AN ABSTRACT OF THE THESIS OF

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(Name) (Degree)

in MATHEMATICS presented on January 25, 1971
(Major) (Date)

Title: INVESTIGATIONS CONCERNING HEAT TRANSFER INVOLVING WEDGE-SHAPED CONFIGURATIONS

Abstract approved: Redacted for privacy
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The temperature field inside a "truncated" wedge is investigated. For this purpose the two dimensional Green's function of the heat equation for this domain is established. Further applications of these results center around a problem of heat conduction investigated by Lebedev.

Investigations Concerning Heat Transfer Involving Wedge-Shaped Configurations

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

June 1971

APPROVED:

Redacted for privacy

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Date thesis is presented January 25, 1971

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INVESTIGATIONS CONCERNING HEAT TRANSFER INVOLVING WEDGE-SHAPED CONFIGURATIONS

I. INTRODUCTION

The heat conduction equation $\kappa \Delta u = \frac{\partial u}{\partial t}$ for a homogeneous solid in a wedge shaped region has been treated in a number of contributions. The majority of these are based on subjecting the heat equation to a Laplace transform and obtaining the final solution in the form of a contour integral, which in turn, can often be expressed as an infinite series by applying the residue theorem [2, 3]. An approach suggested here is based upon a connection between the Green's function of the heat equation and the Helmholtz equation. The region considered is that of a homogeneous solid bounded by two coaxial cylindrical surfaces $\rho = a$ and $\rho = b$, $(a < b)$, and two planes $\varphi = 0$ and $\varphi = \alpha$, under the assumption that the temperature field is independent of the z coordinate (two dimensional problem). The boundary condition requires that the temperature vanish at the boundaries, i.e., $u = 0$ for $\rho = a, b$ and $\varphi = 0, \alpha$, (isothermic problem).

The procedure is as follows: First we determine Green's function G_1 for the modified Helmholtz equation $\Delta u + k^2 u = 0$ (time harmonic case) putting $k = -i\gamma$. Then the Green's function for the heat conduction equation is given by taking the inverse Laplace transform of G_1 with respect to $\sqrt{\gamma}$. A further integration involving the Green's function would lead to the solution of $\kappa \Delta u = \frac{\partial u}{\partial t}$ in the case where an arbitrary initial temperature distribution throughout the solid is given.

In Chapter II we show the connection between the heat conduction equation and the modified Helmholtz equation. We are also concerned with expressing solutions of the heat conduction problems in terms of the eigenvalue and the eigenfunctions of Helmholtz's equation.

Chapter III contains applications to wedge-shaped domains, especially the problem of the heat conduction inside a wedge with a constant initial temperature. This problem is treated in great detail applying the methods explained in Chapter II. The results presented here are equivalent with those given by N. N. Lebedev and I. P. Skalskaya [7] which were obtained by Laplace and Kontorovich-Lebedev transform methods.

II. CONNECTIONS BETWEEN THE HEAT CONDUCTION EQUATION AND HELMHOLTZ'S EQUATION IN TWO AND THREE DIMENSIONS

Helmholtz's Equation

(1) $$\Delta u + k^2 u = 0$$

governs time harmonic wave phenomena [time dependency $e^{i\omega t}$, $k = \frac{\omega}{c}$; ω is the frequency, k is the wave number, and c is the phase velocity]. If solutions of (1) are sought which depend only on the distance between two given points P (point of observation) and Q (location of the source) one has:

(2) $$u = \frac{1}{4} i H_0^{(2)}(k\,\overline{PQ})$$

(two dimensional case independent of z representing the radiation of a divergent cylindrical wave with the axis parallel to the z axis and passing through the point Q in the xy plane)

(3) $$u = -\frac{1}{4\pi}\frac{e^{-ik\overline{PQ}}}{\overline{PQ}}$$

(three dimensional case, representing the radiation of a divergent spherical wave due to a point source located at Q). In (2) $H_0^{(2)}(k\rho)$ denotes the second Hankel function of order zero. The factors $\frac{1}{4}i$

and $-\frac{1}{4\pi}$ in (2) and (3) are chosen such that the "yield" of the source is unity. The expressions (2) and (3) are also referred to as the <u>free space Green's functions</u> of (1) in two or three dimensions. Their property is that they satisfy (1)

$$\Delta u + k^2 u = 0$$

everywhere except if P is at Q.

Furthermore

$$u - \frac{1}{2\pi} \log \overline{PQ}$$

regular as $P \to Q$ in (2)

$$u + \frac{1}{4\pi} \frac{1}{\overline{PQ}}$$

regular as $P \to Q$ in (3).

The differentiations in Δu apply to the point of observation P with fixed Q, while the term regular means the property is continuous everywhere together with its first and second partial derivatives.

The Heat Conduction Equation

Similar expressions with given singular behavior are sought for solutions of the heat conduction equation

(4) $$\Delta v = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \quad (\kappa \text{ is a constant})$$

Such solutions are [10, p. 59]

(5) $$v = v(P, Q, t) = \frac{1}{4\pi\kappa t} e^{-\frac{(\overline{PQ})^2}{4\kappa t}}$$

in two dimensions (line source at Q)

(6) $$v = v(P, Q, t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-\frac{(\overline{PQ})^2}{4\kappa t}}$$

in three dimensions (point source at Q) or, if Cartesian coordinates are introduced, $P(x, y, z)$; $Q(x', y', z')$

(7) $$v(P, Q, t) = \frac{1}{4\pi\kappa t} e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}} \quad \text{(2 dimensions)}$$

(8) $$v(P, Q, t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}$$

(3 dimensions).

The properties of (7) and (8) are respectively

$$\frac{1}{4\pi\kappa t}\iint_{-\infty}^{\infty} e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}\,dxdy = 1$$

$$\frac{1}{(4\pi\kappa t)^{3/2}}\iiint_{-\infty}^{\infty} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}\,dxdydz = 1$$

Furthermore in (7) and (8) for $t = 0$

$$v(P,Q,0) = 0 \quad \text{for all} \quad \overline{PQ} \neq 0$$

$$v(P,Q,0) \rightarrow \infty \quad \text{as} \quad P \rightarrow Q$$

This means that (7) and (8) have the character of a δ function. Again the solutions (7) and (8) are referred to as the free space Green's functions of (6).

The physical meaning of, for instance (8), is the temperature field at a point P at a time t when at $t = 0$, $v(P,Q,0) = 0$ for all P with $\overline{PQ} \neq 0$. This represents the equalization of the temperature due to an instantaneous point source or heat pole located at Q. A similar explanation holds for the instantaneous infree space line source as expressed by (7).

The Modified Helmholtz Equation

For the following it is necessary to replace temporarily the wave number k in (1) by another parameter putting $k = -i\gamma$, where for the time being γ is regarded as real and positive. This means not the Helmholtz's equation (1) but the "modified" Helmholtz equation

$$\Delta \bar{u} - \gamma^2 \bar{u} = 0 \tag{9}$$

is considered and the emphasis will not center so much upon the character of possible solutions $\bar{u}$ of (9) but upon the parameter γ, especially solutions $\bar{u}$ of (9) which are analytic functions of γ. Physically this means the transition from a periodic problem as expressed in (1) to an exponential decay problem characterized by (9). The basic reason is that while solutions of (1) have an oscillatory behavior, the solutions of (9) are monotonic.

Having obtained solutions of (9), one returns to solutions of (1) upon replacing γ by ik, provided that the solution $\bar{u}$ of (9) [considered as a possible analytic function of γ] includes the positive imaginary γ-axis. This method removes the often serious difficulties regarding the convergence of certain intermediate operations. The merits of this procedure have been demonstrated in [9].

Moreover, we wish in general, as in the case of this thesis, to derive solutions of (1), where the boundaries of certain configurations stretch into infinity, since the limiting case of finite boundaries for solutions of (1) are not always suitable.

To illustrate this a simple example is used. Consider the 1st Green's function of $\Delta u + k^2 u = 0$ for the interior of a sphere of radius a under the boundary condition $u = 0$ at the surface $r = a$ of the sphere. The point source Q is the center of the sphere. The 1st Green's function is easily seen to be

$$G_1 = \frac{1}{4\pi r} \frac{\sin[k(r-a)]}{\sin(ka)}$$

One should expect that in the limiting case $a \to \infty$ G_1 tends to the free space Green's function $-\frac{1}{4\pi r} e^{-ikr}$. But G_1 does not tend to a limit as $a \to \infty$. The transition to an exponential decay problem $k \to -i\gamma$ would lead to the 1st Green's function for $\Delta\bar{u} - \gamma^2\bar{u} = 0$ and

$$\bar{G}_1 = \frac{1}{4\pi r} \frac{\sinh[\gamma(r-a)]}{\sinh(\gamma a)}$$

Assuming $\gamma > 0$ we obtain in the limit $a \to \infty$

$$\begin{aligned} (\bar{G}_1)_{a\to\infty} & \frac{1}{4\pi r} [\sinh(\gamma r) - \cosh(\gamma r)] \\ &= -\frac{1}{4\pi r} e^{-\gamma r} \end{aligned}$$

returning to the wave problem with $\gamma = ik$ the above expression becomes

$$(G_1)_{r\to\infty} \to -\frac{1}{4\pi r} e^{-ikr}$$

the free space Green's function for $\Delta u + k^2 u = 0$ as given in (3).

The free space Green's functions of (9) become now

$$\bar{u} = -\frac{1}{2\pi} K_0(\gamma \overline{PQ}) \qquad \text{(line source)} \tag{10}$$

$$\bar{u} = -\frac{1}{4\pi} \frac{e^{-\gamma \overline{PQ}}}{\overline{PQ}} \qquad \text{(point source)} \tag{11}$$

with the properties (K_0 denotes now the modified Hankel function of order zero), $\Delta \bar{u} - \gamma^2 \bar{u} = 0$ everywhere except if P is at Q and

$$u - \frac{1}{2\pi} \log \overline{PQ}$$

regular as $P \to Q$ (line source)

$$u + \frac{1}{4\pi} \frac{1}{\overline{PQ}}$$

regular as $P \to Q$ (point source).

[Note that $K_0(\gamma \overline{PQ}) \sim -\log \overline{PQ}$ for small $\overline{PQ}$.]

Connections Between Heat Conduction-Equation and Modified Helmholtz Equation Solutions

We are now in a position to use the inverse Laplace transforms of certain known formulas [Appendix formulas C'' and C'''].

$$L_\gamma^{-1}[K_0(2\sqrt{\beta\gamma})] = \frac{1}{2} t^{-1} e^{-\beta/t} \qquad \operatorname{Re} \gamma > 0 \tag{12}$$

$$L_\gamma^{-1}[e^{-2\sqrt{\beta\gamma}}] = \pi^{-1/2} t^{-3/1} e^{-\beta/t}, \qquad \text{Re } \gamma > 0 \tag{13}$$

The index γ in (12) and (13) means that the inversion has to be performed with respect to the wave parameter γ or, instead of (12) and (13) using the explicit inversion formulas

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_0(2\sqrt{\beta\gamma})e^{\gamma t} d\gamma = \frac{1}{2}t^{-1} e^{-\beta/t} \tag{14}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-2\sqrt{\beta\gamma}} e^{\gamma t} d\gamma = \pi^{-1/2} t^{3/2} e^{-\beta/t} \tag{15}$$

In both cases $c > 0$. Replace in both formulas t by $4\kappa t$, $\beta = (\overline{PQ})^2$ and substitute the integration variable γ by $\frac{\gamma}{4}$. (Here $\overline{PQ}$ is the distance between two axes or the distance between two points P and Q.) The result is:

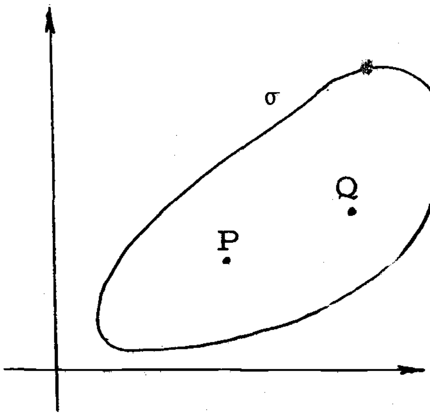
$$\frac{1}{4\pi\kappa t} e^{-\frac{(\overline{PQ})^2}{4\kappa t}} = -L_\gamma^{-1}[-\frac{1}{2\pi} K_0(\sqrt{\gamma}\,\overline{PQ})], \qquad t \rightarrow \kappa t \tag{16}$$

and

$$\frac{1}{(4\pi\kappa t)^{3/2}} e^{-\frac{(\overline{PQ})^2}{4\kappa t}} = -L_\gamma^{-1}[-\frac{1}{4\pi} \frac{e^{-\sqrt{\gamma}\,\overline{PQ}}}{\overline{PQ}}], \qquad t \rightarrow \kappa t \tag{17}$$

In both cases $t \rightarrow \kappa t$ means that after the inversion process, t has to be replaced by κt. The l.h.s. of (16) and (17) represent the two or three dimensional <u>free space Green's functions</u> (5) and (6) of the heat conduction equation (4) while on the r.h.s. the inverse Laplace transforms of the free space Green's functions (10) and (11) of the modified Helmholtz equation (9) appear with γ replaced by $\sqrt{\gamma}$.

Definition of both Green's functions of (4) and (9) for the interior problem. Since only interior problems are considered here, we examine the interior D of a simple closed surface σ. P represents the point of observation and Q characterizes the location of an instantaneous line or point source. Then, for the first and second Green's functions $g_1(P, Q, t)$ and $g_2(P, Q, t)$ of (4)



1) $\Delta g = \frac{1}{\kappa} \frac{\partial g}{\partial t}$, everywhere for $t > 0$

2) For $t = 0$, $g = 0$ everywhere except if P is at Q and at this point g have the character of a δ function (instantaneous source)

3a) $g_1 = 0$ if P is on σ (first Green's functions, isothermic boundary condition)

3b) $\frac{\partial g_2}{\partial n} = 0$ if P is on σ (second Green's function, adiabatic boundary condition). The Δ operation in g is performed with respect to the coordinates of P. In the case of a finite number of discretely distributed instantaneous sources a summation over all sources Q_v has to be performed, while in the case of an initial temperature distribution $f(Q)$ one would have for the temperature field in P at a time $t > 0$

$$(18) \qquad v(P,Q,t) = \int g(P,Q,t)\, f(Q)\, d\tau_Q$$

The integration is taken over all Q and $d\tau_Q$ is a volume element at Q. The definition of $G(P,Q,\gamma)$ of the modified Helmholtz equation (9) for the interior of σ is

1) $\Delta\overline{G} - \gamma^2\overline{G} = 0$ everywhere except if P is at Q

2) $\overline{G} - \frac{1}{2\pi}\log(\overline{PQ})$ regular everywhere inside σ

3a) $\overline{G}_1 = 0$ if P is on σ

3b) $\frac{\partial G_2}{\partial n} = 0$ if P is on σ

As mentioned before the (so far purely real) parameter γ will be emphasized. It is now concluded from (16) and (17) that if the Green's function $\overline{G}(P,Q,\gamma)$ for the modified Helmholtz equation (9) is known, the corresponding Green's functions for the heat equation is

$$(19) \qquad g_{\substack{1\\2}}(P,Q,t) = -L_\gamma^{-1}[\overline{G}_{\substack{1\\2}}(P,Q,\sqrt{\gamma}], \quad t \to \kappa t$$

Expression of Green's function for the heat equation by means of the Eigenvalues and Eigenfunctions for Helmholtz's equation. The well known expansion theorem for the Green's functions of the Helmholtz equation (1) is [10, p. 183)

$$(20) \qquad G = \sum_{\ell} \frac{U_\ell(P)U_\ell^*(Q)}{k^2 - k_\ell^2}$$

Here the U_ℓ are the normalized eigenfunctions of $\Delta u + k^2 u = 0$ and k_ℓ are the eigenvalues for the boundary value problem under consideration. The sum in (20) is a triple sum (three dimension) or a double sum (two dimensional problem). Here $U_\ell^*(Q)$ means the conjugate complex value of the ℓth eigenfunction U_ℓ with respect to the source point Q. Let U_ℓ denote an eigenfunction, not necessarily normalized, then from

$$(21) \qquad N^2 = \int U_\ell(P)U_\ell^*(P)d\tau_P$$

the integral extended over the whole space interior to σ. With this, and replacing k by $-i\gamma$ one obtains for the modified Green's functions

$$(22) \qquad \overline{G}(P, Q, \gamma) = - \sum_{\ell} \frac{1}{N^2} \frac{U_\ell(P)U_\ell^*(Q)}{\gamma^2 + k_\ell^2}$$

But replacing γ by $\sqrt{\gamma}$ and taking the inverse Laplace transform with respect to γ one has, since

$$L_{\gamma}^{-1}\left(\frac{1}{\gamma+k_{\ell}^{2}}\right)_{t=\kappa t} = e^{-k_{\ell}^{2}\kappa t}$$

The Green's functions for the heat equation becomes

$$g(P,Q,t) = \sum_{\ell} \frac{1}{N^{2}} U_{\ell}(P)U_{\ell}^{*}(Q)e^{-k_{\ell}^{2}\kappa t} \tag{23}$$

expressed in terms of the eigenvalues U_{ℓ} and the eigenfunctions of $\Delta u + ku^{2} = 0$.

In (23), the two dimensional eigenfunctions (double sum) has to be taken for the case where the excitation is due to a line source or the case where the temperature field inside a closed surface is due to a heat pole. A separate case (heat pole inside a closed cylindrical surface whose cross section forms a curve C) can be treated as follows. Take the identities [5, p. 94, Form. 46] with $\mu = -\frac{1}{2}$ and $v = 0$ to get

$$-\frac{1}{2\pi}\frac{e^{-\gamma\sqrt{\rho^{2}+z^{2}}}}{\sqrt{\rho^{2}+z^{2}}} = \frac{1}{\pi} - \frac{1}{2\pi}\int_{0}^{\infty} K_{0}(\rho\sqrt{\gamma^{2}+\lambda^{2}}) \cos(\lambda z)d\lambda . \tag{24}$$

The expression on the l. h. s. of (24) represents the free space Green's function (11) of a point source at the origin, while the expression

$-\frac{1}{2\pi} K_0(\rho\sqrt{\gamma^2+\lambda^2})$ represents the free space Green's function (10) of a line source along the z-axis with γ replaced by $\sqrt{\gamma^2+\lambda^2}$. From this relation it can be concluded that the point source solution inside an infinite cylinder (pt. source located in the xy-plane) can be obtained from the line source solution by replacing γ by $\sqrt{\gamma^2+\lambda^2}$, multiplying by $\cos(\lambda z)$, and integrating over λ from 0 to ∞. Hence, from (22), with

$$\int_0^\infty \frac{\cos(xy)}{c^2+x^2}\,dx = \frac{\pi}{2c} e^{-cx}, \qquad c, x > 0$$

one obtains

$$\overline{G} = \frac{\pi}{2\sqrt{\gamma^2+k_\ell^2}}\, e^{-|z|\sqrt{\gamma^2+k_\ell^2}}$$

Further, by C'''' (Appendix)

$$L_\gamma^{-1} \frac{e^{-|z|\sqrt{\gamma+k_\ell^2}}}{\sqrt{\gamma+k_\ell^2}} = (\pi\kappa t)^{-\frac{1}{2}} e^{-\frac{z^2}{4\kappa t}} e^{-k_\ell^2 \kappa t}$$

and hence for this case

$$(25) \qquad g(x,y,z,t) = \frac{1}{2}(\pi\kappa t)^{-\frac{1}{2}} e^{-\frac{(z-z')^2}{4\kappa t}} \sum_\ell \frac{1}{N^2} U_\ell(P) U_\ell^*(Q)\, e^{-k_\ell^2 \kappa t}$$

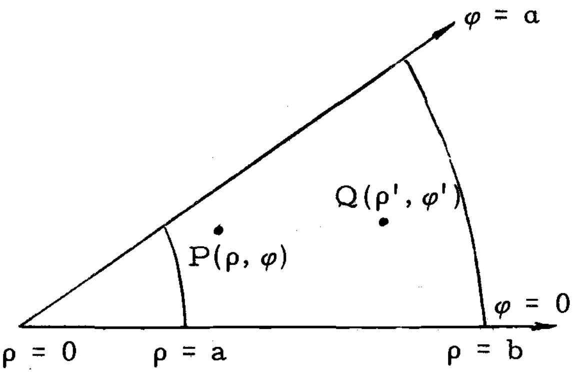
if the point source is at z'. The series is the same as in (23). The case of the line source can be regained from (25) by integrating over all z' from $-\infty$ to $+\infty$. But

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

and with this one regains the line source field.

III. APPLICATIONS

A. The first Green's function for the domain $0 \leqq \varphi \leqq \alpha$, $a \leqq \rho \leqq b$, $-\infty < z < \infty$, and for the two dimensional case (U independent of z) [P = point of observation, Q = location of line source].



$U = U(\rho, \varphi)$; $\Delta u + k^2 u = 0$

A solution of $\Delta u + k^2 u = 0$ by the method of the separation of variables $U = f_1(\rho) f_2(\varphi)$ with

$$f_1(\rho) = \begin{cases} J_\nu(k\rho) \\ Y_\nu(k\rho) \\ H_\nu^{(1),(2)}(k\rho) \end{cases} \quad ; \quad f_2(\varphi) = e^{\pm i\nu\varphi}$$

The J_ν, Y_ν, $H_\nu^{(1),(2)}$ are the Bessel functions of the first order and the two Hankel functions respectively. So far, ν is an arbitrary separation parameter. Hence, a simple solution of $\Delta u + k^2 u = 0$, regular in $a \leqq \rho \leqq b$; $0 \leqq \varphi \leqq \alpha$ is

$$U = [J_\nu(k\rho) + C Y_\nu(k\rho)][A \cos(\nu\varphi) + B \sin(\nu\varphi)] \tag{26}$$

The boundary conditions are

$$U = 0 \quad \text{at} \quad \varphi = 0, \ \varphi = \alpha, \quad \text{for} \quad a \leqq \rho \leqq b$$

$$U = 0 \quad \text{at} \quad \rho = a, \ \rho = b, \quad \text{for} \quad 0 \leqq \varphi \leqq \alpha$$

The first equation leads to $A = 0$, and $\nu = n\frac{\pi}{\alpha}$, $n = 1, 2, 3, \ldots$.
The second equation yields

$$J_\nu(ka) + CY_\nu(ka) = 0$$

$$J_\nu(kb) + CY_\nu(kb) = 0$$

or

(27) $$C = -\frac{J_\nu(ka)}{Y_\nu(ka)} = -\frac{J_\nu(kb)}{Y_\nu(kb)}, \qquad \nu = n\frac{\pi}{\alpha}$$

This leads to the determination of the eigenvalues

(28) $$J_\nu(ka)Y_\nu(kb) - Y_\nu(ka)J_\nu(kb) = 0$$

$$\nu = n\frac{\pi}{\alpha}$$

It is known that [5, p. 62] the function

(29) $$F_\nu(z) = J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$$

has for $\nu, \lambda > 0$, real and simple zeros for z only. Furthermore, while $J_\nu(z)$ and $Y_\nu(z)$ are many valued, the cross product (29) represents an even, regular function of z. This can be seen using the relations [5, p. 80]

$$J_\nu(ze^{\pm im\pi}) = e^{\pm im\pi\nu} J_\nu(z)$$

$$Y_\nu(ze^{\pm im\pi}) = e^{\mp im\pi\nu} Y_\nu(z) + 2i \frac{\sin(m\pi\nu)}{\sin(\nu\pi)} \cos(\pi\nu) J_\nu(z)$$

Hence, for $m = 1$

$$F_\nu(ze^{\pm i\pi}) = F_\nu(z)$$

This means that the solution of (28) has real values $\pm k_{n,m}$ only, if $k_{n,m}$ denotes the mth positive root of (29). Equation (29) for $F_\nu(z)$ can also be represented in a different way using the relation

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

(30) $$F_\nu(z) = i[H_\nu^{(2)}(\lambda z))J_\nu(z) - J_\nu(\lambda z)H_\nu^{(2)}(\lambda z)]$$

And with z replaced by $ze^{-i\frac{\pi}{2}}$ remembering that

$$J_\nu(ze^{-i\frac{\pi}{2}}) = e^{-i\frac{\pi}{2}\nu} I_\nu(z), \quad H_\nu^{(2)}(ze^{-i\frac{\pi}{2}}) = \frac{2i}{\pi} e^{i\frac{\pi}{2}\nu} K_\nu(z)$$

(The I_ν and K_ν are the "modified" Bessel function of the first and second kind respectively.)

(31) $$F_\nu(ze^{-i\frac{\pi}{2}}) = G_\nu(z) = -\frac{2}{\pi}[I_\nu(z)K_\nu(\lambda z) - I_\nu(\lambda z)K_\nu(z)] .$$

Hence the zeros of $G_\nu(z)$ for $\lambda, \nu > 0$, are all purely imaginary. Also by [5, p. 80]

$$G_\nu(ze^{\pm i\pi}) = G_\nu(z)$$

Furthermore, $G_\nu(z)$ is an even regular function of z. Inserting (27) into (26), the non-normalized eigenfunctions become

$$(32) \qquad U = \sin\left(\frac{n\pi}{\alpha}\varphi\right) U_{n,m}(k\rho)$$

with

$$(33) \qquad U_{n,m}(k_{n,m}\rho) = J_\nu(k_{n,m}\rho)Y_\nu(k_{n,m}b) - J_\nu(k_{n,m}b)Y_\nu(k_{n,m}\rho)$$

$$= F_\nu(k_{n,m}\rho, \frac{b}{a})$$

by (29) or also

$$(34) \qquad U_{n,m}(k_{n,m}\rho) = J_\nu(k_{n,m}\rho)Y_\nu(k_{n,m}a) - Y_\nu(k_{n,m}\rho)J_\nu(k_{n,m}a)$$

$$= F_\nu(k_{n,m}\rho, \frac{a}{b})$$

A factor $[Y_\nu(k_{n,m}b)]^{-1}$ and $[Y_\nu(k_{n,m}a)]^{-1}$ respectively in (33) and (34) is suppressed here since these factors do not concern the normalized eigenfunctions. Here $\nu = \frac{n\pi}{\alpha}$ and the eigenvalues $k_{n,m}$ are the solution of

$$(35) \qquad F_\nu(ka, \frac{b}{a}) = J_\nu(ka)Y_\nu(kb) - J_\nu(kb)Y_\nu(ka) = 0$$

Normalization of the eigenfunctions. One forms

$$(36)\qquad N^2 = \int_{\varphi=0}^{a} \int_{\rho=a}^{b} [U_{n,m}(k_{n,m}\rho)]^2 \rho \sin^2(\frac{n\pi}{a})\rho d\rho d\varphi$$

$$= \frac{1}{2} a \int_{\rho=a}^{b} [U_{n,m}(k_{n,m}\rho)]^2 \rho d\rho$$

By (33)

$$\int_a^b U_{n,m}^2 \rho d\rho = \int_a^b [J_\nu(k_{n,m}\rho) Y_\nu(k_{n,m}b) - J_\nu(k_{n,m}b) Y_\nu(k_{n,m}\rho)]^2 \rho d\rho$$

or, introducting ω, and putting $\omega = k_{n,m}$

$$(37)\qquad \int_a^b U_{n,m}^2(\omega\rho)\rho d\rho = Y_\nu^2(\omega b)\int_a^b J_\nu^2(\omega x) x dx + J_\nu^2(\omega b)\int_a^b Y_\nu^2(\omega x) x dx - 2J_\nu(\omega b) Y_\nu(\omega b)\int_a^b J_\nu(\omega x) Y_\nu(\omega x) x dx$$

$$\nu = \frac{n\pi}{a}, \qquad \omega = k_{n,m}$$

In order to evaluate (37) the following formulas are used [5, p. 90]

(38)

$$\int J_\nu^2(\omega x)x\,dx = \frac{1}{2}x^2[J_\nu^2(\omega x) - J_{\nu-1}(\omega x)J_{\nu+1}(\omega x)]$$

$$\int Y_\nu^2(\omega x)x\,dx = \frac{1}{2}x^2[Y_\nu^2(\omega x) - Y_{\nu-1}(\omega x)Y_{\nu+1}(\omega x)]$$

$$\int J_\nu(\omega x)Y_\nu(\omega x)x\,dx = \frac{1}{4}x^2[2J_\nu(\omega x)Y_\nu(\omega x) - J_{\nu+1}(\omega x)Y_{\nu-1}(\omega x) - J_{\nu-1}(\omega x)Y_{\nu+1}(\omega x)]$$

Hence

(39)

$$\begin{aligned}
&\int_a^b [J_\nu(\omega\rho)Y_\nu(\omega b) - J_\nu(\omega b)Y_\nu(\omega\rho)]^2\rho\,d\rho \\
&= \frac{1}{2}b^2Y_\nu^2(\omega b)[J_\nu^2(\omega b) - J_{\nu-1}(\omega b)J_{\nu+1}(\omega b)] \\
&\quad + \frac{1}{2}b^2J_\nu^2(\omega b)[Y_\nu^2(\omega b) - Y_{\nu+1}(\omega b)Y_{\nu-1}(\omega b)] \\
&\quad - \frac{1}{2}b^2J_\nu(\omega b)Y_\nu(\omega b)[2J_\nu(\omega b)Y_\nu(\omega b) - J_{\nu+1}(\omega b)Y_{\nu-1}(\omega b) \\
&\qquad\qquad - J_{\nu-1}(\omega b)Y_{\nu+1}(\omega b)] \\
&\quad - \frac{1}{2}a^2Y_\nu^2(\omega b)[J_\nu^2(\omega a) - J_{\nu-1}(\omega a)J_{\nu+1}(\omega a)] \\
&\quad - \frac{1}{2}a^2J_\nu^2(\omega b)[Y_\nu^2(\omega a) - Y_{\nu-1}(\omega a)Y_{\nu+1}(\omega a)] \\
&\quad + \frac{1}{2}a^2J_\nu(\omega b)Y_\nu(\omega b)[2J_\nu(\omega a)Y_\nu(\omega a) - J_{\nu+1}(\omega a)Y_{\nu-1}(\omega a) \\
&\qquad\qquad - J_{\nu-1}(\omega a)Y_{\nu+1}(\omega a)]
\end{aligned}$$

$$\nu = \frac{n\pi}{\alpha}, \quad \omega = k_{n,m}$$

This somewhat lengthy expression can be considerably simplified by making use of the eigenvalue condition (28) and of certain identities involving cross prodicts of Bessel and Neumann functions [5, p. 80]. These intermediate operations are omitted here. One arrives finally at the results;

$$(40)\qquad \int_a^b [J_\nu(\omega\rho)Y_\nu(\omega b) - J_\nu(\omega b)\, Y_\nu(\omega\rho)]^2 \rho d\rho$$

$$= \frac{2}{\pi^2\omega^2}\left[1 - \frac{J_\nu^2(\omega b)}{J_\nu^2(\omega a)}\right]$$

$$\omega = k_{n,m}; \qquad \nu = \frac{n\pi}{a}.$$

Hence, by (36)

$$(41)\qquad N^2 = \frac{a}{\pi^2\omega^2}\,\frac{J_\nu^2(\omega a) - J_\nu^2(\omega b)}{J_\nu^2(\omega a)}$$

Hence, the normalized eigenfunctions U_ℓ of the domain under consideration are

$$(42)\quad U_{n,m}(\rho,\varphi) = \frac{1}{N}\sin\left(\frac{n\pi}{a}\varphi\right)$$

$$\times\, [J_\nu(k_{n,m}\rho)Y_\nu(k_{n,m}b) - J_\nu(k_{n,m}b)Y_\nu(k_{n,m}\rho)],$$

$$\nu = \frac{n\pi}{a}$$

with N given by (41). Now, with the preceding results it is possible to represent Green's function (23) for the heat conduction equation using (41) and (33) as

$$(43)\qquad g_1(P,Q,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{N^2} U_{n,m}(k_{n,m}\rho)U_{n,m}(k_{n,m}\rho') \times \sin\left(\frac{n\pi}{\alpha}\varphi\right)\sin\left(\frac{n\pi}{\alpha}\varphi^1\right)e^{-k_{n,m}^2 \kappa t}$$

It is of interest to investigate the first Green's function $\overline{G}_1$ for the modified Helmholtz equation as given by (22). This, again leads to a double sum

(44)

$$\overline{G}_1(P,Q,\gamma) = -\frac{2}{\alpha}\sum_{n=1}^{\infty} \sin(n\varphi)\sin(n\varphi') \times \sum_{m=1}^{\infty} \frac{U_{n,m}(k_{n,m}\rho)U_{n,m}(k_{n,m}\rho')}{[\gamma^2+k_{n,m}^2]\int_a^b U_{n,m}^2(k_{n,m}\rho)\rho d\rho}$$

The value of the integral in the denominator of (44) is given by (40). The sum over m in (44) can be given without explicit knowledge of

$$\int_a^b [U_{n,m}(k_{n,m}\rho)]^2\rho d\rho$$

The result is

$$(45)\qquad \sum_{m=1}^{\infty} \frac{U_{n,m}(k_{n,m}\rho)U_{n,m}(k_{n,m}\rho')}{[\gamma^2+k_{n,m}^2]\int_a^b U_{n,m}^2(k_{n,m}\rho)\rho d\rho}$$

$$= [I_\nu(\gamma a)K_\nu(\gamma b)-I_\nu(\gamma b)K_\nu(\gamma a)]^{-1}[I_\nu(\gamma\rho)K_\nu(\gamma a)-I_\nu(\gamma a)K_\nu(\gamma\rho)]$$

$$\times [I_\nu(\gamma\rho')(K_\nu(\gamma b)-I_\nu(\gamma b)K_\nu(\gamma\rho')]$$

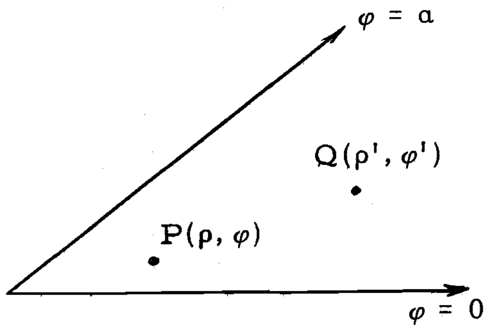
$$\nu = \frac{n\pi}{\alpha} \qquad a < \rho < \rho' < b, \quad \text{i.e.,} \quad \rho < \rho'$$

and the same formula with ρ and ρ' interchanged if $\rho > \rho'$. Thus the Green's function $\overline{G}_1$ given by (44) is reduced to a single series involving the summation over only n, and the zeros $k_{n,m}$ no longer appear. If (45) is inserted in (44), it is necessary, in order to obtain the first Green's function g_1 for the heat conduction equation by (19), to evaluate

$$(46)\qquad L_\gamma^{-1}\{\overline{G}_1(P,Q,\sqrt{\gamma})\}, \qquad t \to \kappa t$$

i.e., the inverse Laplace transform of the right side of (45) with γ replaced by $\sqrt{\gamma}$. This inversion can be performed by using the well known inversion formula of Laplace's transform by means of a complex integral. But this method leads to an infinite series, using the residue theorem and the fact that the expression $[I_\nu(\sqrt{\gamma}\,a)K_\nu(\sqrt{\gamma}\,b) - I_\nu(\sqrt{\gamma}\,b)K_\nu(\sqrt{\gamma}\,b)]$ has an infinite number of simple zeros at $\gamma = -k_{n,m}^2$. Thus this approach would again lead to the double series (43) for the first heat conduction Green's function. The summation of the series (45) can be obtained in the same manner as a similar formula obtained by Buchholz [1]. Since the use of the sum formula (45) does not simplify the heat conduction solution of the problem considered here, its derivation is not included.

B. The case of the domain of a wedge $0 \leqq \varphi \leqq \alpha$, $0 \leqq \rho < \infty$, $-\infty < z < \infty$ is the limiting case treated in A for $a \to 0$ and $b \to \infty$. The two Green's functions $\overline{G}_1$ and $\overline{G}_2$ become



(47)
$$\overline{G}_{\substack{1\\2}} = \overline{G}_1(\rho, \rho', \varphi, \varphi'; \gamma)$$
$$= -\frac{1}{2\alpha} \sum_{0}^{\infty} \varepsilon_n \{\cos[\frac{n\pi}{\alpha}(\varphi-\varphi')] \mp \cos[\frac{n\pi}{\alpha}(\varphi+\varphi')]\}$$
$$\times I_{\frac{n\pi}{\alpha}}(\gamma\rho) K_{\frac{n\pi}{\alpha}}(\pi\rho')$$

$\rho < \rho'$, and the same formula with ρ and ρ' interchanged if $\rho > \rho'$. In order to obtain the solution of the equalizing of the temperature inside a homogeneous solid wedge when the initial temperature inside the wedge is a constant equal to unity, it is necessary to integrate (47) over all line sources inside the wedge. This means it is necessary to obtain

$$\int_{\varphi'=0}^{\alpha} \int_{\rho'=0}^{\infty} \overline{G}_{\frac{1}{2}}(\rho, \rho', \varphi, \varphi'; \gamma)\rho' d\rho' d\varphi'$$

The expression (47) is not suited for this operation since the ρ integration would involve integrals of the form

$$\int_{\rho'=0}^{\rho} I_{\frac{n\pi}{\alpha}}(\gamma\rho')\rho' d\rho' \quad \text{and} \quad \int_{\rho}^{\infty} K_{\frac{n\pi}{\alpha}}(\gamma\rho')\rho' d\rho'$$

These integrals would lead to Lommel's functions [5, p. 90]. It is clear that this complication is due to the fact that the formula (47) is not symmetric in ρ and ρ' and that therefore the ρ' integration from 0 to ∞ has to be split up in two parts. In order to replace (47) by an expression which is symmetric in ρ and ρ' the formula [6, p. 176]

$$(48) \qquad I_\nu(a)K_\nu(b) = 2\pi^{-2} \int_0^\infty \frac{x \sinh(\pi x)}{x^2+\nu^2} K_{ix}(a)K_{ix}(b)dx$$

is used. This formula is valid for arbitrary ν with $\operatorname{Re} \nu > 0$ and for the left side, $b > a$. But the right side is symmetric in a and b. Hence

$$I_{\frac{n\pi}{a}}(\gamma\rho)K_{\frac{n\pi}{a}}(\gamma\rho') = 2\pi^{-2}\int_0^{\infty} \frac{x\sinh(\pi x)}{x^2+(\frac{n\pi}{a})^2} K_{ix}(\gamma\rho)K_{ix}(\gamma\rho')dx$$

Then, interchanging the order of summation and integration

$$\sum_{n=0}^{\infty} \varepsilon_n I_{\frac{n\pi}{a}}(\gamma\rho)K_{\frac{n\pi}{a}}(\gamma\rho') \cos\left(\frac{n\pi}{a}\theta\right)$$

$$= 2\pi^{-2}\int_0^{\infty} x\sinh(\pi x)K_{ix}(\gamma\rho)K_{ix}(\gamma\rho')$$

$$\times \sum_{n=0}^{\infty} \frac{\varepsilon_n \cos(\frac{n\pi}{a}\theta)}{x^2+(\frac{n\pi}{a})^2} \quad dx$$

But the sum is elementary

$$\sum_{n=0}^{\infty} \frac{\varepsilon_n \cos(\frac{n\pi}{a}\theta)}{x^2+(\frac{n\pi}{a})^2} = \frac{a}{x}\,\frac{\cosh[x(a-(\theta)]}{\sinh(ax)}$$

$$0 \leqq \theta \leqq 2a$$

Therefore

$$\sum_{n=0}^{\infty} \varepsilon_n I_{\frac{n\pi}{a}}(\gamma\rho)K_{\frac{n\pi}{a}}(\gamma\rho') \cos\left(\frac{n\pi}{a}\theta\right) =$$

$$= 2\pi^{-2} a \int_0^\infty \frac{\sinh(\pi x)}{\sinh(\alpha x)} K_{ix}(\gamma\rho) K_{ix}(\gamma\rho') \cosh[x(\alpha-\theta)]dx$$

$$\theta = |\varphi \mp \varphi'|$$

Hence, one obtains instead of (47)

$$\overline{G}_{\substack{1\\2}} = -\pi^{-2} \int_0^\infty \frac{\sinh(\pi x)}{\sinh(\alpha x)} K_{ix}(\gamma\rho) K_{ix}(\gamma\rho') \tag{49}$$

$$\times \{\cosh[x(\alpha-|\varphi-\varphi'|] \mp \cosh[x(\alpha-\varphi-\varphi')]\} dx$$

$$0 \leqq |\varphi \mp \varphi'| \leqq 2a$$

Formula (49) is symmetric in ρ and ρ' (see also [9]).

The convergence of the integral (49) for $0 \leq |\varphi \mp \varphi'| \leq 2a$ follows from the following considerations. From [5, p. 88]

$$K_{ix}(a) \sim \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} e^{-\frac{\pi}{2}x} \sin\left[x \log\left(\frac{2x}{a}\right)\right]$$

for large $x \gg a$. Also for large x

$$\frac{\sinh(\pi x)}{\sinh(\alpha x)} \sim e^{x(\pi-\alpha)}$$

$$\cosh[x-(\alpha-|\theta|)] \sim e^{x(\alpha-|\theta|)}, \quad |\theta| < \alpha$$

$$\sim e^{x(|\theta|-\alpha)}, \quad |\theta| > \alpha$$

Hence, $0 \leq |\varphi \mp \varphi'| \leq 2a$.

The expression (49) for the two Green's functions is now used to form

$$(50) \qquad \int_{\varphi'=0}^{a} \int_{\rho'=0}^{\infty} \overline{G}(\rho, \rho', \varphi, \varphi'; \gamma)\rho' d\rho' d\varphi'$$

It can be easily verified that the order of integration can be interchanged. The case of the first Green's function shall be considered only, corresponding to the heat conduction problem with zero temperature at the boundaries $\varphi = 0$ and $\varphi = a$ of the wedge. The ρ' integration follows from the formula [5, p. 51]

$$\int_0^{\infty} t^{\mu-1} K_{\nu}(\beta t)dt = 2^{\mu-2}\beta^{-\mu}\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu)$$

$$\mathrm{Re}\ (\mu \pm \nu) > 0, \quad \mathrm{Re}\ \beta > 0$$

with $\mu = 2$, $\nu = ix$, $\beta = \gamma$

$$\int_0^{\infty} \rho' K_{i\nu}(\gamma\rho')d\rho' = \frac{\pi x}{2\gamma^2 \sinh(\frac{\pi}{2}x)}$$

The integration of the integrand in (49) with respect to φ' (note that in the term $\cosh[x(a-|\varphi-\varphi'|)]$ the integration has to be taken over the interval 0 to φ' and over the interval φ' to a). The result is

$$\int_0^{\alpha} \{\cosh[x(\alpha-|\varphi-\varphi'|)] - \cosh[x(\alpha-\varphi-\varphi')]\}\varphi^1$$

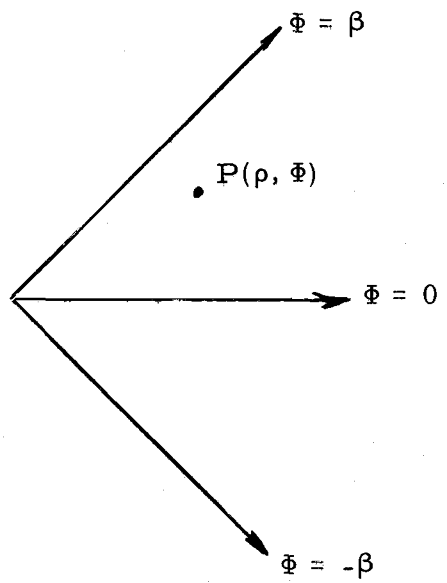
$$= \frac{2}{x}\{\sinh(x\alpha) - \sinh(x\varphi) - \sinh[x(\alpha-\varphi)]\}$$

Therefore (50) can now be written as

$$(51) \qquad \int_{\varphi'=0}^{\alpha} \int_{\rho'=0}^{\infty} \overline{G}_1(\rho, \rho', \varphi, \varphi'; \gamma)\rho' d\rho' d\varphi'$$

$$= \frac{1}{\pi\gamma^2} \int_0^{\infty} \frac{\sinh(\pi x) K_{ix}(\gamma\rho)}{\sinh(\frac{\pi}{2}x)\ \sinh(\alpha x)} \{\sinh(\alpha x) - \sinh(\varphi x) - \sinh[x(\alpha-\varphi)]\}\, dx$$

It can be shown, as before, that this integral is convergent for $0 \leqq \varphi \leqq \alpha$. In order to be in accordance with the investigation in [7], the transformation $\varphi = \Phi + \beta$, $\alpha = 2\beta$ will be made. This leads to a configuration as indicated by the figure below. The interior of the wedge is now given by $0 \leqq \rho < \infty$, $-\beta \leqq \Phi \leqq \beta$. It follows then from (51) and (19) that the temperature distribution $T(\rho, \Phi; t)$, at the time $t > 0$, inside the wedge is the inverse Laplace transform with respect to γ of



$$\frac{2}{\pi\gamma}\int_{x=0}^{\infty}\cosh(\frac{\pi}{2}x)K_{ix}(\rho\sqrt{\gamma})\,[1-\frac{\cosh(x\Phi)}{\cosh(x\beta)}]\,dx; \qquad t \to \kappa t$$

provided that the initial temperature inside the wedge is equal to unity. Since, [6, p. 175]

$$\int_0^{\infty}\cosh(\frac{\pi}{2}x)K_{ix}(\rho\sqrt{\gamma})dx = \frac{1}{2}\pi$$

(independent of ρ and γ) one has for the temperature field

$$(52) \qquad T(\rho, \Phi; t) = L_{\gamma}^{-1}\left\{\frac{1}{\gamma} - \frac{2}{\pi\gamma}\int_0^{\infty}\frac{\cosh(\frac{\pi}{2}x)\cosh(\Phi x)}{\cosh(\beta x)}K_{ix}(\rho\sqrt{\gamma})dx\right\},$$

$$t \to \kappa t$$

and since

$$L_{\gamma}^{-1}(\frac{1}{\gamma}) = U(t) = 0, \quad t < 0$$
$$= 1, \quad t > 0$$

(unit step function), one can also write (52) as

$$(53) \qquad T(\rho, \Phi; t) = 1 - \frac{2}{\pi}L_{\gamma}^{-1}\left\{\frac{1}{\gamma}\int_0^{\infty}K_{ix}(\rho\sqrt{\gamma})\frac{\cosh(\frac{\pi}{2}x)\cosh(\Phi x)}{\cosh(\beta x)}dx\right\},$$

$$t \to \kappa t$$

The integral in (53) can be processed further by means of an integral expression for the modified Hankel function [5, p. 82]

$$(54)\qquad K_{ix}(\rho\sqrt{\gamma})\cosh(\tfrac{1}{2}\pi x) = \int_0^\infty \cos(\rho\sqrt{\gamma}\sinh t)\cos(xt)dt.$$

Inserting (54) into (53), it follows, interchanging the order of integration.

$$(55)\qquad \int_0^\infty K_{ix}(\rho\sqrt{\gamma})\,\frac{\cosh(\frac{\pi}{2}x)\cosh(\Phi x)}{\cosh(\beta x)}\,dx$$

$$= \int_{t=0}^{\infty}[\cos(\rho\sqrt{\gamma}\sinh t)]\int_{x=0}^{\infty}\frac{\cosh(\Phi x)\cos(xt)}{\cosh(\beta x)}\,dx\quad dt$$

The inner integral is known [6, Vol. 1, p. 31]

$$\int_0^\infty \frac{\cosh(x\Phi)}{\cosh(x\beta)}\cos(xt)dx = \frac{\pi}{\beta}\cos(\frac{\pi}{2\beta}\Phi)\,\frac{\cosh(\frac{\pi}{2\beta}t)}{\cos(\frac{\pi}{\beta}\Phi)\cosh(\frac{\pi}{\beta}t)}$$

Then from (55)

$$(56)\qquad \int_0^\infty K_{ix}(\rho\sqrt{\gamma})\,\frac{\cosh(\frac{\pi}{2}x)\cosh(\Phi x)}{\cosh(\beta x)}\,dx$$

$$= \frac{\pi}{\beta}\cos(\frac{\pi}{2\beta}\Phi)\int_0^\infty \frac{\cosh(\frac{\pi}{2\beta}t)\cos(\rho\sqrt{\gamma}\sinh t)}{\cos(\frac{\pi}{\beta}\Phi)+\cosh(\frac{\pi}{\beta}t)}\,dt$$

Now, by (53) and (56)

(57)

$$T(\rho, \Phi; t) = 1 - L_{\gamma}^{-1}\left\{ \frac{2}{\gamma\beta} \cos(\frac{\pi}{2\beta}\Phi) \int_0^{\infty} \frac{\cos(\rho\sqrt{\gamma}\,\overline{\sinh t})\cosh(\frac{\pi}{2\beta}t)}{\cosh(\frac{\pi}{\beta}t) + \cos(\frac{\pi}{\beta}\Phi)}\, dt \right\}$$

$$t \rightarrow \kappa t$$

This represents the temperature distribution inside a wedge of arbitrary angle 2β for the boundary condition $T = 0$ for $\Phi = \pm\beta$. It does not seem possible to perform the Laplace inversion in (57). For the special case $\alpha = \frac{\pi}{\ell}$ $(\ell = 1, 2, 3, \ldots)$ the result given by (57), can be expressed in terms of error functions. For this purpose an expansion in partial fractions [7] is used.

(58)

$$\frac{\pi}{\beta} \frac{\cos(\pi\frac{\Phi}{2\beta})}{\cosh t} \frac{\cosh(\frac{\pi}{2\beta}t)}{\cosh(\frac{\pi}{\beta}t) + \cos(\frac{\pi}{\beta}\Phi)}$$

$$= (-1)^m \frac{\cos(\Phi)}{\sinh^2 t + \cos^2 \Phi}$$

$$+ \sum_{s=1}^{m} (-1)^{s-1} \left\{ \frac{\sin(2s\beta - \beta - \Phi)}{\sinh^2 t + \sin^2(2s\beta - \beta - \Phi)} + \frac{\sin(2s\beta - \beta + \Phi)}{\sinh^2 t + \sin^2(2s\beta - \beta + \Phi)} \right\}$$

for $2\beta = \frac{\pi}{2m+1}$; $m = 0, 1, 2, \; .\; .$

A similar expression exists for the case $2\beta = \frac{\pi}{2m}$. Upon inserting (58) into (57) there occur expressions of the form:

$$(-1)^m \frac{2\cos\Phi}{\gamma\pi} \int_0^\infty \frac{\cos(\rho\sqrt{\gamma}\,x)}{x^2+\cos^2\Phi}\,dx = \frac{(-1)^m}{\gamma}\, e^{-\rho\sqrt{\gamma}\cos\Phi}$$

$$(-1)^{s-1}\frac{2}{\gamma}\sin(2s\beta-\beta\mp\Phi)\int_0^\infty \frac{\cos(\rho\sqrt{\gamma}\,x)dx}{x^2+\sin^2(2s\beta-\beta\mp\Phi)}$$

$$= \frac{(-1)^{s-1}}{\gamma}\, e^{-\rho\sqrt{\gamma}\sin(2s\beta-\beta\mp\Phi)}$$

if the substitution $\sinh t = x$ and [6, Vol. 1, p. 8]

$$\int_0^\infty \frac{\cos(xy)}{x^2+a^2}\,dx = \frac{\pi}{2a}\, e^{-ay}$$

is used. The Laplace inversions with respect to γ of the formula above are known [6, Vol. 1, p. 245] to be respectively

$$L_\gamma^{-1}\left\{\frac{(-1)^m}{\gamma}\, e^{-\rho\sqrt{\gamma}\cos\Phi}\right\}_{\gamma\to\kappa t} = (-1)^m \operatorname{Erfc}\left(\frac{\rho\cos\Phi}{\sqrt{4\kappa t}}\right)$$

$$L_\gamma^{-1}\left\{\frac{(-1)^{s-1}}{\gamma}\, e^{-\rho\sqrt{\gamma}\sin(2s\beta-\beta\mp\Phi)}\right\}_{\gamma\to\kappa t} = (-1)^{s-1}\operatorname{Erfc}\left[\frac{\rho\sin(2s\beta-\beta\mp\Phi)}{\sqrt{4\kappa t}}\right]$$

With these results the temperature distribution becomes finally

(59)

$$T(\rho,\Phi;t) = (-1)^m \operatorname{Erf}\left(\frac{\rho\cos\Phi}{\sqrt{4\kappa t}}\right) + \sum_{s=1}^{m}(-1)^{s-1}\left\{\operatorname{Erf}\left[\frac{\rho\sin(2s\beta-\beta-\Phi)}{\sqrt{4\kappa t}}\right]+\operatorname{Erf}\left[\frac{\rho\sin(2s\beta-\beta+\Phi)}{\sqrt{4\kappa t}}\right]\right\}$$

if one uses $\mathrm{Erfc}(z) = 1 - \mathrm{Erf}(z)$ and $2\beta = \frac{\pi}{2m+1}$, $m = 0, 1, 2, \ldots$.
A similar analysis holds for $2\beta = \frac{\pi}{2\ell}$ for this case see [7].

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APPENDIX

APPENDIX

MacDonald's formulas [5, p. 53] represent the product of two modified Hankel functions of different arguments in the form of a Laplace transform

$$\text{(A)} \qquad \int_0^\infty e^{-\gamma t} e^{-\frac{\alpha^2+\beta^2}{t}} K_\nu\left(2\alpha\frac{\beta}{t}\right)t^{-1}dt = 2K_\nu(2\alpha\sqrt{\gamma})K_\nu(2\beta\sqrt{\gamma}), \quad \operatorname{Re}\gamma > 0,$$

The inversion of the above Laplace integral with respect to γ is:

$$\text{(A')} \qquad L_\gamma^{-1}[K_\nu(2\alpha\sqrt{\gamma})K_\nu(2\beta\sqrt{\gamma})] = \frac{1}{2} e^{-\frac{\alpha^2+\beta^2}{t}} t^{-1}K_\nu\left(2\alpha\frac{\beta}{t}\right)$$

$$\text{(B)} \qquad \int_0^\infty e^{-\gamma t} e^{-\frac{\alpha^2+\beta^2}{t}} I_\nu\left(2\alpha\frac{\beta}{t}\right)t^{-1}dt = 2I_\nu(2\alpha\sqrt{\gamma})K_\nu(2\beta\sqrt{\gamma}),$$

$$\operatorname{Re}\gamma > 0, \quad 0 < \alpha < \beta$$

and the same formula with α and β interchanged if $\beta < \alpha$. The inverse of the above Laplace integral with respect to γ is

$$\text{(B')} \qquad L_\gamma^{-1}[I_\nu(2\alpha\sqrt{\gamma})K_\nu(2\beta\sqrt{\gamma})] = \frac{1}{2} e^{-\frac{\alpha^2+\beta^2}{t}} t^{-1}I_\nu\left(2\alpha\frac{\beta}{t}\right), \quad \alpha < \beta.$$

Similarly for only one modified Bessel function involved [5, p. 82]

$$\text{(C)} \qquad \int_0^\infty e^{-\gamma t} e^{-\frac{\beta}{t}} t^{-\nu-1} dt = 2\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}\nu} K_\nu(2\sqrt{\beta\gamma}), \quad \mathrm{Re}\ (\gamma, \beta) > 0.$$

Its inversion

$$\text{(C')} \qquad L_\gamma^{-1}[\gamma^{\frac{1}{2}\nu} K_\nu(2\sqrt{\beta\gamma})] = \frac{1}{2} \beta^{\frac{1}{2}\nu} t^{-\nu-1} e^{-\frac{\beta}{t}}$$

Special Cases of C'

$\nu = 0$

$$\text{(C'')} \qquad L_\gamma^{-1}[K_0(2\sqrt{\beta\gamma})] = \frac{1}{2} t^{-1} e^{-\frac{\beta}{t}}$$

$\nu = \frac{1}{2}$ [Note that $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}$]

$$\text{(C''')} \qquad L_\gamma^{-1}[e^{-2\sqrt{\beta\gamma}}] = \pi^{-\frac{1}{2}} \beta^{\frac{1}{2}} t^{-\frac{3}{2}} e^{-\frac{\beta}{t}}$$

$$\text{(C'''')} \qquad L_\gamma^{-1}[\gamma^{-\frac{1}{2}} e^{-2\sqrt{\beta\gamma}}] = \pi^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{\beta}{t}}, \qquad \mathrm{Re}\ \gamma > 0.$$