}^{\infty} h_n(x)t^n = \sum_{n=0}^{\infty} b_n f_n(x)t^n,$$

$$(151) \qquad F(x,t) = \sum_{n=0}^{\infty} b_n f_n(x)t^n$$

$$= \sum_{n=0}^{\infty} b_n \frac{C_n}{\mu(x)} \frac{n!}{2\pi i} \int_C \frac{r^n(z)\mu(z)}{(z-x)^{n+1}} dz\, t^n .$$

Usually b_n is chosen so that

$$(152) \qquad B^n = b_n n! C_n ,$$

where B is independent of n. If this choice is made,

$$(153) \qquad F(x,t) = \sum_{n=0}^{\infty} \frac{1}{\mu(x)2\pi i} B^n \int_C \frac{r^n(z)\mu(z)}{(z-x)^{n+1}} dz\, t^n .$$

Assume that the integration and summation can be interchanged.

Then

$$(154) \qquad F(x,t) = \frac{1}{2\pi i \mu(x)} \int_C \frac{\mu(z)}{z-x}\, dz \sum_{n=0}^{\infty} \left[\frac{Br(z)t}{z-x}\right]^n .$$

For $\left|\frac{Br(z)t}{z-x}\right| < 1$ the convergent series is summable since it is geometric. It gives

$$\frac{1}{1-\frac{Br(z)t}{z-x}} = \frac{z-x}{z-x-Br(z)t} .$$

Therefore,

$$(155) \qquad F(x,t) = \frac{1}{2\pi i \mu(x)} \int_C \frac{\mu(z)}{z-x} \frac{z-x}{[z-x-Br(z)t]}\, dz$$

$$= \frac{1}{2\pi i \mu(x)} \int_C \frac{\mu(z)}{z-x-Br(z)t}\, dz.$$

Many times the integral can be evaluated by appealing to Cauchy's residue theorem. If the integrand can be expanded in a Laurent series,

$$\sum_{m=-\infty}^{\infty} c_m (z-z_0)^m,$$

where not all the c_{-m} vanish, there is a singularity at $z = z_0$. c_{-1} is called the residue of the series. If a finite number of singularities z_k, $k = 1, 2, \ldots, m$ lie inside the contour C, the integral has the value

$$2\pi i \sum_{k=1}^{m} c_{-1,k}.$$

Then,

(156a) $$F(x, t) = \frac{1}{\mu(x)} \sum_{k=1}^{m} c_{-1,k}$$

where

(156b) $$c_{-1,k} = c_{-1,k}(x, t), \quad k = 1, 2, \ldots, m.$$

An example will demonstrate the efficacy of this method. Consider the following differential equation;

(157a) $$xy'' + (1-x)y' + ny = 0,$$

the Laguerre differential equation. Here, $r(x) = x$, $g(x) = 1 - x$, and $\lambda = n$ with r and $g \neq 0$ if the open interval $(0, 1)$ is taken for x. So,

(157b) $$\begin{aligned} \mu(x) &= \frac{1}{r(x)} \exp\left[\int^x \frac{g(t)}{r(t)}\, dt\right] \\ &= \frac{1}{x} \exp\left[\int^x \frac{1-x}{x}\, dt\right] \\ &= \frac{1}{x} \cdot x \exp(-x) \\ &= \exp(-x). \end{aligned}$$

C_n is chosen to be $\frac{1}{n!}$, so that

(157c)
$$
\begin{aligned}
B^n &= b_n C_n n! \\
&= b_n \frac{1}{n!} n! \\
&= b_n .
\end{aligned}
$$

An obvious choice is $b_n = 1$ for all n; this makes $B^n = 1$. Therefore,

(158)
$$
\begin{aligned}
F(x, t) &= \frac{1}{2\pi i \mu(x)} \int_C \frac{\mu(z)dz}{z-x-Bf(z)t} \\
&= \frac{1}{2\pi i \exp(-x)} \int_C \frac{e^{-z}}{z-x-zt} dz.
\end{aligned}
$$

The integrand has a singularity at $z_1 - x - z_1 t = 0$ or $z_1 = \frac{x}{1-t}$. Thus, the contour C must enclose x and be such that

$$
\left| \frac{Br(z)t}{z-x} \right| = \left| \frac{zt}{z-x} \right| < 1.
$$

The requisite contour can be found when t is sufficiently small. Let $\phi(z) = \frac{e^{-z}}{z-x-zt}$. Since the integrand has a simple pole at $z = z_1$, $c_{-1} = \lim_{z \to z_1} (z-z_1)\phi(z)$ [6]; that is,

(159)
$$
\begin{aligned}
c_{-1} &= \lim_{z \to z_1} (z-z_1) \frac{e^{-z}}{z-x-zt} \\
&= (1-t)^{-1} e^{-\frac{x}{1-t}} .
\end{aligned}
$$

So,

$$F(x,t) = \frac{1}{\mu(x)}\, c_{-1} \tag{160}$$

$$= \frac{1}{e^{-x}}\,(1-t)^{-1} e^{-\frac{x}{1-t}}$$

$$= (1-t)^{-1} e^{-\frac{xt}{1-t}}$$

$$= \sum_{n=0}^{\infty} f_n(x) t^n .$$

Of course restrictions are placed on t and x according to the preceding developments, but Eq, (160) is the correct generating function for the Laguerre polynomial set of the first kind.

The final method to be discussed is based on the assumption that any element of the generated set can be represented by means of a definite integral, which is already known. Symbolically, this can be written as

$$f_n(x) = \int_a^b g_n(x, \theta)\, d\theta, \qquad n = 0, 1, 2, \ldots, \tag{161}$$

Usually, such a representation, if it exists, is found from integrating an appropriate, unknown function in the complex plane. All the variables involved in Eq. (161) are assumed real. If a generating series of the form

$$\sum_{n=0}^{\infty} f_n(x)t^n$$

is postulated;

(162) $$\sum_{n=0}^{\infty} f_n(x)t^n = \sum_{n=0}^{\infty} \int_a^b g_n(x, \theta)d\theta t^n$$

$$= \int_a^b \sum_{n=0}^{\infty} g_n(x, \theta)t^n d\theta,$$

if uniform convergence in the variables is assumed,

$$= \int_a^b G(x, \theta;\ t)d\theta,$$

if the series is summable in closed form,

$$= F(x, t),$$

if the function is integrable over $[a \leq \theta \leq b]$. These restrictions are relatively severe when taken together, and few functions $g_n(x, \theta)$ are as serviceable as this when they exist.

Some functions $g_n(x, \theta)$ do exist, however, For example,

(163) $$P_n(x) = \frac{1}{n} \int_0^{\pi} \frac{d\theta}{[x^2+\sqrt{x^2-1} \cos \theta]^{n+1}} \quad .$$

If

$$\sum_{n=0}^{\infty} P_n(x)t^n$$

has a generating function,

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi}\frac{d\theta}{[x+\sqrt{x^2-1}\cos\theta]^{n+1}}\,t^n. \tag{164}$$

If integration and summation interchange is assumed valid,

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x)t^n &= \frac{1}{\pi}\int_0^{\pi}\sum_{n=0}^{\infty}[x+\sqrt{x^2-1}\cos\theta]^{-n-1}t^n\,d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{1}{x-t+\sqrt{x^2-1}\cos\theta}\,d\theta\end{aligned} \tag{165}$$

when $\left|\frac{t}{x+\sqrt{x^2-1}\cos\theta}\right| < 1$. The integral can be evaluated since

$$\int\frac{1}{a+b\cos\theta}\,d\theta = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{a^2-b^2}\tan\frac{1}{2}x}{a+b} + C, \tag{166}$$

$$|a| > |b|.$$

So,

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x)t^n &= \frac{1}{\pi}\cdot\frac{\pi}{2}(1-2xt+t^2)^{-1/2}\,2 \\ &= (1-2xt+t^2)^{-1/2},\end{aligned} \tag{167}$$

the generating function for the Legendre polynomial set $\{P_n(x)\}$.

IV. MISCELLANEOUS TOPICS

Let

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n.$$

Consider the situation in which $F(x, t)$ is known and it is desired to find the set $\{f_n(x)\}$. If the assumption is made that $F(x, t)$ is differentiable in t, and the series is uniformly convergent and continuous in both x and t for all orders n, then by Property 4, Appendix B;

$$(168) \qquad f_n(x) = \frac{1}{n!} \frac{\partial^n}{\partial t^n} [F(x, t)] \Big|_{t=0}, \quad n = 0, 1, 2, \ldots .$$

This result is not surprising since the generating equation can be considered a Maclaurin series in t having a parameter x.

If

$$F(x, t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} f_n(x)t^n,$$

$$(169a) \qquad f_n(x) = \frac{1}{n!} \frac{\partial^n}{\partial t^n} [e^{2xt-t^2}] \Big|_{t=0} .$$

The first three values are

$$(169b) \qquad f_0(x) = \frac{1}{0!} [e^{2xt-t^2}] \Big|_{t=0} = 1,$$

$$\text{(169c)} \qquad f_1(x) = \frac{1}{1!}\frac{\partial}{\partial t}(e^{2xt-t^2})\Big|_{t=0} = 2x,$$

and

$$\text{(169d)} \qquad f_2(x) = \frac{1}{2!}\frac{\partial^2}{\partial t^2}(e^{2xt-t^2})\Big|_{t=0} = 2x^2 - 1.$$

Comparison with a table of special functions will show that these terms are members of the set $\{\frac{H_n(x)}{n!}\}$ with $H_n(x)$ the ordinary Hermite polynomial of order n. Further differentiation of Eq. (169a) will confirm this conjecture for higher n.

As the example demonstrates, formula (168) is difficult to apply in practice because of the laborious and tedious calculations that are usually required in finding the t-derivatives of $F(x, t)$. Other methods are normally much easier to use.

Suppose $F(x, t)$ is an expression which can be easily expanded in an infinite series involving expressions containing powers of t. By suitable manipulations and substitutions it may be possible to separate and collect the factors so that a power series in t is achieved. By comparing the coefficient expression with $f_n(x)$, an explicit expression for the latter can be found.

Consider

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} f_n(x)t^n,$$

ν a real parameter. By the binomial theorem;

$$(170) \qquad (1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} \binom{-\nu}{n}(-1)^n(2xt-t^2)^n 1^{-\nu-n}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(2x)^{n-k}}{k!(n-k)!}(\nu)_n t^{n+k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}}{k!(n-2k)!} t^n,$$

if $n+k$ is replaced by n,

$$= \sum_{n=0}^{\infty} f_n^{\nu}(x)t^n.$$

Comparison of coefficients of t^n leads to the implication

$$(171a) \qquad f_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}}{k!(n-2k)!} .$$

This series defines the set of Gegenbauer polynomials $\{C_n^{\nu}(x)\}$.

If $\nu = \frac{1}{2}$ Eq. (171a) becomes

$$(171b) \qquad C_n^{1/2}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(\frac{1}{2})_{n-k}(2x)^{n-2k}}{k!(n-2k)!} = P_n(x) ,$$

the Legendre polynomial of the first kind of order n.

Let

$$F(x, z) = \sum_{n=0}^{\infty} f_n(x) z^n$$

with z a complex variable while x is real. Assume, initially, that $F(x, z)$ is analytic in a finite circular neighborhood about $z = 0$ for all valid values of the parameter x. Then,

$$F(x, z) z^{-k-1} = \sum_{n=0}^{\infty} f_n(x) z^{n-k-1} . \tag{172}$$

Integrate Eq. (172) along a simple closed contour C that incloses $z = 0$ and is contained within the analytic neighborhood. The result is,

$$\begin{aligned} \oint F(x, z) z^{-k-1} dz &= \oint \sum_{n=0}^{\infty} f_n(x) z^{n-k-1} dz \\ &= \sum_{n=0}^{\infty} f_n(x) \oint z^{n-k-1} dz \\ &= 2\pi i f_k(x) \end{aligned} \tag{173}$$

by Cauchy's integral formula and Cauchy's theorem. Therefore,

(174) $$f_n(x) = \frac{1}{2\pi i} \oint F(x, z) z^{-n-1} dz.$$

For Appell polynomials,

$$e^{xz} A(z) = \sum_{n=0}^{\infty} a_n(x) z^n .$$

If $A(z)$ is analytic in a neighborhood about $z = 0$,

(175) $$a_n(x) = \frac{1}{2\pi i} \oint e^{xz} A(z) z^{-n-1} dz, \quad n = 0, 1, 2, \ldots .$$

For Legendre polynomials

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n.$$

The generating function is analytic in a neighborhood of $z = 0$ if $-1 \leq x \leq 1$. Then,

(176) $$P_n(x) = \frac{1}{2\pi i} \oint (1-2xz+z^2)^{-1/2} z^{-n-1} dz, \quad n = 0, 1, 2, \ldots .$$

Formula (174) applies even if $F(x, z)$ is expressed by a Laurent series since Cauchy's theorem will force the additional terms to give no contribution. If a generating function is given by a

Laurent series, it may possess essential singularities about, or at, $z = 0$ yet the generated set may still have an integral representation given by Eq. (174). For example if

$$F(x, z) = e^{-\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n,$$

it can be shown [6] that

$$J_n(x) = \frac{1}{2\pi i} \oint e^{-\frac{x}{2}(z-\frac{1}{z})} z^{-n-1} dz.$$

Another method, which is sometimes used to obtain a general expression for the set $\{f_n(x)\}$ from $F(x, t)$, is based on the technique of power series inversion. It is found [6] that if a power series of the form

$$w = f(z) \tag{177}$$

$$= w_0 + \sum_{n=1}^{\infty} a_n (z-z_0)^n, \quad a_1 \neq 0,$$

is given, the inverse function $z = z(w)$ can be also obtained in a power series with the form

$$z = z_0 + \sum_{n=1}^{\infty} b_n (w-w_0)^n \tag{178a}$$

where

$$b_n = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ \frac{(z-z_0)^n}{[f(z)-w_0]^n} \right\} \Bigg|_{z=z_0} . \tag{178b}$$

Formula (178) is known as the Lagrange expansion.

This result can be applied to $F(x, t)$, for if it is possible to find a function $y = G(t, x, y)$ such that

$$\frac{\partial y}{\partial x} = F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n,$$

it may be possible to apply the Lagrange expansion to y in powers of t if the original function can be converted to the form $t = G_1(x, y)$, a power series in y.

For example, $y = x + \frac{1}{2} t(y^2 - 1)$ gives

$$\frac{dy}{dx} = (1 - 2xt + t^2)^{-1/2},$$

the generating function of

$$\sum_{n=0}^{\infty} P_n(x) t^n.$$

Then,

(179a)
$$t = \frac{2x}{1-y^2} - \frac{2y}{1-y^2}$$

$$= 2x \sum_{n=0}^{\infty} y^{2n} - 2 \sum_{n=0}^{\infty} y^{2n+1}$$

$$= \sum_{n=0}^{\infty} a_n(x) y^n$$

with

(179b) $a_n = 2x, \quad n \text{ even}$

and

(179c) $a_n = -2, \quad n \text{ odd}.$

Applying Eq. (178b) to Eq. (179), the result [6] is

(180a)
$$y = x + \sum_{n=1}^{\infty} b_n t^n$$

with

(180b)
$$b_n = \frac{1}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n.$$

So,

(181a)
$$\frac{\partial y}{\partial x} = 1 + \sum_{n=1}^{\infty} \left\{ \frac{d^n}{dx^n} (x^2-1)^n \right\} \frac{1}{2^n n!} t^n$$

$$= P_0(x) + \sum_{n=1}^{\infty} P_n(x) t^n$$

$$= F(x, t).$$

Compare coefficients to obtain the result

(181b) $$P_0(x) = 1$$

and

(181c) $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \quad n = 1, 2, 3, \ldots,$$

the Rodriques formula for $\{P_n(x)\}$.

Normally generated polynomial sets are not obtained from $F(x, t)$; they are initially determined as series solutions to ordinary second-order differential equations about some ordinary or regular singular point, usually zero.

Generating functions are sometimes useful in solving special integral equations. Consider the following Fredholm equation,

(182) $$f(x) = \int_a^b K(x, t)\phi(t)dt.$$

$f(x)$ and $K(x, t)$ are known; $\phi(t)$ is to be found. If

$$K(x, t) = F(x, t)w(t) = \sum_{n=0}^{\infty} f_n(t)x^n w(t),$$

where $\{f_n(t)\}$ is an orthogonal polynomial set with a weight function $w(t)$ over the interval $[a, b]$, the following statements can be made, assuming, of course, that $f(t)$ is continuous over $[a, b]$

and that appropriate interchanges of summation and integration are allowable;

$$f(x) = \int_a^b \sum_{n=0}^{\infty} f_n(t)x^n w(t)\phi(t)dt \tag{183}$$

$$= \sum_{n=0}^{\infty} \int_a^b f_n(t)w(t)\phi(t)dt\, x^n .$$

If $\{f_m(t)\}$ form a complete set over the interval $[a, b]$, then for any continuous function $\phi(t)$ over $[a, b]$,

$$\phi(t) = \sum_{m=0}^{\infty} a_m f_m(t). \tag{184}$$

Thus, if this further assumption is made,

$$f(x) = \sum_{n=0}^{\infty} \int_a^b f_n(t)w(t) \sum_{m=0}^{\infty} a_m f_m(t)dt\, x^n \tag{185}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_m \int_a^b f_n(t)f_m(t)w(t)dt\, x^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_m \delta_{mn} K_n x^n$$

$$= \sum_{n=0}^{\infty} a_n K_n x^n.$$

If $f(x)$ has derivatives of all orders and $a < 0 < b$,

(186) $$a_r = \frac{f^{(r)}(0)}{r!K_r} .$$

Therefore,

(187) $$\phi(t) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!K_m} f_m(t) .$$

The obvious disadvantages of this method are the specialization of the kernel $K(x, t)$ and the large number of restrictive conditions and assumptions present. Very few integral equations are so tractable. One such is

(188) $$f(x) = \int_{-1}^{1} (1-2xt+x^2)^{-1/2} \phi(t)dt, \quad -1 < x < 1.$$

Here

$$K(x, t) = (1-2xt+x^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(t)x^n.$$

Using the procedure above leads to the result

(189) $$\phi(t) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} \cdot \frac{2m+1}{2} P_m(t).$$

Other examples are given in [6].

As usual assume

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

with the orthogonal set $\{f_n(x)\}$ possessing a weight function $w(x)$ over the x interval $[a, b]$. Then

$$(190) \qquad F(x, t)f_m(x)w(x) = \sum_{n=0}^{\infty} f_n(x)f_m(x)w(x)t^n.$$

If the left side is integrable over $[a, b]$ and Property 2 of Appendix B holds,

$$(191) \qquad \int_a^b F(x, t)f_m(x)w(x)dx = \int_a^b \sum_{n=0}^{\infty} f_n(x)f_m(x)w(x)t^n dx$$

$$= \sum_{n=0}^{\infty} \int_a^b f_n(x)f_m(x)w(x)dxt^n$$

$$= K_m t^m.$$

So,

$$K_m = t^{-m} \int_a^b F(x, t)f_m(x)w(x)dx, \quad t \neq 0.$$

V. OBSERVATIONS

This report has limited itself to techniques which, hopefully, the applied mathematician can utilize when he comes in contact with generating functions or series. For this reason theoretical discussion has been kept to a minimum while topics of interest directed mainly to the pure mathematician, such as the Sheffer classification scheme, the group theoretic approach to special functions, or analytic properties of polynomial expansions, have been neglected. Certainly a rigorous development of the generating function field is to be encouraged and perhaps could be the subject of further research. Yet, for the sake of limitation, a choice had to be made. Personal prejudice dictated the selection of techniques instead of theory.

"Lack of space" prevented discussion of generating functions in three or more variables. Several examples are given in Appendix A. It certainly appears that an approach similar to the one taken in the second chapter could be profitably employed in gaining useful recurrence relations between the elements of the generated sets involved.

Another subject little touched upon was the description of a generated set as a complex-valued contour integral. Truesdell [9] has developed some new contour formulas for expressions which satisfy the F-equation. Another approach found in the literature [6] relies on forcing an integrand to satisfy certain auxillary conditions which the $\{f_n(x)\}$ fulfill. If the set is a solution of an ordinary differential equation having a parameter n, the contour representation may be obtained by finding the explicit form of an unknown function, found in the integrand, which satisfies the equation. Such an approach is in some ways analogous to taking transforms for real variable differential equations.

As a final consideration, some generated sets can be expressed in terms of generating functions containing other special functions. Again, Truesdell [9] has obtained a systematic procedure to find such relations. Although these results are of mostly academic interest, they do reveal surprising dependencies. It would seem that other such relations not covered by Truesdell's scheme could be investigated.

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APPENDIX A

A List of Selected Generating Functions

(A legend showing the reference sources for the more obscure generating functions is given at the end of the list.)

Expression	Generated Set
1. $F(x,t) = \sum_{n=0}^{\infty} b_n f_n(x) t^n$	General generated set $\{f_n(x)\}$
2. $\frac{1}{1-xt} = \sum_{n=0}^{\infty} x^n t^n$	Geometric series
3. $F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(a)}{n!} (t-a)^n$	Taylor series coefficients
4. $(1+t)^x = \sum_{n=0}^{\infty} \binom{x}{n} t^n$	Binomial polynomials
5. $(1-t-t^2)^{-1} = \sum_{n=0}^{\infty} \phi_n t^n$	Fibonacci numbers (J)

Bessel Class

Expression	Generated Set
6. $e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$	Bessel functions of the first kind

Expression	Generated Set
7. $e^{\frac{1}{2}x(t+\frac{1}{t})} = \sum_{n=-\infty}^{\infty} I_n(x)t^n$	Modified Bessel functions of the first kind
8. $\sqrt{\frac{2}{\pi x}}\cos\sqrt{x^2-2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} J_{n-\frac{1}{2}}(x)t^n$	Spherical Bessel functions of the first kind
$\sqrt{\frac{2}{\pi x}}\sin\sqrt{x^2+2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} J_{\frac{1}{2}-n}(x)t^n$	

Legendre Class

Expression	Generated Set
9. $\frac{(2m)!(1-x^2)^{m/2}t^m}{2^m m!(1-2xt+t^2)^{m+\frac{1}{2}}} = \sum_{n=0}^{\infty} P_n^m(x)t^n$	Associated Legendre polynomials of the first kind
10. $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n$	Legendre polynomials of the first kind
11. $(1-2xt+t^2)^{-\frac{1}{2}} \cosh^{-1} \frac{t-x}{\sqrt{x^2-1}} = \sum_{n=0}^{\infty} Q_n(x)t^n$	Associated Legendre polynomials of the second kind

Laguerre Class

Expression	Generated Set
12. $(1-t)^{-1-\alpha} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n$	Associated Laguerre polynomials of the first kind

Expression	Generated Set
13. $(1-t)^{-1}e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x)t^n$	Laguerre polynomials of the first kind

Jacobi Class

14. ${}_0F_1[-;\, 1+\alpha;\, t\frac{(x-1)}{2}]\,{}_0F_1[-;\, 1+\beta;\, t\frac{(x+1)}{2}] = \sum_{n=0}^{\infty} \frac{1}{(1+\alpha)_n(1+\beta)_n} P_n^{(\alpha,\beta)}(x)t^n$	Jacobi polynomials
15. $(1-2xt+t^2)^{-\frac{1}{2}-\alpha} = \sum_{n=0}^{\infty} \frac{(1+2\alpha)_n}{(1+\alpha)_n} P_n^{(\alpha,\alpha)}(x)t^n$	Ultraspherical polynomials
16. $(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n$	Gegenbauer polynomials

Hermite Class

17. $e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$	Hermite polynomials (common form)
18. $e^{xt-\frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{He_n(x)}{n!} t^n$	Hermite polynomials (alternative form)

Tchebicheff Class

19. $(1-xt)(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} T_n(x)t^n$	Tchebicheff polynomials of the first kind

Expression	Generated Set
20. $(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n$	Tchebicheff polynomials of the second kind

Boas and Buck Class

21. $A(t)\psi[xH(t)] = \sum_{n=0}^{\infty} p_n(x)t^n$	Boas and Buck polynomials (generalized Appell polynomials)

Brenke Class

22. $A(t)\psi(xt) = \sum_{n=0}^{\infty} p_n(x)t^n$	Brenke polynomials

Appell Class

23. $e^{xt}A(t) = \sum_{n=0}^{\infty} a_n(x)t^n$	Appell polynomials
24. $\dfrac{t^{\ell}e^{xt}}{(e^t-1)^{\ell}} = \sum_{n=0}^{\infty} \dfrac{1}{n!} B_n^{(\ell)}(x)t^n$	Generalized Bernoulli polynomials (E)
25. $\dfrac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \dfrac{1}{n!} B_n(x)t^n$	Bernoulli polynomials
26. $\dfrac{(1-a)^{\ell}e^{xt}}{(1-ae^t)^{\ell}} = \sum_{n=0}^{\infty} \dfrac{1}{n!} E_n^{(\ell,a)}(x)t^n$	Generalized Euler polynomials (E)

Expression	Generated Set
27. $\dfrac{2e^{xt}}{e^{t}+1} = \sum_{n=0}^{\infty} \dfrac{1}{n!} E_n(x)t^n$	Euler polynomials
28. $e^{xt}I_0(t) = \sum_{n=0}^{\infty} \dfrac{1}{n!} R_n^*(x)t^n$	Reversed Rainville polynomials (B)

Sheffer Class

29. $e^{xg(t)}A(t) = \sum_{n=0}^{\infty} p_n(x)t^n$	Sheffer polynomials of zero type
30. $f_a(t)e^{xt+k(t)} = \sum_{n=0}^{\infty} \dfrac{1}{n!} \phi_n^{(a)}(x)t^n$	Generalized ϕ polynomials (E)
31. $e^{-t}(1+\frac{t}{a})^x = \sum_{n=0}^{\infty} a^{-\frac{1}{2}n}(\frac{1}{n!})^{\frac{1}{2}} p_n(x)t^n$	Poisson-Charlier polynomials (E)
32. $e^{x(1+t-e^t)} = \sum_{n=0}^{\infty} \dfrac{1}{n!} g_n(x)t^n$	Mahler polynomials (E)
33. $e^{at+x(1-e^t)} = \sum_{n=0}^{\infty} \dfrac{1}{n!} g_n^{(a)}(x)t^n$	Toscano polynomials (E)
34. $(1+t)^x(1-t)^x = \sum_{n=0}^{\infty} g_n(x)t^n$	Mittag-Leffler polynomials (E)

Expression	Generated Set
35. $e^{x(e^t-1)} = \sum_{n=0}^{\infty} p_n(x)t^n$	Touchard polynomials (E)
36. $(1-3xt+t^3)^{-\nu} = \sum_{n=0}^{\infty} h_n(x)t^n$	Humbert polynomials (E)
37. $(1+t)^{-m-1} e^{\frac{xt}{1+t}} = \sum_{n=0}^{\infty} \Gamma(m+n+1) S_m^n(x)t^n$	Sonine polynomials (E)
38. $2te^{xt}(\frac{p+t}{p-t} e^{2t}-1)^{-1} = \frac{p}{p+1} + \sum_{n=1}^{\infty} \omega_n^{(p)}(x)\frac{1}{n!} t^n$	Koshliakov polynomials (E)
39. $(1-t)^{-k-1}(1-xt)^{-1} = \sum_{n=0}^{\infty} g_n^{(k)}(x)t^n$	Cesàro polynomials (E)
40. $(1-t)^{-x} e^{xt} = \sum_{n=0}^{\infty} \phi_n(x)t^n$	Sylvester polynomials (E)
41. $(1-t^2)^{-\frac{1}{2}c}(\frac{1+t}{1-t})^x e^{-2xt} = \sum_{n=0}^{\infty} T_n(x)t^n$	Tricomi polynomials (E)

Expression	Generated Set
42. $\left(\frac{1-e^{-t}}{t}\right)^{-x-1} = 1 + (x+1)\sum_{n=0}^{\infty} \psi_n(x)t^{n+1}$	Stirling polynomials (E)
43. $(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} e^{-\frac{4xt}{(1+\sqrt{1-4t})^2}} = \sum_{n=0}^{\infty} R_n(a, x)t^n$	Pseudo-Laguerre polynomials (R)
44. $e^t I_0(xt) = \sum_{n=0}^{\infty} \frac{1}{n!} R_n(x)t^n$	Rainville polynomials (B)
45. $(1-t)^{-1} e^{-\frac{2xt}{(1-t)^2}} I_0\left[-\frac{2xt}{(1-t)^2}\right] = \sum_{n=0}^{\infty} Z_n(x)t^n$	Bateman's $Z_n(x)$ polynomials (R)
46. $[1-x^m+(x-t)^m]^{-\nu} = \sum_{n=0}^{\infty} {}_mC_n^{\nu}(x)t^n$	Devisme 2nd olynomials (E)
47. $(1-2xt)^{-\frac{1}{2}}\left[\frac{1}{2}+\frac{1}{2}(1-2xt)^{\frac{1}{2}}\right]^{2-a} \times e^{\{\frac{1}{2}bx^{-1}[1-(1-2tx)^{\frac{1}{2}}]\}} = \sum_{n=0}^{\infty} y_n(x, a, b)\frac{1}{n!}t^n$	Generalized Bessel polynomials (Krall and Frink) (E)

Expression	Generated Set
48. $e^{ix-i\sqrt{x^2+2ixt}} = \sum_{n=0}^{\infty} \frac{1}{n!} y_{n-1}(\frac{1}{ix})t^n$	Simple Bessel polynomials (Krall and Frink) (J)
49. $(1-t)^{-1} {}_pF_q[a_1,\ldots,a_p; b_1,\ldots,b_q; -\frac{4xt}{(1-t)^2}]$ $= \sum_{n=0}^{\infty} f_n[a_1,\ldots,a_p; b_1,\ldots,b_q; x]t^n$	Sister Celine polynomials (R)
50. $(1-4t)^{-\frac{1}{2}}(\frac{2}{1+\sqrt{1-4t}})^{a-1}$ $\times {}_pF_q[\alpha_1,\ldots,\alpha_p; \beta_1,\ldots,\beta_q; -\frac{4xt}{(1+\sqrt{1-4t})^2}]$ $= \sum_{n=0}^{\infty} S_n(x)t^n$	Shively's first polynomials (R)
51. ${}_0F_1[-; 1; \frac{t-\sqrt{4xt+t^2}}{2}]\,{}_0F_1[-; 1; \frac{t+\sqrt{4xt+t^2}}{2}]$ $= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \sigma_n(x)t^n$	Shively's second polynomials (R)
52. $e^t {}_1F_1[1+x; 1; -t(1-e^{-\lambda})]$ $= \sum_{n=0}^{\infty} \frac{1}{n!} \phi_n(x;\lambda)t^n$	Gottlieb polynomials (R)

Expression	Generated Set
53. $${}_2F_1[\alpha, \beta; \alpha+\beta; 2xt-t^2] = \sum_{n=0}^{\infty} G_n(\alpha, \beta; x)t^n$$	Bedient's $G_n(\alpha, \beta; x)$ polynomials (R)
54. $$(1-2xt)^{-\beta}{}_2F_1[\beta, \gamma-\beta; \gamma; -\frac{t^2}{1-2xt}] = \sum_{n=0}^{\infty} R_n(\beta, \gamma; x)t^n$$	Bedient's $R_n(\beta, \gamma; x)$ polynomials (R)
55. $$(1-t)^{-1}{}_2F_1[\frac{1}{2}, \frac{1}{2}+\frac{1}{2}x; 1; -\frac{4t}{(1-t)^2}] = \sum_{n=0}^{\infty} F_n(x)t^n$$	Bateman's $F_n(x)$ polynomials (R)
56. $$(1-t)^{-1}{}_2F_1[\rho, \frac{1}{2}; p; -\frac{4vt}{(1-t)^2}] = \sum_{n=0}^{\infty} H_n(\rho; p, v)t^n$$	Rice polynomials (R)
57. $$(1-xt)^{-\alpha}(1-yt)^{-\beta} = \sum_{n=0}^{\infty} g_n(x, y)t^n$$	Lagrange polynomials (E)
58. $$(1-3xt+3yt^2-t^3)^{-\nu} = \sum_{n=0}^{\infty} H_n^{\nu}(x, y)t^n$$	Devisme $1^{\underline{st}}$ polynomials (E)

Expression	Generated Set
59. $(1-4t^2)^{-\frac{1}{2}}\exp[y^2-\frac{(y-2xt)^2}{1-4t^2}]$ $= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)H_n(y)t^n$	Hermite polynomials (common form) (E)
60. $(1-t)^{-1-\alpha}\exp[-\frac{(x+y)t}{1-t}]$ $\times\ {}_0F_1[-;1+\alpha;\frac{xyt}{(1-t)^2}]$ $= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)t^n$	Laguerre polynomials (E)

Legend

B - [Boas and Buck, 3]

E - [Erdélyi, 4]

J - [Johnson and Johnson, 5]

R - [Rainville, 7]

APPENDIX B

Properties of Uniformly Convergent Series of the Form $F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$

Let

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

with the continuous generating set $\{f_n(x)\}$, $n = 0, 1, 2, \ldots$ being defined over the real closed interval $[a, b]$. Also, let t be defined over the real closed interval $[c, d]$. The series is then said to converge uniformly to $F(x,t)$ with respect to x if given any $t_1 \in [c, d]$ and any $\varepsilon_1 > 0$, $\exists$ an integer $N_{t_1\varepsilon_1}$, independent of $x \in [a, b]$, $\ni |F(x, t_1) - S_k(x, t_1)| < \varepsilon_1$, whenever $k \geq N_{t_1\varepsilon_1}$, where

$$S_k(x, t_1) = \sum_{n=0}^{k} f_n(x)t^n,$$

the first $k+1$ terms of the series. A similar definition with appropriate modifications in nomenclature describes the uniform convergence of the series with respect to t to $F(x, t)$ for a given $x_1 \in [a, b]$.

Several interesting properties of uniform convergence are given below for the series under consideration. These results are utilized in discussions on the preceding pages.

Property 1. Uniform and separate convergence of x and t in the series

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

implies joint uniform convergence of x and t.

Proof. From the definition of uniform convergence given before, given any $\varepsilon_1 > 0$ and any $t_1 \in [c, d]$, $\exists$ an integer $N_{t_1\varepsilon_1}$, independent of $x \in [a, b]$, $\ni |F(x, t_1) - S_{k_1}(x, t_1)| < \varepsilon_1$, whenever $k_1 \geq N_{t_1\varepsilon_1}$. Also, given any $\varepsilon_2 > 0$ and any $x_1 \in [a, b]$, $\exists$ an integer $N_{x_1\varepsilon_2}$, independent of $t \in [c, d]$, $\ni |F(x_1, t) - S_{k_2}(x_1, t)| < \varepsilon_2$, whenever $k_2 \geq N_{x_1\varepsilon_2}$. Choose $N = \max\{N_{t_1\varepsilon_1}, N_{x_1\varepsilon_2}\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. For all $k \geq N$,

$$|F(x, t) - S_k(x, t)| < \varepsilon \qquad \text{(B-1)}$$

independently of any $x \in [a, b]$ or $t \in [c, d]$, which is the condition of joint uniform convergence of x and t.

Property 2. If $F(x, t)$ is integrable in x over a finite interval $[a, b]$, is integrable in t over a finite interval $[c, d]$, and $f_n(x)$ is integrable over $[a, b]$ for all n, then

$$(\text{B-2}) \qquad \text{A.} \quad \int_a^b F(x,t)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dxt^n$$

and

$$(\text{B-3}) \qquad \text{B.} \quad \int_a^b F(x,t)dt = \sum_{n=0}^{\infty} f_n(x) \int_c^d t^n dt$$

$$= \sum_{n=0}^{\infty} f_n(x)\left(\frac{1}{n+1}\right)[d^{n+1} - c^{n+1}].$$

Proof. For A: Since the series is assumed to be uniformly convergent in x over $[a, b]$, for a given $\varepsilon > 0$ and a given $t_1 \epsilon [c, d]$ $\exists$ an integer $N_{t_1\varepsilon}$ independent of $x \ni$

$$(\text{B-4}) \qquad |F(x, t_1) - S_k(x, t_1)| < \varepsilon$$

for all $k \geq N_{t_1\varepsilon}$. Thus,

$$(\text{B-5}) \qquad \left| \int_a^b F(x, t_1)dx - \sum_{n=0}^{k} \int_a^b f_n(x)dxt_1^n \right|$$

$$= \left| \int_a^b F(x, t_1)dx - \int_a^b \sum_{n=0}^{k} f_n(x)t_1^n dx \right|$$

$$= \left| \int_a^b F(x, t_1)dx - \int_a^b S_k(x, t_1)dx \right| \leq$$

$$\leq \int_a^b |F(x, t_1) - S_k(x, t_1)| dx$$

$$< \varepsilon(b-a),$$

which can be made arbitrarily small, implying A.

The proof of B developes in a similar manner. It should be noticed that t^n is integrable over any finite interval $[c, d]$ for all $n \geq 0$.

<u>Property 3.</u> If each $f_n(x)$ is continuous on $[a, b]$, then $F(x, t)$ is continuous in x on (a, b) for any $t \in [c, d]$. $F(x, t)$ is continuous in t on (c, d) for any $x \in [a, b]$. If $f_n(x)$ is continuous on $[a, b]$ for each n, then $F(x, t)$ is jointly continuous over the interior of the domain $[a \leq x \leq b,\ c \leq t \leq d]$.

<u>Proof.</u> By definition of uniform convergence in x, $\exists\ N_{t_1 \varepsilon_{t_1}} \ni$ for all $k \geq N_{t_1 \varepsilon_{t_1}}$ and for x in (a, b),

$$|F(x, t_1) - S_k(x, t_1)| < \frac{\varepsilon_{t_1}}{3} \tag{B-6}$$

Similarly, for $x_0 \in (a, b)$,

$$|F(x_0, t_1) - S_k(x_0, t_1)| < \frac{\varepsilon_{t_1}}{3}\ . \tag{B-7}$$

Since $S_k(x, t_1)$ is a finite sum for all x in the interval and thus continuous at x_0, $\exists$ a

$$\delta_{t_1} > 0 \ni |S_k(x, t_1) - S_k(x_0, t_1)| < \frac{\varepsilon_{t_1}}{3}$$

for all $x \ni |x-x_0| < \delta_{t_1}$. Then,

$$\text{(B-8)} \qquad |F(x, t_1)-F(x_0, t_1)|$$

$$= |F(x, t_1)-S_k(x, t_1)+S_k(x, t_1)-S_k(x_0, t_1)+S_k(x_0, t_1)-F(x_0, t_1)|$$

$$\leq |F(x, t_1)-S_k(x, t_1)|+|S_k(x, t_1)-S_k(x_0, t_1)|$$

$$+ |S_k(x_0, t_1)-F(x_0, t_1)|$$

$$< \frac{\varepsilon_{t_1}}{3} + \frac{\varepsilon_{t_1}}{3} + \frac{\varepsilon_{t_1}}{3} = \varepsilon_{t_1}.$$

If x_0 is chosen to be a or b the proof must be appropriately modified, for $|x-s| < \delta_{t_1}$ only when $x \geq a$ and $|x-b| < \delta_{t_1}$ only when $x \leq b$. Otherwise, the proof is the same and the results become

$$\lim_{x \to a^+} F(x, t_1) = F(a, t_1)$$

and

$$\lim_{x \to b^-} F(x, t_1) = F(b, t_1).$$

The second statement is proved in the same manner, utilizing the fact that t^n is continuous for all $n \geq 0$. Finally, the last statement is similarly proved after Property 1 is invoked for the last step.

Property 4. If for $x \in [a, b]$ and $t_1 \in [c, d]$,

$$F(x, t_1) = \sum_{n=0}^{\infty} f_n(x) t_1^n$$

converges, $\dfrac{\partial F(x, t_1)}{\partial x}$ exists, each $f_n'(x)$ is continuous, and

$$\sum_{n=0}^{\infty} f_n'(x) t_1^n$$

converges uniformly, then

$$\text{(B-9)} \qquad \frac{\partial F(x, t_1)}{\partial x} = \sum_{n=0}^{\infty} f_n'(x) t_1^n, \quad a < x < b.$$

Similarly, if $t \in [c, d]$ and $x_1 \in [a, b]$,

$$F(x_1, t) = \sum_{n=0}^{\infty} f_n(x_1) t^n$$

converges, $\dfrac{\partial F(x_1, t)}{\partial t}$ exists, and

$$\sum_{n=1}^{\infty} nf_n(x_1)t^{n-1} = \sum_{n=0}^{\infty} (n+1)f_{n+1}(x_1)t^n$$

converges uniformly, then

$$\text{(B-10)} \qquad \frac{\partial F(x_1, t)}{\partial t} = \sum_{n=0}^{\infty} (n+1)f_{n+1}(x_1)t^n, \quad c < t < d.$$

Furthermore, the appropriate results hold for

$$\frac{\partial^k F(x, t_1)}{\partial x^k} \quad \text{and} \quad \frac{\partial^k F(x_1, t)}{\partial t^k}; \quad \text{i.e.,}$$

If each $f^{(k)}(x)$ is continuous, $\dfrac{\partial^k F(x, t_1)}{\partial x^k}$ exists,

$$\frac{\partial^{k-1} F(x, t_1)}{\partial x^{k-1}} = \sum_{n=0}^{\infty} f_n^{(k-1)}(x)t_1^n$$

converges, and

$$\sum_{n=0}^{\infty} f_n^{(k)}(x)t_1^n$$

converges uniformly in $[a, b]$, then

$$\text{(B-11)} \qquad \frac{\partial^k F(x, t_1)}{\partial x^k} = \sum_{n=0}^{\infty} f_n^{(k)}(x)t_1^n, \quad a < x < b.$$

Under the corresponding conditions for $\dfrac{\partial^k F(x_1, t)}{\partial t^k}$, the result is

$$\frac{\partial^k F(x_1, t)}{\partial t^k} = \sum_{n=0}^{\infty} f_{n+k}(x_1)(n+1)_k t^n, \quad c < t < d,$$

where $(n+1)_k = (n+1)(n+2)\ldots(n+k)$, Pochhammer's symbol.

Proof. Let

$$\frac{\partial F(x, t_1)}{\partial x} = \sum_{n=0}^{\infty} f_n'(x) t_1^n .$$

Since the series is assumed to converge uniformly, Property 2 allows it to be written as

$$\text{(B-12)} \qquad \int_a^x \frac{\partial F(x, t_1)}{\partial x} dx = \sum_{n=0}^{\infty} \int_a^x f_n'(x) dx t_1^n, \quad x \in (a, b)$$

$$= \sum_{n=0}^{\infty} [f_n(x) - f_n(a)] t_1^n$$

$$= \sum_{n=0}^{\infty} f_n(x) t_1^n - \sum_{n=0}^{\infty} f_n(a) t_1^n .$$

By Property 3 $\dfrac{\partial F(x, t_1)}{\partial x}$ is continuous so that

$$\text{(B-13)} \qquad \frac{\partial}{\partial x}\int_a^x \frac{\partial F(x, t_1)}{\partial x}\,dx = \frac{\partial F(x, t_1)}{\partial x} = \sum_{n=0}^{\infty} f_n'(x)t_1^n .$$

The other statements are proved by the same method. For the last sentence,

$$\text{(B-14)} \qquad \frac{\partial^k F(x_1, t)}{\partial t^k} = \sum_{n=0}^{\infty} f_n(x_1)\frac{\partial^k t^n}{\partial t^k} = \sum_{n=k}^{\infty} f_n(x_1)(n)^{(k)} t^{n-k} = \sum_{n=0}^{\infty} f_{n+k}(x_1)(n+1)_k t^n, \qquad c < t < d.$$

It should be noted that if

$$\sum_{n=0}^{\infty} f_n^{(k)}(x)t^n$$

is uniformly convergent for all n and k, the requirement that

$$\sum_{n=0}^{\infty} f_n^{(k-1)}(x)t^n$$

converges can be dropped since uniform convergence implies convergence. A similar result holds for the last step.

APPENDIX C

A Uniqueness Theorem Linking Generating Functions with Their Generated Sets

The following theorem is established to ensure that a one-to-one correspondence exists between a given generating function, $F(x, t)$, and the set it generates, $\{f_n(x)\}$, when they satisfy the hypotheses.

Uniqueness Theorem. Let

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n$$

with the series uniformly convergent in the real intervals $[a \leq x \leq b]$ and $[c \leq t \leq d]$ where $c < 0 < d$. Let the series

$$\sum_{n=k}^{\infty} f_n(x)(n)^{(k)} t^{n-k}$$

be uniformly convergent in the same x and t intervals for all $k = 1, 2, \ldots$. Also, assume that the generated set $\{f_n(x)\}$ has continuous derivatives of all orders in the x interval. Then, the representation

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

relating $F(x, t)$ and $\{f_n(x)\}$, is unique over $[a, b]$ and $[c, d]$.

Proof. Assume $F(x, t)$ can be represented by two generated sets, $\{f_n(x)\}$ and $\{g_n(x)\}$, possessing the listed properties for $n = 0, 1, 2, \ldots$. Then,

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n \tag{C-1}$$

and

$$F(x, t) = \sum_{n=0}^{\infty} g_n(x)t^n. \tag{C-2}$$

Subtract (C-2) from (C-1). The result is

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} f_n(x)t^n - \sum_{n=0}^{\infty} g_n(x)t^n \\ &= \sum_{n=0}^{\infty} [f_n(x)-g_n(x)]t^n \\ &= \sum_{n=0}^{\infty} \phi_n(x)t^n, \end{aligned} \tag{C-3}$$

where $\phi_n(x) = f_n(x) - g_n(x)$ is continuously differentiable in all orders, for $f_n(x)$ and $g_n(x)$ have these properties by hypothesis. Take the k^{th} partial derivative of (C-3) with respect to t. The equation becomes

$$\text{(C-4)} \qquad \frac{\partial^k 0}{\partial t^k} = 0 = \frac{\partial^k}{\partial t^k}\left[\sum_{n=0}^{\infty} \phi_n(x) t^n\right] = \sum_{n=0}^{\infty} \phi_n(x) \frac{\partial^k t^n}{\partial t^k},$$

for the series is assumed to be uniformly convergent in all orders. So,

$$\text{(C-5)} \qquad 0 = \sum_{n=k}^{\infty} \phi_n(x)(n)^{(k)} t^{n-k}, \quad n \geq k.$$

Since $t = 0$ lies in the region of validity for t, evaluation there gives

$$\text{(C-6)} \qquad 0 = \sum_{n=k}^{\infty} \phi_n(x)(n)^{(k)}(0)^{n-k} = \phi_k(x) k^{(k)} = \phi_k(x) k!\,.$$

Now, $k! > 0$ because k is an integer and ≥ 0. This fact implies $\phi_k(x) = 0$ for all valid x and k. Consequently,

$$f_n(x) - g_n(x) = 0, \tag{C-7}$$

or

$$f_n(x) = g_n(x), \quad n = 0, 1, 2, \ldots . \tag{C-8}$$

Assume that

$$\sum_{n=0}^{\infty} f_n(x)t^n$$

can be represented by two different generating functions, $F_1(x, t)$ and $F_2(x, t)$, for all valid x and t; i.e.,

$$\begin{aligned} \sum_{n=0}^{\infty} f_n(x)t^n &= F_1(x, t) \\ &= F_2(x, t). \end{aligned} \tag{C-9}$$

Then, $F_1(x, t) = F_2(x, t)$ identically, contradicting the previous statement. Therefore, the generating function is uniquely determined by the series.

It should be noted that the restrictions placed on the series and generated sets by the hypotheses of the theorem are met by the vast majority of sets considered.