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Title: ON THE MULTIPOLE EXPANSION IN THE COMPUTATION OF GRAVITY ANOMALIES

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Donald F. Heinrichs

Techniques for computing gravity anomalies by multipole expansions obtained from surface integrals and volume integrals are derived together with a vertical line element method. The results are compared with exact calculation for right rectangular prisms and right circular cylinders and the effects of block size and separation between the field point and source body are evaluated.

For sources near field points, the multipole expansion of volume integrals consistantly yielded more accurate approximations of the gravity field than either vertical line element or surface integrals. For a given source, the surface integral method compared to vertical line elements gives a better approximation of the field. As distance increases, all three techniques yield accurate gravity values. Improved estimates of the gravity field can be obtained by subdividing the source body into small elements and summing the effect of the

elements. The 2nd-order or quadrupole term of the expansions is dominant for near sources while the 0th-order or monopole term becomes increasingly important with increasing separation of the source and field point.

On the Multipole Expansion in the Computation of Gravity Anomalies

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ON THE MULTIPOLE EXPANSION IN THE COMPUTATION OF GRAVITY ANOMALIES

INTRODUCTION

The use of gravity measurements as an aid in determining geologic structures is well known. Subsurface density changes arising from composition changes or geologic structure yield mass variations, which can be detected by measuring the gravity field. Since the disturbing mass parameters are seldom known exactly, approximation methods are often more useful than attempts at quite complicated exact calculations.

Various approximation methods have been used for the computation of gravity anomalies (e.g. Jakosky, 1940). This paper develops techniques for approximating anomalous mass distributions by multipole expansions and by vertical line elements. Grant and West (1965) and Bodvarsson (1970) have discussed some aspects of multipole expansions. This paper extends their results and evaluates the accuracy of the method for test cases using vertical rectangular prisms and vertical cylinders.

Gravity calculations from either surface integrals or volume integrals are directly related to the moment of inertia of the disturbing body. Both types of integrals can be expanded in a series of powers of the distance from the gravity field point to a fixed point in the mass by using Legendre polynomials. The techniques used is to find the non-vanishing multipole moments or the moments of inertia of the anomalous body. The mass distribution is then determined directly from the corresponding gravity anomalies observed at the surface of the earth. For the simplified test structures used the second and fourth terms of a multipole expansion vanish and the gravity field is described by monopole and quadrupole terms. With increasing distance of the gravity field point from the body, the monopole term becomes dominant with a corresponding decrease in the importance of the quadrupole term. Later sections of this paper discuss the relationship in detail.

The second type of approximation, using vertical line elements, is useful when the source body is at a considerable depth and the surface of the body can be described by a series of small vertical prisms. The effects of "grid size" and distance in calculating good approximations to the source body are discussed.

BASIC THEORY OF GRAVITY FIELD

The gravity field of the earth would be constant if the earth were a uniform, non-rotating sphere. But the earth is an oblate spheroid and rotates. The earth's gravity potential can be expressed as $W = v + \frac{1}{2}\omega^2(x^2 + y^2)$ where v is the acceleration potential, ω is the angular velocity and x, y are the space coordinates of the point outside of the rotating earth.

Isaac Newton showed that the force of attraction of two bodies is proportional to the product of the mass and inversely proportional to the square of the distance between them.

$$F = \frac{Gm_1 m_2}{r^2} \tag{1}$$

where m_1, m_2 are the masses of the attracting bodies, r is the distance between them, and G is a universal constant. The value of the gravitational constant G is,

$$G = 6.67 \times 10^{-8}\ cm^3 - sec^{-2} - gm^{-1}$$

For a non-rotating uniform earth, the force exerted on the mass m_0 is

$$F = \frac{GMm_0}{R^2} \tag{2}$$

where M is the mass of the earth and R is the radius of the earth.

Using Newton's second law, the gravitational acceleration at the earth's surface is

$$g = \frac{F}{m_0} = \frac{GM}{R^2} \tag{3}$$

The value of g varies between 978 cm/sec^2 at the poles and 983 cm/sec^2 at the equator because of the earth's oblate shape.

The force of gravity is the resultant of the attraction of the earth and centrifugal force. The first approximation to the figure of the earth is a sphere and the second approximation is an oblate spheroid. According to the second approximation, gravity of the earth at latitude φ^o is

$$g = g_e[1 + (\frac{5}{2}A - f)\sin^2\varphi] \tag{4}$$

where g_e is gravity at the equator, $\frac{5}{2}A - f$ = gravitational flattening = $\frac{g_p - g_e}{g_e}$, $f = \frac{1}{297}$, $A = \frac{\omega^2 a}{g_e}$, and a is the equatorial radius.

An attracting body has a potential field (Ramsey, 1940). Since the potential is a scalar quantity, it is possible to calculate gravity by using the potential. The gravity field $g(r)$ at any point r in space is defined as the force per unit mass for the continuous mass distribution.

$$g(\vec{r}) = \int_v \frac{G(\vec{r}\,' - \vec{r})\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|^3} dv' \tag{5}$$

Assuming that $\rho(\vec{r}\,')$ has a continuous distribution of mass, the gravity potential is

$$u(\vec{r}) = -\int_V \frac{G\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, dv' \tag{6}$$

Finally the gravity field g(r) is obtained from (5) or from differentiating equation (6). Therefore

$$g(\vec{r}) = \vec{\nabla} u(\vec{r}) \tag{7}$$

The physical concepts of the gravity field and the gravity potential have the same meaning as the electric field intensity and electrostatic potential in electrostatics. It is of importance to discuss Poisson's equation for the gravity potential

$$\nabla^2 u(\vec{r}) = -4\pi G\rho(\vec{r}\,') \tag{8}$$

Poisson's equation holds inside the body and gives density information when the gravity potential is known. When $\rho(\vec{r}\,') = 0$, $\nabla^2 u(\vec{r}) = 0$, which is called Laplace's equation. All gravity problems are solutions of Poisson's equation or Laplace's equation.

STATEMENT OF THE PROBLEM

Multipole methods for the approximate calculations of gravity using surface and volume integrals are derived in this paper and compared with exact solutions to evaluate the accuracy of the computations. There is also another approximation using vertical line elements. The approximation by volume integrals is the most accurate and convenient to use among these three approximations. In general, however, the accuracy of computation of gravity by approximation methods depends upon the dimensions of the body and the distance of the field point from the body.

An exact calculation of gravity for vertical rectangular prisms is obtained from triple integrals. The gravity field of vertical cylinders is obtained from the complete elliptical integrals of the first and the second kind and Heuman's Lambda function. These computations are quite often complicated.

Finally it is also possible to discuss the 0th-order and the 2nd-order approximation obtained from the multipole expansions.

EXACT GRAVITY FIELD CALCULATION FOR VERTICAL PRISMS AND CYLINDERS

Exact Expression of Gravity for a Right Rectangular Prism

This result has been derived by Nagy (1965) but without details. In this paper, the mathematical procedure will be given in more detail.

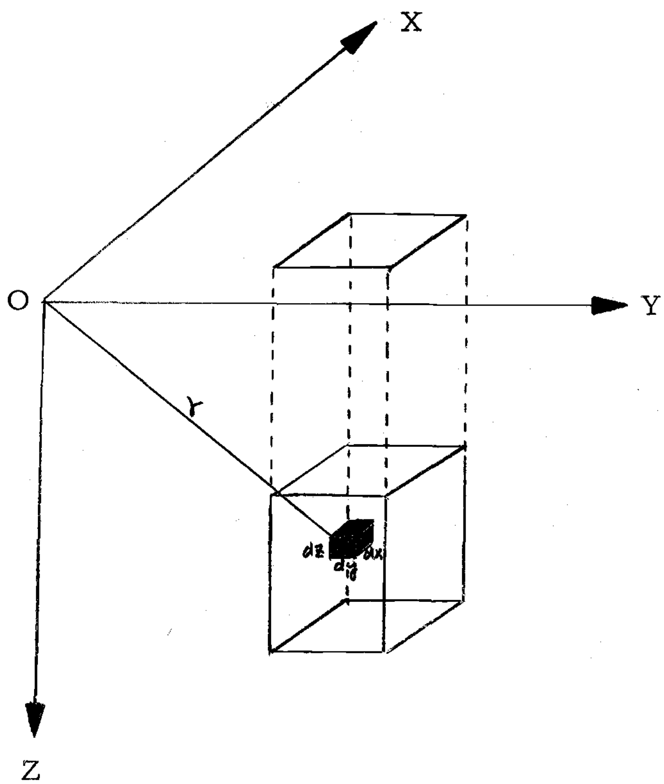


Figure 1. A right rectangular prism and the Cartesian coordinate system.

The gravity field of the volume element, Δv, is given by

$$\Delta F = G\rho \frac{\Delta v}{r^2} \tag{9}$$

where G is the gravitational constant, ρ is the density contrast, and $r = \sqrt{x^2+y^2+z^2}$. The total vertical component of the field is

$$F_z = G\rho \int_v \frac{dv}{r^2} \cos\theta = G\rho \int \frac{z dz}{r^3}$$

$$F_z = G\rho \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} \frac{z dz}{(x^2+y^2+z^2)^{3/2}}$$

Let $$L_1 = \int \frac{z dz}{(x^2+y^2+z^2)^{3/2}} = \int z(x^2+y^2+z^2)^{-3/2} dz$$

and $$u = x^2+y^2+z^2$$

Then $$L_1 = \int \frac{1}{2} du\, u^{-3/2} = -u^{-1/2} = -\frac{1}{(x^2+y^2+z^2)^{1/2}}$$

Let $$L_2 = \int L_1 dy = -\int \frac{1}{\sqrt{x^2+y^2+z^2}} dy$$

From the formula

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln (x + \sqrt{x^2 + a^2}),$$

$$L_2 = -\int \frac{1}{\sqrt{x^2+y^2+z^2}} dy = -\ln (y + \sqrt{x^2+y^2+z^2})$$

Let $$L_3 = \int L_2 dx = - \int \ell n \left(y + \sqrt{x^2+y^2+z^2}\right) dx$$

Now let $w = \ell n \left(y+\sqrt{x^2+y^2+z^2}\right)$ and differentiate with respect to x

$$\frac{dw}{dx} = \frac{1}{y + \sqrt{x^2+y^2+z^2}} \quad \frac{x}{\sqrt{x^2+y^2+z^2}}$$

From the formula, $\int udv = uv - \int vdu$

$$\int \ell n \left(y + \sqrt{x^2+y^2+z^2}\right) dx = x \, \ell n\left(y + \sqrt{x^2+y^2+z^2}\right)$$

$$- \int x \frac{xdx}{\left(y+\sqrt{x^2+y^2+z^2}\right)\left(x^2+y^2+z^2\right)}$$

$$L_3 = - \; x \, \ell n\left(y+\sqrt{x^2+y^2+z^2}\right) - \int \frac{x^2 dx}{\left(y+\sqrt{x^2+y^2+z^2}\right)\sqrt{x^2+y^2+z^2}}$$

Let $$T = \int \frac{x^2 dx}{\left(y+\sqrt{x^2+y^2+z^2}\right)\sqrt{x^2+y^2+z^2}}$$

Let $$R = y + \sqrt{x^2+y^2+z^2}$$

$$x^2 = (R-y)^2 - y^2 - z^2$$

$$= R^2 - 2Ry - z^2$$

$$(R-y)^2 = x^2 + y^2 + z^2$$

$$2(R-y)dR = 2xdx \,, \qquad dx = \frac{(R-y)dR}{\sqrt{(R-y)^2-y^2-z^2}}$$

$$T = \int \frac{x^2}{R(R-y)} \frac{(R-y)}{\sqrt{(R-y)^2 - y^2 - z^2}} \, dR$$

$$= \int \frac{\sqrt{R^2 - 2Ry - z^2}}{R} \, dR$$

From the formulas

$$\int \frac{\sqrt{X}}{x} \, dx = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x\sqrt{X}}$$

where $$X = a + bx + cx^2$$

$$\int \frac{dx}{\sqrt{X}} = \frac{1}{c} \ln \left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}} \right. \qquad c > 0$$

$$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{bx + 2a}{x\sqrt{b^2 - 4ac}} \right) \qquad a > 0$$

Let $$X = -z^2 + (-2y)R + R^2$$

$$\int \frac{\sqrt{R^2 - 2Ry - z^2}}{R} \, dR = \sqrt{R^2 - 2Ry - z^2} - y \int \frac{dR}{\sqrt{R^2 - 2Ry - z^2}} - z^2 \int \frac{dR}{\sqrt{R^2 - 2Ry - z^2}}$$

$$= x - y \int \frac{dR}{\sqrt{R^2 - 2Ry - z^2}} - z^2 \int \frac{dR}{R\sqrt{R^2 - 2Ry - z^2}}$$

where $$\int \frac{dR}{\sqrt{R^2 - 2Ry - z^2}} = \ln \left(\sqrt{R^2 - 2Ry - z^2} \right) + R + (-y))$$

$$= \ln (x + R - y)$$

and
$$\int \frac{dR}{R\sqrt{R^2 - 2Ry - z^2}} = \frac{1}{z}\sin^{-1}\left(\frac{-2yR + 2(-z^2)}{R\sqrt{4y^2+4z^2}}\right)$$
$$= \frac{1}{z}\sin^{-1}\left(\frac{-yR-z^2}{R\sqrt{y^2+z^2}}\right)$$

Therefore

$$T = x - y\ell n(x+R-y) - z\sin^{-1}\left(\frac{-yR-z^2}{R\sqrt{y^2+z^2}}\right)$$

$$L_3 = -\left[x\,\ell n(y+\sqrt{x^2+y^2+z^2}) - x + y\ell n(x+R-y) + z\sin^{-1}\left(\frac{-yR-z^2}{R\sqrt{y^2+z^2}}\right)\right]$$

$$= -\Big[x\,\ell n(y+\sqrt{x^2+y^2+z^2}) - x + y\,\ell n(x+\sqrt{x^2+y^2+z^2})$$
$$+ z\sin^{-1}\left(-\frac{z^2+y^2+y\sqrt{x^2+y^2+z^2}}{(y+\sqrt{x^2+y^2+z^2})\sqrt{y^2+z^2}}\right)\Big]$$

$$L_3 = -\left[x\,\ell n(y+r) - x + y\,\ell n(x+r) - z\sin^{-1}\frac{z^2+y^2+yr}{(y+r)\sqrt{y^2+z^2}}\right]$$

where $r = \sqrt{x^2+y^2+z^2}$

Finally

$$F_z = G\rho \left|{}^{x_2}_{x_1}\right. \left|{}^{y_2}_{y_1}\right. \left|{}^{z_2}_{z_1}\right. x\,\ell n(y+r) + y\,\ell n(x+r) - z\sin^{-1}\frac{z^2+y^2+yr}{(y+r)\sqrt{y^2+z^2}} \Big|\Big|\Big|$$

$$\left|{}^{x_2}_{x_1}\right. \left|{}^{y_2}_{y_1}\right. \left|{}^{z_2}_{z_1}\right. x\,\ell n(y+r) + y\,\ell n(x+r) - z\,\frac{\sin^{-1} z^2+y^2+yr}{(y+r)\sqrt{y^2+z^2}} \Big|\Big|\Big|$$

$$= \left|{}^{y_2}_{y_1}\right. \left|{}^{z_2}_{z_1}\right. x_2\ell n(y+\sqrt{x_2^2+y^2+z^2}) + y\,\ell n(x_2+\sqrt{x_2^2+y^2+z^2})$$

$$- z \sin^{-1} \frac{z^2+y^2+y\sqrt{x^2+y^2+z^2}}{(y+\sqrt{x_2^2+y^2+z^2})} - x_1 \ln (y+\sqrt{x_1^2+y^2+z^2})$$

$$- y \ln (x_1+\sqrt{x_1^2+y^2+z^2}) + z \sin^{-1} \frac{z^2+y^2+y\sqrt{x_1^2+y^2+z^2}}{(y+\sqrt{x_1^2+y^2+z^2})} \Bigg\|$$

$$= \Big|_{z_1}^{z_2} x_2 \ln (y_2+\sqrt{x_2^2+y_2^2+z^2}) + y_2 \ln (x_2+\sqrt{x_2^2+y_2^2+z^2})$$

$$- z \sin^{-1} \frac{z^2+y_2^2+y_2\sqrt{x_2^2+y_2^2+z^2}}{(y_2+\sqrt{x_2^2+y_2^2+z^2})} - x_1 \ln (y_2+\sqrt{x_1^2+y_2^2+z^2})$$

$$- y_2 \ln (x_1+\sqrt{x_1^2+y_2^2+z^2}) + z \sin^{-1} \frac{z+y_2^2+y_2\sqrt{x_1^2+y_2^2+z^2}}{(y_2+\sqrt{x_1^2+y_2^2+z^2})}$$

$$- x_2 \ln (y_1+\sqrt{x_2^2+y_1^2+z^2}) - y_1 \ln (x_2+\sqrt{x_2^2+y_1^2+z^2})$$

$$z \sin^{-1} \frac{z^2+y_1^2+y_1\sqrt{x_2^2+y_1^2+z^2}}{(y_1+\sqrt{x_2^2+y_1^2+z^2}} + x_1 \ln (y_1+\sqrt{x_1^2+y_1^2+z^2})$$

$$+ y_1 \ln (x_1+\sqrt{x_1^2+y_1^2+z^2}) - z \sin^{-1} \frac{z^2+y_1^2+y_1\sqrt{x_1^2+y_1^2+z^2}}{(y_1+\sqrt{x_1^2+y_1^2+z^2})} \Bigg|$$

$$= x_2 \ln (y_2+\sqrt{x_2^2+y_2^2+z_2^2}) + y_2 \ln (x_2+\sqrt{x_2^2+y_2^2+z_2^2})$$

$$- z_2 \sin^{-1} \frac{z_2^2+y_2^2+y_2\sqrt{x_2^2+y_2^2+z_2^2}}{(y_2+\sqrt{x_2^2+y_2^2+z_2^2})} - x_1 \ln (y_2+\sqrt{x_1^2+y_2^2+z_2^2})$$

$$- y_2 \ln (x_1 + \sqrt{x_1^2+y_2^2+z_2^2}) + z_2 \sin^{-1} \frac{z_2^2+y_2^2+y_2 \sqrt{x_1^2+y_2^2+z_2^2}}{(y_2+\sqrt{x_1^2+y_2^2+z_2^2})}$$

$$- x_2 \ln (y_1 + \sqrt{x_2^2+y_1^2+z_2^2}) - y_1 \ln (x_2 + \sqrt{x_2^2+y_1^2+z_2^2})$$

$$+ z_2 \sin^{-1} \frac{z_2^2+y_1^2+y_1 \sqrt{x_2^2+y_1^2+z_2^2}}{(y_1+\sqrt{x_2^2+y_1^2+z_2^2})} + x_1 \ln (y_1 + \sqrt{x_1^2+y_1^2+z_2^2})$$

$$+ y_1 \ln (x_1 + \sqrt{x_1^2+y_1^2+z_2^2}) - z_2 \sin^{-1} \frac{z_2^2+y_1^2+y_1 \sqrt{x_1^2+y_1^2+z_2^2}}{(y_1+\sqrt{x_1^2+y_1^2+z_2^2})}$$

$$- x_2 \ln (y_2 + \sqrt{x_2^2+y_2^2+z_1^2}) - y_2 \ln (x_2 + \sqrt{x_2^2+y_2^2+z_1^2})$$

$$+ z_1 \sin^{-1} \frac{z_1^2+y_2^2+y_2 \sqrt{x_2^2+y_2^2+z_1^2}}{(y_2+\sqrt{x_2^2+y_2^2+z_1^2}} + x_1 \ln (y_2 + \sqrt{x_1^2+y_2^2+z_1^2})$$

$$+ y_2 \ln (x_1 + \sqrt{x_1^2+y_2^2+z_1^2}) - z_1 \sin^{-1} \frac{z_1^2+y_2^2+y_2 \sqrt{x_1^2+y_2^2+z_1^2}}{(y_2+\sqrt{x_1^2+y_2^2+z_1^2})}$$

$$+ x_2 \ln (y_1 + \sqrt{x_2^2+y_1^2+z_1^2}) + y_1 \ln (x_2 + \sqrt{x_2^2+y_1^2+z_1^2})$$

$$- z_1 \sin^{-1} \frac{z_1^2+y_1^2+y_1 \sqrt{x_2^2+y_1^2+z_1^2}}{(y_1+\sqrt{x_2^2+y_1^2+z_1^2})} - x_1 \ln (y_1 + \sqrt{x_1^2+y_1^2+z_1^2})$$

$$- y_1 \ln (x_1 + \sqrt{x_1^2+y_1^2+z_1^2}) + z_1 \sin^{-1} \frac{z_1^2+y_1^2+y_1 \sqrt{x_1^2+y_1^2+z_1^2}}{(y_1+\sqrt{x_1^2+y_1^2+z_1^2})}$$

Let $\Phi = F_z / G\rho$

$$\Phi = x_2 \ell n \left(\frac{y_1 + \sqrt{x_2^2 + y_1^2 + z_2^2}}{y_1 + \sqrt{x_2^2 + y_1^2 + z_1^2}} \quad \frac{y_2 + \sqrt{x_2^2 + y_2^2 + z_1^2}}{y_2 + \sqrt{x_2^2 + y_2^2 + z_2^2}} \right)$$

$$- x_1 \ell n \left(\frac{y_1 + \sqrt{x_1^2 + y_1^2 + z_2^2}}{y_1 + \sqrt{x_1^2 + y_1^2 + z_1^2}} \quad \frac{y_2 + \sqrt{x_1^2 + y_2^2 + z_1^2}}{y_2 + \sqrt{x_1^2 + y_2^2 + z_2^2}} \right)$$

$$+ y_2 \ln \left(\frac{x_1 + \sqrt{x_1^2 + y_2^2 + z_2^2}}{x_1 + \sqrt{x_1^2 + y_2^2 + z_1^2}} \quad \frac{x_2 + \sqrt{x_2^2 + y_2^2 + z_1^2}}{x_2 + \sqrt{x_2^2 + y_2^2 + z_2^2}} \right)$$

$$- y_1 \ell n \left(\frac{x_1 + \sqrt{x_1^2 + y_1^2 + z_2^2}}{x_1 + \sqrt{x_1^2 + y_1^2 + z_1^2}} \quad \frac{x_2 + \sqrt{x_2^2 + y_1^2 + z_1^2}}{x_2 + \sqrt{x_2^2 + y_1^2 + z_2^2}} \right)$$

$$+ z_2 \left(\sin^{-1} \frac{z_2^2 + y_2^2 + y_2 \sqrt{x_2^2 + y_2^2 + z_2^2}}{(y_2 + \sqrt{x_1^2 + y_2^2 + z_2^2}) \sqrt{y_2^2 + z_2^2}} \right.$$

$$- \sin^{-1} \frac{z_2^2 + y_2^2 + y_2 \sqrt{x_1^2 + y_2^2 + z_2^2}}{(y_2 + \sqrt{x_1^2 + y_2^2 + z_2^2}) \sqrt{y_2^2 + z_2^2}}$$

$$- \sin^{-1} \frac{z_2^2 + y_1^2 + y_1 \sqrt{x_2^2 + y_1^2 + z_2^2}}{(y_1 + \sqrt{x_2^2 + y_1^2 + z_2^2}) \sqrt{y_1^2 + z_2^2}}$$

$$+\sin^{-1}\frac{z_2^2+y_1^2+y_1\sqrt{x_1^2+y_1^2+z_2^2}}{(y_1+\sqrt{x_1^2+y_1^2+z_2^2})\sqrt{y_1^2+z_2^2}}\Bigg)$$

$$-z_1\Bigg(\sin^{-1}\frac{z_1^2+y_2^2+y_2\sqrt{x_2^2+y_2^2+z_1^2}}{(y_2+\sqrt{x_2^2+y_2^2+z_1^2})\sqrt{y_2^2+z_1^2}}$$

$$-\sin^{-1}\frac{z_1^2+y_2^2+y_2\sqrt{x_1^2+y_2^2+z_1^2}}{(y_2+\sqrt{x_1^2+y_2^2+z_1^2})\sqrt{y_2^2+z_1^2}}$$

$$-\sin^{-1}\frac{z_1^2+y_1^2+y_1\sqrt{x_2^2+y_1^2+z_1^2}}{(y_1+\sqrt{x_2^2+y_1^2+z_1^2})\sqrt{y_1^2+z_1^2}}$$

$$+\sin^{-1}\frac{z_1^2+y_1^2+y_1\sqrt{x_1^2+y_1^2+z_1^2}}{(y_1+\sqrt{x_1^2+y_1^2+z_1^2})\sqrt{y_1^2+z_1^2}}\Bigg)$$

Hence the total vertical component of the field is

$$F_z = G\rho\,\Phi \tag{10}$$

The coordinate system is taken as the rectangular Cartesian coordinate system for computation of the vertical gravity.

Exact Expression of Gravity for a Right Circular Cylinder

Exact expression of gravity of the right circular cylinder has been derived by Kolbenheyer (1962) and Nabighian (1962).

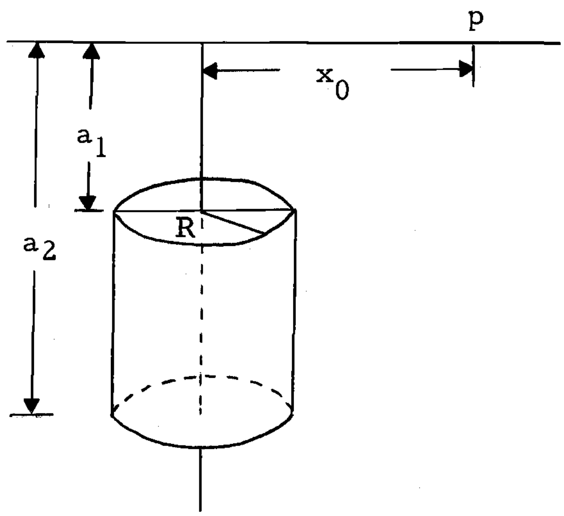


Figure 2. A right circular cylinder and its parameters.

The vertical component of gravity at p which the depth of ai is

$$\Delta g_i = 2G\rho \left[\frac{1 - x^2}{\sqrt{(1+x)^2 + a^2}} K(k_i) + \sqrt{(1+x)^2 + a^2}\, E(k_i) + \frac{\pi}{2} a \lambda_0(\varphi_i, k_i) - \pi a \right] \tag{11}$$

where G is the gravitational constant

ρ is the density

$$x = \frac{x_0}{R}$$

$$a = \frac{a_i}{R}, \quad i = 1, 2$$

K and E are the complete elliptical integrals of the first and the second kind with parameter k_i

$$k_i = \frac{4x}{(1+x)^2 + a^2}$$

$\lambda_0(\varphi_i, k_i)$ is Heuman's Lambda function

$$\varphi_i = \sin^{-1} \frac{a}{\sqrt{(1-x)^2 + a^2}}$$

The vertical gravity is obtained from subtraction of two depths.

$$\Delta g = \Delta g_1 - \Delta g_2 \qquad (12)$$

MULTIPOLE APPROXIMATIONS

Finite Structures (Volume Integrals)

This method has been discussed by Grant et al. (1965). The gravity potential at any point outside a body is

$$u(\vec{r}) = - G \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dv' \tag{13}$$

where G is the gravitational constant and $\rho(\vec{r}')$ is the density throughout a volume v. It is worthwhile to find a simplified expression for equation (13)

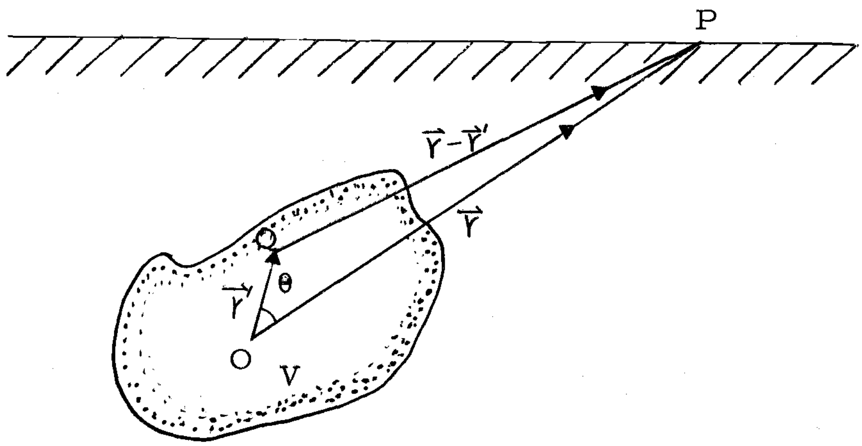


Figure 3. The gravitational potential between the point of observation, P, and a body, V.

To do this we express the distance between a body point and the field point as

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta) \quad r' \leq r \tag{14}$$

where P_ℓ is the Legendre polynomial of order ℓ.

The gravity potential becomes

$$u(r) = -\frac{G}{r}\sum_{\ell=0}^{\infty}\int_V \ell(r')\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta)\, dv' \tag{15}$$

Grant et al. (1965) have discussed this problem in detail.

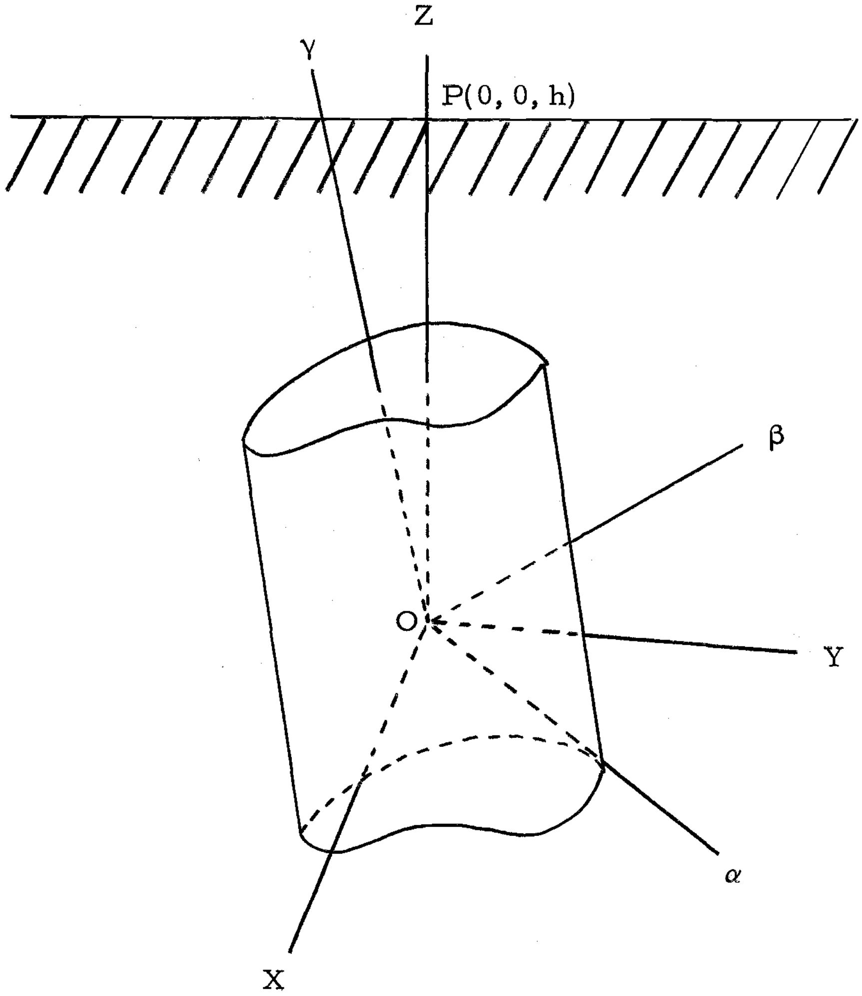
Equation (15) can be rewritten as

$$u(r) = -\sum_{\ell=0}^{\infty}\sum_{m=\ell}^{m=\ell}\frac{4\pi G}{2\ell+1}\int_V \ell(r')(r')^{\ell}\, y_\ell^{m*}(\theta_0, \varphi_0)\, dv' \,\frac{y_\ell^m(\theta,\varphi)}{r^{\ell+1}}$$

$$= -\sum_{\ell=0}^{\infty}\sum_{m=\ell}^{m=\ell} b_\ell^m \frac{y_\ell^m(\theta,\varphi)}{r^{\ell+1}} \tag{16}$$

where $$b_\ell^m = \frac{4\pi G}{2\ell+1}\int_V \rho(r')(r')^{\ell}\, y_\ell^{m*}(\theta_0, \varphi_0)\, dv'$$

The b_ℓ^m constitute multipole moments expressed in imaginary form. The multipole moments can be reduced to simple algebraic forms with a proper choice of the origin and axis of the coordinate system. In the coordinate system (x, y, z) used here, the origin lies at the center of mass and the spatial axes coincide with the body axes α, β, and γ.



α, β, γ Body axes

X, Y, Z Spatial axes

Figure 4a. Generalized coordinate for a vertical finite structure. A special case where the body axes and spatial axes coincide has been used in numerical calculations.

It is convenient to change Equation (16) into a real form since all the multipole moments are real

$$u(r) = -\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{B_\ell^m Y_\ell^m}{r^{\ell+1}} \tag{17}$$

where $B_\ell^m Y_\ell^m = b_\ell^{m*} y_\ell^{m*} + b_\ell^m y_\ell^m$

$$Y_\ell^m(\theta, \varphi) = \cos m\varphi \, P_\ell^{|m|}(\cos\theta)$$

$$B_\ell^m = (2-\delta_{\ell,m})\frac{G(\ell-m)!}{(\ell+m)!}\int_v \rho(r')(r')^\ell \cos m\varphi \, P_\ell^{|m|}(\cos\theta_0)dv'$$

where $\delta_{\ell,m} = \begin{cases} 1 & \ell = m \\ 0 & \ell \neq m \end{cases}$

Since Equation (17) is an infinite number of multipole moments, they can hardly all be determined. Grant et al. (1965) show that (17) is convergent. The first three terms of Equation (17) are usually sufficient to approximate most gravimetric measurement.

$$-u(r) = \sum_{\ell=0}^{2}\sum_{m=0}^{\ell}\frac{B_\ell^m Y_\ell^m}{r^{\ell+1}} \tag{18}$$

The vertical component of gravity $\Delta g(p)$ is obtained by differentiating Equation (18). The coefficients of $\ell = 1$ and the coefficient B_2^1 vanish by symmetry.

Hence

$$-u(r) = \frac{B_0^0 Y_0^0}{r} + \frac{B_2^0 Y_2^0}{r^3} + \frac{B_2^2 Y_2^2}{r^3} \tag{19}$$

where $Y_1^0 = 1$, $Y_2^0 = \frac{3}{2}\left(\cos^2\theta - \frac{1}{3}\right)$,

and $Y_2^2 = 3\cos 2\varphi \sin^2\theta$

Rewriting (19)

$$-u(r) = \frac{B_0^0}{r} + \frac{B_2^0(3\cos^2\theta - 1)}{2r^3} + \frac{3B_2^2\cos^2\varphi\sin^2\theta}{r^3} \tag{20}$$

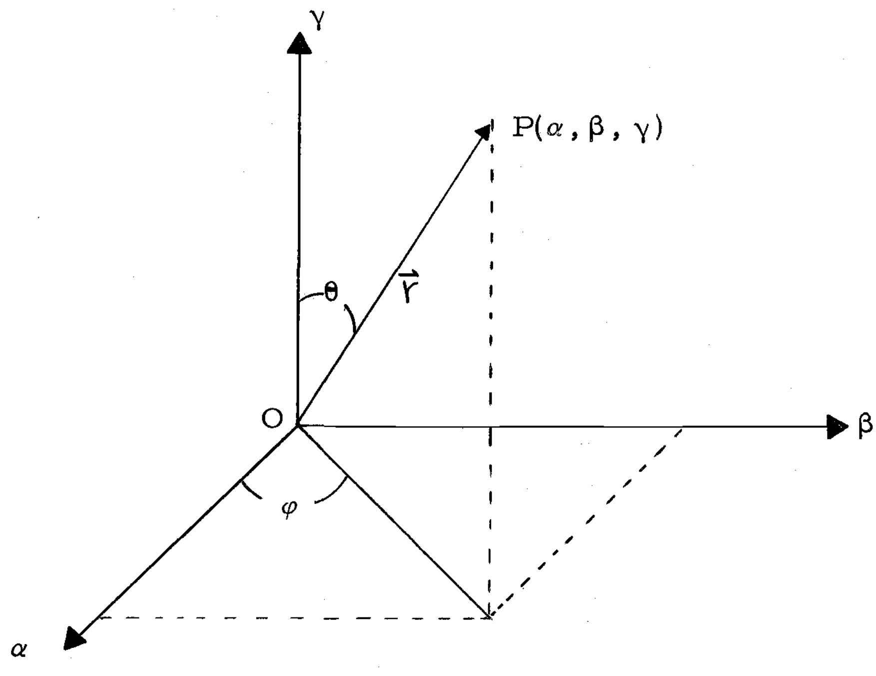


Figure 4b. Spherical polar coordinates

The spherical polar coordinates (r, θ, φ) can be transformed into the rectangular coordinates (Figure 4b)

$$\alpha = r \sin\theta \cos\varphi$$

$$\beta = r \sin\theta \cos\varphi$$

$$\gamma = r \cos\theta$$

where $0 \leq \varphi \leq 2\pi$ and $0 \leq \theta < \pi$

Therefore equation (20) becomes

$$-u(r) = \frac{B_0^0}{r} + \frac{B_2^0(3\gamma^2 - r^2)}{2r^5} + \frac{3B_2^2(\gamma^2 - \beta^2)}{r^5} \tag{21}$$

where $r = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$

Finally the vertical component of gravity $\Delta g_v(P)$ is obtained by differentiating (21)

$$\Delta g_v(P) = \frac{B_0^0 \gamma}{r^3} - \frac{B_2^0}{2r^7}(9\gamma\alpha^2 + 9\gamma\beta^2 - 6\gamma^3) - \frac{15B_2^2}{r^7}(\gamma\alpha^2 - \gamma\beta^2)$$

Hence the vertical gravity at the depth of h as measured from the center of mass is

$$\Delta g_v(P) = \frac{B_0^0 h}{r^3} - \frac{B_2^0 h}{2r^7}[9(\alpha^2 + \beta^2) - 6h^2] - \frac{15B_2^2 h}{r^7}(\alpha^2 - \beta^2) \tag{22}$$

Using equation (22), the dimension and shape of the disturbing body may be estimated by the moments determined from the field measurements if the density is known for the homogeneous body. It is also important to locate the center of gravity of the disturbing body before performing the potential analysis. The coordinate of the center of gravity, $(\bar{x}, \bar{y})$, is expressed as follows:

$$\bar{x} \int_S \Delta g(x, y)\, ds = \int_S x \Delta g(x, y)\, ds$$

$$\bar{y} \int_S \Delta g(x, y)\, ds = \int_S y \Delta g(x, y)\, ds$$

by Gauss' theorem

$$\int \Delta g(x, y)\, ds = 2\pi G M$$

where M is the disturbing mass.

Therefore

$$\bar{x} = \frac{1}{2\pi G M} \int_S x\, \Delta g(x, y)\, ds \qquad (23)$$

$$\bar{y} = \frac{1}{2\pi G M} \int_S y \Delta g(x, y) ds$$

Semi-infinite Structures (surface integrals)

In potential theory (e.g., Kellogg, 1953), the volume integrals are usually used in the mathematical derivations. However, for practical cases, it is often convenient to transform volume integrals over homogeneous bodies into surface integrals. This transformation has been discussed by Bodvarsson (1970). In this paper, the mathematical procedure will be given in more detail.

The acceleration of gravity due to a finite body B at a field point P is

$$\Delta g(P) = G \int_B \nabla_P \left(\frac{1}{r_{PQ}}\right) \rho(Q)\, dv_Q \qquad (24)$$

where G is the gravitational constant, r_{PQ} is the distance from P to the source point Q, and $\rho(Q)$ is the density of body B at Q. If the density ρ is constant inside body B, the density gradient can be expressed by the delta function

$$\nabla \rho(Q) = \rho\, \delta\, (Q - D)\, \vec{n}$$

where D is a point on the surface S of the body B and n is the inward normal vector to the surface S. Therefore

$$\Delta g(P) = G\rho \int_{B'} \frac{\delta(Q-D)\vec{n}(D)}{r_{PQ}}\, dv_Q = G\rho \int_S \frac{\vec{n}(D) da_s}{r_{PS}} \qquad (25)$$

where B' is the bounded volume by the surface which completely encloses the body B and r_{PS} is the distance from the observation point to a point on the surface of the body.

The acceleration in the direction of the unit vector $\vec{K}$ is

$$\Delta g_{\vec{K}}(P) = G\rho \int_S \frac{\vec{K} \cdot \vec{n}(D)}{r_{PS}}\, da_s \qquad (26)$$

Equation (26) shows that the volume integral can be transformed into a surface integral on the condition that the density ρ is constant. Equation (26) can be used to calculate the vertical gravity $\Delta g_v(P)$ in case of structures bounded by vertical lateral faces such as the vertical rectangular prism and the vertical circular cylinder. It is often possible to approximate complicated bodies in nature as simple bodies for geophysical work. From equation (26), $\vec{K} \cdot \vec{n} = 0$ on the vertical lateral faces. Hence it is necessary only to compute of gravity of the top and the bottom for finite bodies. As the gravity difference between the top and the bottom is found easily, this technique is suitable for model calculations. If the structure is semi-infinite, is it only necessary to compute gravity of the top. Thus the surface integral approximation method is useful for semi-infinite bodies in contrast to the volume integral approximation method.

Equation (26) can be expanded in a series of multipoles, which is a useful method in potential theory.

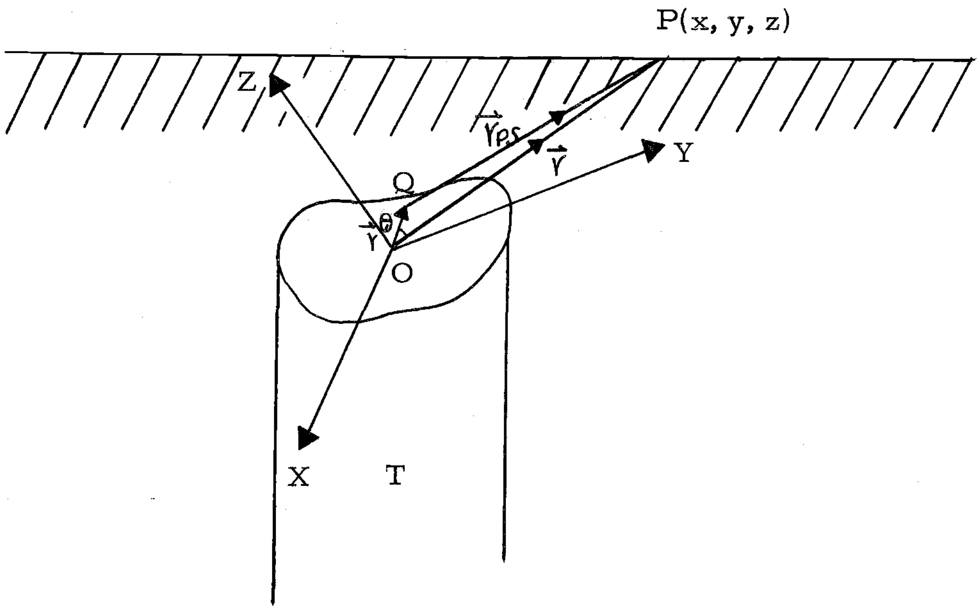


Figure 5. The Cartesian coordinate system for a vertical semi-infinite structure.

It is convenient to assume that the origin of the coordinate system is located at the center of gravity of the surface of body T and that the spatial axes coincide with the principal axes of the body, T.

$$r_{PS} = \sqrt{r^2 + (r')^2 - 2rr'\cos\theta}$$

$$\frac{1}{r_{PS}} = \frac{1}{r\sqrt{1 + \alpha^2 - 2\alpha\mu}} \tag{27}$$

where $\alpha = \frac{r'}{r}$ and $\mu = \cos\theta$. The expansion of equation (27) has been discussed by Kellogg (1953).

Since $|2\alpha\mu - \alpha^2| < 1$, equation (27) can be expanded by the binomial theorem, then

$$\frac{1}{r_{PS}} = \frac{1}{r}\left(P_0(\mu) + P_1(\mu)\,\alpha + P_2(\mu)\,\alpha^2 + \cdots\right)$$

$$= \frac{1}{r}\sum_{n=0}^{\infty} \alpha^n P_n(\mu) \tag{28}$$

where $P_0 = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3}{2}(\mu^2 - \frac{1}{3})$, ...

$$P_n = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2 - 1)^n$$

$P_n(\mu)$ is a Legendre polynomial of nth order.

$$\Delta g_{\vec{K}}(P) = G\rho \int_S \frac{\vec{K}\cdot\vec{n}(S)}{r_{PS}}\, da_s = G \int_S \frac{\rho\, da_s}{r_{PS}}$$

$$= G \int_S \frac{\rho}{r_{PS}} \sum \alpha^n P_n(\mu)\, da_s$$

$$= G\left(\frac{1}{r}\int \rho\, da_s + \frac{1}{r^2}\int \rho r' P_1(\mu) da_s + \frac{1}{r^3}\int \rho (r')^2 P_2(\mu) da_s + \cdots\right)$$

$$= G\,(S_1 + S_2 + S_3 + \cdots) \tag{29}$$

From Equation (29), the first term S_1 is

$$S_1 = \frac{1}{r}\int \rho\, da_s = \frac{m}{r}$$

where m is the sheet mass of body T.

The second term S_2 is

$$S_2 = \frac{1}{r^2}\int \rho r' P_1(\mu) da_s \quad \text{where} \quad P_1(\mu) = \cos\theta = \frac{xx' + yy' + zz'}{rr'}$$

$$= \frac{1}{r^3}\left[\int (\rho xx' + \rho yy' + \rho zz')\, da_s\right]$$

$$= \frac{1}{r^3}(x\bar{x}m + y\bar{y}m + z\bar{z}m)$$

where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are coordinates of the center of gravity of the surface.

The third term S_3 is

$$S_3 = \frac{1}{r^3}\int \rho (r')^2 P_2(\mu)\, da_s$$

$$= \frac{1}{2r^5}\int \rho\left[3(xx' + yy' + zz')^2 - (rr')^2\right] da_s$$

$$= \frac{1}{2r^5} \left[\int 3\rho(xx')^2 da_s + \int 3\rho(yy')^2 da_s + \int 3\rho(zz')^2 da_s \right.$$

$$+ \int 2\cdot 3\rho(xx'yy')da_s + \int 2\cdot 3\rho(yy'zz')\, da_s + \int 2\cdot 3\rho(xx'zz')da_s$$

$$- \left(\int \rho x^2((x')^2+(y')^2+(z')^2)da_s + \int \rho y^2(x')^2+(y')^2+(z')^2)da_s \right.$$

$$\left. \left. + \int \rho\ z^2((x')^2 + (y')^2 + (z')^2)\, da_s \right) \right]$$

$$\equiv \frac{1}{2r^5} [3(x^2M_1 + y^2M_2 + z^2M_3) + 6(xyM_{xy} + yzM_{yz}$$

$$+ xz M_{xz}) - (x^2M_1 + x^2M_2 + x^2M_3 + y^2M_1 + y^2M_2 + y^2M_3 + z^2M_1 + z^2M_2 + z^2M_3)]$$

The products of inertia are zero when the axes coincide with the principal axes of the moments of inertia. M_1, M_2 and M_3 are the moments of inertia of the x-axis, y-axis, and z-axis of the surface of the body T.

After rearrangement

$$S_3 = \frac{1}{2r^5} [(x^2-y^2)(M_1-M_2) + (y^2-z^2)(M_2-M_3) + (x^2-z^2)(M_1-M_3)]$$

If the origin of the coordinates is located at the center of gravity on the surface and the coordinate axes coincide with the principal axes of the moments of inertia, the gravity acceleration at the depth of z_i is

$$\Delta g_i(P) = G[\frac{m}{r_i} + \frac{1}{2r_i^5}((x^2-y^2)(M_1-M_2) + (y^2-z_i^2)(M_2-M_3)$$

$$+ (x^2-z_i^2)(M_1-M_3))] \qquad (30)$$

where $r_i = \sqrt{x^2+y^2+z_i^2}$

This asymptotic expression holds well when the distance r is large compared to the dimensions of the body. This approximate method is more convenient for infinite structures. For mathematical models, the right rectangular prism and the right circular cylinder with flat top and bottom are given. Assuming that the surface of the top and bottom of the body is flat, the mass of the sheet per unit area $c = \rho\vec{K}\cdot\vec{n} = \rho$, and $M_3 = 0$. Hence, the vertical gravity is

$$\Delta g(P) = \Delta g_1 - \Delta g_2 \qquad (31)$$

Convergence and Truncation of Series Expressions

There are different ways to prove the convergence of Equations (14) and (27). It is convenient to prove the convergence with complex functions.

From equation (27)

$$r_{PS} = \sqrt{r^2 + (r')^2 - 2rr'\cos\theta}$$

$$\frac{1}{r_{PS}} = \frac{1}{r}(1 - 2\frac{r'}{r}\cos\theta + \eta^2)^{-\frac{1}{2}}, \quad \eta = \frac{r'}{r}$$

$$r_{PS}^2 = r^2(1 - 2\eta\cos\theta + \eta^2)^{-\frac{1}{2}}$$

$$= r^2(1 - \eta e^{i\theta})(1 - \eta e^{-i\theta})$$

Hence $$\frac{1}{r_{PS}} = \frac{1}{r}(1 - \eta e^{i\theta})^{-\frac{1}{2}}(1 - \eta e^{-i\theta})^{-\frac{1}{2}} \qquad (32)$$

The expansion in powers of η is absolutely convergent if $|\eta e^{i\theta}| < 1$ and $|\eta e^{-i\theta}| < 1$. Since $|e^{i\theta}| = |e^{-i\theta}| = 1$ for all real value of θ, both conditions of equation (32) are satisfied when $|\eta| < 1$.

The expansions are absolutely and uniformly convergent if $|\eta| < \eta_0 < 1$. Hence the series expressions in equations (14) and (27) for the potential can be integrated term by term. But it is too difficult to carry out the infinite integration. The simple and truncated expression gives a good and finite form for many practical purposes. In this paper, the second and the fourth term vanish if the origin is located at the center of gravity or the center of mass and the body axes coincide with the spatial axes in the symmetrical bodies. The fifth term is small as compared to the whole expressions. Hence the first and the third term are mainly discussed in this work.

VERTICAL LINE ELEMENT APPROXIMATION

Various investigators (Vacquier, 1967; Danes, 1960) have used small rectangular prisms to model three-dimensional geologic structures. The source body is divided into a number of small vertical prisms with heights chosen to correspond to the body outlines. The gravity field of the body is determined by summing the gravity effect of the individual prisms at a field point. An extension of this technique is to replace the small prisms with one-dimensional vertical bars located at the center of each element.

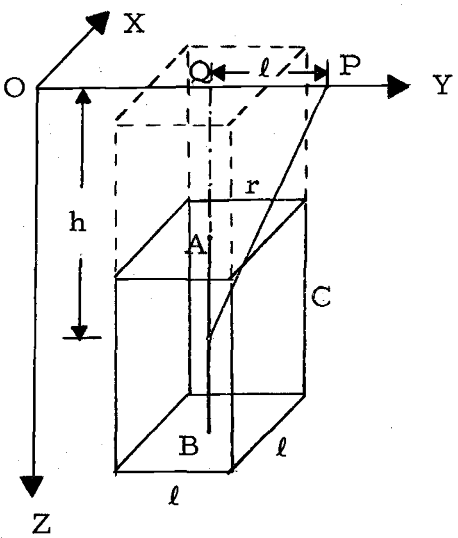


Figure 6. The vertical line element AB for the prism ℓ^2C

The gravity field for regular distribution of a vertical line elements (Figure 6) is given by

$$\Delta F = G\rho \ell^2 \int_{h-\frac{c}{2}}^{h+\frac{c}{2}} \frac{z\,dz}{(\ell^2 + z^2)^{3/2}}$$

$$= \frac{G\rho \ell^2}{r} \left\{ \left(1 - \frac{ch}{r^2}\right)^{-1/2} - \left(1 + \frac{ch}{r^2}\right)^{-1/2} \right\}$$

$$= \frac{G\rho \ell^2}{r} \left\{ \frac{ch}{r^2} + \frac{5}{8}\left(\frac{ch}{r^2}\right)^3 + \frac{63}{128}\left(\frac{ch}{r^2}\right)^5 + \cdots \right\} \quad h << \ell^2 + h^2$$

$$\doteq \frac{G\rho \ell^2}{r^3} + \frac{5}{8}\,\frac{G\rho \ell^2 (ch)^3}{r^7} \tag{33}$$

where $r = \sqrt{\ell^2 + h^2}$, G is the gravitational constant, ρ is the density contrast, ℓ is the separation of two adjacent vertical line elements, h is the depth to the center of element and c is the height of the element.

MODEL CALCULATIONS AND COMPARISON OF METHODS

Comparison of Vertical Line Elements and Volume Multipoles

Grant and West (1965) have discussed the accuracy of numerical calculations obtained from multipole moments for the case of the rectangular vertical prism. From Equation (22) and Figure 6

$$\Delta g_v(P) = \frac{G\rho \ell^2 ch}{r^3} - \frac{G\rho \ell^2 ch}{8r^7}[(c^2-\ell^2)(3\ell^2-2h^2)] \qquad (34)$$

where $r \sqrt{\ell^2+h^2}$ and ℓ is the length of the surface dimension and is also a distance of axis of two adjacent vertical rectangular prisms.

It is of interest to compare this result with the vertical line element technique. From Equation (34) and taking the limit as $\ell \to 0$,

$$\Delta F - \Delta g_v(P) \doteqdot \frac{3}{8}\frac{G\rho \ell^2 (ch)^3}{r^7}$$

Thus the approximation obtained by using vertical line elements is larger than that from volume multipoles by $\sim \frac{3}{8}\frac{G\rho \ell^2 (ch)^3}{r^7}$

For a test model, a rectangular body with horizontal dimensions of 2km, a thickness of 2 km and with the upper surface at a depth of 3 km was constructed. The exact gravity field was calculated from Equation (10). To evaluate and compare the accuracy of the approximate solutions, the source body was successively divided into 1, 16, 100, 625, and 2500 equal prisms. The gravity field using

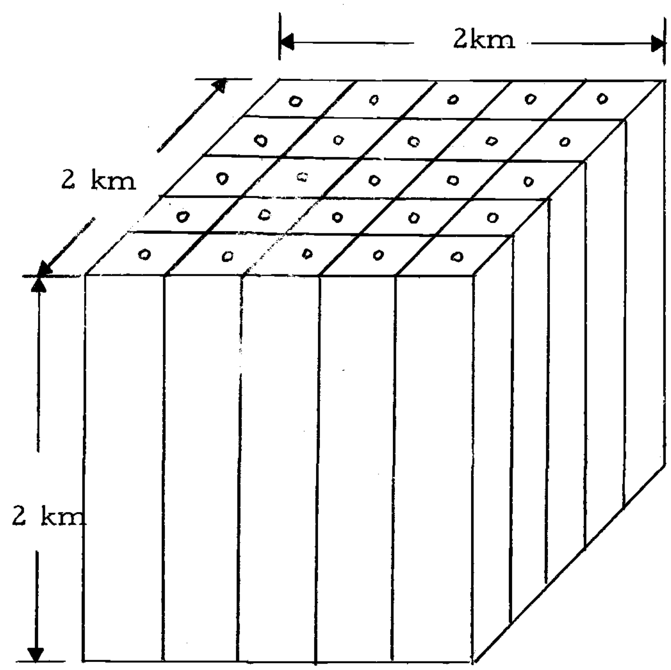


Figure 7a. Subdivisions of the source body.

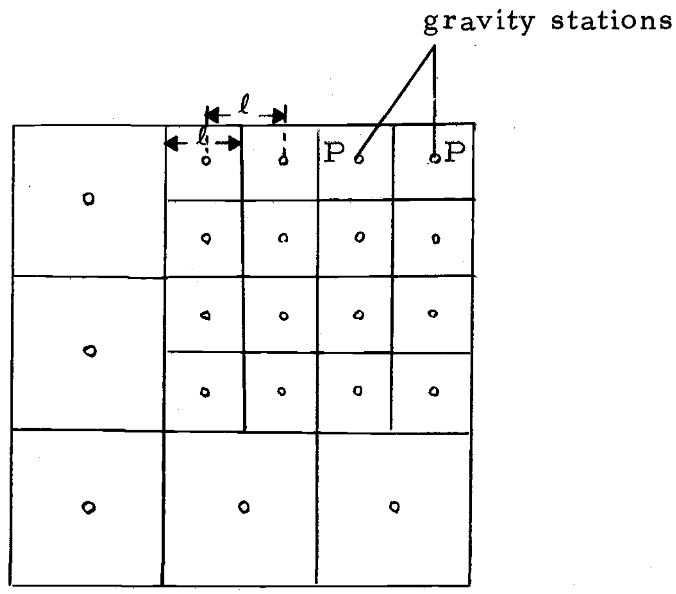


Figure 7b. The vertical gravity of an equal prism at the axis of an adjacent prism.

Table 1. Comparison of Results of Volume Integrals and Vertical Line Elements

No. of Blocks (n^2)	Exact Values (mgal)	Volume Integrals (mgal)	Vertical Line Elements (mgal)	Relative Error of Volume (%)	Relative Error of Line (%)	Dimension c.g.s. Unit
1	3.421	3.415	3.870	0.18	13.12	200,000
16	0.392	0.389	0.449	0.08	14.54	50,000
100	0.066	0.065	0.075	0.02	13.64	20,000
625	0.011	0.011	0.012	0	9.10	8,000
2500	0.003	0.003	0.003	0	0	4,000

h = 3 km depth to the center of mass.

volume multipoles and vertical line elements was calculated for each case and compared with the exact value (Table I).

From Table I, one can see that the approximation from volume integrals using 625 rectangular prisms reproduce the exact field, while the approximation using vertical line elements requires 2500 rectangular prisms for the same accuracy. Therefore the approximation from vertical line elements should not be used unless the body is subdivided into a number of small rectangular prisms because the method is performed from a one-dimensional vertical bar.

Monopole and Quadrupole Approximations

It is interesting to discuss the 0th-order (the monopole term) and the 2nd-order (the quadrupole term) approximation from the multipole expansion.

From Equation (22)

$$g_v(P) = \frac{G\rho \ell^2 cz}{r^3} - \frac{G\rho \ell^2 cz}{8r^7}[(c^2 - \ell^2)(3y^2 - 2z^2)] \qquad (35)$$

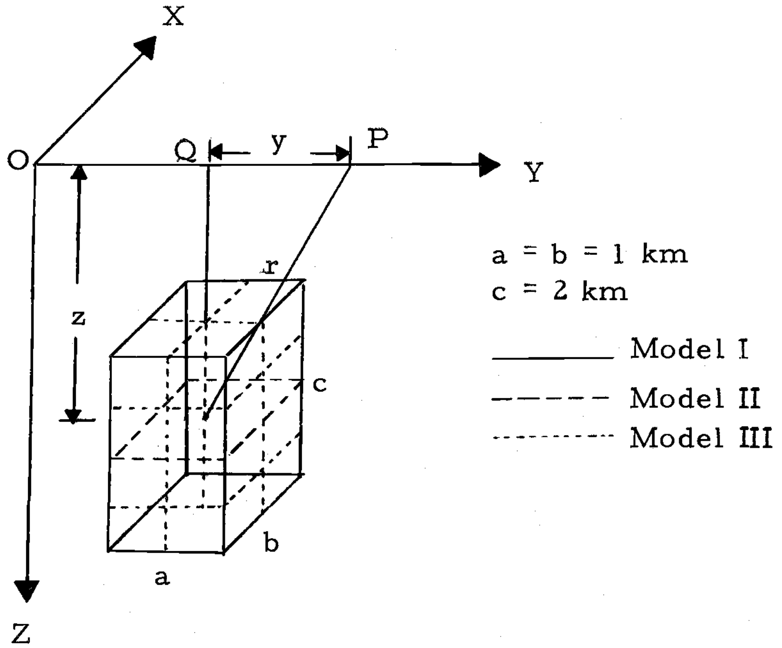


Figure 8. Monopole approximation for a small cube.

Equation (35) is derived from the rectangular prism with the equal length of the surface dimension ($a = b = \ell$). There is only the monopole term when $c = \ell$ or $y : z = \sqrt{2} : \sqrt{3}$. Therefore the monopole approximation for the computation of gravity can be a good approximation if the body is divided into a number of cubes with small surface dimensions relative to the distance to the field point.

The numerical computations of the gravity field due to the arbitrary rectangular block are performed from equation (35). The values of Tables 2, 3, 4 and 5 show how the 0th-order and the 2nd order approximation hold with decreasing the dimension of the subdivision. According to values from these tables, the closer the field point is to the body, the smaller the dimension of the subdivision should be in order to use the monopole approximation. It is not necessary to divide the body into many small regular cubes when the field point is located at great distance. As shown in Table 4, it is possible to use the monopole approximation as $g_z = \frac{G\rho \ell^3 h}{r^3}$ in case the rectangular block is divided into two regular cubes and $r \geq 3d$ where r is the distance of the field point and d is the length of the dimension of the regular cubes. Hence it gives the same formula as a sphere whose total mass is concentrated at the center of mass. It is also obvious that the 2nd-order approximation is more accurate than the 0th-order approximation when the field points get close to the body which is shown on Table 3. If the rectangular block is divided into 16 regular cubes, the gravity values are slightly different from the other table values because there are so many computational procedures, which makes errors. Therefore, it is good to divide the rectangular block into the small cubes where the 2nd-order approximation holds well in order to use the monopole approximation as shown on Table 4.

It is also interesting to discuss the monopole and the quadrupole term. As shown on Figure 11 and 12, the monopole terms are more than 95% of the total gravity. If the field point is located just above, or very close to the body, the monopole terms are over estimated, which means that the approximation from the multipole expansion decreases accuracy. As the field point is moved farther, the

Table 2. Exact Calculation (mgal)

y(km) \ z(km)	1	2	3
0	20.247	3.991	1.609
1	3.975	2.564	1.338
2	1.017	1.148	0.865
3	0.383	0.545	0.518

Table 3. Monopole and Quadrupole Approximations for Model I (= 1 block) (mgal)

z(km)	1		2		3	
y(km)	*a	*b	a	b	a	b
0	13.340	23.345	3.338	3.960	1.482	1.606
1	4.716	4.274	2.386	2.565	1.266	1.338
2	1.193	1.014	1.179	1.151	0.854	0.865
3	0.422	0.382	0.569	0.545	0.524	0.518

a = the 0th-order approx. b = the 2nd-order approx.

Table 4. Monopole Approximation for Model II (= 2 cubic blocks) (mgal)

y(km) \ z(km)	1	2	3
0	29.644	4.032	1.612
1	4.094	2.562	1.338
2	1.021	1.148	0.865
3	0.384	0.545	0.518

Table 5. Monopole Approximation for Model III(= 16 cubic blocks) (mgal)

y(km) \ z(km)	1	2	3
0	37.657	4.235	1.642
1	3.833	2.590	1.353
2	0.995	1.140	0.866
3	0.378	0.541	0.517

monopole term is dominant. Therefore as great distance, the 0th-order approximation can be a good approximation. Similarly if the body is divided into small cubes, it is also possible to use the 0-th order approximation.

According to Figure 11 and 12, the ratio of the monopole term to the total gravity values is increasing from 4 km with increasing distance, while the ratio of the quadrupole term to the total gravity values is decreasing gradually. These graphs show that the other higher-order terms are negligible as compared to these two main terms by knowing A/E and B/E, where E is the exact value and A, B are the monopole and the quadrupole term, respectively.

Volume and Surface Integral Approximations for Vertical Rectangular Prisms and Cylinders

The approximations from volume and surface integrals are computed by Equations (22) and (31). It is assumed that the rectangular prism has $a = 1$ km, $b = c = 2$ km, the depth to the center of the mass, $h = 3$ km, while the cylinder has $R = 1$ km, and $h = 3$ km with unit density, $\rho = 1 gm/cm^3$. For simplified calculations, the field point is moved from the origin $(x = y = 0)$ to Y-axis with $x = 0$ with constant depth. The values of Tables 6 and 7 show the computational results comparing with the exact calculations. As shown on Figures 9 and 10, these approximations converge to the exact values with increasing the distance of the field point.

Table 6. Numerical Calculations for a Right Rectangular Prism

d/h	Exact Value (mgal)	Surface (1) Inte. (mgal)	Volume (2) Inte. (mgal)	Relative (1) Error (%)	Relative (2) Error (%)
0	3.066	3.031	3.088	1.14	0.72
1/6	2.947	2.930	2.960	0.58	0.44
1/3	2.629	2.635	2.626	0.23	0.11
1	1.073	1.074	1.070	0.09	0.28
4/3	0.650	0.650	0.650	0	0
5/3	0.408	0.408	0.408	0	0
2	0.267	0.267	0.267	0	0
7/3	0.182	0.182	0.182	0	0
8/3	0.129	0.129	0.129	0	0
3	0.094	0.094	0.094	0	0
10/3	0.071	0.071	0.071	0	0

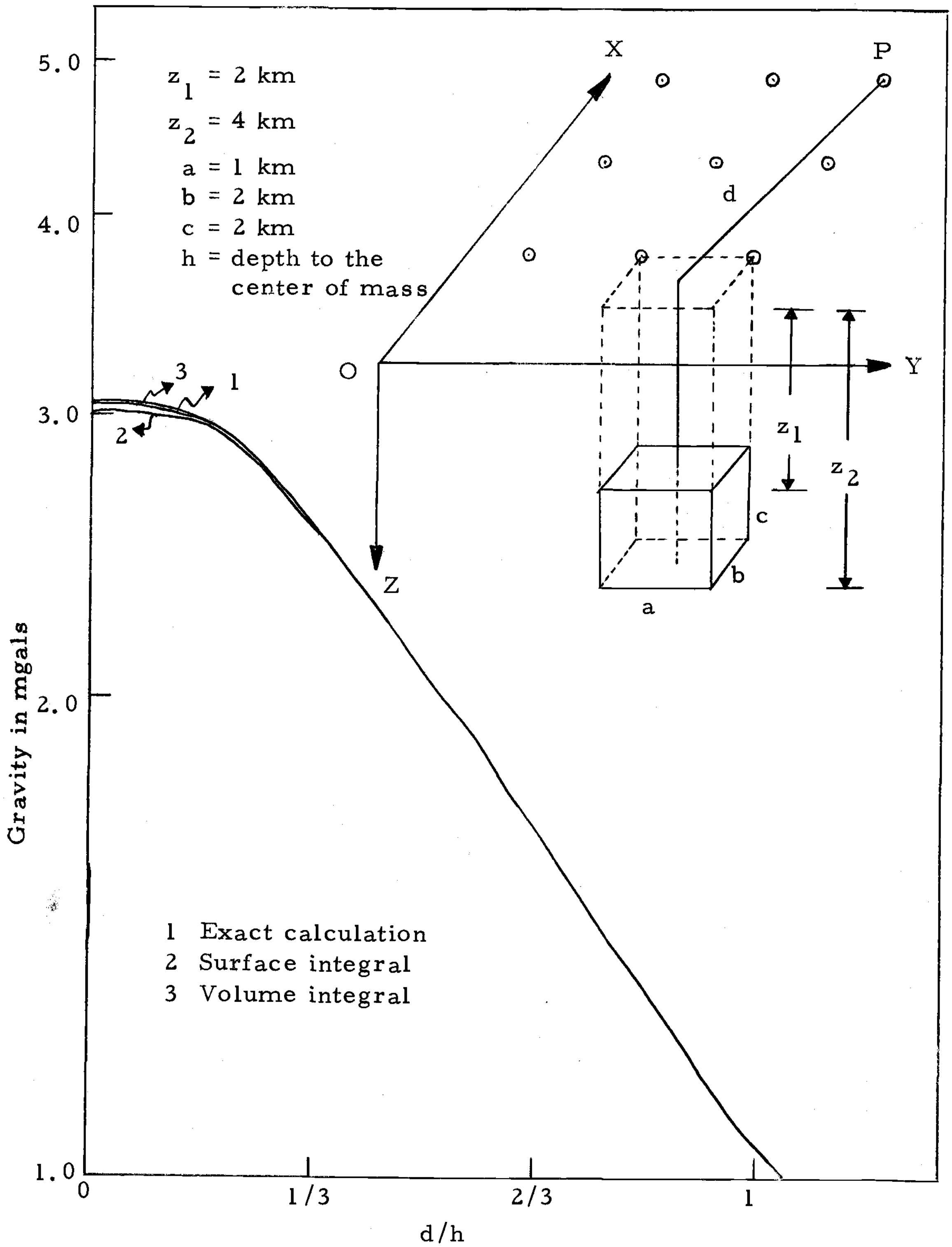


Figure 9. Vertical gravity vs d/h for a right rectangular prism.

Table 7. Numerical Calculations for a Right Circular Cylinder

d/h	Exact Value (mgal)	Surface Inte. (mgal)	Volume Inte. (mgal)	Solid Angle (mgal)	Relative (1) Error (%)	Relative (2) Error (%)	Relative (3) Error (%)
0	14.932	11.787	14.487	14.077	21.06	2.98	5.73
1/3	13.602	12.998	13.518	12.894	4.44	0.62	5.21
2/3	10.293	10.622	10.348	9.968	3.20	0.53	3.16
1	6.758	6.861	6.768	6.748	1.52	0.15	0.15
4/3	4.217	4.228	4.216	4.292	0.26	0.02	1.78
2	1.740	1.737	1.740	1.780	0.17	0	2.30
8/3	0.833	0.831	0.833	0.847	0.02	0	1.68
3	0.605	0.605	0.605	0.614	0	0	1.49
10/3	0.452	0.452	0.452	0.458	0	0	1.33
4	0.270	0.270	0.270	0.273	0	0	1.11
5	0.142	0.142	0.142	0.143	0	0	0.70
6	0.083	0.083	0.083	0.084	0	0	1.20
20/3	0.061	0.061	0.061	0.061	0	0	0

Nettleton (1942) has discussed the solid angle method by using $g_v(p) = \Omega \times G \times \rho \times t$ where Ω is a solid angle, G is the gravitational constant, ρ is the density contrast, and t is the thickness of the cylinder. The solid angle values are obtained from Musket et al. (1956).

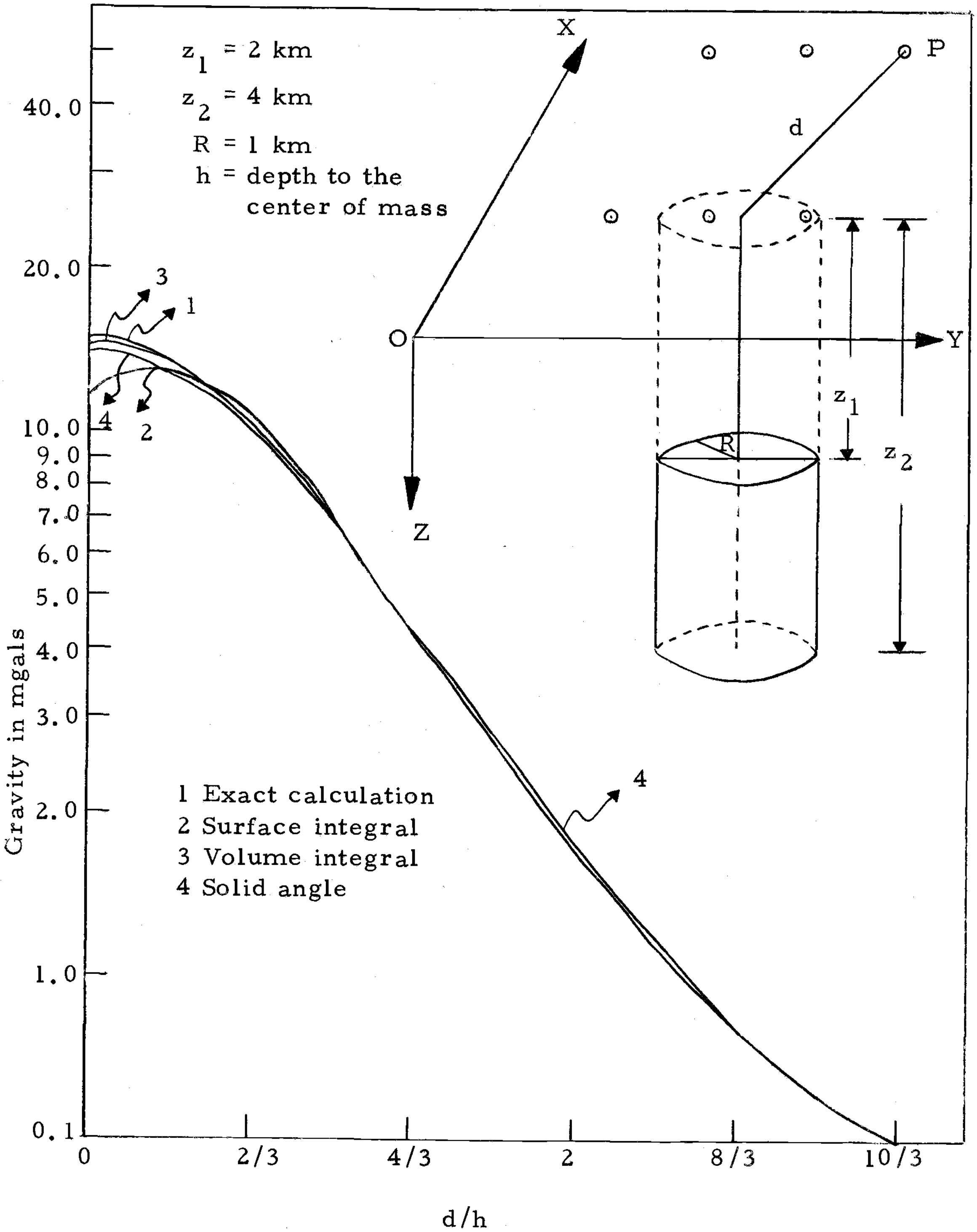


Figure 10. Vertical gravity vs d/h for a right circular cylinder.

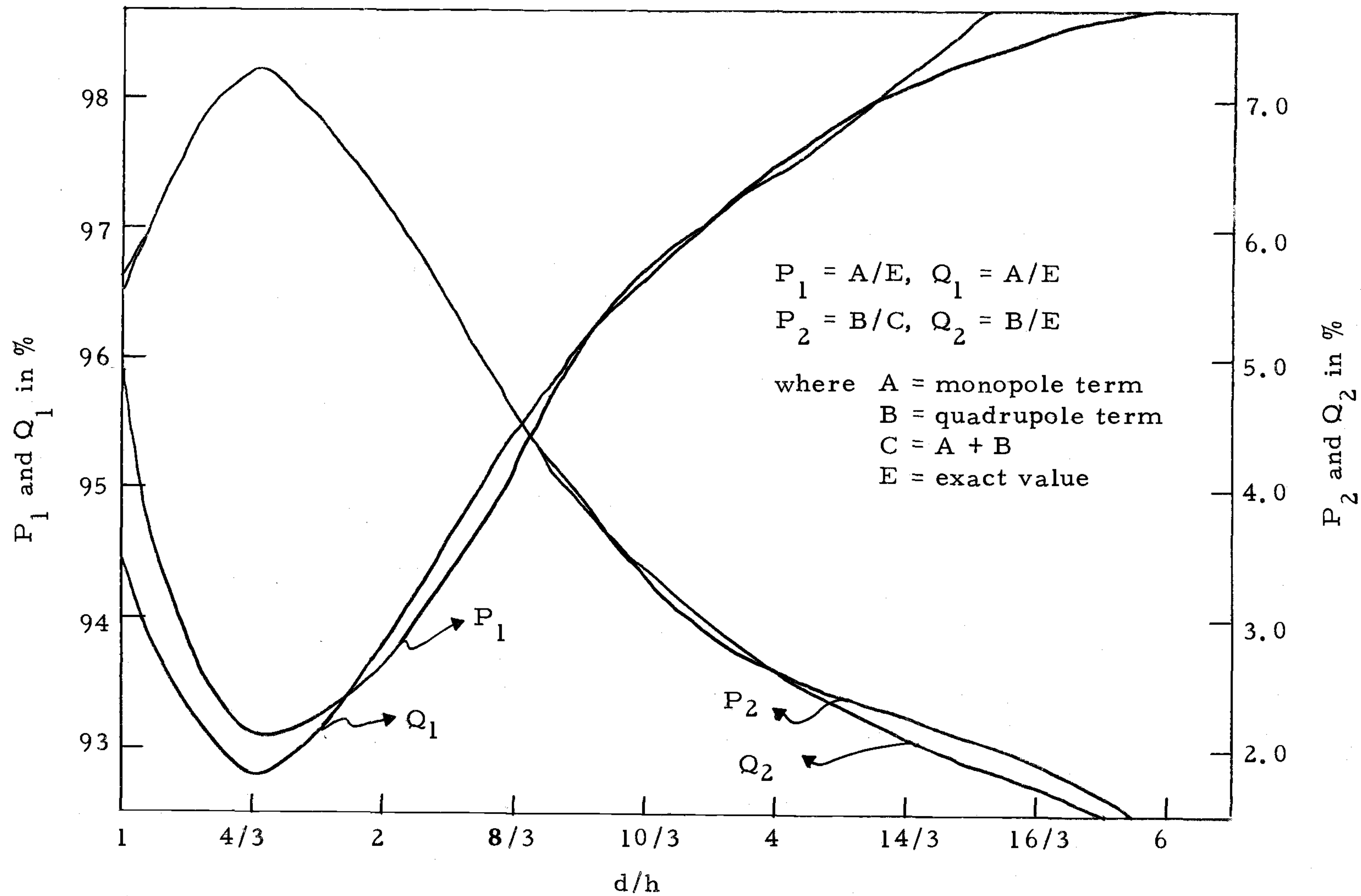


Figure 11. The ratio of the monopole and quadrupole term to the total or exact values vs. d/h for a right circular cylinder by surface integrals.

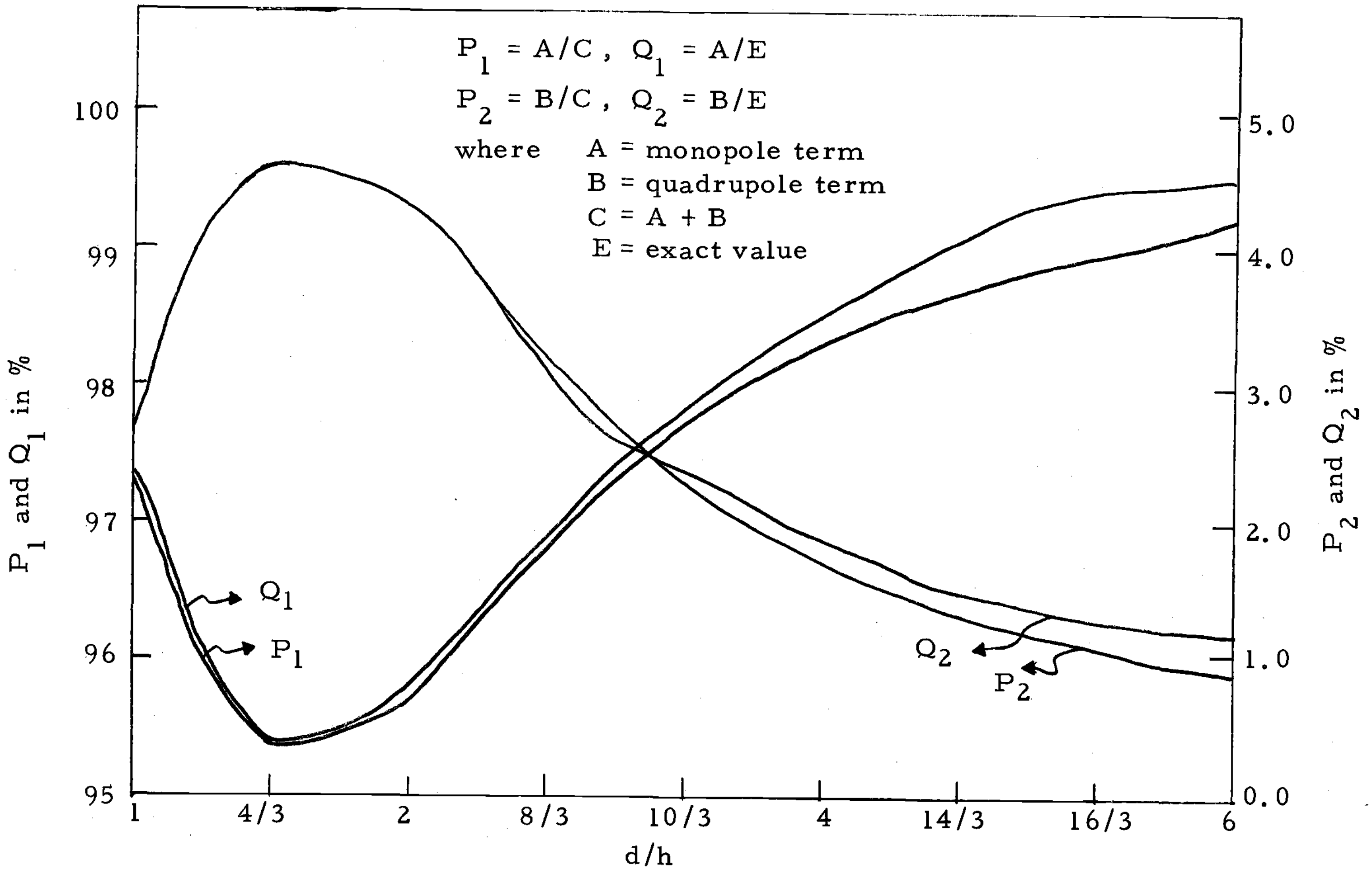


Figure 12. The ratio of the monopole and quadrupole term to the total or exact values vs d/h for a right circular cylinder by volume integrals.

DISCUSSION

Approximations using multipole expansions hold well when the distance to the field point is large compared to the dimensions of the body. The values of Tables 1, 2, 3, 4, and 5 show how these approximation methods hold according to the distance of the field point. Tables 6 and 7 also show that the approximations from surface integrals have larger errors than the volume integral approximations for the same body.

The errors of the approximation from the multipole expansion depend upon the relationship between the body dimension and the distance of the field point. In order to use these approximation, the distance of the field point should be at least three times the length of the body dimension, i.e., $r \geq 3d$.

It is of importance to make many gravity stations at appropriate distances from each other, which is shown on Figures 7a and 7b. When the field point is very close to the cylindrical body, it may be more convenient to use the solid angle approximation method unless the thickness of the cylinder is great.

It is also worthwhile to discuss the vertical line element approximation and the approximation from volume integrals. The values of Table 1 show how many subdivisions are needed to converge to the accuracy for these approximation methods. The vertical line element approximation is mainly a function of the length of the surface dimension and the distance of the field point. It is obvious that the approximation from volume integrals is more accurate than the vertical line element approximation. For the vertical line element approximation, the horizontal dimension of the subdivided rectangular blocks should be so small that they may be approximated by the vertical line bar. In order to obtain successive approximations fast, it is convenient to make a constant and small distance of axis of two adjacent vertical rectangular blocks.

Finally it is interesting to discuss the monopole term and the quadrupole term from surface and volume integrals. According to Figures 11 and 12 it is possible to use the 0th-order approximation (the monopole term) at great distance. Similarly it is also possible to use the monopole approximation when the body is divided into a number of regular cubes with small dimensions compared to the distance to the field point, e.g., Tables 3, 4, and 5.

CONCLUSIONS

Gravity interpretation by the approximate methods is very useful for geophysical prospecting, because the exact calculation is very often complicated, time-consuming, and laborious. Moreover, the approximate methods from potential theory are inverse problems to find out the shape and size of the anomalous body. The anomalous body could be determined directly from the corresponding gravity anomalies observed along the surface.

All models in this paper are mathematical models, which can be good natural shapes, too, because it is convenient to approximate a certain body as a simple body such as a prism or a cylinder in nature.

The approximation by surface integrals is good for the semi-infinite body, while the approximation by volume integrals is good for the finite body. According to the numerical calculations, the latter is more accurate than the former.

The approximation obtained from vertical line elements depends upon the surface unit area and the depth. In order to use this approximation, it is important to make a constant and small distance of axis of two adjacent vertical rectangular blocks.

Since a very fine gravimeter can measure within an accuracy of $\pm$ 2–3 microgals, all the numerical calculation in this paper gives three digits beyond the decimal point by round-off.

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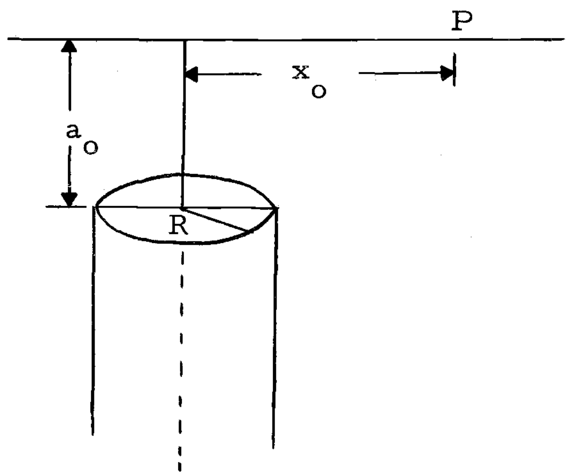
APPENDICES

APPENDIX A

THE EXACT EXPRESSION OF THE GRAVITY OF THE RIGHT VERTICAL CIRCULAR CYLINDER

This method has been discussed by Kolbenheyer (1962), Nabighian (1962), and Parasnis (1961). Kolbenheyer and Nabighian expressed the gravity of the right circular cylinder in closed form by means of complete elliptical normal integrals of the first, the second, and the third type with parameters K and λ which depend upon the dimension of the cylinder and upon the position of the top point. On the other hand, Parasnis (1961) gives the infinite series form. In this paper, Kolbenheyer's (1962) and Nabighian's (1962) methods are chosen because Parasnis' (1961) method is hardly accurate for small value of a (the ratio of depth to radius).



The vertical gravity at P is derived by Kolbenheyer (1962) and Nabighian (1962).

$$\Delta g(P) = 2G\rho \left[\frac{1 - x^2}{\sqrt{(1+x)^2+a^2}} K(k) + \sqrt{(1+x)^2+a^2}\ E(k) + \frac{\pi}{2} a\lambda_0 (\varphi, k) - \pi a \right]$$

where G is the gravitational constant,

ρ is the density contrast

$x = x_0/R$, $a = a_0/R$

K and E are the complete elliptical integrals of the first and the second kind with parameters k

$k^2 = \dfrac{4x}{(1+x)^2 + a^2}$, and $\lambda_0(\varphi, k)$ is Heuman's Lambda function with φ.

$$\varphi = \sin^{-1} \frac{a}{\sqrt{(1-x)^2 + a^2}}$$

K, E, and $\lambda_0(\varphi, k)$ could be obtained from the tables by Byrd and Friedman (1954) or Heuman (1941). But as far as the accuracy is concerned, it is more accurate to compute directly by computer by using Chebishev Approximation developed by Hastings (1955). This paper tells how to use Chebishev Approximation.

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad \text{where} \quad 0 \leq k < 1$$

$$K(k) \doteq (a_0 + a_1\eta + \cdots + a_4\eta^4) + (b_0 + b_1\eta + \cdots + b_4\eta^4) \times \ln\frac{1}{\eta}$$

where $\eta = 1 - k^2$

$a_0 = 1.3862, 9436, 112$ $\qquad$ $b_0 = .5$

$a_1 = .0966, 6344, 259$ $\qquad$ $b_1 = .1249, 8593, 597$

$a_2 = .0359, 0092, 383$ $\qquad$ $b_2 = .0688, 0248, 576$

$a_3 = .0374, 2563, 713$ $\qquad$ $b_3 = .0332, 8355, 346$

$a_4 = .0145, 1196, 212$ $\qquad$ $b_4 = .0044, 1787, 012$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi \quad \text{where } 0 \leq k < 1$$

$$E(k) \doteq (1 + a_1\eta + \cdots + a_5\eta^4) + b_1\eta + \cdots + b_4\eta^4) \times \ln\frac{1}{\eta}$$

where $\eta = 1 - k^2$

$a_1 = .4432, 5141, 463$ $\qquad$ $b_1 = .2499, 8368, 310$

$a_2 = .0626, 0601, 220$ $\qquad$ $b_2 = .0920, 0180, 037$

$a_3 = .0475, 7383, 546$ $\qquad$ $b_3 = .0406, 9697, 526$

$a_4 = .0173, 6506, 451$ $\qquad$ $b_4 = .0052, 6449, 639$

$K(k)$ and $E(k)$ can be computed with error of less than 1.6×10^{-8}.

Heuman's Lambda function is combined by complete elliptical integrals

$$\lambda_0\ (\varphi, k) = \frac{2}{\pi}\ [\ KE(\varphi, k') + EF(\varphi, k')\ ]$$

where K and E are the complete elliptical integrals of the first and the second kind and $F(\varphi, k')$ and $E(\varphi, k')$ are the incomplete elliptical integral of the first and the second kind.

$$k'^2 = 1 - k^2 = \eta$$

$$F(\varphi, k') = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2\varphi}}$$

$$E(\varphi, k') = \int_0^{\varphi} \sqrt{1 - k'^2 \sin^2\varphi}\ d\varphi$$

After the binomial expansion of the incomplete integrals, they can be integrated term by term.

Let $\quad q_2 = \int \sin^2\varphi\ d\varphi \quad$ and $\quad q_4 = \int \sin^4\varphi\ d\varphi$

therefore

$$q_2 = -\frac{\sin\varphi \cos\varphi}{2} + \frac{1}{2}\int d\varphi = \frac{1}{2}(\varphi - \sin\varphi\cos\varphi)$$

$$\varphi_4 = -\frac{\sin^3\varphi\cos\varphi}{4} + \frac{3}{4} q_2$$

$$\vdots$$

$$q_{2m} = -\frac{1}{2m}\sin^{2m-1}\varphi\ \cos\varphi + \frac{2m-1}{2m} q_{2m-2}$$

where $m = 1, 2, \ldots \infty$ and $q_0 = \varphi$

Hence Heuman's Lambda function is

$$\lambda_0(\varphi, k) = \frac{2}{\pi} \{Eq_0 - \frac{1}{2}(2K-E)q_2k'^2 - \frac{1}{2\cdot 4}(4K-3E)q_4k'^4$$

$$- \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}(6K - 5E)q_6k'^6 - \cdots - \frac{1\cdot 1\cdots(2n-3)}{2\cdot 4\cdots 2n}$$

$$[2nK - (2n-1)E]\, q_{2m}k'^{2n} - \cdots\}$$

The numerical calculation in this paper is truncated at 20th term, which is good enough for the actual gravity measurement.

It is very important to know the characteristic of Lambda function:

$$x = 0 \rightarrow \varphi = \frac{\pi}{4} \qquad \text{and} \quad \lambda_0(\varphi, k) = \frac{a}{\sqrt{1 + a^2}}$$

$$0 < x < 1 \rightarrow 0 < \varphi < \frac{\pi}{2} \qquad \text{and} \quad \lambda_0(\varphi, k)$$

$$x = 1 \rightarrow \varphi = \frac{\pi}{2} \qquad \text{and} \quad \lambda_0(\varphi, k) = 1$$

$$x > 1 \rightarrow \frac{\pi}{2} < \varphi < \pi \qquad \text{and} \quad \lambda_0(\pi - \varphi, k) = 2 - \lambda_0(\varphi, k)$$

APPENDIX B

PARASNIS' METHOD

Parasnis' (1961) estimated the gravity anomaly of a right vertical circular cylinder assuming that the anomaly is proportional to the plane area.

$$\text{(i)} \quad \Delta g = 2\pi G\rho \left\{ z_2 - z_1 - \sqrt{z_2^2 + a^2} + \sqrt{z_1^2 + a^2} \right\} \qquad x = 0$$

$$\text{(ii)} \quad \Delta g = \frac{\pi G\rho\, a}{2x} \left\{ \sqrt{z_2^2 + (x-a)} - \sqrt{z_2^2 + (x+a)^2} + \sqrt{z_1^2 + (x+a)^2} - \sqrt{z_1^2 + (x-a)^2} \right\} \qquad x \geq a$$

$$\text{(iii)} \quad \Delta g = 2\pi G\rho \left[z_2 - z_1 - \frac{1}{2}\left\{ F(x+a) + F(x-a) \right\} - \frac{1}{4}\frac{x}{a}\left\{ F(x-a) - F(x+a) \right\} \right] \qquad x \leq a$$

where a is radius, z_1 and z_2 are the depth of the top and bottom, x is the distance from the axis to the observation point, and

$$F(t) = \sqrt{z_2^2 + t^2} - \sqrt{z_1^2 + t^2} .$$