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Title : PREDICTION INTERVALS IN EXPONENTIAL FAMILIES

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Prediction intervals for an outcome of a sufficient statistic, T_y, associated with the probability distribution of a future experiment are developed based on information obtained from n independent, previously conducted trials of an informative experiment. The random outcomes of the informative and future experiments are assumed to be continuous and identically distributed according to a K-parameter exponential family, and the future experiment is conducted independent of the informative experiment.

These intervals are of the general forms, $S_1(t_x)=[L(t_x),\infty)$, $S_2(t_x)=(-\infty,U(t_x)]$ and $S_3(t_x)=[L(t_x),U(t_x)]$ where U(.), L(.) are functions of t_x , the observed value of a sufficient statistic for the joint probability distribution of the random outcomes from the informative experiment.

A general theory and procedure for deriving these prediction intervals is developed using hypothesis testing procedures. Optimal properties of hypothesis tests carry over to similarly defined optimal properties of prediction intervals. The intervals

have the 'similar mean coverage' property (Aitchison, J. and Dunsmore, I.R. (1975)).

The generalized Newton's method and the IMSL routines are used for numerical computation of tables for the examples considered. An application of the saddle point approximation, Barndorff-Nielsen, O. (1983), for finding an approximate conditional density function for sufficient statistics associated with the probability distribution of the experiments is discussed.

PREDICTION INTERVALS IN EXPONENTIAL FAMILIES

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PREDICTION INTERVALS IN EXPONENTIAL FAMILIES

I. INTRODUCTION

Statistical prediction analysis involves an informative and a future experiment. Based on n independent random outcomes, $X_1,\ldots,X_n$ from the informative experiment, it is of interest to construct a region that contains an outcome of the future experiment, with a specified probability. This region is called a prediction region for the future outcome, Y .

A prediction region is similar to a confidence set for a parameter of a distribution. That is, before observing the results of the informative experiment the probability is $1-\alpha$ that we will obtain a region that will contain the outcome, Y, of the future experiment.

Prediction methods are of several types as described by Aitchison ,J. and Dunsmore, I.R. (1975). In 'Decisive Prediction', a prior distribution on the parameter space of the probability distributions and also a utility function are available. The idea is to maximize the expected value of the utility function over all possible prediction regions. The expectation is evaluated with respect to the 'Predictive Density', Aitchison ,J. and Dunsmore, I.R. (1975), of Y given $\underline{x}=(x_1,\ldots,x_n)$. 'Bayesian Informative Prediction' does not require specification of a utility function. The idea is to choose a prediction region so that the predictive density is above some point in that region. 'Informative Tolerance Prediction' does not use a prior distribution or utility function and it has two types, Mean Coverage Tolerance Prediction and Guaranteed Coverage Tolerance Prediction.

In this thesis we will work with the informative mean coverage tolerance prediction. We are interested in developing a technique for obtaining prediction regions by using hypothesis testing methods. In Chapter II, we will formulate the prediction problem for the one parameter exponential family of distributions in terms of hypothesis testing results. Examples of one parameter exponential distributions will be given in Chapter III and prediction intervals will be obtained by using the method of Chapter II. We will suggest an approximation method for some cases. The generalization to the K-parameter exponential distribution will be discussed in Chapter IV and an application of a 'Saddle Point Approximation', Barndorff-Nielsen (1983) for obtaining the density function of a sufficient statistic will be considered. Chapter V has two examples of two-parameter exponential distributions where prediction intervals are obtained using the method described in Chapter IV.

Aitchison, J. and Sculthorpe, D. (1965), give a general framework for Bayesian and non-Bayesian prediction. Hahn, J. G.

(1972) develops simultaneous prediction intervals for the standard deviation of future samples when sampling from a normal distribution. Faulkenberry, D. G. (1973), gives a method for obtaining prediction intervals. The method is based on conditioning on a sufficient statistic associated with probability distributions of the experiments. Statistical prediction intervals for observations of a future experiment is discussed by Olsen, D. E. (1974). In his work discrete probability distributions associated with the experiments are considered. Aitchison, J. and Dunsmore, I.R. (1975) have a detailed discussion on types of prediction methods and Chhikara, R.S. and Guttman, I. (1982) give prediction intervals when sampling from an Inverse-Gaussian distribution. They use Bayesian Informative and Informative Tolerance prediction techniques. Given a set of observations from a general linear model and having prior distribution for parameters, Johnson W. and Geisser S. (1982) develop a method for assessing the influence of specified subset of the data when prediction of future observations is of interest. We will not assume any regression type models and construct prediction intervals which are based only on outcomes of the informative experiment. The method of obtaining prediction intervals in this thesis is developed under the assumption that the probability distribution of experiments are from a continuous exponential family, but the method can be applied to a general case.

II. PREDICTION INTERVALS IN ONE PARAMETER CONTINUOUS EXPONENTIAL FAMILIES

In this chapter we will formulate the problem of deriving a prediction region by using hypothesis testing theory. For the special case of the one parameter continuous exponential family it will be shown that the optimal properties of the hypothesis tests carry over to similarly defined optimal properties of prediction intervals.

2.1. Deriving Prediction Regions Using Hypothesis Tests

Let $X_1,\ldots,X_n$ be independent identically distributed, (iid) random outcomes of an informative experiment and Y be an outcome of a future experiment. In order that the informative experiment should provide information on the future experiment, there must be some link between the two experiments, Aitchison and Dunsmore (1975). This link is through the probability distributions associated with the experiments and is also through an indexing set of parameters of the distributions. The common assumptions are that the experiments are being conducted independently, and the probability distributions are the same.

Let $(P_\theta, \theta \varepsilon \Theta)$ denote the probability distribution associated with the experiments, $f(.:\theta)$ be its density function, and let $\mathcal{X}$ be the sample space. A family of subsets $S(\underline{X})$ of the sample space, $\mathcal{X}$, is said to constitute a family of prediction regions for Y based on $\underline{X}=(X_1,\ldots,X_n)$, if the random set $S(\underline{X})$ covers Y with some specified probability. We have

$$S : \mathcal{X}^n \longrightarrow \mathcal{X}$$

<u>Definition</u> <u>2.1</u>: $S(\underline{X})$ is called a 'Mean Coverage Tolerance Prediction Region' of Cover $(1-\alpha)$ for Y if

$$\inf_{\theta \in \Theta} E_\theta[\ P_\theta\{Y \in S(\underline{X})\}\ ] =$$

$$\inf_{\theta \in \Theta} \int_{\mathcal{X}^n} \int_{S(\underline{X})} f(y:\theta) f(\underline{x}:\theta) d\underline{x} dy = 1-\alpha \ .$$

If $E_\theta[\ P_\theta\{Y \in S(\underline{X})\}\ ] = (1-\alpha)$ for all $\theta \in \Theta$, then $S(\underline{X})$ is called a 'Similar Mean Coverage Tolerance Prediction Region' of Cover $(1-\alpha)$.

We will obtain prediction regions by using hypothesis testing procedures. It will be shown that these prediction regions are similar mean coverage and also have some optimal properties.

To set up the prediction problem using hypothesis tests, let $X_1,\ldots,X_n$ be (iid) outcomes of an informative experiment with density, $(f(x:\theta_x),\ \theta_x \in \Theta_x,\ x \in \mathcal{X}\ )$ and Y be an outcome of a future experiment with density, $(f(y:\theta_y),\ \theta_y \varepsilon\ \Theta_y,\ y \varepsilon \mathcal{X})$, where Θ_x and Θ_y are parameter spaces associated with the probability distributions of the informative and future experiments, respectively.

Define a 'notional null hypothesis', Cox and Hinkley (1974),

$$H_o:\ \theta_y=\theta_x$$

concerning the true parameters of the experiments. Let H_a be an alternative hypothesis and $A_{Ha}(\theta_y,\theta_x)$ denote the acceptance region of a size α test for testing H_o versus H_a.

<u>Theorem</u> <u>2.1:</u> For each sample point $(\underline{x},y)\ \varepsilon\ \mathcal{X}^n \times \mathcal{X}$ let

$$S_{Ha}(\underline{x})=\{\ y:\ (\underline{x},y)\ \varepsilon\ A_{Ha}(\theta_y,\theta_x)\ \}. \qquad (2.1.1)$$

Then $S_{Ha}(\underline{X})$ is a family of prediction regions for Y with confidence level $(1-\alpha)$. If $A_{Ha}(\theta_y,\theta_x)$ is the uniformly most powerful $(1-\alpha)$ level region in a certain class of acceptance regions for testing H_o versus H_a , then $S_{Ha}(\underline{X})$ minimizes

$$P_{(\theta y,\theta x)}\{\ Y\ \varepsilon\ S_{Ha}(\underline{X})\ \}\quad \text{for all}\quad (\theta_y,\theta_x)\ \varepsilon\ \Theta_a$$

where, Θ_a is the parameter space associated with H_a .

Proof: By definition of $S_{Ha}(\underline{X})$

$$y \in S_{Ha}(\underline{x}) \quad \text{iff} \quad (\underline{x},y) \in A_{Ha}(\theta_y,\theta_x)$$

$$S_{Ha} : \mathcal{X}^n \rightarrow \mathcal{X}$$

and hence

$$P_{(\theta y,\theta x)}\{ Y \in S_{Ha}(\underline{X}) \}$$
$$=P_{(\theta y,\theta x)}\{ (\underline{X},Y) \in A_{Ha}(\theta_y,\theta_x) \}=(1-\alpha)$$

for all $(\theta_y,\theta_x) \in \Theta_o$, where $\Theta_o=\{ (\theta_y,\theta_x) : \theta_y=\theta_x) \}$.

Therefore $S_{Ha}(\underline{X})$ is a family of prediction regions for Y with probability, $(1-\alpha)$.

If $S^*_{Ha}(\underline{X})$ is another family of prediction regions for Y with probability, $(1-\alpha)$ and $A^*_{Ha}(\theta_y,\theta_x)=\{ (\underline{x},y) : y \in S^*_{Ha}(\underline{x}) \}$, then

$$P_{(\theta y,\theta x)}\{ (\underline{X},Y) \in A^*_{Ha}(\theta_y,\theta_x) \}$$
$$=P_{(\theta y,\theta x)}\{ Y \in S^*_{Ha}(\underline{X}) \}=(1-\alpha)$$
$$\text{for all } (\theta_y,\theta_x) \in \Theta_o \quad .$$

So $A^*_{Ha}(\theta_y,\theta_x)$ is the acceptance region of a level α test for testing H_o versus H_a . $A_{Ha}(\theta_y,\theta_x)$ is assumed to be the UMP in a certain class of acceptance regions. Therefore

$$P_{(\theta y,\theta x)}\{\ (\underline{X},Y)\ \varepsilon\ A^{*}_{Ha}(\theta_y,\theta_x)\ \}\ \geq$$
$$P_{(\theta y,\theta x)}\{\ (\underline{X},Y)\ \varepsilon\ A_{Ha}(\theta_y,\theta_x)\ \}$$
$$\text{for all}\quad (\theta_y,\theta_x)\ \varepsilon\quad \Theta_a$$

and hence

$$P_{(\theta y,\theta x)}\{\ Y\ \varepsilon\ S^{*}_{Ha}(\underline{X})\ \}\ \geq\ P_{(\theta y,\theta x)}\{\ Y\ \varepsilon\ S_{Ha}(\underline{X})\ \}$$
$$\text{for all}\quad (\theta_y,\theta_x)\ \varepsilon\quad \Theta_a$$

We note that the form of the UMP acceptance region depends on the alternative hypothesis used and as a result the type of prediction region, $S_{Ha}(\underline{X})$, depends on the alternative hypothesis.

While it seems reasonable that using good or optimal hypothesis testing procedures should result in good prediction, there are no formulated criteria available for comparing prediction regions. Similar criteria for comparing acceptance regions will carry over to comparing prediction regions. Comparison of unbiased prediction regions is discussed in section 2.3.

As an example of using an acceptance region to get a prediction region, let $X_1,\ldots,X_n$ be (iid) outcomes of an informative experiment with probability distribution, normal (θ_x,σ^2) and Y be an outcome of a future experiment with probability distribution, normal (θ_y,σ^2) . It is assumed that σ has a known value. The $(1-\alpha)$ level acceptance region for testing

$$H_o : \theta_y = \theta_x$$

$$H_a : \theta_y < \theta_x$$

is

$$A_{Ha}(\theta_y, \theta_x) = \{ (y, \underline{x}) : (1+1/n)^{-1/2}(y-\bar{x})/\sigma \leq Z_{(1-\alpha/2)}$$

where, $\bar{x} = (\sum_{i=1}^{n} x_i)/n$ is the sample mean and $Z_{(1-\alpha/2)}$ is $(1-\alpha/2)$ percentile of the standard normal distribution. From (2.1.1) we have

$$S_{Ha}(\underline{x}) = \{ y : y \leq \bar{x} + \sigma(1+1/n)^{1/2} Z_{(1-\alpha/2)} \}$$

which is an upper limit prediction for Y with confidence level $1-\alpha$.

2.2. The Theory For One Parameter Case

2.2.1. Formulation Of The Prediction Problem In terms Of Hypothesis Testing

Definition 2.2: A family of probability measures, $P = \{P_\theta : \theta \in \Theta\}$ on a sample space, $\mathcal{X}$ is called a K-parameter exponential family if, with respect to some σ-finite measure, μ , it has densities of the form

$$f(x:\theta) = C(\theta)\exp[Q'(\theta)T(x)]h(x) \qquad (2.2.1)$$

where, $C: \Theta \longrightarrow (0,\infty)$, $Q: \Theta \longrightarrow R^K$, $T: \mathcal{X} \longrightarrow R^K$, $h: \mathcal{X} \longrightarrow [0,\infty)$.

Let random outcomes of an informative experiment, $X_1,\dots,X_n$, represent independent and identically distributed, (iid) random variables according to a one parameter continuous exponential family of distributions with density function

$$f(x:\theta_x)=C(\theta_x)\exp[\theta_x T(x)]h(x)$$

and let a random outcome of the corresponding future experiment, Y, be independent of the X_i's and distributed according to the same family with density function

$$f(y:\theta_y)=C(\theta_y)\exp[\theta_y T(y)]h(y)\ .$$

The probability distributions are assumed to be absolutely continuous. Therefore the σ-finite measure, μ , Definition 2.2, is the Lebesque measure and the sample space, $\mathcal{X}$ is the real line.

In some problems the density functions are not in the form of the 'natural parametrization', Ferguson (1967), but they can be reparametrized to get the desired form. The value of a future outcome, Y, is not known but to formulate the prediction problem using hypothesis testing techniques, we pretend at this point that Y is observed like the X_i's. Hypotheses of the following forms are of interest.

1. H_o: $\theta_x=\theta_y$ vs H_{a1}: $\theta y>\theta_x$

2. H_o: $\theta_x=\theta_y$ vs H_{a2}: $\theta_y<\theta_x$

3. H_o: $\theta_x=\theta_y$ vs H_{a3}: $\theta_y\neq\theta_x$

The choice of the alternative hypothesis depends on the form of prediction region (interval) desired.

The joint density function of $\underline{X}$,Y can be written as

$$f(\underline{x},y:\theta_x,\theta_y)=C^n(\theta_x)C(\theta_y)\exp[\theta_x \sum_{i=1}^n T(x_i)+\theta_y T(y)] \quad (2.2.2)$$
$$\pi_{i=1}^n \ h(x_i)h(y) \ .$$

The density function in (2.2.2) is in the form of a two parameter regular exponential family and therefore $\sum_{i=1}^n T(X_i)$ and T(Y) are jointly complete and sufficient statistics for the joint distribution of $\underline{X}$ and Y. Thus it is natural to reduce the problem to the consideration of the sufficient statistics.

Let $T_x= \sum_{i=1}^n T(X_i)$, $T_y=T(Y)$, $T=T_x+T_y$ and $\beta=\theta_y-\theta_x$, then according to Lehmann (1959), we have

$$f(t_x:\theta_x)=C_{tx}(\theta_x)\exp[\theta_x t_x]h_{tx}(t_x) \quad (2.2.3)$$

where, C_{tx}: $\Theta_x \longrightarrow (0,\infty)$, T_x: $R^n \longrightarrow R$, h_{tx}: $R \longrightarrow [0,\infty)$ and

$$f(t_y:\theta_y)=C_{ty}(\theta_y)\exp[\theta_y t_y]h_{ty}(t_y) \quad (2.2.4)$$

where, $C_{ty}: \Theta_y \longrightarrow (0,\infty)$, $T_y:R \longrightarrow R$, $h_{ty}:R \longrightarrow [0,\infty)$.

T_x is independent of T_y. By (2.2.3) and (2.2.4), the joint density of T_x and T_y is

$$f(t_x,t_y:\theta_x,\theta_y)=C_{tx}(\theta_x)C_{ty}(\theta_y)\exp[\theta_x t_x+\theta_y t_y] \\ h_{tx}(t_x)h_{ty}(t_y)$$

which implies

$$f(t_y,t:\theta_x,\beta)=C_{tx}(\theta_x)C_{ty}(\beta+\theta_x)\exp[\beta t_y+\theta_x t] \\ h_{tx}(t-t_y)h_{ty}(t_y) \quad , t_y \leq t \tag{2.2.5}$$

The density function in (2.2.5) is in the form of the two parameter regular exponential family and therefore (T_y,T) are complete sufficient statistics for (β,θ_x). Using (2.2.5), the conditional density of T_y given t is

$$f(t_y \mid t,\beta)= \frac{\exp[\beta t_y]h_{tx}(t-t_y)h_{ty}(t_y)}{\int_{-\infty}^{t}\exp[\beta t_y]h_{tx}(t-t_y)h_{ty}(t_y)} \quad , t_y \leq t \tag{2.2.6}$$

and with the new parametrization hypotheses of interest become

1. H_o: $\beta=\beta_o$ vs H_{a1}: $\beta>\beta_o$
2. H_o: $\beta=\beta_o$ vs H_{a2}: $\beta<\beta_o$
3. H_o: $\beta=\beta_o$ vs H_{a3}: $\beta\neq\beta_o$

where, $\beta_o = 0$ and θ_x is a nuisance parameter. Existence of the nuisance parameter θ_x does not allow us to find UMP tests for the hypotheses. Thus in this thesis we will concentrate on unbiased tests, Ferguson (1967).

__Lemma 2.1__: If $\theta_x=\theta_y(=\theta)$, then the conditional density function of T_y given T=t, under H_o is

$$f(t_y|T=t)= \frac{h_{tx}(t-t_y)h_{ty}(t_y)}{h_t(t)} \qquad (2.2.7)$$

__Proof__: If $\theta_x=\theta_y(=\theta)$, then $\beta=\beta_o$ and using (2.2.6) we get

$$f(t_y \mid T=t)= \frac{h_{tx}(t-t_y)h_{ty}(t_y)}{h_t(t)}$$

where, $h_t(t)= \int_{-\infty}^{t} h_{tx}(t-t_y)h_{ty}(t_y)dt_y$.

The following Theorem is based on the method of finding uniformly most powerful (UMP) unbiased tests in a K-parameter exponential family of distributions, Ferguson (1967). Notations will correspond to the cases where X and Y are from one parameter continuous exponential families and the joint density of T_y and T is given by (2.2.5). The Theorem can be generalized to the K-parameter case which will be stated in Chapter IV.

Theorem 2.2: Let $f_o=f(t_y|T=t)$, (2.2.7), denote the conditional density function of T_y given t under the notional null hypothesis, H_o.

(1) $$\Phi_1(t_y,t)=\begin{cases}1 & \text{if} \quad t_y > z(t)\\ 0 & \text{if} \quad t_y \leq z(t)\end{cases}$$

is the UMP unbiased size α test for testing H_o versus H_{a1} and $z(t)$ can be found from $E_{\beta=\beta o}[\,\Phi_1(T_y,T)|T]=\alpha$ which is equivalent to

$$\int_{z(t)}^{\infty} f_o dt_y=\alpha\ .$$

(2) $$\Phi_2(t_y,t)=\begin{cases}1 & \text{if} \quad t_y < z(t)\\ 0 & \text{if} \quad t_y \geq z(t)\end{cases}$$

is the UMP unbiased size α test for testing H_o versus H_{a2} and $z(t)$ can be found from $E_{\beta=\beta o}[\,\Phi_2(T_y,T)|T]=\alpha$ which is equivalent to

$$\int_{-\infty}^{z(t)} f_o dt_y=\alpha\ .$$

(3) $$\Phi_3(t_y,t)=\begin{cases}1 & \text{if} \quad t_y < z_1(t) \text{ or } t_y > z_2(t)\\ 0 & \text{if} \quad z_1(t) \leq t_y \leq z_2(t)\end{cases}$$

is the UMP unbiased size α test for testing H_o versus H_{a3} and $z_1(t)$, $z_2(t)$ can be found from

$$E_{\beta=\beta o}[\ \Phi_3(T_y,T)\,|T]=\alpha \text{ and } E_{\beta=\beta o}[T_y\ \Phi_3(T_y,T)\,|T]=\alpha\ E_{\beta=\beta o}[T_y\,|T]$$

which are equivalent to

$$\int_{z_1(t)}^{z_2(t)} f_o dt_y=1-\alpha \quad \text{and} \quad \int_{z_1(t)}^{z_2(t)} t_y f_o dt_y=(1-\alpha)\int_{-\infty}^{\infty} t_y f_o dt_y.$$

Sometimes some work can be saved when under H_o there exists an ancillary statistic, V, for the joint distribution of T_y and T. An ancillary statistic in this thesis is defined to be a statistic with a distribution that does not depend on any parameters. If an ancillary statistic exists, instead of finding the conditional density, f_o, we find the density function of V under H_o. For, by Basu's Theorem given in Lehmann (1959), since T is a complete sufficient statistic (C.S.S) for the joint distribution under H_o and V is an ancillary statistic for the same distribution, then they are independent.

The following Lemma, Lehmamn (1959), will be used for finding UMP unbiased tests when there exists an ancillary statistic for the joint distribution of T_y and T under H_o.

<u>Lemma</u> <u>2.2</u>: Suppose there exists an ancillary statistic, $V=G(T_y,T)$ for joint distribution of (T_y,T) under H_o . Let $f_{Ho}(v)$ denote the density function of V under H_o .

(1) $$\Phi_1(v)=\begin{cases}1 & \text{if} \quad v > z\\ 0 & \text{if} \quad v \leq z\end{cases}$$

is the UMP unbiased size α test for testing H_o versus H_{a1} provided $G(T_y,T)$ is increasing in t_y for each t. z can be found from $E_{\beta=\beta o}[\ \Phi_1(V)]=\alpha$ which is equivalent to $\int_z^{\infty} f_{Ho}(v)dv=\alpha$.

(2) $$\Phi_2(v)=\begin{cases}1 & \text{if} \quad v < z\\ 0 & \text{if} \quad v \geq z\end{cases}$$

is the UMP unbiased size α test for testing H_o versus H_{a2} provided $G(T_y,T)$ is increasing function in t_y for each t. z can be found from $E_{\beta=\beta o}[\ \Phi_2(V)]=\alpha$ which is equivalent to $\int_{-\infty}^{z} f_{Ho}(v)dv=\alpha$.

(3) $$\Phi_3(V)=\begin{cases}1 & \text{if} \quad v < z_1 \text{ or } v > z_2\\ 0 & \text{if} \quad z_1 \leq v \leq z_2\end{cases}$$

is the UMP unbiased size α test for testing H_o versus H_{a3} provided $V=G(T_y,T)=a(T)T_y+b(T)$, $a(T)>0$. z_1, z_2 can be found from

$$E_{\beta=\beta o}[\ \Phi_3(V)]=\alpha \quad \text{and} \quad E_{\beta=\beta o}[V\,\Phi_3(V)]=\alpha\ E_{\beta=\beta o}[V]$$

which are equivalent to

$$\int_{z_1}^{z_2} f_{Ho}(v)dv=1-\alpha \quad \text{and} \quad \int_{z_1}^{z_2} vf_{Ho}dv=(1-\alpha) \int_{-\infty}^{\infty} vf_{Ho}dv \ .$$

A similar result holds if $a(T)<0$, Lehmann (1947). Φ_i ($i=1,2,3$) in Theorem 2.2 and Lemma 2.1 are α similar on the boundary sets $\Theta_{bi}= \overline{\Theta}_o \cap \overline{\Theta}_{ai}$ and have Neyman Structure, Ferguson (1967).

According to the following Lemma, Lehmann (1947), z_1, z_2 in Lemma 2.2 (3) exist.

<u>Lemma 2.3</u>: Let $0<\alpha<1$, let $f(x)$ be a density function of a random variable, X with $\int_{-\infty}^{\infty} x^s f(x)dx<\infty$, ($s=0,1$) then, there exist A,B so that

$$\int_B^A x^s f(x)dx = \alpha \int_{-\infty}^{\infty} x^s f(x)dx, \ (s=0,1).$$

<u>Remark:</u> For any size α test, Φ if distribution of V under H_o is symmetric about some point A, then $E_{\beta=\beta o}[V]=A$ and we have

$$E_{\beta=\beta o}[V \ \Phi(V)]=E_{\beta=\beta o}[(V-A) \ \Phi(V)+A \quad (V)]$$
$$=AE_{\beta=\beta o}[\ \Phi(V)]=A\alpha=\alpha \ E_{\beta=\beta o}[V].$$

That is, if V under H_o has a symmetric distribution about some point A, then any test of size α, Φ, which satisfies $E_{\beta=\beta o}[\Phi(V)]=\alpha$ must satisfy $E_{\beta=\beta o}[V\ \Phi(V)]=\alpha\ E_{\beta=\beta o}[V]$ and to find acceptance region for H_o versus H_{a3}, we just solve

$$\int_{-\infty}^{m} f_{Ho}(v)dv=\alpha/2$$

for m where, $m=z_1$ and $2A-m=z_2$. A similar argument to the above one holds when f_o in the Theorem 2.2 is symmetric about some point, A .

2.2.2. Monotone Likelihood Ratio Family of Distributions

Definition 2.3 : A real parameter family of distributions is said to have a monotone likelihood ratio, if densities (probability mass functions), $f(w:\theta)$ exist so that for $\theta_1 < \theta_2$,

$$\frac{f(w:\theta_2)}{f(w:\theta_1)}$$

is a nondecreasing function of w, for all w in the set of existence of $f(w:\theta_1)$ and $f(w:\theta_2)$.

<u>Lemma</u> <u>2.4:</u> Let W be a continuous random variable distributed according to a one parameter exponential family, $\mathcal{P}_\theta$, with the density function

$$f(w:\theta)=C(\theta)\exp[\theta T(w)]h(w)$$

and cumulative distribution function, $F_\theta(w)$, where

$$C: \Theta \longrightarrow R \ , \quad T: \mathcal{W} \longrightarrow R \ , \quad h: \mathcal{W} \longrightarrow R \ , \quad \Theta \subset R$$

and $T(w)$ is nondecreasing in w. Then, the family, $\mathcal{P}_\theta$, has a monotone likelihood ratio in w and for $\theta_2 > \theta_1$, $F_{\theta 2}(w) \leq F_{\theta 1}(w)$, for all w in the set of existence of $f(w:\theta)$.

Consider a random sample, $X_1,\ldots,X_n$ from a one parameter exponential family with density, (2.2.3) and let Y be independent of the X_i, $(i=1,2,\ldots,n)$ from the same family with density, (2.2.4). By Lemma 2.4, the distributions of X and Y have monotone likelihood ratio in T(x) and T(y), respectively. Recall definitions of sufficient statistics T_y, T_x and T from section 2.2.1, and consider the conditional density of T_y given T=t

$$f(t_y \mid t:\beta)= \frac{\exp[\beta t_y]h_{tx}(t-t_y)h_{ty}(t_y)}{\int_{-\infty}^{t} \exp[\beta t_y]h_{tx}(t-t_y)h_{ty}(t_y)} \ . \quad (2.2.8)$$

The distribution associated with the density function in (2.2.8) has a monotone likelihood ratio in t_y. Therefore we conclude that if X (Y) have monotone likelihood ratio family of distributions, then the conditional distribution of T_y given T has also a monotone likelihood ratio. In section 2.2.1 we had $\beta=\theta_y-\theta_x$, but in some cases in which a reparametrization is needed to get natural parametrization for densities (2.2.3) and (2.2.4), β can be defined as $\beta=g(\theta_y-\theta_x)$, with g(.) being nondecreasing and $\beta=0$ whenever $\theta_y=\theta_x$.

<u>Lemma</u> <u>2.5:</u> Let the conditional density function of T_y given t be given by (2.2.8), then

(1) $P_{\beta=\beta 1}\{T_y \in (-\infty, C_2(t)] \mid T=t\} \leq P_{\beta=\beta o}\{T_y \in (-\infty, C_2(t)] \mid T=t\}$
for all $\beta_1 > \beta_o$.

(2) $P_{\beta=\beta 1}\{T_y \in [C_1(t), \infty) \mid T=t\} \leq P_{\beta=\beta o}\{T_y \in [C_1(t), \infty) \mid T=t\}$
for all $\beta_1 < \beta_o$.

(3) $P_{\beta=\beta 1}\{T_y \in [C_1(t), C_2(t)] \mid T=t\} \leq P_{\beta=\beta o}\{T_y \in [C_1(t), C_2(t)] \mid T=t\}$
for all $\beta_1 \neq \beta_o$.

where, $\beta_o=0$ and $C_i(.)$, i=1,2 are functions of t .

<u>Proof</u>: Let $F_\beta^{Ty|t}$ be the cumulative distribution associated with conditional distribution of T_y given t, defined by, $F_\beta^{Ty|t}(C(t))=P_\beta\{T_y \leq C(t) \mid T=t\}$. Use Lemma 2.4 .

2.2.3. The Match-up Of Hypothesis Tests and Prediction Intervals

A. Ancillary Statistic Does Not Exist

Consider notations and results of section 2.2.1 and suppose the joint distribution of T_y and T under H_o does not have an ancillary statistic, V, and UMP unbiased acceptance regions for testing H_o versus H_{ai} are found using f_o. (See Theorem 2.2.)

According to the results of Theorem 2.2, UMP unbiased tests for testing H_o versus H_{ai}, (i=1,2,3) are performed as conditional tests given T=t and as a consequence acceptance regions are found conditionally. According to our hypotheses testing formulation of the problem in section 2.2.1, β is the parameter of interest and θ_x is the nuisance parameter. We denote the conditional acceptance region of H_o versus H_{ai}, (i=1,2,3), by $A_i(\beta|t)$. UMP unbiased tests provide UMP acceptance regions, Lehmann (1959), therefore $A_i(\beta|t)$ are UMP unbiased acceptance regions. From Theorem 2.2 we have

$$
\begin{aligned}
A_1(\beta|t) &= \{ (t_y,t) : t_y \le z(t) \} \\
A_2(\beta|t) &= \{ (t_y,t) : t_y \ge z(t) \} \qquad (2.2.9) \\
A_3(\beta|t) &= \{ (t_y,t) : z_1(t) \le t_y \le z_2(t) \},
\end{aligned}
$$

where z(.) are functions of t. Lemma 2.5 identifies intervals which have larger probabilities when parameters of the distributions are assumed to be the same than if they are different in some direction. The acceptance regions are constructed under H_0 and note that they have the same forms as intervals in Lemma 2.5 . Therefore it would be a natural thing to use acceptance regions to obtain prediction intervals.

Let $A_i(\beta \mid t)$ be any of the acceptance regions in above. Boundaries of these regions are functions of $T=T_x+T_y$ which has unknown value, because T_y is unknown. We define the prediction region for T_y by

$$S_i(t_x)=\{\ t_y:\ (t_y,t)\ \varepsilon\ A_i(\beta \mid t)\ \}$$

so that $S_i(t_x)$ can be obtained by solving $A_i(\beta \mid t)$ for T_y, where $T=T_y+T_x$. Assuming z(.) functions in $A_i(\beta \mid t)$, monotone increasing, the prediction intervals will be of the forms

$$S_1(t_x)=(-\infty,U(t_x)]$$
$$S_2(t_x)=[L(t_x),\infty) \qquad (2.2.10)$$
$$S_3(t_x)=[L(t_x),U(t_x)]$$

where, $L(t_x)$ and $U(t_x)$ are functions of t_x ,α and n.

$S_i(t_x)$ are prediction intervals for T_y, but since T_y is a function of single random variable, Y, then one can use simple mathematical operations to get prediction interval for Y.

If z(.) are not monotone, then the acceptance regions will not map necessarily onto prediction intervals of the same forms. In the following we will give a sufficient condition for z(.) to be increasing function of t.

Definition 2.4: The conditional family of distributions of T_y given t, (under H_o), is said to have a monotone likelihood ratio, if the ratio

$$\frac{f_{Ty\,|t2}(t_y)}{f_{Ty\,|t1}(t_y)}$$

is a nondecreasing function of t_y whenever $t_2 > t_1$, for all t_y in the set of existence of $f_{Ty\quad t1}$ and $f_{Ty\quad t2}$.

Lemma 2.6: Suppose the conditional distribution of T_y given t, under H_o, with density function

$$f_{Ty\,|\,t}(t_y) = \frac{h_{tx}(t-t_y)h_{ty}(t_y)}{h_t(t)}$$

has a monotone likelihood ratio, then z(.) are increasing functions of t.

<u>Proof:</u> Without loss of generality we will consider $z(t)$ in $A_1(\beta|t)$. Proofs for the other cases are similar.

$A_1(\beta|t)$ is constructed under H_o: $\theta_x=\theta_y$ $(\beta=\beta_o)$ and since T is a complete sufficient statistic for the joint distribution of (T_y,T) under H_o, then the following conditional probability statements, under H_o, do not depend on θ_x. (See density function 2.2.5.) We use the method of proof given in Theorem (5), Olsen (1974).

$$P_{\beta=\beta o}\{ (T_y,T) \varepsilon A_1(\beta|t) \}=P_{Ty\,|\,t}\{ T_y \leq z(t) \}=1-\alpha \qquad (2.2.11)$$

for all t. The probability associated with $A_1(\beta|t)$ is exactly $1-\alpha$ for it corresponds to a test which has Neyman Structure, Ferguson, (1967).

Using (2.2.11), we have

$$P_{Ty|t1}\{ T_y \leq z(t_1) \}=P_{Ty|t2}\{ T_y \leq z(t_2) \}=1-\alpha \qquad (2.2.12)$$
$$\text{for any } t_1 \text{ and } t_2.$$

Consider t as a parameter of the conditional distribution of T_y given t, (under H_o) and test K_1: $T=t_1$ vs K_2: $T=t_2$, $t_2>t_1$. By Neyman-Person Lemma, the UMP test of size α is

$$\psi=\begin{cases} 1 & \text{if } R>c \\ 0 & \text{if } R\leq c \end{cases}$$

which is equivalent to

$$\Phi = \begin{cases} 1 & \text{if } t_y > k \\ 0 & \text{if } t_y \leq k \end{cases}$$

where
$$R = \frac{f_{Ty \mid t2}(t_y)}{f_{Ty \mid t1}(t_y)}$$

and c , k are constants to make the tests, size α. Using (2.2.11), (2.2.12) and having $E_{t1}(\Phi)=\alpha$ implies

$$\int_{-\infty}^{k} f_{Ty|t1}(t_y)\, dt_y = \int_{-\infty}^{z(t1)} f_{Ty|t1}(t_y)\, dt_y = 1-\alpha,$$

Φ is unique, so $k=z(t_1)$. We also have

$$E_{t2}(\Phi) = \int_{k}^{\infty} f_{Ty|t2}(t_y)\, dt_y = \int_{z(t_1)}^{\infty} f_{Ty|t2}(t_y)\, dt_y \geq E_{t1}(\Phi)$$

which implies

$$P_{Ty|t2}\{ T_y \leq z(t_1) \} \leq P_{Ty|t1}\{ T_y \leq z(t_1) \}$$

and by using (2.2.12), we get

$$P_{Ty|t2}\{ T_y \leq z(t_1) \} \leq P_{Ty|t2}\{ T_y \leq z(t_2) \}.$$

Therefore $z(t_2) \geq z(t_1)$ and $z(t)$ is an increasing function of t.

Using Lemma 2.6 we conclude that z(.) functions are increasing if

$$\frac{h_{tx}(t_2-t_y)}{h_{tx}(t_1-t_y)}$$

is nondecreasing in t_y, whenever $t_2>t_1$.

B. <u>Ancillary Statistic Exists</u>

Suppose the joint distribution of T_y and T under H_o has an ancillary statistic, $V=G(T_y,T)$. According to Lemma 2.2, UMP unbiased acceptance regions for testing H_o versus H_{ai}, i=1,2,3, are found using the density of V under H_o. From Lemma 2.2 we have

$$A_1(\beta)=\{ v: v \leq z \}$$

$$A_2(\beta)=\{ v: v \geq z \}$$

$$A_3(\beta)=\{ v: z_1 \leq v \leq z_2 \}$$

where, z, z_1 and z_2 are constants not depending on t. $V=G(T_y,T)$ in Lemma 2.2 is assumed to be an increasing function of T_y for given t, therefore $G^{-1}(.)$ will be increasing in V for given t.

Since the conditional density of T_y given t, (2.2.8), has a monotone likelihood ratio in t_y, we conclude that the conditional distribution of V given t has also a monotone likelihood ratio. This can be seen easily when $V=a(T)T_y+b(T)$, $a(T)>0$. From (2.2.8) we have

$$f(v|t:\beta)=\frac{\exp[\beta v/a(t)]h_{tx}\{t-(v-b(t))/a(t)\}h_{ty}\{(v-b(t))/a(t)\}}{\int_{-\infty}^{Q(t)}\exp[\beta v/a(t)]h_{tx}\{t-(v-b(t))/a(t)\}h_{ty}\{(v-b(t))/a(t)\}dv}$$

where, $Q(T)=a(T)T+b(T)$.

Using the above argument and Lemma 2.4 we have

$$P_{\beta=\beta 1}\{\ V\leq C_2(t)\ |T=t\ \}\leq P_{\beta=\beta o}\{\ V\leq C_2(t)\ |\ T=t\ \}\quad \text{for all } \beta_1>\beta_o$$

$$P_{\beta=\beta 1}\{\ V\geq C_1(t)\ |T=t\ \}\leq P_{\beta=\beta o}\{\ V\geq C_1(t)\ |\ T=t\ \}\quad \text{for all } \beta_1<\beta_o$$

$$P_{\beta=\beta 1}\{\ C_1(t)\leq\ V\ \leq C_2(t)\ |\ T=t\ \}\leq P_{\beta=\beta o}\{\ C_1(t)\leq\ V\ \leq C_2(t)\ |\ T=t\ \}$$

for all $\beta_1\neq\beta_o$

where, $C_1(t)$ and $C_2(t)$ are functions of t. Therefore the acceptance regions $A_i(\beta)$ have larger probability under H_o than under H_{ai}. Similar to the previous case we define a prediction region for T_y by

$$S_i(t_x)=\{\ t_y:\ v\ \varepsilon\ A_i(\beta)\ \}$$

so that $A_i(\beta)$ are solved for t_y by writing V as a function of T_x and T_y. The prediction intervals will be of the forms

$$S_1(t_x)=(-\infty,U(t_x)]$$
$$S_2(t_x)=[L(t_x),\infty] \qquad (2.2.13)$$
$$S_3(t_x)=[L(t_x),U(t_x)]$$

provided, $G^{-1}(.)$ is an increasing function of t. For example if $V=a(T)T_y+b(T)$, then we need $(V-b(T))/a(T)$ to be increasing in T.

2.3. Unbiased Prediction Intervals and Optimal Properties

Using the acceptance regions we get prediction intervals which are based on the assumption of equal parameters . If the parameters are not the same it seems natural to require the intervals to have small probability of coverage in some sense.

Definition 2.5: Suppose the $P_{\theta x}$ is probability distribution of a random sample, $X_1,\dots,X_n$ and the $P_{\theta y}$ is probability distribution of a random variable Y, which is independent of $\underline{X}=(X_1,\dots,X_n)$.

(1) $(-\infty,U(\underline{x})]$ is said to be a $(1-\alpha)$ level unbiased prediction interval for Y iff

1. $E_{\theta x}[P_{\theta y}\{Y \in (-\infty,U(\underline{X})]\}]=1-\alpha$ for all $\theta_y=\theta_x$
2. $E_{\theta x}[P_{\theta y}\{Y \in (-\infty,U(\underline{X})]\}]\leq 1-\alpha$ for all $\theta_y>\theta_x$

(2) $[L(\underline{x}),\infty)$ is said to be a $(1-\alpha)$ level unbiased prediction interval for Y iff

1. $E_{\theta x}[P_{\theta y}\{Y \in [L(\underline{X}),\infty)\}]=1-\alpha$ for all $\theta_y=\theta_x$
2. $E_{\theta x}[P_{\theta y}\{Y \in [L(\underline{X}),\infty)\}]\leq 1-\alpha$ for all $\theta_y<\theta_x$

(3) $[L(\underline{x}),U(\underline{x})]$ is said to be a $(1-\alpha)$ level unbiased prediction interval for Y iff

1. $E_{\theta x}[P_{\theta y}\{Y \varepsilon [L(\underline{X}),U(\underline{X})]\}]=1-\alpha$ for all $\theta_y=\theta_x$
2. $E_{\theta x}[P_{\theta y}\{Y \varepsilon [L(\underline{X}),U(\underline{X})]\}]\leq 1-\alpha$ for all $\theta_y\neq\theta_x$

where, $L(\underline{X})$, $U(\underline{X})$ are functions of $\underline{X}$, α and n.

For each of the prediction intervals in Definition 2.5 the first condition is the property of a 'similar mean coverage tolerance prediction region' of cover $(1-\alpha)$. (See Definition 2.1.)

<u>Definition 2.6:</u> $(-\infty,U(\underline{x})]$ is said to be $(1-\alpha)$ level uniformly most accurate unbiased prediction interval for Y, if it is $(1-\alpha)$ level unbiased prediction interval and for any other $(1-\alpha)$ level unbiased prediction interval, $S^*(\underline{x})$ for Y,

$$E_{\theta x}[P_{\theta y}\{Y \varepsilon (-\infty,U(\underline{X})]\}] \leq E_{\theta x}[P_{\theta y}\{Y \varepsilon S^*(\underline{X})\}]$$

for all $\theta_y>\theta_x$. Similar definitions hold for the other cases.

Optimal property of the prediction intervals can be stated as follows. Let $S(\underline{X})$ be a prediction interval for Y and assume it is constructed under the assumption of equal parameters, θ_x and θ_y , so that it has more probability coverage under such assumption. It is natural to define $P_{\theta y}\{Y \varepsilon S(\underline{X})\}$ as a measure of undesirability when in fact the parameters are unequal in some direction. We would like to have a prediction interval which minimizes the measure. $\underline{X}=(X_1,\dots,X_n)$ is random and therefore $P_{\theta y}\{Y \varepsilon S(\underline{X})\}$ is also random and it is reasonable to minimize the expected value of undesirability, $E_{\theta x}[P_{\theta y}\{Y \varepsilon S(\underline{X})\}]$.

<u>Theorem 2.3</u>: Assume distributions of X (Y) have monotone likelihood ratios, so that the conditional distribution of T_y given T has also a monotone likelihood ratio. Consider prediction intervals in (2.2.10), obtained from UMP unbiased acceptance regions $A_i(\beta|t)$. These intervals minimize the corresponding measures of undesirability.

<u>Proof:</u> Without loss of generality we will establish optimal property of $S_1(t_x)=(-\infty,U(t_x)]$. Proofs for the other cases are similar. Let Φ_1^* be any unbiased size α test for testing H_o versus H_{a1}, with $(1-\alpha)$ level unbiased acceptance region, $A_1^*(\beta|t)$ and $(1-\alpha)$ level unbiased prediction interval $S_1^*(t_x)$.

$\Phi_1^*(t_y,t)$ being unbiased test of size α, implies that it is α similar on the boundary set and

$$E_{\theta x,\beta}[\ \Phi_1^*(T_y,T)\ ]=\alpha \qquad \text{for all } (\beta,\theta_x)$$

in the boundary set , Ferguson (1967) .

Power of Φ_1^* is maximized if and only if

$$\max_{\Phi_1^*} \quad E_{\theta x,\beta}[\,\Phi_1^*(T_y,T)\,] \qquad \text{for } \beta>\beta_o \text{ and all } \theta_x$$

$$\text{subject to} \quad E_{(\theta,\beta)}[\,\Phi_1^*(T_y,T)\,]=\alpha\ ,$$

$$\text{for } \beta=\beta_o \text{ and all } \theta=(\theta_x=\theta_y).$$

Since T is a complete sufficient statistic for θ_x under H_o and Φ_1^* is a similar on the boundary set, then Φ_1^* has Neyman Structure, Ferguson (1967).

that is

$$E_{\theta x,\beta}[\,\Phi_1^*(T_y,T)\,]=$$

$$E_{\theta x}[E_\beta\ [\,\Phi_1^*(T_y,T)\mid T\,]]=\alpha$$

for all (β,θ_x) in the boundary set.

Therefore the maximization problem is equivalent to

$$\max_{\Phi 1^*} \quad E_{\theta x}[E_\beta\ [\,\Phi_1^*(T_y,T)\mid T\,]], \quad \text{for } \beta>\beta_o \text{ and all } \theta_x$$

$$\text{subject to} \quad E_\theta[E_\beta\ [\,\Phi_1^*(T_y,T)\mid T\,]]=\alpha,$$

$$\text{for } \beta=\beta_o \text{ and all } \theta=(\theta_x=\theta_y)$$

or

$$\min_{A_1^*(\beta\mid t)} \quad E_{\theta x}[P_\beta\{\,T_y \in A_1^*(\beta\mid t)\,\}] \quad \text{for } \beta>\beta_o \text{ and all } \theta_x$$

$$\text{subject to} \quad E_\theta[P_\beta\{\,T_y \in A_1^*(\beta\mid t)\,\}]=1-\alpha$$

$$\text{for } \beta=\beta_o \text{ and all } \theta=(\theta_x=\theta_y)$$

but $\qquad (t_y,t) \in A_1^*(\beta\mid t) \quad \text{iff} \quad t_y \in S_1^*(t_x)$

and the maximization problem is equivalent to

$$\min_{S_1^*(t_x)} \quad E_{\theta x}[P_{\theta y}[S_1^*(T_x)]] \quad \text{for all } \theta_y > \theta_x \qquad (2.3.1)$$

$$\text{subject to} \quad E_\theta[P_\theta[S_1^*(T_x)]]=1-\alpha \quad \text{for all } \theta=(\theta_x=\theta_y) \ .$$

By Theorem 2.2 we know $\Phi_1(t_y,t)$ is the UMP unbiased for testing H_0 versus H_{a1}, so using $\Phi_1(t_y,t)$ and its corresponding regions in the above equations, we conclude that $S_1(t_x)$ minimizes the expected value of measure of undesirability. By (2.3.1) and unbiasedness property of $S_1^*(t_x)$ we have $E_{\theta x}[P_{\theta y}\{S_1(T_x)\}]\leq 1-\alpha$ for all $\theta_y > \theta_x$, so $S_1(t_x)$ is unbiased prediction interval and has the optimal property.

A similar argument to Theorem 2.3. holds for the case that an ancillary statistic for joint distribution of (T_y,T) under H_0 exists. The prediction intervals in (2.2.13) are also the $(1-\alpha)$ level UMAU prediction intervals for T_y.

III. EXAMPLES: ONE PARAMETER CASE

3.1. The Normal Distribution With Known Mean

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with the density function

$$f(x:\mu,\sigma_x^2)=(2\pi\sigma_x^2)^{-1/2}\ \exp[-(2\sigma_x^2)^{-1}\ (x-\mu)^2]$$

and Y is independent of the X_i, $(i=1,2,\ldots,n)$, with the density function

$$f(y:\mu,\sigma_y^2)=(2\pi\sigma_y^2)^{-1/2}\ \exp[-(2\sigma_y^2)^{-1}(y-\mu)^2]$$

where μ is assumed to have a known value and $\sigma_x>0$, $\sigma_y>0$.

$\sum_{i=1}^n X_i^2$ is a sufficient statistic for the joint distribution of $X_1,\ldots,X_n$ and Y^2 is a sufficient statistic for the distribution of Y . Without loss of generality assume that $\mu=0$.

Using $\theta_x=-(2\pi\sigma_x^2)^{-1}$, $\theta_y=-(2\pi\sigma_y^2)^{-1}$, $\beta=\theta_y-\theta_x$, $T_x=\sum_{i=1}^n X_i^2$, $T_y=Y^2$ and $T=T_x+T_y$ we have

$$f(t_y\ t:\beta)=\frac{\exp[\beta t_y](t-t_y)^{n/2\ -1}\ (t_y)^{-1/2}}{\int_0^t \exp[\beta t_y](t-t_y)^{n/2\ -1}\ (t_y)^{-1/2}\ dt_y}\ , \quad t_y\leq t$$

which has a monotone likelihood ratio in t_y.

Under $H_o:\sigma_x=\sigma_y$, $V=G(T_y,T)=a(T)T_y$, $a(T)=1/T$, is distributed according to Beta(1/2,n/2) distribution, Bickel and Doksum (1977). Therefore V is an ancillary statistic for the joint distribution of (T_y,T) under H_o and we can use Lemma 2.2 instead of Theorem 2.2 . We have

(1) $\beta=\beta_o$ iff $\sigma_y=\sigma_x$

$\beta>(<)\beta_o$ iff $\sigma_y>(<)\sigma_x$, $\beta_o=0$.

(2) V is increasing in t_y for each t. Therefore the conditional distribution of V given t with parameter β, has also a monotone likelihood ratio.

(3) V/a(T) is increasing in T. Therefore acceptance regions will map onto the same form of prediction intervals. (See section 2.2.3 (B).)

<u>One Sided Upper Limit Prediction Interval</u>

(1), (2) and (3) imply that the acceptance region $A_1(\beta)=\{ v: v\leq z \}$ must be used where z , according to Lemma 2.2, is found from

$$\int_z^1 f_{Ho}(v)dv=\alpha \quad , \quad f_{Ho}(v)=\frac{\pi((n+1)/2)}{\pi(n/2)\pi(1/2)} v^{-1/2}(1-v)^{n/2-1}$$

$0 \leq v \leq 1$, $\pi(k) = \int_0^\infty x^{k-1} e^{-x} dx$ for real number, k and $\pi(k) = (k-1)!$ for integer $k > 0$.

By using $A_1(\beta)$ we get

$$S_1(t_x) = [0, t_x z/(1-z)]$$

which is the $(1-\alpha)$ level uniformly most accurate unbiased (UMAU) upper limit prediction interval for T_y. Since T_y/T_x has F-distribution under H_o , then $nz/(1-z)$ is $(1-\alpha)$ percentile of F-distribution with (1,n) degrees of freedom.

One Sided Lower Limit Prediction Interval

A similar argument to the previous one holds and we use $A_2(\beta) = \{ v: v \geq z \}$ where z is found from

$$\int_0^z f_{Ho}(v) dv = \alpha$$

By solving $A_2(\beta)$, the $(1-\alpha)$ level UMAU lower limit prediction interval for T_y is $S_2(t_x) = [t_x z/(1-z), \infty)$. $nz/(1-z)$ is α percentile of F-distribution with (1,n) degrees of freedom.

Two Sided Prediction Interval

According to (1), (2) and (3), we need to use the acceptance region $A_3(\beta)=\{ v: z_1 \leq v \leq z_2 \}$ where z_1 and z_2 are found from

$$\int_{z_2}^{z_1} f_{Ho}(v)dv=1-\alpha \quad \text{and} \quad \int_{z_2}^{z_1} vf_{Ho}(v)dv=(1-\alpha) \int_{0}^{1} vf_{Ho}(v)dv \quad (3.1.1)$$

It is easier to obtain equal-tailed prediction interval. That is, assume the distribution of V is symmetric about some point when H_o is true. In this case second equation in (3.1.1) is not needed (See the remark in section 2.2). To get right values of z_1 and z_2, in this case we need to solve the two equations in (3.1.1), for Beta(1/2,n/2) is not a symmetric distribution. z_1 and z_2 can not be found in a closed form so an approximation method is needed.

The Generalized Newton's Method

Consider a system of two equations

$$\begin{cases} M(a,b)=0 \\ K(a,b)=0 \end{cases}$$

where, M(.,.) and K(.,.) are some functions of a and b. We will use Young and Gregory (1972), to explain the 'Generalized Newton's Method' by which values of a and b for the above equations can be found approximately. We find

$$M_{ao}= \frac{\partial\ M(a,b)}{\partial\ a} \Big|_{a=a_o}$$

$$M_{bo}= \frac{\partial\ M(a,b)}{\partial\ b} \Big|_{b=b_o}$$

$$K_{ao}= \frac{\partial\ K(a,b)}{\partial\ a} \Big|_{a=a_o}$$

$$K_{bo}= \frac{\partial\ K(a,b)}{\partial\ b} \Big|_{b=b_o}$$

where a_o and b_o are initial values for a and b , respectively. If $M(a_o,b_o)\simeq 0$ and $K(a_o,b_o)\simeq 0$, then a_o and b_o are the right values of a and b. Otherwise we find

$$J=K_{ao}\ M_{bo} - M_{ao}\ K_{bo}$$

$$a_1=a_o + (\ M(a_o,b_o)K_{bo} - K(a_o,b_o)M_{bo}\ )/J$$

$$b_1=b_o + (\ K(a_o,b_o)M_{ao} - M(a_o,b_o)K_{ao}\ )/J\ .$$

If $M(a_1,b_1)\simeq 0$ and $K(a_1,b_1)\simeq 0$, then a_1 and b_1 are the right values of a and b. Otherwise we let $a_o=a_1$ and $b_o=b_1$ and repeat the above steps until roots of the equations can be found.

Consider the following facts about the Beta distribution.

<u>Fact I</u>: Suppose a random variable, X, is distributed according to Beta(r,s), s>0, r>0, $0 \leq x \leq 1$. Then

$$(1) \quad f(x) = \frac{\pi(r+s)}{\pi(r)\ \pi(s)}\ x^{r-1}(1-x)^{s-1}$$

$$(2) \quad xf(x) = \frac{r}{r+s} f(y) \quad \text{where, Y} \quad \text{Beta}(r+1,s)$$

$$(3) \quad E[X] = \frac{r}{r+s} \quad .$$

<u>Fact II.</u> Let $X \sim \text{Beta}(r,s)$, $0 \leq X \leq 1$, r>0, s>0 ,

$$M(a,b) = \int_a^b \frac{\pi(r+s)}{\pi(r)\ \pi(s)}\ x^{r-1}(1-x)^{s-1}dx - (1-\alpha) = 0$$

and

$$K(a,b) = \int_a^b \frac{\pi(r+s+1)}{\pi(r+1)\ \pi(s)}\ x^{r}(1-x)^{s-1}dx - (1-\alpha) = 0 \ . \quad \text{Then}$$

$$M_a = \frac{\partial[M(a,b)]}{\partial a} = - \frac{\pi(r+s)}{\pi(r)\ \pi(s)}\ a^{r-1}(1-a)^{s-1}$$

$$M_b = \frac{\pi(r+s)}{\pi(r)\ \pi(s)}\ b^{r-1}(1-b)^{s-1}$$

$$K_a = - \frac{\pi(r+s+1)}{\pi(r+1)\ \pi(s)}\ a^{r}(1-a)^{s-1}$$

and $$K_b = \frac{\pi(r+s+1)}{\pi(r+1)\ \pi(s)}\ b^r(1-b)^{s-1} .$$

Using Fact I equations in (3.1.1) can be written as

$$\int_{z_1}^{z_2} \frac{\pi(n/2+1/2)}{\pi(1/2)\ \pi(n/2)}\ v^{-1/2}(1-v)^{n/2\ -1}dv=1-\alpha \qquad (3.1.2)$$

and

$$\int_{z_1}^{z_2} \frac{\pi(n/2+3/2)}{\pi(3/2)\ \pi(n/2)}\ v^{3/2\ -1}(1-v)^{n/2\ -1}dv=1-\alpha ,$$

respectively.

We apply Fact II and the Generalized Newton's Method along with MDBETA routine in IMSL library to obtain approximate values of z_1 and z_2 in (3.1.2) . Values of z_1 and z_2 are given in table A. Solving $A_3(\beta)$ for t_y, we get

$$S_3(t_x)=[t_x z_1/(1-z_1),t_x z_2/(1-z_2)] \quad .$$

Note that when μ is not zero prediction intervals for $(Y-\mu)^2$ are

$$S_1(t_x)=[0,t_x^* z/(1-z)]$$
$$S_2(t_x)=[t_x^* z/(1-z),\infty)$$
$$S_3(t_x)=[t_x^* z_1/(1-z_1),t_x^* z_2/(1-z_2)]$$

where, $t_x^* = \sum_{i=1}^{n} (x_i-\mu)^2$ and z , z_1 , z_2 are defined as before for each case.

Remark: $X \sim$ Normal (μ,σ^2) iff $\exp(X) \sim$ Lognormal (μ,σ^2).

Using results of Example 3.1 and the relationship between Normal and Lognormal random variables, the prediction intervals can be obtained when sampling is from Lognormal (μ,σ^2) distribution with known μ .

Numerical Example:

Each run of a process produces a large batch of ball bearings whose diameter (mm) are normally distributed with known mean, $\mu=8$ (mm) and unknown variance, σ^2 . From a particular batch a sample of 15 ball bearings is chosen at random and their diameter are found to be

8.07 , 8.15 , 8.06 , 7.79 , 7.85 , 8.02 , 8.07 ,
8.17 , 8.11 , 8.09 , 7.96 , 9.02 , 8.20 , 7.97 , 8.12.

Suppose the process is to be run in future. For a randomly selected ball bearing from a batch, find .90 level, one and two sided prediction limits for its squared deviation of diameter from $\mu=8$ (mm) .

According to the notation used in Example 3.1, we have

$$t_x^* = \sum_{i=1}^{n} (x_i-\mu)^2 = 1.2061 \ , \ n=15 \ .$$

Upper Limit Prediction:

$$S_1(t_x)=[0, t_x^* z/(1-z)]=[\ 0\ ,\ 0.246848]$$

where, $nz/(1-z) = 3.07$ is 90^{th} percentile of F-distribution with (1,15) degrees of freedom.

Lower Limit Prediction:

$$S_1(t_x)=[t_x^* z/(1-z), \infty)=[0.00131\ ,\ \infty)$$

where $nz/(1-z) = 0.0163$ is 10^{th} percentile of F-distribution with (1,15) degrees of freedom.

Two sided Prediction:

$$S_3(t_x)=[t_x^* z_1/(1-z_1), t_x^* z_2/(1-z_2)]=[0.000962\ ,\ 0.601519]$$

where $z_1=0.000797$ and $z_2=0.332769$ are found from Table A for $n=15$ and $\alpha=.10$.

3.2. The Normal Distribution With Known Variance

Let $X_1,\ldots,X_n$ be a random sample from a normal distribution with the density function

$$f(x:\theta_x,\sigma)=(2\pi\sigma^2)^{-1/2} \exp[-1/2\sigma^2\ (x-\theta_x)^2],$$

and Y be independent of the X_i, (i=1,2,...,n) with the density function

$$f(y:\theta_y,\sigma)=(2\pi\sigma^2)^{-1/2}\exp[-1/2\sigma^2\ (y-\theta_y)^2]$$

where σ is known and $X \varepsilon R$, $Y \varepsilon R$.

$T_x=\sum_{i=1}^n X_i$ and $T_y=Y$ are sufficient statistics for the distributions of $(X_1,\ldots,X_n)$ and Y, respectively. Without loss of generality we assume, $\sigma=1$.

$$f(t_y,t_x:\theta_x,\theta_y)=(n^{-1/2}/2\pi)\exp[(-1/2)(n\theta_x^2+\theta_y^2)] \qquad (3.2.1)$$
$$\exp[\theta_x t_x+\theta_y t_y]\exp[(-1/2)(t_x^2/n+t_y^2)]$$

Let $\beta=(\theta_y-\theta_x)/(1/n+1)$, $\gamma=(n\theta_x+\theta_y)/(n+1)$, $U=T_y-T_x/n$ and $T=T_x+T_y$. Then (3.2.1) implies

$$f(u,t:\beta,\gamma)=C(\beta,\gamma)\exp[\beta u+\gamma t]h(u,t) \qquad (3.2.2)$$

where $C(\beta,\gamma)$ and $h(u,t)$ can be obtained from (3.2.1). But we will not need these for deriving the prediction intervals. U is increasing in T_y and is being used for simplicity. The statistic, V defined by

$$V=[(n+1)/n]^{-1/2}\ U$$
$$=[(n+1)/n]^{-1/2}\ T_y-[n(n+1)]^{-1/2}\ T$$
$$=a(T)T_y+b(T)$$

is increasing in T_y for each t and under $H_o:\theta_y=\theta_x$ has standard normal distribution. Therefore V is an ancillary statistic and Lemma 2.2 will be used instead of Theorem 2.2. We have

(1) $$\beta=\beta_o \quad \text{iff} \quad \theta_y=\theta_x$$
$$\beta>(<)\beta_o \quad \text{iff} \quad \theta_y>(<)\theta_x \ , \ \beta_o=0 \ .$$

(2) V is increasing in T_y for each t. Therefore the conditional distribution of V given t with parameter β, has a monotone likelihood ratio. (See section 2.2.3 (B).)

(3) [V-b(T)]/a(T) is an increasing function of T. Therefore acceptance regions will map onto the same form of prediction intervals. (See section 2.2.3 (B).)

(4) The distribution of V under H_o is symmetric about zero. Hence the remark in section 2.2.1 applies.

<u>One Sided Upper Limit Prediction Interval</u>

(1), (2) and (3) imply that $A_1(\beta)=\{v : v\leq z\}$ must be used, where according to Lemma 2.2, z is found from $\int_{-\infty}^{z} f_{Ho}(v)dv=1-\alpha$. Having the value of z from the table of the standard normal distribution and using $A_1(\beta)$, the (1-α) level upper limit UMAU prediction interval for Y is

$$S_1(t_x)=(\ -\infty\ ,(t_x/n)+z(1/n\ +1)^{1/2}].$$

One Sided Lower Limit Prediction Interval

A similar argument to the above one holds. Using $A_2(\beta)\{ v: v \geq z \}$, the (1-α) level lower limit UMAU prediction interval for Y is $S_2(t_x)=[(t_x/n)-z(1/n +1)^{1/2}, \infty)$, where z is a percentile of the standard normal distribution.

Two Sided Prediction Interval

According to (1), (2) and (3), $A_3(\beta)=\{ v: z_1 \leq v \leq z_2 \}$ must be used. From property (4) we have, $z_2=-z_1$ where z_2 is the (1-α/2) percentile of the standard normal distribution. (1-α) level UMAU prediction interval for Y is

$$S_3(t_x)=[(t_x/n)+z_1(1/n +1)^{1/2}),(t_x/n)+z_2(1/n +1)^{1/2}].$$

Remark I. If $\sigma \neq 1$, then

$$S_1(t_x)=(-\infty ,(t_x/n)+\sigma z(1/n +1)^{1/2}]$$
$$S_2(t_x)=[(t_x/n)+\sigma z(1/n +1)^{1/2}, \infty)$$
$$S_3(t_x)=[(t_x/n)+\sigma z_1(1/n +1)^{1/2},(t_x/n)+\sigma z_2(1/n +1)^{1/2}]$$

where z, z_1 and z_2 are defined as before for each case.

Remark II: $X \sim$ Normal (μ,σ^2) iff $\exp(X) \sim$ Lognormal (μ,σ^2)

Usingresultsof Example 3.2 and the relationship between Normal and Lognormal random variables, the prediction intervals can be obtained when sampling is from Lognormal (μ,σ^2) distribution with known σ .

3.3. The Negative Exponential Distribution

Suppose $X_1,\dots,X_n$ is a random sample from a Negative Exponential distribution with the density function

$$f(x:\gamma_x)=1/\gamma_x \exp[-x/\gamma_x] \ , \ \gamma_x>0, \ x>0$$

and Y is independent of the X_i, $(i=1,2,\dots,n)$ with the density function

$$f(y:\gamma_y)=1/\gamma_y \exp[-y/\gamma_y] \ , \ \gamma_y>0, \ y>0 \quad .$$

The Negative Exponential family with the above density functions is a monotone likelihood ratio family.

$(T_x=\sum_{i=1}^{n} X_i, \ T_y=Y)$ is a sufficient statistics for the joint distribution of $(\underline{X},Y)$. Let $\theta_x=-1/\gamma_x$, $\theta_y=-1/\gamma_y$, $\beta=(\theta_y-\theta_x)$ and $T=T_x+T_y$. We have,

$$f(t_y \ t:\beta)= \frac{\exp[\beta t_y](t-t_y)^{n-1}}{\int_0^t \exp[\beta t_y](t-t_y)^{n-1}\, dt_y} \quad , \ t_y\leq t \ .$$

Under null hypothesis, H_o: $\gamma_y=\gamma_x$, $V=G(T_y,T)=a(T)/T$, $a(T)=1/T$, has Beta(1,n) distribution, Bickel and Doksum (1977).

Therefore V is an ancillary statistic for the joint distribution of (T_y,T) under H_o and we can use Lemma 2.2 instead of Theorem 2.2 . We have

(1) $\beta=\beta_o$ iff $\gamma_y=\gamma_x$ iff $\theta_y=\theta_x$

$\beta>(<)\beta_o$ iff $\gamma_y>(<)\gamma_x$ iff $\theta_y>(<)\theta_x$, $\beta_o=0$.

(2) V is increasing in T_y for each T. Therefore the conditional distribution of V given t with parameter β, has a monotone likelihood ratio. (See section 2.2.3 (B).)

(3) V/a(T) is an increasing function of t. Therefore acceptance regions will map onto the same form of prediction intervals. (See section 2.2.3 (B).)

<u>One Sided Upper Limit Prediction Interval</u>

(1), (2) and (3) imply that $A_1(\beta)=\{v : v\leq z\}$ must be used. According to the Lemma 2.2, z is found from $\int_{-\infty}^{z} f_{Ho}(v)dv=1-\alpha$, where $f_{Ho}(v)$ is density of Beta(1,n). Using $A_1(\beta)$, the $(1-\alpha)$ level upper limit UMAU prediction interval for Y is

$$S_1(t_x)=(-\infty,t_xz/(1-z)].$$

Since nT_y/T_x under H_o has F-distribution with (2,2n) degrees of freedom, then $nz/(1-z)$ is $(1-\alpha)$ percentile of F-distribution with (2,2n) degrees of freedom.

One Sided Lower Limit Prediction Interval

A similar argument to the above one holds. Using $A_2(\beta)=\{ v: v \geq z \}$, the $(1-\alpha)$ level lower limit UMAU prediction interval for Y is $S_2(t_x)=[t_x z/(1-z), \infty)$. $nz/(1-z)$ is a percentile of F-distribution with (2,2n) degrees of freedom.

Two Sided Prediction Interval

According to (1), (2) and (3) we need to use the acceptance region $A_3(\beta)=\{ v: z_1 \leq v \leq z_2 \}$. Using Lemma 2.2, z_1 and z_2 are found from

$$\int_{z_2}^{z_1} f_{Ho}(v)dv=1-\alpha \quad \text{and} \quad \int_{z_2}^{z_1} vf_{Ho}(v)dv=(1-\alpha)\ E_{Ho}(V) \qquad (3.3.1)$$

where $E_{Ho}(V)=1/(n+1)$.

Equations in (3.3.1) can be written as

$$(1-z_1)^n - (1-z_2)^n = (1-\alpha)$$

and $$z_1(1-z_1)^n - z_2(1-z_2)^n = 0 \qquad (3.3.2)$$

respectively.

However it is impossible to solve the equations in (3.3.2) and find z_1 and z_2 in terms of n and α, in closed forms. Hence an approximation method like the 'Generalized Newton's Method' given in Example 3.1 must be used. According to the notation used in Example 3.1 we have

$$M(a,b):\ (1-a)^n-(1-b)^n-(1-\alpha)=0$$

$$K(a,b):\ a(1-a)^n-b(1-b)^n=0$$

$$M_a=\frac{\partial[M(a,b)]}{\partial a}=-n(1-a)^{n-1}$$

$$M_b=\frac{\partial[M(a,b)]}{\partial b}=n(1-b)^{n-1}$$

$$K_a=\frac{\partial[K(a,b)]}{\partial a}=(1-a)^{n-1}[(1-a)-na]$$

$$K_b=\frac{\partial[K(a,b)]}{\partial b}=(1-b)^{n-1}[nb-(1-b)]\ . \tag{3.3.3}$$

Using the equations (3.3.3), a program is written for the Generalized Newton's Method and values of z_1, z_2 for the equations (3.3.2) are listed in Table B. Using the acceptance region $A_3(\beta)$, the $(1-\alpha)$ level UMAU two sided prediction interval for T_y is

$$S_3(t_x)=[t_x z_1/(1-z_1),t_x z_2/(1-z_2)]\ .$$

A Numerical Example:

The time (hours) of first failure of an electrical device is assumed to be distributed according to the Negative Exponential distribution. 15 of this particular device are selected from production line at random and the times to first failure of each are found to be

62 , 74 , 19 , 18 , 209 , 409 , 57 , 46 ,
13 , 29 , 231 , 46 , 5 , 25 .

If we are to select another device of the same type, a natural question to be ask is concerned with the maximum time period that the device can work before it fails. Suppose an answer to the question is to be given with .90 level confidence.

According to our notation in Example 3.3,

$$t_x = \sum_{i=1}^{n} x_i = 1243 \quad , \quad n=15$$

and the upper limit prediction interval is

$$S_1(t_x) = [0 \ , \ t_x z/(1-z)] = [0 \ , \ 206.338].$$

where $nz/(1-z)=2.49$ is 90^{th} percentile of F-distribution with (2,30) degrees of freedom. Therefore based on the 15 previously observed values, we predict with .90 confidence that the device might work 206.23 hours before it fails.

If the question is how early a randomly selected device can fail, then we need to find a lower limit prediction interval

$$S_2(t_x)=[t_x z/(1-z),\ \infty)=[8.759\ ,\ \infty\ )$$

where $nz/(1-z)=0.1057$ is 10^{th} percentile of F-distribution with (2,30) degrees of freedom.

For two sided prediction interval, we have

$$S_3(t_x)=[t_x z_1/(1-z_1), t_x z_2/(1-z_2)]=[6.7335\ ,\ 359.3905]$$

where, $z_1=0.005388$ and $z_2=0.224284$ for $n=15$ and $\alpha=.10$ are obtained from Table B.

The above prediction intervals are based on assumption that quality of production of the device remains unchanged over the time. That is to say, the probability distribution associated with the time of failure of the device is exponential with fixed unknown parameter over the time.

3.4. The Weibull Distribution With Known Shape Parameter

Let $X_1,\ldots,X_n$ be random sample from a Weibull distribution with the density function

$$f(x:\mu,\gamma)=(\gamma/\mu)\ x^{\gamma-1}\ \exp[-x^{\gamma}/\mu] \quad \mu>0\ ,\ \gamma>0,\ x>0 \quad (3.4.1)$$

The shape parameter, γ is assumed to be known. The density function in (3.4.1) is a one parameter exponential family and has a monotone likelihood ratio in x.

Define $W=X^{\gamma}$. Then

$$f(w:\mu)=(1/\mu)\exp[-w/\mu] \;,\; \mu>0,\; w>0$$

which is the density function of a Negative Exponential distribution. Using the results of Example 3.3 the prediction intervals for a future outcome based on observed values of $X_1,\ldots,X_n$ are

$$S_2(t_x)=[0,\ \{t_x z/(1-z)\}^{1/\gamma}]$$
$$S_1(t_x)=[\{t_x z/(1-z)\}^{1/\gamma},\infty)$$
$$S_3(t_x)=[(t_x z_1/(1-z_1))^{1/\gamma},(t_x z_2/(1-z_2))^{1/\gamma}]$$

$t_x=\sum_{i=1}^{n} x_i^{\gamma}$ and z, z_1 and z_2 are defined as in the Example 3.3.

IV. GENERALIZATION OF THE PREDICTION PROBLEM TO K-PARAMETER CASE

In this Chapter we will generalize the method of obtaining prediction intervals when the distributions of informative and future experiments are from a K-parameter exponential family.

4.1. Formulation Of The Problem In Terms Of Hypothesis Testing

Let $X_1,\ldots,X_n$ be (iid) random outcomes of an informative experiment with the density function

$$f(x:\theta_x)=C(\theta_x)\exp[\theta_x' T(x)]h(x)$$

where T(X) and θ_x are K-dimensional real-valued vectors. Suppose Y is as yet an unknown outcome of a corresponding future experiment, independent of the X_i,(i=1,2,...,n) and with the density function

$$f(y:\theta_y)=C(\theta_y)\exp[\theta_y' T(y)]h(y)$$

where θ_y is also a K-dimensional real-valued vector.

Since Y is a single random variable then all components of θ_y are not identifiable. We will assume

$$\theta_y'=(\theta_{1y},\gamma'), \quad \theta_x'=(\theta_{1x},\gamma') \quad \text{and} \quad \gamma'=(\theta_2,\ldots,\theta_k) .$$

Hypotheses which will be considered are

$$H_o:\ \theta_{1x}=\theta_{1y} \quad \text{vs} \quad H_{a1}:\ \theta_{1y}>\theta_{1x}$$
$$H_o:\ \theta_{1x}=\theta_{1y} \quad \text{vs} \quad H_{a2}:\ \theta_{1y}>\theta_{1x} \qquad (4.1.1)$$
$$H_o:\ \theta_{1x}=\theta_{1y} \quad \text{vs} \quad H_{a3}:\ \theta_{1y}\neq\theta_{1x}\ .$$

The joint density function of $(\underline{X},Y)$ is

$$f(\underline{x},y:\theta_x,\theta_y)=C^n(\theta_x)C(\theta_y)\exp[\theta_{1x}\sum_{i=1}^{n} T_1(x_i)+\theta_{1y}T_1(y)]$$
$$\exp[\sum_{j=2}^{k} \theta_j(\sum_{i=1}^{n} T_j(x_i)+T_j(y))]\pi_{i=1}^{n} h(x_i)h(y). \quad (4.1.2)$$

Let $T_{jx}=\sum_{i=1}^{n} T_j(x_i)$, $T_{jy}=T_j(y)$, $T_j=T_{jx}+T_{jy}$, $T_x=(T_{1x},\ldots,T_{kx})$, $T_y=(T_{1y},\ldots,T_{ky})$, $T=(T_1,T_2,\ldots,T_k)$ and $\beta=\theta_{1y}-\theta_{1x}$, $(j=2,\ldots,k)$, then the density in (4.1.2) can be written as

$$f(\underline{x},y:\beta,\gamma,\theta_{1x})=C^n(\theta_{1x},\gamma)C(\beta+\theta_{1x},\gamma)\exp[\beta T_{1y}+\theta_{1x}T_1] \qquad (4.1.3)$$
$$\exp[\sum_{j=2}^{k} \theta_j T_j]\ \pi_{i=1}^{n} h(x_i)h(y)$$

which is the density function of a K+1 parameter exponential family. By factorization Theorem, Lehmann (1959), $(T_{1y},T_1,\ldots,T_k)$ are jointly sufficient statistics for $(\beta,\theta_{1x},\gamma)$. Similar to Chapter II, we reduce the problem to the consideration of the sufficient statistics. $T=(T_1,T_2,\ldots,T_k)$ is a complete and sufficient statistic for (θ_x,γ) when H_o is true. Thus for testing the hypothese we need the conditional distribution of T_{1y} given t.

According to Lehmann (1959), we have

$$f(t_{1y}:\theta_y)=C_{t1y}(\theta_y)\exp[\theta_{1y}t_{1y}]h_{t1y}(t_{1y})$$
$$f(t_x:\theta_x)=C_{tx}(\theta_x)\exp[\theta_{1x}t_{1x}+\textstyle\sum_{j=2}^{k}\theta_j t_{jx}]h_{tx}(t_x) \qquad (4.1.4)$$

which implies

$$f(t_{1y}:\theta_y)f(t_x:\theta_x)=f(t_{1y},t_1,t_{2x},\ldots,t_{kx}:\beta,\theta_{1x},\gamma) \qquad (4.1.5)$$
$$=C_{t1y}(\beta+\theta_{1x},\gamma)C_{tx}(\theta_x)\exp[\beta t_{1y}+\theta_{1x}t_1+\textstyle\sum_{j=2}^{k}\theta_j t_{jx}]$$
$$h_{tx}(t_1-t_{1y},t_{2x},\ldots,t_{kx}))h_{t1y}(t_{1y}) \ .$$

Using (4.1.5), the conditional density of T_{1y} given t can be written as

$$f(t_{1y} \quad t:\beta)= \frac{\exp[\beta t_{1y}]h_{tx}(t_1-t_{1y},w_2,\ldots,w_k))h_{t1y}(t_{1y})}{\int_{-\infty}^{t_1}\exp[\beta t_{1y}]h_{tx}(t_1-t_{1y},w_2,\ldots,w_k)h_{t1y}(t_{1y})dt_{1y}} \qquad (4.1.6)$$

where $w_j=t_j - K_j(t_{1y})$, $t_{jk}=K_j(t_{1y})$, $j=2,3,\ldots,k$, for some functions $K_j(.)$, because Y is a single random variable and all components of $T_y=(T_{1y},\ldots,T_{ky})$ can be obtained from T_{1y} .

The conditional density function in (4.1.6) is the density function of a one parameter exponential family and therefore has a monotone likelihood ratio in T_{1y} . (See Section 2.2.2.)

When an exponential family has more than one parameter, it is difficult and sometimes impossible to find $h_{tx}(.)$ for the density function of t_x, (4.1.4).

Saddle Point Approximation, Barndorff-Nielsen (1983)

Let $X_1,\ldots,X_n$ be (iid) random variables with the density function

$$f(x:\theta)=C(\theta)\exp[\theta' T(x)]h(x)$$

where $\theta : \Theta \longrightarrow R^k \qquad T : \mathcal{X} \longrightarrow R^k$

$h : \mathcal{X} \longrightarrow R^k \qquad C : \Theta \longrightarrow (0,\infty).$

The 'Saddle Point Approximation' to the density function of $T_x = \sum_{i=1}^{n} T(X_i)$ is

$$f(t_x:\theta) \simeq (2\pi)^{-k/2} \, |i(\hat{\theta})|^{-1/2} \, \exp[(\theta-\hat{\theta})t_x] \, [C(\theta)C^{-1}(\hat{\theta})]^n \qquad (4.1.7)$$

where $\hat{\theta}$ = Maximum Likelihood Estimator of θ

and $$i(\hat{\theta}) = \frac{-\partial^2 \operatorname{Log} f(\underline{x}:\theta)}{\partial^2 \theta} \Big|_{\theta=\hat{\theta}}$$

is the observed Fisher Information Matrix. The order of the approximation is $O(n^{-1})$.

According to the Saddle-Point Approximation, $h_{tx}(.)$ in (4.1.4) is approximated by

$$|i(\hat{\theta}_x)|^{-1/2} \exp[(-\hat{\theta}_x)t_x]C^{-n}(\hat{\theta}_x)$$

where, $\hat{\theta}_x$ is the Maximum Likelihood Estimator of θ_x based on $X_1,\ldots,X_n$.

EXAMPLE I. The Negative Exponential Distribution

Let $X_1,\ldots,X_n$ be a random sample with the density function

$$f(x:\theta)=\theta \exp[\theta x] \ , \ \theta>0, \ x>0 \ .$$

$T_x=\sum_{i=1}^{n} X_i$ is a sufficient statistic for the joint distribution of $X_1,\ldots,X_n$. We have

$$\hat{\theta}=n(t_x)^{-1}$$

$$|i(\hat{\theta})|^{-1/2}=n^{3/2}(t_x)^{-1}$$

$$C^{-n}(\hat{\theta})=(t_x/n)^{n}$$

Using (4.1.7), the Saddle-Point Approximation to the density function of T_x is

$$f(t_x:\theta)\cong \frac{(\hat{\theta})^{n}\ t_x^{n-1}\exp[-\hat{\theta}t_x]}{(2\pi)^{1/2}\ n^{(n-1/2)}\exp[-n]} \qquad (4.1.8)$$

which is the exact density function of T_x except for the denominator part which is an approximation to $\pi(n)$ according to Stirling's formula. The negative exponential distribution belongs to a one parameter exponential family. Therefore we could have obtained the exact distribution of T_x which is Gamma (n, θ) .

Example II. The Gamma Distribution With Unknown Parameters

Let $X_1,\ldots,X_n$ be a random sample from a Gamma distribution with the density

$$f(x:p,\gamma)=(\gamma)^p \exp[(p-1)\mathrm{Log}(x) - \gamma x]\ \pi^{-1}(p)$$

where, $x>0$, $p>0$, $\gamma>0$. We assume that p is large so that the Stirling's Approximation to $\pi(p)$ can be used.

$T_x=(T_{1x}=\sum_{i=1}^{n} X_i\ ,\ T_{2x}=\sum_{i=1}^{n} \mathrm{Log}(X_i)\ )$ is a two-dimensional sufficient statistic for the joint distribution of $X_1,\ldots,X_n$ and we have

$$\hat{p}=.5[\mathrm{Log}(t_{1x}/n) - t_{2x}/n]^{-1}$$

$$\hat{\gamma}=(n\hat{p})/t_{1x}$$

$$|i(\hat{p},\hat{\gamma})|^{-1/2} = (2\hat{p})^{1/2}\ \hat{\gamma}\ /n$$

$$C^{-n}(\hat{p},\hat{\gamma})= (\pi(\hat{p})/\hat{\gamma}^{\hat{p}})^n \quad .$$

Using (4.1.7), $h_{tx}(.,.)$ is approximated by

$$h_{tx}(t_{1x},t_{2x}) \cong \frac{(\hat{p})^{(3-n)/2}\ (t_{1x})^{n\hat{p}-1}\ \exp[-\hat{p}t_{2x}]}{(n)^{n\hat{p}}} \quad .$$

For this problem the exact form of $h_{tx}(.,.)$ can not be obtained.

Similar arguments to the Theorem 2.2 and Lemma 2.2 will be applied to obtain UMP unbiased tests and as a result UMP unbiased acceptance regions.

Theorem 4.1 : Consider the conditional density of T_{1y} given T, (4.1.6), and the hypotheses of interest, (4.1.1).

(1) $A_1(\beta \mid t)=\{ (t_{1y},t) : t_{1y} \leq z(t) \}$ is the (1−α) level UMP unbiased acceptance region for H_o versus H_{a1} and z(t) is found

$$\int_{z(t)}^{\infty} f_o dt_{1y} = \alpha \ .$$

(2) $A_2(\beta \mid t)=\{ (t_{1y},t) : t_{1y} \geq z(t) \}$ is the (1−α) level UMP unbiased acceptance region for H_o versus H_{a2} and z(t) is found from

$$\int_{-\infty}^{z(t)} f_o dt_{1y} = \alpha \ .$$

(3) $A_3(\beta \mid t)=\{ (t_{1y},t) : z_1(t) \leq t_{1y} \leq z_2(t) \}$ is the (1−α) level UMP unbiased acceptance region for H_o versus H_{a3} and $z_1(t)$, $z_2(t)$ are found from

$$\int_{z_1(t)}^{z_2(t)} f_o dt_{1y}=1-\alpha \quad \text{and} \quad \int_{z_1(t)}^{z_2(t)} t_{1y} f_o dt_{1y}=(1-\alpha) \int_{-\infty}^{\infty} t_{1y} f_o dt_{1y} \ ,$$

where f_o is the conditional distribution of T_{1y} given T , under H_o , which can be obtained by letting $\beta=0$ in (4.1.6).

If there exists an ancillary statistic, $V=G(T_{1y},T)$ for the joint distribution of (T_{1y},T), under H_o , then by Basu's Theorem, V and T are independent and distribution of V under H_o can be used to obtain the UMP unbiased acceptance regions.

<u>Theorem</u> <u>4.2</u>: Suppose there exists an ancillary statistic, $V=G(T_{1y},T)$ for the joint distribution of (T_{1y},T), under H_o. Let $f_{Ho}(v)$ denote the density of V under H_o.

(1) $A_1(\beta)=\{ v : v \leq z \}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for testing H_o versus H_{a1} and z is found from

$$\int_z^\infty f_{Ho}(v)dv = \alpha ,$$

provided V is increasing in T_{1y} for each t .

(2) $A_2(\beta)=\{ v : v \geq z \}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for testing H_o versus H_{a2} and z is found from

$$\int_{-\infty}^z f_{Ho}(v)dv = \alpha ,$$

provided V is increasing in T_{1y} for each t .

(3) $A_3(\beta)=\{ v : z_1 \leq v \leq z_2 \}$ is the $(1-\alpha)$ level UMP unbiased acceptance region for H_o versus H_{a3} and z_1, z_2 are found from

$$\int_{z_1}^{z_2} f_{Ho}(v)dv=(1-\alpha) \quad \text{and} \quad \int_{z_1}^{z_2} vf_{Ho}(v)dv=(1-\alpha)\int_{-\infty}^{\infty} vf_{Ho}(v)dv$$

provided, $V=G(T_{1y},T)=a(T)T_{1y}+b(T)$, $a(t)>0$.

4.2. Obtaining The Prediction Intervals

A. Ancillary Statistic Does Not Exist

Since the conditional density of T_{1y} given t, (4.1.6), has a monotone likelihood ratio in T_{1y}, therefore according to Lemma 2.4, the acceptance regions in Theorem 4.1 have more probability under H_o than under the corresponding alternative hypotheses. Similar to Chapter II, we use $A_i(\beta \mid t)$ to obtain prediction intervals. Define

$$S_i(t_x)=\{ t_{1y} : (t_{1y},t) \varepsilon A_i(\beta \mid t) \}, \ i=1,2,3 \quad .$$

That is, the acceptance regions must be solved for t_{1y} by writing $t=(t_{1y}+t_{1x},t_{2y}+t_{2x},\ldots,t_{ky}+t_{kx})$,to obtain the corresponding prediction intervals. We get

$$\begin{aligned} &S_1(t_x)=(-\infty,U(t_x)] \\ &S_1(t_x)=[L(t_x),\infty) \qquad (4.2.1) \\ &S_1(t_x)=[L(t_x),U(t_x)] \end{aligned}$$

provided z(.) functions in the acceptance regions are increasing. Note that t can be written as function of t_{1y} and t_x. Using (4.1.6) and a similar argument to Lemma 2.6, a sufficient condition for z(.) to be increasing is that

$$\frac{h_{tx}(t_1^*-t_{1y},w_2^*,\ldots,w^*_k)}{h_{tx}(t_1-t_{1y},w_2,\ldots,w_k)}$$

must be nondecreasing in t_{1y} , whenever $t_1^*>t_1$,...., $t_k^*>t_k$, where $w_j=t_j - K_j(t_{1y})$, $t_{jy}=K_j(t_{1y})$ and $w_j^*=t_j^* - K_j(t_{1y})$, $j=2,3,\ldots,k$, for some functions $K_j(.)$ (See the density function in (4.1.6)). L(.) and U(.) are function of $t_x=(t_{1x},\ldots,t_{kx})$, n and α .

The prediction intervals in (4.2.1) are obtained from the $(1-\alpha)$ level UMP unbiased acceptance regions and they have similar properties to the ones mentioned in Chapter II.

(1) $E_{(\theta 1x,\gamma)}[\ P_{(\theta 1y,\gamma)}\ \{\ S_i(T_x)\ \}\ ]=1-\alpha$,

for all $\theta_{1y}=\theta_{1x}$ and all $\gamma'=(\theta_2,\ldots,\theta_k)$. That is, $S_i(T_x)$, i=1,2,3 are 'Similar Mean Coverage Tolerance Prediction Regions' of cover $(1-\alpha)$, Aitchison and Dunsmore, (1975).

(2) $E_{(\theta 1x,\gamma)}[\ P_{(\theta 1y,\gamma)}\ \{\ S_1(T_x)\ \}\ ]\leq 1-\alpha$,

for all $\theta_{1y} > \theta_{1x}$ and all $\gamma'=(\theta_2,\ldots,\theta_k)$.

$E_{(\theta 1x,\gamma)}[\ P_{(\theta 1y,\gamma)}\ \{\ S_2(T_x)\ \}\ ]\leq 1-\alpha$,

for all $\theta_{1y} < \theta_{1x}$ and all $\gamma'=(\theta_2,\ldots,\theta_k)$.

$$E_{(\theta 1x,\acute{\gamma})}[\ P_{(\theta 1y,\acute{\gamma})}\ \{\ S_3(T_x)\ \}\ ] \le 1-\alpha \ ,$$

for all $\theta_{1y} \neq \theta_{1x}$ and all $\acute{\gamma}=(\theta_2,\ldots,\theta_k)$.

(3) $S_i(T_x)$ are 'Uniformly Most Accurate Unbiased' (UMAU), Prediction intervals. (See Definition 2.6.)

B. Ancillary Statistic Exists

In Theorem 4.2 we assumed that the ancillary statistic, $V=G(T_{1y},T)$, is an increasing function of T_{1y} for any T. Since the conditional distribution of T_{1y} given T (4.1.6), has a monotone likelihood ratio in T_{1y}, then the conditional distribution of V given t also has a monotone likelihood ratio in V for each t. By Lemma 2.4, the acceptance regions in Theorem 4.2 have more probability under H_o than under the corresponding alternative hypotheses and they will be used to derive the prediction intervals. Define

$$S_i(t_x)=\{\ t_{1y} : v \in A_i(\beta)\ \},\ i=1,2,3 \ .$$

We have

$$S_1(t_x)=(-\infty,U(t_x)]$$
$$S_1(t_x)=[L(t_x),\infty) \qquad (4.2.2)$$
$$S_1(t_x)=[L(t_x),U(t_x)] \ .$$

That is, $S_i(t_x)$ is obtained by writing

$$t=(t_{1x}+t_{1y},t_{2x}+t_{2y},\ldots,t_{kx}+t_{ky})$$

in $V=G(T,T_{1y})$ and solving $A_i(\beta)$ for t_{1y}. Acceptance regions will map onto the same form of prediction intervals, provided $G^{-1}(.)$ is increasing in T_{1y}. The prediction intervals in (4.2.2) are obtained by using $(1-\alpha)$ level UMP unbiased acceptance regions and therefore have the same properties as the prediction intervals in (4.2.1).

V. EXAMPLES: K-PARAMETER CASE (K=2)

As examples of K-parameter exponential families, the Normal and the Inverse-Gaussian distributions will be considered and the method of Chapter IV will be used to obtain the (1−α) level UMAU prediction intervals for a future outcome.

5.1. The Normal Distribution With Unknown Parameters

Let $X_1,\dots,X_n$ be a random sample from a normal distribution with the density function

$$f(x:\mu_x,\sigma^2)=(2\pi\sigma^2)^{-1}\ \exp[-(2\sigma^2)^{-1}(x-\mu_x)]\ ,\ \mu_x\ \varepsilon\ R,\ x>0,\ \sigma>0$$

and Y be independent of the $X_i,(i=1,2,\dots,n)$ with the density function

$$f(y:\mu_y,\sigma^2)=(2\pi\sigma^2)^{-1}\ \exp[-(2\sigma^2)^{-1}(y-\mu_y)]\ ,\ \mu_y\ \varepsilon\ R,\ x>0,\ \sigma>0\ .$$

We assume that μ_x, μ_y and σ are unknown parameters and prediction intervals for Y are of interest. To get a natural parametrization let

$\theta_{1x}=\mu_x\sigma^{-2}$, $\theta_{1y}=\mu_y\sigma^{-2}$, $\gamma=-(2\sigma^2)^{-1}$, $T_{1x}=\sum_{i=1}^{n} X_i$, $T_{2x}=\sum_{i=1}^{n} X_i^2$, $T_{1y}=Y$, $T_{2y}=Y^2$, $T_j=T_{jx}+T_{jy}$,$(j=1,2)$, then the joint density of $(\underline{X},Y)$ can be written as

$$f(\underline{x},y:\theta_{1x},\theta_{1y},\gamma)=(-\pi/\gamma)^{-(n+1)/2}\exp[(n\theta_{1x}^2+\theta_{1y}^2)/4\gamma]$$
$$\exp[\theta_{1y}t_{1y}+\theta_{1x}t_{1x}+\gamma t_2]\qquad .\qquad (5.1.1)$$

To simplify the problem let

$$\beta=(1/n+1)^{-1}(\theta_{1y}-\theta_{1x})$$

and

$$\theta=(n+1)^{-1}(n\theta_{1x}+\theta_{1y}) \quad .$$

Then (5.1.1) is equivalent to

$$f(\underline{x},y:\beta,\gamma,\theta)=C(\beta,\gamma,\theta)\exp[\beta(t_{1y}-t_{1x}/n)+\theta t_1+\gamma t_2] \qquad (5.1.2)$$

where C(.,..,.) is a function of parameters, β , γ and θ.

Under H_o : $\mu_y=\mu_x$ $(\theta_{1y}=\theta_{1x})$, (T_1,T_2) are jointly complete and sufficient statistics for (θ,γ) .

Define

$$U=T_{1y}-T_{1x}/n$$

$$V=G(U,T)=BU[T_2-(n+1)^{-1}(T_1^2+nU^2)]^{-1} \qquad (5.1.3)$$

$$B=(n(n-1)/(n+1))^{1/2}$$

V is increasing in U and it can be shown that under H_o it has student's t-distribution with (n-1) degrees of freedom. Therefore V is an ancillary statistic for the joint distribution of (U,T_1,T_2) when H_o is true, and conditions of Theorem 4.2 (1), (2) are satisfied. We have

(1) $\mu_y=\mu_x$ iff $\theta_{1y}=\theta_{1x}$ iff $\beta=\beta_o$

$\mu_y>(<)\mu_x$ iff $\theta_{1y}>(<)\theta_{1x}$ iff $\beta>(<)\beta_o$, $\beta_o=0$.

(2) (5.1.2) implies that the conditional distribution of U given

$t=(t_1,t_2)$ has a monotone likelihood ratio. Since V is increasing in U, then the conditional distribution of V given T also has a monotone likelihood ratio. Thus acceptance regions based on V need to be used to obtain prediction intervals.

(3) Distribution of V is symmetric about zero. Therefore the remark given in section 2.2.1 applies in this case.

<u>One Sided Upper Limit Prediction Interval</u>

(1) and (2) imply that the acceptance region, $A_1(\beta)=\{\ v\ :v\leq z\ \}$ must be used. According to Theorem 4.2, z is found from $\int_z^\infty f_{Ho}(v)dv=\alpha$, $f_{Ho}(v)$ is the density of t-distribution with (n-1) degrees of freedom. We have

$$V \varepsilon A_1(\beta) \quad \text{iff} \quad T_{1y} \leq t_{1x}/n + z\{(n+1)Q/n(n-1)\}^{1/2}$$

where $Q=\sum_{i=1}^{n} (x_i-\bar{x})^2$.

<u>One Sided Lower Limit Prediction Interval</u>

A similar argument to the above one holds. Using $A_2(\beta)$ in Theorem 4.2 we get

$$S_2(t_x)=[t_{1x}/n + z\{(n+1)Q/n(n-1)\}^{1/2},\infty)$$

where z is α percentile of t-distribution with (n-1) degrees of freedom.

Two Sided Prediction Interval

The Statistic V in (5.1.3) is not a linear function of U. Therefore the condition of Theorem 4.2 (3) is not satisfied. Define

$$W=U[T_2-T_1^2/(n+1)]^{-1/2} \quad ,$$

W is linear in U for each $t=(t_1,t_2)$ and is related to V by

$$V=BW[1-\ nW^2/(n+1)]^{-1/2}$$

where $$B=[n(n-1)/(n+1)]^{1/2} \quad .$$

Under H_o , W is independent of T and is an ancillary statistic because it is a function of another ancillary statistic, V. The condition of Theorem 4.2 (3) satisfies for W. But for hypotheses testing purposes, W and V are equivalent test statistics, for V is an increasing function of W. Therefore the acceptance regions associated with the two tests are equivalent and as a result , $A_3(\beta)=\{\ v:\ z_1 \leq v \leq z_2\ \}$ can be used where, z_1 and z_2 are $\alpha/2$ and $(1-\alpha/2)$ percentile of t-distribution with $(n-1)$ degrees of freedom, for the distribution of V is symmetric about zero when H_o is true. Using $A_3(\beta)$, we have

$$S_3(t_x)=[t_{1x}/n + z_1\{(n+1)Q/n(n-1)\}^{1/2} \ ,$$
$$t_{1x}/n + z_2\{(n+1)Q/n(n-1)\}^{1/2}] \ .$$

5.2. The Inverse-Gaussian Distribution With Unknown Parameters

The Inverse-Gaussian distribution or first passage time distribution which has a skewed and unimodel density function is a two parameter exponential family. Chhikara (1975), discusses hypothesis testing in single and two sample cases for the Inverse-Gaussian distribution. Chhikara and Folks (1975), discuss sampling distribution and statistical inference related to this distribution. A very useful background about this distribution can be found in a paper by Folks and Chhikara (1978), in which they present test of hypothesis, estimation, confidence interval, regression and analysis of variance based on the Inverse-Gaussian distribution. For application, the distribution has been considered as a model for emptiness of dam by Hasofer (1964), and Lancaster (1972), applied it as a model for duration of strikes. When no obvious choice of distribution for a data with considerable skewness is suggested, Chhikara and Folks (1978), suggest the Inverse-Gaussian over the Lognormal distribution.

Let $X_1,\ldots,X_n$ be a random sample from an Inverse-Gaussian distribution with the density function

$$f(x:\mu_x,\theta)=[\theta/(2\pi x^3)]^{1/2} \exp[-\theta(x-\mu_x)^2/(2\mu_x^2 x)] \ ,$$

$$x>0,\ \theta>0,\ \mu_x>0$$

and Y be independent of the $X_i,(i=1,2,\ldots,n)$ with the density function

$$f(y:\mu_y,\theta)=[\theta/(2\pi y^3)]^{1/2}\exp[-\theta(y-\mu_y)^2/(2\mu_y{}^2y)] ,$$
$$y>0,\ \theta>0,\ \mu_y>0 .$$

We assume that μ_x , μ_y and θ are unknown parameters and prediction intervals for Y based on the outcomes of $X_1,\ldots,X_n$, are of interest. Let

$$T_{1x}=\sum_{i=1}^{n} X_i ,\quad T_{1y}=Y ,\quad T_{2x}=\sum_{i=1}^{n} X_i^{-1} ,\quad T_{2y}=Y^{-1} , \qquad (5.2.1)$$
$$\theta_{1x}=-(\theta/2)\mu_x{}^{-2} ,\ \theta_{1y}=-(\theta/2)\mu_y^{-2} ,\ \gamma=-\theta/2 ,\ T_j=T_{jx}+T_{jy}\ (j=1,2) ,$$
$$\beta=[n/(n+1)][\theta_{1y}-\theta_{1x}] ,\ \lambda=(n\theta_{1x}+\theta_{1y})(n+1)^{-1} \text{ and } U=T_{1y}-T_{1x}/n .$$

The joint density function of $(\underline{X},Y)$ can be written as

$$f(\underline{x},y:\beta,\lambda,\gamma)=C(\beta,\lambda,\gamma)\exp[\beta u+\lambda t_1+\gamma t_2][\textstyle\prod_{i=1}^{n} (x_i)\ y]^{-3/2} \qquad (5.2.2)$$

where $C(.,.,.)$ is a function of parameters, γ , λ and β. The joint distribution of (U,T_1,T_2) is a three parameter exponential family and Under $H_o:\mu_y=\mu_x\ (\theta_{1y}=\theta_{1x})$, $T=(T_1,T_2)$ is a complete sufficient statistic for (λ,γ) . Therefore the conditional distribution of U given $t=(t_1,t_2)$ is a one parameter exponential.

Define

$$W=[(n(n+1)]^{1/2}\ (n+1)U\ [A-n(n+1)^2U^2]^{-1/2}$$

where $$A=(T_1T_2-(n+1)^2)(T_1+U)(T_1-nU) .$$

According to Chhikara (1975), the conditional density of W given $t=(t_1,t_2)$ under H_o, is

$$f_{Ho}(w|t_1,t_2)=(n-1)^{-1/2}(beta(1/2,(n-1)/2))^{-1}(1+w^2/(n-1))^{-n/2} \quad (5.2.3)$$
$$[1+ (n-1)(n+1)^{-1} w(t_1t_2-(n+1)^2)^{1/2} [(t_1t_2-(n-1)^2)w^2+4n(n-1)]^{-1/2}]$$

where beta(.,.) is the beta function and $-\infty<w<\infty$.

W is nondecreasing in U. But W is not ancillary statistic for the joint distribution of (U,T_1,T_2) when H_o is true, for the density function in (5.2.3) depends on t_1 and t_2 (Basu's Theorem). Therefore, $f_{Ho}(w|t_1,t_2)$ need to be used to get the UMP unbiased acceptance regions (Theorem 4.1). We have

(1) $\mu_y=\mu_x$ iff $\theta_{1y}=\theta_{1x}$ iff $\beta=\beta_o$
$\mu_y>(<)\mu_x$ iff $\theta_{1y}>(<)\theta_{1x}$ iff $\beta>(<)\beta_o$, $\beta_o=0$.

(2) U is increasing in T_{1y} and the conditional distribution of U given $t=(t_1,t_2)$ has a monotone likelihood ratio, because it is one parameter exponential. Since W is increasing in U, then the conditional distribution of W given t also has a monotone likelihood ratio. Thus the acceptance regions based on W need to be used to obtain prediction intervals. (Lemma 2.6.)

(3) W is not an ancillary statistic.

One sided Prediction Interval

To get one sided upper limit prediction interval, (1) and (2) imply that the acceptance region $A_1(\beta|t)=\{ w : w\le z(t) \}$ need to be used, where from Theorem 4.1 we have

$$\int_{z(t)}^{\infty} f_{Ho}(w|t_1,t_2)dw=\alpha \quad . \qquad (5.2.4)$$

The expression, (5.2.4) is equivalent to

$$F_{st}[z(t)]-(n-1)(n+1)^{-1} N^{(n-2)/2} \{1-F_{st}[z^*(t)]\}=1-\alpha \qquad (5.2.5)$$

where

$$N=(t_1t_2-(n-1)^2)/(t_1t_2-(n+1)^2)$$

$$z^*(t)=[z^2(t)+4n(n-1)]/[t_1t_2-(n-1)^2]^{1/2} \quad .$$

$F_{st}(.)$ is the cumulative distribution of the Student's t-distribution with (n-1) degrees of freedom, Chhikara (1975). From (5.2.5) we can not find z . This fact is mentioned in a paper by Chhikara and Guttman (1982), where they show that one sided prediction intervals can be obtained if 'Bayesian informative prediction' approach is used.

A similar argument to the above one holds when the lower limit prediction interval is of interest.

Two sided Prediction Interval

To find the two sided prediction interval, according to

Theorem 4.1 (3), we need to solve

$$\int_{z_1(t)}^{z_2(t)} f_{Ho}(w \mid t)dw = (1-\alpha)$$

and

$$\int_{z_1(t)}^{z_2(t)} wf_{Ho}(w \mid t)dw = (1-\alpha) \quad \int_{-\infty}^{\infty} wf_{Ho}(w \mid t)dw \ .$$

The function $L(w)=wf_{Ho}(w|t)$ is an odd function of w and is symmetric about w=0. Therefore

$$\int_{-\infty}^{\infty} wf_{Ho}(w \mid t)dw = 0$$

and as a result $z_1(t)=-z_2(t)$. Based on this fact and (5.2.5) we conclude that $z_1(t)$ and $z_2(t)$ must be found from

$$F_{st}[z_2(t)]-F_{st}[z_1(t)]=1-\alpha \quad . \qquad (5.2.6)$$

That is, $z_1(t)$, $z_2(t)$ are independent of t and are $\alpha/2$ and $(1-\alpha/2)$ percentiles of t-distribution with (n-1) degrees of freedom, respectively.

According to Theorem 4.1, we need to use the acceptance region $A_3(\beta|t)=\{ w: z_1 \leq w \leq z_2 \}$. But W^2 under H_o has F-distribution with (1,n-1) degrees of freedom, Chhikara, (1975). Therefore

$$z_1 \leq w \leq z_2 \qquad \text{iff} \qquad w^2 \leq m \,,$$

where m is $(1-\alpha)$ percentile of F-distribution with $(1,n-1)$ degrees of freedom. Hence instead of the acceptance region, $A_3(\beta \mid t)$, we can work with the equivalent region which is based on F-distribution.

According to Chhikara and Guttman (1982), inverting $w^2 \leq m$ provides the two sided prediction interval for $T_{1y}=Y$,

$$S_3(t_x)=\{[n/t_{1x} +(nAm/2(n-1))] \pm$$
$$[(n(n+1)Am)/(n-1)t_{1x})+(nAm)^2/4(n-1)^2]^{1/2}\}^{-1}$$

where

$$A=\sum_{i=1}^{n} (1/x_i - x^{-1}) \;.$$

They mention that there is a positive probability to get a negative real number for the lower limit of $S_3(t_x)$, and since $T_{1y}>0$, the solution must be restricted to the positive real line.

1. If $((n-1)-t_{1x}Am/n)>0$, we get two sided prediction interval.
2. If $((n-1)-t_{1x}Am/n)<0$, lower limit will be obtained using + sign in $S_3(t_x)$ and upper limit would be ∞ .

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APPENDICES

APPENDIX A

The following program uses 'The Generalized Newton's Method' and MDBETA routine of IMSL library to find values of z_1 and z_2 for the two sided prediction interval, when the distribution of informative and future experiments are normal with known mean, μ=0. (Example 3.1.)

```
      PROGRAM A
      REAL S,R,XO,YO,AL,A,B,C,GX,GY,HX,HY,J,P1,P2
      REAL P3,P4,HXY,GXY,TOL,X1,Y1,L,K
      INTEGER SI,RI
      REAL*8 G1,G2,G3,G4,G5,DGAMMA
      WRITE(*,*) 'VALUES OF AL,R,XO,YO ARE='
      READ(*,'(F4.2,1X,F4.2,1X,F10.8,1X,F10.8)') AL,R,XO,YO
      DO 60 N=2,100
      S=N*R
      A=R+S
      B=R+S+1
      C=R+1
      G1=DGAMMA(R)
      G2=DGAMMA(S)
      G3=DGAMMA(C)
      G4=DGAMMA(A)
      G5=DGAMMA(B)
      SI=INT(S)
      RI=INT(R)
 20   GX=(-G5/(G3*G2))*((XO**RI)*((1-XO)**(SI-1)))
      GY=(G5/(G3*G2))*((YO**RI)*((1-YO)**(SI-1)))
      HX=(-G4/(G1*G2))*((XO**(RI-1))*((1-XO)**(SI-1)))
      HY=(G4/(G1*G2))*((YO**(RI-1))*((1-YO)**(SI-1)))
      J=GX*HY-HX*GY
      CALL MDBETA (YO,R,S,P1,IER)
      CALL MDBETA (XO,R,S,P2,IER)
      CALL MDBETA (YO,C,S,P3,IER)
      CALL MDBETA (XO,C,S,P4,IER)
      HXY=P1-P2-(1-AL)
      GXY=P3-P4-(1-AL)
      X1=XO+(HXY*GY-GXY*HY)/J
      Y1=YO+(GXY*HX-HXY*GX)/J
      TOL=1.E-3
      K=ABS(GXY)
```

```
      L=ABS(HXY)
      IF(K.GT.TOL) GO TO 25
      IF(L.GT.TOL) GO TO 25
      GO TO 30
25    XO=X1
      YO=Y1
      GO TO 20
30    WRITE (*,40) N,XO,YO
40    FORMAT(10X,I2,4X,F8.6,4X,F8.6)
60    CONTINUE
      END
```

TABLE A

Values of z_1 and z_2 to find (1-α) level prediction interval $S_3(t_x)=[t_x z_1/(1-z_1), t_x z_2/(1-z_2)]$, where $t_x=\sum_{i=1}^{n} x_i^2$ and $X_1,...,X_n$ are (iid) normal with mean, μ=0 . (ꞌxample 3.1.)

n	α=.01 z_1	α=.01 z_2	α=.05 z_1	α=.05 z_2
2	.000044	.993295	.001158	.966407
3	.000043	.969367	.000878	.902746
4	.000035	.926041	.000701	.831825
5	.000029	.878583	.000582	.764553
6	.000025	.830216	.000497	.704107
7	.000022	.783645	.000433	.650803
8	.000019	.740034	.000384	.604028
9	.000017	.699760	.000344	.562940
10	.000016	.662814	.000312	.526716
11	.000014	.629010	.000285	.494631
12	.000013	.598095	.000263	.466066
13	.000012	.569798	.000243	.440505
14	.000011	.543853	.000227	.417517
15	.000011	.520017	.000212	.396752
16	.000010	.498064	.000200	.377912
17	.000009	.477817	.000188	.360749
18	.000009	.459083	.000178	.345050
19	.000008	.441685	.000169	.330635
20	.000008	.425523	.000161	.317362
21	.000008	.410471	.000153	.305102
22	.000007	.396431	.000147	.293746
23	.000007	.383293	.000140	.283197
24	.000007	.370973	.000135	.273370
25	.000006	.359406	.000129	.264196
26	.000006	.348539	.000124	.255619
27	.000006	.338277	.000120	.247570
28	.000006	.328625	.000116	.240018
29	.000006	.319459	.000112	.232900
30	.000005	.310813	.000108	.226197
31	.000005	.302601	.000105	.219862
32	.000005	.294814	.000102	.213874
33	.000005	.287421	.000099	.208204
34	.000005	.280380	.000096	.202822
35	.000005	.273668	.000093	.197710
36	.000005	.267281	.000090	.192852

n	α=.01 z_1	α=.01 z_2	α=.05 z_1	α=.05 z_2
37	.000004	.261161	.000088	.188222
38	.000004	.255333	.000086	.183811
39	.000004	.249732	.000084	.179596
40	.000004	.244392	.000082	.175576
41	.000004	.239275	.000080	.171730
42	.000004	.234358	.000078	.168048
43	.000004	.229628	.000076	.164517
44	.000004	.225112	.000074	.161136
45	.000004	.220753	.000073	.157890
46	.000004	.216576	.000071	.154774
47	.000003	.212514	.000069	.151770
48	.000003	.208620	.000068	.148884
49	.000003	.204877	.000067	.146108
50	.000003	.201254	.000065	.143431
51	.000003	.197755	.000064	.140851
52	.000003	.197755	.000063	.138357
53	.000003	.191112	.000062	.135958
54	.000003	.187970	.000061	.133640
55	.000003	.184937	.000059	.131401
56	.000003	.181953	.000058	.129227
57	.000003	.179085	.000057	.127126
58	.000003	.179085	.000056	.125096
59	.000003	.173645	.000055	.123132
60	.000003	.171017	.000055	.121222
61	.000003	.171017	.000054	.119371
62	.000003	.166026	.000053	.117579
63	.000003	.163641	.000052	.115840
64	.000003	.161326	.000051	.114152
65	.000003	.161326	.000050	.112507
66	.000002	.156861	.000050	.110913
67	.000002	.156861	.000049	.109362
68	.000002	.152659	.000048	.107857
69	.000002	.150631	.000048	.106389
70	.000002	.148661	.000047	.104961
71	.000002	.148661	.000046	.103574
72	.000002	.144877	.000046	.102220
73	.000002	.143045	.000045	.100900
74	.000002	.143045	.000044	.099616
75	.000002	.139548	.000044	.098366
76	.000002	.137821	.000043	.097137
77	.000002	.137821	.000043	.095948
78	.000002	.134565	.000042	.094785
79	.000002	.133022	.000042	.093658
80	.000002	.133022	.000041	.092546
81	.000002	.129951	.000041	.091460
82	.000002	.128491	.000040	.090407

n	α=.01 z_1	α=.01 z_2	α=.05 z_1	α=.05 z_2
83	.000002	.128491	.000040	.089373
84	.000002	.125680	.000039	.088369
85	.000002	.124273	.000039	.087376
86	.000002	.122933	.000038	.086412
87	.000002	.122933	.000038	.085461
88	.000002	.120335	.000037	.084545
89	.000002	.120335	.000037	.083638
90	.000002	.117838	.000036	.082755
91	.000002	.117838	.000036	.081885
92	.000002	.115407	.000036	.081034
93	.000002	.114282	.000035	.080210
94	.000002	.114282	.000035	.079394
95	.000002	.112046	.000035	.078602
96	.000002	.112046	.000034	.077808
97	.000002	.109838	.000034	.077045
98	.000002	.109838	.000034	.076293
99	.000002	.107786	.000033	.075563
100	.000002	.106778	.000033	.074840

n	α=.10 z_1	α=.10 z_2	α=.20 z_1	α=.20 z_2
2	.004301	.932366	.016681	.863320
3	.003332	.843863	.012584	.748045
4	.002651	.759054	.009963	.653086
5	.002197	.685161	.008234	.577110
6	.001872	.622300	.007005	.515941
7	.001630	.568951	.006092	.465995
8	.001442	.523445	.005391	.424597
9	.001293	.484338	.004831	.389800
10	.001172	.450456	.004377	.360181
11	.001071	.420869	.003999	.334686
12	.000987	.394836	.003682	.312521
13	.000914	.371773	.003413	.293082
14	.000852	.351209	.003179	.275901
15	.000797	.332769	.002976	.260609
16	.000749	.316147	.002797	.246914
17	.000706	.301089	.002636	.234579
18	.000668	.287386	.002494	.223412
19	.000634	.274862	.002367	.213254
20	.000604	.263376	.002252	.203977
21	.000576	.252806	.002148	.195470
22	.000550	.243048	.002052	.187643
23	.000527	.234010	.001965	.180417

n	α=.10 z_1	α=.10 z_2	α=.20 z_1	α=.20 z_2
24	.000505	.225616	.001885	.173725
25	.000486	.217800	.001812	.167511
26	.000467	.210507	.001743	.161726
27	.000450	.203682	.001680	.156324
28	.000434	.197288	.001621	.151273
29	.000420	.191277	.001566	.146536
30	.000406	.185623	.001516	.142087
31	.000393	.180291	.001467	.137899
32	.000381	.175257	.001423	.133951
33	.000370	.170497	.001380	.130223
34	.000359	.165987	.001340	.126696
35	.000349	.161707	.001303	.123355
36	.000339	.157645	.001267	.120186
37	.000330	.153779	.001233	.117174
38	.000322	.150099	.001201	.114311
39	.000314	.146588	.001171	.111582
40	.000306	.143240	.001142	.108982
41	.000298	.140042	.001114	.106500
42	.000292	.136982	.001088	.104129
43	.000285	.134051	.001063	.101860
44	.000278	.131246	.001039	.099689
45	.000272	.128555	.001016	.097608
46	.000266	.125973	.000995	.095612
47	.000261	.123488	.000974	.093695
48	.000255	.121102	.000954	.091854
49	.000250	.118807	.000935	.090084
50	.000245	.116596	.000916	.088381
51	.000241	.114466	.000898	.086741
52	.000236	.112410	.000881	.085160
53	.000232	.110430	.000865	.083637
54	.000227	.108519	.000849	.082167
55	.000223	.106673	.000833	.080749
56	.000219	.104885	.000819	.079377
57	.000215	.103157	.000804	.078051
58	.000212	.101487	.000791	.076769
59	.000208	.099871	.000777	.075530
60	.000205	.098302	.000765	.074328
61	.000201	.096782	.000752	.073164
62	.000198	.095310	.000740	.072037
63	.000195	.093883	.000729	.070944
64	.000192	.092497	.000717	.069883
65	.000189	.091149	.000706	.068853
66	.000186	.089842	.000696	.067853
67	.000184	.088571	.000685	.066882
68	.000181	.087337	.000675	.065939
69	.000178	.086136	.000666	.065021

n	α=.10 z_1	α=.10 z_2	α=.20 z_1	α=.20 z_2
70	.000176	.084967	.000656	.064129
71	.000173	.083831	.000647	.063262
72	.000171	.082723	.000638	.062417
73	.000169	.081644	.000629	.061594
74	.000166	.080593	.000621	.060792
75	.000164	.079571	.000613	.060013
76	.000162	.078569	.000605	.059250
77	.000160	.077596	.000597	.058509
78	.000158	.076646	.000589	.057785
79	.000156	.075723	.000582	.057081
80	.000154	.074816	.000575	.056391
81	.000152	.073932	.000568	.055718
82	.000150	.073071	.000561	.055063
83	.000148	.072228	.000554	.054421
84	.000147	.071407	.000548	.053796
85	.000145	.070599	.000541	.053183
86	.000143	.069813	.000535	.052584
87	.000142	.069039	.000529	.051998
88	.000140	.068290	.000523	.051427
89	.000138	.067552	.000517	.050867
90	.000137	.066832	.000511	.050320
91	.000135	.066124	.000506	.049783
92	.000134	.065432	.000500	.049257
93	.000133	.064759	.000495	.048745
94	.000131	.064095	.000490	.048242
95	.000130	.063448	.000484	.047750
96	.000128	.062806	.000479	.047264
97	.000127	.062184	.000474	.046792
98	.000126	.061573	.000470	.046328
99	.000125	.060976	.000465	.045875
100	.000123	.060388	.000460	.045429

APPENDIX B

The following program uses the 'Generalized Newton's Method' to find values of z_1 and z_2 for the two sided prediction interval, when the distribution of informative and future experiments are Negative Exponential. (Example 3.4.)

```
      PROGRAM B
      REAL  K,L,AL,XO,YO,X1,Y1,GX,GY,HX,HY,GXY,HXY,A,TOL
      INTEGER  N,I
      WRITE  (*,*)'VALUES OF AL,XO,YO ARE= '
      READ (*,'(F4.2,1X,F8.6,1X,F8.6)') AL,XO,YO
      DO 60 N=2,100
 20   GX=-N*((1-XO)**(N-1))
      GY=N*((1-YO)**(N-1))
      HX=((1-XO)**(N-1))*(1-XO-N*XO)
      HY=((1-YO)**(N-1))*(N*YO-(1-YO))
      A=GX*HY-HX*GY
      GXY=((1-XO)**N)-((1-YO)**N)-(1-AL)
      HXY=XO*((1-XO)**N)-YO*((1-YO)**N)
      X1=XO+(HXY*GY-GXY*HY)/A
      Y1=YO+(GXY*HX-HXY*GX)/A
      TOL=1.E-5
      K=ABS(GXY)
      L=ABS(HXY)
      IF(K.GT.TOL) GO TO 25
      IF(L.GT.TOL) GO TO 25
      GO TO 30
 25   XO=X1
      YO=Y1
      GO TO 20
 30   WRITE(*,40) N,XO,YO
 40   FORMAT(4X,I3,4X,F8.6,4X,F8.6)
 60   CONTINUE
      END
```

TABLE B

Values of z_1 and z_2 to find $(1-\alpha)$ level prediction interval $S_3(t_x)=[t_x z_1/(1-z_1), t_x z_2/(1-z_2)]$, where $t_x=\sum_{i=1}^{n} x_i$ and $X_1,\ldots,X_n$ are (iid) negative exponential. (³xample 3.3.)

	α=.01		α=.05	
n	z_1	z_2	z_1	z_2
2	.003276	.941193	.016119	.865752
3	.002411	.859294	.011774	.752865
4	.001900	.778032	.009247	.659005
5	.001565	.705910	.007605	.583406
6	.001330	.643602	.006455	.522246
7	.001155	.590076	.005606	.472169
8	.001021	.544232	.004953	.430557
9	.000915	.504565	.004436	.395516
10	.000828	.470030	.004017	.365644
11	.000757	.439755	.003669	.339902
12	.000697	.413029	.003378	.317502
13	.000645	.389291	.003128	.297842
14	.000601	.367885	.002914	.280453
15	.000562	.348887	.002726	.264936
16	.000528	.331708	.002562	.251068
17	.000498	.316116	.002416	.238570
18	.000471	.301908	.002286	.227251
19	.000447	.288906	.002169	.216951
20	.000426	.276962	.002063	.207541
21	.000406	.265961	.001967	.198910
22	.000388	.255790	.001880	.190965
23	.000371	.246368	.001800	.183629
24	.000356	.237605	.001727	.176833
25	.000342	.229442	.001659	.170521
26	.000329	.221823	.001597	.164644
27	.000317	.214687	.001539	.159157
28	.000306	.207992	.001485	.154023
29	.000296	.201700	.001435	.149209
30	.000286	.195780	.001388	.144687
31	.000277	.190191	.001344	.140429
32	.000269	.184914	.001302	.136416
33	.000261	.179924	.001263	.132624
34	.000253	.175190	.001227	.129038
35	.000246	.170700	.001192	.125641
36	.000239	.166437	.001160	.122416

n	α=.01 z_1	α=.01 z_2	α=.05 z_1	α=.05 z_2
37	.000233	.162375	.001129	.119354
38	.000227	.158509	.001100	.116441
39	.000221	.154827	.001072	.113666
40	.000216	.151305	.001045	.111021
41	.000210	.147943	.001020	.108496
42	.000205	.144727	.000996	.106082
43	.000201	.141644	.000973	.103773
44	.000196	.138694	.000951	.101563
45	.000192	.135859	.000931	.099446
46	.000188	.133141	.000911	.097414
47	.000184	.130532	.000891	.095465
48	.000180	.128016	.000873	.093590
49	.000176	.125600	.000855	.091789
50	.000173	.123270	.000838	.090055
51	.000170	.121032	.000822	.088386
52	.000166	.118864	.000807	.086777
53	.000163	.116780	.000791	.085227
54	.000160	.114764	.000777	.083730
55	.000157	.112713	.000763	.082285
56	.000155	.110936	.000749	.080889
57	.000152	.109023	.000736	.079539
58	.000149	.107272	.000724	.078234
59	.000147	.105571	.000712	.076971
60	.000144	.103924	.000700	.075747
61	.000142	.102325	.000689	.074562
62	.000140	.100778	.000678	.073415
63	.000138	.099270	.000667	.072300
64	.000135	.097813	.000657	.071221
65	.000133	.096396	.000647	.070173
66	.000131	.095018	.000637	.069155
67	.000129	.093676	.000627	.068165
68	.000128	.092377	.000618	.067205
69	.000126	.091110	.000609	.066270
70	.000124	.089883	.000601	.065362
71	.000122	.088681	.000592	.064478
72	.000120	.087518	.000584	.063617
73	.000119	.086380	.000576	.062779
74	.000119	.086380	.000568	.061963
75	.000116	.084230	.000561	.061167
76	.000116	.084230	.000554	.060392
77	.000113	.082149	.000546	.059638
78	.000113	.082149	.000539	.058901
79	.000110	.080166	.000533	.058181
80	.000110	.080166	.000526	.057479
81	.000107	.078274	.000520	.056794

n	α=.01 z_1	α=.01 z_2	α=.05 z_1	α=.05 z_2
82	.000107	.078274	.000513	.056125
83	.000105	.076473	.000507	.055473
84	.000105	.076473	.000501	.054834
85	.000102	.074753	.000495	.054211
86	.000102	.074753	.000490	.053601
87	.000100	.073110	.000484	.053004
88	.000100	.073110	.000479	.052422
89	.000098	.071535	.000473	.051840
90	.000098	.071535	.000468	.051293
91	.000095	.070028	.000463	.050747
92	.000095	.070028	.000458	.050213
93	.000093	.068584	.000453	.049690
94	.000093	.068584	.000448	.049177
95	.000091	.067198	.000443	.048675
96	.000091	.067198	.000439	.048182
97	.000090	.065863	.000434	.047690
98	.000090	.065863	.000430	.047228
99	.000088	.064584	.000426	.046764
100	.000088	.064584	.000421	.046309

n	α=.10 z_1	α=.10 z_2	α=.20 z_1	α=.20 z_2
2	.031962	.807394	.063480	.722374
3	.023287	.683340	.046211	.592487
4	.018275	.587757	.036275	.499781
5	.015025	.514046	.029839	.431370
6	.012752	.456108	.025336	.379105
7	.011074	.409582	.022011	.337982
8	.009785	.371498	.019457	.304826
9	.008765	.339796	.017432	.277548
10	.007936	.313020	.015789	.254723
11	.007251	.290118	.014429	.235350
12	.006674	.270313	.013284	.218704
13	.006183	.253023	.012308	.204243
14	.005758	.237783	.011465	.191577
15	.005388	.224284	.010730	.180386
16	.005063	.212229	.010084	.170428
17	.004775	.201398	.009511	.161510
18	.004518	.191615	.008999	.153476
19	.004287	.182736	.008540	.146203
20	.004078	.174641	.008125	.139587
21	.003889	.167231	.007749	.133543
22	.003717	.160423	.007406	.128000

n	α=.10 z_1	α=.10 z_2	α=.20 z_1	α=.20 z_2
23	.003559	.154147	.007092	.122898
24	.003414	.148341	.006804	.118188
25	.003280	.142957	.006538	.113824
26	.003157	.137950	.006292	.109771
27	.003042	.133280	.006064	.105997
28	.002936	.128916	.005852	.102473
29	.002837	.124828	.005654	.099176
30	.002744	.120992	.005470	.096084
31	.002657	.117384	.005296	.093179
32	.002575	.113985	.005134	.090445
33	.002498	.110777	.004981	.087866
34	.002426	.107744	.004837	.085430
35	.002358	.104873	.004701	.083126
36	.002293	.102150	.004573	.080943
37	.002232	.099566	.004451	.078871
38	.002174	.097108	.004336	.076903
39	.002119	.094769	.004226	.075030
40	.002067	.092541	.004122	.073247
41	.002017	.090414	.004023	.071546
42	.001970	.088383	.003928	.069922
43	.001925	.086441	.003838	.068371
44	.001881	.084583	.003752	.066886
45	.001840	.082803	.003670	.065465
46	.001801	.081096	.003591	.064103
47	.001763	.079458	.003516	.062797
48	.001726	.077885	.003443	.061542
49	.001692	.076372	.003374	.060337
50	.001658	.074918	.003307	.059178
51	.001626	.073518	.003243	.058063
52	.001595	.072169	.003181	.056989
53	.001565	.070869	.003122	.055954
54	.001537	.069615	.003065	.054956
55	.001509	.068404	.003010	.053993
56	.001482	.067235	.002957	.053063
57	.001457	.066105	.002905	.052164
58	.001432	.065012	.002856	.051296
59	.001408	.063955	.002808	.050456
60	.001384	.062931	.002761	.049643
61	.001362	.061940	.002717	.048855
62	.001340	.060980	.002673	.048093
63	.001319	.060049	.002631	.047353
64	.001299	.059146	.002590	.046637
65	.001279	.058270	.002551	.045941
66	.001260	.057419	.002513	.045266
67	.001241	.056593	.002475	.044611
68	.001223	.055790	.002439	.043974

	α=.10		α=.20	
n	z_1	z_2	z_1	z_2
69	.001205	.055010	.002404	.043355
70	.001188	.054251	.002370	.042753
71	.001172	.053513	.002337	.042168
72	.001155	.052795	.002305	.041599
73	.001140	.052096	.002274	.041045
74	.001124	.051414	.002243	.040505
75	.001109	.050751	.002213	.039979
76	.001095	.050104	.002185	.039467
77	.001081	.049474	.002156	.038968
78	.001067	.048859	.002129	.038481
79	.001054	.048260	.002102	.038006
80	.001041	.047675	.002076	.037543
81	.001028	.047104	.002051	.037091
82	.001015	.046546	.002026	.036650
83	.001003	.046002	.002002	.036219
84	.000991	.045470	.001978	.035798
85	.000980	.044950	.001955	.035387
86	.000968	.044442	.001932	.034985
87	.000957	.043946	.001910	.034592
88	.000947	.043460	.001889	.034208
89	.000936	.042985	.001867	.033832
90	.000926	.042520	.001847	.033464
91	.000916	.042066	.001827	.033105
92	.000906	.041620	.001807	.032753
93	.000896	.041184	.001788	.032409
94	.000887	.040757	.001769	.032071
95	.000877	.040340	.001750	.031741
96	.000868	.039930	.001732	.031417
97	.000859	.039529	.001714	.031100
98	.000851	.039135	.001697	.030789
99	.000842	.038750	.001680	.030484
100	.000834	.038371	.001663	.030185