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Title: PREDICTION INTERVALS IN EXPONENTIAL FAMILIES

Abstract approved: G. David Faulkenberry

Prediction intervals for an outcome of a sufficient statistic, $T_y$, associated with the probability distribution of a future experiment are developed based on information obtained from $n$ independent, previously conducted trials of an informative experiment. The random outcomes of the informative and future experiments are assumed to be continuous and identically distributed according to a $k$-parameter exponential family, and the future experiment is conducted independent of the informative experiment.

These intervals are of the general forms, $S_1(t_x) = [L(t_x), \omega]$, $S_2(t_x) = (\omega, U(t_x)]$ and $S_3(t_x) = [L(t_x), U(t_x)]$ where $U(.)$, $L(.)$ are functions of $t_x$, the observed value of a sufficient statistic for the joint probability distribution of the random outcomes from the informative experiment.

A general theory and procedure for deriving these prediction intervals is developed using hypothesis testing procedures. Optimal properties of hypothesis tests carry over to similarly defined optimal properties of prediction intervals. The intervals
have the 'similar mean coverage' property (Aitchison, J. and Dunsmore, I.R. (1975)).

The generalized Newton's method and the IMSL routines are used for numerical computation of tables for the examples considered. An application of the saddle point approximation, Barndorff-Nielsen, O. (1983), for finding an approximate conditional density function for sufficient statistics associated with the probability distribution of the experiments is discussed.
PREDICTION INTERVALS IN EXPONENTIAL FAMILIES

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PREDICTION INTERVALS IN EXPONENTIAL FAMILIES

I. INTRODUCTION

Statistical prediction analysis involves an informative and a future experiment. Based on \( n \) independent random outcomes, \( X_1, \ldots, X_n \) from the informative experiment, it is of interest to construct a region that contains an outcome of the future experiment, with a specified probability. This region is called a prediction region for the future outcome, \( Y \).

A prediction region is similar to a confidence set for a parameter of a distribution. That is, before observing the results of the informative experiment the probability is \( 1-\alpha \) that we will obtain a region that will contain the outcome, \( Y \), of the future experiment.

Prediction methods are of several types as described by Aitchison, J. and Dunsmore, I.R. (1975). In 'Decisive Prediction', a prior distribution on the parameter space of the probability distributions and also a utility function are available. The idea is to maximize the expected value of the utility function over all possible prediction regions. The expectation is evaluated with respect to the 'Predictive Density', Aitchison, J. and Dunsmore, I.R. (1975), of \( Y \) given \( x=(x_1, \ldots, x_n) \). 'Bayesian Informative Prediction' does not require specification of a
utility function. The idea is to choose a prediction region so that the predictive density is above some point in that region. 'Informative Tolerance Prediction' does not use a prior distribution or utility function and it has two types, Mean Coverage Tolerance Prediction and Guaranteed Coverage Tolerance Prediction.

In this thesis we will work with the informative mean coverage tolerance prediction. We are interested in developing a technique for obtaining prediction regions by using hypothesis testing methods. In Chapter II, we will formulate the prediction problem for the one parameter exponential family of distributions in terms of hypothesis testing results. Examples of one parameter exponential distributions will be given in Chapter III and prediction intervals will be obtained by using the method of Chapter II. We will suggest an approximation method for some cases. The generalization to the K-parameter exponential distribution will be discussed in Chapter IV and an application of a 'Saddle Point Approximation', Barndorff-Nielsen (1983) for obtaining the density function of a sufficient statistic will be considered. Chapter V has two examples of two-parameter exponential distributions where prediction intervals are obtained using the method described in Chapter IV.

Aitchison, J. and Sculthorpe, D. (1965), give a general framework for Bayesian and non-Bayesian prediction. Hahn, J. G.
(1972) develops simultaneous prediction intervals for the standard deviation of future samples when sampling from a normal distribution. Faulkenberry, D. G. (1973), gives a method for obtaining prediction intervals. The method is based on conditioning on a sufficient statistic associated with probability distributions of the experiments. Statistical prediction intervals for observations of a future experiment is discussed by Olsen, D. E. (1974). In his work discrete probability distributions associated with the experiments are considered. Aitchison, J. and Dunsmore, I.R. (1975) have a detailed discussion on types of prediction methods and Chhikara, R.S. and Guttman, I. (1982) give prediction intervals when sampling from an Inverse-Gaussian distribution. They use Bayesian Informative and Informative Tolerance prediction techniques. Given a set of observations from a general linear model and having prior distribution for parameters, Johnson W. and Geisser S. (1982) develop a method for assessing the influence of specified subset of the data when prediction of future observations is of interest. We will not assume any regression type models and construct prediction intervals which are based only on outcomes of the informative experiment. The method of obtaining prediction intervals in this thesis is developed under the assumption that the probability distribution of experiments are from a continuous exponential family, but the method can be applied to a general case.
II. Prediction Intervals in One Parameter

Continuous Exponential Families

In this chapter we will formulate the problem of deriving a prediction region by using hypothesis testing theory. For the special case of the one parameter continuous exponential family it will be shown that the optimal properties of the hypothesis tests carry over to similarly defined optimal properties of prediction intervals.

2.1. Deriving Prediction Regions Using Hypothesis Tests

Let $X_1, ..., X_n$ be independent identically distributed, (iid) random outcomes of an informative experiment and $Y$ be an outcome of a future experiment. In order that the informative experiment should provide information on the future experiment, there must be some link between the two experiments, Aitchison and Dunsmore (1975). This link is through the probability distributions associated with the experiments and is also through an indexing set of parameters of the distributions. The common assumptions are that the experiments are being conducted independently, and the probability distributions are the same.

Let $(P_\theta, \theta \in \Theta)$ denote the probability distribution associated with the experiments, $f(\cdot; \theta)$ be its density function,
and let $\mathcal{X}$ be the sample space. A family of subsets $S(\mathcal{X})$ of the sample space, $\mathcal{X}$, is said to constitute a family of prediction regions for $Y$ based on $\mathbf{X}=(X_1,\ldots,X_n)$, if the random set $S(\mathbf{X})$ covers $Y$ with some specified probability. We have

$$S : \mathcal{X}^n \rightarrow \mathcal{X}$$

**Definition 2.1:** $S(\mathcal{X})$ is called a 'Mean Coverage Tolerance Prediction Region' of Cover $(1-\alpha)$ for $Y$ if

$$\inf_{\theta \in \Theta} \mathbb{E}_\theta \left[ P_\theta (Y \in S(\mathbf{X})) \right] = \inf_{\theta \in \Theta} \int \int \int_{\mathcal{X}} f(y;\theta)f(\mathbf{x};\theta)dx_1dxdy = 1-\alpha.$$ 

If $\mathbb{E}_\theta \left[ P_\theta (Y \in S(\mathbf{X})) \right] = (1-\alpha)$ for all $\theta \in \Theta$, then $S(\mathcal{X})$ is called a 'Similar Mean Coverage Tolerance Prediction Region' of Cover $(1-\alpha)$.

We will obtain prediction regions by using hypothesis testing procedures. It will be shown that these prediction regions are similar mean coverage and also have some optimal properties.

To set up the prediction problem using hypothesis tests, let $X_1,\ldots,X_n$ be (iid) outcomes of an informative experiment with density, $(f(x;\theta_x), \theta_x \in \Theta_x, x \in \mathcal{X})$ and $Y$ be an outcome of a
future experiment with density, \( f(y; \Theta_y), \Theta_y \in \Theta_y, \ y \in \mathcal{X} \),
where \( \Theta_x \) and \( \Theta_y \) are parameter spaces associated with the probability distributions of the informative and future experiments, respectively.

Define a 'notional null hypothesis', Cox and Hinkley (1974),
\[
H_0: \Theta_y = \Theta_x
\]
concerning the true parameters of the experiments. Let \( H_a \) be an alternative hypothesis and \( A_{H_a}(\Theta_y, \Theta_x) \) denote the acceptance region of a size \( \alpha \) test for testing \( H_0 \) versus \( H_a \).

**Theorem 2.1:** For each sample point \((x, y) \in \mathcal{X}^n \times \mathcal{X}\) let

\[
S_{H_a}(x) = \{ y: (x, y) \in A_{H_a}(\Theta_y, \Theta_x) \}.
\]

(2.1.1)

Then \( S_{H_a}(x) \) is a family of prediction regions for \( Y \) with confidence level \( (1-\alpha) \). If \( A_{H_a}(\Theta_y, \Theta_x) \) is the uniformly most powerful \( (1-\alpha) \) level region in a certain class of acceptance regions for testing \( H_0 \) versus \( H_a \), then \( S_{H_a}(x) \) minimizes

\[
P(\Theta_y, \Theta_x) \{ Y \in S_{H_a}(x) \} \text{ for all } (\Theta_y, \Theta_x) \in \Theta_a
\]

where, \( \Theta_a \) is the parameter space associated with \( H_a \).
Proof: By definition of $S_{Ha}(X)$

$$y \in S_{Ha}(X) \iff (x,y) \in A_{Ha}(\theta_y, \theta_x)$$

and hence

$$S_{Ha} : \mathcal{X} \overset{n}{\rightarrow} \mathcal{X}$$

$$P(\theta_y, \theta_x)\{ Y \in S_{Ha}(X) \} = P(\theta_y, \theta_x)\{ (X,Y) \in A_{Ha}(\theta_y, \theta_x) \} = (1-\alpha)$$

for all $(\theta_y, \theta_x) \in \Theta_0$, where $\Theta_0 = \{(\theta_y, \theta_x) : \theta_y = \theta_x\}$.

Therefore $S_{Ha}(X)$ is a family of prediction regions for $Y$ with probability, $(1-\alpha)$.

If $S^*_{Ha}(X)$ is another family of prediction regions for $Y$ with probability, $(1-\alpha)$ and $A^*_{Ha}(\theta_y, \theta_x) = \{(X,Y) : y \in S^*_{Ha}(X)\}$, then

$$P(\theta_y, \theta_x)\{ (X,Y) \in A^*_{Ha}(\theta_y, \theta_x) \} = P(\theta_y, \theta_x)\{ Y \in S^*_{Ha}(X) \} = (1-\alpha)$$

for all $(\theta_y, \theta_x) \in \Theta_0$.

So $A^*_{Ha}(\theta_y, \theta_x)$ is the acceptance region of a level $\alpha$ test for testing $H_0$ versus $H_a$. $A_{Ha}(\theta_y, \theta_x)$ is assumed to be the UMP in a certain class of acceptance regions. Therefore
We note that the form of the UMP acceptance region depends on the alternative hypothesis used and as a result the type of prediction region, $S_{Ha}(X)$, depends on the alternative hypothesis.

While it seems reasonable that using good or optimal hypothesis testing procedures should result in good prediction, there are no formulated criteria available for comparing prediction regions. Similar criteria for comparing acceptance regions will carry over to comparing prediction regions. Comparison of unbiased prediction regions is discussed in section 2.3.

As an example of using an acceptance region to get a prediction region, let $X_1, \ldots, X_n$ be (iid) outcomes of an informative experiment with probability distribution, normal $(\theta_x, \sigma^2)$ and $Y$ be an outcome of a future experiment with probability distribution, normal $(\theta_y, \sigma^2)$. It is assumed that $\sigma$ has a known value. The $(1-\alpha)$ level acceptance region for testing
\[ H_0 : \theta_y = \theta_x \]
\[ H_a : \theta_y < \theta_x \]

is

\[ A_{Ha}(\theta_y, \theta_x) = \{ (y, \bar{x}) : \frac{(1+1/n)^{-1/2}(y-x)}{\sigma} \leq Z_{1-a/2} \} \]

where, \( \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \) is the sample mean and \( Z_{1-a/2} \) is \((1-a/2)\) percentile of the standard normal distribution. From (2.1.1) we have

\[ S_{Ha}(\bar{x}) = \{ y : y \leq \bar{x} + \sigma(1+1/n)^{1/2}Z_{1-a/2} \} \]

which is an upper limit prediction for \( Y \) with confidence level \( 1-a \).

2.2. The Theory For One Parameter Case

2.2.1. Formulation Of The Prediction Problem

In terms Of Hypothesis Testing

Definition 2.2: A family of probability measures, \( \mathcal{P} = \{ P_\theta : \theta \in \Theta \} \) on a sample space, \( \mathcal{X} \) is called a \( K \)-parameter exponential family if, with respect to some \( \sigma \)-finite measure, \( \mu \), it has densities of the form

\[ f(x; \theta) = C(\theta) \exp[Q(\theta)T(x)]h(x) \quad (2.2.1) \]

where, \( C : \Theta \to (0, \infty) \), \( Q : \Theta \to \mathbb{R}^K \), \( T : \mathcal{X} \to \mathbb{R}^K \), \( h : \mathcal{X} \to [0, \infty) \).
Let random outcomes of an informative experiment, $X_1, ..., X_n$, represent independent and identically distributed, (iid) random variables according to a one parameter continuous exponential family of distributions with density function

$$f(x; \theta_x) = C(\theta_x) \exp[\theta_x T(x)] h(x)$$

and let a random outcome of the corresponding future experiment, $Y$, be independent of the $X_i$'s and distributed according to the same family with density function

$$f(y; \theta_y) = C(\theta_y) \exp[\theta_y T(y)] h(y) .$$

The probability distributions are assumed to be absolutely continuous. Therefore the $\sigma$-finite measure, $\mu$, Definition 2.2, is the Lebesgue measure and the sample space, $X$ is the real line.

In some problems the density functions are not in the form of the 'natural parametrization', Ferguson (1967), but they can be reparametrized to get the desired form. The value of a future outcome, $Y$, is not known but to formulate the prediction problem using hypothesis testing techniques, we pretend at this point that $Y$ is observed like the $X_i$'s. Hypotheses of the following forms are of interest.
1. \( H_0: \theta_x = \theta_y \) vs \( H_{a1}: \theta_y > \theta_x \)
2. \( H_0: \theta_x = \theta_y \) vs \( H_{a2}: \theta_y < \theta_x \)
3. \( H_0: \theta_x = \theta_y \) vs \( H_{a3}: \theta_y \neq \theta_x \)

The choice of the alternative hypothesis depends on the form of prediction region (interval) desired.

The joint density function of \( X, Y \) can be written as

\[
f(x, y; \theta_x, \theta_y) = C^n(\theta_x)C(\theta_y)\exp[\theta_x \sum_{i=1}^n T(x_i) + \theta_y T(y)]
\]

\[
\pi_{i=1}^n h(x_i)h(y).
\]

The density function in (2.2.2) is in the form of a two parameter regular exponential family and therefore \( \sum_{i=1}^n T(X_i) \) and \( T(Y) \) are jointly complete and sufficient statistics for the joint distribution of \( X \) and \( Y \). Thus it is natural to reduce the problem to the consideration of the sufficient statistics.

Let \( T_x = \sum_{i=1}^n T(X_i), T_y = T(Y), T = T_x + T_y \) and \( \beta = \theta_y - \theta_x \), then according to Lehmann (1959), we have

\[
f(t_x; \theta_x) = C_{t_x}(\theta_x)\exp[\theta_x t_x]h_{t_x}(t_x)
\]

where, \( C_{t_x}: \Theta_x \rightarrow (0, \infty), T_x: \mathbb{R}^n \rightarrow \mathbb{R}, h_{t_x}: \mathbb{R} \rightarrow [0, \infty) \) and

\[
f(t_y; \theta_y) = C_{t_y}(\theta_y)\exp[\theta_y t_y]h_{t_y}(t_y)
\]
where, \( C_{ty}: \Theta_y \rightarrow (0, \infty), T_y: R \rightarrow R, h_{ty}: R \rightarrow [0, \infty) \).

\( T_x \) is independent of \( T_y \). By (2.2.3) and (2.2.4), the joint density of \( T_x \) and \( T_y \) is

\[
f(t_x, t_y; \theta_x, \theta_y) = c_{tx}(\theta_x)c_{ty}(\theta_y)\exp[\theta_x t_x + \theta_y t_y] h_{tx}(t_x)h_{ty}(t_y)
\]

which implies

\[
f(t_y, t; \theta_x, \beta) = c_{tx}(\theta_x)c_{ty}(\beta + \theta_x)\exp[\beta t_y + \theta_x t] (2.2.5)
\]

\[
h_{tx}(t-t_y)h_{ty}(t_y), t_y \leq t
\]

The density function in (2.2.5) is in the form of the two parameter regular exponential family and therefore \((T_y, T)\) are complete sufficient statistics for \((\beta, \theta_x)\). Using (2.2.5), the conditional density of \( T_y \) given \( t \) is

\[
f(t_y | t, \beta) = \frac{\exp[\beta t_y]h_{tx}(t-t_y)h_{ty}(t_y)}{\int_{-\infty}^{t} \exp[\beta t_y]h_{tx}(t-t_y)h_{ty}(t_y), t_y \leq t}
\]

and with the new parametrization hypotheses of interest become

1. \( H_0: \beta = \beta_0 \) vs \( H_{a1}: \beta > \beta_0 \)
2. \( H_0: \beta = \beta_0 \) vs \( H_{a2}: \beta < \beta_0 \)
3. \( H_0: \beta = \beta_0 \) vs \( H_{a3}: \beta \neq \beta_0 \)
where, $\beta_0 = 0$ and $\theta_x$ is a nuisance parameter. Existence of the nuisance parameter $\theta_x$ does not allow us to find UMP tests for the hypotheses. Thus in this thesis we will concentrate on unbiased tests, Ferguson (1967).

Lemma 2.1: If $\theta_x = \theta_y (= \theta)$, then the conditional density function of $T_y$ given $T=t$, under $H_0$ is

$$f(t_y | T=t) = \frac{h_{tx}(t-t_y)h_{ty}(t_y)}{h_t(t)}$$

(2.2.7)

Proof: If $\theta_x = \theta_y (= \theta)$, then $\beta = \beta_0$ and using (2.2.6) we get

$$f(t_y | T=t) = \frac{h_{tx}(t-t_y)h_{ty}(t_y)}{h_t(t)}$$

where, $h_t(t) = \int_{-\infty}^{t} h_{tx}(t-t_y)h_{ty}(t_y) dt_y$.

The following Theorem is based on the method of finding uniformly most powerful (UMP) unbiased tests in a $K$-parameter exponential family of distributions, Ferguson (1967). Notations will correspond to the cases where $X$ and $Y$ are from one parameter continuous exponential families and the joint density of $T_y$ and $T$ is given by (2.2.5). The Theorem can be generalized to the $K$-
parameter case which will be stated in Chapter IV.

**Theorem 2.2:** Let \( f_0 = f(t_y | T=t) \), (2.2.7), denote the conditional density function of \( T_y \) given \( t \) under the notional null hypothesis, \( H_0 \).

(1) \[ \Phi_1(t_y, t) = \begin{cases} 1 & \text{if } t_y > z(t) \\ 0 & \text{if } t_y \leq z(t) \end{cases} \]

is the UMP unbiased size \( \alpha \) test for testing \( H_0 \) versus \( H_{a1} \) and \( z(t) \) can be found from \( E_{\beta=\beta_0} [ \Phi_1(T_y, T) | T] = \alpha \) which is equivalent to

\[ \int_{z(t)}^{\infty} f_0 dt_y = \alpha . \]

(2) \[ \Phi_2(t_y, t) = \begin{cases} 1 & \text{if } t_y < z(t) \\ 0 & \text{if } t_y \geq z(t) \end{cases} \]

is the UMP unbiased size \( \alpha \) test for testing \( H_0 \) versus \( H_{a2} \) and \( z(t) \) can be found from \( E_{\beta=\beta_0} [ \Phi_2(T_y, T) | T] = \alpha \) which is equivalent to

\[ \int_{-\infty}^{z(t)} f_0 dt_y = \alpha . \]

(3) \[ \Phi_3(t_y, t) = \begin{cases} 1 & \text{if } t_y < z_1(t) \text{ or } t_y > z_2(t) \\ 0 & \text{if } z_1(t) \leq t_y \leq z_2(t) \end{cases} \]
is the UMP unbiased size $\alpha$ test for testing $H_0$ versus $H_{a3}$ and 
$z_1(t), z_2(t)$ can be found from

$$E_{\beta=\beta_0} \left[ \Phi_3(T_y, T) \mid T \right] = \alpha \quad \text{and} \quad E_{\beta=\beta_0} \left[ T_y \Phi_3(T_y, T) \mid T \right] = \alpha \quad E_{\beta=\beta_0} [T_y \mid T]$$

which are equivalent to

$$\int_{z_1(t)}^{z_2(t)} f_0 dt_y = 1 - \alpha \quad \text{and} \quad \int_{z_1(t)}^{z_2(t)} t_y f_0 dt_y = (1 - \alpha) \quad \int_{-\infty}^{\infty} t_y f_0 dt_y.$$

Sometimes some work can be saved when under $H_0$ there exists
an ancillary statistic, $V$, for the joint distribution of $T_y$ and $T$. An ancillary statistic in this thesis is defined to be a statistic
with a distribution that does not depend on any parameters. If
an ancillary statistic exists, instead of finding the conditional
density, $f_0$, we find the density function of $V$ under $H_0$. For, by
Basu's Theorem given in Lehmann (1959), since $T$ is a complete
sufficient statistic (C.S.S) for the joint distribution under $H_0$
and $V$ is an ancillary statistic for the same distribution,
then they are independent.

The following Lemma, Lehmann (1959), will be used for
finding UMP unbiased tests when there exists an ancillary statistic
for the joint distribution of $T_y$ and $T$ under $H_0$. 
Lemma 2.2: Suppose there exists an ancillary statistic, \( V = G(T_Y, T) \) for joint distribution of \((T_Y, T)\) under \( H_0 \). Let \( f_{H_0}(v) \) denote the density function of \( V \) under \( H_0 \).

\[
\begin{align*}
\Phi_1(v) &= \begin{cases} 
1 & \text{if } v > z \\
0 & \text{if } v \leq z
\end{cases} \\
\end{align*}
\]

is the UMP unbiased size \( \alpha \) test for testing \( H_0 \) versus \( H_{a1} \) provided \( G(T_Y, T) \) is increasing in \( T_Y \) for each \( t \). \( z \) can be found from

\[
E_{\beta=\beta_0} [\Phi_1(V)] = \alpha \text{ which is equivalent to } \int_{z}^{\infty} f_{H_0}(v) dv = \alpha.
\]

\[
\begin{align*}
\Phi_2(v) &= \begin{cases} 
1 & \text{if } v < z \\
0 & \text{if } v \geq z
\end{cases} \\
\end{align*}
\]

is the UMP unbiased size \( \alpha \) test for testing \( H_0 \) versus \( H_{a2} \) provided \( G(T_Y, T) \) is increasing function in \( T_Y \) for each \( t \). \( z \) can be found from

\[
E_{\beta=\beta_0} [\Phi_2(V)] = \alpha \text{ which is equivalent to } \int_{-\infty}^{z} f_{H_0}(v) dv = \alpha.
\]

\[
\begin{align*}
\Phi_3(v) &= \begin{cases} 
1 & \text{if } v < z_1 \text{ or } v > z_2 \\
0 & \text{if } z_1 \leq v \leq z_2
\end{cases} \\
\end{align*}
\]

is the UMP unbiased size \( \alpha \) test for testing \( H_0 \) versus \( H_{a3} \) provided \( V = G(T_Y, T) = a(T)T_Y + b(T) \), \( a(T) > 0 \). \( z_1, z_2 \) can be found from

\[
E_{\beta=\beta_0} [\Phi_3(V)] = \alpha \text{ and } E_{\beta=\beta_0}[V \Phi_3(V)] = \alpha
\]

\[
E_{\beta=\beta_0}[V]
\]
which are equivalent to

\[
\int_{z_1}^{z_2} f_{H_0}(v) dv = 1 - \alpha \quad \text{and} \quad \int_{z_1}^{z_2} v f_{H_0}(v) dv = (1 - \alpha) \int_{-\infty}^{\infty} v f_{H_0}(v) dv.
\]

A similar result holds if \( a(T) < 0 \), Lehmann (1947). \( \Phi_i \) (i=1,2,3) in Theorem 2.2 and Lemma 2.1 are similar on the boundary sets \( \Theta_{b_i} = \Theta_o \cap \Theta_{a_i} \) and have Neyman Structure, Ferguson (1967).

According to the following Lemma, Lehmann (1947), \( z_1, z_2 \) in Lemma 2.2 (3) exist.

**Lemma 2.3:** Let \( 0 < \alpha < 1 \), let \( f(x) \) be a density function of a random variable, \( X \) with \( \int_{-\infty}^{\infty} x^s f(x) dx < \infty \), \( s=0,1 \) then, there exist \( A,B \) so that

\[
\int_{A}^{\infty} x^s f(x) dx = \alpha \int_{-\infty}^{\infty} x^s f(x) dx, \quad (s=0,1).
\]

**Remark:** For any size \( \alpha \) test, \( \Phi \) if distribution of \( V \) under \( H_0 \) is symmetric about some point \( A \), then \( E_{\beta=\beta_0}[V] = A \) and we have

\[
E_{\beta=\beta_0}[V \cdot \Phi(V)] = E_{\beta=\beta_0}[(V-A) \cdot \Phi(V) + A] \quad (V)
\]

\[
= AE_{\beta=\beta_0} [\Phi(V)] = A \alpha = \alpha E_{\beta=\beta_0}[V].
\]
That is, if \( V \) under \( H_0 \) has a symmetric distribution about some point \( A \), then any test of size \( \alpha \), \( \Phi \), which satisfies 
\[ E_{\beta=\beta_0} [\Phi(V)] = \alpha \]
must satisfy 
\[ E_{\beta=\beta_o}[V \Phi(V)] = \alpha \]
\( E_{\beta=\beta_0}[V] \) and to find acceptance region for \( H_0 \) versus \( H_{a3} \), we just solve

\[ \int_{-\infty}^{m} f_{H_0}(v) dv = \alpha / 2 \]

for \( m \) where, \( m = z_1 \) and \( 2A-m = z_2 \). A similar argument to the above one holds when \( f_o \) in the Theorem 2.2 is symmetric about some point, \( A \).

2.2.2. Monotone Likelihood Ratio Family of Distributions

Definition 2.3: A real parameter family of distributions is said to have a monotone likelihood ratio, if densities (probability mass functions), \( f(w;\theta) \) exist so that for \( \theta_1 < \theta_2 \),

\[
\frac{f(w;\theta_2)}{f(w;\theta_1)}
\]

is a nondecreasing function of \( w \), for all \( w \) in the set of existence of \( f(w;\theta_1) \) and \( f(w;\theta_2) \).
Lemma 2.4: Let $W$ be a continuous random variable distributed according to a one parameter exponential family, $f_{\theta}$, with the density function

$$f(w; \theta) = C(\theta) \exp[\theta T(w)] h(w)$$

and cumulative distribution function, $F_{\theta}(w)$, where

$$C: \theta \rightarrow \mathbb{R}, \quad T: \mathbb{W} \rightarrow \mathbb{R}, \quad h: \mathbb{W} \rightarrow \mathbb{R}, \quad \theta \subset \mathbb{R}$$

and $T(w)$ is nondecreasing in $w$. Then, the family, $f_{\theta}$, has a monotone likelihood ratio in $w$ and for $\theta_2 > \theta_1$, $F_{\theta_2}(w) < F_{\theta_1}(w)$, for all $w$ in the set of existence of $f(w; \theta)$.

Consider a random sample, $X_1, ..., X_n$ from a one parameter exponential family with density, (2.2.3) and let $Y$ be independent of the $X_i$, $(i=1, 2, ..., n)$ from the same family with density, (2.2.4). By Lemma 2.4, the distributions of $X$ and $Y$ have monotone likelihood ratio in $T(x)$ and $T(y)$, respectively. Recall definitions of sufficient statistics $T_y$, $T_x$ and $T$ from section 2.2.1, and consider the conditional density of $T_y$ given $T=t$

$$f(t_y | t; \beta) = \frac{\exp[\beta t_y] h_{tx}(t-t_y) h_{ty}(t_y)}{\int_{-\infty}^{t} \exp[\beta t_y] h_{tx}(t-t_y) h_{ty}(t_y)} \tag{2.2.8}$$
The distribution associated with the density function in (2.2.8) has a monotone likelihood ratio in $t_y$. Therefore we conclude that if $X$ ($Y$) have monotone likelihood ratio family of distributions, then the conditional distribution of $T_y$ given $T$ has also a monotone likelihood ratio. In section 2.2.1 we had $\beta = \theta_y - \theta_x$, but in some cases in which a reparametrization is needed to get natural parametrization for densities (2.2.3) and (2.2.4), $\beta$ can be defined as $\beta = g(\theta_y - \theta_x)$, with $g(.)$ being nondecreasing and $\beta = 0$ whenever $\theta_y = \theta_x$.

**Lemma 2.5:** Let the conditional density function of $T_y$ given $t$ be given by (2.2.8), then

1. $P_{\beta = \beta_1} (T_y \in (-\infty, C_2(t)) | T=t) \leq P_{\beta = \beta_0} (T_y \in (-\infty, C_2(t)) | T=t)$ for all $\beta_1 > \beta_0$.

2. $P_{\beta = \beta_1} (T_y \in [C_1(t), \infty) | T=t) \leq P_{\beta = \beta_0} (T_y \in [C_1(t), \infty) | T=t)$ for all $\beta_1 < \beta_0$.

3. $P_{\beta = \beta_1} (T_y \in [C_1(t), C_2(t)] | T=t) \leq P_{\beta = \beta_0} (T_y \in [C_1(t), C_2(t)] | T=t)$ for all $\beta_1 \neq \beta_0$.

where, $\beta_0 = 0$ and $C_i(\cdot), i=1,2$ are functions of $t$.

**Proof:** Let $F_{t_y}^T|t$ be the cumulative distribution associated with conditional distribution of $T_y$ given $t$, defined by,

$$F_{t_y}^T|t(C(t))=P_{\beta} (T_y \leq C(t)|T=t) \text{ . Use Lemma 2.4 .}$$
2.2.3. The Match-up Of Hypothesis Tests and Prediction Intervals

A. Ancillary Statistic Does Not Exist

Consider notations and results of section 2.2.1 and suppose the joint distribution of $T_y$ and $T$ under $H_0$ does not have an ancillary statistic, $V$, and UMP unbiased acceptance regions for testing $H_0$ versus $H_{a_1}$ are found using $f_0$. (See Theorem 2.2.)

According to the results of Theorem 2.2, UMP unbiased tests for testing $H_0$ versus $H_{a_1}$, $(i=1,2,3)$ are performed as conditional tests given $T=t$ and as a consequence acceptance regions are found conditionally. According to our hypotheses testing formulation of the problem in section 2.2.1, $\beta$ is the parameter of interest and $\Theta_x$ is the nuisance parameter. We denote the conditional acceptance region of $H_0$ versus $H_{a_i}$, $(i=1,2,3)$, by $A_i(\beta \mid t)$. UMP unbiased tests provide UMP acceptance regions, Lehmann (1959), therefore $A_i(\beta \mid t)$ are UMP unbiased acceptance regions. From Theorem 2.2 we have

\begin{align}
A_1(\beta \mid t) &= \{ (t_y, t) : t_y \leq z(t) \} \\
A_2(\beta \mid t) &= \{ (t_y, t) : t_y \geq z(t) \} \\
A_3(\beta \mid t) &= \{ (t_y, t) : z_1(t) \leq t_y \leq z_2(t) \},
\end{align}

(2.2.9)
where \( z(.) \) are functions of \( t \). Lemma 2.5 identifies intervals which have larger probabilities when parameters of the distributions are assumed to be the same than if they are different in some direction. The acceptance regions are constructed under \( H_0 \) and note that they have the same forms as intervals in Lemma 2.5. Therefore it would be a natural thing to use acceptance regions to obtain prediction intervals.

Let \( A_1(\beta \mid t) \) be any of the acceptance regions in above. Boundaries of these regions are functions of \( T = T_x + T_y \) which has unknown value, because \( T_y \) is unknown. We define the prediction region for \( T_y \) by

\[
S_i(t_x) = \{ t_y : (t_y, t) \in A_i(\beta \mid t) \}
\]

so that \( S_i(t_x) \) can be obtained by solving \( A_i(\beta \mid t) \) for \( T_y \), where \( T = T_y + T_x \). Assuming \( z(.) \) functions in \( A_i(\beta \mid t) \), monotone increasing, the prediction intervals will be of the forms

\[
\begin{align*}
S_1(t_x) &= (-\infty, U(t_x)] \\
S_2(t_x) &= [L(t_x), \infty) \\
S_3(t_x) &= [L(t_x), U(t_x)]
\end{align*}
\]

where, \( L(t_x) \) and \( U(t_x) \) are functions of \( t_x, \alpha \) and \( n \).
$S_1(t_x)$ are prediction intervals for $T_y$, but since $T_y$ is a function of single random variable, $Y$, then one can use simple mathematical operations to get prediction interval for $Y$.

If $z(.)$ are not monotone, then the acceptance regions will not map necessarily onto prediction intervals of the same forms. In the following we will give a sufficient condition for $z(.)$ to be increasing function of $t$.

**Definition 2.4:** The conditional family of distributions of $T_y$ given $t$, (under $H_0$), is said to have a monotone likelihood ratio, if the ratio

$$\frac{f_{T_y|t_2}(t_y)}{f_{T_y|t_1}(t_y)}$$

is a nondecreasing function of $t_y$ whenever $t_2 > t_1$, for all $t_y$ in the set of existence of $f_{T_y|t_1}$ and $f_{T_y|t_2}$.

**Lemma 2.6:** Suppose the conditional distribution of $T_y$ given $t$, under $H_0$, with density function

$$f_{T_y|t}(t_y) = \frac{h_{tx}(t-t_y)h_{ty}(t_y)}{h_t(t)}$$

has a monotone likelihood ratio, then $z(.)$ are increasing functions of $t$. 
Proof: Without loss of generality we will consider \( z(t) \) in \( A_1(\beta | t) \).

Proofs for the other cases are similar.

\( A_1(\beta | t) \) is constructed under \( H_0: \theta_x=\theta_y (\beta=\beta_0) \) and since

\( T \) is a complete sufficient statistic for the joint distribution

of \( (T_y,T) \) under \( H_0 \), then the following conditional probability

statements, under \( H_0 \), do not depend on \( \theta_x \). (See density function

2.2.5.) We use the method of proof given in Theorem (5), Olsen

(1974).

\[
P_{\beta=\beta_0}(T_y,T) \in A_1(\beta | t) = P_{T_y | t} \{ T_y \leq z(t) \} = 1 - \alpha \quad (2.2.11)
\]

for all \( t \). The probability associated with \( A_1(\beta | t) \) is exactly

\( 1 - \alpha \) for it corresponds to a test which has Neyman Structure,

Ferguson, (1967).

Using (2.2.11), we have

\[
P_{T_y | t_1} \{ T_y \leq z(t_1) \} = P_{T_y | t_2} \{ T_y \leq z(t_2) \} = 1 - \alpha \quad (2.2.12)
\]

for any \( t_1 \) and \( t_2 \).

Consider \( t \) as a parameter of the conditional distribution of

\( T_y \) given \( t \), (under \( H_0 \)) and test \( K_1: T=t_1 \) vs \( K_2: T=t_2 \), \( t_2>t_1 \).

By Neyman-Person Lemma, the UMP test of size \( \alpha \) is

\[
\psi = \begin{cases} 
1 & \text{if } R > c \\
0 & \text{if } R \leq c
\end{cases}
\]

which is equivalent to
\[ \phi = \begin{cases} 
1 & \text{if } t_y > k \\
0 & \text{if } t_y \leq k 
\end{cases} \]

where
\[ R = \frac{f_{Ty} | t_2(t_y)}{f_{Ty} | t_1(t_y)} \]

and \( c, k \) are constants to make the tests, size \( \alpha \). Using (2.2.11), (2.2.12) and having \( E_{t_1}(\phi) = \alpha \) implies

\[ \int_{-\infty}^{k} f_{Ty} | t_1(t_y) \, dt_y = \int_{-\infty}^{z(t_1)} f_{Ty} | t_1(t_y) \, dt_y = 1 - \alpha, \]

\( \phi \) is unique, so \( k = z(t_1) \). We also have

\[ E_{t_2}(\phi) = \int_{k}^{\infty} f_{Ty} | t_2(t_y) \, dt_y = \int_{z(t_1)}^{\infty} f_{Ty} | t_2(t_y) \, dt_y \geq E_{t_1}(\phi) \]

which implies

\[ P_{Ty} | t_2( T_y \leq z(t_1) ) \leq P_{Ty} | t_1( T_y \leq z(t_1) ) \]

and by using (2.2.12), we get

\[ P_{Ty} | t_2( T_y \leq z(t_1) ) \leq P_{Ty} | t_2( T_y \leq z(t_2) ). \]

Therefore \( z(t_2) \geq z(t_1) \) and \( z(t) \) is an increasing function of \( t \).

Using Lemma 2.6 we conclude that \( z(.) \) functions are increasing if
B. Ancillary Statistic Exists

Suppose the joint distribution of $T_y$ and $T$ under $H_0$ has an ancillary statistic, $V=G(T_y, T)$. According to Lemma 2.2, UMP unbiased acceptance regions for testing $H_0$ versus $H_{ai}$, $i=1,2,3$, are found using the density of $V$ under $H_0$. From Lemma 2.2 we have

$$A_1(\beta) = \{ v : v < z \}$$
$$A_2(\beta) = \{ v : v > z \}$$
$$A_3(\beta) = \{ v : z_1 < v < z_2 \}$$

where, $z$, $z_1$ and $z_2$ are constants not depending on $t$. $V=G(T_y, T)$ in Lemma 2.2 is assumed to be an increasing function of $T_y$ for given $t$, therefore $G^{-1}(\cdot)$ will be increasing in $V$ for given $t$.

Since the conditional density of $T_y$ given $t$, (2.2.8), has a monotone likelihood ratio in $t_y$, we conclude that the conditional distribution of $V$ given $t$ has also a monotone likelihood ratio. This can be seen easily when $V=a(T)T_y+b(T)$, $a(T)>0$. From (2.2.8) we have
\[
f(\nu | t; \beta) = \frac{\exp[\beta \nu / a(t)] h_{tx}[t - (\nu - b(t))/a(t)] h_{ty}((\nu - b(t))/a(t))}{\int_{-\infty}^{Q(t)} \exp[\beta \nu / a(t)] h_{tx}[t - (\nu - b(t))/a(t)] h_{ty}((\nu - b(t))/a(t)) d\nu}
\]

where, \( Q(T) = a(T)T + b(T) \).

Using the above argument and Lemma 2.4 we have

\[
\begin{align*}
P_{\beta} &= \beta_1 \{ V \leq C_2(t) \mid T = t \} \leq P_{\beta_0} \{ V \leq C_2(t) \mid T = t \} \quad \text{for all } \beta_1 > \beta_0 \\
P_{\beta} &= \beta_1 \{ V \geq C_1(t) \mid T = t \} \leq P_{\beta_0} \{ V \geq C_1(t) \mid T = t \} \quad \text{for all } \beta_1 < \beta_0 \\
P_{\beta} &= \beta_1 \{ C_1(t) \leq V \leq C_2(t) \mid T = t \} \leq P_{\beta_0} \{ C_1(t) \leq V \leq C_2(t) \mid T = t \} \\
&\text{for all } \beta_1 \neq \beta_0
\end{align*}
\]

where, \( C_1(t) \) and \( C_2(t) \) are functions of \( t \). Therefore the acceptance regions \( A_i(\beta) \) have larger probability under \( H_0 \) than under \( H_{a1} \). Similar to the previous case we define a prediction region for \( T_y \) by

\[
S_1(t_x) = \{ t_y : \nu \in A_i(\beta) \}
\]

so that \( A_i(\beta) \) are solved for \( t_y \) by writing \( \nu \) as a function of \( T_x \) and \( T_y \). The prediction intervals will be of the forms

\[
\begin{align*}
S_1(t_x) &= (-\infty, U(t_x)] \\
S_2(t_x) &= [L(t_x), \infty] \\
S_3(t_x) &= [L(t_x), U(t_x)]
\end{align*}
\]

(2.2.13)
provided, $G^{-1}(.)$ is an increasing function of $t$. For example if $V = a(T)T + b(T)$, then we need $(V - b(T))/a(T)$ to be increasing in $T$.

2.3. **Unbiased Prediction Intervals**

and Optimal Properties

Using the acceptance regions we get prediction intervals which are based on the assumption of equal parameters. If the parameters are not the same it seems natural to require the intervals to have small probability of coverage in some sense.

**Definition 2.5:** Suppose the $P_{\theta x}$ is probability distribution of a random sample, $X_1,...,X_n$ and the $P_{\theta y}$ is probability distribution of a random variable $Y$, which is independent of $x=(x_1,...,x_n)$.

(1) $(-\infty, U(x)]$ is said to be a $(1-\alpha)$ level unbiased prediction interval for $Y$ iff

1. $E_{\theta x}[P_{\theta y}[Y \leq -\infty, U(x)]] = 1 - \alpha$ for all $\theta_y = \theta_x$
2. $E_{\theta x}[P_{\theta y}[Y \leq U(x)]] \leq 1 - \alpha$ for all $\theta_y > \theta_x$

(2) $[L(x), \infty)$ is said to be a $(1-\alpha)$ level unbiased prediction interval for $Y$ iff

1. $E_{\theta x}[P_{\theta y}[Y \geq L(x), \infty]] = 1 - \alpha$ for all $\theta_y = \theta_x$
2. $E_{\theta x}[P_{\theta y}[Y \geq L(x), \infty]] \leq 1 - \alpha$ for all $\theta_y < \theta_x$
(3) \([L(x), U(x)]\) is said to be a \((1-\alpha)\) level unbiased prediction interval for \(Y\) iff

1. \(E_{\Theta_X}[P_{\Theta_Y}[Y \in [L(x), U(x)]]] = 1-\alpha\) \quad \text{for all } \Theta_Y = \Theta_X

2. \(E_{\Theta_X}[P_{\Theta_Y}[Y \in [L(x), U(x)]]] \leq 1-\alpha\) \quad \text{for all } \Theta_Y \neq \Theta_X

where, \(L(x), U(x)\) are functions of \(x, \alpha\) and \(n\).

For each of the prediction intervals in Definition 2.5 the first condition is the property of a 'similar mean coverage tolerance prediction region' of cover \((1-\alpha)\). (See Definition 2.1.)

**Definition 2.6:** \((-\infty, U(x)]\) is said to be \((1-\alpha)\) level uniformly most accurate unbiased prediction interval for \(Y\), if it is \((1-\alpha)\) level unbiased prediction interval and for any other \((1-\alpha)\) level unbiased prediction interval, \(S^*(x)\) for \(Y\),

\[E_{\Theta_X}[P_{\Theta_Y}[Y \in (-\infty, U(x)]]] \leq E_{\Theta_X}[P_{\Theta_Y}[Y \in S^*(x)]]\]

for all \(\Theta_Y > \Theta_X\). Similar definitions hold for the other cases.

Optimal property of the prediction intervals can be stated as follows. Let \(S(x)\) be a prediction interval for \(Y\) and assume it is constructed under the assumption of equal parameters, \(\Theta_x\) and \(\Theta_y\), so that it has more probability coverage under such assumption. It is natural to define \(P_{\Theta_Y} [Y \in S(x)]\) as a measure of undesirability when in fact the parameters are unequal in some
direction. We would like to have a prediction interval which minimizes the measure. \( X=(X_1,...,X_n) \) is random and therefore \( P_{\Theta Y}(Y \in S(X)) \) is also random and it is reasonable to minimize the expected value of undesirability, \( E_{\Theta X}[P_{\Theta Y}(Y \in S(X))] \).

**Theorem 2.3:** Assume distributions of \( X (Y) \) have monotone likelihood ratios, so that the conditional distribution of \( Ty \) given \( T \) has also a monotone likelihood ratio. Consider prediction intervals in (2.2.10), obtained from UMP unbiased acceptance regions \( A_1(\beta | t) \). These intervals minimize the corresponding measures of undesirability.

**Proof:** Without loss of generality we will establish optimal property of \( S_1(t_x)=[-\infty, U(t_x)] \). Proofs for the other cases are similar. Let \( \Phi_1^* \) be any unbiased size \( \alpha \) test for testing \( H_0 \) versus \( H_{a1} \), with \( (1-\alpha) \) level unbiased acceptance region, \( A_1^*(\beta | t) \) and \( (1-\alpha) \) level unbiased prediction interval \( S_1^*(t_x) \).

\( \Phi_1^*(t_y,t) \) being unbiased test of size \( \alpha \), implies that it is \( \alpha \) similar on the boundary set and

\[
E_{\Theta X, \beta} [ \Phi_1^*(T_y,T) ] = \alpha \quad \text{for all } (\beta, \Theta_X)
\]

in the boundary set, Ferguson (1967).

Power of \( \Phi_1^* \) is maximized if and only if
\[
\max_{\phi_1} E_{\theta_x, \beta} [ \phi_1^*(T_y, T) ] \quad \text{for } \beta > \beta_0 \text{ and all } \theta_x
\]

subject to \( E(\theta, \beta) [ \phi_1^*(T_y, T) ] = \alpha \),

for \( \beta = \beta_0 \) and all \( \theta = (\theta_x = \theta_y) \).

Since \( T \) is a complete sufficient statistic for \( \theta_x \) under \( \mathcal{H}_0 \) and \( \phi_1^* \) is a similar on the boundary set, then \( \phi_1^* \) has Neyman Structure, Ferguson (1967),

that is

\[
E_{\theta_x, \beta} [ \phi_1^*(T_y, T) ] =
E_{\theta_x} [ E_{\beta} [ \phi_1^*(T_y, T) | T ] ] = \alpha
\]

for all \((\beta, \theta_x)\) in the boundary set.

Therefore the maximization problem is equivalent to

\[
\max_{\phi_1} E_{\theta_x} [ E_{\beta} [ \phi_1^*(T_y, T) | T ] ], \quad \text{for } \beta > \beta_0 \text{ and all } \theta_x
\]

subject to \( E_{\theta} [ E_{\beta} [ \phi_1^*(T_y, T) | T ] ] = \alpha \),

for \( \beta = \beta_0 \) and all \( \theta = (\theta_x = \theta_y) \)

or

\[
\min_{A_1^*(\beta | t)} E_{\theta_x} [ P_{\beta} [ T_y \leq A_1^*(\beta | t) ] ] \quad \text{for } \beta > \beta_0 \text{ and all } \theta_x
\]

subject to \( E_{\theta} [ P_{\beta} [ T_y \leq A_1^*(\beta | t) ] ] = 1 - \alpha \)

for \( \beta = \beta_0 \) and all \( \theta = (\theta_x = \theta_y) \)

but \((t_y, t) \in A_1^*(\beta | t) \) iff \( t_y \leq S_1^*(t_x) \)
and the maximization problem is equivalent to

\[
\min_{S^*_1(t_x)} \mathbb{E}_{\theta_x}[P_{\theta_y}[S^*_1(T_x)]] \quad \text{for all } \theta_y > \theta_x \quad (2.3.1)
\]

subject to \( \mathbb{E}_{\theta}[P_{\theta}[S^*_1(T_x)]] = 1 - \alpha \) for all \( \theta = (\theta_x = \theta_y) \).

By Theorem 2.2 we know \( \Phi^*_1(t_y, t) \) is the UMP unbiased for testing \( H_0 \) versus \( H_{a_1} \), so using \( \Phi^*_1(t_y, t) \) and its corresponding regions in the above equations, we conclude that \( S^*_1(t_x) \) minimizes the expected value of measure of undesirability. By (2.3.1) and unbiasedness property of \( S^*_1(t_x) \) we have \( \mathbb{E}_{\theta_x}[P_{\theta_y}[S^*_1(T_x)]] \leq 1 - \alpha \) for all \( \theta_y > \theta_x \), so \( S^*_1(t_x) \) is unbiased prediction interval and has the optimal property.

A similar argument to Theorem 2.3 holds for the case that an ancillary statistic for joint distribution of \( (T_y, T) \) under \( H_0 \) exists. The prediction intervals in (2.2.13) are also the \( (1-\alpha) \) level UMAU prediction intervals for \( T_y \).
III. EXAMPLES: ONE PARAMETER CASE

3.1. The Normal Distribution With Known Mean

Suppose \( X_1, X_2, \ldots, X_n \) is a random sample from a normal distribution with the density function

\[
f(x; \mu, \sigma_x^2) = (2\pi \sigma_x^2)^{-1/2} \exp\left[-\frac{(x-\mu)^2}{2\sigma_x^2}\right]
\]

and \( Y \) is independent of the \( X_i, (i=1,2,\ldots,n) \), with the density function

\[
f(y; \mu, \sigma_y^2) = (2\pi \sigma_y^2)^{-1/2} \exp\left[-\frac{(y-\mu)^2}{2\sigma_y^2}\right]
\]

where \( \mu \) is assumed to have a known value and \( \sigma_x > 0, \sigma_y > 0 \).

\[\sum_{i=1}^{n} X_i^2 \] is a sufficient statistic for the joint distribution of \( X_1, \ldots, X_n \) and \( Y^2 \) is a sufficient statistic for the distribution of \( Y \). Without loss of generality assume that \( \mu = 0 \).

Using \( \theta_x = -(2\pi \sigma_x^2)^{-1} \), \( \theta_y = -(2\pi \sigma_y^2)^{-1} \), \( \beta = \theta_y - \theta_x \), \( T_x = \sum_{i=1}^{n} X_i^2 \), \( T_y = Y^2 \) and \( T = T_x + T_y \) we have

\[
f(t_y \mid t; \beta) = \frac{\exp[\beta t_y] (t-t_y)^{n/2} - 1 (t_y)^{-1/2}}{\exp[\beta t_y] (t-t_y)^{n/2} - 1 (t_y)^{-1/2}} dt_y, \quad t_y < t
\]
which has a monotone likelihood ratio in $t_y$.

Under $H_0: \sigma_x = \sigma_y$, $V = G(T_y, T) = a(T)T_y$, $a(T) = 1/T$, is distributed according to Beta$(1/2, n/2)$ distribution, Bickel and Doksum (1977). Therefore $V$ is an ancillary statistic for the joint distribution of $(T_y, T)$ under $H_0$ and we can use Lemma 2.2 instead of Theorem 2.2. We have

1. $\beta = \beta_0$ iff $\sigma_y = \sigma_x$
2. $\beta > (\beta_0)$ iff $\sigma_y > (\sigma_x)$, $\beta_0 > 0$.

(V is increasing in $t_y$ for each $t$. Therefore the conditional distribution of $V$ given $t$ with parameter $\beta$, has also a monotone likelihood ratio.

($V/a(T)$ is increasing in $T$. Therefore acceptance regions will map onto the same form of prediction intervals. (See section 2.2.3 (B).)

One Sided Upper Limit Prediction Interval

(1), (2) and (3) imply that the acceptance region $A_1(\beta) = \{ v: v \leq z \}$ must be used where $z$, according to Lemma 2.2, is found from

$$\int_z^1 f_{H_0}(v)dv = \alpha , \quad f_{H_0}(v) = \frac{\Gamma((n+1)/2)}{\pi^{n/2}(n/2)(1/2)^{n/2-1}} v^{-1/2}(1-v)^{n/2-1}$$
$0 \leq v \leq 1$, $\pi(k) = \int_0^\infty x^{k-1}e^{-x}dx$ for real number, $k$ and $\pi(k)=(k-1)!$ for integer $k>0$.

By using $A_1(\beta)$ we get

$$S_1(t_x)=[0,t_xz/(1-z)]$$

which is the $(1-\alpha)$ level uniformly most accurate unbiased (UMAU) upper limit prediction interval for $T_y$. Since $T_y/T_x$ has $F$-distribution under $H_0$, then $nz/(1-z)$ is $(1-\alpha)$ percentile of $F$-distribution with $(1,n)$ degrees of freedom.

**One Sided Lower Limit Prediction Interval**

A similar argument to the previous one holds and we use $A_2(\beta) = \{ v : v \geq z \}$ where $z$ is found from

$$\int_0^z f_{H_0}(v)dv = \alpha$$

By solving $A_2(\beta)$, the $(1-\alpha)$ level UMAU lower limit prediction interval for $T_y$ is $S_2(t_x)=[t_xz/(1-z),\infty)$. $nz/(1-z)$ is a percentile of $F$-distribution with $(1,n)$ degrees of freedom.
Two Sided Prediction Interval

According to (1), (2) and (3), we need to use the acceptance region $A_3(\beta) = \{ v : z_1 \leq v \leq z_2 \}$ where $z_1$ and $z_2$ are found from

$$\int_{z_2}^{z_1} f_H(v) dv = 1-\alpha \quad \text{and} \quad \int_{z_2}^{z_1} v f_H(v) dv = (1-\alpha) \int_0^1 v f_H(v) dv \quad (3.1.1)$$

It is easier to obtain equal-tailed prediction interval. That is, assume the distribution of $V$ is symmetric about some point when $H_0$ is true. In this case second equation in (3.1.1) is not needed (See the remark in section 2.2). To get right values of $z_1$ and $z_2$, in this case we need to solve the two equations in (3.1.1), for Beta(1/2, n/2) is not a symmetric distribution. $z_1$ and $z_2$ can not be found in a closed form so an approximation method is needed.

The Generalized Newton's Method

Consider a system of two equations

$$\begin{cases} M(a, b) = 0 \\ K(a, b) = 0 \end{cases}$$

where, $M(\ldots)$ and $K(\ldots)$ are some functions of $a$ and $b$. We will use Young and Gregory (1972), to explain the 'Generalized Newton's Method' by which values of $a$ and $b$ for the above equations can
be found approximately. We find

\[ M_{ao} = \frac{\partial M(a,b)}{\partial a} \bigg|_{a=a_0} \]

\[ M_{bo} = \frac{\partial M(a,b)}{\partial b} \bigg|_{b=b_0} \]

\[ K_{ao} = \frac{\partial K(a,b)}{\partial a} \bigg|_{a=a_0} \]

\[ K_{bo} = \frac{\partial K(a,b)}{\partial b} \bigg|_{b=b_0} \]

where \( a_0 \) and \( b_0 \) are initial values for \( a \) and \( b \), respectively.

If \( M(a_0,b_0) = 0 \) and \( K(a_0,b_0) = 0 \), then \( a_0 \) and \( b_0 \) are the right values of \( a \) and \( b \). Otherwise we find

\[ J = K_{ao} M_{bo} - M_{ao} K_{bo} \]

\[ a_1 = a_0 + \frac{M(a_0,b_0)K_{bo} - K(a_0,b_0)M_{bo}}{J} \]

\[ b_1 = b_0 + \frac{K(a_0,b_0)M_{ao} - M(a_0,b_0)K_{ao}}{J}. \]

If \( M(a_1,b_1) = 0 \) and \( K(a_1,b_1) = 0 \), then \( a_1 \) and \( b_1 \) are the right values of \( a \) and \( b \). Otherwise we let \( a_0 = a_1 \) and \( b_0 = b_1 \) and repeat the above steps until roots of the equations can be found.
Consider the following facts about the Beta distribution.

**Fact I:** Suppose a random variable, \( X \), is distributed according to
\[
\text{Beta}(r, s), \; s > 0, \; r > 0, \; 0 \leq x \leq 1.
\]
Then

1. \[
    f(x) = \frac{\pi(r+s)}{\pi(r) \pi(s)} x^{r-1}(1-x)^{s-1}
\]
2. \[
    xf(x) = \frac{\Gamma(r+s)}{\Gamma(r+s+1)} f(y) \quad \text{where, } Y \sim \text{Beta}(r+1, s)
\]
3. \[
    E[X] = \frac{\Gamma(r+s+1)}{\Gamma(r+s)}
\]

**Fact II.** Let \( X \sim \text{Beta}(r, s), \; 0 \leq X \leq 1, \; r > 0, \; s > 0 \),

\[
M(a, b) = \int_a^b \frac{\pi(r+s)}{\pi(r) \pi(s)} x^{r-1}(1-x)^{s-1} dx - (1-a) = 0
\]

and

\[
K(a, b) = \int_a^b \frac{\pi(r+s+1)}{\pi(r+1) \pi(s)} x^{r}(1-x)^{s-1} dx - (1-a) = 0.
\]

Then

\[
M_a = \frac{\partial [M(a, b)]}{\partial a} = - \frac{\pi(r+s)}{\pi(r) \pi(s)} a^{r-1}(1-a)^{s-1}
\]

\[
M_b = \frac{\pi(r+s)}{\pi(r) \pi(s)} b^{r-1}(1-b)^{s-1}
\]

\[
K_a = \frac{\pi(r+s+1)}{\pi(r+1) \pi(s)} a^{r}(1-a)^{s-1}
\]
and 

\[ K_b = \frac{\pi(r+s+1)}{\pi(r+1) \pi(s)} b^r(1-b)^{s-1}. \]

Using Fact I equations in (3.1.1) can be written as

\[
\int_{z_1}^{z_2} \frac{\pi(n/2+1/2)}{\pi(1/2) \pi(n/2)} v^{-1/2} (1-v)^{n/2} -1 dv = 1-\alpha \quad (3.1.2)
\]

and

\[
\int_{z_1}^{z_2} \frac{\pi(n/2+3/2)}{\pi(3/2) \pi(n/2)} v^{3/2} (1-v)^{n/2} -1 dv = 1-\alpha ,
\]

respectively.

We apply Fact II and the Generalized Newton's Method along with MDBETA routine in IMSL library to obtain approximate values of \(z_1\) and \(z_2\) in (3.1.2). Values of \(z_1\) and \(z_2\) are given in table A. Solving \(A_3(\beta)\) for \(t_\gamma\), we get

\[ S_3(t_x) = [t_x z_1/(1-z_1), t_x z_2/(1-z_2)] . \]

Note that when \(\mu\) is not zero prediction intervals for \((Y-\mu)^2\) are

\[ S_1(t_x) = [0, t_x^* z/(1-z)] \]
\[ S_2(t_x) = [t_x^* z/(1-z), \infty) \]
\[ S_3(t_x) = [t_x^* z_1/(1-z_1), t_x^* z_2/(1-z_2)] \]
where, \( t_x = \sum_{i=1}^{n} (x_i - \mu)^2 \) and \( z, z_1, z_2 \) are defined as before for each case.

**Remark:** \( X \sim \text{Normal} (\mu, \sigma^2) \) iff \( \exp(X) \sim \text{Lognormal} (\mu, \sigma^2) \).

Using results of Example 3.1 and the relationship between Normal and Lognormal random variables, the prediction intervals can be obtained when sampling is from Lognormal \( (\mu, \sigma^2) \) distribution with known \( \mu \).

**Numerical Example:**

Each run of a process produces a large batch of ball bearings whose diameter (mm) are normally distributed with known mean, \( \mu = 8 \) (mm) and unknown variance, \( \sigma^2 \). From a particular batch a sample of 15 ball bearings is chosen at random and their diameter are found to be

\[
8.07, 8.15, 8.06, 7.79, 7.85, 8.02, 8.07, \\
8.17, 8.11, 8.09, 7.96, 9.02, 8.20, 7.97, 8.12.
\]

Suppose the process is to be run in future. For a randomly selected ball bearing from a batch, find .90 level, one and two sided prediction limits for its squared deviation of diameter from \( \mu = 8 \) (mm).

According to the notation used in Example 3.1, we have
\[ t_x^* = \sum_{i=1}^{n} (x_i - \mu)^2 = 1.2061, \ n = 15. \]

**Upper Limit Prediction:**

\[ S_1(t_x) = [0, t_x^*/(1-z)] = [0, 0.246848] \]

where, \( nz/(1-z) = 3.07 \) is the 90th percentile of the F-distribution with (1,15) degrees of freedom.

**Lower Limit Prediction:**

\[ S_1(t_x) = [t_x^*/(1-z), \infty) = [0.00131, \infty) \]

where, \( nz/(1-z) = 0.0163 \) is the 10th percentile of the F-distribution with (1,15) degrees of freedom.

**Two-sided Prediction:**

\[ S_3(t_x) = [t_x^*/(1-z_1), t_x^*/(1-z_2)] = [0.000962, 0.601519] \]

where, \( z_1 = 0.000797 \) and \( z_2 = 0.332769 \) are found from Table A for \( n = 15 \) and \( \alpha = 0.10 \).

3.2. **The Normal Distribution With Known Variance**

Let \( X_1, \ldots, X_n \) be a random sample from a normal distribution with the density function

\[ f(x; \theta_x, \sigma) = (2\pi\sigma^2)^{-1/2} \exp[-1/2\sigma^2 (x - \theta_x)^2]. \]
and Y be independent of the $X_i$, (i=1,2,...,n) with the density function

$$f(y;\theta_y,\sigma) = (2\pi\sigma^2)^{-1/2} \exp[-1/2\sigma^2 (y-\theta_y)^2]$$

where $\sigma$ is known and X $\in$ R, Y $\in$ R.

$T_x = \sum_{i=1}^n X_i$ and $T_y = Y$ are sufficient statistics for the distributions of $(X_1,...,X_n)$ and Y, respectively. Without loss of generality we assume, $\sigma = 1$.

$$f(t_y, t_x; \theta_x, \theta_y) = (n^{-1/2}/2\pi) \exp[(-1/2)(n\theta_x^2 + \theta_y^2)]$$

$\exp[\theta_x t_x + \theta_y t_y] \exp[(-1/2)(t_x^2/n + t_y^2)]$ (3.2.1)

Let $\beta = (\theta_y - \theta_x)/(1/n+1)$, $\gamma = (n\theta_x + \theta_y)/(n+1)$, $U = T_y - T_x/n$ and $T = T_x + T_y$. Then (3.2.1) implies

$$f(u, t; \beta, \gamma) = C(\beta, \gamma) \exp[\beta u + \gamma t] h(u, t)$$

(3.2.2)

where $C(\beta, \gamma)$ and $h(u, t)$ can be obtained from (3.2.1). But we will not need these for deriving the prediction intervals. $U$ is increasing in $T_y$ and is being used for simplicity. The statistic, $V$ defined by

$$V = [(n+1)/n]^{-1/2} U$$

$$= [(n+1)/n]^{-1/2} T_y - [n(n+1)]^{-1/2} T$$

$$= a(T) T_y + b(T)$$
is increasing in $T_y$ for each $t$ and under $H_0: \theta_y = \theta_x$ has standard normal distribution. Therefore $V$ is an ancillary statistic and Lemma 2.2 will be used instead of Theorem 2.2. We have

\begin{align*}
(1) & \quad \beta = \beta_0 \quad \text{iff} \quad \theta_y = \theta_x \\
& \quad \beta > \langle \langle \beta \rangle_0 \quad \text{iff} \quad \theta_y > \langle \langle \theta_x \rangle, \beta_0 = 0.
\end{align*}

(2) $V$ is increasing in $T_y$ for each $t$. Therefore the conditional distribution of $V$ given $t$ with parameter $\beta$, has a monotone likelihood ratio. (See section 2.2.3 (B).)

(3) $[V-b(T)]/a(T)$ is an increasing function of $T$. Therefore acceptance regions will map onto the same form of prediction intervals. (See section 2.2.3 (B).)

(4) The distribution of $V$ under $H_0$ is symmetric about zero. Hence the remark in section 2.2.1 applies.

**One Sided Upper Limit Prediction Interval**

(1), (2) and (3) imply that $A_1(\beta) = \{v : v \leq z\}$ must be used, where according to Lemma 2.2, $z$ is found from $\int_{-\infty}^{z} f_{H_0}(v)dv = 1-\alpha$. Having the value of $z$ from the table of the standard normal distribution and using $A_1(\beta)$, the $(1-\alpha)$ level upper limit upper limit UMAU prediction interval for $Y$ is

\[ S_1(t_x) = ( -\infty , (t_x/n)+z(1/n +1)^{1/2}]. \]
One Sided Lower Limit Prediction Interval

A similar argument to the above one holds. Using $A_2(\beta)\{ \nu: \nu \geq z \}$, the $(1-\alpha)$ level lower limit UMAU prediction interval for $Y$ is

$$S_2(t_x) = [(t_x/n) - z(1/n + 1)^{1/2}, \infty), \text{ where } z \text{ is a percentile of the standard normal distribution.}$$

Two Sided Prediction Interval

According to (1), (2) and (3), $A_3(\beta)\{ \nu: z_1 \leq \nu \leq z_2 \}$ must be used. From property (4) we have, $z_2 = -z_1$ where $z_2$ is the $(1-\alpha/2)$ percentile of the standard normal distribution. $(1-\alpha)$ level UMAU prediction interval for $Y$ is

$$S_3(t_x) = [(t_x/n) + z_1(1/n + 1)^{1/2}, (t_x/n) + z_2(1/n + 1)^{1/2}].$$

Remark I. If $\sigma \neq 1$, then

$$S_1(t_x) = (\infty, (t_x/n) + \sigma z(1/n + 1)^{1/2})$$

$$S_2(t_x) = [(t_x/n) + \sigma z(1/n + 1)^{1/2}, \infty)$$

$$S_3(t_x) = [(t_x/n) + \sigma z_1(1/n + 1)^{1/2}, (t_x/n) + \sigma z_2(1/n + 1)^{1/2}]$$

where $z$, $z_1$ and $z_2$ are defined as before for each case.

Remark II: $X \sim \text{Normal} (\mu, \sigma^2)$ iff $\exp(X) \sim \text{Lognormal} (\mu, \sigma^2)$

Using results of Example 3.2 and the relationship between Normal and Lognormal random variables, the prediction intervals
can be obtained when sampling is from Lognormal $(\mu, \sigma^2)$ distribution with known $\sigma$.

3.3. The Negative Exponential Distribution

Suppose $X_1, \ldots, X_n$ is a random sample from a Negative Exponential distribution with the density function

$$f(x; \gamma_x) = \frac{1}{\gamma_x} \exp\left[-\frac{x}{\gamma_x}\right] \quad \gamma_x > 0, \ x > 0$$

and $Y$ is independent of the $X_i$, $(i=1,2,\ldots,n)$ with the density function

$$f(y; \gamma_y) = \frac{1}{\gamma_y} \exp\left[-\frac{y}{\gamma_y}\right] \quad \gamma_y > 0, \ y > 0$$

The Negative Exponential family with the above density functions is a monotone likelihood ratio family.

$$(T_x = \sum_{i=1}^{n} X_i, \ T_y = Y)$$ is a sufficient statistics for the joint distribution of $(X,Y)$. Let $\theta_x = -1/\gamma_x$, $\theta_y = -1/\gamma_y$, $\beta = (\theta_y - \theta_x)$ and $T = T_x + T_y$. We have,

$$f(t_y; t; \beta) = \frac{\exp[\beta t_y](t-t_y)^{n-1}}{\int_0^t \exp[\beta t_y](t-t_y)^{n-1} \, dt_y} \quad t_y \leq t$$

Under null hypothesis, $H_0: \gamma_y = \gamma_x$, $V = G(T_y, T) = a(T)/T$, $a(T) = 1/T$, has Beta(1, n) distribution, Bickel and Doksum (1977).
Therefore \( V \) is an ancillary statistic for the joint distribution of \( (T_y, T) \) under \( H_0 \) and we can use Lemma 2.2 instead of Theorem 2.2. We have

\[
\begin{align*}
(1) & \quad \beta = \beta_0 \quad \text{iff} \quad \gamma_y = \gamma_x \quad \text{iff} \quad \theta_y = \theta_x \\
& \quad \beta > (\) \beta_0 \quad \text{iff} \quad \gamma_y > (\) \gamma_x \quad \text{iff} \quad \theta_y > (\) \theta_x \quad , \quad \beta_0 = 0 .
\end{align*}
\]

(2) \( V \) is increasing in \( T_y \) for each \( T \). Therefore the conditional distribution of \( V \) given \( t \) with parameter \( \beta \), has a monotone likelihood ratio. (See section 2.2.3 (B).)

(3) \( V/a(T) \) is an increasing function of \( t \). Therefore acceptance regions will map onto the same form of prediction intervals. (See section 2.2.3 (B).)

**One Sided Upper Limit Prediction Interval**

(1), (2) and (3) imply that \( A_1(\beta) = \{ v : v \leq z \} \) must be used. According to the Lemma 2.2, \( z \) is found from

\[
\int_{-\infty}^{z} f_{H_0}(v) dv = 1 - \alpha ,
\]

where \( f_{H_0}(v) \) is density of Beta\((1,n)\). Using \( A_1(\beta) \), the \((1-\alpha)\) level upper limit UMAU prediction interval for \( Y \) is

\[
S_1(t_x) = (-\infty, t_x z/(1-z)].
\]

Since \( nT_y/T_x \) under \( H_0 \) has \( F \)-distribution with \((2,2n)\) degrees of freedom, then \( nz/(1-z) \) is \((1-\alpha)\) percentile of \( F \)-distribution with \((2,2n)\) degrees of freedom.
One Sided Lower Limit Prediction Interval

A similar argument to the above one holds. Using
\[ A_2(\beta) = \{ v: v \geq z \} \], the \((1-\alpha)\) level lower limit UMAU prediction interval for \( Y \) is
\[ S_2(t_x) = \left[ t_x - \frac{z}{\sqrt{n(1-z)}}, \infty \right) \]. \( nz/(1-z) \) is a percentile of \( F \)-distribution with \((2,2n)\) degrees of freedom.

Two Sided Prediction Interval

According to (1), (2) and (3) we need to use the acceptance region \( A_3(\beta) = \{ v: z_1 \leq v \leq z_2 \} \). Using Lemma 2.2, \( z_1 \) and \( z_2 \) are found from

\[ \int_{z_2}^{z_1} f_{H_0}(v) dv = 1-\alpha \quad \text{and} \quad \int_{z_2}^{z_1} v f_{H_0}(v) dv = (1-\alpha) E_{H_0}(V) \quad (3.3.1) \]

where \( E_{H_0}(V) = 1/(n+1) \).

Equations in (3.3.1) can be written as

\[ (1-z_1)^n - (1-z_2)^n = (1-\alpha) \]
\[ z_1(1-z_1)^n - z_2(1-z_2)^n = 0 \quad (3.3.2) \]

respectively.

However it is impossible to solve the equations in (3.3.2) and find \( z_1 \) and \( z_2 \) in terms of \( n \) and \( \alpha \), in closed forms. Hence an approximation method like the 'Generalized Newton's Method' given in Example 3.1 must be used. According to the notation used in Example 3.1 we have
Using the equations (3.3.3), a program is written for the Generalized Newton's Method and values of \( z_1, z_2 \) for the equations (3.3.2) are listed in Table B. Using the acceptance region \( A_3(\beta) \), the \((1-\alpha)\) level UNAUA two sided prediction interval for \( T_y \) is

\[
S_3(t_x) = [t_xz_1/(1-z_1), t_xz_2/(1-z_2)] .
\]

A Numerical Example:

The time (hours) of first failure of an electrical device is
assumed to be distributed according to the Negative Exponential distribution. 15 of this particular device are selected from production line at random and the times to first failure of each are found to be

\[ 62, 74, 19, 18, 209, 409, 57, 46, 13, 29, 231, 46, 5, 25. \]

If we are to select another device of the same type, a natural question to be ask is concerned with the maximum time period that the device can work before it fails. Suppose an answer to the question is to be given with .90 level confidence.

According to our notation in Example 3.3,

\[ t_x = \sum_{i=1}^{n} x_i = 1243, \quad n=15 \]

and the upper limit prediction interval is

\[ S_1(t_x) = [0, t_x z/(1-z)] = [0, 206.338]. \]

where \( nz/(1-z)=2.49 \) is 90\textsuperscript{th} percentile of F-distribution with (2,30) degrees of freedom. Therefore based on the 15 previously observed values, we predict with .90 confidence that the device might work 206.23 hours before it fails.

If the question is how early a randomly selected device can fail, then we need to find a lower limit prediction interval
\[ S_2(t_x) = \left[ \frac{tx}{(1-z)}, \frac{tx}{z} \right] = [8.759, \infty) \]

where \( nz/(1-z) = 0.1057 \) is the 10\textsuperscript{th} percentile of F-distribution with (2, 30) degrees of freedom.

For two-sided prediction interval, we have

\[ S_3(t_x) = \left[ \frac{txz_1}{(1-z_1)}, \frac{txz_2}{(1-z_2)} \right] = [6.7335, 359.3905] \]

where, \( z_1 = 0.005388 \) and \( z_2 = 0.224284 \) for \( n = 15 \) and \( a = 0.10 \) are obtained from Table B.

The above prediction intervals are based on the assumption that the quality of production of the device remains unchanged over the time. That is to say, the probability distribution associated with the time of failure of the device is exponential with fixed unknown parameter over the time.

3.4. The Weibull Distribution With Known Shape Parameter

Let \( X_1, \ldots, X_n \) be a random sample from a Weibull distribution with the density function

\[
f(x; \mu, \gamma) = \left(\frac{\gamma}{\mu}\right) x^{\gamma-1} \exp\left[-x^{\gamma}/\mu\right] \quad \mu > 0, \quad \gamma > 0, \quad x > 0 \quad (3.4.1)
\]

The shape parameter, \( \gamma \) is assumed to be known. The density function in (3.4.1) is a one-parameter exponential family and has
a monotone likelihood ratio in x.

Define \( W = X' \). Then

\[
f(w; \mu) = \frac{1}{\mu} \exp[-w/\mu], \quad \mu > 0, \quad w > 0
\]

which is the density function of a Negative Exponential distribution.

Using the results of Example 3.3 the prediction intervals for a future outcome based on observed values of \( X_1, \ldots, X_n \) are

\[
S_2(t_x) = [0, \left(\frac{t_x z/(1-z)}{1/\gamma}\right)^{1/\gamma}]
\]

\[
S_1(t_x) = \left[\left(\frac{t_x z/(1-z)}{1/\gamma}\right)^{1/\gamma}, \infty\right)
\]

\[
S_3(t_x) = \left[\left(\frac{t_x z_1/(1-z_1)}{1/\gamma}, \left(\frac{t_x z_2/(1-z_2)}{1/\gamma}\right)\right]
\]

\( t_x = \sum_{i=1}^{n} x_i' \) and \( z, z_1 \) and \( z_2 \) are defined as in the Example 3.3.
IV. GENERALIZATION OF THE PREDICTION PROBLEM TO K-PARAMETER CASE

In this Chapter we will generalize the method of obtaining prediction intervals when the distributions of informative and future experiments are from a K-parameter exponential family.

4.1. Formulation Of The Problem In Terms Of Hypothesis Testing

Let $X_1, \ldots, X_n$ be (iid) random outcomes of an informative experiment with the density function

$$f(x; \theta_x) = C(\theta_x) \exp[\theta_x^T T(x)] h(x)$$

where $T(X)$ and $\theta_x$ are K-dimensional real-valued vectors. Suppose $Y$ is as yet an unknown outcome of a corresponding future experiment, independent of the $X_i, (i=1,2,\ldots,n)$ and with the density function

$$f(y; \theta_y) = C(\theta_y) \exp[\theta_y^T T(y)] h(y)$$

where $\theta_y$ is also a K-dimensional real-valued vector.

Since $Y$ is a single random variable then all components of $\theta_y$ are not identifiable. We will assume

$$\theta_y^* = (\theta_{1y}, \gamma), \quad \theta_x^* = (\theta_{1x}, \gamma) \quad \text{and} \quad \gamma = (\theta_2, \ldots, \theta_K).$$
Hypotheses which will be considered are

\[ H_0: \theta_{1x} = \theta_{1y} \quad \text{vs} \quad H_{a1}: \theta_{1y} > \theta_{1x} \]
\[ H_0: \theta_{1x} = \theta_{1y} \quad \text{vs} \quad H_{a2}: \theta_{1y} > \theta_{1x} \quad (4.1.1) \]
\[ H_0: \theta_{1x} = \theta_{1y} \quad \text{vs} \quad H_{a3}: \theta_{1y} \neq \theta_{1x} \]

The joint density function of \((X, Y)\) is

\[
f(x, y; \theta_x, \theta_y) = c^n(\theta_x)c(\theta_y)\exp[\theta_{1x} \sum_{i=1}^{n} T_1(x_i) + \theta_{1y} T_1(y)] \\
\exp[ \sum_{j=2}^{K} \theta_j (\sum_{i=1}^{n} T_j(x_i) + T_j(y))] \prod_{i=1}^{n} h(x_i)h(y). \quad (4.1.2)
\]

Let \( T_{jx} = \sum_{i=1}^{n} T_j(x_i), \quad T_{jy} = T_j(y), \quad T_j = T_{jx} + T_{jy}, \)
\( T_x = (T_{1x}, \ldots, T_{kx}), \quad T_y = (T_{1y}, \ldots, T_{ky}), \quad T = (T_1, T_2, \ldots, T_k) \) and
\( \beta = \theta_{1y} - \theta_{1x}, \quad (j = 2, \ldots, k), \) then the density in (4.1.2) can be written as

\[
f(x, y; \beta, \gamma, \theta_{1x}) = c^n(\theta_{1x}, \gamma)c(\beta + \theta_{1x}, \gamma)\exp[\beta T_{1y} + \theta_{1x} T_1] \\
\exp[ \sum_{j=2}^{K} \theta_j T_j] \prod_{i=1}^{n} h(x_i)h(y). \quad (4.1.3)
\]

which is the density function of a \( K+1 \) parameter exponential family. By factorization Theorem, Lehmann (1959),
\((T_{1y}, T_1, \ldots, T_k)\) are jointly sufficient statistics for \((\beta, \theta_{1x}, \gamma)\).

Similar to Chapter II, we reduce the problem to the consideration of the sufficient statistics. \( T = (T_1, T_2, \ldots, T_k) \) is a complete and sufficient statistic for \((\theta_x, \gamma)\) when \( H_0 \) is true. Thus for testing the hypothesis we need the conditional distribution of \( T_{1y} \) given \( t \).
According to Lehmann (1959), we have

\[
\begin{align*}
  f(t_{ly};_\theta_y) &= C_{tly}(\theta_y) \exp[\theta_{1y} t_{ly}] h_{tly}(t_{ly}) \\
  f(t_x;_\theta_x) &= C_{tx}(\theta_x) \exp[\theta_{1x} t_{lx} + \sum_{j=2}^{k} \theta_j t_{jx}] h_{tx}(t_x)
\end{align*}
\]

(4.1.4)

which implies

\[
\begin{align*}
  f(t_{ly};_\theta_y) f(t_x;_\theta_x) &= f(t_{ly}, t_1, t_2, \ldots, t_{kx}; \beta, \theta_{1x}, \gamma) \tag{4.1.5} \\
  &= C_{tly}(\beta+\theta_{1x}, \gamma) C_{tx}(\theta_x) \exp[\beta t_{ly} + \theta_{1x} t_{lx} + \sum_{j=2}^{k} \theta_j t_{jx}] \\
  &\quad h_{tx}(t_1-t_{ly}, t_2, \ldots, t_{kx}) h_{tly}(t_{ly})
\end{align*}
\]

Using (4.1.5), the conditional density of \( T_{ly} \) given \( t \) can be written as

\[
\begin{align*}
  f(t_{ly}; t; \beta) &= \int_{-\infty}^{t_1} \frac{\exp[\beta t_{ly}] h_{tx}(t_1-t_{ly}, w_2, \ldots, w_k) h_{tly}(t_{ly})}{\exp[\beta t_{ly}] h_{tx}(t_1-t_{ly}, w_2, \ldots, w_k) h_{tly}(t_{ly}) dt_{ly}} \\
  &= \frac{\exp[\beta t_{ly}] h_{tx}(t_1-t_{ly}, w_2, \ldots, w_k) h_{tly}(t_{ly})}{\int_{-\infty}^{t_1} \exp[\beta t_{ly}] h_{tx}(t_1-t_{ly}, w_2, \ldots, w_k) h_{tly}(t_{ly}) dt_{ly}} \tag{4.1.6}
\end{align*}
\]

(4.1.6)

where \( w_j = t_j - K_j(t_{ly}), t_{jk} = K_j(t_{ly}), j=2,3,\ldots,k \), for some functions \( K_j(\cdot) \), because \( Y \) is a single random variable and all components of \( T_y=(T_{ly}, \ldots, T_{ky}) \) can be obtained from \( T_{ly} \).

The conditional density function in (4.1.6) is the density function of a one parameter exponential family and therefore has a monotone likelihood ratio in \( T_{ly} \). (See Section 2.2.2.)

When an exponential family has more than one parameter, it is difficult and sometimes impossible to find \( h_{tx}(\cdot) \) for the density function of \( t_x \), (4.1.4).
Saddle Point Approximation, Barndorff-Nielsen (1983)

Let $X_1, \ldots, X_n$ be (iid) random variables with the density function

$$f(x; \theta) = C(\theta) \exp[\theta^T T(x)] h(x)$$

where

$$\begin{align*}
\theta : \Theta & \rightarrow \mathbb{R}^k \\
T : \mathcal{X} & \rightarrow \mathbb{R}^k \\
h : \mathcal{X} & \rightarrow \mathbb{R}^k \\
C : \Theta & \rightarrow (0, \infty).
\end{align*}$$

The 'Saddle Point Approximation' to the density function of $T_n = \sum_{i=1}^n T(X_i)$ is

$$f(t_x; \theta) \approx (2\pi)^{-k/2} |i(\hat{\theta})|^{-1/2} \exp[(\theta - \hat{\theta})t_x] (4.1.7)$$

$$[C(\theta)C^{-1}(\hat{\theta})]^n$$

where $\hat{\theta}$ is the Maximum Likelihood Estimator of $\theta$.

and

$$i(\hat{\theta}) = -\frac{\partial^2 \log f(x; \theta)}{\partial^2 \theta} \bigg|_{\theta = \hat{\theta}}$$

is the observed Fisher Information Matrix. The order of the approximation is $O(n^{-1})$.

According to the Saddle-Point Approximation, $h_{tx}(.)$ in (4.1.4) is approximated by

$$|i(\hat{\theta}_x)|^{-1/2} \exp[(-\hat{\theta}_x)t_x] C^{-B}(\hat{\theta}_x)$$

where, $\hat{\theta}_x$ is the Maximum Likelihood Estimator of $\theta_x$ based on $X_1, \ldots, X_n$. 
EXAMPLE I. The Negative Exponential Distribution

Let $X_1, \ldots, X_n$ be a random sample with the density function

$$f(x; \theta) = \theta \exp[\theta x], \theta > 0, x > 0.$$  

$T_x = \sum_{i=1}^{n} X_i$ is a sufficient statistic for the joint distribution of $X_1, \ldots, X_n$. We have

$$\hat{\theta} = n(t_x)^{-1}$$

$$|i(\hat{\theta})|^{-1/2} = n^{3/2}(t_x)^{-1}$$

$$C^{-n}(\hat{\theta}) = (t_x/n)^n$$

Using (4.1.7), the Saddle-Point Approximation to the density function of $T_x$ is

$$f(t_x; \theta) \approx \frac{(\theta)^n t_x^{n-1} \exp[-\theta t_x]}{(2\pi)^{1/2} n^{(n-1)/2} \exp[-n]} (4.1.8)$$

which is the exact density function of $T_x$ except for the denominator part which is an approximation to $\pi(n)$ according to Stirling's formula. The negative exponential distribution belongs to a one parameter exponential family. Therefore we could have obtained the exact distribution of $T_x$ which is Gamma $(n, \theta)$.  

Example II. The Gamma Distribution With Unknown Parameters

Let \( X_1, \ldots, X_n \) be a random sample from a Gamma distribution with the density

\[
f(x; p, \gamma) = (\gamma)^p \exp\left((p-1)\log(x) - \gamma x\right) \pi^{-1}(p)
\]

where, \( x > 0 \), \( p > 0 \), \( \gamma > 0 \). We assume that \( p \) is large so that the Stirling's Approximation to \( \pi(p) \) can be used.

\( T_x = (T_1, T_2) \) is a two-dimensional sufficient statistic for the joint distribution of \( X_1, \ldots, X_n \) and we have

\[
\hat{p} = 0.5 \left[ \log \left( \frac{T_1}{n} \right) - \frac{T_2}{n} \right]^{-1}
\]

\[
\hat{\gamma} = \frac{(n \hat{p})}{T_1}
\]

\[
|i(p, \gamma)|^{-1/2} = (2 \hat{p})^{1/2} \hat{\gamma} / n
\]

\[
C^{-n}(p, \gamma) = \left( \pi(p) / \hat{\gamma} \right)^n
\]

Using (4.1.7), \( h_{tx}(\ldots) \) is approximated by

\[
h_{tx}(t_1, t_2) \approx \frac{(\hat{p})^{3-n}/2 (t_1)^{n \hat{p}-1} \exp[-\hat{p} t_2]}{(n)^{n \hat{p}}}
\]

For this problem the exact form of \( h_{tx}(\ldots) \) cannot be obtained.
Similar arguments to the Theorem 2.2 and Lemma 2.2 will be applied to obtain UMP unbiased tests and as a result UMP unbiased acceptance regions.

**Theorem 4.1**: Consider the conditional density of $T_{1y}$ given $T$, (4.1.6), and the hypotheses of interest, (4.1.1).

1. $A_1(\beta | t) = \{(t_{1y}, t) : t_{1y} \leq z(t)\}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for $H_0$ versus $H_{a_1}$ and $z(t)$ is found from
   \[ \int_{z(t)}^{\infty} f_{o dt_{1y}} = \alpha. \]

2. $A_2(\beta | t) = \{(t_{1y}, t) : t_{1y} \geq z(t)\}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for $H_0$ versus $H_{a_2}$ and $z(t)$ is found from
   \[ \int_{-\infty}^{z(t)} f_{o dt_{1y}} = \alpha. \]

3. $A_3(\beta | t) = \{(t_{1y}, t) : z_1(t) \leq t_{1y} \leq z_2(t)\}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for $H_0$ versus $H_{a_3}$ and $z_1(t)$, $z_2(t)$ are found from
   \[ \int_{z_1(t)}^{z_2(t)} f_{o dt_{1y}} = 1-\alpha \text{ and } \int_{z_1(t)}^{z_2(t)} t_{1y} f_{o dt_{1y}} = (1-\alpha) \int_{-\infty}^{\infty} t_{1y} f_{o dt_{1y}}, \]
   where $f_o$ is the conditional distribution of $T_{1y}$ given $T$, under $H_0$, which can be obtained by letting $\beta = 0$ in (4.1.6).
If there exists an ancillary statistic, $V = G(T_{1y}, T)$ for the joint distribution of $(T_{1y}, T)$, under $H_0$, then by Basu's Theorem, $V$ and $T$ are independent and distribution of $V$ under $H_0$ can be used to obtain the UMP unbiased acceptance regions.

**Theorem 4.2:** Suppose there exists an ancillary statistic, $V = G(T_{1y}, T)$ for the joint distribution of $(T_{1y}, T)$, under $H_0$. Let $f_{H_0}(v)$ denote the density of $V$ under $H_0$.

1. $A_1(\beta) = \{ v : v < z \}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for testing $H_0$ versus $H_{a1}$ and $z$ is found from

   $$\int_{z}^{\infty} f_{H_0}(v) dv = \alpha,$$

   provided $V$ is increasing in $T_{1y}$ for each $t$.

2. $A_2(\beta) = \{ v : v > z \}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for testing $H_0$ versus $H_{a2}$ and $z$ is found from

   $$\int_{-\infty}^{z} f_{H_0}(v) dv = \alpha,$$

   provided $V$ is increasing in $T_{1y}$ for each $t$.

3. $A_3(\beta) = \{ v : z_1 \leq v \leq z_2 \}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for $H_0$ versus $H_{a3}$ and $z_1, z_2$ are found from
\[ \int_{z_1}^{z_2} f_{H_0}(v) dv = (1-a) \quad \text{and} \quad \int_{z_1}^{z_2} v f_{H_0}(v) dv = (1-a) \int_{-\infty}^{\infty} v f_{H_0}(v) dv \]

provided, \( V = G(T_1 T, T) = a(T)T + b(T), a(t) > 0 \).

4.2. Obtaining The Prediction Intervals

A. Ancillary Statistic Does Not Exist

Since the conditional density of \( T_1 T \) given \( t \), (4.1.6), has a monotone likelihood ratio in \( T_1 T \), therefore according to Lemma 2.4, the acceptance regions in Theorem 4.1 have more probability under \( H_0 \) than under the corresponding alternative hypotheses.

Similar to Chapter II, we use \( A_1(\beta | t) \) to obtain prediction intervals. Define

\[ S_i(t_x) = \{ t_{1y} : (t_{1y}, t) \in A_i(\beta | t) \}, i = 1, 2, 3 \]

That is, the acceptance regions must be solved for \( t_{1y} \) by writing \( t = (t_{1y} + t_{1x}, t_{2y} + t_{2x}, \ldots, t_{ky} + t_{kx}) \), to obtain the corresponding prediction intervals. We get

\[ S_1(t_x) = (-\infty, U(t_x)] \]
\[ S_1(t_x) = [L(t_x), \infty) \] \quad (4.2.1)
\[ S_1(t_x) = [L(t_x), U(t_x)] \]

provided \( z(.) \) functions in the acceptance regions are
increasing. Note that \( t \) can be written as function of \( t_{1y} \) and \( t_x \).

Using (4.1.6) and a similar argument to Lemma 2.6, a sufficient condition for \( z(.) \) to be increasing is that

\[
\frac{h_{tx}(t_{11}, w_2, \ldots, w_k)}{h_{tx}(t_{11}, w_2, \ldots, w_k)}
\]

must be nondecreasing in \( t_{1y} \), whenever \( t_{11} > t_1, \ldots, t_k > t_k \),

where \( w_j = t_j - K_j(t_{1y}) \), \( t_{jy} = K_j(t_{1y}) \), and \( w_j = t_j - K_j(t_{1y}) \),

\( j = 2, 3, \ldots, k \), for some functions \( K_j(.) \) (See the density function in (4.1.6)). \( L(.) \) and \( U(.) \) are function of

\( t_x = (t_{1x}, \ldots, t_{kx}) \), \( n \) and \( \alpha \).

The prediction intervals in (4.2.1) are obtained from the

\( (1-\alpha) \) level UMP unbiased acceptance regions and they have similar properties to the ones mentioned in Chapter II.

(1) \[
E(\theta_1x, \gamma) \{ P(\theta_{1y}, \gamma) \{ S_i(T_x) \} \} = 1 - \alpha, \]

for all \( \theta_1y = \theta_{1x} \) and all \( \gamma = (\theta_2, \ldots, \theta_k) \). That is, \( S_i(T_x) \), \( i = 1, 2, 3 \) are 'Similar Mean Coverage Tolerance Prediction Regions' of cover \( (1-\alpha) \), Aitchison and Dunsmore, (1975).

(2) \[
E(\theta_1x, \gamma) \{ P(\theta_{1y}, \gamma) \{ S_1(T_x) \} \} < 1 - \alpha, \]

for all \( \theta_1y > \theta_{1x} \) and all \( \gamma = (\theta_2, \ldots, \theta_k) \).

\[
E(\theta_1x, \gamma) \{ P(\theta_{1y}, \gamma) \{ S_2(T_x) \} \} < 1 - \alpha, \]

for all \( \theta_1y < \theta_{1x} \) and all \( \gamma = (\theta_2, \ldots, \theta_k) \).
E(\theta_1, \gamma) [ \mathbb{P}(\theta_1, \gamma) \{ S_3(T_x) \} ] \leq 1 - \alpha,
for all \theta_1 \neq \theta_{1x} and all \gamma = (\theta_2, \ldots, \theta_k).

(3) \text{S}_i(T_x) \text{ are 'Uniformly Most Accurate Unbiased' (UMAU),}
Prediction intervals. (See Definition 2.6.)

B. Ancillary Statistic Exists

In Theorem 4.2 we assumed that the ancillary statistic,
\( V = G(T_{iy}, T) \), is an increasing function of \( T_{iy} \) for any \( T \).
Since the conditional distribution of \( T_{iy} \) given \( T \) (4.1.6),
has a monotone likelihood ratio in \( T_{iy} \), then the conditional
distribution of \( V \) given \( t \) also has a monotone likelihood ratio in
\( V \) for each \( t \). By Lemma 2.4, the acceptance regions in Theorem 4.2
have more probability under \( \mathbb{H}_0 \) than under the corresponding
alternative hypotheses and they will be used to derive the prediction
intervals. Define

\[ S_i(t_x) = \{ t_{iy} : v \in A_i(\beta) \}, i=1,2,3 \, . \]

We have

\[ S_1(t_x) = (-\infty, U(t_x)) \]
\[ S_1(t_x) = [L(t_x), \infty) \] (4.2.2)
\[ S_1(t_x) = [L(t_x), U(t_x)] \, . \]

That is, \( S_1(t_x) \) is obtained by writing
\[ t = (t_{1x} + t_{1y}, t_{2x} + t_{2y}, \ldots, t_{kx} + t_{ky}) \]

in \( V = G(T, T_{1y}) \) and solving \( A_1(\beta) \) for \( t_{1y} \). Acceptance regions will map onto the same form of prediction intervals, provided \( G^{-1}(\cdot) \) is increasing in \( T_{1y} \). The prediction intervals in (4.2.2) are obtained by using \( (1-\alpha) \) level UMP unbiased acceptance regions and therefore have the same properties as the prediction intervals in (4.2.1).
V. EXAMPLES: K-PARAMETER CASE (K=2)

As examples of K-parameter exponential families, the Normal and the Inverse-Gaussian distributions will be considered and the method of Chapter IV will be used to obtain the (1-α) level UMAU prediction intervals for a future outcome.

5.1. The Normal Distribution With Unknown Parameters

Let \( X_1, \ldots, X_n \) be a random sample from a normal distribution with the density function

\[
   f(x; \mu_x, \sigma^2) = \frac{1}{(2\pi \sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}(x - \mu_x)^2\right], \quad \mu_x \in \mathbb{R}, \quad x > 0, \quad \sigma > 0
\]

and \( Y \) be independent of the \( X_i, (i=1,2,\ldots,n) \) with the density function

\[
   f(y; \mu_y, \sigma^2) = \frac{1}{(2\pi \sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}(y - \mu_y)^2\right], \quad \mu_y \in \mathbb{R}, \quad y > 0, \quad \sigma > 0
\]

We assume that \( \mu_x, \mu_y \) and \( \sigma \) are unknown parameters and prediction intervals for \( Y \) are of interest. To get a natural parametrization let

\[
   \begin{align*}
   \theta_{1x} &= \mu_x \sigma^{-2}, \quad \theta_{1y} = \mu_y \sigma^{-2}, \quad \gamma = -(2\sigma^2)^{-1}, \\
   T_{1x} &= \sum_{i=1}^{n} X_i, \\
   T_{2x} &= \sum_{i=1}^{n} X_i^2, \quad T_{1y} = Y, \quad T_{2y} = Y^2, \quad T_j = T_{1x} + T_{1y}, (j=1,2),
   \end{align*}
\]

then the joint density of \((X, Y)\) can be written as

\[
   f(x, y; \theta_{1x}, \theta_{1y}, \gamma) = (\pi \gamma)^{-(n+1)/2} \exp\left\{\frac{-(n \theta_{1x}^2 + \theta_{1y}^2)}{4\gamma}\right\} \exp\left\{\theta_{1y} t_{1y} + \theta_{1x} t_{1x} + \gamma t_2\right\} \quad (5.1.1)
\]
To simplify the problem let

$$\beta = (1/n + 1)^{-1}(\theta_{1y} - \theta_{1x})$$

and

$$\theta = (n+1)^{-1}(n\theta_{1x} + \theta_{1y})$$

Then (5.1.1) is equivalent to

$$f(x,y;\beta,\gamma,\Theta) = C(\beta,\gamma,\Theta)\exp[\beta(t_{1y} - t_{1x}/n) + \Theta t_1 + \gamma t_2] \quad (5.1.2)$$

where $C(\ldots,\ldots)$ is a function of parameters, $\beta$, $\gamma$ and $\Theta$.

Under $H_0$:

- $\mu_y = \mu_x$ (i.e., $\theta_{1y} = \theta_{1x}$),
- $(T_1, T_2)$ are jointly complete and sufficient statistics for $(\Theta, \gamma)$.

Define

$$U = T_{1y} - t_{1x}/n$$

$$V = G(U,T) = BU[T_2 - (n+1)^{-1}(T_1^2 + nU^2)]^{-1} \quad (5.1.3)$$

$$B = (n(n-1)/(n+1))^{1/2}$$

$V$ is increasing in $U$ and it can be shown that under $H_0$ it has student's $t$-distribution with $(n-1)$ degrees of freedom.

Therefore $V$ is an ancillary statistic for the joint distribution of $(U, T_1, T_2)$ when $H_0$ is true, and conditions of Theorem 4.2 (1), (2) are satisfied. We have

1. $\mu_y = \mu_x$ iff $\theta_{1y} = \theta_{1x}$ iff $\beta = \beta_0$

2. $\mu_y > (\mu_x)$ iff $\theta_{1y} > (\theta_{1x}$ iff $\beta > (\beta_0$, $\beta_0 > 0$.

(2) (5.1.2) implies that the conditional distribution of $U$ given
\[ t=(t_1, t_2) \] has a monotone likelihood ratio. Since \( V \) is increasing in \( U \), then the conditional distribution of \( V \) given \( T \) also has a monotone likelihood ratio. Thus acceptance regions based on \( V \) need to be used to obtain prediction intervals.

(3) Distribution of \( V \) is symmetric about zero. Therefore the remark given in section 2.2.1 applies in this case.

**One Sided Upper Limit Prediction Interval**

(1) and (2) imply that the acceptance region, \( A_1(\beta) = \{ v : v \leq z \} \) must be used. According to Theorem 4.2, \( z \) is found from

\[
\int_{z}^{\infty} f_{\alpha_0}(v) dv = \alpha, \quad f_{\alpha_0}(v) \text{ is the density of t-distribution with } (n-1) \text{ degrees of freedom.}
\]

We have

\[
V \in A_1(\beta) \iff T_{1y} \leq t_{1z}/n + z((n+1)Q/n(n-1))^{1/2}
\]

where \( Q = \sum_{i=1}^{n} (x_i - \bar{x})^2 \).

**One Sided Lower Limit Prediction Interval**

A similar argument to the above one holds. Using \( A_2(\beta) \) in Theorem 4.2 we get

\[
S_2(t_{x}) = [t_{1x}/n + z((n+1)Q/n(n-1))^{1/2}, \infty)
\]

where \( z \) is a percentile of t-distribution with \( (n-1) \) degrees of freedom.
Two Sided Prediction Interval

The Statistic $V$ in (5.1.3) is not a linear function of $U$. Therefore the condition of Theorem 4.2 (3) is not satisfied.

Define

$$W = U \left[ T_2 - T_1^2 / (n+1) \right]^{-1/2}$$

$W$ is linear in $U$ for each $t=(t_1, t_2)$ and is related to $V$ by

$$V = BW \left[ 1 - n^2 / (n+1) \right]^{-1/2}$$

where

$$B = [n(n-1)/(n+1)]^{1/2}$$

Under $H_0$, $W$ is independent of $T$ and is an ancillary statistic because it is a function of another ancillary statistic, $V$. The condition of Theorem 4.2 (3) satisfies for $W$. But for hypotheses testing purposes, $W$ and $V$ are equivalent test statistics, for $V$ is an increasing function of $W$. Therefore the acceptance regions associated with the two tests are equivalent and as a result, $A_3(\beta) = \{ v : z_1 \leq v \leq z_2 \}$ can be used where, $z_1$ and $z_2$ are $\alpha/2$ and $(1-\alpha/2)$ percentile of $t$-distribution with $(n-1)$ degrees of freedom, for the distribution of $V$ is symmetric about zero when $H_0$ is true. Using $A_3(\beta)$, we have

$$S_3(t_x) = t_{1x} / n + z_1 \left[ (n+1)Q / n(n-1) \right]^{1/2},$$

$$t_{1x} / n + z_2 \left[ (n+1)Q / n(n-1) \right]^{1/2}.$$
5.2. **The Inverse-Gaussian Distribution**

**With Unknown Parameters**

The Inverse-Gaussian distribution or first passage time distribution which has a skewed and unimodel density function is a two parameter exponential family. Chhikara (1975), discusses hypothesis testing in single and two sample cases for the Inverse-Gaussian distribution. Chhikara and Folks (1975), discuss sampling distribution and statistical inference related to this distribution. A very useful background about this distribution can be found in a paper by Folks and Chhikara (1978), in which they present test of hypothesis, estimation, confidence interval, regression and analysis of variance based on the Inverse-Gaussian distribution. For application, the distribution has been considered as a model for emptiness of dam by Hasofer (1964), and Lancaster (1972), applied it as a model for duration of strikes. When no obvious choice of distribution for a data with considerable skewness is suggested, Chhikara and Folks (1978), suggest the Inverse-Gaussian over the Lognormal distribution.

Let $X_1, ..., X_n$ be a random sample from an Inverse-Gaussian distribution with the density function

$$f(x; \mu_x, \theta) = \left[\frac{\theta}{(2\pi x^3)}\right]^{1/2} \exp\left[-\frac{\theta (x-\mu_x)^2}{2\mu_x^2x}\right],$$

$x>0, \theta>0, \mu_x>0$
and \( Y \) be independent of the \( X_i, (i=1,2,...,n) \) with the density function

\[
f(y; \mu_y, \theta) = \left( \frac{\theta}{2\pi \mu_y^2} \right)^{1/2} \exp\left\{ -\frac{\theta (y - \mu_y)^2}{2\mu_y^2} \right\},
\]
\( y > 0, \theta > 0, \mu_y > 0 \).

We assume that \( \mu_x, \mu_y \) and \( \theta \) are unknown parameters and prediction intervals for \( Y \) based on the outcomes of \( X_1, ..., X_n \) are of interest. Let

\[
T_{1x} = \sum_{i=1}^{n} x_i, \quad T_{1y} = y, \quad T_{2x} = \sum_{i=1}^{n} x_i^{-1}, \quad T_{2y} = y^{-1},
\]
(5.2.1)
\( \theta_{1x} = -(\theta/2)\mu_x^{-2}, \quad \theta_{1y} = -(\theta/2)\mu_y^{-2}, \quad \gamma = -\theta/2, \quad T_j = T_{jx} + T_{jy} (j=1,2), \)
\( \beta = [n/(n+1)](\theta_{1y} - \theta_{1x}), \quad \lambda = (n\theta_{1x} + \theta_{1y})(n+1)^{-1} \) and \( U = T_{1y} - T_{1x}/n \).

The joint density function of \((X,Y)\) can be written as

\[
f(x,y; \beta, \lambda, \gamma) = C(\beta, \lambda, \gamma) \exp\left[ \beta u + \lambda t_1 + \gamma t_2 \right] \prod_{i=1}^{n} \pi(x_i) y^{-3/2} \quad (5.2.2)
\]
where \( C(.,.,.,.) \) is a function of parameters, \( \gamma, \lambda, \beta \). The joint distribution of \((U,T_1,T_2)\) is a three parameter exponential family and Under \( H_0: \mu_y = \mu_x (\theta_{1y} = \theta_{1x}) \), \( T = (T_1,T_2) \) is a complete sufficient statistic for \((\lambda,\gamma)\). Therefore the conditional distribution of \( U \) given \( t = (t_1,t_2) \) is a one parameter exponential.

Define

\[
W = [(n(n+1))^{1/2}(n+1)U[A-n(n+1)^2U^2]^{-1/2}
\]
where \( A = (T_1T_2-(n+1)^2)(T_1+U)(T_1-nU) \).
According to Chhikara (1975), the conditional density of $W$ given $t=(t_1, t_2)$ under $H_0$, is

$$f_{H_0}(w|t_1, t_2) = (n-1)^{-1/2}(\text{beta}(1/2, (n-1)/2))^{-1}(1+w^2/(n-1))^{-n/2}$$

where \(\text{beta}(., .)\) is the beta function and \(-\infty < w < \infty\).

$W$ is nondecreasing in $U$. But $W$ is not ancillary statistic for the joint distribution of $(U, T_1, T_2)$ when $H_0$ is true, for the density function in (5.2.3) depends on $t_1$ and $t_2$ (Basu's Theorem). Therefore, $f_{H_0}(w|t_1, t_2)$ need to be used to get the UMP unbiased acceptance regions (Theorem 4.1). We have

(1) \(\mu_y = \mu_x\) iff \(\theta_{ly} = \theta_{lx}\) iff \(\beta = \beta_0\)

\(\mu_y > (<) \mu_x\) iff \(\theta_{ly} > (<) \theta_{lx}\) iff \(\beta > (<) \beta_0\), \(\beta_0 = 0\).

(2) $U$ is increasing in $T_{1y}$ and the conditional distribution of $U$ given $t=(t_1, t_2)$ has a monotone likelihood ratio, because it is one parameter exponential. Since $W$ is increasing in $U$, then the conditional distribution of $W$ given $t$ also has a monotone likelihood ratio. Thus the acceptance regions based on $W$ need to be used to obtain prediction intervals. (Lemma 2.6.)

(3) $W$ is not an ancillary statistic.
One sided Prediction Interval

To get one sided upper limit prediction interval, (1) and (2) imply that the acceptance region \( A_1(\beta|t) = \{ w : w \leq z(t) \} \) need to be used, where from Theorem 4.1 we have

\[
\int_{z(t)}^{\infty} f_{H_0}(w|t_1, t_2)dw = \alpha \quad . \tag{5.2.4}
\]

The expression, (5.2.4) is equivalent to

\[
F_{st}(z(t)) = (n-1)(n+1)^{-1} \frac{N(n-2)/2}{1-F_{st}(z^*(t))} = 1 - \alpha \quad . \tag{5.2.5}
\]

where

\[
z^*(t) = \left[ z^2(t) + 4n(n-1) \right] / \left[ t_1 t_2 - (n-1)^2 \right]^{1/2} \quad .
\]

\( F_{st}(\cdot) \) is the cumulative distribution of the Student's t-distribution with \((n-1)\) degrees of freedom, Chhikara (1975).

From (5.2.5) we cannot find \( z \). This fact is mentioned in a paper by Chhikara and Guttman (1982), where they show that one sided prediction intervals can be obtained if 'Bayesian informative prediction' approach is used.

A similar argument to the above one holds when the lower limit prediction interval is of interest.

Two sided Prediction Interval

To find the two sided prediction interval, according to
Theorem 4.1 (3), we need to solve

\[ \int_{z_1(t)}^{z_2(t)} f_{H_0}(w \mid t) \, dw = (1-\alpha) \]

and

\[ \int_{z_1(t)}^{z_2(t)} w f_{H_0}(w \mid t) \, dw = (1-\alpha) \int_{-\infty}^{\infty} w f_{H_0}(w \mid t) \, dw. \]

The function \( L(w) = w f_{H_0}(w \mid t) \) is an odd function of \( w \) and is symmetric about \( w=0 \). Therefore

\[ \int_{-\infty}^{\infty} w f_{H_0}(w \mid t) \, dw = 0 \]

and as a result \( z_1(t) = -z_2(t) \). Based on this fact and (5.2.5) we conclude that \( z_1(t) \) and \( z_2(t) \) must be found from

\[ F_{st}[z_2(t)] - F_{st}[z_1(t)] = 1-\alpha \quad (5.2.6) \]

That is, \( z_1(t) \), \( z_2(t) \) are independent of \( t \) and are \( \alpha/2 \) and \((1-\alpha/2)\) percentiles of \( t \)-distribution with \((n-1)\) degrees of freedom, respectively.

According to Theorem 4.1, we need to use the acceptance region \( A_3(\beta \mid t) = \{ w : z_1 \leq w \leq z_2 \} \). But \( W^2 \) under \( H_0 \) has \( F \)-distribution with \((1,n-1)\) degrees of freedom, Chikara, (1975). Therefore
\[ z_1 \leq w \leq z_2 \quad \text{iff} \quad w^2 \leq m, \]

where \( m \) is \((1-\alpha)\) percentile of \( F \)-distribution with \((1,n-1)\) degrees of freedom. Hence instead of the acceptance region, \( A_3(\beta \mid t) \), we can work with the equivalent region which is based on \( F \)-distribution.

According to Chhikara and Guttman (1982), inverting \( w^2 \leq m \) provides the two sided prediction interval for \( T_{1y} = Y \),

\[
S_3(t_x) = \left\{ \left[ \frac{n}{t_{1x}} + \left( \frac{nA}{2(n-1)} \right) \right] + \left[ \frac{(n+1)A}{(n-1)t_{1x}} + \frac{nA^2}{4(n-1)^2} \right]^{1/2} \right\}^{1/2} \]

where

\[
A = \sum_{i=1}^{n} \left( \frac{1}{x_i} - x^{-1} \right). \]

They mention that there is a positive probability to get a negative real number for the lower limit of \( S_3(t_x) \), and since \( T_{1y} > 0 \), the solution must be restricted to the positive real line.

1. If \( \left( (n-1) - t_{1x}A/n \right) > 0 \), we get two sided prediction interval.
2. If \( \left( (n-1) - t_{1x}A/n \right) < 0 \), lower limit will be obtained using + sign in \( S_3(t_x) \) and upper limit would be \( \infty \).
VI. BIBLIOGRAPHY


APPENDIX A

The following program uses 'The Generalized Newton's Method' and MDBETA routine of IMSL library to find values of $z_1$ and $z_2$ for the two sided prediction interval, when the distribution of informative and future experiments are normal with known mean, $\mu=0$.

(Example 3.1.)

PROGRAM A

REAL S,R,XO,YO,AL,A,B,C,GX,GY,HX,HY,J,P1,P2
REAL P3,P4,HXY,GXY,TOL,X1,Y1,L,K
INTEGER SI,RI
REAL*8 G1,G2,G3,G4,G5,DGAMMA

WRITE(*,*) 'VALUES OF AL,R,X0,YO ARE='
READ(*,'(F4.2,1X,F4.2,1X,F10.8,1X,F10.8)') AL,R,X0,Y0
DO 60 N=2,100
S=N*R
A=R+S
B=R+S+1
C=R+1
G1=DGAMMA(R)
G2=DGAMMA(S)
G3=DGAMMA(C)
G4=DGAMMA(A)
G5=DGAMMA(B)
SI=INT(S)
RI=INT(R)

20 GX=(-G5/(G3*G2))*((X0**RI)*((1-X0)**(SI-1)))
GY=(G5/(G3*G2))*((YO**RI)*((1-Y0)**(SI-1)))
HX=(-G4/(G1*G2))*((X0**(RI-1))*((1-X0)**(SI-1)))
HY=(G4/(G1*G2))*((YO**(RI-1))*((1-Y0)**(SI-1)))
J=GX*HY-HX*GY
CALL MDBETA (YO,R,S,P1,IER)
CALL MDBETA (XO,R,S,P2,IER)
CALL MDBETA (YO,C,S,P3,IER)
CALL MDBETA (XO,C,S,P4,IER)
HXY=P1-P2-(1-AL)
GXY=P3-P4-(1-AL)
X1=XO+(HXY*GY-GXY*HY)/J
Y1=YO+(GXY*HX-HXY*GX)/J
TOL=1.E-3
K=ABS(GXY)
L=ABS(HXY)
IF(K.GT.TOL) GO TO 25
IF(L.GT.TOL) GO TO 25
GO TO 30
25
X0=X1
Y0=Y1
GO TO 20
30 WRITE (*,40) N,X0,Y0
40 FORMAT(10X,I2,4X,F8.6,4X,F8.6)
60 CONTINUE
END
Values of $z_1$ and $z_2$ to find $(1-\alpha)$ level prediction interval $S_3(t_x)=[t_xz_1/(1-z_1), t_xz_2/(1-z_2)]$, where
\[ t_x = \sum_{i=1}^{n} x_i^2 \] and $X_1, \ldots, X_n$ are (iid) normal with mean, $\mu=0$.

(Example 3.1.)

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APPENDIX B

The following program uses the 'Generalized Newton's Method' to find values of $z_1$ and $z_2$ for the two sided prediction interval, when the distribution of informative and future experiments are Negative Exponential. (Example 3.4.)

PROGRAM B
REAL K,L,AL,XO,YO,X1,Y1,GX,GY,HX,HY,GXY,HXY,A,TOL
INTEGER N,I
WRITE (*,*)'VALUES OF AL,XO,YO ARE='
READ (*,'(F4.2,1X,F8.6,1X,F8.6)') AL,XO,YO
DO 60 N=2,100
20 GX=N*((1-XO)**(N-1))
GY=N*((1-YO)**(N-1))
HX=((1-XO)**(N-1))*(1-XO-YO)
HY=((1-YO)**(N-1))*(N*YO-(1-YO))
A=GX*HY-HX*GY
GXY=((1-XO)**N)-(1-YO)**N-(1-AL)
HXY=XO*((1-XO)**N)-YO*((1-YO)**N)
X1=XO+(HXY*GY-GXY*HY)/A
Y1=YO+(GXY*HX-HXY*GX)/A
TOL=1.E-5
K=ABS(GXY)
L=ABS(HXY)
IF(K.GT.TOL) GO TO 25
IF(L.GT.TOL) GO TO 25
GO TO 30
25 XO=X1
YO=Y1
GO TO 20
30 WRITE(*,40) N,XO,YO
40 FORMAT(4X,I3,4X,F8.6,4X,F8.6)
60 CONTINUE
END
TABLE B

Values of $z_1$ and $z_2$ to find $(1-a)$ level prediction
interval $S_3(t_x) = [t_x z_1 / (1-z_1), t_x z_2 / (1-z_2)]$, where
$t_x = \sum_{i=1}^{n} x_i$ and $X_1, \ldots, X_n$ are (iid) negative exponential.

Example 3.3.)

\begin{tabular}{llllll}
\hline
$n$ & $z_1$ & $z_2$ & $z_1$ & $z_2$ \\
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3 & .002411 & .859294 & .011774 & .752865 \\
4 & .001900 & .778032 & .009247 & .659005 \\
5 & .001565 & .705910 & .007605 & .583406 \\
6 & .001330 & .643602 & .006455 & .522246 \\
7 & .001155 & .590076 & .005606 & .472169 \\
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9 & .000915 & .504565 & .004436 & .395516 \\
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11 & .000757 & .439755 & .003669 & .339902 \\
12 & .000697 & .413029 & .003378 & .317502 \\
13 & .000645 & .389291 & .003128 & .297842 \\
14 & .000601 & .367885 & .002914 & .280453 \\
15 & .000562 & .348887 & .002726 & .264936 \\
16 & .000528 & .331708 & .002562 & .251068 \\
17 & .000498 & .316116 & .002416 & .238570 \\
18 & .000471 & .301908 & .002286 & .227251 \\
19 & .000447 & .288906 & .002169 & .216951 \\
20 & .000426 & .276962 & .002063 & .207541 \\
21 & .000406 & .265961 & .001967 & .198910 \\
22 & .000388 & .255790 & .001880 & .190965 \\
23 & .000371 & .246368 & .001800 & .183629 \\
24 & .000356 & .237605 & .001727 & .176833 \\
25 & .000342 & .229442 & .001659 & .170521 \\
26 & .000329 & .221823 & .001597 & .164644 \\
27 & .000317 & .214687 & .001539 & .159157 \\
28 & .000306 & .207992 & .001485 & .154023 \\
29 & .000296 & .201700 & .001435 & .149209 \\
30 & .000286 & .195780 & .001388 & .144687 \\
31 & .000277 & .190191 & .001344 & .140429 \\
32 & .000269 & .184914 & .001302 & .136416 \\
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