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Theoretical bases and techniques are discussed in this paper for practical numerical analysis and synthesis of linear feedback control systems. Criterion are based on the superposition integrals.

A synthesis method is introduced to find the impulse response of a system or a part of the system. The existing system can be analyzed by using these impulse responses for unity or non-unity feedback systems.

A method for compensation of an existing system is also introduced, and the transfer function of the required compensating network can be computed directly from its computed impulse function.

Several examples for each kind of problem have been computed by using IBM 1620. Information of how to determine the required time increment is given to assure computation accuracy.

DIGITAL COMPUTER ANALYSIS AND SYNTHESIS OF LINEAR FEEDBACK CONTROL SYSTEMS USING SUPERPOSITION INTEGRALS

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DIGITAL COMPUTER ANALYSIS AND SYNTHESIS OF LINEAR FEEDBACK CONTROL SYSTEMS USING SUPERPOSITION INTEGRALS

I. INTRODUCTION

The numerical method of analysis and synthesis of linear networks started in 1947 when Professor Tustin published his time series method for analyzing the behavior of linear systems (14). The results of this method are coincident with some later developed numerical methods using superposition integrals.

Truxal (13), Ragazzini and Bergen (10) in 1954 introduced the Z-transformation method developed originally for the sampled-data system to the analysis of linear systems. Ba Hli (2) applied Tustin's method to obtain the approximate impulse response of an open loop system in 1953, which along with Kautz's (7) work of 1954 gave a general idea of time domain synthesis.

In 1955 and 1956, Cruickshank (5), Boxer and Thaler (4) gave a different approximation method for converting Laplace transform output into time response. Stout (11) suggested his step-by-step method for transient analysis of control systems in 1957; Naumov (9) in his paper of 1961 set up an approximate method for calculating the time response of unity feedback control systems from its Laplace transform transfer function.

Adams (1) in 1962 has proved the possibility of digital computer analysis for unity feedback system from its given transfer function.

Sometimes, the transfer function of an existing system is not known. If the system is to be linear or nearly linear, it is possible to find its impulse response from the input signal and the output response.

The main interests of this paper are how to use the digital computer to find:

1. The time response of linear, open loop, unity feedback and non-unity feedback systems from their impulse responses and a given input signal.
2. The impulse response of an existing open or closed loop system from the input signal and output transient response.
3. The impulse response of a desired compensating network for improving an existing system from the system input signal, output time response and the impulse response.
4. The transfer function of a network from its impulse response provided the steady state value of the impulse response is zero.

Numerical methods of trapezoidal rule and extrapolation are used in calculation of functions from superposition integrals.

II. ANALYSIS

A. Open Loop System:

Suppose an open loop system as shown in Figure 1 has its linear or linearized transfer function represented by G(s), with input E(s) and output C(s). Then:

$$C(s) = E(s)\ G(s) \tag{1}$$

The transfer function can usually be represented as:

$$G(s) = \frac{A(s)}{B(s)} \tag{2}$$

where A(s) and B(s) are polynomials of s, and because of the physical nature, the degree of B(s) is either equal to or greater than the degree of A(s).

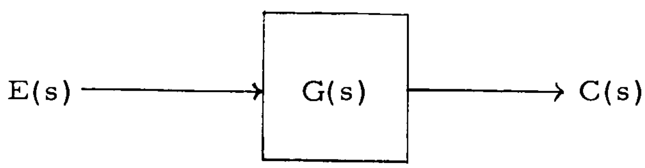

Figure 1. Open loop system I

First let us consider the case where the degree of B(s) is greater than the degree of A(s). The impulse response of this system will be a continuous time function, say:

$$g(t) = L^{-1}\ \{G(s)\} \tag{3}$$

Since this system is linear, the superposition theorem can be applied, and the transient time response of the output can be represented as:

$$c(t) = \int_0^t e(t-T)\ g(T)\ dT \tag{4}$$

or

$$c(t) = \int_0^t e(T)\ g(t-T)\ dT \tag{5}$$

where

$$e(t) = L^{-1}\{E(s)\}, \tag{6}$$

the time function of the input voltage.

If the numerical values of $e(t)$ and $g(t)$ are known, the approximate values of the equation (4) or (5) can be calculated by numerical methods. The simplest numerical integration method that can be applied to this problem is the trapezoidal rule.

Let the numerical values of $e(t)$ and $g(t)$ at equally spaced time increments be represented as:

Time	0	h	2h	3h	4h	. . .
e(t)	e_1	e_2	e_3	e_4	e_5	. . .
g(t)	g_1	g_2	g_3	g_4	g_5	. . .

The approximate values of the integral when calculated by trapezoidal rule are:

$$c(nh) = h\left(\frac{e_1 g_{n+1}}{2} + e_2 g_n + e_3 g_{n-1} + \ldots + e_n g_2 + \frac{e_{n+1} g_1}{2}\right) \tag{7}$$

where $n = 0, 1, 2, \ldots$

Let $a_n = h\, g_n$, and: $c(nh) = c_{n+1}$, then:

$$c_2 = \frac{e_1 a_2}{2} + \frac{e_2 a_1}{2}$$

$$c_3 = \frac{e_1 a_3}{2} + e_2 a_2 + \frac{e_3 a_1}{2}$$

. . . .

$$c_n = \frac{e_1 a_n}{2} + e_2 a_{n-1} + e_3 a_{n-2} + \ldots + e_{n-1} a_2 + \frac{e_n a_1}{2} \qquad (8)$$

The function $a_n = h\, g_n$ is called the weighting function.

This method was first introduced by Tustin, (14) where he used $\frac{1}{4} e_1 a_1$ as the initial value of the output. This approach has also been adopted by Adams (1). Obviously, this is not a close approximation especially when the initial values of the input and system impulse response are high.

The initial value of the output is always zero for a transfer function with a continuous impulse response, since by the initial-value theorem,

$$\lim_{t \to 0} c(t) = \lim_{s \to \infty} s\, C(s) = \lim_{s \to \infty} R(s)\; s\, \frac{A(s)}{B(s)} \qquad (9)$$

The degree of the numerator of $R(s)$ is always less than its denominator, since any deterministic input signal $r(t)$ may be considered to be composed of steps, ramps, parabolas or any combination of their functions. Therefore, from Equation (9), the initial value of $c(t)$ is always zero.

Now, suppose that the degree of $A(s)$ is equal to the degree of $B(s)$, and

$$G(s) = \frac{a_1 s^n + a_2 s^{n-1} + \ldots + a_n s + a_{n+1}}{b_1 s^n + b_2 s^{n-1} + \ldots + b_n s + b_{n+1}} \tag{10}$$

Then, by division,

$$G(s) = \frac{a_1}{b_1} + \frac{U(s)}{V(s)} = K + \frac{U(s)}{V(s)} \tag{11}$$

where K is a constant and the degree of $U(s)$ is one less than the degree of $V(s)$.

Equation (11) can be considered as two transfer functions connected in parallel as shown in Figure 2.

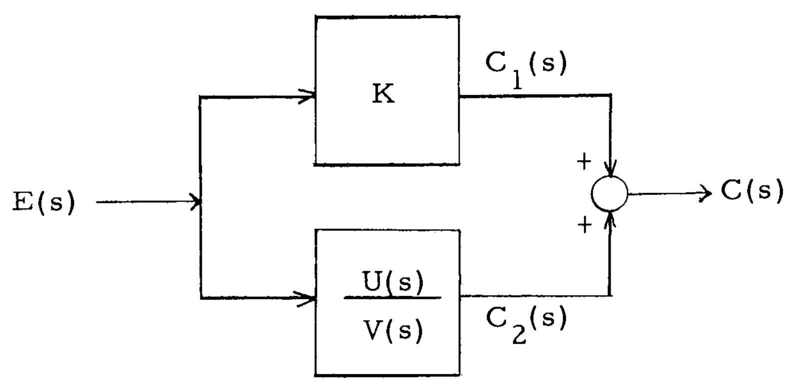

Figure 2. Open loop system II

From Equation (11) and Figure 2,

$$C(s) = KE(s) + E(s)\,\frac{U(s)}{V(s)} \tag{12}$$

where: $$C_1(s) = K\,E(s) \tag{13}$$

and $$c_1(t) = K\,e(t) \tag{14}$$

which can be found by:

$$c_{1n} = K\,e_n \tag{15}$$

$$C_2(s) = E(s)\,\frac{U(s)}{V(s)} \tag{16}$$

Values for $c_2(t)$ can be found by the previous method of continuous impulse response, and:

$$c_n = c_{1n} + c_{2n} \tag{17}$$

Since $c_{21} = 0$,

$$c_1 = c_{11} = Ke_1 \tag{18}$$

B. Closed Loop System With Unity Feedback

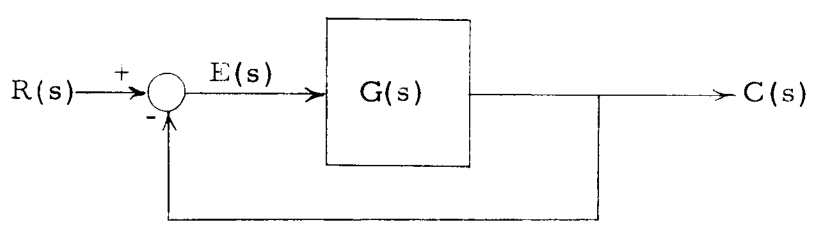

Figure 3. Unity feedback system

In the unity feedback system as shown in Figure 3,

$$E(s) = R(s) - C(s) \tag{19}$$

or:

$$e(t) = r(t) - c(t) \tag{20}$$

where $r(t)$ is the input time function.

In order to calculate $c(t)$ from Equation (8), Adams (1) suggested a linear extrapolation method to estimate c'_n from c_{n-1} and c_{n-2}. Then e_n is found by

$$e_n = r_n - c'_n \tag{21}$$

This e_n is used in Equation (8) to find the final approximate c_n.

Adams' method does not give a very good result. First, the value of e_1 calculated from the improper initial value c_1 created some error when applying Equation (8). Second, the linear extrapolation introduced significant error especially when the slope of the output response is changing rapidly.

Two different methods have been studied here in the attempt to find a better solution:

In the first method, a higher order extrapolation formula is used to find c'_n, such as a Newton's 4th order extrapolation formula:

$$c'_n = 4\,c_{n-1} - 6\,c_{n-2} + 4\,c_{n-3} - c_{n-4} \tag{22}$$

For the starting points, lower order extrapolation formulas have to be used. Then by applying Equations (21) and (8), a second approximation c''_n can be calculated, and this c''_n may be used again by applying Equations (21) and (8) to find a more precise value of c_n. This c_n may be used as the final approximated value of the output and the e_n required in successive calculations is obtained by:

$$e_n = r_n - c_n \tag{23}$$

The remaining problem in this method is how to determine the initial value of the output. The overall transfer function of the unity feedback system:

$$\frac{G(s)}{1 + G(s)} = \frac{A(s)}{A(s) + B(s)} \tag{24}$$

The degree of the numerator and denominator are the same as the degree of the numerator and denominator of the transfer function itself. If the transfer function has a continuous impulse response, the initial value of the output c_1 is always zero.

The results of the above method are satisfactory as shown in Table 2. This method is discarded later however, since it required laborious calculations.

The second method is derived from Equations (8) and (23). Since:

$$c_n = \frac{e_1 a_n}{2} + e_2 a_{n-1} + \ldots + e_{n-1} a_2 + \frac{(r_n - c_n) a_1}{2} \tag{25}$$

or:

$$c_n = \frac{1}{1+\frac{a_1}{2}}\left(\frac{e_1 a_n}{2} + e_2 a_{n-1} + \ldots + e_{n-1} a_2 + \frac{r_n a_1}{2}\right) \tag{26}$$

In order to facilitate computations, a computation table as shown in Table 1, is made. The computation sequence is: first row, first column; second row, second column, third row and so forth.

The same example as used by Adams for $G(s) = \frac{1}{1+s}$ and $h = 0\ 1$ sec. is calculated by desk calculator, using both the first and the second method. The results are listed in Table 2, for comparison.

Table 1. Unity-feedback Analysis Computation

Time	0	h	2h	3h	4h	. . .
Input	r_1	r_2	r_3	r_4	r_5	. . .
Weighting function	a_1	a_2	a_3	a_4	a_5	. . .
		$\frac{a_1}{2} r_2$	$\frac{a_1}{2} r_3$	$\frac{a_1}{2} r_4$	$\frac{a_1}{2} r_5$	. . .
		$\frac{e_1}{2} a_2$	$\frac{e_1}{2} a_3$	$\frac{e_1}{2} a_4$	$\frac{e_1}{2} a_5$	. . .
			$e_2 a_2$	$e_2 a_3$	$e_2 a_4$	. . .
				$e_3 a_2$	$e_3 a_3$	. . .
					$e_4 a_2$	. . .
Summation of column q_n		q_2	q_3	q_4	q_5	. . .
$c_n = \frac{q_n}{D}$ *	$c_1 = 0$	c_2	c_3	c_4	c_5	. . .
$e_n = r_n - c_n$	$e_1 = r_1$	e_2	e_3	e_4	e_5	. . .

* $D = 1 + \frac{a_1}{2}$

Table 2. Comparison of Analysis Results

Results from $G(s) = \frac{1}{1+s}$, $R(s) = U(t)$, $h = 0.1$ sec.

Time	Adams method	True value	2nd method	1st method
0	0.25	0	0	0
0.1	0.09274	0.09063	0.09070	0.09048
0.2	0.16888	0.16484	0.16496	0.16502
0.3	0.22855	0.22559	0.22575	0.22579
0.4	0.27642	0.27533	0.27552	0.27555
0.5	0.31619	0.31606	0.31628	0.31632
0.6	0.34887	0.34940	0.34962	
0.7		0.37670	0.37693	
0.8		0.39905	0.39926	
0.9		0.41735	0.41758	
1.0		0.43233	0.43257	

Nevertheless, the above mentioned methods can only be applied when the numerator of G(s) is less than the degree of its denominator, or when the transfer function G(s) has a continuous impulse response. If the degree of the numerator of G(s) is equal to the degree of its denominator, G(s) can be manipulated as Figure 2 and Equation (11) and:

$$c_n = \frac{e_1 a_n}{2} + e_2 a_{n-1} + \ldots + e_{n-1} a_2 + \frac{(r_n - c_n) a_1}{2} + K(r_n - c_n) \qquad (27)$$

or:

$$cn = \frac{1}{1 + \frac{a_1}{2} + K} \left[\frac{e_1 a_n}{2} + e_2 a_{n-1} + \ldots + e_{n-1} a_2 + \left(\frac{a_1}{2} + K\right) r_n \right] \qquad (28)$$

The value of

$$c_1 = K e_1 = K (r_1 - c_1) \tag{29}$$

or:

$$c_1 = \frac{K r_1}{1 + K} \tag{30}$$

Therefore, the initial value of the output is always less than the initial value of the input signal when $K > 0$. When $K = 0$, Equation (28) equals Equation (26) and $c_1 = 0$.

C. Closed Loop System With Non-unity Feedback

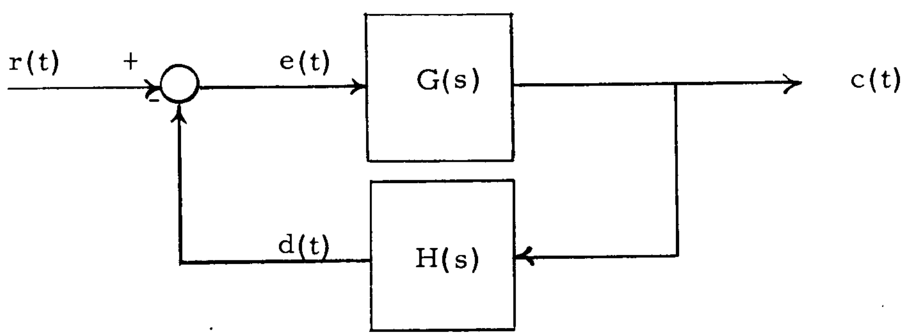

Figure 4. Non-unity feedback system I

If we know the transfer functions G(s) and H(s), this whole system can be considered either as an open loop system of transfer function:

$$G'(s) = \frac{G(s)}{1 + G(s)\ H(s)} \tag{31}$$

or can be transformed to a unity feedback system with the open loop transfer function:

$$G'(s) = \frac{G(s)}{1 + G(s)\ H(s) - G(s)} . \tag{32}$$

Then, if the impulse response of either of the above transfer functions can be calculated, $c(t)$ can be calculated by one of the previous methods. Suppose the impulse responses $g(t)$ and $h(t)$ are given as:

Time	0	h	2h	3h	4h	. . .
g(t)	g_1	g_2	g_3	g_4	g_5	. . .
h(t)	h_1	h_2	h_3	h_4	h_5	. . .

at equally spaced time increments, and it is known that they are both continuous. Then: $c_1 = d_1 = 0$, $e_1 = r_1$, and:

$$c_n = \frac{e_1 a_n}{2} + e_2 a_{n-1} + \ldots + e_{n-1} a_2 + \frac{e_n a_1}{2} \tag{33}$$

$$d_n = \frac{c_1 b_n}{2} + c_2 b_{n-1} + \ldots + c_{n-1} b_2 + \frac{c_n b_1}{2} \tag{34}$$

$$e_n = r_n - d_n \tag{35}$$

where:

$$a_n = h\, g_n \quad \text{and} \quad b_n = h\, h_n$$

Solving Equations (33), (34) and (35),

$$c_n = \frac{1}{1 + \frac{1}{4} a_1 b_1} \Big[\frac{1}{2} a_1 r_n + \frac{1}{2} e_1 a_n + e_2 a_{n-1} + \ldots + e_{n-1} a_2 - \frac{1}{2} a_1 \big(\frac{1}{2} c_1 b_n + c_2 b_{n-1} + \ldots + c_{n-1} b_2 \big) \Big] \tag{36}$$

If the degree of the numerators of both $G(s)$ and $H(s)$ are equal to the degree of their denominators, Figure 4 can be transformed to Figure 5.

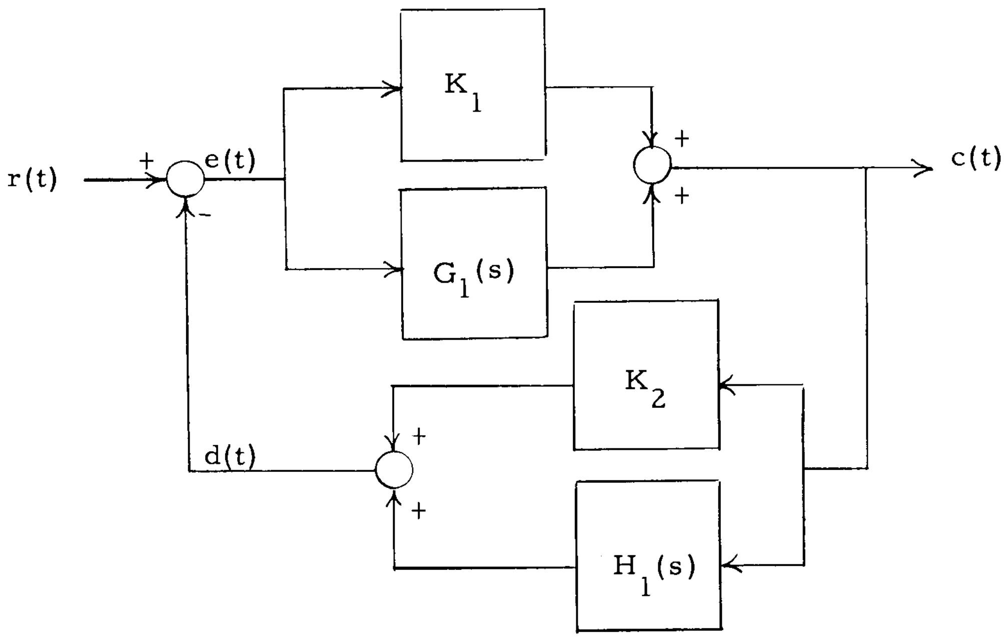

Figure 5. Non-unity feedback system II

and:

$$d_n = \frac{1}{2} c_1 b_n + c_2 b_{n-1} + \dots + c_{n-1} b_2 + \frac{1}{2} c_n b_1 + K_2 c_n \qquad (37)$$

$$c_n = \frac{1}{2} e_1 a_n + e_2 a_{n-1} + \dots + e_{n-1} a_2 + \left(\frac{1}{2} a_1 + K_1\right)(r_n - d_n) \qquad (38)$$

where $a_n = h\, g_{1n}$ and $b_n = h\, h_{1n}$

Therefore:

$$c_n = \frac{1}{1 + \left(\frac{1}{2} a_1 + K_1\right)\left(\frac{1}{2} b_1 + K_2\right)} \left\{\left(\frac{1}{2} a_1 + K_1\right) r_n + \frac{1}{2} e_1 a_n + e_2 a_{n-1} + \dots + \right.$$

$$\left. e_{n-1} a_2 - \left(\frac{1}{2} a_1 + K_1\right)\left(\frac{1}{2} c_1 b_n + c_2 b_{n-1} + \dots + c_{n-1} b_2\right)\right\} \qquad (39)$$

The value of

$$c_1 = \frac{r_1 K_1}{1 + K_1 K_2} \tag{40}$$

When $K_1 = K_2 = 0$, Equation (39) is equal to Equation (36). Table 3 shows the computation procedure.

Table 3. Non-unity Feedback Analysis Computation

r_1	r_2	r_3	r_4	. . .
$\frac{1}{2}a_1$	a_2	a_3	a_4	. . .
$\frac{1}{2}b_1$	b_2	b_3	b_4	. . .
$c_1 =$	$(\frac{1}{2}a_1+K_1)r_2$	$(\frac{1}{2}a_1+K_1)r_3$	$(\frac{1}{2}a_1+K_1)r_4$	. . .
$\frac{r_1K_1}{1+K_1K_2}$	$\frac{1}{2}e_1a_3$	$\frac{1}{2}e_1a_3$	$\frac{1}{2}e_1a_4$	. . .
	$-(\frac{1}{2}a_1+K_1)\,p_2$	e_2a_2	e_2a_3	. . .
		$-(\frac{1}{2}a_1+K_1)\,p_3$	e_3a_2	. . .
			$-(\frac{1}{2}a_1+K_1)\,p_4$	. . .
+)				
Summation	q_2	q_3	q_4	. . .
$c_n = q_n/F$	c_2	c_3	c_4	. . .
			c_3b_2	. . .
		c_2b_2	c_2b_3	. . .
+)	$\frac{1}{2}c_1b_2$	$\frac{1}{2}c_1c_3$	$\frac{1}{2}c_1b_4$	. . .
Summation	p_2	p_3	p_4	. . .
+)	$\frac{1}{2}b_1c_2$	$\frac{1}{2}b_1c_3$	$\frac{1}{2}b_1c_4$	. . .
	d_2	d_3	d_4	. . .
$e_n=r_n-d_n$	e_2	e_3	e_4	. . .

where: $e_1 = r_1 - c_1K_2 = \dfrac{r_1}{1 + K_1K_2}$

and: $F = 1 + (\frac{1}{2}a_1 + K_1)(\frac{1}{2}b_1 + K_2)$

III. SYNTHESIS

In Figure 1, if e(t) and c(t) are given, and if we know that the impulse response of the system is a continuous function of time, then from Equation (8):

$$a_n = \frac{2}{e_1}\left(c_n - \frac{1}{2}a_1 e_n - a_2 e_{n-1} - \ldots - a_{n-1}e_2\right) \tag{41}$$

where $e_1 \neq 0$

Therefore, the impulse response can be calculated in terms of the input signal and output time response.

In order to start the computation of Equation (41), the initial value of the impulse response has to be found. Different starting procedures have been studied. The only satisfactory procedure is to use a high order extrapolation formula for finding the initial value by simultaneous equations. For example, if Newton's binomial formula of order 4 is used:

$$\begin{aligned}
0 &= a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 \\
c_2 &= \frac{1}{2}e_2 a_1 + \frac{1}{2}e_1 a_2 \\
c_3 &= \frac{1}{2}e_3 a_1 + e_2 a_2 + \frac{1}{2}e_1 a_3 \\
c_4 &= \frac{1}{2}e_4 a_1 + e_3 e_2 + e_2 a_3 + \frac{1}{2}e_1 a_4 \\
c_5 &= \frac{1}{2}e_5 a_1 + e_4 a_2 + e_3 a_3 + e_2 a_4 + \frac{1}{2}e_1 a_5
\end{aligned} \tag{42}$$

Solving the above equations - by using a digital computer, the first five values of the impulse response can be found, and the successive values can be computed by applying Equation (41).

To determine experimentally whether the transfer function of this system has equal degree in its numerator and denominator, the best way is to apply an input signal with non-zero initial value. A non-zero initial value for the output indicates that numerator and denominator are of the same degree.

Since for an input signal of non-zero initial value, the degree of the deonominator of its Laplace transform will be one greater than its numerator, let it be represented by

$$E(s) = \frac{P(s)}{Q(s)} = \frac{a(s^{m-1} + p_1 s^{m-2} + \ldots + p_{m-1})}{s^m + q_1 s^{m-1} + \ldots + q_m} \tag{43}$$

The transfer function of the system is

$$G(s) = K + \frac{U(s)}{V(s)} \tag{11}$$

then by the initial-value theorem,

$$\lim_{t \to 0} c(t) = \lim_{s \to \infty} a \frac{s^m + p_1 s^{m-1} + \ldots + p_{m-1} s}{s^m + q_1 s^{m-1} + \ldots + q_m} \left(K + \frac{U(s)}{V(s)}\right) \tag{44}$$

$$= \lim_{s \to \infty} a \left(1 + \frac{P'(s)}{Q(s)}\right)\left(K + \frac{C(s)}{D(s)}\right)$$

$$= aK \tag{45}$$

Since $a = e_1$:

$$c_1 = \begin{cases} 0 & \text{if } K = 0 \\ Ke_1 & \text{if } K \neq 0 \end{cases} \tag{46}$$

Therefore:

$$K = \frac{c_1}{e_1} \tag{47}$$

After K is found from Figure 2 and Equations (14) and (15):

$$c_2(t) = c(t) - K\,e(t) \tag{48}$$

or:

$$c_{2n} = c_n - K\,e_n \tag{49}$$

Then the impulse response of the term $\frac{U(s)}{V(s)}$ in Equation (11) can be found by the method of Equations (41) and (42).

In the case where it is impossible to use an input signal with non-zero initial value or when doing compensating synthesis as will be discussed later, the constant K in Equation (11) can be computed by the use of the second value of the input signal e_2 provided that e_2 is correct and is not too small:

When $e_1 = 0$, from Equation (27), :

$$c_2 = \frac{1}{2}a_1 e_2 + K\,e_2 = (\frac{1}{2}a_1 + K)\;e_2 \tag{50}$$

or $$\frac{1}{2}a_1 + K = \frac{c_2}{e_2} \tag{51}$$

After the rest of a's have been found by:

$$a_{n-1} = \frac{1}{e_2}[c_n - (K+\frac{1}{2}a_1)e_n - a_2e_{n-1} - \cdots - a_{n-2}e_3] \tag{52}$$

The value a_1 can be determined by using a high order extrapolation formula similar to that used in Equation (42), then:

$$K = (K+\frac{1}{2}a_1) - \frac{1}{2}a_1 \tag{53}$$

When K is zero, Equations (50) - (53) can still be used.

The above is a general method for finding the impulse response of an open loop system. If the system is a unity feedback system as shown in Figure 3, and $r(t)$ and $c(t)$ are given as discrete values at equally spaced time increments, then:

$$e(t) = r(t) - c(t) \tag{20}$$

or

$$e_n = r_n - c_n \tag{21}$$

If the system is a non-unity feedback system as shown in Figures 4 or 5, and $r(t)$, $h(t)$, and $c(t)$ are given as discrete values at equally spaced time increments, then

$$d(t) = \int_0^t c(T)h(t-T)dT \tag{54}$$

The open loop analysis method mentioned before can be used to find $d(t)$, and then $e(t)$ is found by:

$$e(t) = r(t) - d(t) \tag{55}$$

or

$$e_n = r_n - d_n \tag{35}$$

From the computed $e(t)$ and the given $c(t)$, the required $g(t)$ can be found.

If the impulse response found from the above is mainly for analysis use, there is no need to convert it to the frequency domain. The method for finding the equivalent transfer function of certain system with given impulse response will be discussed in the next section.

IV. COMPENSATION OF CONTROL SYSTEM

A. Compensation

Suppose a control system as shown in Figure 3 or 4 is given. If the time response of the system when computed by the analysis method mentioned before does not give a satisfactory result, usually, a series or feedback compensating circuit can be used to improve the output response as shown in Figure 6 and Figure 7

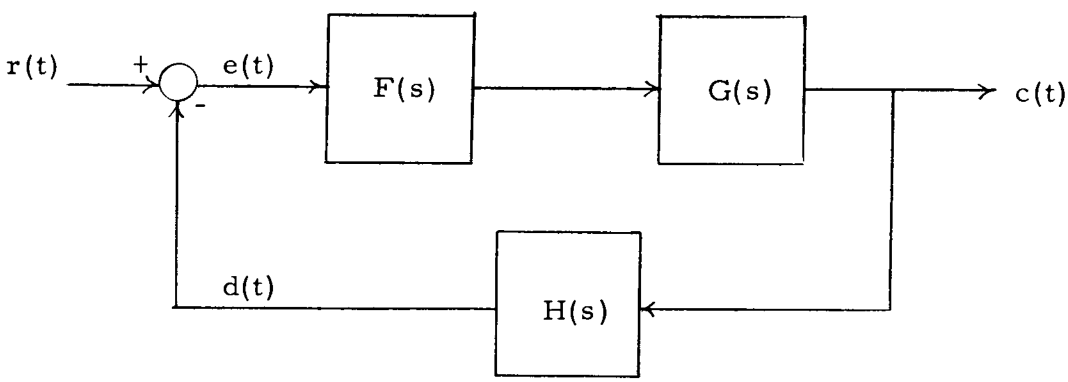

Figure 6. Series compensating system

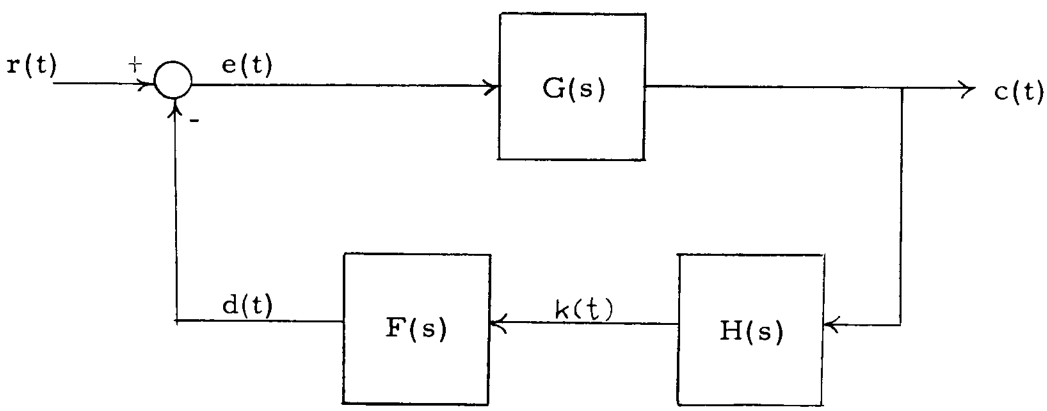

Figure 7. Feedback compensating system

For the series compensating method, if r(t), g(t) and h(t) are given for a desired c(t), d(t) can be found by Equation (54 and e(t) can be obtained from Equation (35). In order to facilitate the computation, the compensating circuit should be arranged as shown in Figure 8.

For a linear system, this change does not effect the result. The values of k(t) can be then found by applying the open loop analysis method, and f(t) is obtained by the synthesis method mentioned before.

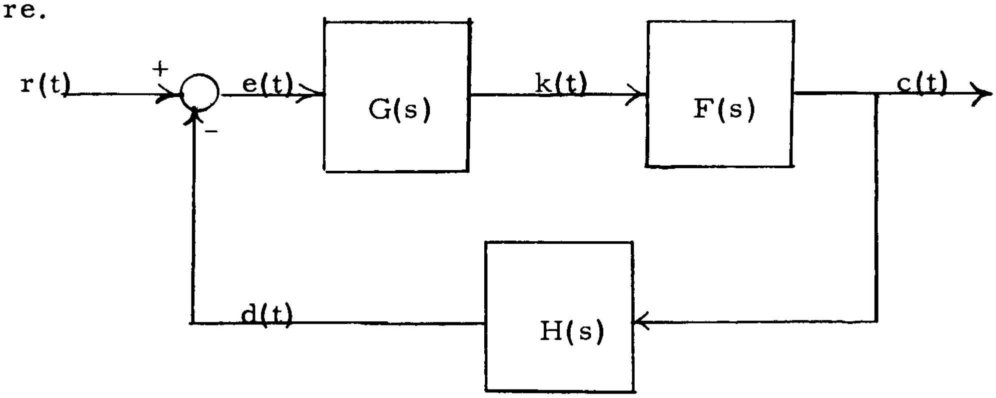

Figure 8. Rearranged series compensating system

For the feedback compensating system, since:

$$c(t) = \int_0^t e(T)g(t-T)dT \tag{5}$$

and:

$$c(t) = \int_0^t k(T)h(t-T)dT \tag{56}$$

e(t) and k(t) can be computed by the method used for synthesis. Then, d(t) is found by Equation (35). Now f(t) can be computed by the synthesis method.

B. Calculation of the Compensating Transfer Function

In order to find the transfer function from its impulse response, the superposition integral is used again. Since:

$$G(s) = \int_o^\infty g(t)\, e^{-st}\, dt \tag{57}$$

and from:

$$e^x = 1 + x + \frac{1}{2}x^2 + \ldots \tag{58}$$

the transfer function can be represented as:

$$G(s) = h\left(\frac{1}{2}g_1 + g_2 e^{-hs} + g_3 e^{-2hs} + g_4 e^{-3hs} + \ldots\right) \tag{59}$$

$$\begin{aligned} &= \frac{1}{2}a_1 + a_2\left(1 - hs + \frac{1}{2}h^2 s^2 - \frac{1}{6}h^3 s^3 + \ldots\right) \\ &\quad + a_3\left(1 - 2hs + \frac{1}{2}(2hs)^2 - \frac{1}{6}(2hs)^3 + \ldots\right) \\ &\quad + \ldots\ldots\ldots \\ &\quad + a_n\left[1 - (n-1)hs + \frac{1}{2}((n-1)hs)^2 - \frac{1}{6}((n-1)hs)^3 + \ldots\right] \end{aligned} \tag{60}$$

$$\begin{aligned} &= \left(\frac{1}{2}a_1 + \sum_{n=2}^{\infty} a_n\right) - \left(\sum_{n=2}^{\infty} a_n (n-1)h\right)s + \left(\sum_{n=2}^{\infty} a_n ((n-1)h)^2\right)\frac{s^2}{2} \\ &\quad - \left(\sum_{n=2}^{\infty} a_n ((n-1)h)^3\right)\frac{s^3}{6} + \ldots \end{aligned} \tag{61}$$

$$= Y_1 + Y_2 s + Y_3 s^2 + Y_4 s^3 + \ldots \tag{62}$$

where, $a_n = h\, g_n$, and h = time increment, as used before.

In Equation (62), Y_1 is the total area between the impulse response curve and the time axis. It is just the summation of all the values of the weighting function. The other Y's can also be calculated from the weighting function.

In order to avoid laborious computation, it is assumed that the transfer function of the continuous impulse response has a denominator of degree 2 and a numerator of degree 1:

$$G(s) = \frac{X_1 + X_2 s}{1 + X_3 s + X_4 s^2} \tag{63}$$

Compare Equation (62) and (63), :

$$\begin{aligned} X_1 &= Y_1 \\ X_2 &= Y_2 + Y_1 X_3 \\ 0 &= Y_3 + Y_1 X_4 + Y_2 X_3 \\ 0 &= Y_4 + Y_2 X_4 + Y_3 X_3 \end{aligned} \tag{64}$$

Solving the above equations, the continuous transfer function $G(s)$ can be determined. If the system has a parallel constant transfer function K as indicated in Equation (11), the total transfer function will be

$$G(s) = \frac{(K+X_1) + (KX_3+X_2) s + KX_4 s^2}{1 + X_3 s + X_4 s^2} \tag{65}$$

Therefore, as long as the impulse response or its weighting function is known, the transfer function can be calculated and represented as either Equation (63) or (65). This quadratic transfer function can then be checked with the original system. The new system can be further compensated by adding additional networks of the form of Equations (63) or (65). These networks are found by applying the above method. With this compensating procedure, ordinary systems can always be compensated.

V. EXAMPLES AND THEIR ACCURACY

A. Analysis

Several examples similar to those used in Adams' paper have been computed by using the IBM 1620. Different time increments were used in order to find the best choice, i.e. satisfactory results and not too laborious computation. The conclusions are:

1. Find the time from the given impulse response curve where the value has decreased to about 1/100 of the maximum value; say T.

2. Choose the time increment h equal to T divided by a convenient number around 60 to 80, and choose T as the computing range. Since beyond T, the response will be approaching steady state.

3. Compute the output response, from which find a new T. The value of T should be just large enough to include the entire transient period. This T will be divided by a convenient number around 70 to find a new h; then compute again. The final results will give a response accurate to the third place.

The IBM 1620 takes about one minute acutal computing time (input time are not included) to compute the output of an open loop or an unity feedback system. It will take four minutes to compute a non-unity feedback system output. This is about 1/3 to 1/4 the time required by Adams' method to attain comparable accuracy using the same digital computer.

B. Synthesis

In order to find the impulse response only, a similar procedure as stated above can be used. If the response value at T has decreased to about 1/100 of its maximum value and the divisor is around 70, the error will be less than 1/500 of the response maximum value.

In order to find the equivalent transfer function of the impulse response, the computing range should be longer. The value of the impulse response at T should be around 1/10000 to 1/100000 of its maximum in order to find accurate coefficients. The divisor is chosen as 100.

It takes about one minute to compute the impulse response on the IBM 1620. The equivalent transfer function can be calculated in about nine minutes.

The coefficients of a transfer function computed for different ranges are compared in Table 4. The transfer function is

$$\frac{1 + s}{4 + 2s + s^2}$$

Table 4. Comparison of Computed Transfer Functions

T (sec)	X_1	X_2	X_3	X_4
True value	.25000000	.25000000	.50000000	.25000000
10.0	.25081436	.25639184	.51969386	.24160654
5.0	.25356454	.097199090	-.04270133	.34020567
2.5	.22511990	.41002583	.89874220	.31479567

VI. CONCLUSION

The methods represented in this paper furnish a practical way to analyze almost any kind of linear control system provided that it is stable. They also give an accurate method to compensating an existing system. The synthesis method mentioned in this paper provides a simple procedure for directly converting a computed impulse response to its transfer function. The impulse response can be obtained for any kind of an input signal and a reasonable output response. A complete synthesis program written in Fortran is contained in the appendix for reference.

Common types of non-linear and time varying systems can also be solved by numerical methods. Analysis methods appear in some of the papers contained in the attached bibliography.

It is hoped that high speed digital computers can be used along with analog-digital converters to solve routine problems for control engineers. The methods presented in this paper give a fundamental technique for this purpose.

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APPENDICES

APPENDIX 1

OPEN LOOP ANALYSIS PROGRAM

Input E
Weighting Function A
Computed Output C
Read in H = Time increment in sec
M = Number of intervals needed

```
   E(1) = E(1)/2.
   A(1) = A(1)/2.
   DO 6  I = 2, M
6  C(1) = E(1) *A(1)
   DO 8  I = 2, M
   K = 2
   DO 7  J = I , M
   C(J) = C(J) + E (I-1)*A(K)
7  K = K + 1
8  CONTINUE
   C(1) = 4. * (C(2)+C(4)) -6. *C(3) - C(5)
   PUNCH 54
   DO 9 I = 1, M
9  PUNCH 52, I, C(1)
52 FORMAT (5X I4, 5X E14. 8)
54 FORMAT (/10X 7H OUTPUT/)
   END
```

APPENDIX 2

UNITY FEEDBACK SYSTEM ANALYSIS PROGRAM

Input R weighting function A computed output C
Read in H = time increments in sec
M = number of intervals needed

```
   A(1) = A(1)/2.
   N = 1
   C(1) = 0.
   PUNCH 54
   PUNCH 52, N, C(1)
   DO 5  I = 2, M
5  C(I) = A(1)*R(I)
   E(1) = 0.5*R(1)
   D = 1. +A(1)
   DO 8  I = 2, M
   K = 2
   DO 7  J = I , M
   C(J) = C(J)+E(I-1)*A(K)
7  K = K +1
   C(I) = C(I)/D
   PUNCH 52, I, C(I)
8  E(I) = R(I) - C(I)
52 FORMAT (5X I4, 5X E14.8)
54 FORMAT (/10X 7H OUTPUT/)
   END
```

APPENDIX 3

NON-UNITY FEEDBACK SYSTEM ANALYSIS PROGRAM

Input R Weighting functions A and B computed output C
Read in H = time increments in sec
M = number of intervals needed

```
   A(1) = A(1)/2.
   B(1) = B(1)/2.
   N = 1
   C(1) = 0.
   D(1) = 0.
   E(1) = R(1)*0.5
   PUNCH 54
   PUNCH 52, N, C(1)
   DO 5  I = 2, M
   D(I) = 0.
5  C(I) = A(1)*R(I)+E(1)*A(I)
   F = 1. +A(1)*B(1)
   DO 7  I = 2, M
   C(I) = C(I) - A(1)*D(I)
   C(I) = C(I)/F
   D(I) = D(I)+B(1)*C(I)
   E(I) = R(I) - D(I)
   L = I+1
   K = 2
   DO 6  J = L, M
   D(J) = D(J)+C(I)*B(K)
   C(J) = C(J)+E(I)*A(K)
6  K = K+1
7  PUNCH 52, I, C(I)
52 FORMAT (5X I4, 5X E14.8)
54 FORMAT (/10X 7H OUTPUT/)
   END
```

APPENDIX 4

OPEN LOOP SYNTHESIS PROGRAM

Input E Output C Computed weighting function A
Computed transfer function
Numerator $GN(1)+GN(2)S+GN(3)S^2$
Denominator $GD(1)+GD(2)S+GD(3)S^2$

```
      READ 51, DELTA
      IF (E(1)) 11, 15, 12,
 11   P = -E(1)
      GO  TO 13
 12   P = E(1)
 13   IF (P-DELTA) 15, 15, 14
 14   SK = C(1)/E(1)
      PUNCH 64
      PUNCH 59, SK
      DO 4 I = 1, M
  4   C(I) = C(I) - SK*E(I)
      E(1) = E(1)/2.
      DIMENSION X (5, 6),  Y(4, 5)
      DO 22 I = 2, 5
 22   X(I, 1) = C(I)*2.
      X(1, 1) = 0.
      X(1, 2) = 1.
      DO 23 I = 2, 5
 23   X(I, 2) = E(I)
      X(1, 3) = -4.
      DO 24  I = 2, 5
 24   X(I, 3) = E(I-1)*2.
      X(1, 4) = 6.
      X(3, 4) = E(1)*2.
      X(4, 4) = E(2)*2.
      X(5, 4) = E(3)*2.
      X(1, 5) = -4.
      X(4, 5) = E(1)*2.
      X(5, 5) = E(2)*2.
      X(1, 6) = 1.
      X(5, 6) = E(1)*2.
      DO 31 I = 1, 5
 31   Y(1, I) = X(5, I) -X(1, I)*X(5, 6)
      DO 32  I = 1, 4
```

```
32   Y(2,I) = X(4,I)*Y(1,5)-Y(1,I)*X(4,5)
     DO 33  I = 1,3
33   Y(3,I) = X(3,I)*Y(2,4)-Y(2,I)*X(3,4)
     DO 34  I = 1,2
34   Y(4,I) = X(2,I)*Y(3,3)-Y(3,I)*X(2,3)
     A(1) = Y(4,1)/Y(4,2)
     A(2) = (X(2,1)-A(1)*X(2,2))/X(2,3)
     A(3) = (X(3,1)-A(1)*X(3,2)-A(2)*X(3,3))/X(3,4)
     A(4) = (X(4,1)-A(1)*X(4,2)-A(2)*X(4,3)-A(3)*X(4,4))/X(4,5)
     A(5) = X(5,1)-A(1)*X(5,2)-A(2)*X(5,3)-A(3)*X(5,4)-A(4)*X(5,5)
     A(5) = A(5)/X(5,6)
     PUNCH 55
     DO 35 I = 1,5
35   PUNCH 52,I,A(I)
     DO 36 I = 6,M
36   A(I) = 0.
     A(1) = 0.5*A(1)
     DO 38  I = 1,5
     DO 37  J = 6,M
     K = J-I+1
37   A(J) = A(J) + A(I)*E(K)
38   CONTINUE
     DO 40  I = 6,M
     A(I) = (C(I) - A(I))/E(1)
     PUNCH 52,I,A(I)
     K = 2
     L = I+1
     DO 39  J = L,M
     A(J) = A(J)+A(I)*E(K)
39   K = K+1
40   CONTINUE
     GO TO 20
15   DO 16 I = 1,M
16   A(I) = 0.
     DO 18 I = 1,M
     A(I) = (C(I+1)-A(I))/E(2)
     K = 3
     L = I+1
     DO 17 J = L,M
     A(J) = A(J)+A(I)*E(K)
17   K = K+1
18   CONTINUE
```

```
      AT = 4.*(A(2)+A(4))-6.*A(3)-A(5)
      PUNCH 55
      N = 1
      PUNCH 52, N, AT
      DO 19  I = 2, M
19    PUNCH 52, I, A(I)
      SK = A(1) - AT/2.
      PUNCH 64
      PUNCH 59, SK
      A(1) = AT/2.
20    PUNCH 56
      A(1) = A(1)*2.
      DO 21 I = 1, M
      D(I) = A(I)/H
21    PUNCH 52, I, D(I)
      A(1) = A(1)/2.
      DIMENSION U(4), V(4), W(100), Z(100)
      DIMENSION GD(3), GN(3)
      DO 101  I = 1, 4
      V(I) = 0.
101   U(I) = 0.
      V(1) = A(1)
      W(1) = 0.
      DO 102  I = 2, M
      V(1) = V(1) + A(I)
      W(I) = W(I - 1) + H
      Z(I) = A(I)*W(I)
      V(2) = V(2) + Z(I)
      Z(I) = Z(I)*W(I)
      V(3) = V(3) + Z(I)
      Z(I) = W(I)*Z(I)
102   V(4) = V(4) + Z(I)
      V(2) = -V(2)
      V(3) = 0.5*V(3)
      V(4) = -V(4)/6.
      U(1) = V(1)
      S = V(2)*V(3)
      U(3) = (S-V(1)*V(4))/(V(1)*V(3)-V(2)*V(2))
      U(4) = (V(3)+V(2)*U(3))/(-V(1))
      U(2) = V(2)+V(1)*U(3)
      PUNCH 60
      DO 104  I = 1, 4
104   PUNCH 52, I, V(I)
      PUNCH 61
```

```
      DO 105 I = 1,4
  105 PUNCH 52,I,U(I)
      GN(1) = SK+U(1)
      GN(2) = SK*U(3)+U(2)
      GN(3) = SK*U(4)
      GD(1) = 1.
      GD(2) = U(3)
      GD(3) = U(4)
      PUNCH 62
      DO 106  I = 1,3
  106 PUNCH 52,I GN(I)
      PUNCH 63
      DO 107 I = 1,3
  107 PUNCH 52,I,GD(I)
   51 FORMAT (E8.0,E8.0,E8.0)
   52 FORMAT (5X I4,  5X E14.8)
   55 FORMAT (/1X 27H COMPUTED WEIGHTING FUNCTION/)
   56 FORMAT (/2X 26H COMPUTED IMPULSE RESPONSE/)
   59 FORMAT ( 20X E14.8/)
   60 FORMAT (/10X 2H Y/)
   61 FORMAT (/10X 2H X/)
   62 FORMAT (/10X 23H NUMERATOR COEFFICIENTS/)
   63 FORMAT (/10X 25H DENOMINATOR COEFFICIENTS/)
   64 FORMAT (/5X 39H PARALLEL CONSTANT TRANSFER
                FUNCTION IS)
      END
```

APPENDIX 5

EXAMPLE OF UNITY FEEDBACK SYSTEM ANALYSIS

Input: $r(t) = U(t)$
Impulse response: $g(t) = t\ e^{-t}$
Time increment: $h = 0.1$ sec.

Time	Computed Output	True Output	Error
0	.00000000	.00000000	.00000000
1	.00452418	.00467502	-.00015083
2	.01719474	.01746637	-.00027162
3	.03630559	.03667131	-.00036571
4	.06034332	.06077974	-.00043641
5	.08797965	.08846652	-.00048686
6	.11806193	.11858195	-.00052002
7	.14960215	.15014081	-.00053866
8	.18176502	.18231033	-.00054531
9	.21385542	.21439766	-.00054224
10	.24530550	.24583702	-.00053152
11	27566192	.27617691	-.00051499
12	.30457336	.30506759	-.00049423
13	.33177835	.33224893	-.00047058
14	.35709365	.35753887	-.00044522
15	.38040354	.38082260	-.00041906
16	.40164947	.40204243	-.00039296
17	.42082101	.42118848	-.00036747
18	.43794698	.43829017	-.00034319
19	.45308824	.45340859	-.00032035
20	.46633035	.46662967	-.00029932
21	.47777788	.47805806	-.00028018
22	.48754889	.48781192	-.00026303
23	.49577037	.49601827	-.00024790
24	.50257431	.50280910	-.00023479
25	.50809456	.50831814	-.00022358
26	.51246381	.51267802	-.00021421
27	.51581165	.51601817	-.00020652
28	.51826257	.51846297	-.00020040
29	.51993487	.52013048	-.00019561
30	.52093928	.52113143	-.00019215
31	.52137883	.52156853	-.00018970

Time	Computed Output	True Output	Error
32	.52134782	.52153607	-.00018825
33	.52093220	.52111977	-.00018757
34	.52020916	.52039672	-.00018756
35	.51924759	.51943563	-.00018804
36	.51810804	.51829704	-.00018900
37	.51684348	.51703371	-.00019023
38	.51549941	.51569114	-.00019173
39	.51411461	.51430797	-.00019336
40	.51272155	.51291661	-.00019506
41	.51134689	.51154370	-.00019681
42	.51001221	.51021076	-.00019855
43	.50873444	.50893465	-.00020021
44	.50752641	.50772818	-.00020177
45	.50639729	.50660055	-.00020326
46	.50535327	.50555787	-.00020460
47	.50439781	.50460362	-.00020581
48	.50353217	.50373904	-.00020687
49	.50275566	.50296351	-.00020785
50	.50206631	.50227494	-.00020863
51	.50146073	.50167001	-.00020928
52	.50093470	.50114451	-.00020981
53	.50048331	.50069356	-.00021025
54	.50010121	.50031180	-.00021059
55	.49978281	.49999361	-.00021080
56	.49952225	.49973321	-.00021096
57	.49931382	.49952483	-.00021101
58	.49915181	.49936283	-.00021102
59	.49903076	.49924172	-.00021096
60	.49894542	.49915629	-.00021087
61	.49889086	.49910162	-.00021076
62	.49886250	.49907311	-.00021061
63	.49885611	.49906654	-.00021043
64	.49885611	.49906654	-.00021043
65	.49889413	.49910418	-.00021005
66	.49893191	.49914177	-.00020986
67	.49897844	.49918808	-.00020964
68	.49903123	.49924068	-.00020945
69	.49908821	.49929750	-.00020929

APPENDIX 6

EXAMPLE OF UNITY FEEDBACK SYSTEM SYNTHESIS

Input: $r(t) = U(t)$
Output: $c(t) = 0.5 + 0.707\, e^{-t} \sin(t - 3\pi/r)$
Time increment: $h = 0.1$ sec.

Time	Computed Impulse Response	True Impulse Response	Error
0	.00168218	.00000000	.00168218
1	.09182608	.09048374	.00134233
2	.16488101	.16374614	.00133487
3	.22314089	.22224546	.00089543
4	.26889979	.26812801	.00077178
5	.30386574	.30326532	.00060042
6	.32981960	.32928698	.00053262
7	.34801640	.34760971	.00040669
8	.35983960	.35946317	.00037643
9	.36619360	.36591269	.00028091
10	.36815380	.36787944	.00027436
11	.36635680	.36615819	.00019861
12	.36164060	.36143305	.00020755
13	.35443620	.35429132	.00014488
14	.34539860	.34523574	.00016286
15	.33480380	.33469523	.00010857
16	.32316660	.32303443	.00013217
17	.31064500	.31056199	.00008301
18	.29764840	.29763800	.00011040
19	.28424440	.28418037	.00006403
20	.27076420	.27067056	.00009364
21	.25720680	.25715849	.00004831
22	.24384740	.24376694	.00008046
23	.23063040	.23059534	.00003506
24	.21779200	.21772309	.00006891
25	.20523640	.20521249	.00002391
26	.19317000	.19311130	.00005870
27	.18146880	.18145488	.00001392
28	.17031680	.17026817	.00004863
29	.15957240	.15956734	.00000506
30	.14940120	.14936120	.00004000
31	.13965000	.13965252	-.00000252
32	.13046980	.13043905	.00003075

Time	Computed Impulse Response	True Impulse Response	Error
33	.12170600	.12171445	-.00000845
34	.11349160	.11346911	.00002249
35	.10567620	.10569084	-.00001464
36	.09838240	.09836540	.00001699
37	.09145580	.09147704	-.00002124
38	.08502060	.08500893	.00001166
39	.07891760	.07894345	-.00002585
40	.07327120	.07326255	.00000864
41	.06791640	.06794796	-.00003156
42	.06298640	.06298142	.00000497
43	.05831020	.05834480	-.00003460
44	.05402240	.05402029	.00000210
45	.04995380	.04999048	-.00003668
46	.04623800	.04523844	-.00000044
47	.04270940	.04274780	-.00000238
49	.03644940	.03648825	-.00003885
50	.03368640	.03368973	-.00000333
51	.03105340	.03109340	-.00004000
52	.02868080	.02868613	-.00000533
53	.02641780	.02645544	-.00003764
54	.02438240	.02438953	-.00000713
55	.02244040	.02247724	-.00003684
56	.02070080	.02070803	-.00000723
57	.01903520	.01907200	-.00003680
58	.01755460	.01755981	-.00000521
59	.01612460	.01616272	-.00003812
60	.01486900	.01487251	-.00000351
61	.01364520	.01368149	-.00003629
62	.01257900	.01258246	-.00000346
63	.01153260	.01156872	-.00003612
64	.01063180	.01063396	-.00000216
65	.00973860	.00977235	-.00003375
66	.00897600	.00897842	-.00000242
67	.00821420	.00824710	-.00003290
68	.00757400	.00757367	.00000032
69	.00692100	.00695371	-.00003271