AN ABSTRACT OF THE THESIS OF

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(Name) (Degree)

in MATHEMATICS presented on April 23, 1969
(Major) (Date)

Title: ON THE DISTRIBUTION FUNCTION OF THE INTERVAL BETWEEN ZERO-CROSSINGS OF A STATIONARY GAUSSIAN PROCESS

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Abstract approved: ______________________
Professor William M. Stone

The distribution function of the interval between successive zeros of stationary Gaussian processes is discussed in general. The zeros of a stationary Gaussian process can be interpreted as a stationary point process.

Using the stationary point process methods we obtain an exact expression for the density function of the forward recurrence time of the stationary Gaussian process, first considered by Wong [27], with zero mean and covariance function

$$\rho(\tau) = EX(t+\tau)X(t) = \frac{3}{2} e^{-\frac{|\tau|}{\sqrt{3}}} \left[1 - \frac{1}{3} e^{-\frac{2|\tau|}{\sqrt{3}}}\right].$$

An exact expression for the probability of no zero-crossings in time t following an arbitrary zero-crossing is obtained. For the

probability of no zero-crossings in time t following an arbitrarily chosen sample point an expression is obtained for small values of t. A physical situation that gives rise to a random process with the above covariance function is also discussed.

Finally we discuss the difficulties that impede obtaining a closed expression for the distribution function $F_I(t)$ of the interval between successive zero-crossings. We also obtain a series expansion for $F_I(t)$ in terms of multiple integrals. The fourth order information is discussed in some detail and a reduction formula for the four-variate integral involved is given.

On the Distribution Function of the Interval Between Zero-Crossings of a Stationary Gaussian Process

by

Farag Abdel-Salam Attia

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1969

APPROVED:

Redacted for Privacy

Professor of Mathematics

in charge of major

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Date thesis is presented April 23, 1969

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ACKNOWLEDGMENT

I am very much indebted to my major professor, Professor William M. Stone, for his excellent guidance and critical reading of this work.

I would like to extend my gratitude to the Government of the United Arab Republic for giving me the opportunity for graduate study.

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ON THE DISTRIBUTION FUNCTION OF THE INTERVAL BETWEEN ZERO-CROSSINGS OF A STATIONARY GAUSSIAN PROCESS

I. STATIONARY POINT PROCESSES

1.1. Preliminary Remarks

Definition: A stationary point process is defined by the following requirement:

The joint distribution of the number of events in a set of k fixed intervals is invariant under translation for all $k = 1, 2, \ldots$.

If we consider the case $k = 1$, i.e., a fixed, closed on the right interval $(t', t'']$, an immediate consequence of the definition would be that the distribution of the number of events in an interval depends only on the length of the interval. It is easy to show, by considering two adjacent intervals, that the expected number of events in an interval is proportional to the length of the interval.

A definition of stationarity weaker than the above requires only that the first and second order properties of the process are invariant under a time shift. This is particularly important when we base the statistical analysis on the first and second order properties.

In what follows we shall assume that the probability of more than one event occurring in a small interval of length $\Delta\tau$ is $o(\Delta\tau)$ as $\Delta\tau \to 0$. That is, we shall not allow multiple events. Furthermore, we consider all the events to be indistinguishable with the information at hand, except by where they happen in time or space.

1.2. The Forward Recurrence Time

In the definition of stationarity mentioned in the section above we considered the number of events in fixed time intervals following an arbitrarily chosen time origin. Instead we may consider the sequence of random intervals $\{I_i\}$ between successive events following an arbitrarily chosen event. Thus the events subsequent to the chosen event occur at times $I_1, I_1 + I_2, \ldots,$ after that event. Then the sequence $\{I_i\}$ is a stationary sequence of random variables since the series of events is stationary. That is, the joint distribution of any k of the intervals $I_1, I_2, \ldots,$ is invariant under a translation along the time axis for all $k = 1, 2, \ldots$.

To relate the two considerations, the number of events in fixed time intervals following an arbitrarily chosen time origin and the sequence of random intervals between successive events subsequent to an arbitrarily chosen event, we examine the distribution of the forward recurrence time, that is the interval measured from the arbitrary time origin to the next event. This in general, is different from I_1, which is the time from an arbitrarily chosen event to the next event.

Suppose that starting from an arbitrarily chosen event we consider the interval $(0, I_1 + \ldots + I_n)$, where n is large. We suppose that the mean value of a typical interval I is $E(I) = \mu$, and that the variance of I, $Var(I) = \sigma_I^2$, exists. Choose an arbitrary sample point over this interval and let W be the time from this sampling point up to the next event. The sample point will fall in one of the random intervals $I_1, I_2, \ldots, I_n$, say L_o. This L_o will, in general, have a different distribution from any of the I_i, $i = 1, 2, \ldots, n$, which can be seen from the following:

If $n_t dt$ is the number of the $\{I_i\}$ with lengths in $(t, t+dt]$, the probability of L_o having length in $(t, t+dt]$ is (see McFadden [13])

$$\frac{tn_t dt}{\sum_{i=1}^{n} I_i} = \frac{tn_t dt}{n} \frac{1}{\frac{1}{n}\sum_{i=1}^{n} I_i} .$$

But since

$$\frac{n_t}{n} \rightarrow f_I(t) \quad \text{and} \quad \sum_{i=1}^{n} \frac{1}{n} I_i \rightarrow E(I) = \mu \quad \text{as} \quad n \rightarrow \infty$$

we have

$$f_{L_o}(t)dt = \frac{tf_I(t)dt}{\mu} . \tag{1.2.1}$$

This procedure is known as length-biased sampling.

The bias in the sampling is illustrated by considering the mean value

$$E(L_o) = \int_0^\infty x f_{L_o}(x)dx.$$

By use of (1. 2. 1) we get

$$E(L_o) = \frac{1}{\mu}\int_0^\infty x^2 f_I(x)dx = \frac{1}{\mu} E(I^2).$$

Since the variance of I is given by

$$\sigma_I^2 = E(I-\mu)^2 = E(I^2) - \mu^2$$

we have

$$E(L_o) = \mu + \frac{1}{\mu}\sigma_I^2 = \mu[1+\frac{\sigma_I^2}{\mu^2}] = \mu[1+C^2(I)] , \qquad (1. 2. 2)$$

where $C(I)$ is the coefficient of variance of the $\{I_i\}$.

Now consider $L_o = t_o$, a fixed value. The sampling point is uniformly distributed over a time interval of length t_o, so the conditional distribution of W is given by

$$f_W(t|L_o=t_o) = \begin{cases} \frac{1}{t_o}, & 0 \leq t \leq t_o, \\ 0, & t_o < t. \end{cases} \qquad (1. 2. 3)$$

Thus we have

$$f_W(t) = \int_0^\infty f_W(t|L_o=t_o)f_{L_o}(t_o)dt_o$$

and substituting from (1. 2. 1) and (1. 2. 3) we have

$$f_W(t) = \int_t^\infty \frac{1}{t_o} \frac{t_o f_I(t_o)}{\mu}\, dt_o = \frac{1}{\mu} \int_t^\infty f_I(t_o) dt_0 = \frac{1}{\mu} R_I(t), \qquad (1.2.4)$$

where $R_I(t) = \text{Prob}(I > t)$ is the survivor function for I. It is obvious that

$$R_I(t) = 1 - F_I(t),$$

where $F_I(t)$ is the marginal distribution function of the random variable I. Since $F_I(t)$ is a non-decreasing function of t it follows from (1.2.4) that $f_W(t)$ is a non-increasing function of t with

$$f_W(0_+) = \frac{1}{\mu}\,. \qquad (1.2.5)$$

We consider next the times between events which follow the forward recurrence time W. Denote by $\{L_i\}$ this sequence of events starting with L_o. It is obvious that the joint distribution of any set of the L_i, $i = 1, 2, \ldots,$ given that $L_o = t_o$, is the same as the joint distribution of the corresponding set of the I_i, given that $I_o = t_o$. That is

$$f_{L_o}(t_o | L_o = t_o) = f_{I_i}(t_i | I_o = t_o), \quad i = 1, 2, \ldots . \qquad (1.2.6)$$

Consequently, using (1.2.1), we have the joint density function

$$f_{L_o, L_i}(t_o, t_i) = f_{L_i}(t_i \mid L_o = t_o) f_{L_o}(t_o)$$

$$= f_{I_i}(t_i \mid X_o = t_o) \frac{t_o f_{I_o}(t_o)}{\mu} \qquad (1.2.7)$$

$$= \frac{t_o}{\mu} f_{I_o, I_i}(t_o, t_i), \quad i = 1, 2, \ldots .$$

The last relation generalizes (1. 2. 1) and shows that the bias which appears in the randomly selected interval L_o persists in the following intervals.

Using (1. 2. 7) we can get the expected value of L_i as follows:

$$E(L_i) = \int_0^\infty \int_0^\infty t_i f_{L_o, L_i}(t_o, t_i) dt_o dt_i$$

$$= \frac{1}{\mu} \int_0^\infty \int_0^\infty t_o t_i f_{I_o, I_i}(t_o, t_i) dt_o dt_i \qquad (1.2.8)$$

$$= \frac{1}{\mu} E(I_o I_i) = \mu + \rho_i \frac{\sigma_I^2}{\mu} = \mu[1 + C^2(I)\rho_i], \quad i = 1, 2, \ldots ,$$

where $\rho_i = \mathrm{Corr}(I_o, I_i)$ is the serial correlation coefficient for the stationary sequence of random variables $I_o, I_1, \ldots$.

Finally, we have a useful extension of (1. 2. 4) connecting the partial sums of the sequences $W, L_1, L_2, \ldots$ and $I_1, I_2, \ldots$.

Let

$$G_1(t) = \text{Prob}[W \leq t] = F_W(t),$$

$$G_i(t) = \text{Prob}[W + L_1 + \ldots + L_{i-1} \leq t], \quad i \geq 2,$$

and

$$F_i(t) = \text{Prob}[I_1 + \ldots + I_i \leq t], \quad (i=0, 1, 2, \ldots)$$

where $F_o(t)$ is the unit step function. Then the density function always exists and

$$g_i(t) = \frac{F_{i-1}(t) - F_i(t)}{\mu}, \quad i = 1, 2, \tag{1.2.9}$$

(see McFadden [13]).

1.3. Fundamental Relationship Between Counts of Events and Time Intervals Between Events

Let N_t be the number of events in an interval of length t, following the arbitrarily selected point where observation of the process begins. The following relationship relate N_t and the sequence of random variables $W, L_1, L_2, \ldots$: $N_t = 0$ if and only if $W > t$, $N_t < n$ if and only if $W + L_1 + \ldots + L_{n-1} > t$, $n = 2, 3, \ldots$, so that

$$\text{Prob}[N_t = 0] = \text{Prob}[W > t) = R_W(t), \tag{1.3.1}$$

and

$$\text{Prob}[N_t < n] = \text{Prob}[W + L_1 + \ldots + L_{n-1} > t], \quad n \geq 2. \tag{1.3.2}$$

Thus, given the distribtuion of the counts, it is possible to get the distribution of W and the L_i.

The times between events which are of practical and statistical interest are the I_i rather than the L_i, which are the ones usually observed. Since, according to (1. 2. 8), the sequence $\{L_i\}$ is not a stationary sequence it is difficult to deal with.

As mentioned before, N_t, the number of events in an interval of length t, is associated with the random sequence $W, L_1, L_2, \ldots$. Denote by $N_t^{(f)}$ the number of events in an interval $(0, t]$, which starts with but does not include an arbitrary event. That is, $N_t^{(f)}$ is associated with the random sequence $\{I_i\}$. Thus, we have

$$\mathrm{Prob}[N_t^{(f)} < r] = \mathrm{Prob}[I_1 + I_2 + \ldots + I_r > t], \quad r = 1, 2, \ldots \; . \tag{1. 3. 3}$$

To obtain the relationship between the counting processes N_t and $N_t^{(f)}$ consider the generating function (see Khintchine [9, p. 34])

$$\begin{aligned} \varphi_f(\zeta, t) &= \sum_{k=0}^{\infty} \zeta^k \,\mathrm{Prob}[N_t^{(f)} = k] \\ &= \sum_{k=0}^{\infty} \zeta^k [F_k(t) - F_{k+1}(t)] = \frac{\mu}{\zeta} \sum_{k=1}^{\infty} \zeta^k g_k(t), \end{aligned} \tag{1. 3. 4}$$

where we have used (1. 2. 9) in the second and third step.

Thus

$$\int_0^t \varphi_f(\zeta, u)du = \frac{\mu}{\zeta}\sum_{k=1}^{\infty} \zeta^k G_k(t). \tag{1.3.5}$$

Also, the generating function of N_t,

$$\varphi(\zeta, t) = \sum_{k=0}^{\infty} \zeta^k \text{ Prob}[N_t = k] = \sum_{k=0}^{\infty} \zeta^k[G_k(t)-G_{k+1}(t)]$$

$$= 1 + (1-\frac{1}{\zeta}) \sum_{k=1}^{\infty} \zeta^k G_k(t). \tag{1.3.6}$$

From (1. 3. 5) and (1. 3. 6) we see that

$$\varphi(\zeta, t) = 1 + \frac{\zeta-1}{\mu}\int_0^t \varphi_f(\zeta, u)du, \tag{1.3.7}$$

which is the desired relationship.

Now let us consider the first moment of $N_t^{(f)}$:

$$E(N_t^{(f)}) = M_f(t) = \sum_{k=1}^{\infty} k \text{ Prob}[N_t^{(f)} = k]$$

$$= \sum_{k=1}^{\infty} \text{Prob}[N_t^{(f)} \geq k] = \sum_{k=1}^{\infty} F_k(t), \tag{1.3.8}$$

where $F_k(t)$ is the distribution function of $I_1 + I_2 + \ldots + I_k$. If the derivative of $M_f(t)$ exists for all t

$$m_f(t) = \frac{d}{dt} M_f(t) = \sum_{k=1}^{\infty} f_k(t). \tag{1. 3. 9}$$

If we assume that a strong law of large numbers (Parzen [14], p. 371) holds for the sequence $\{I_i\}$ then

$$\frac{I_1 + \ldots + I_k}{k} \sim E(I) \tag{1. 3. 10}$$

holds for sufficiently large k. From this and (1. 3. 3) we have, for large t,

$$M_f(t) \sim \frac{t}{E(I)}, \tag{1. 3. 11}$$

which agrees with the physical interpretation of $E(I)$ as a long run average. For the stationary counting process N_t we have

$$M(t) = \frac{t}{E(I)} \tag{1. 3. 12}$$

for all t, since $M(t) = E(N_t)$ is proportional to t for all t. Finally,

$$m(t) = \frac{d}{dt} M(t) = \frac{1}{E(I)} = \frac{1}{\mu}. \tag{1. 3. 13}$$

1.4. Moments of the Interval Between Events and of Counts of Events

For the sequence $\{I_i\}$ we consider the mean value

$$\mu = E(I) = \int_0^\infty x f_I(x)dx = \int_0^\infty R_I(x)dx, \tag{1.4.1}$$

the variance

$$\sigma_I^2 = \int_0^\infty x^2 f_I(x)ds - [E(I)]^2 = 2\int_0^\infty xR_I(x)dx - \mu^2, \tag{1.4.2}$$

and the ratio

$$C^2(I) = \frac{\sigma_I^2}{[E(I)]^2} = \frac{\sigma_I^2}{\mu^2}. \tag{1.4.3}$$

This $C(I)$, which is called the coefficient of variation of I, measures roughly departures from the exponential distribution, for which $C(I) = 1$.

The autocorrelation sequence is defined by

$$\rho_k = \frac{Cov(I_i, I_{i+k})}{\sigma_I^2}, \quad k = 0 \pm 1, \pm 2, \ldots . \tag{1.4.4}$$

Wold [26, p. 66], showed that a sequence $\{\rho_k\}$ is the autocorrelation sequence for a stationary random sequence $\{I_i\}$ if and only if ρ_k can be represented in the form

$$\rho_k = \int_{-\pi}^{\pi} \cos k\omega dF(\omega), \quad k = 0, \pm 1, \pm 2, \ldots, \tag{1.4.5}$$

where $F(\omega)$ is the spectral distribution function. If $F(\omega)$ is absolutely continuous its derivative $f(\omega)$ exists and is called the sepctral density function. Then (1.4.5) can be written as

$$\rho_k = \int_{-\pi}^{\pi} f(\omega) \cos k\omega \, d\omega, \quad k = 0, 1, 2, \ldots, \tag{1.4.6}$$

the set of Fourier coefficients associated with $f(\omega)$.

From (1.4.6) we see that the inverse relationship is

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \rho_k e^{-ki\omega}$$

$$= \frac{1}{2\pi} [1+2 \sum_{k=1}^{\infty} \rho_k \cos k\omega], \quad -\pi \leq \omega \leq \pi, \tag{1.4.7}$$

since $\rho_k = \rho_{-k}$ for real sequences $\{I_i\}$. It follows from (1.4.7) that $f(\omega)$ is an even function of ω.

To obtain higher moments of $\{I_i\}$ we introduce the Laplace and Laplace-Stieltjes transormations with respect to t and x as appropriate for the following functions:

$$p(n, t) = \text{Prob}[N_t = n], \tag{1.4.8}$$

$F_k(x)$ which, as defined before, is the distribution function of $I_1 + \ldots + I_k$, and $g_n(x)$, defined by (1. 2. 8). Transformation pairs are defined by

$$p^*(n, s) = \int_0^\infty e^{-st} p(n, t)dt,$$

$$f_n^*(s) = \int_0^\infty e^{-sx} dF_n(x) \qquad (1.4.9)$$

$$g_n^*(s) = \int_0^\infty e^{-sx} g_n(x)dx .$$

The functions $f_n^*(s)$ and $g_n^*(s)$ are moment generating functions for the random variables $I_1 + \ldots + I_n$ and $W + L_1 + \ldots + L_{n-1}$ respectively (see McFadden [13]). Transforming (1. 2. 8) and (1. 3. 1) we obtain

$$\begin{aligned} g_1^*(s) &= \frac{1-f_1^*(s)}{\mu s}, \\ g_{n+1}^*(s) &= \frac{f_n^*(s)-f_{n+1}^*(s)}{\mu s}, \quad n \geq 1, \\ p^*(0, s) &= \frac{1-g_1^*(s)}{\mu s}, \\ p^*(n, s) &= \frac{g_n^*(s)-g_{n+1}^*(s)}{s}, \quad n \geq 1. \end{aligned} \qquad (1.4.10)$$

Obvious rearrangements lead to

$$p^*(0, s) = \frac{1}{\mu s^2}[f_1^*(s) - 1] + \frac{1}{s},$$

$$p^*(1, s) = \frac{1}{\mu s^2}[f_2^*(s) - 2f_1^*(s) + 1], \tag{1.4.11}$$

$$p^*(n, s) = \frac{1}{\mu s^2}[f_{n+1}^*(s) - 2f_n^*(s) + f_{n-1}^*(s)], \quad n \geq 2;$$

and the finite sum representations

$$f_n^*(s) = 1 - \mu s \sum_{k=1}^{n} g_k^*(s), \quad n \geq 1, \tag{1.4.12}$$

and

$$g_n^*(s) = 1 - s \sum_{k=0}^{n-1} p^*(k, s), \quad n \geq 1. \tag{1.4.13}$$

Substituting for $g_k^*(s)$ in (1.4.12) leads to

$$\begin{aligned} f_n^*(s) &= 1 - \mu n s + \mu s^2 \sum_{j=1}^{n} \sum_{k=0}^{j-1} p^*(k, s) \\ &= 1 - \mu n s + \mu s^2 \sum_{k=0}^{n-1} (n-k) p^*(k. s), \quad n \geq 1. \end{aligned} \tag{1.4.14}$$

Now we can use the above relationships to express the higher moments of the random variable I in terms of the moments of $p(0, t)$. By definition $E(I) = \mu$; if we let $n = 1$ in (1.4.14), then differentiate twice with respect to s we arrive at

$$f_1^{*''}(s) = 2\mu p^*(0, s) + 4\mu s p^{*'}(0, s) + \mu s^2 p^{*''}(0, s). \qquad (1.4.15)$$

If $p^*(0, s)$ behaves well enough so that $p^{*'}(0, s)$ and $p^{*''}(0, s)$ both tend to zero as $s \to 0$,

$$E(I^2) = f_1^{*''}(0) = 2\mu \int_0^\infty p(0, t)dt. \qquad (1.4.16)$$

For the nth moment of the random variable I we differentiate (1.4.14) $n-2$ times with respect to s, arriving at

$$f_1^{*(n)}(s) = n(n-1)\mu p^{*(n-2)}(0, s) + 2n\mu s p^{*(n-1)}(0, s) + \mu s^2 p^{*(n)}(0, s),$$
$$n \geq 2. \qquad (1.4.17)$$

Again if $p^*(0, s)$ behaves well enough so the last two terms vanish as $s \to 0$, then

$$E(I^n) = n(n-1)\mu \int_0^\infty t^{n-2} p(0, t)dt, \quad n \geq 2, \qquad (1.4.18)$$

which shows that the nth moment of I is related to the (n-2)th moment of $p(0, t)$.

II. STATIONARY GAUSSIAN PROCESSES

2.1. Preliminary Remarks

In this chapter we shall consider some of the random variables associated with a Gaussian process $X(t)$. In particular, we shall consider the real-valued stationary Gaussian process $X(t)$. The choice of the random variables is motivated by usefulness in application. For example, some such random variables are the number of times $X(t)$ crosses the t-axis, or some arbitrary level α, or takes on an extreme value above some other level β, or the time intervals between such events. These and associated random variables have considerable importance in communication theory and in other fields.

Without loss of generality we can assume $X(t)$ to be a zero-mean process, for if not we would consider the process $X(t) - EX(t)$.

Since $X(t)$ is stationary Gaussian it is also strictly stationary (see Wozencraft and Jacobs [28, p. 184]). Denote by $\rho_X(\tau)$ and $S_X(\omega)$ the covariance function and the spectral function, respectively, of the process $X(t)$; the relationship is

$$\rho_X(\tau) = \int_0^\infty \cos \omega\tau \, dS_X(\omega) \tag{2.1.1}$$

(see Doob [4, p. 536]).

Finally, we shall denote by λ_k the kth spectral moment,

$$\lambda_k = \int_0^\infty \omega^k dS_X(\omega). \qquad (2.1.2)$$

2.2. Crossings, Upcrossings and Downcrossings

In considering the crossings of our process $X(t)$ of an arbitrary level α we have to distinguish between "genuine" crossings and tangencies. Ylvisaker [30] showed that the number of tangencies to the level α by $X(t)$ in an interval T is equal to zero with probability one.

In general, we shall consider cases where $X(t)$ does not have a sample derivative for all t, though we shall always assume that $X(t)$ is continuous. A sufficient condition for continuity is that $X(t)$ be separable (Cramér and Leadbetter [3, p. 174]). Thus the tangencies to the level α should be defined in terms of the continuity of $X(t)$ rather than in terms of differentiability.

Now we are in a position to introduce the following definitions concerning crossings, upcrossings, downcrossings and tangencies of a continuous function $f(t)$ on the unit interval to the level α.

Definitions: (i) $f(t)$ is said to have an upcrossing of the level α at t_o if there exists a positive ϵ such that $f(t) \leq \alpha$ in $(t_o-\epsilon, t_o)$, and $f(t) \geq \alpha$ in $(t_o, t_o+\epsilon)$. Denote the number of upcrossings by $f(t)$ in $[0, 1]$ by U_α.

(ii) $f(t)$ is said to have a downcrossing of the level α at t_o if there exists an $\epsilon > 0$ such that $f(t) \geq \alpha$ in $(t_o - \epsilon, t_o)$ and $f(t) \leq \alpha$ in $(t_o, t_o + \epsilon)$. Let D_α denote the number of downcrossings by $f(t)$ in $[0, 1]$ of the level α.

(iii) $f(t)$ is said to have a crossing of the level α at t_o if in each neighborhood of t_o there exist points t_1 and t_2 such that

$$[f(t_1) - \alpha][f(t_2) - \alpha] < 0.$$

Let C_α denote the number of crossings by $f(t)$ in $[0, 1]$.

(iv) The point t_o is called a tangency of $f(t)$ to the level α if $f(t_o) = \alpha$ and there is a neighborhood of t_o on which $f(t) - \alpha$ does not change sign.

Some of the above definitions were used by Ylvisaker [30]. If instead of the unit interval we use the interval $[t_1, t_2]$ the above definitions apply and, in this case, give $U_\alpha[t_1, t_2]$, $D_\alpha[t_1, t_2]$ and $C_\alpha[t_1, t_2]$ for the upcrossings, downcrossings and crossings respectively.

2.3. Mean Number of Crossings in Time T and Mean Duration of an Excursion

The general formulas for determining the probability of a crossing of the level α, within a time interval of length t, by the trajectory of our process $X(t)$, and for the mean time that a realization of $X(t)$ stays above or below a given level, were first given by Rice [17, 18].

The mean number of crossings of the level α by $X(t)$ within a time interval of length t, is given by

$$EC_\alpha(0, t) = \frac{t}{\pi}\sqrt{\frac{\lambda_2}{\lambda_o}}\, e^{-\alpha^2/2\lambda_o} . \qquad (2.3.1)$$

For a fixed time instant T, $EC_\alpha(0, t)$ has a maximum at $\alpha = 0$ and approaches zero for large values of α, in agreement with the underlying physical picture.

The above formula goes back to Rice [17, 18], who obtained it under the hypothesis that S_X, the spectral function, has finitely many points of increase. The conditions under which (2.3.1) was derived were weakened by Itô [6], Ylvisaker [30], and others.

The next theorem (see Cramér and Leadbetter [3, p. 177]) gives a necessary and sufficient condition under which $\lambda_2 < \infty$, and, consequently, $EC_\alpha(0, t) < \infty$.

<u>Theorem:</u> (i) $$\lim_{\tau \to 0} \frac{2[1-\rho_X(\tau)]}{\tau^2} = \lambda_2 = \int_{-\infty}^{\infty} \omega^2 dS_X(\omega) \leq \infty,$$
and

(ii) the second derivative $\rho_X''(\tau)$ exists and is finite at $\tau = 0$ if an only if $\lambda_2 < \infty$. If $\lambda_2 < \infty$ then $\rho_X''(0) = -\lambda_2$ and $\rho_X''(\tau)$

exists for all τ.

If $\lambda_2 < \infty$ it is equal to $\rho_X''(0)$. Also, from (2.1.1) and (2.1.2) we have

$$\lambda_o = \int_{-\infty}^{\infty} dS_X(\omega) = \rho(0). \tag{2.3.2}$$

and in this case (2.3.1) may be written in the more familiar form

$$EC_a(0, t) = \frac{t}{\pi}\sqrt{-\frac{\rho''(0)}{\rho(0)}}\; e^{-a^2/2\rho(0)}. \tag{2.3.3}$$

The mean time during which a realization of $X(t)$ stays above a given level a is given by Rice [17, 18].

$$\pi\sqrt{-\frac{\rho(0)}{\rho''(0)}}\; e^{-a^2/2\rho(0)}[1-\Phi(\frac{a}{\rho(0)})], \tag{2.3.4}$$

where $\Phi(x)$ is Laplace's integral function.

2.4. The Variance of the Number of Zeros

The number of zero crossings, $N = C_o(0, T)$, by the process $X(t)$ in time duration T is a random variable. Variance is

$$\mathrm{Var}\, N = E(N-EN)^2 = EN^2 - (EN)^2;$$

since EN is given by (2.3.1) it is sufficient to compute the second moment EN^2. Let A be the covariance matrix of the random variables $X(t_1)$, $X(t_2)$, $X'(t_1)$, $X'(t_2)$:

$$A = \begin{bmatrix} 1 & \rho(\tau) & 0 & \rho'(\tau) \\ \rho(\tau) & 1 & -\rho'(\tau) & 0 \\ 0 & -\rho'(\tau) & \lambda_2 & -\rho''(\tau) \\ \rho'(\tau) & 0 & -\rho''(\tau) & \lambda_2 \end{bmatrix},$$

$$\tau = t_2 - t_1.$$

If $M_{ij}(\tau)$ is the cofactor of the ij element of A then Steinberg, et al., [22], have obtained

$$EN^2 = EN + \frac{2}{\pi^2} \int_0^T (T-\tau) \frac{[M_{33}^2(\tau)-M_{34}^2(\tau)]^{1/2}}{[1-\rho^2(\tau)]^{3/2}} \cdot \left[1 + \frac{M_{34}(\tau)}{[M_{33}^2(\tau)-M_{34}^2(\tau)]^2} \tan^{-1} \frac{M_{34}(\tau)}{[M_{33}^2(\tau)-M_{34}^2(\tau)]^{1/2}} \right] d\tau . \tag{2.4.1}$$

The last expression is quite suitable for machine computation.

The following theorem, Cramér and Leadbetter [3, p. 209], gives a sufficient condition for the finiteness of the second moment EN^2.

<u>Theorem</u>: If, for some $\delta > 0$,

$$\int_0^\delta \frac{\lambda_2 + \rho''(\tau)}{\tau} d\tau < \infty , \tag{2.4.2}$$

where λ_2 is the second spectral moment and $\rho(T)$ is the covariance function of the process $X(t)$, then the second moment EN^2 is finite.

2.5. Shot Noise Signals Through Linear Systems

Throughout the last four sections of this chapter we have considered the zero crossings of a stationary Gaussian process and some of the related random variables. Direct engineering applications of these ideas occur in connection with electronics equipment which is designed to measure frequency by averaging the number of zero crossings in a short time interval. The study of the zero crossings statistics provides additional information which, with the power spectrum or the autocorrelation function, characterizes a particular random noise. The firing of missiles from a rolling vessel and rocket guidance systems are two of the many areas of interest in these matters.

In this section we shall discuss a physical example that motivates the study of a stationary Gaussian process. Suppose that the input to a linear system consists of a large number n of impulses occurring at random times and with random amplitudes. Physical examples are random gusts on airplane surfaces, or random disturbances in a vacuum tube circuit, leading to output voltage fluctuations known as the shot effect. Let $h(t)$ be the weighting function of the system; by definition, $h(t)$ measures the response of the system to a unit impulse function t time-units after the impulse occurs. For physical realizability $h(t) = 0$ for $t < 0$.

Let us assume that the various impulses occur at randomly occurring points on the time axis, and these successive events are independent. When λ, the number of impulses per second, is small each impulse stands out as separate entity. As λ increases the effects of individual impulses overlap. The former situation is called impulse noise, the latter situation is called thermal noise or random noise, and is the limiting form to which much noise approximates when there is a superposition of large numbers of small effects.

For fixed i let $A(t_i)$ be the set of possible amplitudes occurring at time t_i. Assume that $A(t_i)$ form a set of mutually independent random variables with common mean and mean square values that are independent of the times t_i. Then for arbitrary i and t_i,

$$EA(t_i) = \overline{a}\,, \tag{2.5.1}$$

and

$$EA(t_i)A(t_k) = \begin{cases} \overline{a^2}\,, & i = k, \\ (\overline{a})^2, & i \neq k. \end{cases} \tag{2.5.2}$$

The input random process is expressed by $Y(t)$ and a realization is given by

$$Y(t) = \sum_{i=1}^{n} A(t_i)\delta(t-t_i), \tag{2.5.3}$$

where $\delta(t-t_i)$ is an impulse function at $t = t_i$. The output random process in an interval $(-T, T)$, after passing through a linear system with weighting function $h(t)$, is expressed by $X(t)$ with

$$X(t) = \int_{-T}^{T} Y(t)h(t-\tau)d\tau = \sum_{i=1}^{2\lambda T} A(t_i)h(t-t_i), \tag{2.5.4}$$

since $n = 2\lambda T$ impulses will occur in $(-T, T)$ if the average density of impulses is λ.

The mean of the input is given by

$$EY(t) = \lim_{T\to\infty}\left[\frac{E\int_{-T}^{T} Y(t)dt}{2T}\right] = \lim_{T\to\infty}\left[\frac{E\sum_{i=1}^{2\lambda T} A(t_i)}{2T}\right] = \lim_{T\to\infty}\left[\frac{2\lambda T\, EA(t_i)}{2T}\right] = \lambda\bar{a}, \tag{2.5.5}$$

where the average is taken before the limiting operation on T in order to secure the existence of the limit. The same argument is used in (2.5.6) and (2.5.7) below.

For (2.5.4) it follows that

$$EX(t) = \lim_{T\to\infty} \left[(2\lambda T\bar{a})\frac{1}{2T}\int_{-T}^{T} h(t-\tau)d\tau\right] = \lambda\bar{a}\int_{-\infty}^{\infty} h(t-\tau)d\tau. \tag{2.5.6}$$

We note that $EX(t) = 0$ for all t if $\bar{a} = 0$, as might be expected.

The power spectral density of the input is defined by Papoulis [14, p. 343],

$$S_Y(\omega) = \lim_{T\to\infty} \left[\frac{E\left|\int_{-T}^{T} Y(t)e^{-j\omega t}dt\right|^2}{2\pi T} \right]$$
$$= \lim_{T\to\infty} \left[\frac{E \sum_{i,k=1}^{2\lambda T} A(t_i)A(t_k)e^{-j\omega t_i} e^{-j\omega t_k}}{2\pi T} \right] \tag{2.5.7}$$

In agreement with much observed noise data we can assume that $\overline{a} = 0$. With this hypothesis (2.5.7) may be written in the form

$$S_Y(\omega) = \lim_{T\to\infty} \frac{2\lambda T \overline{a^2}}{2\pi T} = \frac{\lambda \overline{a^2}}{\pi} \; . \tag{2.5.8}$$

Thus $S_Y(\omega)$ is a constant, which implies that the input random process has the characteristics of a white noise source.

Corresponding to $S_Y(\omega)$ the input autocorrelation function is

$$\rho_Y(\tau) = \lambda \overline{a^2} \delta(\tau), \tag{2.5.9}$$

where $\delta(\tau)$ is the usual unit impulse function at $\tau = 0$. The corresponding output spectral density is given by Wozencraft and Jacobs [28],

$$S_X(\omega) = \frac{\lambda \overline{a^2}}{\pi} |H(j\omega)|^2, \tag{2.5.10}$$

where $H(j\omega)$ is the frequency response function of the linear system, defined by

$$H(j\omega) = \int_0^\infty h(t)e^{-j\omega t}dt. \tag{2.5.11}$$

For a stationary process the input-output autocorrelation functions are related by

$$\rho_X(T) = \int_0^\infty \int_0^\infty h(\alpha)h(\beta)\rho_Y(T+\alpha-\beta)d\alpha d\beta \tag{2.5.12}$$

(see Wozencraft and Jacobs [28, Eq. (3.108)]). Here $h(t) = 0$ for $t < 0$, for physical realizability. Substituting from (2.5.9) into (2.5.12) we arrive at

$$\rho_X(T) = \lambda \overline{a^2} \int_0^\infty h(\alpha)h(\alpha+T)d\alpha. \tag{2.5.13}$$

Let us now consider the output of a low-pass filter with weighting function of the form

$$h(t) = \begin{cases} e^{-c_1 t} - e^{-c_2 t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{2.5.14}$$

where c_1 and c_2 are positive constants, $c_2 > c_1$. Let the input to the filter be a large number of impulses occurring at random times with finite density λ and with random independent amplitudes of finite mean square pulse strength $\overline{a^2}$. From (2.5.13) and (2.5.14) we obtain the covariance function of the output,

$$\rho_X(T) = \lambda \overline{a^2} \int_0^\infty [e^{-c_1 a} - e^{-c_2 a}][e^{-c_1(a+T)} - e^{-c_2(a+T)}]\,da,$$

$$= \lambda \overline{a^2} \left[\frac{c_2 - c_1}{2c_1(c_1+c_2)} e^{-c_1|T|} - \frac{c_2 - c_1}{2c_2(c_1+c_2)} e^{-c_2|T|} \right]. \qquad (2.5.15)$$

III. ZERO-CROSSINGS OF A STATIONARY GAUSSIAN PROCESS

3.1. Preliminary Remarks

Throughout this chapter we shall consider and derive some probabilities of interest in connection with real valued stationary Gaussian processes, zero mean and covariance function

$$\rho(\tau) = 1 - \frac{\tau^2}{2} + \frac{a}{6}|\tau|^3 + O(\tau^4). \tag{3.1.1}$$

First we have the definition employed by Slepian [21].

Definition: The continuous covariance function $r(\tau)$ is said to be of class α if, as τ approaches zero,

$$r(\tau) = 1 - \frac{|\tau|^\alpha}{(\alpha+1)} + o(|\tau|^\alpha), \tag{3.1.2}$$

and if $r(\tau)$ is strictly monotone in some neighborhood, $0 < \tau < \tau_o$, of the origin.

In Chapter II, p. 19, we proved that, for the Gaussian process $X(t)$,

$$\lim_{\tau \to 0} \frac{2[1-\rho(\tau)]}{\tau^2} = \lambda_2, \tag{3.1.3}$$

and that if $\lambda_2 < \infty$ then $\rho''(0)$ exists and is equal to $-\lambda_2$. The covariance function (3.1.1) satisfies the condition (3.1.2) for $\alpha = 2$.

Since

$$\lim_{\tau \to 0} \frac{2[1-\rho(\tau)]}{\tau^2} = 1$$

if $\rho(\tau)$ is given by (3. 1. 1) it follows from (3. 1. 3) that $\rho''(0) = -1$.

The covariance function given by (3. 1. 1) has the first derivative zero at the origin, and since $\rho''(0) = -1$ it is clear that $\rho(\tau)$ is strictly monotone in the neighborhood of the origin. So, $\rho(\tau)$ given by (3. 1. 1) is of class 2.

3. 2. The Interval Between Successive Zero-Crossings of the Process X(t)

One of the outstanding problems--still unsolved in its entirety--in the mathematical theory of noise is the determination of the distribution of the randomly varying intervals between axis crossings.

For Gaussian noise Rice [17, 18] has given an approximate result and has discussed the difficulties which impede a more accurate solution. The first exact solution for a very special case of the problem was obtained by Wong [27]. In what follows we shall study the process considered by Wong [27], discuss some of the results obtained by him and obtain some new results concerning probabilities of interest.

Let I be a random variable denoting the interval between successive zero-crossings of $X(t)$. Let $F_I(t) = \mathrm{Prob}[I \leq t]$ be the distribution function of I, and let $f_I(t) = \frac{dF_I(t)}{dt}$ be the corresponding density function.

Wong [27] considered the real-valued stationary Gaussian process $X(t)$, mean zero and covariance function

$$\rho(\tau) = EX(t+\tau)X(t) = \frac{3}{2}\, e^{-|\tau|/\sqrt{3}}\left[1 - \frac{1}{3}\, e^{-2|\tau|/\sqrt{3}}\right] \tag{3.2.1}$$

which is a special case of (3.1.1), $a = 4/\sqrt{3}$. By using recent results in the theory of Brownian motion Wong [27] was able to obtain the following exact expressions for distribution and density functions respectively:

$$F_I(t) = 1 - \frac{3}{2\pi}\left\{\frac{[1-2r^2(t)]^{3/2}}{3-2r^2(t)}\, \pi_1\left[-\frac{3}{4}+\frac{1}{2}r^2(t),\, r(t)\right] + \frac{2\sqrt{1-2r^2(t)}}{3-2r^2(t)}\, K[r(t)]\right\} \tag{3.2.2}$$

and

$$f_I(t) = \frac{\sqrt{3}}{4}\left\{\frac{[1-2r^2(t)]^{1/2}}{[1-r^2(t)][1+2r^2(t)]}\, E[r(t)] + \frac{[1-2r^2(t)]^{1/2}}{3-2r^2(t)}\left[\frac{K[r(t)]-E[r(t)]}{r^2(t)}\right]\right.$$
$$\left. + \frac{8[1-2r^2(t)]^{3/2}}{[3-2r^2(t)]^2[1+2r^2(t)]}\left(\pi_1\left[-\frac{3}{4}+\frac{1}{2}r^2(t),\, r(t)\right] - K[r(t)]\right)\right\}, \tag{3.2.3}$$

where

$$r(t) = \sqrt{\frac{1}{2}(1-e^{-t/\sqrt{3}})}\,. \tag{3.2.4}$$

The three complete elliptic functions are defined as

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi}\, d\varphi,$$

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}},$$

$$\pi_1(\nu, k) = \int_0^{\pi/2} \frac{d\varphi}{(1+\nu \sin^2 \varphi)\sqrt{1-k^2 \sin^2 \varphi}}, \qquad k^2 < 1,$$

the first, second and third forms respectively.

It should be noted that (3. 2. 2) and (3. 2. 3) were obtained under the condition $X(0) = 0$ in the horizontal window sense of Kac and Slepian [8].

Remarks are now in order about the physical situation that gives rise to the random process considered by Wong [27]. We will now show that the covariance function given by (3. 2. 1) stems from the example discussed in Section 2. 5. That is, pass a random process $Y(t)$ --consisting of a large number of impulses occurring at random times with random amplitudes $A(t)$, such that $EA(t) = \bar{a} = 0$ and $\overline{a^2}$ is independent of time--through a lowpass filter with response function

$$h(t) = \begin{cases} e^{-c_1 t} - e^{-c_2 t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{3. 2. 5}$$

where c_1, c_2 are positive constants with $c_2 > c_1$. We proved that the output random process $X(t)$ has the covariance function of (2.5.15), which may be written out as

$$\rho_X(t) = \frac{\lambda \overline{a^2}}{2} \frac{c_2 - c_1}{c_2 + c_1} \left[\frac{e^{-c_1|t|}}{c_1} - \frac{e^{-c_2|t|}}{c_2} \right]. \tag{3.2.6}$$

For $c_1 = 1/\sqrt{3}$, $c_2 = \sqrt{3}$ and $\lambda \overline{a^2} = 2\sqrt{3}$ this takes the form

$$\rho_X(\tau) = \frac{3}{2} e^{-|\tau|/\sqrt{3}}[1 - \frac{1}{3} e^{-2|\tau|/\sqrt{3}}], \tag{3.2.7}$$

which may be regarded as the covariance function associated with output of a lowpass filter with weighting function of the form (2.5.14), pure noise input.

3.3. First Moment of the Interval I Between Successive Zeros

The expected value of the random variable I as given by

$$E(I) = \int_0^\infty t f_I(t) dt = \int_0^\infty [1 - F_I(t)] dt$$

$$= \frac{3}{2\pi} \int_0^\infty \left\{ \frac{[1-2r^2(t)]^{3/2}}{3-2r^2(t)} \pi_1[-\frac{3}{4} + \frac{1}{2} r^2(t), r(t)] + \frac{2\sqrt{1-2r^2(t)}}{3-2r^2(t)} K[r(t)] \right\} dt. \tag{3.3.1}$$

As it stands it is a difficult integral to evaluate. A simple approach is to calculate the mean value of the random variable I; in Chapter II we showed that the mean number of zero-crossings in time t of the process $X(t)$ is given by

$$M(t) = E[C_0(0, t)] = \frac{t}{\pi} \sqrt{-\frac{\rho''(0)}{\rho(0)}} . \tag{3.3.2}$$

For an $X(t)$ with the covariance function of (3.2.1) the mean number of zero crossings in time t is

$$M(t) = E[C_0(0, t)] = \frac{t}{\pi} . \tag{3.3.3}$$

Substitution into (1.3.11) leads to

$$E(I) = \pi, \tag{3.3.4}$$

which, as indicated previously, holds for $t \gg 1$. This result agrees with that of Rice [19].

Equation (3.3.1) suggests that

$$\frac{2\pi^2}{3} = \int_0^\infty \left\{ \frac{[1-2r^2(t)]^{3/2}}{3-2r^2(t)} \pi_1[-\frac{3}{4}+\frac{1}{2}r^2(t), r(t)] + \frac{2\sqrt{1-2r^2(t)}}{3-2r^2(t)} K[r(t)] \right\} dt . \tag{3.3.5}$$

Rainal [16], established by numerical analysis that this is indeed the case.

3.4. Zero-Crossings as a Stationary Point Process

In Chapter I we defined and studied the stationary point process. We now apply the theory of stationary point processes to the zero-crossings of the Gaussian process $X(t)$. To do that we have to show that the zero-crossings of $X(t)$ form a stationary point process in the sense of the definition of Chapter I.

Since $X(t)$ is a stationary Gaussian process it follows that $X(t)$ is also strictly stationary (Wozencraft and Jacobs [28, p. 184]). That is, all joint density functions are invariant to any translation in time origin. Hence, the instants at which $X(t)$ crosses the time axis form a stationary point process in the sense of the definition of Chapter I.

The interpretation of the zero-crossings in terms of the stationary point process is sometimes very convenient. In this section we make the identification and obtain some useful results. In practice the random sequence $W, L_1, L_2, \ldots$ is the one observed, rather than the random sequence $I_1, I_2, \ldots$, which has been considered so far. For that reason it is of statistical and practical importance to obtain some probabilities of interest with respect to the former random sequence.

We proved in Chapter I that the density function of the forward recurrence time W is related to the distribution function of the random variable I by the relation

$$f_I(t) = \frac{R_I(t)}{E(I)} = \frac{1-F_I(t)}{E(I)} .$$

Using (3. 2. 2) and (3. 3. 4) we arrive at

$$f_W(t) = \frac{3}{2\pi^2} \left\{ \frac{[1-2r^2(t)]^{3/2}}{3-2r^2(t)} \pi_1[-\frac{3}{4}+\frac{1}{2}r^2(t), r(t)] + \frac{2\sqrt{1-2r^2(t)}}{3-2r^2(t)} K[r(t)] \right\} . \tag{3.4.1}$$

The probability $Q(t)$ that a random process does not cross the zero axis in a given time interval of length t has important applications in communication theory. In a recent paper Strakhov and Kurz [23] obtained a family of upper bounds on $Q(t)$. However, they found that only one member of the family provides useful results in the case of a stationary Gaussian process $X(t)$. In particular,

$$Q(t) \leq q_2,$$

where

$$q_2 = \int_{-1}^{1} z(t) dF_z(t)$$

and

$$z(t) = \frac{1}{t} \int_0^t \operatorname{sgn}[X(T)] dT.$$

$F_z(t)$ is the cumulative distribution function associated with the random variable $z(t)$.

In what follows we shall obtain an exact expression for the probability that $X(t)$ does not cross the zero-axis in a time interval t.

The probability of no zeros in time t following an arbitrary zero-crossing can be obtained in an exact form, using (3. 2. 2):

$$\mathrm{Prob}[N_t^{(f)}=0] = \mathrm{Prob}[I>t] = 1 - F_I(t)$$

$$= \frac{3}{2\pi}\left\{\frac{[1-2r^2(t)]^{3/2}}{3-2r^2(t)}\,\pi_1[-\tfrac{3}{4}+\tfrac{1}{2}r^2(t), r(t)] + \frac{2\sqrt{1-2r^2(t)}}{3-2r^2(t)}\,K[r(t)]\right\}. \tag{3. 4. 2}$$

On the other hand, the probability that no zero-crossings will occur in an interval of length t following an arbitrarily chosen sample point is given by

$$\mathrm{Prob}[N_t=0] = \mathrm{Prob}[W>t] = 1 - F_W(t)$$

$$= 1 - \int_0^t f_W(\alpha)d\alpha, \tag{3. 4. 3}$$

where $f_W(t)$ is given by (3. 4. 1). The integral in the last equation can not be obtained in a closed form for all values of t, though it can be obtained in an exact form--as we shall see in the next section--for small values of t.

Finally, we consider the probability $P[T]$ that $X(t)$ be non-negative for $0 \leq t \leq T$. This is of interest as a means of describing the duration of the excursions of the process from its mean. From its definition it is clear that $P[T]$ is a nonincreasing function of T which assumes the value one-half for $T = 0$.

Since $N_t = 0$ implies that either $X(t) > 0$ or $X(t) < 0$ in the interval of time duration T then

$$P[T] = \frac{1}{2}\,\text{Prob}[N_T=0] = \frac{1}{2}[1-F_W(T)]. \tag{3.4.4}$$

3.5 Some Results for Small t

If we let $p(t)dt$ be the probability that $X(t)$ has at least one zero in the interval $(t, t+dt)$ then

$$p(t) = \frac{1}{\pi}$$

(see Slepian [20]). Thus, for small t,

$$\text{Prob}[N_t \geq 1] = \int_0^t p(\alpha)d\alpha = \frac{t}{\pi} \tag{3.5.1}$$

This last result agrees with Korolyuk's theorem (see Leadbetter [10]). Using (3.4.3) we have

$$\text{Prob}[N_t \geq 1] = 1-\text{Prob}[N_t=0] = 1-[1-F_W(t)] = F_W(t). \tag{3.5.2}$$

Comparing (3.5.1) and (3.5.2) we have, small t only,

$$F_W(t) = \frac{t}{\pi}. \tag{3.5.3}$$

Substituting into (3.4.4) we arrive at

$$P[T] = \frac{1}{2}[1-F_W(T)] = \frac{1}{2}[1-\frac{T}{\pi}], \tag{3.5.4}$$

small t only.

Slepian [21] obtained several lower bounds for $P[T]$. In particular, he obtained for $\rho(T)$, a nonnegative covariance function of Class 2, the bound

$$P[T] \geq \begin{cases} \frac{1}{2}[1 - \frac{T}{\pi}], & 0 \leq T \leq \frac{\pi}{2}, \\ \frac{1}{4}[\frac{3}{2} - \frac{T}{\pi}], & \frac{\pi}{2} \leq T < \frac{3\pi}{2}. \end{cases} \tag{3.5.5}$$

The value we obtained for $P[T]$ in (3.5.4) agrees with (3.5.5) for small t and gives the lower bound.

IV. THE METHOD OF ENVELOPES

4.1. Preliminary Remarks

Suppose $\xi(t)$ is a function of time which oscillates about an arbitrary level; we may try to find a nonnegative function $A(t)$ which is easier to investigate, such that $|\xi(t)| \leq A(t)$ for all t and $|\xi(t)| = A(t)$ for some t. A function $A(t)$ of that type may be denoted as an envelope of $\xi(t)$. For example, if $\xi(t)$ is of the simple form

$$\xi(t) = C \cos(\omega t + \theta)$$

it is clear that $|\xi(t)| \leq C$ and thus we can define the envelope function as $A(t) = C$.

When the waveform is a realization of a stochastic process it is not obvious how the envelope should be defined. Rice [17, 18], Bunimovitch [1], Dugundji [5] and others have given precise mathematical definitions for the envelope in such cases. In what follows we shall present these definitions and investigate the question of equivalence.

Rice's Definition. First we write the real waveform $X(t)$ as

$$X(t) = \sum_n C_n \cos(\omega_n t + \theta_n), \qquad (4.1.1)$$

then select a frequency q called the "midband frequency". Using this selected frequency (4. 1. 1) can be written in the form

$$X(t) = \sum_n C_n \cos[(\omega_n - q)t + \theta_n + qt]$$

$$= A_c \cos qt - A_s \sin qt,$$

where

$$A_c = \sum_n C_n \cos[(\omega_n - q)t + \theta_n], \quad A_s = \sum_n C_n \sin[(\omega_n - q)t + \theta_n].$$

The envelope $A(t)$ is now defined to be

$$A(t) = [A_c^2 + A_s^2]^{1/2}. \qquad (4.\ 1.\ 2)$$

Dugundji's Definition. If $X(t)$ is a real waveform let $\hat{X}(t)$ be its Hilbert transform. That is,

$$\hat{X}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(\alpha)}{t - \alpha} d\alpha,$$

where the principal value of the integral is always used. Define the "pre-envelope" function to be the complex valued function

$$X^*(t) = X(t) + i\hat{X}(t).$$

The envelope function $A(t)$ is then defined by

$$A(t) = |X^*(t)|. \qquad (4.\ 1.\ 3)$$

Using the definition (4. 1. 3) Dugundji [5] proved that $A(t)$ as defined by (4. 1. 2) is independent of the choice of the midband frequency and that his definition of the envelope agrees with that of Rice [17, 18] whenever the latter is applicable.

Zakai [31] has pointed out that Dugundji's results remain true when the time averages are replaced by ensemble averages, regardless of ergodicity. The only requirement is that the process be wide sense stationary.

Throughout this chapter we shall consider stationary Gaussian processes of the form

$$X(t) = A(t) \cos \Phi(t), \tag{4. 1. 4}$$

where the envelope $A(t)$ and the phase angle $\Phi(t)$ are independent random variables. We can regard $X(t)$ as the projection of $A(t)$ on the x-axis. The projection of $A(t)$ on the y-axis at the same instant of time,

$$Y(t) = A(t) \sin \Phi(t) \tag{4. 1. 5}$$

is the Hilbert transform of $X(t)$, as given by the definite integral. Dugundji [5] has established that $X(t)$ and $Y(t)$ have the same distribution, the same correlation function $\rho(\tau)$, the same spectral density function $S(\omega)$, and they are uncorrelated. That is,

$$\rho_X(\tau) = \rho_Y(\tau) = \rho(\tau), \tag{4. 1. 6}$$

and

$$r(0) = \rho_{XY}(0) = 0. \tag{4.1.7}$$

Without loss of generality we can consider $X(t)$ to have zero mean and variance $\sigma_X^2 = 1$.

4.2. Distribution Densities of the Envelope and the Phase

The probability that $A(t)$ lies between a and $a + da$ is given by

$$\operatorname{Prob}(a \leq A \leq a + da) = \iint\limits_{a \leq x^2 + y^2 \leq a + da} f(x, y)\,dx\,dy, \tag{4.2.1}$$

where $f(x, y)$ is the joint distribution function of $X(t)$ and $Y(t)$. That is

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{x^2 + y^2}{2}}, \quad -\infty < x < \infty, \quad -\infty < y < \infty. \tag{4.2.2}$$

Actually $f(x, y)$ should be written $f_{X,Y}(x, y)$, but for simplicity of notation we dropped the subscripts and we shall continue doing so throughout this chapter. A change to polar coordinates leads to the Rayleigh distribution,

$$f(a) = a\, e^{-a^2/2}, \quad 0 \leq a < \infty. \tag{4.2.3}$$

The phase angle $\Phi(t)$ is uniformly distributed over the interval $[0, 2\pi]$; that is,

$$f(\varphi) = \frac{1}{2\pi}, \quad 0 \leq \varphi \leq 2\pi . \tag{4. 2. 4}$$

4. 3. Second Order Densities of the Envelope and the Phase

The system of random variables

$$\begin{aligned} X_1 &= X(t), \qquad X_2 = X(t+T), \\ Y_1 &= Y(t), \qquad Y_2 = Y(t+T), \end{aligned} \tag{4. 3. 1}$$

is by assumption a system of Gaussian variables with zero mean, hence the four dimensional density function $f(x_1, x_2, y_1, y_2)$ is uniquely determined by the covariance matrix

$$M = \begin{bmatrix} 1 & \rho(T) & 0 & r(T) \\ \rho(T) & 1 & -r(T) & 0 \\ 0 & -r(T) & 1 & \rho(T) \\ r(T) & 0 & \rho(T) & 1 \end{bmatrix}, \tag{4. 3. 2}$$

where we have used the fact that $\rho(-T) = -\rho(T)$. The four dimensional density function corresponding to the above covariance matrix is given by

$$f(x_1, x_2, y_1 \cdot y_2) = \frac{1}{(2\pi)^2 |M|^{1/2}} \exp\left(-\frac{1}{2} CM^{-1} C^T\right),$$

where $|M|$ is the determinant of M,

$$C = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$$

and C^T is the transpose of C. Hence

$$f(x_1,x_2,y_1,y_2)=\frac{1}{4\pi^2 a^2}\exp\left[-\frac{x_1^2+x_2^2+y_1^2+y_2^2}{2a^2}+\frac{\rho}{a^2}(x_1x_2+y_1y_2)+\frac{r}{a^2}(x_1y_2-x_2y_1)\right] \tag{4. 3. 3}$$

where

$$a^2 = 1 - \rho^2(\mathrm{T}) - r^2(\mathrm{T}) = 1 - \rho^2 - r^2. \tag{4. 3. 4}$$

To obtain the four dimensional density function of the system of random variables

$$A_1 = A(t), \qquad A_2 = A(t+\mathrm{T}),$$

$$\Phi_1 = \Phi(t), \qquad \Phi_2 = \Phi(t+\mathrm{T})$$

we make the substitution

$$x_1 = a_1 \cos \varphi_1, \quad x_2 = a_2 \cos \varphi_2,$$

$$y_1 = a_1 \sin \varphi_1, \quad y_2 = a_2 \sin \varphi_2,$$

into (4. 3. 3), arriving at

$$f(a_1,a_2,\varphi_1,\varphi_2) = \frac{a_1 a_2}{4\pi^2 \alpha^2} \exp[-\frac{a_1^2+a_2^2}{2\alpha^2} + \frac{a_1 a_2}{\alpha^2}\rho\cos(\varphi_2-\varphi_1) + \frac{a_1 a_2}{\alpha^2} r\sin(\varphi_2-\varphi_1)],$$

$$0 \leq a_1, a_2, \tag{4. 3. 5}$$

or, using the notation,

$$\gamma = \arctan\frac{r}{\rho},$$

the density function takes on the form

$$f(a_1,a_2,\varphi_1,\varphi_2) = \frac{a_1 a_2}{4\pi^2 \alpha^2} \exp[-\frac{a_1^2+a_2^2}{2\alpha^2} + \frac{a_1 a_2}{\alpha^2}\sqrt{1-\alpha^2}\cos(\varphi_2-\varphi_1-\gamma)],$$

$$0 \leq a_1, a_2, \tag{4. 3. 6}$$

Integration with respect to φ_1 and φ_2 over the 2π-square leads to

$$f(a_1,a_2) = \frac{a_1 a_2 \exp(-\frac{a_1^2+a_2^2}{2\alpha^2})}{4\pi^2\alpha^2}\int_0^{2\pi}\int_0^{2\pi}\exp[\frac{a_1 a_2}{\alpha^2}\sqrt{1-\alpha^2}\cos(\varphi_2-\varphi_1-\gamma)]d\varphi_1 d\varphi_2$$

$$= \frac{a_1 a_2}{\alpha^2}\exp(-\frac{a_1^2+a_2^2}{2\alpha^2}) I_0[\frac{a_1 a_2}{\alpha^2}\sqrt{1-\alpha^2}], \tag{4. 3. 7}$$

where

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}d\theta,$$

is the Bessel function of the first kind of order zero. Expression

(4. 3. 7) was first obtained by Rice [17, 18].

Integration of (4. 3. 6) with respect to a_1 and a_2 leads to

$$f(\varphi_1, \varphi_2) = \int_0^\infty \int_0^\infty f(a_1, a_2, \varphi_1, \varphi_2) da_1 da_2$$

$$= \frac{1}{4\pi^2 \alpha^2} \int_0^\infty \int_0^\infty a_1 a_2 \exp\left[-\frac{a_1^2 + a_2^2}{2\alpha^2} + \frac{a_1 a_2}{\alpha^2}\sqrt{1-a^2}\cos(\varphi_2 - \varphi_1 - \gamma)\right] da_1 da_2. \qquad (4.\ 3.\ 8)$$

With the help of the transformation

$$a_1 = \alpha\sqrt{z}\, e^{\theta/2}, \quad a_2 = \alpha\sqrt{z}\, e^{-\theta/2},$$

(4. 3. 8) may be written as

$$f(\varphi_1, \varphi_2) = \frac{\alpha^2}{4\pi^2} \int_0^\infty z\, e^{z\sqrt{1-a^2}\cos(\varphi_2 - \varphi_1 - \gamma)} \left(\frac{1}{2}\int_{-\infty}^\infty e^{-z\cosh\theta} d\theta\right) dz,$$

and since

$$K_o(z) = \int_0^\infty e^{-z\cosh\theta} d\theta, \quad z \geq 0,$$

is the modified Bessel function of the second kind (see Watson [24, p. 182]) we can write

$$f(\varphi_1, \varphi_2) = \frac{\alpha^2}{4\pi^2} \int_0^\infty z\, e^{z\sqrt{1-a^2}\cos(\varphi_2 - \varphi_1 - \gamma)} K_o(z) dz.$$

This in turn can be evaluated (see Watson [24, p. 410]) to yield the desired end result

$$f(\varphi_1, \varphi_2) = \frac{\alpha^2}{4\pi^2}\left\{\frac{1}{1-\beta^2} + \beta\frac{\frac{\pi}{2}+\arcsin\beta}{(1-\beta^2)^{3/2}}\right\} \tag{4. 3. 9}$$

where

$$\beta = \sqrt{1-\alpha^2}\cos(\varphi_2-\varphi_1-\gamma), \tag{4. 3. 10}$$

a relation obtained by MacDonald [12].

4. 4. The Density Functions of the Time Derivatives of the Phase Angle and of the Envelope

We shall assume that the envelope function and the phase angle are differentiable or, equivalently, that the original random process $X(t)$ is differentiable. The time derivative of the phase $\Phi(t)$ is, by definition,

$$\dot{\Phi}(t) = \lim_{\tau\to 0}\frac{\Phi(t+\tau)-\Phi(t)}{\tau} .$$

To determine the density function of $\dot{\Phi}(t)$ we pass from the density (4. 3. 9) of the system of random variables $\Phi_1 = \Phi(t)$ and $\Phi_2 = \Phi(t+\tau)$ to the density function of the system or random variables $\frac{\Phi_2-\Phi_1}{\tau}$ and Φ_1, then integrate the expression obtained with respect to all possible values of Φ_1 and, finally, pass to the limit as $\tau \to 0$.

Carrying out the procedure described we arrive at

$$f(\varphi) = \lim_{\tau \to 0} \frac{\alpha^2 \tau}{2\pi} \left[\frac{1}{1-\beta^2} + \beta \frac{\frac{\pi}{2} + \arcsin \beta}{(1-\beta^2)^{3/2}} \right], \qquad (4.4.1)$$

a result obtained by Bunimovitch [1]. As $\tau \to 0$, $\alpha \to 0$, $\varphi_2 \to \varphi_1$ and $\gamma \to 0$, so β, as defined by (4.3.10) tends to unity and the expression on the right side of (4.4.1) becomes indeterminant. To resolve this difficulty we expand α^2, β, $\arcsin \beta$ and $(1-\beta^2)$ in powers of τ, then allow $\tau \to 0$. Taking into account that

$$\rho(0) = 1, \quad r(0) = 0, \quad \dot{\rho}(0) = 0$$

and that $\ddot{\rho}(0)$ exists, a fact which is implied by the assumption that $X(t)$ is differentiable, we get

$$\alpha^2 \simeq -[\ddot{\rho}(0) + \dot{r}^2(0)]\tau^2, \qquad (4.4.2)$$

$$\beta^2 \simeq 1 - [\dot{\varphi} - \dot{r}(0)]^2\tau^2 - [\ddot{\rho}(0) + \dot{r}^2(0)]\tau^2, \qquad (4.4.3)$$

$$1 - \beta^2 \simeq [\dot{\varphi} - \dot{r}(0)]^2\tau^2 - [\ddot{\rho}(0) + \dot{r}^2(0)]\tau^2. \qquad (4.4.4)$$

Substituting the last three expressions into (4.4.2) and taking the limit we get

$$f(\varphi) = \frac{-\ddot{\rho}(0) - \dot{r}^2(0)}{2\{[\dot{\varphi} - \dot{r}(0)]^2 + [-\ddot{\rho}(0) - \dot{r}^2(0)]\}^{3/2}}. \qquad (4.4.5)$$

By assumption $\ddot{\rho}(0)$ exists, so by the theorem of Chapter II,

$$\ddot{\rho}(0) = -\lambda_2,$$

where λ_2 is the second spectral moment. Also, since

$$r(\tau) = \int_{-\infty}^{\infty} S_X(\omega)\sin\omega\tau\, d\omega$$

we have

$$\dot{r}(0) = \int_{-\infty}^{\infty} \omega\, S_X(\omega)d\omega = \lambda_1.$$

Thus the quantities $\ddot{\rho}(0)$ and $\dot{r}(0)$, determining the form of the density $f(\dot{\varphi})$, are uniquely determined by the spectral density $S_X(\omega)$ of the original random process $X(t)$. Equation (4. 4. 5) may now be written in the form

$$f(\dot{\varphi}) = \frac{\lambda_2 - \lambda_1^2}{2[(\dot{\varphi}-\lambda_1)^2 + (\lambda_2 - \lambda_1^2)]^{3/2}} \tag{4. 4. 6}$$

The difference

$$\Delta^2 = \lambda_2 - \lambda_1^2 \tag{4. 4. 7}$$

occurring in (4. 4. 6) may be represented as

$$\lambda_2 - \lambda_1^2 = \int_{-\infty}^{\infty} (\omega - \lambda_1)^2 S_X(\omega)d\omega, \tag{4. 4. 8}$$

so may be considered as the "mean width of the spectrum".

Formula (4. 4. 6) shows that the $f(\dot{\varphi})$ is symmetric about the value $\dot{\varphi} = \lambda_1$ and not about the origin. Accordingly the probability of increase of the phase, $\text{Prob}(\dot{\varphi} > 0)$, is not equal to the probability of its decrease, $\text{Prob}(\dot{\varphi} < 0)$, where

$$\text{Prob}(\dot{\varphi} > 0) = \frac{1}{2}\int_0^\infty \frac{\lambda_2 - \lambda_1^2}{[(\dot{\varphi}-\lambda_1)^2 + (\lambda_2 - \lambda_1^2)]^{3/2}}\, d\dot{\varphi}$$

$$= \frac{1}{2}\left(1 + \frac{\lambda_1}{\sqrt{\lambda_2}}\right), \tag{4. 4. 9}$$

and

$$\text{Prob}(\dot{\varphi} < 0) = \frac{1}{2}\int_{-\infty}^0 \frac{\lambda_2 - \lambda_1^2}{[(\dot{\varphi}-\lambda_1) + (\lambda_2 - \lambda_1^2)]^{3/2}}\, d\dot{\varphi}$$

$$= \frac{1}{2}\left(1 - \frac{\lambda_1}{\sqrt{\lambda_2}}\right) \tag{4. 4. 10}$$

To calculate the density function of the time derivative of the envelope function we consider the mean value theorem in the form

$$a_2(t+T) = a_1(t) + T\dot{a}_1(t). \tag{4. 4. 11}$$

Equation (4. 3. 7) can be written in the form

$$f(a_1, a_2)$$

$$= \frac{a_1(a_1+T\dot{a}_1)}{\alpha^2} \exp\left(-\frac{2a_1^2 + 2a_1\dot{a}_1 T + \dot{a}_1^2 T^2}{2\alpha^2}\right) I_o\left[\frac{a_1(a_1+T\dot{a}_1)}{\alpha^2}\sqrt{1-\alpha^2}\right]. \tag{4. 4. 12}$$

To obtain the joint density function $f(a_1, \dot{a}_1)$ of the envelope and of its time derivative we have to multiply $f(a_1, a_2)$ by T and then let $T \to 0$, as in the derivations of (4. 3. 13). That is,

$$f(a_1, \dot{a}_1)$$

$$= \lim_{T \to 0} \frac{T a_1(a_1+T\dot{a}_1)}{\alpha^2} \exp\left(-\frac{2a_1^2+2a_1\dot{a}_1T+\dot{a}_1^2T^2}{2\alpha^2}\right) I_o\left[\frac{a_1(a_1+T\dot{a}_1)}{\alpha^2}\sqrt{1-\alpha^2}\right]. \tag{4.4.13}$$

As $T \to 0$, α^2 tends to zero as T^2 so the argument of the Bessel function increases without limit. For large values of x the asymptotic representation of $I_o(x)$ is in the form

$$I_o(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \tag{4.4.14}$$

where the error decreases as x increases (see Whittaker and Watson [25, p. 373]).

Equation (4. 4. 13) can now be written in the form

$$f(a, \dot{a}) = \lim_{T \to 0} \frac{T a(a+T\dot{a})}{\alpha^2} \exp\left(-\frac{2a^2+2a\dot{a}T+\dot{a}^2T^2}{2\alpha^2}\right) \exp\left(\frac{a(a+Ta)}{\alpha^2}\sqrt{1-\alpha^2}\right)$$

$$\cdot \frac{\alpha}{\sqrt{2\pi(1-\alpha^2)a(a+T\dot{a})}}, \tag{4.4.15}$$

where the subscript is omitted from a_1 and $\dot{a}_1$. Since $\ddot{\rho}(0) = -\lambda_2$ and $\dot{r}(0) = \lambda_1$ the limiting procedure yields

$$f(a, \dot{a}) = a e^{-a^2/2} \frac{1}{\sqrt{2\pi}\,\Delta} e^{-\dot{a}^2/2\Delta^2}, \tag{4.4.16}$$

where Δ is defined by (4. 4. 8). Since the density function of the amplitude is given by

$$f(a) = a\, e^{-a^2/2}, \qquad a > 0, \tag{4.4.17}$$

expression (4. 4. 16) may be written in the form

$$f(a, \dot{a}) = f(a)f(\dot{a}) ,$$

where

$$f(\dot{a}) = \frac{1}{\sqrt{2\pi}\,\Delta}\, e^{-\dot{a}^2/2\Delta^2}, \qquad -\infty < a < \infty, \tag{4.4.18}$$

is the density function of the time derivative of the amplitude $A(t)$.

4. 5. Density Distribution of the Interval Between Successive Zero-Crossings of a Narrow Band Stationary Gaussian Process

The random functions occurring in applications are never narrow band since it is impossible to point out a range of frequencies outside of which $S_X(\omega) = 0$. However, if the spectral density has a sharp maximum the spectrum can be considered as approximately narrow band.

The density function $f_I(t)$ of the random interval I between successive zero-crossings can only be determined for a special case, as we saw in Chapter III. However, it is easy to obtain an approximate result in the case of narrow band noise.

If $X(t)$ is a narrow band stationary Gaussian process and λ_1 is the average frequency of the energy spectrum of the process, then

$X(t)$ can be written in the form (see Bunimovitch [2])

$$X(t) = A(t) \cos \Phi(t) = A(t) \cos [\lambda_1 t + \Omega(t)], \tag{4.5.1}$$

where $A(t)$ and $\Phi(t)$ are functions which vary slowly relative to $\cos \lambda_1 t$. The random function $\Omega(t)$ varies slowly over one period of time and, consequently, beginning with an arbitrary instant of time in (4.5.1), we have for the determination of the time T during which the phase changes by 2π,

$$\lambda_1 T + \dot{\Omega} T \simeq 2\pi. \tag{4.5.2}$$

That is,

$$T \simeq \frac{2\pi}{\lambda_1 + \dot{\Omega}}.$$

For the interval of time I during which $X(t)$ is above (below) the zero level we can write the approximation

$$I = \frac{T}{2} \simeq \frac{\pi}{\lambda_1 + \dot{\Omega}} = \frac{\pi}{\dot{\Phi}}. \tag{4.5.3}$$

Thus the density function of the random variable I is given by

$$f_I(t) = \frac{\pi}{t^2} f(\dot{\varphi}) \Big|_{\dot{\varphi} = \frac{\pi}{t}}.$$

Substituting for $f(\dot{\varphi})$ from (4.4.6) we have

$$f_I(t) = \frac{\pi \Delta^2}{2[(\frac{\pi}{t} - \lambda_1)^2 + \Delta^2]^{3/2}},$$

where Δ^2 is given by (4.4.7).

V. THE INTERVAL BETWEEN ZERO-CROSSINGS OF A STATIONARY GAUSSIAN PROCESS

5.1. Preliminary Remarks

In the present chapter we shall consider stationary Gaussian processes, first considering an arbitrary random process.

The determination of the distribution function, $F_I(t)$, of the random interval, I, between successive zero-crossings of $X(t)$ is difficult to obtain in a closed form, as we mentioned in Section 3.2.

Approximate expressions, as well as upper and lower bounds for $F_I(t)$, have been obtained by Rice [17, 18], McFadden [13a, 13b], Slepian [21, 21a], Longuet-Higgins [11a, 11b] and Strakhov and Kurtz [23], among others.

We shall discuss here some of the difficulties that impede the obtaining of a closed expression for $F_I(t)$. Also, we shall obtain a series expansion for $F_I(t)$ in terms of multiple integrals. The multiple integrals involved in obtaining the first, second and third order information were discussed by Kac [7a], Cramér [2a], Kamat [8a] and Nabeya [13c]. Finally, we shall discuss in some detail the fourth order information and show that the four-variate integral involved can be reduced to a double integral, much easier for numerical computations.

5.2. The Probability of No Zero-Crossings in a Given Time Interval

Let $Q(T)$ be the probability that a random process $X(t)$ does not cross the zero axis in a given time interval (t_o, t_o+T). Consider a Δ-partition of the (t_o, t_o+T) segment by the points.

$$t_o < t_1 < \ldots t_{n-1} < t_o + T = t_n,$$

and let

$$t_i - t_{i-1} = \Delta_i, \quad i = 1, 2, \ldots, n.$$

Let us introduce the random variables e_i, defined as follows:

$e_i = 1$ if at least one zero of $X(t)$ occurs in $(t_{i-1}, t_i]$,

$e_i = 0$ if no zeros of $X(t)$ occur in $(t_{i-1}, t_i]$.

Thus

$$\text{Prob}\,[e_{i_1} = 1, e_{i_2} = 1, \ldots, e_{i_k} = 1]$$

$$= q_n(t_{i_1}, t_{i_2}, \ldots, t_{i_k})\Delta_{i_1}\Delta_{i_2}\ldots\Delta_{i_k} \tag{5.2.1}$$

is the probability that $X(t)$ has at least one zero in each of the intervals $(t_{i_1}, t_{i_1}+\Delta_{i_1}), \ldots, (t_{i_k}, t_{i_k}+\Delta_{i_k})$. Since the events $e_i = 1$ and $e_i = 0$, $(i = 1, 2, \ldots, n)$ are mutually exclusive the total reliable event expressed, such that the value of e_i is not indicated, is given by

$$\text{Prob}\,[e_1 = 0, \ldots, e_{i-1} = 0, e_i = 1, e_{i+1} = 1, \ldots, e_n = 1]$$

$$+ \text{Prob}\,[e_i = 0, \ldots, e_{i-1} = 0, e_i = 0, e_{i+1} = 1, \ldots, e_n = 1]$$

$$= \text{Prob}\,[e_1 = 0, \ldots, e_{i-1} = 0, e_{i+1} = 1, \ldots, e_n = 1]. \qquad (5.2.2)$$

Thus

$$\text{Prob}\,[e_1 = 0, \ldots, e_{i-1} = 0, e_i = 0, e_{i+1} = 1, \ldots, e_n = 1]$$

$$= \text{Prob}\,[e_1 = 0, \ldots, e_{i-1} = 0, e_{i+1} = 1, \ldots, e_n = 1]$$

$$- \text{Prob}\,[e_1 = 0, \ldots, e_{i-1} = 0, e_i = 1, e_{i+1} = 1, \ldots, e_n = 1]. \qquad (5.2.3)$$

To obtain an expression for Prob $[e_1 = 0, \ldots, e_n = 0]$, in which only probabilities of the form of (5.2.1) will appear we use the formula (5.2.3) n times on Prob $[e_1 = 0, \ldots, e_n = 0]$. This expression is the sum of 2^n terms in which each e_i $(i = 1, 2, \ldots, n)$ either does not appear or is equal to one. Each Prob $[e_{i_1} = 1, e_{i_2} = 1, e_{i_3} = 1, \ldots, e_{i_k} = 1]$ $(0 < i_1 < i_2 < \ldots < i_k \leq n)$ is encountered only once and its sign is determined by k. That is

$$\text{Prob}\,[e_1 = 0, \ldots, e_n = 0] = 1 - \sum_{i_1} \text{Prob}\,(e_{i_1} = 1) + \sum_{i_1 < i_2} \text{Prob}\,(e_{i_1} = 1, e_{i_2} = 1)$$

$$- \ldots + (-1)^n \text{Prob}\,(e_1 = 1, \ldots, e_n = 1) \qquad (5.2.4)$$

By definition

$$Q(T) = \lim_{\max(\Delta_1, \ldots, \Delta_n) \to 0} \text{Prob}\,[e_1 = 0, \ldots, e_n = 0] \tag{5.2.5}$$

We now show that

$$Q(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^T \cdots \int_0^T q_k(t_1, \ldots, t_k)\,dt_1 \ldots dt_k \tag{5.2.6}$$

by following a method used by Hilbert [5d, p. 8] to prove a formula similar to (5.2.6) given by Fredholm.

From (5.2.4) and (5.2.6)

$$\text{Prob}\,[e_1 = 0, \ldots, e_n = 0]$$

$$= 1 - \sum_{i_1} q_1(t_{i_1})\Delta_{i_1} + \sum_{i_1 < i_2} q_2(t_{i_1}, t_{i_2})\Delta_{i_1}\Delta_{i_2} - \ldots$$

$$+ (-1)^n q_n(t_1, \ldots, t_n)\Delta_1 \ldots \Delta_n$$

$$= 1 - d_{1,n} + d_{2,n} - \ldots + (-1)^n d_{n,n} \tag{5.2.7}$$

where

$$d_{k,n} = \sum_{i_1 < \ldots < i_k}^{n} q_k(t_{i_1}, \ldots, t_{i_k})\Delta_{i_1} \ldots \Delta_{i_k}$$

$$= \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} q_k(t_{i_1}, t_{i_2}, \ldots, t_{i_k})\Delta_{i_1}\Delta_{i_2} \ldots \Delta_{i_k} \tag{5.2.8}$$

Denote

$$\delta_k = \frac{1}{k!}\int_0^T \cdots \int_0^T q_k(t_1, \ldots, t_k)dt_1 \ldots dt_k \qquad (5.2.9)$$

Since $q_k(t_1, \ldots, t_k)$ as defined by (5. 2. 1) is a density function and thus is Riemann integrable,

$$\lim_{n\to\infty} d_{k,n} = \delta_k \qquad (k = 1, 2, \ldots) \qquad (5.2.10)$$

To prove (5. 2. 6), we have to show that

$$\lim_{n\to\infty} (1-d_{1,n}+\ldots \pm d_{n,n}) = 1 - \delta_1 + \delta_2 - \ldots$$

The infinite series

$$1 - \delta_1 + \delta_2 - \delta_3 + \ldots$$

is convergent, since the integral of $q_k(t_1, \ldots, t_k)$ over the k-dimensional cube is always less than or equal to unity for all positive integers k.

Thus given a positive ϵ there exists a positive integer N such that for $n \geq N$

$$|(1-\delta_1+\ldots \pm\delta_N) - (1-\delta_1+\ldots \pm\delta_n)| < \frac{\epsilon}{3} \qquad (5.2.11)$$

and

$$|Q(T) - (1-d_{1,N}+\ldots \pm d_{N,N})| < \frac{\epsilon}{3}$$

For every k the sequence $\{d_{k,n}\}_{n=1}^{\infty}$ is Cauchy, thus there exists a positive integer M_1 such that for $m \geq M_1$

$$|(1-d_{1,N}+\ldots\pm d_{N,N}) - (1-d_{1,N+m}+\ldots\pm d_{N,N+m})| < \frac{\epsilon}{6} \tag{5.2.12}$$

Also because of (5.2.10) there exists a positive integer M_2 such that for $m \geq M_2$

$$|(1-d_{1,N+m}+\ldots\pm d_{N,N+m}) - (1-\delta_1+\ldots\pm\delta_N)| < \frac{\epsilon}{6}$$

From the last two inequalities we have

$$|(1-d_{1,N}+\ldots\pm d_{N,N}) - (1-\delta_1+\ldots\pm\delta_N)| < \frac{\epsilon}{3}$$

The inequalities (5.2.11), (5.2.12) and (5.2.13) complete the proof.

By generalizing an expression given by Rice [17, 18] for the probability that $X(t)$ has a downcrossing (t_1, t_1+dt_1) and an upcrossing in (t_2, t_2+dt_2) (see also McFadden [13b]) $q_n(t_1, t_2, \ldots, t_n)$ can be written in the form

$$q_n(t_1,\ldots,t_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |x_1\ldots x_n| f(0,\ldots,0; x_1',\ldots,x_n')dx_1'\ldots dx_n' \tag{5.2.14}$$

where $f(x_1,\ldots,x_n,x_1',\ldots,x_n')$ is the joint probability density function of the $2n$ random variable

$$X_i = X(t_i), \quad X_i' = \frac{dX(t)}{dt}\Big|_{t=t_i}, \quad i = 1,\ldots,n.$$

The distribution function $F_I(t)$ of the interval between successive zero-crossings is related to $Q(t)$ by

$$F_I(t) = \mathrm{Prob}(I \leq t) = 1 - \mathrm{Prob}(I > t) = 1 - Q(t). \qquad (5.2.15)$$

Thus the density function $f_I(t)$ of the random variable I is given by

$$f_I(t) = \frac{dF_I(t)}{dt} = -\frac{d}{dt}Q(t). \qquad (5.2.16)$$

Rice's formula ([17, 18] 3-4.11) can be obtained from (5.2.6) by differentiating with respect to T. He also took as an approximation for $f_I(\tau)$

$$f_I(\tau) \simeq q_2(\tau)$$

when $\tau = t_2 - t_1$ is "small."

Expression (5.2.8) for $Q(T)$ was obtained for an arbitrary random process. $Q(T)$ can be obtained if we compute $q_n(t_1, t_2, \ldots, t_n)$ for all positive integers n. As we shall see in the next section, $q_n(t_1, \ldots, t_n)$ can only be expressed in terms of elementary functions for $n \leq 3$ when $X(t)$ is a stationary Gaussian process. Also, if we require that $X(0) = 0$ in the horizontal window sense (see Kac and Slepian [8]) this would complicate some probabilities of interest. In this case the random variables $X(t_1), X(t_2), \ldots, X(t_n)$

are no longer jointly Gaussian and their joint density is given by

$$2\pi \int_{-\infty}^{\infty} dx' |x'| f(x', 0, x_1, \ldots, x_n),$$

where $f(x', x_0, x_1, \ldots, x_n)$ is a Gaussian joint density of the unconditioned random variables $X'(0), X(0), X(t_1), \ldots, X(t_n)$.

5.3. The Density Function $q_n(t_1, \ldots, t_n)$ for a Stationary Gaussian Process

Throughout the rest of this chapter we shall consider $X(t)$ to be a stationary Gaussian process with covariance function

$$EX(t)X(t+T) = \rho(T) = \rho_T . \tag{5.3.1}$$

The covariance matrix of the $2n$ random variables X_i and X_i' $(i = 1, 2, \ldots, n)$ is given by

$$R = \begin{bmatrix} \rho_{11} & \cdots & \rho_{1n} & \rho'_{11} & \cdots & \rho'_{1n} \\ \vdots & & & & & \\ \rho_{n1} & \cdots & \rho_{nn} & \rho'_{n1} & \cdots & \rho'_{nn} \\ -\rho'_{11} & \cdots & -\rho'_{1n} & -\rho''_{11} & \cdots & -\rho''_{1n} \\ -\rho'_{n1} & \cdots & -\rho'_{nn} & -\rho''_{n1} & \cdots & -\rho''_{nn} \end{bmatrix} \tag{5.3.2}$$

where

$$\rho_{ij} = \rho(t_i - t_j).$$

The covariance matrix R can be written in the partioned form

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \tag{5.3.3}$$

where

$$R_{11} = \begin{bmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & & \vdots \\ \rho_{n1} & \cdots & \rho_{nn} \end{bmatrix}, \qquad R_{12} = \begin{bmatrix} \rho'_{11} & \cdots & \rho'_{1n} \\ \vdots & & \vdots \\ \rho'_{n1} & \cdots & \rho'_{nn} \end{bmatrix},$$

$$R_{21} = -R_{12}, \quad \text{and} \quad R_{22} = \begin{bmatrix} -\rho''_{11} & \cdots & -\rho''_{1n} \\ \vdots & & \vdots \\ -\rho''_{n1} & \cdots & -\rho''_{nn} \end{bmatrix}.$$

For a Gaussian process it is known that

$$f(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = \frac{1}{(2\pi)^n |R|^{1/2}} \exp\left[-\frac{1}{2} y^T R^{-1} y\right] \tag{5.3.4}$$

where y^T is the transpose of

$$y = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x'_1 \\ \vdots \\ x'_n \end{bmatrix},$$

R^{-1} is the inverse of R and $|R|$ is the determinant of R.

Substituting from (5. 3. 4) into (5. 2. 9) we get

$$q_n(t_1,\dots,t_n) = \frac{1}{(2\pi)^n |R|^{1/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |x_1'\dots x_n'| \exp[-\tfrac{1}{2} Z'^T R_{22}^{-1} Z'] dx_1'\dots dx_n' \tag{5.3.5}$$

where

$$Z'^T = (x_1', \dots, x_n').$$

Let us denote the inverse matrix of R by $S = R^{-1}$. The matrix S, partitioned in the same manner as R, can be written in the form

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{5.3.6}$$

Denote $S_{22}^{-1} = M = [\mu_{ij}]$. By using a result on compound matrices (see [5a, p. 21]) we have

$$S\begin{pmatrix} n+1 & n+2 & \dots & 2n \\ n+1 & n+2 & \dots & 2n \end{pmatrix} = \frac{(-1)^{2n} R\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}}{R\begin{pmatrix} 1 & 2 & \dots & 2n \\ 1 & 2 & \dots & 2n \end{pmatrix}},$$

or equivalently,

$$|S_{22}| = \frac{|R_{11}|}{|R|}.$$

That is,

$$|M| = \frac{|R|}{|R_{11}|}. \tag{5.3.7}$$

The matrix $M = [\mu_{ij}]$ is the covariance matrix of $(X_1', X_2', \ldots, X_n')$, given that $X_1 = X_2 = \ldots = X_n = 0$. For, if $f(x_1', x_2', \ldots, x_n' | x_1, \ldots, x_n)$ denotes the conditional probability density of $(X_1', \ldots, X_n')$ for given values of $(X_1, \ldots, X_n)$, we have

$$f(x_1', \ldots, x_n' | x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n, x_1', \ldots, x_n')}{f(x_1, \ldots, x_n)},$$

where

$$f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} |R_{11}|^{1/2}} \exp\left[-\frac{1}{2} Z^T R_{11}^{-1} Z\right],$$

and

$$Z^T = (x_1, \ldots, x_n).$$

Thus when $X_i = 0$ $(i = 1, 2, \ldots, n)$ we have, by using (5.3.4) and (5.3.7),

$$f(x_1', \ldots, x_n' | 0, 0, \ldots, 0) = \frac{1}{(2\pi)^{n/2} |M|^{1/2}} \exp\left[-\frac{1}{2} Z'^T M^{-1} Z'\right], \tag{5.3.8}$$

By substituting from (5.3.4) $y = Z'$ which is a generalization of the two dimensional case given by Cramér ([2a, p. 15]).

The expression on the right hand side of the last equation is recognized as the Gaussian density function, $G(Z', M)$ of the random variables $X_1', \ldots, X_n'$, with covariance matrix $M = [\mu_{ij}]$. Using (5.3.8) Equation (5.3.5) can be written in the form

$$q_n(t_1,\ldots,t_n) = \frac{1}{(2\pi)^{n/2}|R_{11}|^{1/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |x'_1 \ldots x'_n| G(Z', M) dx'_1 \ldots dx'_n \, . \tag{5.3.9}$$

Let

$$\nu_{ij} = \frac{\mu_{ij}}{(\mu_{ii}\mu_{jj})^{1/2}} \quad \text{and} \quad \xi_i = \frac{x'_i}{(\mu_{ij})^{1/2}}$$

so that $N = [\nu_{ij}]$ is the covariance matrix of the new random variables ξ_i, $(i = 1, 2, \ldots, n)$.

Equation (5.3.9) is now written in the form

$$q_n(t_1,\ldots,t_n) = \frac{(\mu_{11}\mu_{22}\cdots\mu_{nn})^{1/2}}{(2\pi)^{n/2}|R_{11}|^{1/2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |\xi_1 \ldots \xi_n| G(\xi, N) d\xi_1 \ldots d\xi_n . \tag{5.3.10}$$

Thus the problem is reduced to evaluating integrals of the form

$$J_n = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |\xi_1 \ldots \xi_n| G(\xi, N) d\xi_1 \ldots d\xi_n , \tag{5.3.11}$$

where $G(\xi, N)$ is the joint Gaussian density function of $\xi_1, \xi_2, \ldots, \xi_n$ with covariance matrix $N = [\nu_{ij}]$.

5.4. First, Second and Third Order Information

For $n = 1$,

$$J_1 = \int_{-\infty}^{\infty} |\xi| \frac{e^{-\xi^2/2}}{(2\pi)^{1/2}} d\xi = \sqrt{\frac{2}{\pi}} . \tag{5.4.1}$$

Equation (5.3.10) gives for $n = 1$

$$q_1(t) = \left(\frac{-\rho_o''}{2\pi\rho_o}\right)^{1/2} J_1 = \frac{1}{\pi}\left(-\frac{\rho_o''}{\rho_o}\right)^{1/2} . \tag{5.4.2}$$

This result was first obtained by Kac [7a].

For $n = 2$

$$J_2 = \int_{-\infty}^{\infty}\int |\xi_1\xi_2| \frac{\exp -\frac{1}{2}(\xi_1^2+\xi_2^2-2\nu_{12}\xi_1\xi_2)}{2\pi(1-\nu_{12}^2)^{1/2}} d\xi_1 d\xi_2 .$$

The double integral on the first quadrant is

$$\int_0^{\infty}\int \xi_1\xi_2 \frac{\exp -\frac{1}{2}(\xi_1^2+\xi_2^2-2\nu_{12}\xi_1\xi_2)}{2\pi(1-\nu_{12}^2)^{1/2}} d\xi_1 d\xi_2$$

$$= \frac{1}{2\pi}[(1-\nu_{12})^{1/2} + \nu_{12}\cos^{-1}(-\nu_{12})]$$

If we replace ν_{12} by $-\nu_{12}$, φ by $\pi - \varphi$ (where $\varphi = \cos^{-1}\nu_{12}$), we obtain corresponding expressions for, say

$$\int_0^{\infty}\int_{-\infty}^{0} \xi_1\xi_2 \frac{\exp -\frac{1}{2}(\xi_1^2+\xi_2^2+2\nu_{12}\xi_1\xi_2)}{2\pi(1-\nu_{12}^2)^{1/2}} d\xi_1 d\xi_2,$$

which is the double integral on the third quadrant of the (ξ_1, ξ_2) plane. On adding these expressions we have, finally,

$$J_2 = \frac{4}{2\pi(1-\nu_{12}^2)^{1/2}} \operatorname{cosec}^2\varphi[1+(\frac{\pi}{2}-\varphi)\cot\varphi]$$

$$= \frac{4}{2\pi(1-\nu_{12}^2)^{1/2}}(1-\nu_{12}^2)\left[1+(\frac{\pi}{2}+\cos^{-1}\nu_{12})\frac{\nu_{12}}{(1-\nu_{12}^2)}\right]$$

$$= \frac{2}{\pi}[(1-\nu_{12}^2)^{1/2} + \nu_{12}(\frac{\pi}{2}+\cos^{-1}\nu_{12})]. \tag{5. 4. 3}$$

Substituting in (5. 3. 10) for $n = 2$ we have

$$q_2(t_1, t_2) = \frac{(\mu_{11}\mu_{22})^{1/2}}{\pi^2(\rho_0^2-\rho_{12}^2)^{1/2}}[(1-\nu_{12}^2)^{1/2}+\nu_{12}(\frac{\pi}{2}+\cos^{-1}\nu_{12})]. \tag{5. 4. 4}$$

This expression for $q_2(t_1, t_2)$ agrees with that of Rice ([17, 18] Equation 3. 4-10). See also Cramer and Leadbetter [3, p. 212].

For $n = 3$ the $q_3(t_1, t_2, t_3)$ can be obtained by using some integrals calculated by Nabeya [13c] and Kamat [8a], which give

$$J_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_1\xi_2\xi_3 G(\xi, N)d\xi_1 d\xi_2 d\xi_3$$

$$= (\frac{2}{\pi})^{3/2}[\,|N|^{1/2} + (\nu_{23}+\nu_{13})\sin^{-1}\nu_{23\cdot 1} + (\nu_{31}+\nu_{23}\nu_{21})\sin^{-1}\nu_{31\cdot 2}$$

$$+ (\nu_{12}+\nu_{31}\nu_{32})\sin^{-1}\nu_{12\cdot 3}], \tag{5. 4. 5}$$

where $\nu_{ij\cdot k}$ are the partial correlation coefficients defined as

$$\nu_{ij\cdot k} = \frac{\nu_{ij}-\nu_{ki}\nu_{kj}}{[(1-\nu_{ki}^2)(1-\nu_{kj}^2)]^{1/2}}, \quad i, j, k = 1, 2, 3. \tag{5. 4. 6}$$

Substituting from (5. 4. 5) into (5. 3. 10) for $n = 3$ we have

$$q_3(t_1, t_2, t_3) = (\tfrac{2}{\pi})^{3/2} \frac{(\mu_{11}\mu_{23}\mu_{33})^{1/2}}{(2\pi)^{3/2}|R_{11}|^{1/2}} \left[|N|^{1/2} + \sum (\nu_{23}+\nu_{12}\nu_{13}) \sin^{-1} \nu_{23\cdot 1} \right]$$

$$= \frac{(\mu_{11}\mu_{22}\mu_{33})^{1/2}}{\pi^3 |R_{11}|^{1/2}} \left[|N|^{1/2} + \sum (\nu_{23}+\nu_{12}\nu_{13}) \sin^{-1} \nu_{23\cdot 1} \right] \tag{5.4.7}$$

and the significance of the $\sum$ is outlined in (5. 4. 5).

5. 5. The Four-Variate Case

For the four-variate Gaussian distribution the contribution from the positive orthant is given by

$$J_4^{(0)} = \frac{1}{(2\pi)^2 |N|^{1/2}} \iiiint_0^\infty \xi_1 \xi_2 \xi_3 \xi_4 \exp\left(- \frac{1}{2|N|} \sum_{i,j=1}^{4} \nu_{ij} \xi_i \xi_j \; d\xi_1 d\xi_2 d\xi_3 d\xi_4, \right) \tag{5.5.1}$$

where $\nu_{ii} = 1, \quad i = 1, 2, 3, 4.$

The transformation

$$x_i = \frac{\xi_i}{(2|N|)^{1/2}}$$

carries (5. 5. 1) into

$$J_4^{(0)} = \frac{4|N|^{7/2}}{\pi^2} P_4 \tag{5.5.2}$$

where

$$P_4 = \iiiint_0^{\infty} x_1x_2x_3x_4 \exp\left(-\sum_{i,j=1}^{4} \nu_{ij}x_ix_j \; dx_1dx_2dx_3dx_4\right) \qquad (5.5.3)$$

Thus

$$J_4 = \iiiint_{-\infty}^{\infty} x_1x_2x_3x_4 \exp\left(-\frac{1}{2|N|} \sum_{i,j=1}^{4} \nu_{ij}x_ix_j \; dx_1dx_2dx_3dx_4\right)$$

$$= \frac{8|N|^{7/2}}{\pi^2} \sum_{i=1}^{5} P_4^{(i)}, \qquad (5.5.4)$$

where $P_4^{(i)}$, $i = 1, 2, \ldots, 5$, are obtained from (5.5.3) with the signs of $(\pm\nu_{ij})$, $i \neq j$, in each $P_4^{(i)}$ taken in accordance with the following sign pattern:

i	ν_{12}	ν_{13}	ν_{14}	ν_{23}	ν_{24}	ν_{34}
1	+	+	+	+	+	+
2	+	+	-	+	-	-
3	+	-	+	-	+	-
4	-	+	+	-	-	+
5	-	-	-	+	+	+

(5.5.5)

The quadruple integral P_4, as given by (5.5.3), can be obtained by differentiation with respect to ν_{ij} from the following four-variate integral

$$F_4 = \iiiint_0^{\infty} \exp\left(-\sum_{i,j=1}^{4} \nu_{ij}x_ix_j \; dx_1dx_2dx_3dx_4\right). \qquad (5.5.6)$$

For instance,

$$P_4^{(1)} = \frac{1}{4}\frac{\partial^2 F_4}{\partial\nu_{12}\partial\nu_{34}}. \tag{5.5.7}$$

The four-variate integrals $P_4^{(i)}$, $i = 1, 2, \ldots, 5$, are obtained essentially by the same differentiation, the difference being merely in the insertion of the appropriate signs of $(\pm\nu_{ij})$, $i \neq j$, in accordance with (5.5.5).

5.6. A Reduction Formula for F_4

The four-variate integral F_4 is a well-known integral in probability theory. Gupta [56, par. 6] gave an excellent survey of the attempts that had been made to evaluate F_4 and pointed out that it cannot be obtained in closed form.

In this section we shall show that the four-variate integral F_4 can be reduced to a double integral. The transformation

$$\begin{aligned}
x_1 &= r\cos\theta_1\cos\theta_2\cos\theta_3, & 0 &\le \theta_1 \le \tfrac{\pi}{2},\\
x_2 &= r\cos\theta_1\cos\theta_2\sin\theta_3, & 0 &\le \theta_2 \le \tfrac{\pi}{2},\\
x_3 &= r\cos\theta_1\sin\theta_2, & 0 &\le \theta_3 \le \tfrac{\pi}{2},\\
x_4 &= r\sin\theta_1, & 0 &\le r \le \infty,
\end{aligned}$$

carries (5.5.6) into

$$F_4 = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} \exp[-r^2(1+2\nu_{12}c_1^2c_2^2c_3s_3 + 2\nu_{13}c_1^2c_2c_3s_2$$

$$+ 2\nu_{14}c_1s_1c_2c_3 + 2\nu_{23}c_1^2c_2s_2s_3 + 2\nu_{24}c_1s_1c_2s_3 + 2\nu_{34}c_1s_1s_2)]$$

$$\times\ r^3c_1^2c_2drd\theta_1d\theta_2d\theta_3.$$

Integration with respect to r yields

$$F_4 = \frac{1}{2}\iiint_0^{\pi/2} [\,1+2\nu_{12}c_1^2c_2^2c_3s_3 + 2\nu_{13}c_1^2c_2c_3s_2 + 2\nu_{14}c_1s_1c_2c_3 + 2\nu_{23}c_1^2c_2s_2s_3$$

$$+ 2\nu_{24}c_1s_1c_2s_3 + 2\nu_{34}c_1s_1s_2]^{-2}c_1^2c_2d\theta_1d\theta_2d\theta_3. \quad (5.6.1)$$

The substitution $t = \tan\theta_1$ in (5.6.1) leads to

$$F_4 = \frac{1}{2}\int_0^{\infty}dt\int_0^{\pi/2}d\theta_2\int_0^{\pi/2}d\theta_3c_2[\,(1+t^2) + 2\nu_{12}c_2c_2c_3s_3 + 2\nu_{13}c_2c_3s_2$$

$$+ 2\nu_{14}tc_2c_3 + 2\nu_{23}c_2s_2s_3 + 2\nu_{24}tc_2s_3 + 2\nu_{34}ts_2]^{-2} \quad (5.6.2)$$

The integration with respect to t is easily eavluated, giving

$$\int_0^{\infty} c_2dt[1+2\nu_{12}c_2^2c_3s_3 + 2\nu_{13}c_2s_2c_3 + 2\nu_{23}c_2s_2s_3$$

$$+ 2t(\nu_{14}c_2c_3 + \nu_{24}c_2s_3 + \nu_{34}s_2) + t^2]^{-2}$$

$$= \frac{1+\beta}{(1+2\alpha-\beta)(1+\alpha+\beta)} + \frac{1}{2(1+2\alpha-\beta^2)^{3/2}}\tan^{-1}\frac{1+\beta}{(1+2\alpha-\beta^2)^{1/2}} -$$

$$- \frac{\beta}{(1+2\alpha-\beta^2)(1+2\alpha)} - \frac{1}{2(1+2\alpha-\beta^2)^{3/2}} \tan^{-1} \frac{\beta}{(1+2\alpha-\beta^2)^{1/2}}, \tag{5.6.3}$$

where

$$\begin{aligned} \alpha &= \alpha(\theta_2, \theta_3) = \nu_{12} c_2^2 c_3 s_3 + \nu_{13} c_2 s_2 c_3 + \nu_{23} c_2 s_2 s_3, \\ \beta &= \beta(\theta_2, \theta_3) = \nu_{14} c_2 c_3 + \nu_{24} c_2 s_3 + \nu_{34} s_2. \end{aligned} \tag{5.6.4}$$

Substituting (5.6.3) into (5.6.2) we finally have

$$F_4 = \int_0^{\pi/2} d\theta_2 \int_0^{\pi/2} d\theta_3 \left(\frac{1+\beta}{1+\alpha+\beta} + \frac{1}{2(1+2\alpha-\beta^2)^{1/2}} \tan^{-1} \frac{1+\beta}{(1+2\alpha-\beta^2)^{1/2}} \right.$$
$$\left. - \frac{\beta}{1+2\alpha} - \frac{1}{2(1+2\alpha-\beta^2)^{1/2}} \tan^{-1} \frac{\beta}{(1+2\alpha-\beta^2)^{1/2}} \right) \tag{5.6.5}$$

Finally, from (5.3.10) and (5.6.5),

$$q_4(t_1, t_2, t_3, t_4) = \frac{(\mu_{11}\mu_{22}\mu_{33}\mu_{44})^{1/2}}{(2\pi)^2 |R_{11}|^{1/2}} \int_0^{\pi/2} d\theta_2 \int_0^{\pi/2} d\theta_3 \frac{1}{1+2\alpha+\beta^2}$$
$$\times \left(\frac{1+\beta}{1+\alpha+\beta} + \frac{1}{2(1+2\alpha-\beta^2)^{1/2}} \tan^{-1} \frac{1+\beta}{(1+2\alpha-\beta^2)^{1/2}} \right.$$
$$\left. - \frac{\beta}{1+2\alpha} - \frac{1}{2(1+2\alpha-\beta^2)^{1/2}} \tan^{-1} \frac{\beta}{(1+2\alpha-\beta^2)^{1/2}} \right) \tag{5.6.6}$$

5.7. Some Special Cases

If all ν_{ij} $(i, j = 1, \ldots, 4)$ are all small Plackett [15a] gave the approximation

$$F_4 \simeq -\frac{1}{8} + \frac{1}{4\pi^2}\,[\cos^{-1}(-\nu_{12})\cos^{-1}(-\nu_{34}) + \cos^{-1}(-\nu_{13})\cos^{-1}(-\nu_{24})$$

$$+ \cos^{-1}(-\nu_{14})\cos^{-1}(-\nu_{23})], \qquad (5.7.1)$$

and by (5.5.7)

$$P_4^{(i)} \simeq \frac{1}{16\pi^2[(1-\nu_{12}^2)(1-\nu_{34}^2)]^{1/2}} \qquad (5.7.2)$$

In this case

$$P_4^{(i)} \simeq \frac{1}{16\pi^2[(1-\nu_{12}^2)(1-\nu_{34}^2)]^{1/2}}, \qquad i = 1, 2, \ldots, 5.$$

Substituting in (5.5.4)

$$J_4 \simeq \frac{5|N|^{7/2}}{2\pi^4[(1-\nu_{12}^2)(1-\nu_{34}^2)]^{1/2}}, \qquad (5.7.3)$$

where

$$|N| = \begin{vmatrix} 1 & \nu_{12} & 0 & 0 \\ \nu_{12} & 1 & \nu_{23} & 0 \\ 0 & \nu_{23} & 1 & \nu_{34} \\ 0 & 0 & \nu_{34} & 1 \end{vmatrix} = (1-\nu_{34}^2)(1-\nu_{12}^2) - \nu_{23}^2. \qquad (5.7.4)$$

For the covariance matrix N to be positive definite

$$(1-\nu_{34}^2)(1-\nu_{12}^2) > \nu_{23}^2 .$$

From (5.7.3) and (5.7.4) we finally have

$$J_4 \simeq \frac{5[(1-\nu_{12}^2)(1-\nu_{34}^2)-\nu_{23}^2]^{7/2}}{2\pi^4[(1-\nu_{12}^2)(1-\nu_{34}^2)]^{1/2}}\,, \tag{5.7.5}$$

and correspondingly

$$q_4(t_1, t_2, t_3, t_4) \simeq \frac{5(\mu_{11}\mu_{22}\mu_{33}\mu_{44})^{1/2}[(1-\nu_{12}^2)(1-\nu_{34}^2)-\nu_{23}^2]^{7/2}}{8\pi^6|R_{11}|^{1/2}[(1-\nu_{12}^2)(1-\nu_{34}^2)]^{1/2}}\,.$$

If, in addition, $\nu_{23} = 0$ we have the simple approximation

$$q_4(t_1, t_2, t_3, t_4) \simeq \frac{5(\mu_{11}\mu_{22}\mu_{33}\mu_{44})^{1/2}[(1-\nu_{12}^2)(1-\nu_{34}^2)]^3}{8\pi^6|R_{11}|^{1/2}}\,.$$

A second special case is the diagonal matrix, which means that the random variables ξ_1, ξ_2, ξ_3 and ξ_4 are uncorrelated and

$$J_4 = \frac{1}{(2\pi)^2}\int_0^\infty xe^{-x^2/2} = \frac{1}{4\pi^2}\,,$$

and

$$q_4(t_1, t_2, t_3, t_4) = \frac{(\mu_{11}\mu_{22}\mu_{33}\mu_{44})^{1/2}}{16\pi^4|R_{11}|^{1/2}}\,.$$

The final special case is obtained when $\nu_{ij} = 1$, $i, j = 1,2,3,4$. In this case $\xi_1, \xi_2, \xi_3,$ and ξ_4 all reduce to the same Gaussian variate, giving

$$J_4 = \int_0^\infty \xi^4 \frac{e^{-\xi^2/2}}{2\pi} \, d\xi = \left(\frac{1}{\pi}\right)^{1/2}$$

and

$$q_4(t_1, t_2, t_3, t_4) = \frac{(\mu_{11}\mu_{22}\mu_{33}\mu_{44})^{1/2}}{4\pi^{5/2}|R_{11}|^{1/2}} .$$

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